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DIVISIBILITY OF REDUCTION IN GROUPS OF RATIONAL NUMBERS

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Abstract. Given a multiplicative group of non–zero rational numbers and a positive integer m, we consider the problem of determining the density of the set of primes p for which the order of the reduction modulo p of the group is divisible by m. In the case when the group is finitely generated the density is explicitly computed. Some example of groups with infinite rank are considered.

1. Introduction

It is a well known result due to Hasse [?] and others that the probability that 2 generates a subgroup of $\mathbb{F}_p^*$ with even order is 17/24 while the probability that 3 generates a subgroup of $\mathbb{F}_p^*$ with even order is 2/3. So, it might not be a surprise to read that the probability that 2 and 3 together generate a subgroup of $\mathbb{F}_p^*$ with even order is 195/224 and that the probability that 3 and 5 together generate a subgroup of $\mathbb{F}_p^*$ with even order is 6/7. In general, groups of rational numbers containing 2 have a slightly higher tendency, then those not containing 2, to generate subgroups of $\mathbb{F}_p^*$ with even order. This phenomenon is related to the fact that the size of the Galois group of x^8-2 is half of the size of the Galois group of $x^8-\ell$ where ℓ is an odd prime. This paper deals with these properties in a fairly general context.

Let $\Gamma \subset \mathbb{Q}^*$ be a multiplicative subgroup and define the *support* $\operatorname{Supp}\Gamma$ of Γ to be the set of primes p such that the p–adic valuation of some elements of Γ is nonzero. In the special case of finitely generated Γ (see [?]) it is easy to see that $\operatorname{Supp}\Gamma$ is finite. For any prime $p \notin \operatorname{Supp}\Gamma$, we denote by Γ_p the reduction of Γ modulo p. That is

$$\Gamma_p = \{g (\operatorname{mod} p) : \ g \in \Gamma\}.$$

It is clear that since $p \notin \operatorname{Supp}\Gamma$, $\Gamma_p \subseteq \mathbb{F}_p^*$ is a subgroup. As usual we also denote by $\operatorname{ind}_p(\Gamma)$ and $\operatorname{ord}_p(\Gamma)$ the index and the order of Γ_p. That is

$$\operatorname{ord}_p(\Gamma) = \#\Gamma_p \qquad \text{and} \qquad \operatorname{ind}_p(\Gamma) = [\mathbb{F}_p^* : \Gamma_p] = (p-1)/\operatorname{ord}_p(\Gamma).$$

Here, for $m \in \mathbb{Z}$, we consider the function

$$A_\Gamma(x,m) = \#\{p \le x : \ p \notin \operatorname{Supp}\Gamma, \ m \mid \operatorname{ord}_p(\Gamma)\}.$$

The special case of Γ generated by a rational number in $\mathbb{Q}^* \setminus \{1,-1\}$ has been extensively considered in the literature. For a complete and updated account we refer to Moree's survey paper [?, Sections 9.2 and 9.3]. Moree [?], Wiertelak [?] and the author [?], give several asymptotic formulas for $A_{\langle g\rangle}(x,m)$ with $g \in \mathbb{Q}^* \setminus \{1,-1\}$. More general results have been considered by Moree [?] and by Chinen and Murata [?]. In this paper we propose the following:

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Theorem 1. *Let $\Gamma\subset\mathbb{Q}^*$ be a finitely generated group of rank r and let $m\in\mathbb{N}$. Then, as $x\to\infty$, uniformly in m,*

$$A_\Gamma(x,m)=\varrho_{\Gamma,m}\frac{x}{\log x}+O_\Gamma\left(\tau(m)m\times x\left(\frac{(\log\log x)^2}{\log x}\right)^{1+\frac{1}{3r+3}}\right),$$

where if $\gamma(f,t)=\prod_{\ell\mid f}\ell^{v_\ell(t)+1}$,

$$\mathcal{S}_m=\{n\in\mathbb{N}\colon\ \mathrm{Rad}(n)\mid m \text{ and } m\mid n\}$$

and if $\mathbb{Q}(\zeta_k,\Gamma^{1/h})$ is the extension of $\mathbb{Q}$ generated by $\zeta_k=e^{2\pi i/k}$ and by the h–th roots of all the elements of Γ, then

$$\varrho_{\Gamma,m}=\sum_{n\in\mathcal{S}_m}\sum_{\substack{d\mid n\\ f\mid n}}\frac{\mu(d)\mu(f)}{[\mathbb{Q}(\zeta_{nd},\Gamma^{1/\gamma(f,\frac{n}{m})}):\mathbb{Q}]}.$$

In the case when $\Gamma\subset\mathbb{Q}^+$, the group of strictly positive rational numbers, we express $\varrho_{\Gamma,m}$ in terms of the orders of the groups

$$\Gamma(t)=\Gamma\mathbb{Q}^{*t}/\mathbb{Q}^{*t}:$$

Theorem 2. *Assume that Γ is a finitely generated subgroup of $\mathbb{Q}^+$ and that $m\in\mathbb{N}$. For any squarefree integer η, let $t_\eta=\infty$ if either m is odd or for all $t\geq 0$, $\eta^{2^t}\mathbb{Q}^{*2^{t+1}}\notin\Gamma(2^{t+1})$ and $t_\eta=\min\left\{t\in\mathbb{N}:\eta^{2^t}\mathbb{Q}^{*2^{t+1}}\in\Gamma(2^{t+1})\right\}$ otherwise. Furthermore let $s_\eta=v_2\left(\frac{\delta(\eta)}{m}\right)$, where $\delta(\eta)$ is the discriminant of $\mathbb{Q}(\sqrt{\eta})$ and let $\sigma_\Gamma=\prod_{\ell\in\mathrm{Supp}\,\Gamma}\ell$. Then*

$$\varrho_{\Gamma,m}=\frac{1}{\varphi(m)}\prod_{\substack{\ell\mid m\\ \ell>2}}\left(1-\sum_{j\geq 1}\frac{\ell-1}{\ell^j|\Gamma(\ell^j)|}\right)\left(1-\sum_{\eta\mid\gcd(m,\sigma_\Gamma)}\psi_\eta\right),$$

where

$$\psi_\eta=\begin{cases}0 & \text{if } t_\eta=\infty;\\ \displaystyle\sum_{k>t_\eta}\frac{1}{2^k\,|\Gamma(2^k)|} & \text{if } s_\eta\leq t_\eta<\infty;\\ -\frac{1}{2^{s_\eta}|\Gamma(2^{s_\eta})|}+\displaystyle\sum_{k>s_\eta}\frac{1}{2^k\,|\Gamma(2^k)|} & \text{if } s_\eta>t_\eta.\end{cases}$$

Remarks.

(1) The condition $\Gamma\subset\mathbb{Q}^+$ is not essential. It is mainly due to the fact that the group $(\Gamma\cap\mathbb{Q}(\zeta_m)^{*2^\alpha})\cdot\mathbb{Q}^{*2^\alpha}/\mathbb{Q}^{*2^\alpha}$ is easy to describe when $\Gamma\subset\mathbb{Q}^+$. This is done in Corollary **??**. However, similar expressions for $\varrho_{\Gamma,m}$ as in Theorem **??** should be derived also for groups containing negative numbers and in particular containing -1.

(2) It is plain that the Generalized Riemann Hypothesis for the Dedekind zeta functions of the fields $\mathbb{Q}(\zeta_m,\Gamma^{1/d})$ $(d\mid m)$ allows a sharper error term in Theorem **??**. In fact, applying Generalized Riemann Hypothesis, and proceeding along the lines of the proof of Theorem **??** and applying [**?**, Lemma 5] rather than Lemma **??**, it can be showed that, as $x\to\infty$, uniformly in m,

$$A_\Gamma(x,m)=\varrho_{\Gamma,m}\,\mathrm{li}(x)+O_\Gamma\left(\tau(m)^3x^{3/4}\log x\right).$$

(3) All the series involved in the expression for $\varrho_{\Gamma,m}$ are convergent since they are bounded by geometric series. In the case when Γ is finitely generated with rank r, for every prime power ℓ^j, the following identity holds (see (**??**))

$$|\Gamma(\ell^j)| = \ell^{\max\{0, j-v_\ell(\Delta_1), \cdots, (r-1)j-v_\ell(\Delta_{r-1}), rj-v_\ell(\Delta_r)\}},$$

where for $i = 1, \ldots, r$, Δ_i is the i–th exponent of Γ (defined in (**??**)). Therefore

$$\sum_{j>v_\ell(\Delta_r)} \frac{1}{\ell^j|\Gamma(\ell^j)|} = \ell^{v_\ell(\Delta_r)} \sum_{j>v_\ell(\Delta_r)} \frac{1}{\ell^{(r+1)j}} = \frac{\ell^{-rv_\ell(\Delta_r)}}{\ell^{r+1}-1}. \tag{1}$$

This implies that $\varrho_{\Gamma,m} \in \mathbb{Q}^+$. Another immediate consequence of (**??**) is that if $\gcd(m, \Delta_{r-1}) = 1$ and either m is odd or $\gcd(m, \sigma_\Gamma) = 1$, then

$$\varrho_{\Gamma,m} = \frac{1}{\varphi(m)} \prod_{\ell|m} \left(1 - \frac{\ell-1}{\ell^r-1}\left[1 - \frac{\ell^r(\ell-1)}{\ell^{rv_\ell(\Delta_r)}(\ell^{r+1}-1)}\right]\right). \tag{2}$$

(4) If one sets $\Delta_0 = 1$, then (**??**) holds also for $r = 1$. More precisely, if $\Gamma = \langle a \rangle$, where $a \in \mathbb{Q}^* \setminus \{\pm 1\}$, $a = b^h$ where b is not the power on any rational number so that $h = \Delta_1$, we write (in a unique way) $b = a_1 a_2^2$, where a_1 is a squarefree integer. Then

$$1 - \sum_{j\geq 1} \frac{\ell-1}{\ell^j \left|\frac{\langle b^h\rangle \mathbb{Q}^{*\ell^j}}{\mathbb{Q}^{*\ell^j}}\right|} = \frac{1}{\ell^{v_\ell(h)}} \frac{\ell}{\ell+1} \quad \text{and} \quad \sum_{k>r_{a_1}} \frac{1}{2^k \left|\frac{\langle b^h\rangle \mathbb{Q}^{*2^k}}{\mathbb{Q}^{*2^k}}\right|} = \frac{1}{32^{v_2(h)}}$$

since $r_{a_1} = v_2(h)$, $r_1 = 0$ and since $s_{a_1} = v_2\left(\frac{\delta(a_1)}{m}\right)$. By Theorem **??** we obtain that $\varrho_{\langle b^h\rangle,m}$ equals:

$$\frac{1}{m} \prod_{\ell|m} \frac{\ell^{2-v_\ell(h)}}{\ell^2-1} \times \begin{cases} \frac{1}{2} & \text{if } [2, a_1] \mid m \text{ and } v_2(\delta(a_1)) \leq v_2(mh); \\ 1 + \frac{1}{2^{2v_2(\frac{\delta(a_1)}{hm})}} & \text{if } [2, a_1] \mid m \text{ and } v_2(\delta(a_1)) > v_2(mh); \\ 1 & \text{if } [2, a_1] \nmid m. \end{cases}$$

This formula is consistent with the formula in [**?**, Theorem 1.3].

(5) An immediate consequence of the previous remark is that $\varrho_{\Gamma,m} \neq 0$ for any group Γ and for any m. In fact $\varrho_{\langle a\rangle,m} > 0$ for any $a \in \mathbb{Q}^*$ and if $\Gamma' \subset \mathbb{Q}^*$ is a subgroup with $\Gamma' \subset \Gamma$, then $\operatorname{ord}_p \Gamma' \mid \operatorname{ord}_p \Gamma$ for any prime $p \notin \operatorname{Supp}\Gamma$. Therefore $\varrho_{\Gamma,m} \geq \varrho_{\Gamma',m} > 0$.

(6) In the special case when $\Gamma = \langle d_1, d_2 \rangle$ with $d_1, d_2 \in \mathbb{Q}^+$ multiplicatively independent so that $\operatorname{rank}\Gamma = 2$, we have that, for $\ell \geq 3$,

$$\Gamma(\ell^j) = \begin{cases} 1 & \text{if } j \leq v_\ell(\Delta_1); \\ \ell^{j-v_\ell(\Delta_1)} & \text{if } v_\ell(\Delta_1) < j \leq v_\ell(\Delta_2/\Delta_1); \\ \ell^{2j-v_\ell(\Delta_2)} & \text{if } j > v_\ell(\Delta_2/\Delta_1). \end{cases}$$

Hence

$$1 - \sum_{j\geq 1} \frac{\ell-1}{\ell^j\,|\Gamma(\ell^j)|} = \frac{1}{\ell^{v_\ell(\Delta_1)}} \cdot \frac{\ell}{\ell+1} + \frac{1}{\ell^{2v_\ell(\Delta_2/\Delta_1)}} \cdot \left(\frac{\ell^{v_\ell(\Delta_1)}}{\ell+1} - \frac{1}{\ell^2+\ell+1}\right).$$

This identity can be used in Theorem **??** to explicitly compute $\varrho_{\langle d, d_2\rangle,m}$ in the case when m is odd or when $\gcd(m, \sigma_{\langle d, d_2\rangle}) = 1$.

(7) If $\Gamma\subset\mathbb{Q}^*$ is the multiplicative subgroup generated by r distinct prime numbers $p_1,\dots,p_r$, then $|\Gamma(\ell^j)|=\ell^{rj}$ for all j, and if η is a divisor of $\gcd(m,p_1\cdots p_r)$, then $t_\eta=0$. We deduce that

$$\varrho_{\langle p_1,\cdots,p_r\rangle,m}=\frac{1}{\varphi(m)}\prod_{\substack{\ell\mid m,\\ \ell>2}}\frac{\ell(\ell^r-1)}{\ell^{r+1}-1}\times\left(1-\frac{\psi}{2^{r+1}-1}\right),$$

where $u_k=\#\{\eta\in\mathbb{N}\colon\ \eta\mid\gcd(m,p_1\cdots p_r),\ \eta\equiv k(\bmod 4)\}$

$$\psi=\psi_{\langle p_1,p_2,\cdots,p_r\rangle,m}=\begin{cases}0 & \text{if } 2\nmid m;\\ u_1+\left(\frac{1}{2^r}-1\right)\left[\frac{u_2}{2^{r+1}}+u_3\right] & \text{if } 2\|m;\\ u_1+\left(\frac{1}{2^r}-1\right)u_2+u_3 & \text{if } 4\|m;\\ u_1+u_2+u_3 & \text{if } 8\mid m.\end{cases}\tag{3}$$

Several computations of the densities $\varrho_{\langle p_1,\cdots,p_r\rangle,m}$ are presented in Section **??**.

(8) Among the various consequences of Theorem **??**, one can also compute the density of the set of primes for which $\mathrm{ord}_p\,\Gamma$ is k–free (i.e. not divisible by the k–power of any prime). More precisely, if $k\geq 2$ and Γ is finitely generated with rank r, then

$$\#\{p\leq x\colon\ p\notin\operatorname{Supp}\Gamma,\ \mathrm{ord}_p(\Gamma)\text{ is }k\text{–free}\}$$
$$=\left(\beta_{\Gamma,k}+O_{k,\Gamma}\left(\frac{(\log\log x)^3}{\log^{(k-1)/((k+1)(3r+3))}x}\right)\right)\frac{x}{\log x},$$

where

$$\beta_{\Gamma,k}=\sum_{m=1}^{\infty}\mu(m)\varrho_{\Gamma,m^k}.$$

In the special case when $\Gamma=\langle p_1,\cdots,p_r\rangle\subset\mathbb{Q}^*$, where p_j is prime for all $j=1,\dots,r$ and $p_j<p_{j+1}$ for all $j=1,\dots,r-1$, we have that

$$\beta_{\Gamma,k}=\beta_{r,k}\times\tilde\beta_{\Gamma,k},$$

where

$$\beta_{r,k}=\prod_{\ell>2}\left(1-\frac{\ell^r-1}{\ell^{k-2}(\ell-1)(\ell^{r+1}-1)}\right).$$

and $\tilde\beta_{\Gamma,k}\in\mathbb{Q}^+$. Furthermore, if $k\geq 3$ or $p_1\geq 3$, then $\tilde\beta_{\Gamma,k}$ equals

$$1-\frac{1}{2^{k-1}}\left[1-\frac{\gcd(2,p_1)}{2^{r+1}-1}\prod_{j=1}^{r}\left(1-\frac{p_j^r-1}{p_j^{k-2}(p_j-1)(p_j^{r+1}-1)-(p_j^r-1)}\right)\right],$$

while, if $k=2$ and $p_1=2$, $\tilde\beta_{\Gamma,k}$ equals

$$\frac{1}{2}+\frac{1}{2(2^{r+1}-1)}\prod_{j=1}^{r}\left(1-\frac{p_j^r-1}{(p_j-1)(p_j^{r+1}-1)-(p_j^r-1)}\right).$$

The proof of the above statement is carried out along the lines of [**?**, Theorem 1.2]. Indeed one starts from the identity

$$\#\{p\leq x\colon\ p\notin\operatorname{Supp}\Gamma,\ \mathrm{ord}_p(\Gamma)\text{ is }k\text{–free}\}=\sum_{m=1}^{\infty}\mu(m)A_\Gamma(x,m^k).$$

The mail term is obtained by applying Theorem **??** to the values of $m\leq\log^{1/(2k(3r+3))}x$. For $\log^{1/(2k(3r+3))}x<m\leq\log^2 x$, one uses the bound $A_\Gamma(x,m^k)\leq\pi(x,m^k,1)$ and the Brun–Titchmarch Theorem. We will omit further details.

Like for most of the results regarding properties of the index and the order of subgroups of $\mathbb{F}_p^*$, the techniques are those of the pioneering work by C. Hooley [**?**], where Artin's Conjecture for primitive roots is established as one of the consequences of the Generalized Riemann Hypothesis.

The first to consider higher rank groups in relation to the Lang–Trotter Conjecture, were Gupta and Ram Murty in [**?**]. Their approach led to the quasi–resolution of the Artin's Conjecture by Gupta, Ram Murty and Heath–Brown [**?**, **?**].

2. Notational conventions

Throughout the paper, the letters p and ℓ always denote *prime numbers*. As usual, we use $\pi(x)$ to denote the *number of* $p \leq x$ and

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$$

denotes the *logarithmic integral* function.

φ, μ and τ are respectively the *Euler*, the *Möbius* and the *number of divisors* functions. An integer is said *squarefree* if it is not divisible for the square of any prime number and more generally it is said *k–free* if it is not divisible by the k–th power of any prime number.

For $n \in \mathbb{N}$, $\mathrm{Rad}(n)$ denotes the *radical of* n, the largest squarefree integer dividing n. For $\alpha \in \mathbb{Q}^*$, $v_\ell(\alpha)$ denotes the *ℓ–adic valuation* of α and if $\eta \in \mathbb{Q}^*$, $\delta(\eta)$ denotes the *field discriminant* of $\mathbb{Q}(\sqrt{\eta})$. So, if

$$\delta_0(\alpha) = \mathrm{sgn}(\alpha) \prod_{\substack{\ell \\ v_\ell(\alpha) \equiv 1 \bmod 2}} \ell,$$

Then $\delta(\eta) = \delta_0(\eta)$ if $\delta_0(\eta) \equiv 1 (\mathrm{mod}\, 4)$ and $\delta(\eta) = 4\delta_0(\eta)$ otherwise.

For functions F and $G > 0$ the notations $F = O(G)$ and $F \ll G$ are equivalent to the assertion that the inequality $|F| \leq c\,G$ holds with some constant $c > 0$. In what follows, all constants implied by the symbols O and $\ll$ may depend (where obvious) on the small real parameter ϵ but are absolute otherwise; we write O_ρ and $\ll_\rho$ to indicate that the implied constant depends on a given parameter ρ.

3. Finitely generated subgroups of $\mathbb{Q}^*$.

Let Γ be a finitely generated subgroup of $\mathbb{Q}^*$ of rank r and let $(a_1, \ldots, a_r)$ be a $\mathbb{Z}$-basis of Γ. We write $\mathrm{Supp}(\Gamma) = \{p_1, \ldots, p_s\}$. Then we can construct the $s \times r$–matrix with coefficients in $\mathbb{Z}$:

$$M(a_1, \ldots, a_r) = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,r} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \alpha_{s,1} & \cdots & \alpha_{s,r} \end{pmatrix},$$

defined by the property that $|a_i| = p_1^{\alpha_{1,i}} \cdots p_s^{\alpha_{s,i}}$. It is clear that the rank of $M(a_1, \ldots, a_r))$ equals r. This of course implies $r \leq s$. For all $i = 1, \ldots, r$, we define the *i–th exponent* of Γ by

$$\Delta_i = \Delta_i(\Gamma) = \gcd\left(\det A : A \text{ is a } i \times i\text{–minor of } M(a_1, \ldots, a_r)\right)). \tag{4}$$

So Δ_i is the the non-negative greatest common divisor of all the minors of size i of $M(a_1, \ldots, a_r)$. We also set $\Delta_k = \Delta_k(\Gamma) = 1$ for $k \leq 0$ and $\Delta_k = \Delta_k(\Gamma) = 0$ for $k > r$. It can be shown (see [**?**, Section 3]) that $\Delta_1, \ldots, \Delta_r$ are well defined and do not depend on the choice of the basis $(a_1, \ldots, a_r)$ and on the ordering of the support $\{p_1,\dots,p_s\}$. Furthermore, from the Dedekind formula expansion for determinants, we deduce that

$$\Delta_i\Delta_j \mid \Delta_{i+j} \quad \forall i,j\geq 0.$$

For $m\in\mathbb{N}$, we have the following identity (see [**?**, Proposition 2, page 129 and preceding pages])

$$|\Gamma(m)| = |\Gamma\mathbb{Q}^{*m}/\mathbb{Q}^{*m}| = \frac{\varepsilon_{m,\Gamma}\times m^r}{\gcd\left(m^r, m^{(r-1)}\Delta_1,\dots,m\Delta_{r-1},\Delta_r\right)}, \tag{5}$$

where

$$\varepsilon_{m,\Gamma} = \begin{cases} 1 & \text{if } m \text{ is odd or if } -1\notin\Gamma\mathbb{Q}^{*m};\\ 2 & \text{if } m \text{ is even and } -1\in\Gamma\mathbb{Q}^{*m}.\end{cases} \tag{6}$$

Finally, from (**??**) and (**??**), we deduce the bounds:

$$2m^r \geq |\Gamma(m)| \geq \frac{m^r}{\Delta_r(\Gamma)}. \tag{7}$$

4. Locally finite subgroups of $\mathbb{Q}^*$

The case when Γ is not finitely generated is also of interest. In order to apply the machinery used for finitely generated groups, we shall make some necessary assumptions. We say that Γ has *thin support* if $\operatorname{Supp}\Gamma$ has 0 density in the set of prime numbers. This hypothesis assures that $\operatorname{ord}_p(\Gamma)$ is defined for almost all primes p. Furthermore we say that Γ is *locally finite* if $\Gamma(m)=\Gamma\mathbb{Q}^{*m}/\mathbb{Q}^{*m}$ is finite for every $m\in\mathbb{N}$.

If Γ is locally finite, we know that the exponent of finite group $\Gamma(m)$ is a divisor of m. We denote by $r_\Gamma(m)$ the finite group rank of $\Gamma(m)$. That means that

$$\Gamma(m)\cong\frac{\mathbb{Z}}{m_1\mathbb{Z}}\oplus\cdots\oplus\frac{\mathbb{Z}}{m_r\mathbb{Z}},$$

where $r=r_\Gamma(m)$, $m_1\mid m_2\mid\cdots\mid m_r\mid m$, $m_1>1$. If $\eta_1\mathbb{Q}^{*m},\dots,\eta_{r_{\Gamma(m)}}\mathbb{Q}^{*m}$ is a set of generators for $\Gamma(m)$, we define the *m–th local support* as

$$\operatorname{Supp}_m\Gamma=\{p\in\operatorname{Supp}\Gamma:\ v_p(\eta_j)\neq 0,\ \text{for some } j=1,\dots,r_{\Gamma(m)}\}.$$

and

$$\sigma_{\Gamma,m}=\prod_{p\in\operatorname{Supp}_m\Gamma}p.$$

Furthermore it is easy to check that

$$\Gamma(m)=\langle\eta_1,\dots,\eta_{r_{\Gamma(m)}}\rangle\mathbb{Q}^{*m}/\mathbb{Q}^{*m}.$$

So we can apply the identity of (**??**) obtaining

$$|\Gamma(m)|=\frac{\varepsilon_{m,\Gamma}\times m^{r_{\Gamma(m)}}}{\gcd\left(m^{r_{\Gamma(m)}},m^{r_{\Gamma(m)}-1}\Delta_1(\tilde\Gamma),\dots,m\Delta_{r_{\Gamma(m)}-1}(\tilde\Gamma),\Delta_{r_{\Gamma(m)}}(\tilde\Gamma)\right)},$$

where $\tilde\Gamma=\langle\eta_1\dots\eta_{r_{\Gamma(m)}}\rangle$ and $\varepsilon_{m,\Gamma}$ is defined in (**??**).

The free subgroup of $\mathbb{Q}^*$ generated by any fixed set of primes S with zero density is a thin support subgroup. However, if S is infinite, such subgroup is not locally finite. Here we consider the following family of locally finite, thin support, not finitely generated subgroups of $\mathbb{Q}^*$:

Definition 1. *Let S be a set of primes with 0 density and write*

$$S = \{p_1, p_2, \cdots\},$$

where $p_i \leq p_{i+1}$ for all $i \in \mathbb{N}$. Let Γ_S be the subgroup of $\mathbb{Q}^*$ generated by the $k!$–powers of the p_k's. That is

$$\Gamma_S = \langle p_1, p_2^{2!}, \ldots, p_k^{k!}, \ldots \rangle.$$

It is plain that Γ_S is a free $\mathbb{Z}$–module of infinite rank. Furthermore $S = \operatorname{Supp}\Gamma_S$ so that Γ_S has thin support. However, for every $m \in \mathbb{N}$, we have the identity:

$$\Gamma_S(m) = \frac{\Gamma_S \mathbb{Q}^{*m}}{\mathbb{Q}^{*m}} = \frac{\langle p_1, p_2^{2!}, \ldots, p_{m-1}^{(m-1)!} \rangle \mathbb{Q}^{*m}}{\mathbb{Q}^{*m}}.$$

Hence

Proposition 1. *Let $m \in \mathbb{N}$ and let S be a set of prime numbers. Then Γ_S is locally finite and satisfies the following properties:*

1. $r_{\Gamma_S(m)} = r(m) = \max\{k \in \mathbb{N} \colon\ m \nmid k!\} \leq m-1$;
2. *if ℓ is prime, then $r(\ell^\alpha) \leq \alpha\ell - 1$;*
3. $r(\ell^\alpha) = \alpha\ell - 1$ *for $\alpha \leq \ell$.*
4. $\#\Gamma_S(m) = \prod_{j \leq r(m)} \frac{m}{\gcd(m, j!)}$ *is a multiplicative function.*
5. $\operatorname{Supp}_m \Gamma_S = \{p_1, \ldots, p_{r_{\Gamma_S(m)}}\} \subset \{p_1, \ldots, p_{m-1}\}$.

Proof. The first statement is clear from the definition and for the second observe that $v_\ell((\alpha\ell)!)$ satisfies

$$v_\ell((\alpha\ell)!) = \alpha + \sum_{j \geq 1} \left[\frac{\alpha}{\ell^j}\right] \geq \alpha.$$

This observation also implies that $r(\ell^\alpha) = \alpha\ell - 1$ for $\alpha \leq \ell$. As for the fourth statement, it is enough to observe that

$$\frac{\Gamma_S \mathbb{Q}^{*m}}{\mathbb{Q}^{*m}} \cong \bigoplus_{j=1}^{\infty} \frac{\langle p_j^{j!} \rangle \mathbb{Q}^{*m}}{\mathbb{Q}^{*m}}$$

and to apply the fact that

$$\# \frac{\langle p_j^{j!} \rangle \mathbb{Q}^{*m}}{\mathbb{Q}^{*m}} = \frac{m}{\gcd(m, j!)}$$

is a multiplicative function of m which is identically 1 if $j > r(m)$. The last statement is also clear from the definition of $\operatorname{Supp}_m \Gamma_S$. $\square$

Theorem 3. *Let S be a set of prime numbers with 0 density and let $m \in \mathbb{N}$ be either an odd number or such that $\gcd(m, \sigma_{\Gamma_S, m}) = 1$. Then, as $x \to \infty$,*

$$A_{\Gamma_S}(x, m) \sim \frac{\chi_{\Gamma_S, m}}{\varphi(m)} \cdot \frac{x}{\log x},$$

where

$$\chi_{\Gamma_S, m} = \prod_{\ell \mid m} \left(1 - \sum_{\alpha \geq 1} \frac{\ell - 1}{\ell^{\alpha + \sum_{j \geq 1} \max\{0, \alpha - v_\ell(j!)\}}}\right).$$

We will omit the proof of Theorem **??** since it is similar to the proof of Theorem **??**, where the main ingredient Lemma **??** is replaced with Lemma **??**.

Remarks.

(1) When Γ is not finitely generated, the rationality of $\varrho_{\Gamma,m}$ does not hold in general. In fact if ℓ is an odd prime, $\Gamma = \langle p_1, p_2^{a_2}, \dots, p_k^{a_k}, \dots\rangle$ where $\{p_1, p_2 \dots\}$ is a zero density set of primes and $a_k = \ell^{\beta_k} k!/\ell^{v_\ell(k!)}$ where $\beta_1 = 0$ and for $k \geq 2$, β_k is defined by

$$\beta_k = j \text{ if and only if } j! - j < k \leq (j+1)! - j - 1,$$

then Γ has thin support and it is locally finite. Furthermore

$$\Gamma(\ell^j) = \ell^{\max\{k\in\mathbb{N}\ \beta_k<j\}} = \ell^{j!-j}.$$

Hence

$$\varrho_{\Gamma,\ell} = 1 - (\ell-1)\sum_{j\geq 1}\frac{1}{\ell^{j!}}.$$

is rationally depedent to the Liuville transcendental number.

(2) The conditions that either m is odd or that $\gcd(m, \sigma_{\Gamma_S,m}) = 1$ in the statement of Theorem **??** can be removed at the cost of complicating the expression for $\chi_{\Gamma_S,m}$.

(3) It was proven in [?] that if $\Gamma \subset \mathbb{Q}^*$ is a finitely generated subgroup, the Generalized Riemann Hypothesis implies that the set of primes for which $\mathrm{ind}_p(\Gamma) = 1$ has a density δ_Γ that equals

$$\prod_{\ell>2}\left(1 - \frac{1}{|\Gamma(\ell)|(\ell-1)}\right)\left(1 - \frac{1}{|\Gamma(2)|}\sum_{\substack{\xi\in\Gamma(2)\\ \xi\equiv 1 \bmod 4}}\prod_{\ell\mid\xi}\frac{1}{1-|\Gamma(\ell)|(\ell-1)}\right).$$

This formula also holds for thin support, locally finite subgroups. In particular if $S = \{p_1, p_2, \dots\}$ is a set of prime numbers with zero density, then

$$\Gamma_S(\ell) = \frac{\langle p_1, p_2^{2!}, \dots, p_{\ell-1}^{(\ell-1)!}\rangle {\mathbb{Q}^*}^\ell}{{\mathbb{Q}^*}^\ell}.$$

and $|\Gamma_S(\ell)| = \ell^{r(\ell)} = \ell^{\ell-1}$ by 3. in Proposition **??**. Therefore

$$\delta_{\Gamma_S} = \prod_\ell\left(1 - \frac{1}{\ell^{\ell-1}(\ell-1)}\right)\times(1+\tau_{p_1}),$$

where

$$\tau_{p_1} = \begin{cases}\frac{1}{p_1^{p_1-1}(p_1-1)-1} & \text{if } p_1 \equiv 1 \bmod 4;\\ 0 & \text{otherwise.}\end{cases}$$

Example: Let $\mathbb{G} = \{3, 5, 11, 17, 29, \dots\}$ denotes the set of (youngest) twin primes which is well known to have density 0 and we will also assume to be infinite. Hence

$$\Gamma_{\mathbb{G}} = \langle 3, 5^2, 11^6, 17^{24}, 29^{120}, \dots\rangle.$$

In the following table we compare:

- the values of $\varrho_{\Gamma_{\mathbb{G}},m}$ (1^{st} row);
- the values of $\frac{A_{\Gamma_{\mathbb{G}},m}(10^6,3)}{\pi(10^6)}$ (2^{nd} row);
- the values of $\frac{A_{\Gamma_{\mathbb{G}},m}(10^6,3)}{\#\{p\leq 10^6\colon\ p\notin\mathbb{G}\}}$ (3^{rd} row).

$m = 2, \dots, 13$. Note that the numbers are truncated (not approximated) to the nineth decimal digit.

m	2	3	4	5	6	7
	0.733383118	0.462912155	0.366691559	0.249679999	0.447527842	0.166665452
	0.681724375	0.462725165	0.314364697	0.214757063	0.447743891	0.145086499
	0.760844529	0.516428520	0.350849505	0.239681524	0.499708537	0.161925072
m	8	9	10	11	12	13
	0.183345779	0.154304051	0.178962194	0.099999999	0.108035882	0.083333333
	0.156959413	0.154564447	0.161800300	0.088397156	0.107302096	0.074052842
	0.175175943	0.172503021	0.180578659	0.098656429	0.119755456	0.082647330

Finally, $\delta_{\Gamma_{\mathbb{G}}} = 0.47203266462865646291\cdots$ while

$$\frac{|\{p\le 10^6:\ p\notin\mathbb{G}, \mathrm{ind}_p(\Gamma_{\mathbb{G}})=1\}|}{\pi(10^6)}=\frac{33059}{78498}=0.4211444878\cdots$$

and

$$\frac{|\{p\le 10^6:\ p\notin\mathbb{G}, \mathrm{ind}_p(\Gamma_{\mathbb{G}})=1\}|}{|\{p\le 10^6:\ p\notin\mathbb{G}\}|}=\frac{33059}{70335}=0.4700220374\cdots.$$

5. The degree $[\mathbb{Q}(\zeta_m,\Gamma^{1/d}):\mathbb{Q}]$.

Let $\Gamma\subset\mathbb{Q}^*$ be a locally finite subgroup and let m and d be positive integers with $d\mid m$. We denote by K_m the m–th cyclotomic field. So $K_m=\mathbb{Q}(\zeta_m)$, where $\zeta_m=e^{2\pi i/m}$ is the primitive m–th root of unity. Furthermore we denote $K_m(\Gamma^{1/d})$ the subfield of $\mathbb{C}$ generated over K_m by the d–th roots of all elements of Γ. It is well known that $K_m(\Gamma^{1/d})$ is a finite Galois extension of $\mathbb{Q}$ and that there is an isomorphism

$$\mathrm{Gal}(K_m(\Gamma^{1/d})/K_m)\cong\Gamma(K_m^*)^d/(K_m^*)^d. \tag{8}$$

Details on the theory of Kummer's extensions can be found in Lang's book [**?**, Theorem 8.1]. The goal of this section is to prove the following:

Lemma 1. *Let $\Gamma\subset\mathbb{Q}^*$ be a locally finite subgroup. Let m and d be positive integers with $d\mid m$, set $\alpha=v_2(d)$ be the 2–adic valuation and let $k_{m,d}(\Gamma)$ denote the degree of the extension $K_m(\Gamma^{1/d})/\mathbb{Q}$. Then the degree*

$$k_{m,d}(\Gamma)=\frac{\varphi(m)\times|\Gamma(d)|}{|\mathcal{H}_{m,\alpha}|},$$

where

$$\mathcal{H}_{m,\alpha}=(\Gamma\cap {K_m^*}^{2^\alpha})\mathbb{Q}^{*2^\alpha}/\mathbb{Q}^{*2^\alpha}.$$

It is clear that if d is odd, so that $\alpha=0$, then $|\mathcal{H}_{m,0}|=1$. In the following statement we will describe explicitly $\mathcal{H}_{m,\alpha}$ is the case when Γ contains only positive numbers.

Corollary 1. *Given the Hypothesis of Lemma* **??**, *also assume that $\Gamma\subset\mathbb{Q}^+$ and that d is even so that $\alpha>0$. Then*

$$\mathcal{H}_{m,\alpha}=\{\eta\in\mathbb{N}:\ \eta\mid\gcd(m,\sigma_{\Gamma,m}),\ \eta^{2^{\alpha-1}}\cdot\mathbb{Q}^{*2^\alpha}\in\Gamma(2^\alpha),\ \delta(\eta)\mid m\}.$$

Proof of Corollary **??**. First note that if $\zeta\in\Gamma$, then $\zeta\in{K_m^*}^{2^\alpha}$ if and only if $\sqrt[2^\alpha]{\zeta}\in K_m^*$. Since, for $\zeta>0$, $\mathbb{Q}[\sqrt[2^\alpha]{\zeta}]$ is a Galois extension of $\mathbb{Q}$ only if its degree over $\mathbb{Q}$ is less or equal than 2, we deduce that $\zeta\cdot\mathbb{Q}^{*2^\alpha}=\eta^{2^{\alpha-1}}\cdot\mathbb{Q}^{*2^\alpha}$ for a unique squarefree $\eta\in\mathbb{N}$. Furthermore $\mathbb{Q}(\sqrt{\eta})\subset K_m$ if and only if $\delta(\eta)\mid m$ (see for example Weiss [**?**, page 264]). Finally the conditions $\delta(\eta)\mid m$ and η squarefree imply in particular that $\eta\mid\mathrm{Rad}(m)$ and this completes the proof. □

Proof of Lemma **??**. By the multiplicative property of the degree, we have that

$$k_{m,d}(\Gamma)=[K_m(\Gamma^{1/d}):\mathbb{Q}]=\varphi(m)\times\left|\mathrm{Gal}(K_m(\Gamma^{1/d})/K_m)\right|.$$

By (**??**), since $\Gamma(K_m^*)^d/(K_m^*)^d$ is an abelian torsion group with exponent dividing d, we have that

$$k_{m,d}(\Gamma) = \varphi(m) \prod_{\substack{\ell \text{ prime} \\ \ell^\alpha \| d}} [K_m(\Gamma^{1/\ell^\alpha}) : K_m] = \varphi(m) \prod_{\substack{\ell \text{ prime} \\ \ell^\alpha \| d}} |\Gamma {K_m^*}^{\ell^\alpha} / {K_m^*}^{\ell^\alpha}|.$$

Now we apply the standard Isomorphism Theorems of finite groups and obtain that:

$$\frac{\Gamma {K_m^*}^{\ell^\alpha}}{{K_m^*}^{\ell^\alpha}} \cong \frac{\Gamma}{\Gamma \cap {K_m^*}^{\ell^\alpha}} \cong \frac{\Gamma {\mathbb{Q}^*}^{\ell^\alpha}/{\mathbb{Q}^*}^{\ell^\alpha}}{(\Gamma \cap {K_m^*}^{\ell^\alpha}){\mathbb{Q}^*}^{\ell^\alpha}/{\mathbb{Q}^*}^{\ell^\alpha}}.$$

If ℓ is odd, then $\Gamma \cap {K_m^*}^{\ell^\alpha} = \Gamma \cap {\mathbb{Q}^*}^{\ell^\alpha}$. Therefore

$$k_{m,d}(\Gamma) = \frac{\varphi(m)}{|\mathcal{H}_{m,v_2(d)}|} \times \prod_{\substack{\ell \text{ prime} \\ \ell^\alpha \| d}} |\Gamma(\ell^\alpha)| = \frac{\varphi(m)}{|\mathcal{H}_{m,v_2(d)}|} \times |\Gamma(d)|,$$

where $\mathcal{H}_{m,\alpha} = (\Gamma \cap {K_m^*}^{2^\alpha}){\mathbb{Q}^*}^{2^\alpha}/{\mathbb{Q}^*}^{2^\alpha}$ and this concludes the proof. □

6. Chebotarev Density Theorem for $\mathbb{Q}(\zeta_m, \Gamma^{1/d})$.

In this section we apply the celebrated Chebotarev density Theorem to the fields $\mathbb{Q}(\zeta_m, \Gamma^{1/d})$. We start by stating the result proven in [**?**] which, for simplicity, we specialize to the case of extensions of $\mathbb{Q}$ and trivial conjugacy classes:

Lemma 2 (Effective, "unconditional" Chebotarev Density Theorem.)**.** *Assume that $L/\mathbb{Q}$ is a Galois extension and denote by n_L and d_L the degree and the discriminant of L. Then there exist constants c_1 and c_2 such that if*

$$\log x > 10 n_L \log^2 d_L,$$

then

$$\#\{p \le x\colon\ p \nmid d_L, p \text{ split totally in } L/\mathbb{Q}\} = \frac{\mathrm{li}(x)}{n_L} + O\left(\frac{\mathrm{li}(x^{\beta_0})}{n_L} + \frac{x}{e^{c_1\sqrt{\frac{\log x}{n_L}}}}\right)$$

and $\beta_0 \ge \frac{1}{2}$ satisfies:

$$\beta_0 \le \max\{1 - \frac{1}{4\log d_L}, 1 - \frac{1}{c_2 d_L^{1/n_L}}\}.$$

In order to apply the above result, we need a sufficiently sharp estimate for $\log d_L$. An adequate one can be found in [**?**].

Lemma 3. *Assume that $L/\mathbb{Q}$ is a Galois extension and denote by n_L and d_L the degree and the discriminant of L. Then*

$$\frac{n_L}{2}\log(\mathrm{Rad}(d_E)) \le \log d_L \le (n_L - 1)\log(\mathrm{Rad}(d_E)) + n_L \log n_L.$$

Consider the Galois extension $\mathbb{Q}(\zeta_m, \Gamma^{1/d})$, where $d \mid m$ and where $\Gamma \subset \mathbb{Q}^*$ is a locally finite subgroup. So, by Lemma **??**,

$$n_{\mathbb{Q}(\zeta_m, \Gamma^{1/d})} = k_{m,d}(\Gamma) \le m|\Gamma(d)|.$$

Also note that the primes that ramify in such an extension are exactly those that either divide m or those in $\mathrm{Supp}_d\,\Gamma$. Therefore $\mathrm{Rad}(d_{\mathbb{Q}(\zeta_m,\Gamma^{1/d})}) = \mathrm{lcm}(\mathrm{Rad}(m), \sigma_{\Gamma,d})$ and, by Lemma **??**,

$$\log(d_{\mathbb{Q}(\zeta_m,\Gamma^{1/d})}) \le 2m|\Gamma(d)|\log(m|\Gamma(d)|\sigma_{\Gamma,m}).$$

The conditions of uniformity of Lemma **??** are satisfied if

$$(m|\Gamma(d)|)^3 \log^2(m|\Gamma(d)|\sigma_{\Gamma,m}) \le c\log x$$

for some $c > 0$. We set $\pi_\Gamma(x, n, d)$ to be the number of primes up to x that are unramified and split completely in $K_n(\Gamma^{1/d})$.

If we specialize the previous discussion to the case when Γ is a finitely generated group and we use the upper bound in (**??**), we obtain:

Lemma 4. *Assume that $\Gamma \subset \mathbb{Q}^*$ is a fixed finitely generated subgroup of rank r. Let $m, d \in \mathbb{N}$ be integers such that $d \mid m$. Then there exists constants c_1 and c_2 depending only on Γ such that, uniformly for*

$$m \leq c_1 \left(\frac{\log x}{(\log \log x)^2} \right)^{1/(3r+3)},$$

as $x \to \infty$,

$$\pi_\Gamma(x, m, d) = \frac{1}{k_{m,d}(\Gamma)} \operatorname{li}(x) + O_\Gamma\left(\frac{x}{e^{c_2 \sqrt[6]{\log x} \cdot \sqrt[3]{\log \log x}}} \right). \quad \square$$

If we specialize the previous discussion to the case when $\Gamma = \Gamma_S$, where S if a set of primes with zero density, we obtain:

Lemma 5. *Let S be a set of prime numbers with density zero. Let $m, d \in \mathbb{N}$ be integers such that $d \mid m$. Assume also that $\log \sigma_{\Gamma,m} \leq m^m$. Then, there exist absolute positive constants c_1 and $c_2 < 1$ such that for $x \to \infty$, uniformly for*

$$m \leq c_1 \frac{\log \log x}{\log \log \log x}$$

we have

$$\pi_{\Gamma_S}(x, m, d) = \frac{1}{k_{m,d}(\Gamma_S)} \operatorname{li}(x) + O(x \exp(-(\log x)^{c_2}) \,. \quad \square$$

7. Proofs of Theorems ?? and ??

It is a criterion due to Dedekind that an odd prime $p \notin \operatorname{Supp} \Gamma$ splits totally in $K_n(\Gamma^{1/d})$ if and only d divides the index $\operatorname{ind}_p(\Gamma)$ and $p \equiv 1 (\operatorname{mod} n)$. Therefore

$$\pi_\Gamma(x, n, d) = \#\{p \leq x\colon\ p \notin \operatorname{Supp} \Gamma,\ p \equiv 1 (\operatorname{mod} n),\ d \mid \operatorname{ind}_p(\Gamma)\}. \tag{9}$$

The following combinatorial identity allows us to apply the Chebotarev Density Theorem.

Lemma 6. *Let $m \in \mathbb{Z}$ and $\Gamma \leq \mathbb{Q}^*$. We have the identity*

$$A_\Gamma(x, m) = \sum_{n \in \mathcal{S}_m} \sum_{d \mid n} \sum_{f \mid m} \mu(d) \mu(f) \pi_\Gamma(x, nd, \gamma(f, n/m)),$$

where

$$\mathcal{S}_m = \{n \in \mathbb{N}\colon\ \operatorname{Rad}(n) \mid m \text{ and } m \mid n\}$$

and

$$\gamma(f, k) = \prod_{\ell \mid f} \ell^{v_\ell(k)+1}.$$

Note that with the notation above $\gamma(f, n/m) \mid nd$. In fact for every $\ell \mid f$, $v_\ell(n) - v_\ell(m) + 1 \leq v_\ell(n) + v_\ell(d)$ since $v_\ell(m) \geq 1$.

Proof. Let p be a prime such that $p \notin \operatorname{Supp} \Gamma$ and $m \mid \operatorname{ord}_p(\Gamma)$. Then $m \mid p - 1$ and there exists a unique $n \in \mathcal{S}_m$ such that $p \equiv 1 \bmod n$ and $(\frac{p-1}{n}, m) = 1$ (indeed $n = \prod_{\ell \mid m} \ell^{v_\ell(p-1)}$). Hence

$$A_\Gamma(x, m) = \sum_{n \in \mathcal{S}_m} B_\Gamma(x, m),$$

where $B_\Gamma(x,m)$ equals

$$\#\left\{p\le x:\ p\notin \operatorname{Supp}\Gamma,\ m\,|\,\operatorname{ord}_p(\Gamma),\ p\equiv 1(\operatorname{mod} n),\ (\frac{p-1}{n},m)=1\right\}. \tag{10}$$

Now note that if p is a prime with $p\notin \operatorname{Supp}\Gamma$, $p\equiv 1(\operatorname{mod} n)$ and $(\frac{p-1}{n},m)=1$, then

$$m\mid \operatorname{ord}_p(\Gamma)\quad\Longleftrightarrow\quad (\operatorname{ind}_p(\Gamma),n)\left|\frac{n}{m}\right. .$$

Indeed from the hypothesis that $n\in\mathcal{S}_m$ and from

$$n=(p-1,n)=(\operatorname{ind}_p(\Gamma),n)(\operatorname{ord}_p(\Gamma),n)$$

we deduce that $m\mid \operatorname{ord}_p(\Gamma)$ if and only if $m\mid(\operatorname{ord}_p(\Gamma),n)$ i.e. $(\operatorname{ind}_p(a),n)\mid\frac{n}{m}$. So we can rewrite $B_\Gamma(x,m)$ in (**??**) as

$$\#\left\{p\le x:\ p\notin \operatorname{Supp}\Gamma,\ (\operatorname{ind}_p(\Gamma),n)\left|\frac{n}{m},\right.\ \ p\equiv 1(\operatorname{mod} n),\ (\frac{p-1}{n},m)=1\right\}.$$

Next we apply the inclusion–exclusion formula to the conditions $p\equiv 1(\bmod n)$ and $(\frac{p-1}{n},m)=1$, so that $A_\Gamma(x,m)$ equals

$$\sum_{n\in\mathcal{S}_m}\sum_{d|m}\mu(d)\#\left\{p\le x:\ p\notin \operatorname{Supp}\Gamma,\ (\operatorname{ind}_p(\Gamma),n)\left|\frac{n}{m}\right.,\ p\equiv 1(\operatorname{mod} nd)\right\}.$$

Finally observe that, if $\gamma(f,n/m)$ is the quantity defined in the statement of the lemma, then

$$\sum_{\substack{f|n\\ \gamma(f,\frac{n}{m})|\operatorname{ind}_p(\Gamma)}}\mu(f)=\prod_{\substack{\ell|n\\ v_\ell(\frac{n}{m})<v_\ell(\operatorname{ind}_p(\Gamma)))}}(1+\mu(\ell))=\begin{cases}1 & \text{if } (\operatorname{ind}_p(\Gamma),n)\left|\frac{n}{m}\right. ;\\ 0 & \text{otherwise.}\end{cases}$$

So $A_\Gamma(x,m)$ equals

$$\sum_{n\in\mathcal{S}_m}\sum_{\substack{d|m\\ f|n}}\mu(d)\mu(f)\#\left\{p\le x:\ p\notin \operatorname{Supp}\Gamma,\ \gamma(f,\frac{n}{m})\,|\,\operatorname{ind}_p(\Gamma),\ p\equiv 1(\operatorname{mod} nd)\right\}.$$

Applying the definition in (**??**) and the fact that n and m have the same radical, we deduce the claim. □

Proof of Theorem **??**. Let us start from the identity of Lemma **??** and rewrite it as:

$$\begin{aligned}A_\Gamma(x,m)\ &=\ \sum_{\substack{n\in\mathcal{S}_m,\\ nm\le y}}\sum_{\substack{d|m\\ f|n}}\mu(d)\mu(f)\pi_\Gamma\left(x,nd,\gamma\left(f,\frac{n}{m}\right)\right)+\\ &\quad\ O\left(\sum_{\substack{n\in\mathcal{S}_m,\\ nm>y}}\sum_{\substack{d|m\\ f|n}}\pi_\Gamma\left(x,nd,\gamma\left(f,\frac{n}{m}\right)\right)\right)\\ &=\ \Sigma_1+O(\Sigma_2).\end{aligned}$$

Note that Lemma **??** implies that if $y = c_1(\log x/\log^2\log x)^{1/(3r+3)}$, then

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{n\in\mathcal{S}_m,\\ nm\le y}}\sum_{\substack{d|m\\ f|n}}\mu(d)\mu(f)\pi_\Gamma\left(x,nd,\gamma\left(f,\frac{n}{m}\right)\right)\\
&= \sum_{\substack{n\in\mathcal{S}_m,\\ nm\le y}}\sum_{\substack{d|m\\ f|n}}\left(\frac{\mu(d)\mu(f)\operatorname{li}(x)}{k_{dn,\gamma(f,\frac{n}{m})}(\Gamma)}+O_\Gamma\left(\frac{x}{e^{c_2\sqrt[6]{\log x}\cdot\sqrt[3]{\log\log x}}}\right)\right)\\
&= \varrho_{\Gamma,m}\operatorname{li}(x)+E(x,y,m),
\end{aligned}$$

where

$$\begin{aligned}
E(x,y,m) &\ll \sum_{\substack{n\in\mathcal{S}_m,\\ nm\le y}}\frac{\tau(n)\tau(m)x}{e^{c_2\sqrt[6]{\log x}\cdot\sqrt[3]{\log\log x}}}+\sum_{\substack{n\in\mathcal{S}_m,\\ nm>y}}\sum_{\substack{d|m\\ f|n}}\frac{\mu^2(d)\mu^2(f)}{k_{dn,\gamma(f,n/m)}(\Gamma)}\operatorname{li}(x)\\
&\ll \frac{\tau(m)}{m}\frac{xy\log y}{e^{c_2\sqrt[6]{\log x}\cdot\sqrt[3]{\log\log x}}}+\tau(m)\frac{m}{\varphi(m)}\frac{x}{\log x}\sum_{\substack{n\in\mathcal{S}_m,\\ n>y/m}}\frac{1}{\varphi(n)},
\end{aligned}$$

since $k_{dn,\gamma(f,n/m)}\ge d\varphi(n)$. The choice made for y implies that the first term is negligible. For the second term observe that the Rankin Method (see [**?**, Lemma 3.3]) implies that for any $c\in(0,1)$, uniformly in m,

$$\sum_{\substack{n\in\mathcal{S}_m\\ n\ge T}}\frac{1}{n}\ll_c\frac{1}{T^c}. \tag{11}$$

Hence

$$\begin{aligned}
\tau(m)\frac{m}{\varphi(m)}\frac{x}{\log x}\sum_{\substack{n\in\mathcal{S}_m,\\ n>y/m}}\frac{1}{\varphi(n)} &= \tau(m)\left(\frac{m}{\varphi(m)}\right)^2\frac{x}{\log x}\sum_{\substack{n\in\mathcal{S}_m,\\ n>y/m}}\frac{1}{n}\\
&\le \tau(m)\left(\frac{m}{\varphi(m)}\right)^2\frac{x}{\log x}\frac{m^c}{y^c}\\
&\ll \frac{\tau(m)m^cx(\log\log x)^{\frac{2c}{3r+3}+2}}{(\log x)^{1+\frac{c}{3r+3}}}.
\end{aligned}$$

Now let us deal with Σ_2. We have that

$$\sum_{\substack{n\in\mathcal{S}_m,\\ nm>y}}\sum_{\substack{d|m\\ f|n}}\pi_\Gamma\left(x,nd,\gamma\left(f,\frac{n}{m}\right)\right)\ll$$

$$\tau(m)\left(\sum_{\substack{n\in\mathcal{S}_m,\\ y<nm\le z}}\sum_{d|m}\pi\left(x,nd,1\right)+\sum_{\substack{n\in\mathcal{S}_m,\\ nm>z}}\sum_{d|m}\#\{k\le x:\ nd\mid k\}\right),$$

where z is a suitable parameter that will be determined momentarily. By the Brun–Tichmarch Theorem and the trivial estimate, the above is

$$\ll\frac{\tau(m)m}{\varphi(m)}x\left(\frac{1}{\log(x/z)}\sum_{\substack{n\in\mathcal{S}_m,\\ nm>y}}\frac{1}{\varphi(n)}+\sum_{\substack{n\in\mathcal{S}_m,\\ nm>z}}\frac{1}{n}\right).$$

Applying one more (**??**), we obtain the estimate

$$\Sigma_2\ll\tau(m)\left(\frac{m}{\varphi(m)}\right)^2m^cx\left(\frac{1}{\log(x/z)y^c}+\frac{1}{z^c}\right)$$

Finally setting $z=\log^{2+1/c}x$ and $c=1-1/\log\log x$ we obtain the claim. $\square$

Proof of Theorem **??**. We use the formulas for the degrees $k_{nd,\gamma(f,\frac{n}{m})}(\Gamma)$ of Lemma **??** and of Corollary **??** which in this case reads as:

$$k_{nd,\gamma(f,\frac{n}{m})}(\Gamma)=\frac{d\varphi(n)}{|\mathcal{H}_{nd,v_2(\gamma(f,\frac{n}{m}))}|}\prod_{\ell|f}\left|\Gamma(\ell^{v_\ell(n/m)+1})\right|,$$

where $\mathcal{H}_{nd,v_2(\gamma(f,\frac{n}{m}))}$ is trivial if f is odd while if $2\mid f$, then $v_2(\gamma(f,\frac{n}{m}))=v_2(\frac{n}{m})+1$ and

$$\mathcal{H}_{nd,v_2(\frac{n}{m})+1}=\left\{\eta\in\mathbb{N}\colon\ \eta\,|\,\mathrm{Rad}(m),\ \eta^{2^{v_2(\frac{n}{m})}}\mathbb{Q}^{*2^{v_2(\frac{n}{m})+1}}\in\Gamma(2^{v_2(\frac{n}{m})+1}),\ \delta(\eta)\,|\,nd\right\}.$$

Thus, if for brevity we write $v=v_2(\frac{n}{m})$, the sum defining $\varrho_{\Gamma,m}$ in the statement of Theorem **??**, equals

$$\begin{aligned}&\sum_{n\in\mathcal{S}_m}\frac{1}{\varphi(n)}\sum_{d|n}\frac{\mu(d)}{d}\sum_{f|n}\mu(f)\prod_{\ell|f}\left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}+\\ &+\sum_{\substack{\eta|\mathrm{Rad}(m)\\ \eta\neq1}}\ \sum_{\substack{n\in\mathcal{S}_m\\ \eta^{2^v}\mathbb{Q}^{*2^{v+1}}\in\Gamma(2^{v+1})}}\frac{1}{\varphi(n)}\sum_{\substack{d|n\\ \delta(\eta)|nd}}\frac{\mu(d)}{d}\sum_{\substack{f|n\\ f\text{ even}}}\mu(f)\prod_{\ell|f}\left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}.\\ &=\ S_1+S_2,\end{aligned}$$

say. To compute S_1, we use the identity

$$\frac{1}{\varphi(n)}\sum_{d|n}\frac{\mu(d)}{d}=\frac{1}{n}.$$

So that

$$\begin{aligned}S_1&=\sum_{n\in\mathcal{S}_m}\frac{1}{n}\prod_{\ell|m}\left(1-\left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}\right)\\ &=\prod_{\ell|m}\sum_{j\geq v_\ell(m)}\frac{1}{\ell^j}\left(1-\left|\Gamma(\ell^{j-v_\ell(m)+1})\right|^{-1}\right)\\ &=\frac{1}{m}\prod_{\ell|m}\sum_{j\geq0}\frac{1}{\ell^j}\left(1-\left|\Gamma(\ell^{j+1})\right|^{-1}\right)\\ &=\frac{1}{\varphi(m)}\prod_{\ell|m}\left(1-(\ell-1)\sum_{j\geq1}\frac{1}{\ell^j\,|\Gamma(\ell^j)|}\right).\end{aligned}$$

We also deduce that for m odd,

$$\varrho_{\Gamma,m}=\frac{1}{\varphi(m)}\prod_{\ell|m}\left(1-\sum_{j\geq1}\frac{\ell-1}{\ell^j\,|\Gamma(\ell^j)|}\right)$$

In order to compute S_2, we need to use the following lemma:

Lemma 7. *With the notation above, let*

$$S=\frac{1}{\varphi(n)}\sum_{\substack{d|n\\ \delta(\eta)|nd}}\frac{\mu(d)}{d}.$$

Then

$$S=\frac{\tau_{\eta,n}}{n},\qquad\text{where}\qquad\tau_{\eta,n}=\begin{cases}1 & \text{if }\delta(\eta)\mid n;\\ -1 & \text{if }\delta(\eta)\nmid n\text{ but }\delta(\eta)\mid 2n;\\ 0 & \text{if }\delta(\eta)\nmid 2n.\end{cases}$$

Proof of Lemma **??**. Set $\delta(\eta) = x2^\beta$ with x odd squarefree and $\beta \in \{0, 2, 3\}$. Further set $n = n'2^\alpha$ with n' odd.

The condition $\delta(\eta) \mid n$ implies that $\delta(\eta) \mid nd$ for all possible d and in such a case, we have that $S = \frac{1}{n}$ by the multiplicativity of the involved functions.

The condition $\delta(\eta) \nmid n$, $\delta(\eta) \mid 2n$ is equivalent to $x \mid n'$ and $\beta = \alpha + 1$, which in particular implies that n is even. Therefore, in this case, by multiplicativity,

$$S = \frac{1}{n'} \times \frac{1}{2^{\alpha-1}} \sum_{\substack{\gamma\in\{0,1\},\\ \beta\le\alpha+\gamma}} \frac{(-1)^\gamma}{2^\gamma} = -\frac{1}{n}.$$

Finally, if the condition $\delta(\eta) \nmid 2n$ is satisfied, since $x \nmid n'$, for all squarefree $d \mid n$, we have that $\delta(\eta) \nmid nd$ so, in such a case, $S = 0$. So we can assume that $x \mid n'$, $\beta > \alpha + 1$ and that $\beta \in \{2, 3\}$. It follows that

$$S = \frac{1}{n'} \times \frac{1}{\varphi(2^\alpha)} \sum_{\substack{\gamma\in\{0,1\},\\ \alpha+1<\beta\le\alpha+\gamma}} \frac{(-1)^\gamma}{2^\gamma} = 0,$$

since the conditions on γ in the sum are never satisfied. This concludes the proof. $\square$

Next note that $S_2 = 0$ unless m is even. In the latter case we write

$$S_2 = \sum_{\substack{\eta|\mathrm{Rad}(m)\\ \eta\neq1}} S_\eta,$$

where, by Lemma **??**,

$$S_\eta = \sum_{\substack{n\in\mathcal{S}_m\\ \eta^{2^{v_2(n/m)}}\mathbb{Q}^{*2^{v_2(\frac{n}{m})+1}}\in\Gamma(2^{v_2(n/m)+1})}} \frac{\tau_{\eta,n}}{n} \sum_{\substack{f|n\\ f \text{ even}}} \mu(f) \prod_{\ell|f} \left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}.$$

Next we use the fact that $S_\eta = 0$ unless $\delta(\eta) \mid 2n$ and this happens only if $\eta \mid m$. Furthermore $S_\eta = 0$ unless there exists $t \ge 0$ such that $\eta^{2^t}\mathbb{Q}^{*2^{t+1}} \in \Gamma(2^{t+1})$. We will set t_η to be the least of such t so that $t_\eta = \infty$ if there is no t with such a property. Furthermore if $s \ge t_\eta$, then $\eta^{2^s}\mathbb{Q}^{*2^{s+1}} \in \Gamma(2^{s+1})$.

Hence, for m even, we can rewrite

$$S_2 = \sum_{\substack{\eta|\mathrm{Rad}(m),\\ \eta\neq1,\\ t_\eta<\infty}} S_\eta.$$

We deduce that if S_η is one of the summands above, then it equals

$$-\sum_{\substack{n\in\mathcal{S}_m\\ v_2(n/m)\ge t_\eta}} \frac{\tau_{\eta,n}}{n\left|\Gamma(2^{v_2(n/m)+1})\right|} \prod_{\substack{\ell|n\\ \ell>2}} \left(1 - \left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}\right) =$$

$$\sum_{\substack{n\in\mathcal{S}_m\\ v_2(\delta(\eta))\le v_2(n)+1\\ v_2(n/m)\ge t_\eta}} \frac{\epsilon_\eta(v_2(n))}{n\left|\Gamma(2^{v_2(n/m)+1})\right|} \prod_{\substack{\ell|n\\ \ell>2}} \left(1 - \left|\Gamma(\ell^{v_\ell(n/m)+1})\right|^{-1}\right),$$

where $\epsilon_\eta(j)=1$ if $j=v_2(\delta(\eta)/2)$ and $\epsilon_\eta(j)=-1$ if $j>v_2(\delta(\eta)/2)$. So S_η equals

$$\begin{aligned} & S_1\times 2^{v_2(m)-1}\left(1-\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}\right)^{-1}\times\sum_{\substack{j\geq t_\eta+v_2(m)\\ j\geq v_2(\frac{\delta(\eta)}{2})}}\frac{\epsilon_\eta(j)}{2^j\,\left|\Gamma(2^{j-v_2(m)+1})\right|}\\ = & \; S_1\times\left(1-\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}\right)^{-1}\times\sum_{k\geq\max\{t_\eta+1,v_2(\delta(\eta)/m)\}}\frac{\epsilon_\eta(k+v_2(m/2))}{2^k\,|\Gamma(2^k)|}.\end{aligned}$$

Hence,

$$\varrho_{\Gamma,m}=\frac{1}{\varphi(m)}\prod_{\ell\mid m}\left(1-\sum_{j\geq 1}\frac{\ell-1}{\ell^j\,|\Gamma(\ell^j)|}\right)\times\nu_{\Gamma,m},$$

where, if m is odd, $\nu_{\Gamma,m}=1$ and, if m is even, $\nu_{\Gamma,m}$ equals

$$1+\left(1-\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}\right)^{-1}\sum_{\substack{\eta\mid\mathrm{Rad}(m)\\ \eta\neq 1\\ t_\eta<\infty}}\;\sum_{\substack{k\geq t_\eta+1\\ k\geq v_2(\delta(\eta)/m)}}\frac{\epsilon_\eta(k+v_2(m/2))}{2^k\,|\Gamma(2^k)|}.$$

If we add to the last sum above the term $\eta=1$ and we observe that

$$-\sum_{\substack{k\geq t_1+1\\ k\geq v_2(\delta(1)/m)}}\frac{\epsilon_1(k+v_2(m/2))}{2^k\,|\Gamma(2^k)|}=\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}$$

since $t_1=0,\delta(1)=1$ and $\epsilon_1(k+v_2(m/2))=-1$, we mildly simplify the formula for $\nu_{\Gamma,m}$ when m is even, obtaining:

$$\begin{aligned}\nu_{\Gamma,m} &= \left(1-\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}\right)^{-1}\left(1+\sum_{\substack{\eta\mid\mathrm{Rad}(m)\\ t_\eta<\infty}}\;\sum_{\substack{k\geq t_\eta+1\\ k\geq s_\eta}}\frac{\epsilon_\eta(k+v_2(\frac{m}{2}))}{2^k\,|\Gamma(2^k)|}\right).\\ &= \left(1-\sum_{j\geq 1}\frac{1}{2^j\,|\Gamma(2^j)|}\right)^{-1}\left(1-\sum_{\eta\mid\mathrm{Rad}(m)}\psi_\eta\right),\end{aligned}$$

where $s_\eta=v_2(\frac{\delta(\eta)}{m})$ and

$$\psi_\eta = \begin{cases} 0 & \text{if } t_\eta=\infty;\\ \displaystyle\sum_{k>t_\eta}\frac{1}{2^k\,|\Gamma(2^k)|} & \text{if } s_\eta\leq t_\eta<\infty;\\ -\frac{1}{2^{s_\eta}|\Gamma(2^{s_\eta})|}+\displaystyle\sum_{k>s_\eta}\frac{1}{2^k\,|\Gamma(2^k)|} & \text{if } s_\eta>t_\eta,\end{cases}$$

and this completes the proof. □

8. Numerical Data

In this section we compare numerical data. The density $\varrho_{\Gamma,m}$ can be explicitely computed once a set of generators of Γ is given. In particular, the following Pari-GP [?] code allows to compute $\varrho_{\langle p_1,\cdots,p_r\rangle,m}$ =`rho(m,p_1···p_r)`.

```
rho(m,q)={local(a,A,b,B,l,r,rh);
          r=omega(q);rh=gcd(2,m)/m;
          B=divisors(m);b=matsize(B)[2];
          for(k=1,b,l=B[k];
              if(isprime(l)&(l>2),
                 rh=rh*(l^2*(l^r-1)/(l-1)/(l^(r+1)-1))));
          A=divisors(gcd(m,q));a=matsize(A)[2];
          u1=0;u3=0;u2=0;
          for(j=1,a,l=A[j];
              if(l%4==1,u1++);if(l%4==3,u3++);if(l%4==2,u2++));
          psi=if(m%2==1,0,
              if(m%4==2,u1+(2^(-r)-1)*(u3+u2/2^(r+1)),
              if(m%8==4,u1+u3+(2^(-r)-1)*u2,u1+u3+u2)));
          rh*(1-psi/(2^(r+1)-1))}
```

The first table compares the values of $\varrho_{\Gamma_r,m}$ as in Theorem **??** (second row) and $\frac{A_{\Gamma_r}(10^9,m)}{\pi(10^9)}$ (first row) with $\Gamma_r=\langle 2,\dots,p_r\rangle$, $r\le 7$ (p_i is the i–th prime) and $m=2,\dots,16$. All values have been truncated to 7 decimal digits.

$m\backslash\Gamma_r$	1	2	3	4	5	6	7
2	0.7083259	0.8705329	0.9369869	0.9686946	0.9843725	0.9921912	0.9960977
	0.7083333	0.8705357	0.9369791	0.9686869	0.9843672	0.9921865	0.9960936
3	0.3750162	0.4615489	0.4874978	0.4958546	0.4986178	0.4995315	0.4998315
	0.3750000	0.4615384	0.4875000	0.4958677	0.4986263	0.4995425	0.4998475
4	0.4166745	0.4821469	0.4958488	0.4989975	0.4997547	0.4999387	0.4999818
	0.4166666	0.4821428	0.4958333	0.4989919	0.4997519	0.4999384	0.4999846
5	0.2083311	0.2419332	0.2483914	0.2496736	0.2499273	0.2499772	0.2499875
	0.2083333	0.2419354	0.2483974	0.2496798	0.2499359	0.2499871	0.2499974
6	0.2656511	0.4574280	0.4869920	0.4957940	0.4986109	0.4995309	0.4998313
	0.2656250	0.4574175	0.4869921	0.4958052	0.4986186	0.4995415	0.4998474
7	0.1458489	0.1637375	0.1662449	0.1665994	0.1666516	0.1666582	0.1666592
	0.1458333	0.1637426	0.1662500	0.1666071	0.1666581	0.1666654	0.1666664
8	0.0833265	0.1785587	0.2166697	0.2338669	0.2420661	0.2460616	0.2480390
	0.0833333	0.1785714	0.2166666	0.2338709	0.2420634	0.2460629	0.2480392
9	0.1249966	0.1538451	0.1625054	0.1652942	0.1662133	0.1665179	0.1666177
	0.1250000	0.1538461	0.1625000	0.1652892	0.1662087	0.1665141	0.1666158
10	0.1475587	0.2106102	0.2170853	0.2340359	0.2421145	0.2460758	0.2480397
	0.1475694	0.2106134	0.2170890	0.2340434	0.2421216	0.2460806	0.2480442
11	0.0916644	0.0992460	0.0999258	0.0999871	0.0999930	0.0999937	0.0999937
	0.0916666	0.0992481	0.0999316	0.0999937	0.0999994	0.0999999	0.0999999
12	0.1562485	0.2142815	0.2396969	0.2469355	0.2490664	0.2497065	0.2498959
	0.1562500	0.2142857	0.2396875	0.2469341	0.2490658	0.2497098	0.2499084
13	0.0773848	0.0828743	0.0832971	0.0833291	0.0833317	0.0833320	0.0833320
	0.0773809	0.0828779	0.0832983	0.0833306	0.0833331	0.0833333	0.0833333
14	0.1033220	0.1425403	0.1557674	0.1665792	0.1666493	0.1666580	0.1666592
	0.1032986	0.1425438	0.1557727	0.1665861	0.1666555	0.1666651	0.1666664
15	0.0781280	0.1116612	0.1210907	0.1238016	0.1246141	0.1248689	0.1249475
	0.0781250	0.1116625	0.1210937	0.1238082	0.1246246	0.1248792	0.1249606
16	0.0416661	0.0892749	0.1083288	0.1169345	0.1210315	0.1230292	0.1240151
	0.0416666	0.0892857	0.1083333	0.1169354	0.1210317	0.1230314	0.1240196

The next table compares the values of $\varrho_{\tilde\Gamma_r,m}$ as in Theorem **??** (second row) and $\frac{A_{\tilde\Gamma_r}(10^9,m)}{\pi(10^9)}$ (first row) with $\tilde\Gamma_r=\langle 3,\dots,p_{r+1}\rangle$, $r\le 7$ and $2\le m\le 16$.

$m\backslash\Gamma_r$	1	2	3	4	5	6	7
2	0.6666655	0.8571448	0.9333310	0.9677335	0.9841212	0.9921209	0.9960788
	0.6666666	0.8571428	0.9333333	0.9677419	0.9841269	0.9921259	0.9960784
3	0.3749919	0.4615306	0.4874732	0.4958573	0.4986160	0.4995291	0.4998312
	0.3750000	0.4615384	0.4875000	0.4958677	0.4986263	0.4995425	0.4998475
4	0.3333555	0.4285866	0.4666680	0.4838841	0.4920754	0.4960635	0.4980383
	0.3333333	0.4285714	0.4666666	0.4838709	0.4920634	0.4960629	0.4980392
5	0.2083280	0.2419252	0.2484011	0.2496762	0.2499270	0.2499777	0.2499876
	0.2083333	0.2419354	0.2483974	0.2496798	0.2499359	0.2499871	0.2499974
6	0.3124943	0.4450448	0.4834115	0.4948565	0.4983659	0.4994672	0.4998148
	0.3125000	0.4450549	0.4834375	0.4948680	0.4983790	0.4994810	0.4998322
7	0.1458220	0.1637352	0.1662398	0.1666008	0.1666509	0.1666581	0.1666592
	0.1458333	0.1637426	0.1662500	0.1666071	0.1666581	0.1666654	0.1666664
8	0.1666562	0.2142934	0.2333303	0.2419403	0.2460312	0.2480318	0.2490220
	0.1666666	0.2142857	0.2333333	0.2419354	0.2460317	0.2480314	0.2490196
9	0.1250027	0.1538590	0.1625073	0.1652946	0.1662161	0.1665172	0.1666171
	0.1250000	0.1538461	0.1625000	0.1652892	0.1662087	0.1665141	0.1666158
10	0.1388773	0.1728045	0.2152763	0.2335623	0.2419895	0.2460393	0.2480265
	0.1388888	0.1728110	0.2152777	0.2335715	0.2420015	0.2460503	0.2480366
11	0.0916609	0.0992403	0.0999244	0.0999869	0.0999931	0.0999936	0.0999937
	0.0916666	0.0992481	0.0999316	0.0999937	0.0999994	0.0999999	0.0999999
12	0.0624985	0.1648314	0.2112409	0.2319473	0.2414047	0.2458287	0.2479503
	0.0625000	0.1648351	0.2112500	0.2319381	0.2413984	0.2458378	0.2479635
13	0.0773695	0.0828785	0.0832960	0.0833287	0.0833318	0.0833320	0.0833320
	0.0773809	0.0828779	0.0832983	0.0833306	0.0833331	0.0833333	0.0833333
14	0.0972166	0.1403456	0.1648538	0.1662621	0.1665672	0.1666369	0.1666534
	0.0972222	0.1403508	0.1648645	0.1662712	0.1665754	0.1666449	0.1666613
15	0.0781188	0.1116473	0.1210896	0.1238047	0.1246196	0.1248686	0.1249482
	0.0781250	0.1116625	0.1210937	0.1238082	0.1246246	0.1248792	0.1249606
16	0.0833204	0.1071366	0.1166656	0.1209677	0.1230143	0.1240113	0.1245069
	0.0833333	0.1071428	0.1166666	0.1209677	0.1230158	0.1240157	0.1245098

The next table compares the values of $\beta_{\Gamma_r,k}$ (i.e. the density of primes p with $\mathrm{ord}_p(\Gamma_r)$ k–free) (first row) and $\frac{\#\{p\le 10^9, p\notin \mathrm{Supp}\,\Gamma, \mathrm{ord}_p(\Gamma) \text{ is } k\text{–free}\}}{\pi(10^9)}$ (second row) for $k=2,\dots,7$ and $r=1,\dots,7$.

$k\backslash\Gamma_r$	1	2	3	4	5	6	7
	0.4643728	0.3916870	0.3783724	0.3751626	0.3743029	0.3740588	0.3739871
2	0.4643773	0.3916738	0.3783458	0.3751487	0.3742881	0.3740453	0.3739753
	0.8669787	0.7640822	0.7275550	0.7117925	0.7044658	0.7009347	0.6992045
3	0.8669801	0.7640826	0.7275397	0.7117918	0.7044620	0.7009346	0.6992023
	0.9429226	0.8922523	0.8729475	0.8644050	0.8603871	0.8584410	0.8574845
4	0.9429270	0.8922653	0.8729480	0.8644003	0.8603827	0.8584393	0.8574853
	0.9742393	0.9493687	0.9396381	0.9352925	0.9332389	0.9322416	0.9317506
5	0.9742428	0.9493723	0.9396454	0.9352960	0.9332398	0.9322460	0.9317542
	0.9879809	0.9757187	0.9708684	0.9686929	0.9676621	0.9671607	0.9669135
6	0.9879833	0.9757210	0.9708738	0.9687015	0.9676725	0.9671724	0.9669251
	0.9942653	0.9881936	0.9857800	0.9846948	0.9841798	0.9839289	0.9838052
7	0.9942667	0.9881987	0.9857830	0.9846992	0.9841872	0.9839368	0.9838137
	0.9972219	0.9942060	0.9930041	0.9924629	0.9922058	0.9920804	0.9920185
8	0.9972247	0.9942058	0.9930081	0.9924704	0.9922122	0.9920868	0.9920254

Example. Let $\Gamma = \langle 3^3\cdot 11^{15}, 3^3\cdot 11^3, 3^7\cdot 13^7, 2^2\cdot 5^2\cdot 11\cdot 13\rangle$. Then $\mathrm{Supp}(\Gamma) = (2,3,5,11,13)$ and the matrix associated to Γ is

$$M = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 2 \\ 15 & 3 & 0 & 1 \\ 0 & 0 & 7 & 1 \end{pmatrix},$$

so $\Delta_4(\Gamma) = 2^3\cdot 3^2\cdot 7, \Delta_3(\Gamma) = 2\cdot 3$ and $\Delta_2(\Gamma)=\Delta_1(\Gamma)=1$. Hence if $\ell\nmid 42$,

$$1-\sum_{j\ge 1}\frac{\ell-1}{\ell^j|\Gamma(\ell^j)|} = \frac{\ell(\ell^4-1)}{\ell^5-1}$$

while

$$1-\sum_{j\ge 1}\frac{2}{3^j|\Gamma(3^j)|} = \frac{2^4\times 21}{3\times 11^2} \quad\text{and}\quad 1-\sum_{j\ge 1}\frac{6}{7^j|\Gamma(7^j)|} = \frac{2\times 11\times 127}{2801}.$$

Furthermore if η is squarefree and t_η is finite (i.e. $\eta^{2^t}\mathbb{Q}^{*2^{t+1}} \in \Gamma(2^{t+1})$ for some $t \geq 0$), then $\eta \mid 2 \times 3 \times 5 \times 11 \times 13$. More precisely, after some calculations, one obtains that:

$$t_\eta = \begin{cases} 0 & \text{if } \eta \in \{1, 33, 39, 143\}; \\ 1 & \text{if } \eta \in \{30, 110, 130, 4290\}; \\ 2 & \text{if } \eta \in \{3, 11, 10, 13, 330, 390, 1430\}; \\ \infty & \text{otherwise.} \end{cases}$$

So by (**??**)

$$\sum_{j \geq j_0} \frac{1}{2^j |\Gamma(2^j)|} = \begin{cases} \frac{33}{2^3 \times 31} & \text{if } j_0 = 1 \\ \frac{1}{2^2 \times 31} & \text{if } j_0 = 2 \\ \frac{1}{2^7 \times 31} & \text{if } j_0 = 3 \end{cases}$$

We conclude that

$$\psi_\eta = \begin{cases} \frac{33}{2^3 \times 31} & \text{if } \eta \in \{1, 33\} \text{ or if } \eta \in \{39, 143\} \text{ and } 4 \mid m; \\ \frac{1}{2^2 \times 31} & \text{if } \eta \in \{30, 110, 130, 4290\} \text{ and } 4 \mid m; \\ \frac{1}{2^7 \times 31} & \text{if } \eta \in \{3, 11, 10, 13, 330, 390, 1430\}; \\ -\frac{29}{2^3 \times 31} & \text{if } \eta \in \{39, 143\} \text{ and } 2 \| m; \\ -\frac{15}{2^6 \times 31} & \text{if } \eta \in \{30, 110, 130, 4290\} \text{ and } 2 \| m; \\ 0 & \text{otherwise.} \end{cases}$$

The following table compares the values of $\varrho_{\Gamma,m}$ as in Theorem **??** (second row) and $\frac{A_\Gamma(10^9, m)}{\pi(10^9)}$ (first row) with Γ and $m = 2, \ldots, 25$. The numbers are truncated (not approximated) to the seventh decimal digit.

m	2	3	4	5	6	7	8
	0.86691300	0.46280353	0.43348907	0.24967274	0.40110378	0.16624556	0.21673147
	0.86693548	0.46280992	0.43346774	0.24967990	0.40110970	0.16625015	0.21673387

m	9	10	11	12	13	14	15
	0.15427696	0.21638900	0.09998758	0.20057942	0.08332899	0.14412518	0.11554303
	0.15426997	0.21639344	0.09999379	0.20055485	0.08333064	0.14412815	0.11555433

m	16	17	18	19	20	21	22
	0.10836781	0.06248592	0.13371134	0.05554725	0.10819549	0.07695901	0.08666158
	0.10836694	0.06249929	0.13374211	0.05555515	0.10822818	0.07694221	0.08666296

m	23	24	25	26	27	28	29
	0.04544655	0.10028492	0.04993461	0.07222781	0.05141541	0.07206581	0.03571052
	0.04545439	0.10027743	0.04993598	0.07222128	0.05142332	0.07206407	0.03571423

m	30	31	32	33	34	35	36
	0.10098433	0.03332901	0.05418229	0.04627953	0.05417804	0.04149951	0.066869103
	0.10099355	0.03333329	0.05418346	0.04627811	0.05418285	0.04150932	0.066871057

m	37	38	39	40	41	42	43
	0.02777853	0.04815382	0.03856533	0.05408612	0.02500475	0.06670581	0.023815314
	0.02777776	0.04816273	0.03856624	0.05409836	0.02499999	0.06668454	0.023809517

Conclusion. Average values of $\mathrm{ord}_p(\Gamma)$ in the sense of Kurlberg and Pomerance [**?**] or weighted sum of $\mathrm{ind}_p(\Gamma)$ in the sense of [**?**] can also be considered. For example, if $m \in \mathbb{N}$, in [**?**] Susa and the author consider the problem of enumerating primes p such that $\mathrm{ind}_p(\Gamma) = m$.

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