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Irreducibility and components rigid in moduli of the Hilbert scheme of smooth curves

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Abstract
Denote by $\mathcal{H}_{d,g,r}$ the Hilbert scheme of smooth curves, that is the union of components whose general point corresponds to a smooth irreducible and non-degenerate curve of degree d and genus g in $\mathbb{P}^r$. A component of $\mathcal{H}_{d,g,r}$ is rigid in moduli if its image under the natural map $\pi : \mathcal{H}_{d,g,r} \dashrightarrow \mathcal{M}_g$ is a one point set. In this note, we provide a proof of the fact that $\mathcal{H}_{d,g,r}$ has no components rigid in moduli for $g > 0$ and $r = 3$, from which it follows that the only smooth projective curves embedded in $\mathbb{P}^3$ whose only deformations are given by projective transformations are the twisted cubic curves. In case $r \geq 4$, we also prove the non-existence of a component of $\mathcal{H}_{d,g,r}$ rigid in moduli in a certain restricted range of d, $g > 0$ and r. In the course of the proofs, we establish the irreducibility of $\mathcal{H}_{d,g,3}$ beyond the range which has been known before.

Keywords Hilbert scheme · Algebraic curves · Linear series · Gonality

Mathematics Subject Classification Primary 14C05 · 14C20

1 Basic set up, terminologies and preliminary results

Given non-negative integers d, g and r, let $H_{d,g,r}$ be the Hilbert scheme parametrizing curves of degree d and genus g in $\mathbb{P}^r$ and let $\mathcal{H}_{d,g,r}$ be the Hilbert scheme of smooth curves, that is

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the union of components of $H_{d,g,r}$ whose general point corresponds to a smooth irreducible and non-degenerate curve of degree d and genus g in $\mathbb{P}^r$. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g and consider the natural rational map

$$\pi : \mathcal{H}_{d,g,r} \dashrightarrow \mathcal{M}_g$$

which sends each point $c \in \mathcal{H}_{d,g,r}$ representing a smooth irreducible non-degenerate curve C in $\mathbb{P}^r$ to the corresponding isomorphism class $[C] \in \mathcal{M}_g$.

In this article, we concern ourselves with the question regarding the existence of an irreducible component $\mathcal{Z}$ of $\mathcal{H}_{d,g,r}$ whose image under the map π is just a one point set in $\mathcal{M}_g$, which we call a **component rigid in moduli**.

It is a folklore conjecture that such components should not exist, except when $g = 0$. It is also expected [9, 1.47] that there are no rigid curves in $\mathbb{P}^r$, that is curves that admit no deformations other than those given by projectivities of $\mathbb{P}^r$, except for rational normal curves.

In the next two sections, we provide a proof of the fact that $\mathcal{H}_{g+1,g,3}$ is irreducible and $\mathcal{H}_{d,g,3}$ does not have a component rigid in moduli if $g > 0$. This in turn implies that there are no rigid curves in $\mathbb{P}^3$ except for twisted cubic curves. In the subsequent section we also prove that, for $r \geq 4$, $\mathcal{H}_{d,g,r}$ does not carry any component rigid in moduli in a certain restricted range with respect to $d, g > 0$ and r. In proving the results, we utilize several classical theorems including the so-called Accola–Griffiths–Harris' bound on the dimension of a component consisting of birationally very ample linear series in the variety of special linear series on a smooth algebraic curve. We work over the field of complex numbers.

For notation and conventions, we usually follow those in [2]; e.g. $\pi(d, r)$ is the maximal possible arithmetic genus of an irreducible and non-degenerate curve of degree d in $\mathbb{P}^r$. Before proceeding, we recall several related results that are rather well known; cf. [1].

For any given isomorphism class $[C] \in \mathcal{M}_g$ corresponding to a smooth irreducible curve C, there exist a neighborhood $U \subset \mathcal{M}_g$ of $[C]$ and a smooth connected variety $\mathcal{M}$ which is a finite ramified covering $h : \mathcal{M} \to U$, together with varieties $\mathcal{C}$, $\mathcal{W}^r_d$ and $\mathcal{G}^r_d$ which are proper over $\mathcal{M}$ with the following properties:

(1) $\xi : \mathcal{C} \to \mathcal{M}$ is a universal curve, that is for every $p \in \mathcal{M}$, $\xi^{-1}(p)$ is a smooth curve of genus g whose isomorphism class is $h(p)$,
(2) $\mathcal{W}^r_d$ parametrizes pairs (p, L), where L is a line bundle of degree d with $h^0(L) \geq r+1$,
(3) $\mathcal{G}^r_d$ parametrizes couples $(p, \mathcal{D})$, where $\mathcal{D}$ is possibly an incomplete linear series of degree d and of dimension r, which is denoted by g^r_d, on $\xi^{-1}(p)$.

Let $\mathcal{G}$ be the union of components of $\mathcal{G}^r_d$ whose general element $(p, \mathcal{D})$ corresponds to a very ample linear series $\mathcal{D}$ on the curve $C = \xi^{-1}(p)$. Note that the open subset of $\mathcal{H}_{d,g,r}$ consisting of points corresponding to smooth curves is a $\mathbb{P}GL(r+1)$-bundle over an open subset of $\mathcal{G}$.

We also make a note of the following fact which is basic in the theory; cf. [1] or [8, Chapter 2].

Proposition 1.1 *There exists a unique component $\mathcal{G}_0$ of $\mathcal{G}$ which dominates $\mathcal{M}$ (or $\mathcal{M}_g$) if the Brill–Noether number $\rho(d, g, r) := g - (r+1)(g - d + r)$ is non-negative. Furthermore in this case, for any possible component $\mathcal{G}'$ of $\mathcal{G}$ other than $\mathcal{G}_0$, a general element $(p, \mathcal{D})$ of $\mathcal{G}'$ is such that $\mathcal{D}$ is a special linear system on $C = \xi^{-1}(p)$.*

Remark 1.2 In the Brill–Noether range, that is $\rho(d, g, r) \geq 0$, the unique component $\mathcal{G}_0$ of $\mathcal{G}$ (and the corresponding component $\mathcal{H}_0$ of $\mathcal{H}_{d,g,r}$ as well) which dominates $\mathcal{M}$ or $\mathcal{M}_g$ is called the "principal component". We call the other possible components "exceptional components".

We recall the following well-known fact on the dimension of a component of the Hilbert scheme $\mathcal{H}_{d,g,r}$; cf. [8, Chapter 2.a] or [9, 1.E].

Theorem 1.3 *Let $c \in \mathcal{H}_{d,g,r}$ be a point representing a curve C in $\mathbb{P}^r$. The tangent space of $\mathcal{H}_{d,g,r}$ at c can be identified as*

$$T_c\mathcal{H}_{d,g,r} = H^0\left(C, N_{C/\mathbb{P}^r}\right),$$

where $N_{C/\mathbb{P}^r}$ is the normal sheaf of C in $\mathbb{P}^r$. Moreover, if C is a locally complete intersection, in particular if C is smooth, then

$$\chi(N_{C/\mathbb{P}^r}) \le \dim_c \mathcal{H}_{d,g,r} \le h^0\left(C, N_{C/\mathbb{P}^r}\right),$$

where $\chi(N_{C/\mathbb{P}^r}) = h^0\left(C, N_{C/\mathbb{P}^r}\right) - h^1\left(C, N_{C/\mathbb{P}^r}\right)$.

For a locally complete intersection $c \in \mathcal{H}_{d,g,r}$, we have

$$\chi(N_{C/\mathbb{P}^r}) = (r+1)d - (r-3)(g-1)$$

which is denoted by $\lambda(d, g, r)$.

The following bound on the dimension of the variety of special linear series on a fixed smooth algebraic curve shall become useful in subsequent sections.

Theorem 1.4 (Accola–Griffiths–Harris Theorem; [8, p.73]). *Let C be a curve of genus g, $|D|$ a birationally very ample special g^r_d, that is a special linear system of dimension r and degree d inducing a birational morphism from C onto a curve of degree d in $\mathbb{P}^r$. Then in a neighborhood of $|D|$ on $J(C)$, either* $\dim W^r_d(C) = 0$ *or*

$$\dim W^r_d(C) \le h^0(\mathcal{O}_C(2D)) - 3r \le \begin{cases} d - 3r + 1 & \text{if} \quad d \le g \\ 2d - 3r - g + 1 & \text{if} \quad d \ge g \end{cases}.$$

where $J(C)$ denotes the Jacobian variety of C.

We will also use the following lemmas that are a simple application of the dimension estimate of multiples of the hyperplane linear system on a curve of degree d in $\mathbb{P}^r$; cf. [2, p.115] or [8, Chapter 3.a].

Lemma 1.5 *Let $r \ge 3$ and let C be a smooth irreducible non-degenerate curve of degree d and genus g in $\mathbb{P}^r$. Then*

$$r \le \begin{cases} \frac{d+1}{3} & \text{if} \quad d \le g \\ \frac{1}{3}(2d - g + 1) & \text{if} \quad d \ge g \end{cases}.$$

Proof Set $m = \lfloor \frac{d-1}{r-1} \rfloor$. Suppose $d \le g$ and assume that $r > \frac{d+1}{3}$, so that $1 \le m \le 3$. If $m = 1$ we have $g \le \pi(d, r) = d - r \le g - r$, a contradiction. If $m = 2$ we get $g \le \pi(d, r) = 2d - 3r + 1 < d \le g$, a contradiction. Then $m = 3$ and $g \le \pi(d, r) = 3d - 6r + 3 \le d - 1 < g$, again a contradiction. Therefore $r \le \frac{d+1}{3}$ when $d \le g$.

Suppose now $d \ge g$, so that $2d > 2g - 2$ and $H^1(\mathcal{O}_C(2)) = 0$. By [2, p.115] we have $h^0(\mathcal{O}_C(2)) \ge 3r$, whence, by Riemann–Roch, $2d - g + 1 \ge 3r$, that is $r \le \frac{1}{3}(2d - g + 1)$. □

In the next result we will use the second and third Castelnuovo bounds $\pi_1(d, r)$ and $\pi_2(d, r)$.

Lemma 1.6 *Let $r \geq 4$ and let C be a smooth irreducible non-degenerate curve of degree d and genus $g \geq 2$ in $\mathbb{P}^r$. Assume that either*

(i) $d \geq 2r+1$ *and* $g > \pi_1(d,r)$
or
(ii) *C is linearly normal, $r \geq 8, d \geq 2r+3$ and either $g > \pi_2(d,r)$ or $g = \pi_1(d,r)$.*

Then C admits a degeneration $\{C_t \subset \mathbb{P}^r\}_{t \in \mathbb{P}^1}$ to a singular stable curve.

Proof Under hypothesis (i) it follows by [8, Theorem 3.15] that C lies on a surface of degree $r-1$ in $\mathbb{P}^r$. Under hypothesis (ii) it follows by [17, Theorem 2.10] or [8, Theorem 3.15] respectively, that C lies on a surface of degree $r-1$ or r in $\mathbb{P}^r$.

To do the case of the surface of degree $r-1$, we recall the following notation (see [10, Sect. 5.2]). Let $e \geq 0$ be an integer and let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$. On the ruled surface $X_e = \mathbb{P}\mathcal{E}$, let C_0 be a section in $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ and let f be a fiber. Then any curve $D \sim aC_0 + bf$ has arithmetic genus $\frac{1}{2}(a-1)(2b-ae-2)$. Write $r-1 = 2n-e$ for some $n \geq e$ and let $S_{n,e}$ be the image of X_e under the linear system $|C_0 + nf|$. This linear system embeds X_e when $n > e$, while, when $n = e$, it contracts C_0 to a point and is an isomorphism elsewhere, thus $S_{e,e}$ is a cone. As is well-known (see for example [8, Proposition 3.10]), every irreducible surface of degree $r-1$ in $\mathbb{P}^r$ is either the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or an $S_{n,e}$.

If $C \subset v_2(\mathbb{P}^2)$ then $C \sim aQ$ where Q is a conic and $d = 2a \geq 11$, so that $a \geq 6$. Let C_1 be general in $|(a-1)Q|$ and let C_2 be general in $|Q|$. Now C_1 and C_2 are smooth irreducible, $C_1 \cdot C_2 = a-1 \geq 5$ and they intersect transversally. Therefore C specializes to the singular stable curve $C_1 \cup C_2$.

Now suppose that $C \subset S_{n,e}$.

We deal first with the case $n > e$.

We have $C \sim aC_0 + bf$ for some integers a, b such that $a \geq 2$ (because $g \geq 2$) and, as C is smooth irreducible, we get by [10, Corollary V.2.18(b)] that either $e = 0, b \geq 2$ (if $b = 1$ then $g = 0$) or $e > 0, b \geq ae$. Consider first the case $a \geq 3$. If $e = 0$ or if $e > 0$ and $b > ae$, let C_1 be general in $|C - f|$ and C_2 be a general fiber. As above C_1 and C_2 are smooth irreducible, intersect transversally and $C_1 \cdot C_2 = a \geq 3$. Therefore $C_1 \cup C_2$ is a singular stable curve and C specializes to it. If $e > 0$ and $b = ae$ let C_1 be general in $|C_0 + ef|$ and C_2 be general in $|(a-1)(C_0 + ef)|$. As above both C_1 and C_2 are smooth irreducible, intersect transversally and $C_1 \cdot C_2 = e(a-1) \geq 3$ unless $a = 3, e = 1, b = 3$, which is not possible since then $g = 1$. Therefore $C_1 \cup C_2$ is a singular stable curve and C specializes to it. Now suppose that $a = 2$. Then $g = b - e - 1 \geq 2$ and therefore $b \geq e+3$. Let $C_1 = C_0$ and let C_2 be general in $|C_0 + bf|$. As above C_1 and C_2 are smooth irreducible, intersect transversally and $C_1 \cdot C_2 = b - e \geq 3$. Therefore $C_1 \cup C_2$ is a singular stable curve and C specializes to it. This concludes the case $n > e$.

If $n = e$ let $\widetilde{C}$ be the strict transform of C on X_e. Then $C \cong \widetilde{C} \sim aC_0 + bf$ for some integers a, b. We get $d = \widetilde{C} \cdot (C_0 + ef) = b$ and, as C is smooth, $d - ae = \widetilde{C} \cdot C_0 = 0, 1$. Since $e = r-1$, setting $\eta = 0, 1$ we get $d = a(r-1) + \eta$. Note that if $a \leq 2$ we have $d \leq 2r-1$, a contradiction. Hence $a \geq 3$. If $\eta = 1$ let $\widetilde{C}_1$ be general in $|\widetilde{C} - f|$ and $\widetilde{C}_2$ be a general fiber. As above $\widetilde{C}_1$ and $\widetilde{C}_2$ are smooth irreducible, intersect transversally and $\widetilde{C}_1 \cdot \widetilde{C}_2 = a \geq 3$. Therefore $\widetilde{C}_1 \cup \widetilde{C}_2$ is a singular stable curve and $\widetilde{C}$ specializes to it. On the other hand $\widetilde{C}, \widetilde{C}_1$ and $\widetilde{C}_2$ get mapped isomorphically in $\mathbb{P}^r$, therefore also C specializes to a singular stable curve. If $\eta = 0$ let $\widetilde{C}_1$ be general in $|C_0 + ef|$ and $\widetilde{C}_2$ be general in $|(a-1)(C_0 + ef)|$. Again $\widetilde{C}_1$ and $\widetilde{C}_2$ are smooth irreducible, intersect transversally, $\widetilde{C}_1 \cdot \widetilde{C}_2 = (a-1)(r-1) \geq 6$ and they get mapped isomorphically in $\mathbb{P}^r$, therefore C specializes to a singular stable curve image of $\widetilde{C}_1 \cup \widetilde{C}_2$.

This concludes the case of the surface of degree $r-1$.

We now consider the case $r \geq 8$, $d \geq 2r+3$, C is linearly normal and lies on a surface of degree r in $\mathbb{P}^r$. By a classical theorem of del Pezzo and Nagata (see [16, Thorem 8]) we have that such a surface is either a cone over an elliptic normal curve in $\mathbb{P}^{r-1}$ or the 3-Veronese surface $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ or the image of $X_e, e=0,1,2$ with the linear system $|2C_0+2f|$, $|2C_0+3f|$ or $|2C_0+4f|$ respectively. The case $v_3(\mathbb{P}^2)$ is done exactly as the case $v_2(\mathbb{P}^2)$ above, while the cases $e=0,1$ are done exactly as the case $S_{n,e}, n>e$ above, since the linear systems $|2C_0+2f|$, $|2C_0+3f|$ are very ample and therefore a degeneration on X_e gives a degeneration in $\mathbb{P}^8$. In the case $e=2$ let $\widetilde{C}$ be the strict transform of C on X_2. Then $C \cong \widetilde{C} \sim aC_0+bf$ for some integers a, b. We get $d=\widetilde{C}\cdot(2C_0+4f)=2b$ and, as C is smooth, $b-2a=\widetilde{C}\cdot C_0=\eta$. Also if $a \leq 2$ we have $d \leq 10$, a contradiction. Hence $a \geq 3$. Now exactly as in the case $n=e$ above we conclude that C specializes to a singular stable curve.

It remains to do the case when C is contained in the cone over an elliptic normal curve in $\mathbb{P}^{r-1}$. Let $E \subset \mathbb{P}^{r-1}$ be a linearly normal smooth irreducible elliptic curve of degree r, set $\mathcal{E}=\mathcal{O}_E \oplus \mathcal{O}_E(-1)$ and let $\pi : \mathbb{P}\mathcal{E} \to E$ be the standard map. By [10, Example V.2.11.4] the cone is the image of $\mathbb{P}\mathcal{E}$ under the linear system $|C_0+\pi^*\mathcal{O}_E(1)|$, which contracts C_0 to the vertex and is an isomorphism elsewhere. In particular it follows that $C_0+\pi^*\mathcal{O}_E(1)$ is big and base-point-free. Let $\widetilde{C}$ be the strict transform of C on $\mathbb{P}\mathcal{E}$, so that $C \cong \widetilde{C} \sim aC_0+\pi^*M$ for some integer a and some line bundle M on E of degree b. As before we have $d=\widetilde{C}\cdot(C_0+rf)=b$ and, as C is smooth, $d-ar=\widetilde{C}\cdot C_0=\eta$, so that $a \geq 3$.

Assume that $\eta=0$. We claim that $M \cong \mathcal{O}_E(a)$. In fact if $M \ncong \mathcal{O}_E(a)$ we compute

$$\begin{aligned} h^0(\mathbb{P}\mathcal{E}, aC_0+\pi^*M) &= h^0(E, \pi_*(aC_0+\pi^*M)) \\ &= h^0((\operatorname{Sym}^a \mathcal{E})\otimes M) = h^0\left(\bigoplus_{i=0}^{a} M(-i)\right) = \sum_{i=0}^{a-1}(a-i)r \end{aligned}$$

while

$$h^0(\mathbb{P}\mathcal{E}, (a-1)C_0+\pi^*M) = h^0\left(\bigoplus_{i=0}^{a-1} M(-i)\right) = \sum_{i=0}^{a-1}(a-i)r$$

and therefore the linear system $|aC_0+\pi^*M|$ has C_0 as base component. But $\widetilde{C}$ is irreducible and $\widetilde{C} \neq C_0$, whence a contradiction. Hence $M \cong \mathcal{O}_E(a)$ and $\widetilde{C} \sim a(C_0+\pi^*\mathcal{O}_E(1))$. Let $\widetilde{C}_1$ be general in $|C_0+\pi^*\mathcal{O}_E(1)|$ and let $\widetilde{C}_2$ be general in $|(a-1)(C_0+\pi^*\mathcal{O}_E(1))|$. Now $\widetilde{C}_1$ and $\widetilde{C}_2$ are smooth irreducible by Bertini's theorem, intersect transversally and $\widetilde{C}_1\cdot\widetilde{C}_2=(a-1)r \geq 16$. Therefore $\widetilde{C}_1 \cup \widetilde{C}_2$ is a singular stable curve and $\widetilde{C}$ specializes to it. On the other hand $\widetilde{C}, \widetilde{C}_1$ and $\widetilde{C}_2$ get mapped isomorphically in $\mathbb{P}^r$, therefore also C specializes to a singular stable curve.

Finally let us do the case $\eta=1$. Then $M(-a)$ has degree 1 and thefore there is a point $P \in E$ such that $M \cong \mathcal{O}_E(a)(P)$. Let $F=\pi^*(P)$ be a fiber and let $\widetilde{C}_1$ be general in $|\widetilde{C}-F|$. Note that $\widetilde{C}-F \sim a(C_0+\pi^*\mathcal{O}_E(1))$ and therefore $\widetilde{C}_1$ is smooth irreducible. Again $\widetilde{C}_1$ and $\widetilde{C}_2$ intersect transversally and $\widetilde{C}_1\cdot\widetilde{C}_2=a \geq 3$. Hence $\widetilde{C}_1 \cup \widetilde{C}_2$ is a singular stable curve and $\widetilde{C}$ specializes to it. Also $\widetilde{C}, \widetilde{C}_1$ and $\widetilde{C}_2$ get mapped isomorphically in $\mathbb{P}^r$, therefore also C specializes to a singular stable curve. □

2 Irreducibility of $\mathcal{H}_{g+1,g,3}$ for small genus g

The irreducibility of $\mathcal{H}_{g+1,g,3}$ has been known for $g \geq 9$; cf. [11, Theorem 2.6 and Theorem 2.7]. In this section we prove that any non-empty $\mathcal{H}_{g+1,g,3}$ is irreducible of expected dimension for $g \leq 8$, whence for all g without any restriction on the genus g.

Proposition 2.1 *$\mathcal{H}_{g+1,g,3}$ is irreducible of expected dimension $4(g+1)$ if $g \geq 6$ and is empty if $g \leq 5$. Moreover,* $\dim \pi(\mathcal{H}_{g+1,g,3}) = 3g-3$ *if* $g \geq 8$, $\dim \pi(\mathcal{H}_{8,7,3}) = 17$ *and* $\dim \pi(\mathcal{H}_{7,6,3}) = 13$.

Proof By the Castelnuovo genus bound, one can easily see that there is no smooth non-degenerate curve in $\mathbb{P}^3$ of degree $g+1$ and genus g if $g \leq 5$. Hence $\mathcal{H}_{g+1,g,3}$ is empty for $g \leq 5$. We now treat separately the other cases.
(i) $g = 6$: A smooth curve C of genus 6 with a very ample g^3_7 is trigonal; $|K - g^3_7| = g^1_3$. Furthermore, C has a unique trigonal pencil by Castelnuovo–Severi inequality and the g^3_7 is unique as well. Conversely a trigonal curve of genus 6 has a unique trigonal pencil and the residual series $g^3_7 = |K - g^1_3|$ is very ample which is the unique g^3_7. Hence $\mathcal{G} \subset \mathcal{G}^3_7$ is birationally equivalent to the irreducible locus of trigonal curves $\mathcal{M}^1_{g,3}$. Therefore it follows that $\mathcal{H}_{7,6,3}$ is irreducible which is a $\mathbb{P}GL(4)$-bundle over the irreducible locus $\mathcal{M}^1_{g,3}$ and $\dim \mathcal{H}_{7,6,3} = \dim \mathcal{M}^1_{g,3} + \dim \mathbb{P}GL(4) = (2g+1) + 15 = 28$.
(ii) $g = 7$: First we note that a smooth curve C of degree 8 in $\mathbb{P}^3$ of genus 7 does not lie on a quadric surface; there is no integer solution to the equation $a + b = 8$, $(a-1)(b-1) = 7$ assuming C is of type (a, b) on a quadric surface. We then claim that C is residual to a line in a complete intersection of two cubic surfaces; from the exact sequence $0 \to \mathcal{I}_C(3) \to \mathcal{O}_{\mathbb{P}^3}(3) \to \mathcal{O}_C(3) \to 0$, one sees that $h^0(\mathbb{P}^3, \mathcal{I}_C(3)) \geq 2$ and hence C lies on two irreducible cubics. Note that $\deg C = g + 1 = 8 = 3 \cdot 3 - 1$ and therefore C is a curve residual to a line in a complete intersection of two cubics, that is $C \cup L = X$ where L is a line and X is a complete intersection of two cubics. Upon fixing a line $L \subset \mathbb{P}^3$, we consider the linear system $\mathcal{D} = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{I}_L(3)))$ consisting of cubics containing the line L. Note that any 4 given points on L impose independent conditions on cubics and hence $\dim \mathcal{D} = \dim \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(3))) - 4 = 19 - 4 = 15$. Since our curve C is completely determined by a pencil of cubics containing a line $L \subset \mathbb{P}^3$, we see that $\mathcal{H}_{8,7,3}$ is a $\mathbb{G}(1, 15)$ bundle over $\mathbb{G}(1, 3)$, the space of lines in $\mathbb{P}^3$. Hence $\mathcal{H}_{8,7,3}$ is irreducible of dimension $\dim \mathbb{G}(1, 15) + \dim \mathbb{G}(1, 3) = 28 + 4 = 32 = 4 \cdot 8$. By taking the residual series $|K_C - g^3_8| = g^1_4$ of a very ample g^3_8, we see that $\mathcal{H}_{8,7,3}$ maps into the irreducible closed locus $\mathcal{M}^1_{g,4}$ consisting 4-gonal curves, which is of dimension $2g + 3$. We also note that $\dim W^3_8(C) = \dim W^1_4(C) = 0$. For if $\dim W^1_4(C) \geq 1$, then C is either trigonal, bielliptic or a smooth plane quintic by Mumford's theorem; cf. [2, p.193]. Because $g = 7$, C cannot be a smooth plane quintic. If C is trigonal with the trigonal pencil g^1_3, one may deduce that $|g^1_3 + g^1_5|$ is our very ample g^3_8 by the base-point-free pencil trick [2, p.126]; $\ker \nu \cong H^0(C, \mathcal{F} \otimes \mathcal{L}^{-1})$ where $\nu : H^0(C, \mathcal{F}) \otimes H^0(C, \mathcal{L}) \to H^0(C, \mathcal{F} \otimes \mathcal{L})$ is the natural cup-product map with $\mathcal{F} = g^3_8$, $\mathcal{L} = g^1_3$ and $\mathcal{F} \otimes \mathcal{L}^{-1}$ turns out to be a g^1_5. Therefore it follows that C is a smooth curve of type $(3, 5)$ on a smooth quadric in $\mathbb{P}^3$. However, we have already ruled out the possibility for C lying on a quadric. If C is bi-elliptic with a two sheeted map $\phi : C \to E$ onto an elliptic curve E, one sees that $|K - g^3_8| = g^1_4 = |\phi^*(p + q)|$ by Castelnuovo–Severi inequality and hence $g^3_8 = |K - \phi^*(p + q)|$ where $p, q \in E$. Therefore for any $r \in E$, we have $|g^3_8 - \phi^*(r)| = |K - \phi^*(p + q + r)| = |K - g^2_6| = g^2_6$ whereas g^3_8 is very ample, a contradiction. Furthermore we see that $\mathcal{H}_{8,7,3}$ dominates the locus $\mathcal{M}^1_{g,4}$, for otherwise the inequality $\dim \pi(\mathcal{H}_{8,7,3}) < \dim \mathcal{M}^1_{4,g} = 2g + 3$ which would lead to the inequality

$$\dim \mathcal{H}_{8,7,3} = 32 \le \dim \mathbb{P}GL(4) + \dim W^3_8(C) + \dim \pi(\mathcal{H}_{8,7,3}) < 15 + 17,$$

which is an absurdity.
(iii) $g = 8$: Since we have the non-negative Brill–Noether number $\rho(d, g, 3) = \rho(9, 8, 3) = 0$, there exists the principal component of $\mathcal{H}_{9,8,3}$ dominating $\mathcal{M}_8$ by Proposition 1.1. Because almost the same argument as in the proof of [11, Theorem 2.6] works for this case, we provide only the essential ingredient and important issue adopted for our case $g = 8$. Indeed, the crucial step in the proof of [11, Theorem 2.6] was [11, Lemma 2.4] (for a given $g \ge 9$) in which the author used a rather strong result [11, Lemma 2.3]; e.g. if $\dim W^1_5(C) = 1$ on a fixed curve C of genus $g = 9$ then $\dim W^1_4(C) = 0$. However a similar statement for $g = 8$ was not known at that time. In other words, it was not clear at all that the condition $\dim W^1_5(C) = 1$ would imply $\dim W^1_4(C) = 0$ for a curve C of genus $g = 8$. However by the results of Mukai [15] and Ballico et al. [3, Theorem 1] it has been shown that the above statement holds for a curve of genus 8. Therefore the same proof as in [11, Lemma 2.4, Theorem 2.6] works (even without changing any paragraphs or notation therein). The authors apologize for not being kind enough to provide a full proof; otherwise this article may become unnecessarily lengthy and tedious. □

3 Non-existence of components of $\mathcal{H}_{d,g,3}$ rigid in moduli

In this section, we give a strictly positive lower bound for the dimension of the image $\pi(\mathcal{Z})$ of an irreducible component $\mathcal{Z}$ of the Hilbert scheme $\mathcal{H}_{d,g,3}$ under the natural map $\pi : \mathcal{H}_{d,g,r} \dashrightarrow \mathcal{M}_g$, which will in turn imply that $\mathcal{H}_{d,g,3}$ has no components rigid in moduli. The non-existence of a such component of $\mathcal{H}_{d,g,3}$ has certainly been known to some people (e.g. cf. [14, p.3487]). However the authors could not find an adequate source of a proof in any literature.

We start with the following fact about the irreducibility of $\mathcal{H}_{d,g,3}$ which has been proved by Ein [7, Theorem 4] and Keem and Kim [12, Theorems 1.5 and 2.6].

Theorem 3.1 *$\mathcal{H}_{d,g,3}$ is irreducible for $d \ge g + 3$ and for $d = g + 2$, $g \ge 5$.*

Using Proposition 1.1 and Theorem 3.1, one can prove the following rather elementary facts, well known to experts and included for self-containedness, when the genus or the degree of the curves under consideration is relatively low.

Proposition 3.2 *For $1 \le g \le 4$, every non-empty $\mathcal{H}_{d,g,3}$ is irreducible of dimension $\lambda(d, g, 3) = 4d$. Moreover, $\mathcal{H}_{d,g,3}$ dominates $\mathcal{M}_g$.*

Proof For $d \ge g + 3$, we have $\rho(d, g, 3) = g - 4(g - d + 3) \ge 0$ and hence there exists a principal component $\mathcal{H}_0$ which dominates $\mathcal{M}_g$ by Proposition 1.1. Since $\mathcal{H}_{d,g,3}$ is irreducible for $d \ge g + 3$ by Theorem 3.1, it follows that $\mathcal{H}_{d,g,3} = \mathcal{H}_0$ dominates $\mathcal{M}_g$. Therefore it suffices to prove the statement when $d \le g + 2$.

If $1 \le g \le 3$, one has $\pi(d, 3) < g$ for $d \le g + 2$ and hence $\mathcal{H}_{d,g,3} = \emptyset$.

If $g = 4$ we have $d \le 6$. Since $\pi(d, 3) \le 2$ for $d \le 5$, one has $\mathcal{H}_{d,4,3} = \emptyset$ for $d \le 5$ and hence we just need to consider $\mathcal{H}_{6,4,3}$. We note that a smooth curve in $\mathbb{P}^3$ of degree 6 and genus 4 is a canonical curve, that is a curve embedded by the canonical linear series and vice versa. Hence $\mathcal{G} \subset \mathcal{G}^3_6$ is birationally equivalent to the irreducible variety $\mathcal{M}_4$ and it follows that $\mathcal{H}_{6,4,3}$ is irreducible which is a $\mathbb{P}GL(4)$-bundle over an open subset of $\mathcal{M}_4$ or $\mathcal{G}$. □

Proposition 3.3 *The Hilbert schemes* $\mathcal{H}_{8,8,3}$, $\mathcal{H}_{8,9,3}$ *and* $\mathcal{H}_{9,g,3}$ *for* $g = 9, 12$ *are irreducible, while* $\mathcal{H}_{9,11,3}$ *is empty and* $\mathcal{H}_{9,10,3}$ *has two irreducible components.*

Moreover, under the natural map $\pi : \mathcal{H}_{d,g,3} \dashrightarrow \mathcal{M}_g$, *we have*

(i) $\dim \pi(\mathcal{H}_{8,8,3}) = 17$*;*
(ii) $\dim \pi(\mathcal{H}_{8,9,3}) = 18$*;*
(iii) $\dim \pi(\mathcal{Z}) = 21$ *both when* $\mathcal{Z} = \mathcal{H}_{9,9,3}$ *and when* $\mathcal{Z}$ *is one of the two irreducible components of* $\mathcal{H}_{9,10,3}$*;*
(iv) $\dim \pi(\mathcal{H}_{9,12,3}) = 23$.

Proof To see that $\mathcal{H}_{9,11,3} = \emptyset$ we use [8, Corollary 3.14]. In fact note that there is no pair of integers $a \geq b \geq 0$ such that $a + b = 9$, $(a-1)(b-1) = 11$. Since the second Castelnuovo bound $\pi_1(9, 3) = 10$ and $\pi(9, 3) = 12$, it follows that $\mathcal{H}_{9,11,3} = \emptyset$.

As for the other cases, we start with a few general remarks. We will first prove that the Hilbert schemes $\mathcal{H}_{d,g,3}$ or the components $\mathcal{Z} \subset \mathcal{H}_{d,g,3}$ to be considered are irreducible, generically smooth and that their general point represents a smooth irreducible non-degenerate linearly normal curve $C \subset \mathbb{P}^3$. Moreover we will show that the standard multiplication map

$$\mu_0 : H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_C(-1)) \to H^0(\omega_C)$$

is surjective. From the above it will then follow, by well-known facts about the Kodaira-Spencer map (see e.g. [18, Proof of Proposition 3.3], [13, Proof of Theorem 1.2], that if N_C is the normal bundle of C, then

$$\dim \pi(\mathcal{Z}) = 3g - 3 + \rho + h^1(N_C) = 4d - 15 + h^1(N_C) \tag{3.1}$$

and this will give the results in (i)–(iv).

In general, given a smooth surface $S \subset \mathbb{P}^3$ containing C, we have the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_S(1)) \otimes H^0(\omega_S(C)(-1)) & \xrightarrow{\nu} & H^0(\omega_S(C)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_C(-1)) & \xrightarrow{\mu_0} & H^0(\omega_C) \\ & & \downarrow \\ & & H^1(\omega_S) = 0 \end{array} \tag{3.2}$$

so that μ_0 is surjective when ν is.

Now consider a smooth irreducible curve C of type (a, b), with $a \geq b \geq 3$, on a smooth quadric surface $Q \subset \mathbb{P}^3$. In the exact sequence

$$0 \to N_{C/Q} \to N_C \to N_{Q|C} \to 0$$

we have that $H^1(N_{C/Q}) = 0$ since $C^2 = 2g - 2 + 2d > 2g - 2$ and $N_{Q|C} \cong \mathcal{O}_C(2)$, so that

$$h^1(N_C) = h^1(\mathcal{O}_C(2)). \tag{3.3}$$

Moreover we claim that

$$C \text{ is linearly normal and } \mu_0 \text{ is surjective.} \tag{3.4}$$

In fact from the exact sequence

$$0 \to \mathcal{O}_Q(1-a, 1-b) \to \mathcal{O}_Q(1) \to \mathcal{O}_C(1) \to 0$$

and the fact that $H^i(\mathcal{O}_Q(1-a, 1-b)) = 0$ for $i = 0, 1$, we see that $h^0(\mathcal{O}_C(1)) = h^0(\mathcal{O}_Q(1)) = 4$. Setting $S = Q$ in (3.2) we find that ν is the surjective multiplication map of bihomogeneous polynomials

$$H^0(\mathcal{O}_Q(1,1)) \otimes H^0(\mathcal{O}_Q(a-3, b-3)) \to H^0(\mathcal{O}_Q(a-2, b-2))$$

on $\mathbb{P}^1 \times \mathbb{P}^1$. This proves (3.4).

To see (i) note that, since $\pi_1(8,3) = 7$ and $\pi(8,3) = 9$, it follows by [8, Corollary 3.14] that $\mathcal{H}_{8,8,3}$ is irreducible of dimension 32 and its general point represents a curve of type $(5,3)$ on a smooth quadric. Moreover $H^1(\mathcal{O}_C(2)) = 0$ since $2d = 16 > 2g - 2 = 14$ and therefore $H^1(N_C) = 0$ by (3.3) and $\mathcal{H}_{8,8,3}$ is smooth at the point representing C. Now (3.4) and (3.1) give (i).

To see (ii) observe that, by [5, Example (10.4)], $\mathcal{H}_{8,9,3}$ is smooth irreducible of dimension 33 and its general point represents a curve of type $(4,4)$ on a smooth quadric. Moreover $h^1(\mathcal{O}_C(2)) = h^1(\omega_C) = 1$ and therefore $h^1(N_C) = 1$ by (3.3). Hence (3.4) and (3.1) give (ii).

Finally, to prove (iii) and (iv), consider $\mathcal{H}_{9,g,3}$ for $g = 9$, 10 or 12.

By [6, Theorem 5.2.1] we know that $\mathcal{H}_{9,9,3}$ is irreducible of dimension 36 and its general point represents a curve C residual of a twisted cubic D in the complete intersection of a smooth cubic S and a quartic T. In the exact sequence

$$0 \to N_{C/S} \to N_C \to N_{S|C} \to 0$$

we have that $H^1(N_{C/S}) = 0$ since $C^2 = 25 > 2g - 2 = 16$ and $N_{S|C} \cong \mathcal{O}_C(3)$ that has degree 27, so that again $H^1(N_{S|C}) = 0$ and therefore also $H^1(N_C) = 0$. Hence $\mathcal{H}_{9,9,3}$ is smooth at the point representing C. Moreover, as D is projectively normal, C is also projectively normal. It remains to prove that μ_0 is surjective, whence, by (3.2), that ν is surjective. To this end observe that, if H is the hyperplane divisor of S, then $K_S + C - H \sim 2H - D$. A general element $D' \in |2H - D|$ is again a twisted cubic and we get the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S) & \xrightarrow{\nu_1} & H^0(\mathcal{O}_S(H)) \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_S(H)) \otimes H^0(\mathcal{O}_S(D')) & \xrightarrow{\nu} & H^0(\mathcal{O}_S(H + D')) \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_{D'}(H)) \otimes H^0(\mathcal{O}_{D'}(D')) & \xrightarrow{\tau} & H^0(\mathcal{O}_{D'}(H + D')) \\
\downarrow & & \downarrow \\
H^0(\mathcal{O}_S(H)) \otimes H^1(\mathcal{O}_S) = 0 & & H^1(\mathcal{O}_S(H)) = 0
\end{array}$$

from which we see that ν is surjective since ν_1 is and so is τ, being the standard multiplication map

$$H^0(\mathcal{O}_{\mathbb{P}^1}(3)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(4)).$$

Now (3.1) gives (iii) for $g = 9$.

In the case $g = 10$ it follows by [5, Example (10.4)], that $\mathcal{H}_{9,10,3}$ has two generically smooth irreducible components $\mathcal{Z}_1$ and $\mathcal{Z}_2$, both of dimension 36, and their general point represents a curve C of type (6, 3) on a smooth quadric for $\mathcal{Z}_1$ and a complete intersection of two cubics for $\mathcal{Z}_2$. In the first case, from the exact sequence

$$0 \to \mathcal{O}_Q(-4, -1) \to \mathcal{O}_Q(2) \to \mathcal{O}_C(2) \to 0$$

and the fact that $H^1(\mathcal{O}_Q(2)) = H^2(\mathcal{O}_Q(-4, -1)) = 0$, we get $h^1(\mathcal{O}_C(2)) = 0$ and therefore $h^1(N_C) = 0$ by (3.3). Then (3.4) and (3.1) give (iii) for $\mathcal{Z}_1$. As for $\mathcal{Z}_2$, we have that $N_C \cong \mathcal{O}_C(3)^{\oplus 2}$ and $\omega_C \cong \mathcal{O}_C(2)$, whence $H^1(N_C) = 0$. Moreover, if S is one of the two cubics containing C, we have the diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^3}(1)) & \xrightarrow{\nu_1} & H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \\ \downarrow & & \downarrow \alpha \\ H^0(\mathcal{O}_S(1)) \otimes H^0(\mathcal{O}_S(1)) & \xrightarrow{\nu} & H^0(\mathcal{O}_S(2)). \end{array}$$

As is well known both α and ν_1 are surjective, whence so is ν and then μ_0 by (3.2). Therefore (3.1) gives (iii) for $\mathcal{Z}_2$.

Finally, since $\pi(9, 3) = 12$, it follows by [8, Corollary 3.14] that $\mathcal{H}_{9,12,3}$ is irreducible of dimension 38 and its general point represents a curve of type (5, 4) on a smooth quadric. Moreover, from the exact sequence

$$0 \to \mathcal{O}_Q(-3, -2) \to \mathcal{O}_Q(2) \to \mathcal{O}_C(2) \to 0$$

and the fact that $H^i(\mathcal{O}_Q(2)) = 0$ for $i = 1, 2$, we get $h^1(\mathcal{O}_C(2)) = h^2(\mathcal{O}_Q(-3, -2)) = h^0(\mathcal{O}_Q(1, 0)) = 2$ and therefore $h^1(N_C) = 2$ by (3.3). Hence $h^0(N_C) = 38$ and $\mathcal{H}_{9,12,3}$ is smooth at the point representing C. Now (3.4) and (3.1) give (iv). □

We can now prove our result for $r = 3$.

Theorem 3.4 *Let $\mathcal{Z}$ be an irreducible component of $\mathcal{H}_{d,g,3}$ and $g \geq 5$. Then, under the natural map $\pi : \mathcal{H}_{d,g,3} \dashrightarrow \mathcal{M}_g$, the following possibilities occur:*

(i) *$\mathcal{Z}$ dominates $\mathcal{M}_g$;*
(ii) *$\mathcal{Z} = \mathcal{H}_{7,6,3}$ and* $\dim \pi(\mathcal{Z}) = 13$*;*
(iii) *$\mathcal{Z} = \mathcal{H}_{8,7,3}$ or $\mathcal{H}_{8,8,3}$ and* $\dim \pi(\mathcal{Z}) = 17$*;*
(iv) *$\mathcal{Z} = \mathcal{H}_{8,9,3}$ and* $\dim \pi(\mathcal{Z}) = 18$*;*
(v) *$\mathcal{Z} = \mathcal{H}_{9,9,3}$ or $\mathcal{Z} \subset \mathcal{H}_{9,10,3}$ and* $\dim \pi(\mathcal{Z}) = 21$*;*
(vi) $\dim \pi(\mathcal{Z}) \geq 23$.

Proof We first make the following general remark, which will also be used in the proof of Theorem 4.1.

Let $r \geq 3$, let $\mathcal{Z}$ be an irreducible component of $\mathcal{H}_{d,g,r}$ not dominating $\mathcal{M}_g$ and let C be a smooth irreducible non-degenerate curve of degree d and genus g in $\mathbb{P}^r$ corresponding to a general point $c \in \mathcal{Z}$. We claim that $\mathcal{O}_C(1)$ is special.

In fact $\mathcal{Z}$ is a $\mathbb{P}GL(r+1)$-bundle over an open subset of a component $\mathcal{G}_1$ of $\mathcal{G}$. If $\mathcal{O}_C(1)$ is non-special then, by Riemann-Roch, $d \geq g + r$ and $\mathcal{G}_1$ must coincide with $\mathcal{G}_0$ of Proposition 1.1. But $\mathcal{G}_0$ dominates $\mathcal{M}_g$, so that also $\mathcal{Z}$ does, a contradiction.

Therefore $\mathcal{O}_C(1)$ is special. Set $\alpha = \dim|\mathcal{O}_C(1)|$, so that $\alpha \geq r$.

We now specialize to the case $r = 3$.

First we notice that, in cases (ii)–(v), using Propositions 2.1 and 3.3, $\mathcal{H}_{d,g,3}$ is irreducible, except for $\mathcal{H}_{9,10,3}$, and the dimension of the image under π of each component is as listed. Also we have $d \geq 7$, for if $d \leq 6$ then $g \leq \pi(6,3) = 4$.

Assume that $\mathcal{Z}$ is not as in (i), (ii) or the first case of (iii) and that $\dim \pi(\mathcal{Z}) \leq 22$. By Theorem 3.1, Proposition 2.1 and Remark 1.2, we can assume that $d \leq g$.

For any component $\mathcal{G}_1 \subseteq \mathcal{G} \subseteq \mathcal{G}^3_d$, there exists a component $\mathcal{W}$ of $\mathcal{W}^\alpha_d$ and a closed subset $\mathcal{W}_1 \subseteq \mathcal{W} \subseteq \mathcal{W}^\alpha_d$ such that $\mathcal{G}_1$ is a Grassmannian $\mathbb{G}(3,\alpha)$-bundle over a non-empty open subset of $\mathcal{W}_1$. Thus we have

$$\begin{aligned}\lambda(d,g,3) = 4d &\leq \dim \mathcal{Z}\\ &\leq \dim \pi(\mathcal{Z}) + \dim W^\alpha_d(C) + \dim \mathbb{G}(3,\alpha) + \dim \mathbb{P}GL(4)\\ &\leq \dim W^\alpha_d(C) + 4\alpha + 25. \end{aligned} \tag{3.5}$$

By Lemma 1.5 we have $\alpha \leq \frac{d+1}{3}$.

If $\dim W^\alpha_d(C) = 0$, (3.5) gives

$$4d \leq \frac{4}{3}(d+1) + 25$$

therefore $d \leq 9$.

If $\dim W^\alpha_d(C) \geq 1$ then Theorem 1.4 implies $\alpha \leq \frac{d}{3}$. By (3.5) and Theorem 1.4 again, we find

$$4d \leq d + \alpha + 26 \leq \frac{4d+78}{3}$$

that is again $d \leq 9$.

If $d = 7$ we find the contradiction $7 \leq g \leq \pi(7,3) = 6$. If $d = 8$ it follows that $8 \leq g \leq \pi(8,3) = 9$ and we get that $\mathcal{Z}$ is as in the second case of (iii) or as in case (iv). If $d = 9$ we find that $9 \leq g \leq \pi(9,3) = 12$. Since, by Proposition 3.3, $\mathcal{H}_{9,11,3}$ is empty and $\dim \pi(\mathcal{H}_{9,12,3}) = 23$, we get case (v). □

Remark 3.5 (i) There are no reasons to believe that our estimate on the lower bound of $\dim \pi(\mathcal{Z})$ is sharp. On the other hand, it would be interesting to have a better estimate (hopefully sharp) on the lower bound of $\dim \pi(\mathcal{Z})$ and come up with (irreducible or reducible) examples of Hilbert scheme $\mathcal{H}_{d,g,3}$ with a component $\mathcal{Z}$ achieving the bound.

(ii) If $d \leq g^{\frac{2}{3}}$ there is a better lower bound for the dimension of components $\mathcal{Z}$ of $\mathcal{H}_{d,g,3}$ in [4, Theorem 1.3]. This leads, in this case, to a better lower bound of $\dim \pi(\mathcal{Z})$.

Theorem 3.4 and Proposition 3.2 yield the following immediate corollary.

Corollary 3.6 (i) *$\mathcal{H}_{d,g,3}$ has no component that is rigid in moduli if $g > 0$.*

(ii) *Let $C \subset \mathbb{P}^r$ be a smooth irreducible and non-degenerate curve of genus g whose only deformations are given by projective transformations. If $g = 0$ or $r \leq 3$ then C is a rational normal curve.*

Proof Since (i) is immediate from Theorem 3.4 and Proposition 3.2 and since (ii) is trivial for $r = 2$, we only need to check (ii) for $g = 0$ and $r \geq 3$. Now C belongs to a unique component $\mathcal{H}$ of the Hilbert scheme $\mathcal{H}_{d,0,r}$ with $\dim \mathcal{H} \leq (r+1)^2 - 1$. On the other hand $\dim \mathcal{H} \geq \lambda(d, 0, r) = (r+1)d + r - 3$, hence $(r+1)^2 - 1 \geq (r+1)d + r - 3$ giving that $d < r + 1$. Therefore $d = r$ and C is the rational normal curve. □

4 Non-existence of components of $\mathcal{H}_{d,g,r}$ rigid in moduli with $r \geq 4$

In this section, we prove the non-existence of a component of $\mathcal{H}_{d,g,r}$ rigid in moduli in a certain restricted range of d, $g > 0$ and $r \geq 4$.

Theorem 4.1 *$\mathcal{H}_{d,g,r}$ has no components rigid in moduli if $g > 0$ and*

(i) $d > \min\{\frac{17g+72}{64}, \frac{4g+15}{15}, \max\{\frac{g+18}{4}, \frac{17g+44}{64}\}\}$ *if* $r = 4$*;*
(ii) $d > \min\{\frac{9g+20}{20}, \frac{10g+17}{22}, \max\{\frac{2g+25}{5}, \frac{9g+10}{20}\}\}$ *and* $d > \frac{g+22}{3}$ *for* $101 \leq g \leq 113$, *if* $r = 5$*;*
(iii) $d > \min\{\frac{13g+20}{22}, \frac{3g+3}{5}, \max\{\frac{g+10}{2}, \frac{13g+10}{22}\}, \max\{\frac{g+10}{2}, \frac{3g-1}{5}\}\}$ *if* $r = 6$*;*
(iv) $d > \min\{\frac{19g+24}{27}, \max\{\frac{4g+39}{7}, \frac{76g+71}{108}\}\}$ *if* $r = 7$*;*
(v) $d > \min\{\frac{4g+1}{5}, \frac{5g-4}{6}\}$ *if* $r = 8$*;*
(vi) $d > \min\{\frac{9g-5}{10}, \frac{29g+3}{33}\}$ *and* $(d, g) \neq (30, 34)$ *if* $r = 9$*;*
(vii) $d > \min\{\frac{21g-4}{22}, \frac{17g+12}{18}\}$ *if* $r = 10$*;*
(viii) $d > g$ *if* $r = 11$*;*
(ix) $d > \frac{2(r-5)g-r+14}{r+1}$ *if* $r \geq 12$.

Proof Suppose that there is a component $\mathcal{Z}$ of $\mathcal{H}_{d,g,r}$ rigid in moduli and let C be a smooth irreducible non-degenerate curve of degree d and genus g in $\mathbb{P}^r$ corresponding to a general point $c \in \mathcal{Z}$.

Let $\alpha = \dim |\mathcal{O}_C(1)|$, so that $\alpha \geq r$ and note that, as in the proof of Theorem 3.4, using Proposition 1.1, we have that $\mathcal{O}_C(1)$ is special. In particular $d \leq 2g - 2$ and $g \geq 2$.

Moreover we claim that C does not admit a degeneration $\{C_t \subset \mathbb{P}^\alpha\}_{t \in \mathbb{P}^1}$ to a singular stable curve. In fact such a degeneration gives a rational map $\mathbb{P}^1 \dashrightarrow \overline{\mathcal{M}_g}$ whose image contains two distinct points, namely the points representing C and the singular stable curve. Hence the image must be a curve and therefore the curves in the pencil $\{C_t \subset \mathbb{P}^\alpha\}_{t \in \mathbb{P}^1}$ cannot be all isomorphic. Now we have a projection $p : \mathbb{P}^\alpha \dashrightarrow \mathbb{P}^r$ that sends $C \subset \mathbb{P}^\alpha$ isomorphically to $p(C) = C \subset \mathbb{P}^r$. Thus the pencil gets projected and gives rise to a deformation $p(C_t) \subset \mathbb{P}^r$ of C in $\mathcal{Z}$ (recall that C represents a general point of $\mathcal{Z}$). For general t we have that $p(C_t)$ is therefore smooth, whence $p(C_t) \cong C_t$. Since $\mathcal{Z}$ is rigid in moduli we get the contradiction $C \cong C_t$ for general t. This proves the claim and now, by Lemma 1.6, we can and will assume that $g \leq \pi_1(d, \alpha)$ when $d \geq 2\alpha + 1$ and that $g \leq \pi_2(d, \alpha)$, $g < \pi_1(d, \alpha)$ when $\alpha \geq 8$ and $d \geq 2\alpha + 3$.

Recall again that for any component $\mathcal{G}_1 \subseteq \mathcal{G} \subseteq \mathcal{G}^r_d$, there exists a component $\mathcal{W}$ of $\mathcal{W}^\alpha_d$ and a closed subset $\mathcal{W}_1 \subseteq \mathcal{W} \subseteq \mathcal{W}^\alpha_d$ such that $\mathcal{G}_1$ is a Grassmannian $\mathbb{G}(r, \alpha)$-bundle over a non-empty open subset of $\mathcal{W}_1$. By noting that $\mathcal{W}_1$ is a sub-locus inside $W^\alpha_d(C)$ in our current situation, we come up with an inequality similar to (3.5):

$$\begin{aligned}\lambda(d, g, r) &= (r+1)d - (r-3)(g-1)\\ &\leq \dim \mathcal{Z}\\ &\leq \dim \mathcal{W}_1 + \dim \mathbb{G}(r, \alpha) + \dim \mathbb{P}GL(r+1)\end{aligned}$$

$$= \dim \mathcal{W}_1 + (r+1)(\alpha - r) + r^2 + 2r \qquad (4.1)$$
$$\leq \dim W^{\alpha}_d(C) + (r+1)\alpha + r.$$

This leads to the following four cases.

CASE 1: $d < g$ and $\dim W^{\alpha}_d(C) = 0$.

We have $\alpha \leq (d+1)/3$ by Lemma 1.5 and (4.1) gives

$$d < g, \alpha \leq (d+1)/3 \text{ and } (r+1)(d-\alpha) - 3 \leq (r-3)g. \qquad (4.2)$$

CASE 2: $d < g$ and $\dim W^{\alpha}_d(C) \geq 1$.

By Theorem 1.4 we get $\alpha \leq d/3$. By (4.1) and Theorem 1.4 again, we find

$$d < g, \alpha \leq d/3 \text{ and } rd - (r-2)\alpha - 4 \leq (r-3)g. \qquad (4.3)$$

CASE 3: $d \geq g$ and $\dim W^{\alpha}_d(C) = 0$.

We have $\alpha \leq (2d - g + 1)/3$ by Lemma 1.5 whence, in particular, $d \geq (g + 3r - 1)/2$. Now (4.1) gives

$$d \geq g, \alpha \leq (2d - g + 1)/3 \text{ and } (r+1)(d-\alpha) - 3 \leq (r-3)g. \qquad (4.4)$$

CASE 4: $d \geq g$ and $\dim W^{\alpha}_d(C) \geq 1$.

By Theorem 1.4 we have $\alpha \leq (2d-g)/3$ whence, in particular, $d \geq (g+3r)/2$. By (4.1) and Theorem 1.4 again, we find

$$d \geq g, \alpha \leq (2d-g)/3 \text{ and } (r-1)d - (r-2)\alpha - 4 \leq (r-4)g. \qquad (4.5)$$

The plan is to show that, given the hypotheses, the inequalities (4.2)–(4.5) contradict $g \leq \pi_1(d,\alpha)$ when $d \geq 2\alpha + 1$ or $g \leq \pi_2(d,\alpha)$, $g < \pi_1(d,\alpha)$ when $\alpha \geq 8$ and $d \geq 2\alpha + 3$.

To this end let us observe that $d \geq 2\alpha + 3$ in cases (4.2)–(4.5): In fact this is obvious in cases (4.2) and (4.3), while in cases (4.4) and (4.5), using $g \leq 2d - 3\alpha + 1$ and $g \leq 2d - 3\alpha$ respectively, if $d \leq 2\alpha + 2$, we get $4\alpha \leq 3r - 14$ and $4\alpha \leq 2r - 10$, both contradicting $\alpha \geq r$. Therefore in the sequel we will always have that $g \leq \pi_1(d,\alpha)$ and that either $\alpha \leq 7$ or $\alpha \geq 8$ and $g \leq \pi_2(d,\alpha)$, $g < \pi_1(d,\alpha)$.

We now recall the notation. Set

$$m_1 = \lfloor \frac{d-1}{\alpha} \rfloor, m_2 = \lfloor \frac{d-1}{\alpha+1} \rfloor, \varepsilon_1 = d - m_1\alpha - 1, \varepsilon_2 = d - m_2(\alpha+1) - 1$$

and

$$\mu_1 = \begin{cases} 1 & \text{if} \quad \varepsilon_1 = \alpha - 1 \\ 0 & \text{if} \quad 0 \leq \varepsilon_1 \leq \alpha - 2 \end{cases}, \mu_2 = \begin{cases} 2 & \text{if } \varepsilon_2 = \alpha \\ 1 & \text{if} \quad \alpha - 2 \leq \varepsilon_2 \leq \alpha - 1 \\ 0 & \text{if} \quad 0 \leq \varepsilon_2 \leq \alpha - 3 \end{cases}$$

so that

$$\pi_1(d,\alpha) = \binom{m_1}{2}\alpha + m_1(\varepsilon_1 + 1) + \mu_1, \pi_2(d,\alpha) = \binom{m_2}{2}(\alpha+1) + m_2(\varepsilon_2 + 2) + \mu_2.$$

We now deal with the case $r \geq 11$ (and hence $\alpha \geq 11$).

We start with (4.2). If $\alpha \geq d/3$ then either $\alpha = (d+1)/3$ or $\alpha = d/3$. Then $m_2 = 2$, $\mu_2 = 0$ and $\pi_2(d,\alpha) \leq d < g$. Therefore $\alpha \leq (d-1)/3$ and (4.2) gives $d \leq \frac{3(r-3)g - r + 8}{2(r+1)}$, contradicting (viii)–(ix).

Similarly, in (4.3), if $\alpha = d/3$ then $m_2 = 2$, $\mu_2 = 0$ and $\pi_2(d,\alpha) = d - 1 < g$. Therefore $\alpha \leq (d-1)/3$ and (4.3) gives $d \leq \frac{3(r-3)g - r + 14}{2(r+1)}$, contradicting (viii)–(ix).

Now in (4.4), if $\alpha \geq (2d-g)/3$ then either $\alpha = (2d-g)/3$ or $\alpha = (2d-g+1)/3$. We find $m_2 = 2, \mu_2 = 0$ and $\pi_2(d,\alpha) \leq g-1$. Therefore $\alpha \leq (2d-g-1)/3$ and (4.4) gives $d \leq \frac{2(r-5)g-r+8}{r+1}$, contradicting (viii)–(ix).

Instead in (4.5), if $\alpha = (2d-g)/3$ we find $m_2 = 2, \mu_2 = 0$ and $\pi_2(d,\alpha) = g-1$. Therefore $\alpha \leq (2d-g-1)/3$ and (4.5) gives $d \leq \frac{2(r-5)g-r+14}{r+1}$, contradicting (viii)–(ix). This concludes the case $r \geq 11$.

Assume now that $4 \leq r \leq 10$.

We first claim that (4.4) and (4.5) do not occur. In fact note that we have

$$d \geq \max\{r+2, g, (g+3r-1)/2\} \text{ in (4.4) and } d \geq \max\{r+2, g, (g+3r)/2\} \text{ in (4.5)}. \quad (4.6)$$

Plugging in $\alpha \leq (2d-g+1)/3$ in (4.4) and $\alpha \leq (2d-g)/3$ in (4.5) we get

$$d \leq \frac{2(r-5)g+r+10}{r+1} \text{ in case (4.4) and } d \leq \frac{2(r-5)g+12}{r+1} \text{ in case (4.5)}$$

and it is easily seen that these contradict (4.6).

Therefore, in the sequel, we consider only (4.2) and (4.3).

If $\alpha \leq 7$ (whence $r \leq 7$), we see that (4.2) gives

$$d \leq \frac{(r-3)g+7r+10}{r+1} \quad (4.7)$$

and (4.3) gives

$$d \leq \frac{(r-3)g+7r-10}{r}. \quad (4.8)$$

Now assume $\alpha \geq 8$, so that $g \leq \pi_2(d,\alpha)$, $g < \pi_1(d,\alpha)$. Set $i = d+1-3\alpha$ and $j = d-3\alpha$.

Then (4.2) implies $d < g$, $i \geq 0$,

$$\alpha(m_1-1)[\frac{r-3}{2}m_1 - r - 1] + (\varepsilon_1+1)[(r-3)m_1 - r - 1] + 3 + \mu_1(r-3) > 0 \quad (4.9)$$

and

$$(\alpha+1)(m_2-1)[\frac{r-3}{2}m_2 - r - 1] + (\varepsilon_2+1)[(r-3)m_2 - r - 1] - r + 2 + (m_2+\mu_2)(r-3) \geq 0. \quad (4.10)$$

On the other hand (4.3) implies $d < g$, $j \geq 0$,

$$\alpha[(r-3)\binom{m_1}{2} - m_1 r + r - 2] + (\varepsilon_1+1)[(r-3)m_1 - r] + 4 + \mu_1(r-3) > 0 \quad (4.11)$$

and

$$(\alpha+1)[(r-3)\binom{m_2}{2} - m_2 r + r - 2] + (\varepsilon_2+1)[(r-3)m_2 - r] - r + 6 + (m_2+\mu_2)(r-3) \geq 0. \quad (4.12)$$

Suppose now $r = 4$. It is easily seen that (4.9) implies $m_1 \geq 9$ and $i \geq 7\alpha + 1$, so that $\alpha \leq \frac{d}{10}$. Plugging in (4.2) we contradict (i). On the other hand (4.11) implies $m_1 \geq 8$, so that $\alpha \leq \frac{d-1}{8}$, and also $j \geq \frac{11\alpha-2}{2}$, so that $\alpha \leq \frac{2d+2}{17}$. Moreover (4.12) implies $m_2 \geq 8$, so that $\alpha \leq \frac{d-9}{8}$, and also $j \geq \frac{11\alpha+12}{2}$, so that $\alpha \leq \frac{2d-12}{17}$. Plugging in (4.3) and using (4.7) and (4.8) we contradict (i) and the case $r = 4$ is concluded.

If $r = 5$ it is easily seen that (4.9) implies $m_1 \geq 5$ and $i > 3\alpha + 1$, so that $\alpha < \frac{d}{6}$. Also (4.10) implies $m_2 \geq 5$, so that $i \geq 3\alpha + 5$, hence $\alpha \leq \frac{d-4}{6}$. Plugging in (4.2) we contradict (ii). On the other hand (4.11) implies $m_1 \geq 5$, so that $\alpha \leq \frac{d-1}{5}$, and also $j \geq \frac{12\alpha-4}{5}$, so that $\alpha \leq \frac{5d+4}{27}$. Moreover (4.12) implies $m_2 \geq 5$ and also $j \geq \frac{12\alpha+16}{5}$, so that $\alpha \leq \frac{5d-16}{27}$. Plugging in (4.3) and using (4.7) and (4.8) we contradict (ii) and the case $r = 5$ is done.

When $r = 6$ we see that (4.9) implies $m_1 \geq 4$ and $\alpha \leq \frac{d-1}{4}$, but also $i > \frac{8\alpha+2}{5}$, so that $\alpha < \frac{5d+3}{23}$. Also (4.10) implies $m_2 \geq 4$, so that $i \geq \frac{8\alpha+20}{5}$, hence $\alpha \leq \frac{5d-15}{23}$. Plugging in (4.2) we contradict (iii). On the other hand (4.11) implies $m_1 \geq 4$, so that $\alpha < \frac{d-1}{4}$, and also $j > \frac{4\alpha-2}{3}$, so that $\alpha < \frac{3d+2}{13}$. Moreover (4.12) implies $m_2 \geq 4$ so that $\alpha \leq \frac{d-5}{4}$, and also $j \geq \frac{4\alpha+7}{3}$, so that $\alpha \leq \frac{3d-7}{13}$. Plugging in (4.3) and using (4.7) and (4.8) we contradict (iii) and we have finished the case $r = 6$.

Now assume that $r = 7$. We have that (4.9) implies $m_1 \geq 3$ and $i \geq \alpha + 1$, so that $\alpha \leq \frac{d}{4}$. Plugging in (4.2) we contradict (iv). On the other hand (4.11) implies $m_1 \geq 3$ and $j > \frac{4\alpha-4}{5}$, so that $\alpha < \frac{5d+4}{19}$. Moreover (4.12) implies $m_2 \geq 3$ and $j \geq \frac{4\alpha+1}{5}$, so that $\alpha \leq \frac{5d-1}{19}$. Plugging in (4.3) and using (4.7) and (4.8) we contradict (iv). This concludes the case $r = 7$.

If $r = 8$ we find from (4.10) that $m_2 \geq 3$ and $i \geq \frac{\alpha+6}{2}$, so that $\alpha \leq \frac{2d-4}{7}$. Plugging in (4.2) we contradict (v). On the other hand (4.12) implies $m_2 \geq 3$ so that $\alpha \leq \frac{d-4}{3}$ and also $j \geq \frac{3\alpha+11}{7}$, so that $\alpha \leq \frac{7d-11}{24}$. Plugging in (4.3) we contradict (v) and the case $r = 8$ is proved.

Now let $r = 9$. Then (4.10) gives $m_2 \geq 3$ and $i \geq \frac{2\alpha+23}{8}$, so that $\alpha \leq \frac{8d-15}{26}$. Plugging in (4.2) we contradict (vi). On the other hand (4.12) gives $m_2 \geq 2$ and $j \geq 3$, so that $\alpha \leq \frac{d-3}{3}$. We also get $m_2 = 2$ if and only if $(d, g) = (30, 33), (30, 34)$. Then, if $(d, g) = (30, 33), (30, 34)$, we see that (4.12) implies $m_2 \geq 3$ and $j \geq \frac{2\alpha+14}{9}$, so that $\alpha \leq \frac{9d-14}{29}$. Plugging in (4.3) we contradict (vi) and we are done with the case $r = 9$.

Finally let us do the case $r = 10$. We see that (4.10) gives $m_2 \geq 2$ and $i \geq 4$, so that $\alpha \leq \frac{d-3}{3}$. Plugging in (4.2) we contradict (vii). On the other hand (4.11) gives $m_1 \geq 3$ and $j > \frac{\alpha-4}{11}$, so that $\alpha < \frac{11d+4}{34}$. Also (4.12) implies $m_2 \geq 2$ and $j \geq 2$, so that $\alpha \leq \frac{d-2}{3}$. Plugging in (4.3) we contradict (vii) and we are done with the case $r = 10$. □

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