Strip Planarity Testing for Embedded Planar Graphs*

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Abstract. In this paper we introduce and study the *strip planarity testing* problem, which takes as an input a planar graph $G(V, E)$ and a function $\gamma : V \rightarrow \{1, 2, \ldots, k\}$ and asks whether a planar drawing of G exists such that each edge is represented by a curve that is monotone in the y-direction and, for any $u, v \in V$ with $\gamma(u) < \gamma(v)$, it holds that $y(u) < y(v)$.
The problem has strong relationships with some of the most deeply studied variants of the planarity testing problem, such as *clustered planarity*, *upward planarity*, and *level planarity*. Most notably, we provide a polynomial-time reduction from strip planarity testing to clustered planarity.
We show that the strip planarity testing problem is polynomial-time solvable if G has a prescribed combinatorial embedding.

Keywords: planarity, clustered planarity, upward planarity, level planarity, embedded graphs

1 Introduction

Testing the planarity of a given graph is one of the oldest and most deeply investigated problems in algorithmic graph theory. A celebrated result of Hopcroft and Tarjan [26] states that the planarity testing problem is solvable in linear time.

A number of interesting variants of the planarity testing problem have been considered in the literature [34]. Such variants mainly focus on testing, for a given planar graph G, the existence of a planar drawing of G satisfying certain constraints. For example the *partial embedding planarity* problem [3,29] asks whether a planar drawing $\mathcal{G}$ of a given planar graph G exists in which the drawing of a subgraph H of G in $\mathcal{G}$ coincides with a given drawing $\mathcal{H}$ of H. *Clustered planarity testing* [14,19,30], *upward planarity testing* [7,23,27], *level planarity testing* [31], *embedding constrained planarity testing* [24], *radial level planarity testing* [6], and *clustered level planarity testing* [5,20] are further examples of problems falling in this category.

In this paper we introduce and study the *strip planarity testing* problem, which is defined as follows. The input of the problem consists of a planar graph $G(V, E)$ and of

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a function $\gamma : V \rightarrow \{1, 2, \ldots, k\}$. The problem asks whether a *strip planar* drawing of (G, γ) exists, i.e. a planar drawing of G such that each edge is represented by a curve that is monotone in the y-direction and, for any $u, v \in V$ with $\gamma(u) < \gamma(v)$, it holds $y(u) < y(v)$. The name "strip" planarity comes from the fact that, if a strip planar drawing Γ of (G, γ) exists, then k disjoint horizontal strips $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be drawn in Γ so that γ_i lies below γ_{i+1}, for $1 \leq i \leq k-1$, and so that γ_i contains a vertex x of G if and only if $\gamma(x) = i$, for $1 \leq i \leq k$. It is not difficult to argue that strips $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be given as part of the input, and the problem is to decide whether G can be planarly drawn so that each edge is represented by a curve that is monotone in the y-direction and each vertex x of G with $\gamma(x) = i$ lies in the strip γ_i. That is, arbitrarily predetermining the placement of the strips does not alter the possibility of constructing a strip planar drawing of (G, γ). Further, the strip planarity of an instance would not change if we required edges to be straight-line segments. In fact, by results of Eades et al. [17] and of Pach and Tóth [33], a planar drawing in which edges are y-monotone curves can be converted into a planar straight-line drawing in which each vertex maintains its y-coordinate.

1.1 Strip Planarity and Other Planarity Variants

Before describing our results, we discuss the strong relationships of the strip planarity testing problem with three famous graph drawing problems.

Strip planarity and clustered planarity. The *c-planarity testing* problem, introduced by Feng et al. in [19], takes as an input a *clustered graph* $C(G, T)$, that is a planar graph G together with a rooted tree T, whose leaves are the vertices of G. Each internal node μ of T is a *cluster* and is associated with the set V_μ of vertices of G in the subtree of T rooted at μ. The problem asks whether a *c-planar drawing* exists, that is a planar drawing of G together with a drawing of each cluster μ of $C(G, T)$ as a simple closed region R_μ so that: (i) if $v \in V_\mu$, then $v \in R_\mu$; (ii) if $V_\nu \subset V_\mu$, then $R_\nu \subset R_\mu$; (iii) if $V_\nu \cap V_\mu = \emptyset$, then $R_\nu \cap R_\mu = \emptyset$; and (iv) each edge of G intersects the boundary of R_μ at most once. Determining the time complexity of testing the c-planarity of a given clustered graph is a long-standing open problem.

Surprisingly, no c-planarity testing algorithm is known even in the case in which the clustered graph $C(G, T)$ is *flat* and *embedded*. That is, every cluster is a child of the root of T and a combinatorial embedding for G (i.e., an order of the edges incident to each vertex) is fixed in advance; then, the c-planarity testing problem asks whether a c-planar drawing exists in which G has the prescribed combinatorial embedding. This natural variant of the c-planarity testing problem is well-studied [11,12,14,28,30], due to the fact that several NP-hard graph drawing problems are polynomial-time solvable in the fixed embedding scenario [7,23,36] and that testing c-planarity of embedded flat clustered graphs generalizes testing c-planarity of the notable class of triconnected flat clustered graphs. Yet determining the time complexity of testing c-planarity for this innocent-looking case eludes an answer.

An instance (G, γ) of the strip planarity testing problem naturally defines a flat clustered graph $C(G, T)$, where T consists of a root having k children $\mu_1, \ldots, \mu_k$ and, for every $1 \leq j \leq k$, cluster μ_j contains every vertex x of G such that $\gamma(x) = j$. The c-planarity of $C(G, T)$ is a necessary condition for the strip planarity of (G, γ), since

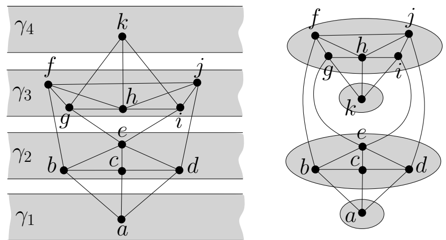


Fig. 1. A negative instance (G, γ) of the strip planarity testing problem whose associated clustered graph $C(G, T)$ is c-planar. Vertices are drawn in the strip (left drawing) and in the cluster (right drawing) they belong to.

suitably bounding the strips in a strip planar drawing of (G, γ) provides a c-planar drawing of $C(G, T)$. On the other hand, the c-planarity of $C(G, T)$ is not sufficient for the strip planarity of (G, γ) (see Fig. 1). However, we will prove that a different reduction from (G, γ) yields a flat clustered graph $C(G, T)$ whose c-planarity is in fact a necessary and sufficient condition for the strip planarity of (G, γ); in other words, we will prove that the strip planarity testing problem reduces in polynomial time to the c-planarity testing problem for flat clustered graphs. Furthermore, it turns out that strip planarity testing *coincides* with a special case of a problem posed by Cortese et al. [12,13] and related to c-planarity testing. The problem asks whether a graph G can be planarly embedded "inside" an host graph H, which can be thought as having "fat" vertices and edges, with each vertex and edge of G drawn inside a prescribed vertex and a prescribed edge of H, respectively. The strip planarity testing problem coincides with this problem in the case in which H is a path.

Strip planarity and level planarity. The *level planarity testing* problem takes as an input a planar graph $G(V, E)$ and a function $\gamma : V \to \{1, 2, \ldots, k\}$ and asks whether a planar drawing of G exists such that each edge is represented by a curve that is monotone in the y-direction and each vertex $u \in V$ is drawn on the horizontal line $y = \gamma(u)$. The level planarity testing (and embedding) problem is known to be solvable in linear time [31], although a sequence of incomplete characterizations by forbidden subgraphs [21,25] (see also [18]) has revealed that the problem is not yet fully understood. Level drawings are widely used in applicative contexts in which a hierarchical graph has to be visualized while conveying its hierarchical information; see the seminal work by Sugiyama et al. [35] and the recent survey by Healy and Nikolov [37, Chapter 13].

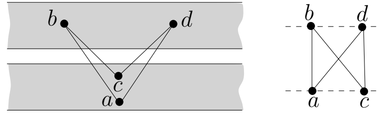


Fig. 2. A positive instance (G, γ) of the strip planarity testing problem that is not level planar.

The similarity of the level planarity testing problem with the strip planarity testing problem is evident: They have the same input, they both require planar drawings with y-monotone edges, and they both constrain the vertices to lie in specific regions of the plane; they only differ in that such regions are horizontal lines in one case, and horizontal strips in the other one. Clearly the level planarity of an instance (G, γ) is a sufficient condition for the strip planarity of (G, γ), as a level planar drawing is also a strip planar drawing. However, it is easy to construct instances (G, γ) that are strip planar and yet not level planar, even if we require that the instances are *strict*, i.e., no edge (u, v) is such that $\gamma(u) = \gamma(v)$. See Fig. 2. Hence, our new drawing style enlarges the spectrum of hierarchical graphs that can be visualized in a planar hierarchical fashion. We remark that the approach of [31] seems to be not applicable to test the strip planarity of a graph. Namely, Jünger et al. [31] visit the instance (G, γ) one level at a time, representing with a PQ-tree [8] the possible orders of the vertices in level i that are consistent with a level planar embedding of the subgraph of G induced by levels $\{1, 2, \dots, i\}$. However, when visiting an instance (G, γ) of the strip planarity testing problem one strip at a time, PQ-trees seem to be not powerful enough to represent the possible orders of the vertices in strip i that are consistent with a strip planar embedding of the subgraph of G induced by strips $\{1, 2, \dots, i\}$.

Strip planarity and upward planarity. The *upward planarity testing* problem asks whether a given directed graph $\overrightarrow{G}$ admits an *upward planar drawing*, i.e., a drawing which is planar and such that each edge is represented by a curve monotonically increasing in the y-direction, according to its orientation. Testing the upward planarity of a directed graph $\overrightarrow{G}$ is an $\mathcal{NP}$-hard problem [23], however it is polynomial-time solvable, e.g., if $\overrightarrow{G}$ has a fixed embedding [1,7], or if it has a single source [27], or if it has a series-parallel structure [16].

A strict instance (G, γ) of the strip planarity testing problem naturally defines a directed graph $\overrightarrow{G}$, by directing an edge (u, v) of G from u to v if $\gamma(u) < \gamma(v)$. It is easy to argue that the upward planarity of $\overrightarrow{G}$ is a necessary and not always sufficient condition for the strip planarity of (G, γ) (see Fig.s 3(a) and 3(b)). Roughly speaking, in an upward planar drawing different parts of the graph are free to "nest" one into the other, while in a strip planar drawing, such a nesting is only allowed if coherent with the strip assignment.

1.2 Our Results

In this paper, we show that the strip planarity testing problem is cubic-time solvable for planar graphs with a fixed combinatorial embedding. In this setting, the graph is given together with a combinatorial embedding and any strip planar drawing is required to respect such an embedding. This result enlarges the spectrum of graph drawing problems for which a polynomial-time solution is known only if the input has a prescribed combinatorial embedding (e.g., *upward planarity testing* [7,23] and *bend minimization in orthogonal drawings* [23,36]). Our approach considers each of the linearly-many plane embeddings corresponding to the given combinatorial embedding separately. For each of them, we perform a sequence of modifications to the input instance (G, γ) (such modifications consist mainly of insertions of graphs inside the faces of G) that

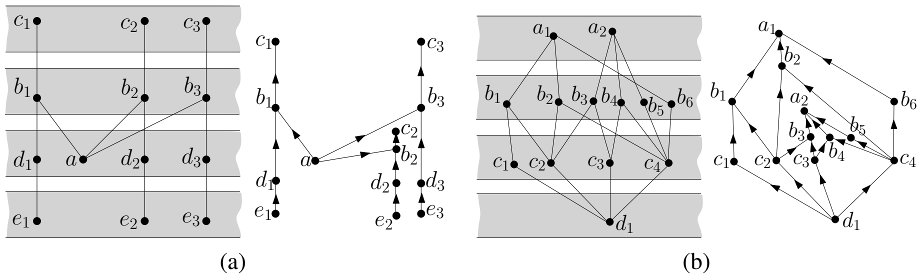


Fig. 3. Two negative instances (G_1, γ_1) (a) and (G_2, γ_2) (b) of the strip planarity testing problem whose associated directed graphs are upward planar, where G_1 is a tree and G_2 is a subdivision of a triconnected plane graph.

ensure that the instance satisfies progressively stronger constraints while not altering its strip planarity. Eventually, the strip planarity of (G, γ) becomes equivalent to the upward planarity of its associated directed plane graph, which can be tested in quadratic time [7].

We also show a polynomial-time reduction from the strip planarity testing problem (for graphs without a fixed plane embedding) to the c-planarity testing problem for flat clustered graphs, deepening the relationship between such problems. This reduction further justifies the study of the relationships between upward planarity and strip planarity. In fact, if we were able to prove that the upward planarity and the strip planarity problems have the same computational complexity (up to polynomial factors) not only in the fixed embedding scenario but also in the variable embedding one, we could infer that the c-planarity problem is NP-hard.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries; in Section 3 we show a quadratic-time algorithm to test the strip planarity of graphs with fixed plane embedding; in Section 4 we show a polynomial-time reduction from the strip planarity testing problem to the c-planarity testing problem; finally, in Section 5 we conclude and present open problems.

2 Preliminaries

In this section we present some definitions and terminology used throughout the paper.

A *drawing* of a graph is a mapping of each vertex to a distinct point of the Euclidean plane and of each edge to a Jordan arc between the endpoints of the edge. Even when not specified, throughout the paper we will only consider the Euclidean metric. A *planar drawing* is such that no two edges intersect except, possibly, at common endpoints. A planar drawing of a graph determines a clockwise order of the edges incident to each vertex. Two drawings of the same graph are *equivalent* if they determine the same clockwise orders around each vertex. A *planar embedding* (or *combinatorial embedding*) is an equivalence class of planar drawings. A planar drawing partitions the plane into path-connected regions, called *faces*, which are the complement of the union of the points and arcs to which the vertices and edges of the graph are mapped, respectively. The unbounded face is the *outer face*, while the other faces are *internal*.

Two planar drawings with the same combinatorial embedding have faces delimited by the same sequences of edges. However, such drawings could still differ for their outer faces. A *plane embedding* of a graph G is a combinatorial embedding of G together with a choice for its outer face. A *plane graph* is a graph together with a fixed plane embedding.

A plane graph G is *simple* if it contains neither parallel edges nor self-loops. Otherwise, G is a multi-graph. In the remainder of the paper, we always assume the considered plane graphs to be simple, unless otherwise specified.

In this paper we will show how to test in quadratic time whether a graph with a prescribed *plane* embedding is strip planar. Since an n-vertex graph with a fixed combinatorial embedding has $O(n)$ choices for its outer face, this implies that the strip planarity of a graph with a prescribed *combinatorial* embedding can be tested in cubic time. In the remainder of this section and in Section 3, we will assume all the considered graphs to have a prescribed plane embedding, even when not explicitly mentioned.

A graph is *connected* if there is a path between every pair of vertices. A graph G with at least k vertices is k*-connected* if removing any $k-1$ vertices leaves G connected. A *cutvertex* is a vertex whose removal disconnects the graph. A *block* of a graph $G(V,E)$ is a maximal (both in terms of vertices and in terms of edges) 2-connected subgraph of G. Also, for sake of readability, we denote by $|G|$ the number of vertices of a graph G.

2.1 Geometric Tools

In the paper, we exploit the following two geometric lemmata.

The first lemma revolves around the following setting. Consider a planar drawing Γ of a graph G, consider a face f of G in Γ and denote by c a y-monotone curve on the boundary of f in Γ. Let u_c and v_c be the end-points of c, with $y_u = y(u_c) < y_v = y(v_c)$. Refer to Fig. 4(a). We have the following.

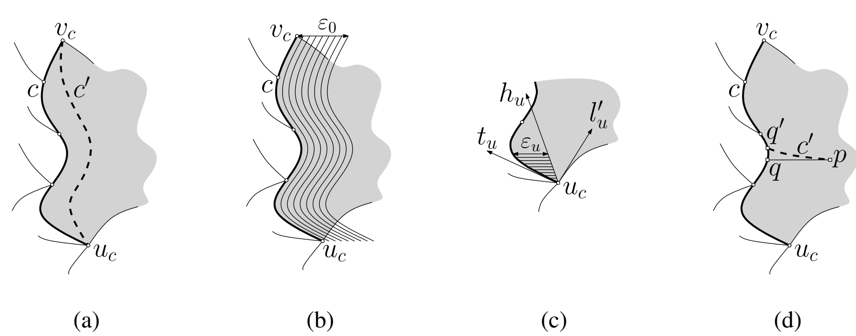


Fig. 4. (a) Illustration for the statement of Lemma 1. (b) The family of y-monotone curves $\{c'(\varepsilon)|\varepsilon > 0\}$. (c) A close look-up at the neighborhood of vertex u_c for the computation of ε_u. The gray-shaded area is part of f. (d) Illustration for the statement of Lemma 2.

Lemma 1. *There exists a simple y-monotone curve c' with end-points u_c and v_c such that: (i) the interior of c' lies in the interior of f, and (ii) the interior of the region delimited by c and by c' has no intersection with Γ.*

Proof: Assume that f is to the right of c, as the case in which it is to the left is analogous.

First, consider a family of y-monotone curves $\{c'(\varepsilon)|\varepsilon > 0\}$, where $c'(\varepsilon)$ is obtained as the translation of c by a vector $(\varepsilon, 0)$; see Fig. 4(b). Observe that, as $\varepsilon \to 0$, we have that $c'(\varepsilon)$ tends to c. This, together with the fact that c has f to its right, implies that there exists a positive ε_0 with the property that, for every $0 < \varepsilon \leq \varepsilon_0$, the only intersections between $c'(\varepsilon)$ and Γ happen between $c'(\varepsilon)$ and curves in Γ incident to u_c and leaving u_c upwards and to the right of c (or incident to v_c and leaving v_c downwards and to the right of c).

Second, let s_u be the slope of the tangent t_u to c at u_c; refer to Fig. 4(c). If there exists a curve in Γ leaving u_c upwards and to the right of c, then let l'_u be the first such curve (in clockwise order after c) and let s'_u be the slope of the tangent to l'_u at u_c. If such a curve does not exist, then let $s'_u = 0$. Consider the half-line h_u with slope $(s_u + s'_u)/2$. Note that h_u intersects the interior of f. In fact, h_u leaves u_c to the right of c and to the left of l'_u, if the latter curve is defined. Hence, there exists an $\varepsilon_u > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_u$, there exists a point $p(\varepsilon)$ on h_u with the property that the horizontal distance between $p(\varepsilon)$ and a point of c is ε, and the open straight-line segment between $p(\varepsilon)$ and u_c lies in f.

Third and analogously, let s_v be the slope of the tangent to c at v_c. If there exists a curve in Γ leaving v_c downwards and to the right of c, then let l'_v be the first such curve (in counter-clockwise order after c) and let s'_v be the slope of the tangent to l'_v at v_c. If such a curve does not exist, then let $s'_v = 0$. Consider the half-line h_v with slope $(s_v + s'_v)/2$. Note that h_v intersects the interior of f. Hence, there exists an $\varepsilon_v > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_v$, there exists a point $q(\varepsilon)$ on h_v with the property that the horizontal distance between $q(\varepsilon)$ and a point of c is ε, and the open straight-line segment between $q(\varepsilon)$ and v_c lies in f.

Now, let $\varepsilon_m = \min\{\varepsilon_0, \varepsilon_u, \varepsilon_v\}$. Then, by construction, the curve c' composed of (i) the straight-line segment between u_c and $p(\varepsilon_m)$, (ii) the part of $c'(\varepsilon_m)$ between $p(\varepsilon_m)$ and $q(\varepsilon_m)$, and (iii) the straight-line segment between $q(\varepsilon_m)$ and v_c satisfies the statement. □

In the proofs exhibited in this paper, when we draw c' inside f as just explained, we say that c' *follows* c *inside* f.

The second lemma deals with a similar setting. Consider a planar drawing Γ of a graph G, consider a face f of G in Γ and denote by c a y-monotone curve on the boundary of f in Γ. Let u_c and v_c be the end-points of c, with $y_u = y(u_c) < y_v = y(v_c)$. Let p be a point which is in the interior or on the boundary of f, that does not belong to c, and such that $y_u < p < y_v$. Refer to Fig. 4(d).

Lemma 2. *Assume that the interior of the horizontal straight-line segment $\overline{pq}$ connecting p and the unique point q of c with $y(q) = y(p)$ has no intersection with Γ. Then there exists a y-monotone curve c' such that: (i) the end-points of c' are p and a point*

q' of c with $y(p) < y(q')$, (ii) the interior of c' lies in the interior of f, and (iii) the interior of c' has no intersection with Γ.

Proof: Assume that f is to the right of c, as the case in which it is to the left is analogous. Consider the horizontal half-line ℓ starting at p through q. By the assumption, ℓ is directed leftwards. Rotate ℓ in clockwise direction of an angle $\varepsilon > 0$, while keeping it fixed at p, obtaining a half-line ℓ_ε. Denote by q_ε the first (while traversing ℓ_ε from p) intersection of ℓ_ε with c, if any. Observe that, as $\varepsilon \to 0$, we have that ℓ_ε tends to ℓ and q_ε tends to q. This, together with the assumptions that $\overline{pq}$ has no intersection with Γ, that its interior belongs to f, and that $y_u < p < y_v$, implies that there exists a positive ε^* with the property that, for every $0 < \varepsilon \leq \varepsilon^*$, the curve c' defined as the straight-line segment $\overline{pq_\varepsilon}$ satisfies the required properties. □

In the proofs exhibited in this paper, when we draw c' inside f as just explained, we say that c' *moves slightly upward from p to c inside f*. A symmetric version of this lemma allows us to define a curve c' that *moves slightly downward from p to c inside f*.

2.2 Strip Planarity

We now define some concepts related to strip planarity.

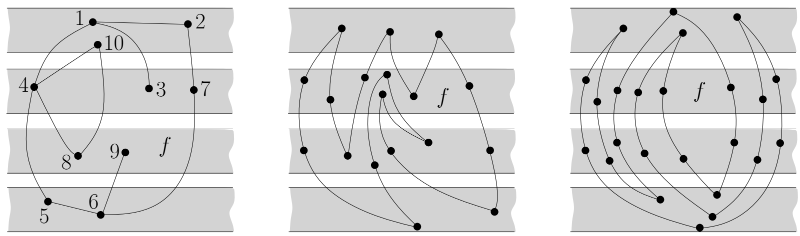
Definition 1. *An instance (G, γ) of strip planarity is* strict *if it contains no intra-strip edge, where an edge (u, v) is* intra-strip *if $\gamma(u) = \gamma(v)$.*

Definition 2. *An instance (G, γ) of strip planarity is* proper *if, for every edge (u, v) of G, it holds $\gamma(v) - 1 \leq \gamma(u) \leq \gamma(v) + 1$.*

For any face f of G, we denote by $C_f = (u_0, u_1, \ldots, u_l)$ the walk delimiting the boundary of f. Recall that G is not necessarily 2-connected, hence f might not be delimited by a simple cycle; also, if a vertex incident to f is a cut-vertex, it might appear several times in C_f. Consider any vertex occurrence u_j with $0 \leq j \leq l$. See Fig. 5(a). We say that u_j is a *local minimum* for f if $\gamma(u_j) \leq \gamma(u_{j-1})$ and $\gamma(u_j) \leq \gamma(u_{j+1})$, where indices are modulo $l + 1$. Analogously, we say that u_j is a *local maximum* for f if $\gamma(u_j) \geq \gamma(u_{j-1})$ and $\gamma(u_j) \geq \gamma(u_{j+1})$, where indices are modulo $l + 1$. Observe that several occurrences of the same vertex might be local minima or maxima for f. In the remainder of the paper, we often say "the number of minima and maxima" of an instance (G, γ) of strip planarity, as a short form for "the number of distinct pairs (v_j, g) such that vertex occurrence v_j is a local minimum or maximum for face g of G". Further, we say that u_j is a *global minimum* for f (a *global maximum* for f) if $\gamma(u_j) \leq \gamma(u_i)$ (resp. $\gamma(u_j) \geq \gamma(u_i)$), for every $i \neq j$ with $0 \leq i \leq l$.

Let (G, γ) be a 2-connected strict proper instance of the strip planarity testing problem. A path $(u_1, \ldots, u_j)$ in G is *monotone* if $\gamma(u_i) = \gamma(u_{i-1}) + 1$, for every $2 \leq i \leq j$. Consider any face f; since G is 2-connected, C_f is a simple cycle. A global minimum u_m and a global maximum u_M for f are *consecutive* in f if no global minimum and no global maximum exists in one of the two paths connecting u_m and u_M in C_f. A local minimum u_m and a local maximum u_M for a face f are *visible* if one of the paths P connecting u_m and u_M in C_f is such that, for every vertex u of P, it holds $\gamma(u_m) < \gamma(u) < \gamma(u_M)$.

We conclude the section with the following definitions.



(a) (b) (c)

Fig. 5. (a) The boundary of a face f in an instance of strip planarity. The walk delimiting f is $C_f = (1, 2, 7, 6, 9, 6, 5, 4, 8, 10, 4, 1, 3, 1)$. Both occurrences of vertex 6 are local and global minima for f; vertex 8 is a local minimum and not a global minimum for f; vertex 7 is neither a local minimum nor a local maximum for f. (b) The boundary of a face f in a quasi-jagged instance of strip planarity. (c) The boundary of a face f in a jagged instance of strip planarity.

Definition 3. *An instance (G, γ) of strip planarity is* quasi-jagged *if it is* 2*-connected, strict, proper and if, for every face f of G and for any two visible local minimum u_m and local maximum u_M for f, one of the two paths connecting u_m and u_M in C_f is monotone (see Fig. 5(b)).*

Definition 4. *An instance (G, γ) of strip planarity is* jagged *if it is* 2*-connected, strict, proper and if, for every face f of G, any local minimum for f is a global minimum for f, and every local maximum for f is a global maximum for f (see Fig. 5(c)).*

Roughly speaking, a jagged instance is such that the boundary of any face consists of a sequence of monotone paths, each connecting a global minimum and a global maximum. The property of quasi-jagged instances is weaker: If a local minimum u_m and a local maximum u_M for a face f are visible (i.e., one of the two paths connecting u_m and u_M on the boundary of f is composed of vertices all lying on intermediate strips between $\gamma(u_m)$ and $\gamma(u_M)$), then one of the two paths connecting u_m and u_M on the boundary of f is monotone (however, u_m and u_M might not be a global minimum and a global maximum). Observe that a jagged instance (G, γ) is also quasi-jagged.

3 How To Test Strip Planarity

In this section we describe an algorithm to test strip planarity. In Sections 3.1– 3.6, we will assume every considered strip planarity instance to be connected. We will show in Section 3.7 how to extend our polynomial-time algorithm to non-connected instances.

In Section 3.1 we show how to reduce a general instance to an equivalent strict instance. In Section 3.2 we show how to reduce a strict instance to an equivalent strict proper instance. In Section 3.3 we show how to reduce a strict proper instance to an equivalent 2-connected strict proper instance. In Section 3.4 we show how to reduce a 2-connected strict proper instance to an equivalent quasi-jagged instance. In Section 3.5 we show how to reduce a quasi-jagged instance to an equivalent jagged instance. Finally, in Section 3.6 we show that testing the strip planarity of a jagged instance is equivalent to test the upward planarity of the associated directed graph.

3.1 From a General Instance to a Strict Instance

In this section we show how to reduce a general instance of the strip planarity testing problem to an equivalent strict instance.

Lemma 3. *Let (G, γ) be an instance of the strip planarity testing problem with n vertices, k strips, and r minima and maxima.*

There exists an $O(n)$-time algorithm that either decides that (G, γ) is not strip planar or constructs an equivalent strict instance (G^*, γ^*) with at most n vertices, k strips, and at most r minima and maxima. Graph G^* might be a multi-graph; however, G^* has no self-loops and no parallel edges between vertices belonging to the same strip.

We construct (G^*, γ^*) from (G, γ) by repeatedly *contracting* intra-strip edges.

Consider an instance (H, γ) of strip planarity in which H is a plane graph. The operation of contracting an edge (u, v) in (H, γ) results in a new instance (H', γ'), where u and v are identified to be the same vertex w, with $\gamma'(w) = \gamma(u) = \gamma(v)$ and with $\gamma'(x) = \gamma(x)$ for every $x \neq w$ in H'. The edges incident to w are all the edges incident to u and v, except for the contracted edge; the clockwise order of the edges incident to w is: All the edges incident to u in H in the same clockwise order starting at (u, v), and then all the edges incident to v in H in the same clockwise order starting at (v, u).

While performing an edge contraction in (H, γ), multiple parallel edges might arise in (H', γ'). In the following we show how to ensure that any intermediate graph (G', γ') that is constructed from (G, γ) by contracting some intra-strip edges is a plane multi-graph with no self-loops and with no parallel intra-strip edges; that is, the only parallel edges connect vertices belonging to distinct strips. Observe that the starting plane graph G indeed satisfies these properties.

Consider any intra-strip edge (u, v) in (G, γ). Since (G, γ) has no parallel intra-strip edges, the contraction of (u, v) does not result in the creation of self-loops incident to w in (G', γ'). However, the contraction of (u, v) might result in the creation of parallel edges, which happens if G has a cycle (u, v, z). Consider each vertex z such that G has a cycle (u, v, z). If z belongs to a strip different from the one of u and v, then the two created parallel edges (w, z) are inter-strip. Otherwise, $\gamma(z) = \gamma(u) = \gamma(v)$; then, we check whether all the vertices of G that lie inside cycle (u, v, z) belong to the strip corresponding to $\gamma(u)$. If this is the case, we remove from (G', γ') all the vertices (and their incident edges) inside cycle (u, v, z) as well as one of the two copies of edge (w, z). Otherwise, we conclude that (G, γ) is not strip planar. Namely, if (u, v, z) contains in its interior a vertex not belonging to $\gamma(u)$, then it is not possible to draw plane graph G with the edges of (u, v, z) being y-monotone curves and with each vertex drawn in the strip it belongs to.

If we did not conclude that (G, γ) is not strip planar, then we have an instance (G', γ') such that G' is a plane multigraph with no self-loops and whose only parallel edges connect vertices belonging to distinct strips. In the next claim, we prove that the described operation does not alter the strip planarity of the instance.

Claim 1 *(G', γ') is strip planar if and only if (G, γ) is strip planar.*

Proof: We first prove the sufficiency. Consider any strip planar drawing Γ of (G, γ) (see Fig. 6(a)). Assume that $2 \leq \gamma(u) \leq k-1$. Denote by $p_1, p_2, \ldots, p_h$ and by $q_1, q_2, \ldots, q_l$ the left-to-right order of the intersection points of the edges of G with the lines delimiting the strip corresponding to $\gamma(u)$ from the top and from the bottom, respectively. Insert dummy vertices at points $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_l$. Each of these vertices splits an edge of G into two dummy edges, one inside the strip corresponding to $\gamma(u)$ and one outside it. Insert dummy edges (p_1, q_1), (p_h, q_l), (p_i, p_{i+1}), for $1 \leq i \leq h-1$, and (q_i, q_{i+1}), for $1 \leq i \leq l-1$, in the strip corresponding to $\gamma(u)$. Contract edge (u, v) into a single vertex w; further, for each vertex z with $\gamma(z) = \gamma(u) = \gamma(v)$ such that G has a cycle (u, v, z), remove all the vertices (and their incident edges) inside cycle (u, v, z) as well as one of the two copies of edge (w, z). Denote by L the subgraph of G' induced by the vertices inside or on the boundary of cycle $(p_1, p_2, \ldots, p_h, q_l, q_{l-1}, \ldots, q_1)$. Observe that L is a simple plane graph, given that the only parallel edges of G' are inter-strip. Triangulate the internal faces of L by inserting dummy vertices and edges, so that no edge connects two vertices p_i and p_j or q_i and q_j with $j \geq i+2$ (see Fig. 6(b)). Construct a convex straight-line drawing of L in which vertices $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_l$ have the same positions they have in Γ (see Fig. 6(c)). Such a drawing always exists [10]. Slightly perturb the positions of the vertices different from $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_l$, so that no two vertices have the same y-coordinate. As a consequence, the edges of L different from (p_i, p_{i+1}), for $1 \leq i \leq h-1$, and (q_i, q_{i+1}), for $1 \leq i \leq l-1$, are y-monotone curves. Removing the inserted dummy vertices and edges results in a strip planar drawing of (G', γ') (see Fig. 6(d)).

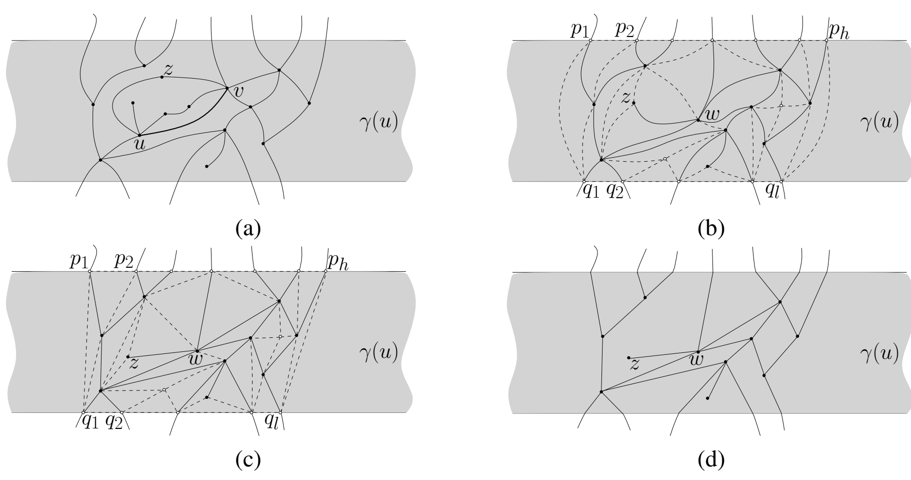


Fig. 6. (a) A strip planar drawing Γ of (G, γ). (b) Modifications performed on the part of G inside the strip corresponding to $\gamma(u)$, resulting in an internally-triangulated simple plane graph L. (c) A convex straight-line drawing of L. (d) A strip planar drawing of (G', γ').

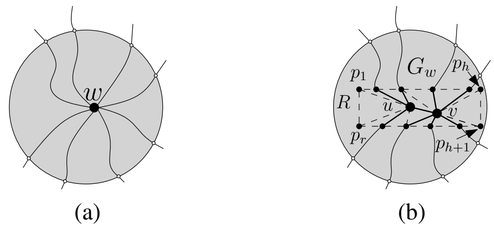


Fig. 7. (a) A disk D containing w. (b) Drawing graph G_w inside rectangle R.

The cases in which $\gamma(u) = 1$ or $\gamma(u) = k$ can be handled analogously to the case in which $2 \leq \gamma(u) \leq k - 1$; however, when $\gamma(u) = 1$ (the case in which $\gamma(u) = k$ is symmetric), points $q_1, q_2, \ldots, q_l$ are not defined. Then, we also insert points p_0 and p_{h+1} to the left of p_1 and to the right of p_h, respectively, and we insert a dummy vertex d in the strip corresponding to $\gamma(u)$ and dummy edges (p_0, d), (p_{h+1}, d), and (p_i, p_{i+1}), for $0 \leq i \leq h$. The remainder of the construction is the same as in the case in which $2 \leq \gamma(u) \leq k - 1$.

We now prove the necessity. Consider any strip planar drawing Γ' of (G', γ'). Slightly perturb the positions of the vertices in Γ', so that no two vertices have the same y-coordinate. Consider a disk D containing w, small enough so that it contains no vertex different from w, and it contains no part of an edge that is not incident to w (see Fig. 7(a)).

Consider a rectangle R enclosing w in Γ' that is entirely contained inside D in such a way that each intersection point between the boundary of R and a curve incident to w lies either on the top or on the bottom side of R. Let $p_1, \ldots, p_h, p_{h+1}, \ldots, p_r$ be a set of points such (i) p_1, p_h, p_{h+1}, and p_r lie on the top-left, top-right, bottom-right, bottom-left corner of R, respectively; (ii) points $p_2, \ldots, p_{h-1}$ are the intersection points between edges incident to w and the top side of R, in this left-to-right order along the top side of R; (iii) points $p_{h+2}, \ldots, p_{r-1}$ are the intersection points between edges incident to w and the bottom side of R, in this right-to-left order along the bottom side of R. Replace each point p_i, for $i = 1, \ldots, r$, with a dummy vertex d_i. Draw a straight-line segment between d_i and d_{i+1}, for each $i = 1, \ldots, r$ (where $d_{r+1} = d_1$). Then, remove from Γ' vertex w and all the curves connecting w to a dummy vertex. Initialize an auxiliary graph G_w to a cycle $C = (d_1, \ldots, d_r, d_1)$. Add to G_w two adjacent vertices u and v, and a set of edges defined as follows. For each dummy vertex d_i, add to G_w an edge (d_i, u) (resp., an edge (d_i, v)) if the edge (x, w) in G' that is split by d_i corresponds to an edge (x, u) (resp., to an edge (x, v)) in G. Observe that, all the dummy vertices adjacent to u (to v) appear consecutively in C, since all the edges incident to w corresponding to edges incident to u (and to v, as a consequence) appear consecutively around w in G'. Hence, G_w is a planar graph. Triangulate each face of G_w, except for the one delimited by C, without introducing any edge connecting vertices of C.

Add to Γ a planar straight-line drawing Γ_w of G_w such that the outer-face of G_w is delimited by cycle C and is represented by rectangle R in such a way that each dummy vertex d_i, for $i = 1, \dots, r$, of G_w lies on point p_i in Γ. This can be done by using any of the algorithms to construct straight-line planar drawings of a planar triconnected graph with a prescribed convex outer-face [10], since G_w is triconnected (and, hence, G_w does not contain chords between vertices of C), by construction, and cycle C is drawn as a rectangle.

Finally, remove all the edges belonging to cycle C from Γ and remove each dummy vertex by joining the two curves incident to it into a single curve, which now represents an edge of G incident to u or to v. Observe that such curves are y-monotone as they are the union of a y-monotone curve and of a straight-line segment lying entirely either above or below such a curve. Hence, Γ is a strip planar drawing of (G, γ). □

Lemma 3 easily follows from Claim 1. First, (G', γ') has at least one intra-strip edge fewer than (G, γ); hence, $O(n)$ repetitions of the above described operation eventually lead either to decide that (G, γ) is not strip planar or to construct an equivalent strict instance (G^*, γ^*). Further, G' has fewer vertices than G (hence G^* has at most n vertices). Moreover, the number of strips of (G', γ') is k (hence (G^*, γ^*) has k strips as well). Finally, the number of minima and maxima of (G', γ') is at most r (and the same holds for (G^*, γ^*)). To prove the running time, first observe that with an $O(n)$-time preprocessing we can determine whether each edge of G is intra-strip or not. Second, using the data structure described in [32], testing whether there exists a common neighbor z of u and v in a planar graph can be done in $O(1)$ time. This data structure is constructed with a linear-time preprocessing and can be updated in constant time. Finally, checking whether the plane subgraph contained within a triangle (u, v, z) is composed of vertices all belonging to the same strip as u, v, and z can be done in linear time in the size of this subgraph. However, once this test has been performed we either conclude that the instance is negative or we construct a new instance in which such a subgraph has been removed. Hence, the total running time is still linear in the size of G.

3.2 From a Strict Instance to a Strict Proper Instance

In this section we show how to reduce a strict instance of the strip planarity testing problem to an equivalent strict proper instance.

Lemma 4. *Let (G, γ) be a strict instance of the strip planarity testing problem with n vertices, k strips, and r minima and maxima, such that G is a plane multi-graph with no self-loops and with no intra-strip parallel edges.*

There exists an $O(kn)$-time algorithm that constructs an equivalent strict proper instance (G^*, γ^*) with $O(kn)$ vertices, $O(k)$ strips, and r minima and maxima, such that G^* is a simple plane graph.

Proof: For each two consecutive strips of (G, γ), we add a new strip between them. Namely, we construct an instance (G', γ') such that $G' = G$ and $\gamma'(v) = 2\gamma(v) - 1$, for each vertex v of G. Graph G^* is the graph obtained from G' by subdividing each edge (u, v) of G' such that $\gamma'(u) = \gamma'(v) + j$ with $j \geq 2$. More formally, the edge is replaced by path $(v = u_1, u_2, \dots, u_{j+1} = u)$ such that $\gamma^*(u_{i+1}) = \gamma^*(u_i) + 1$, for every $1 \leq i \leq j$.

Observe that (G^*, γ^*) has $O(kn)$ vertices, $2k-1$ strips, and the same number r of minima and maxima as (G, γ). The construction of (G^*, γ^*) can clearly be performed in $O(kn)$ time. Also, graph G^* is simple, since all the intra-strip parallel edges of G have been replaced by paths. Finally, the equivalence between the strip planarity of (G, γ) and the one of (G^*, γ^*) can be easily proved by interpreting the drawing of an edge (u, v) of G as a drawing of path $(v = u_1, u_2, \ldots, u_{j+1} = u)$ of G^* (with vertices $u_2, \ldots, u_j$ placed inside the corresponding strips) and vice versa. □

3.3 From a Strict Proper Instance to a 2-Connected Strict Proper Instance

In this section we show how to reduce a strict proper instance (G, γ) of the strip planarity testing problem to an equivalent 2-connected strict proper instance (G^*, γ^*).

The idea for the proof of the upcoming lemma is that, if G contains a cutvertex c, it can be augmented with a new vertex w, with $\gamma(w) = \gamma(c)$, and with new edges connecting w with two consecutive neighbors v_i and v_{i+1} of c, where v_i and v_{i+1} belong to different blocks of G. This augmentation does not change the strip planarity of the instance, and its repetition eventually leads to the desired instance (G^*, γ^*).

Lemma 5. *Let (G, γ) be a strict proper instance of the strip planarity testing problem with n vertices, k strips, and r minima and maxima.*

There exists an $O(n)$-time algorithm that constructs an equivalent 2-connected strict proper instance (G^*, γ^*) with $O(n)$ vertices, k strips, and $O(r)$ minima and maxima.

Proof: Let $(G(V, E), \gamma)$ be a strict proper instance of the strip planarity testing problem. First, we associate each edge $e \in E$ with the unique block of G it belongs to and with its two incident faces. This computation can be performed in total $O(n)$ time [38]. Denote by b the number of blocks of G. Note that, if $b = 1$, then (G, γ) is a 2-connected strict proper instance. Otherwise, consider any cutvertex c of G.

Let $e_1, e_2, \ldots, e_m, e_{m+1} = e_1$ be the clockwise order of the edges incident to c; let $e_i = (c, v_i)$ and $e_{i+1} = (c, v_{i+1})$ be two edges belonging to distinct blocks B_p and B_q of G, respectively, for some $1 \leq i \leq m$. Let f be the face of G that is to the left of e_i when traversing this edge from v_i to c. Insert a vertex w and edges (w, v_i) and (w, v_{i+1}) inside f. Let $V' = V \cup \{w\}$ and let $E' = E \cup \{(w, v_i), (w, v_{i+1})\}$; also, let G' be the graph (V', E'). Let $\gamma' : V' \to \{1, 2, \ldots, k\}$ be defined as follows: $\gamma'(u) = \gamma(u)$ for every vertex $u \in V$, and $\gamma'(w) = \gamma(c)$.

We claim that (G', γ') is an instance of the strip planarity testing problem that is equivalent to (G, γ). We first prove that the claim implies the lemma, and we then prove the claim. Refer to Fig. 8.

First, (G', γ') is proper and strict, given that (G, γ) is proper and strict, that $\gamma'(w) = \gamma'(c) = \gamma'(v_i) \pm 1$, and that $\gamma'(w) = \gamma'(c) = \gamma'(v_{i+1}) \pm 1$; further, the number of blocks of G' is equal to $b - 1$, since blocks B_p and B_q of G belong to the same block of G'. Hence, the repetition of the above augmentation eventually leads to a 2-connected strict proper instance (G^*, γ^*) that is equivalent to (G, γ) and that has $|G^*| = b - 1 + n \in O(n)$ vertices. The fact that (G^*, γ^*) contains $O(r)$ minima and maxima descends from the fact that G^* has the same faces of G, except for the two faces obtained by splitting

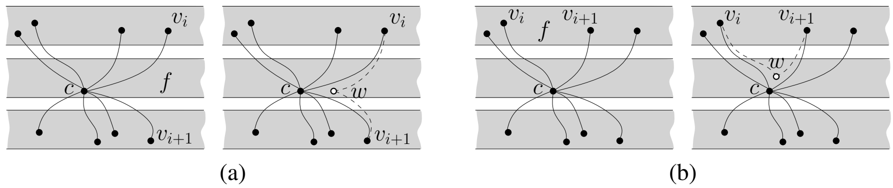


Fig. 8. Inserting vertex w and edges (w, v_i) and (w, v_{i+1}) inside f if (a) $\gamma(v_i) = \gamma(c) + 1 = \gamma(v_{i+1}) + 2$ and if (b) $\gamma(v_i) = \gamma(v_{i+1}) = \gamma(c) + 1$.

face f with path (v_i, w, v_{i+1}), and that w is incident to exactly two faces of G^*. Finally, the augmentation of (G, γ) to (G', γ') can be easily performed in $O(1)$ time (observe that, after the augmentation is performed, blocks B_p and B_q are given the same name, that is now associated to every edge in each of these blocks, in $O(1)$ time). Hence, the total running time is $O(n)$ given that $b \in O(n)$.

We now prove the claim. One direction is trivial. Namely, if (G', γ') is strip planar, then (G, γ) is strip planar, given that G is a subgraph of G' and given that $\gamma(u) = \gamma'(u)$ for every $u \in V$. We prove the other direction. Assume that (G, γ) is strip planar and let Γ be any strip planar drawing of (G, γ). We distinguish two cases:

1. In the first case, $\gamma(v_i) = \gamma(c) + 1$ and $\gamma(v_{i+1}) = \gamma(c) - 1$ (the case in which $\gamma(v_i) = \gamma(c) - 1$ and $\gamma(v_{i+1}) = \gamma(c) + 1$ is symmetric), as in Fig. 8(a). Let δ be the curve composed of the curves representing (v_i, c) and (v_{i+1}, c). Use Lemma 1 to draw a curve δ' that follows δ inside f. Place w at any point of δ inside $\gamma(c)$ (observe that δ' cuts $\gamma(c)$ as its end-points are inside strips $\gamma(c) - 1$ and $\gamma(c) + 1$). The resulting drawing Γ' of G' is strip planar. Namely, each vertex u of G' lies inside $\gamma'(u)$ (by assumption if $u \in V$ and by construction if $u = w$); further, each edge e of G' is represented by a y-monotone curve (by assumption if $e \in E$ and by Lemma 1 if e is incident to w); finally, Γ' is planar because Γ is planar (by assumption) and because edges (v_i, w) and (v_{i+1}, w) do not cross any edge of G, by Lemma 1.

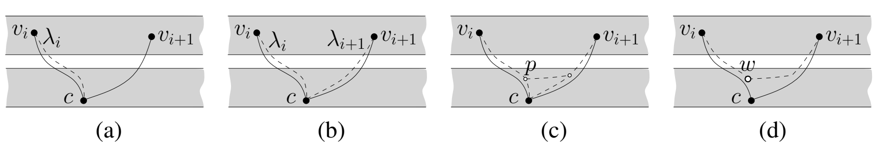


Fig. 9. Illustration for the proof of Lemma 5.

2. In the second case, $\gamma(v_i) = \gamma(v_{i+1}) = \gamma(c) + 1$ (the case in which $\gamma(v_i) = \gamma(v_{i+1}) = \gamma(c) - 1$ is symmetric), as in Fig. 8(b). Use Lemma 1 to draw a curve λ_i that follows the curve representing (c, v_i) inside f. See Fig. 9(a). Observe that λ_i partitions f into two faces; let f' be the face that contains the curve representing

(c, v_{i+1}) on its boundary. Use Lemma 1 to draw a curve λ_{i+1} that follows the curve representing (c, v_{i+1}) inside f'. See Fig. 9(b). Observe that λ_{i+1} partitions f' into two faces; let f'' be the face that contains λ_i and λ_{i+1} on its boundary. Let p be a point on λ_i inside strip $\gamma(c)$ and such that, for every vertex u with $y(u) > y(w)$, we have that $y(p) < y(u)$. Then the interior of the horizontal straight-line segment between p and the unique point of λ_{i+1} with y-coordinate equal to $y(p)$ lies in the interior of f''. Use Lemma 2 to draw a curve λ^* that moves slightly upward from p to λ_{i+1}. See Fig. 9(c). Place w at p. Represent edge (w, v_i) as the part of λ_i between p and v_i; also, represent edge (w, v_{i+1}) as the curve composed of λ^* and of the part of λ_{i+1} between its intersection with λ^* and v_{i+1}. See Fig. 9(d). The resulting drawing Γ' of G' is strip planar. Namely, each vertex u of G' lies inside $\gamma'(u)$ (by assumption if $u \in V$ and by construction if $u = w$); further, each edge e of G' is represented by a y-monotone curve (by assumption if $e \in E$ and by Lemmata 1 and 2 if e is incident to w); finally, Γ' is planar because Γ is planar (by assumption) and because edges (v_i, w) and (v_{i+1}, w) do not cross any edge of G, by Lemmata 1 and 2.

This concludes the proof of the claim and hence of the lemma. □

3.4 From a 2-Connected Strict Proper Instance to a Quasi-Jagged Instance

In this section we show how to reduce a 2-connected strict proper instance of the strip planarity testing problem to an equivalent quasi-jagged instance.

Lemma 6. *Let (G, γ) be a 2-connected strict proper instance of the strip planarity testing problem with n vertices, k strips, and r minima and maxima.*

There exists an $O(kr+n)$-time algorithm that constructs an equivalent quasi-jagged instance (G^*, γ^*) with $O(kr + n)$ vertices, k strips, and $O(r)$ minima and maxima.

Consider any face f of G containing two visible local minimum and maximum u_m and u_M, respectively, such that no path connecting u_m and u_M in C_f is monotone. Insert a monotone path connecting u_m and u_M inside f. Denote by (G^+, γ^+) the resulting instance of the strip planarity testing problem.

In the next claim, we prove that this augmentation does not alter the strip planarity of the instance. Note that drawing a monotone path connecting u_m and u_M inside f might not be possible in a given strip planar drawing Γ of (G, γ). In fact, any y-monotone curve between u_m and u_M might be forced to cross in Γ parts of G that are "intertwined" with the path P that connects u_m and u_M and such that $\gamma(u_m) < \gamma(v) < \gamma(u_M)$ holds for every internal vertex v of P. For this reason, we first perform a horizontal scaling of a portion of the drawing which moves away" from P the parts of G that are intertwined with P.

Claim 2 *(G^+, γ^+) is strip planar if and only if (G, γ) is strip planar.*

Proof: One direction of the equivalence is trivial, namely if (G^+, γ^+) is strip planar, then (G, γ) is strip planar, since G is a subgraph of G^+ and $\gamma(v) = \gamma^+(v)$ for every vertex v in G.

We prove the other direction. Consider a strip planar drawing Γ of (G, γ). Slightly perturb the positions of the vertices in Γ so that no two of them have the same y-coordinate. Denote by P and Q the two paths connecting u_m and u_M along C_f. Since u_m and u_M are visible, it holds $\gamma(u_m) < \gamma(v) < \gamma(u_M)$ for every internal vertex v of P or for every internal vertex v of Q. Assume that $\gamma(u_m) < \gamma(v) < \gamma(u_M)$ holds for every internal vertex v of P, the other case being analogous. We also assume w.l.o.g. that face f is to the right of P when traversing such a path from u_m to u_M. We modify Γ, if necessary, while maintaining its strip planarity so that a y-monotone curve $\mathcal{C}$ connecting u_m and u_M can be drawn inside f.

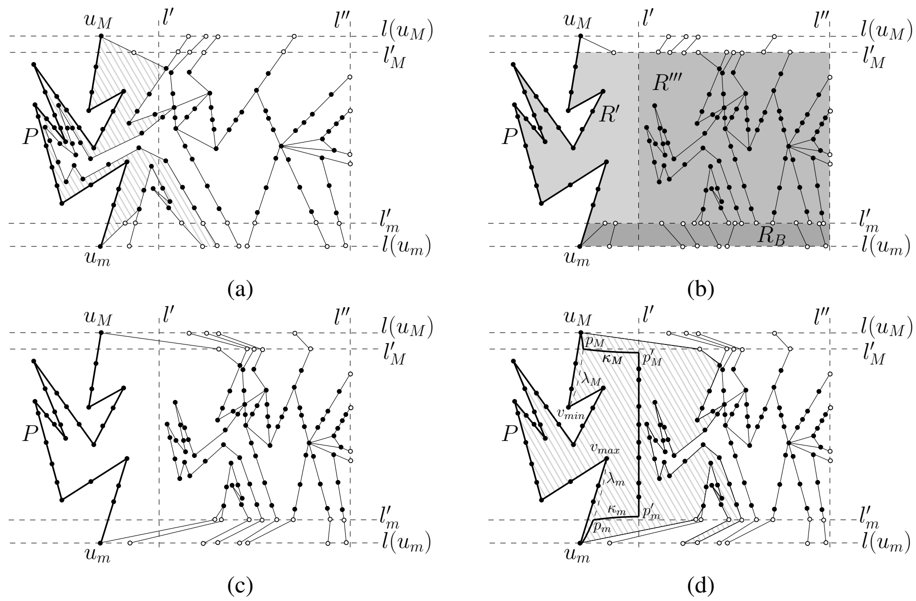


Fig. 10. (a) Drawing Γ inside region R. The part of face f inside R is colored gray. Path P is represented by a thick line. Intersection points of edges with lines l'', $l(u_m)$, $l(u_M)$, l'_m, and l'_M are represented by white circles. (b) Drawing Γ inside region R after the horizontal scaling. Regions R', R''', and R_B are represented as light, medium, and dark gray regions, respectively. (c) Reconnecting parts of edges that have been disconnected by the scaling. (d) Drawing of a monotone path connecting u_m and u_M inside f.

We introduce some notation. Refer to Fig. 10(a). Let $l(u_m)$ and $l(u_M)$ be the horizontal lines through u_m and u_M, respectively. Let l' and l'' be vertical lines entirely lying right of P, with l'' to the right of l'. Denote by D the distance between l' and l''. Denote by R the bounded region of the plane delimited by P, by $l(u_m)$, by $l(u_M)$, and by l''. Denote by l'_m (by l'_M) an horizontal line above $l(u_m)$ (resp. below $l(u_M)$) and sufficiently close to $l(u_m)$ (resp. to $l(u_M)$) so that the strip delimited by l'_m and $l(u_m)$ (resp. by l'_M and $l(u_M)$) does not contain any vertex of G other than u_m (resp. other than u_M). Finally, we define some regions inside R. Let R' be the bounded region of the plane delimited by P, by l'_m, by l'_M, and by l'; let R'' be the bounded region of the plane delimited by P, by l'_m, by l'_M, and by l''; let R''' be the bounded region of the plane delimited by l', by l'_m, by l'_M, and by l'' (observe that $R'' = R' \cup R'''$); let R_B be the bounded region of the plane delimited by P, by l'_m, by $l(u_m)$, and by l''; and let R_A be the bounded region of the plane delimited by P, by l'_M, by $l(u_M)$, and by l''. We are going to modify Γ in such a way that no vertex and no part of an edge lies in the interior of R'. The part of Γ outside R is not modified in the process.

We perform a horizontal scaling of the part of Γ that lies in the interior of R'' (the vertices of P stay still). This is done in such a way that every intersection point of an edge with l'' keeps the same position, and the distance between l'' and every point in the part of Γ that used to lie inside R'' becomes strictly smaller than D. See Fig. 10(b). Hence, the part of Γ that used to lie inside R'' is now entirely contained in R'''. However, some edges of G (namely those that used to intersect l'_m and l'_M) are now disconnected; e.g., if an edge of G used to intersect l'_m, now such an edge contains a line segment inside R''', which has been scaled, and a line segment inside R_B, whose drawing has not been modified by the scaling. By construction R_B does not contain any vertex in its interior. Hence, the line segments that lie in R_B form in Γ a planar y-monotone matching between a set A of points on l'_m and a set B of points on $l(u_m)$. As a consequence of the scaling, the position of the points in A has been modified, however their relative order on l'_m has not been modified. Thus, we can delete the line segments in R_B and reconnect the points in B with the new positions of the points in A on l'_m so that each edge is y-monotone and no two edges intersect. See Fig. 10(c). After performing an analogous modification in R_A, we obtain a planar y-monotone drawing Γ' of G in which no vertex and no part of an edge lies in the interior of R'. Since no vertex changed its y-coordinate and every edge is y-monotone, Γ' is a strip planar drawing of (G, γ).

Finally, we draw a y-monotone curve $\mathcal{C}$ connecting u_m and u_M. See Fig. 10(d). This is done as follows. Let v_{max} be the local maximum of f on P such that u_m and v_{max} are visible, and let v_{min} be the local minimum of f on P such that u_M and v_{min} are visible. Observe that, $v_{max} \neq u_M$ and $v_{min} \neq u_m$ since P is not monotone. Apply Lemma 1 to construct (i) a y-monotone curve λ_m between u_m and v_{max} inside f and (ii) a y-monotone curve λ_M between u_M and v_{min} inside f. Denote by p_m and by p_M the intersection points between λ_m and l'_m, and between λ_M and l'_M, respectively. Observe that, the portion of l'_m between p_m and its intersection with l' and the portion of l'_M between p_M and its intersection with l' lie inside f. Hence we can use Lemma 2 to draw a curve κ_m that moves slightly upward from p_m to a point p'_m on l' and a curve κ_M that moves slightly downward from p_M to a point p'_M on l' inside f. Curve $\mathcal{C}$ is composed of the portion of λ_m from u_m to p_m, the portion of κ_m from p_m to p'_m, the vertical segment from p'_m to p'_M, the portion of κ_M from p'_M to p_M, and the portion of λ_M from p_M to u_M. Place each vertex x of the monotone path connecting u_m and u_M on $\mathcal{C}$ at a suitable y-coordinate, so that x lies in the strip corresponding to $\gamma(x)$. Since $\mathcal{C}$ is y-monotone and since it entirely lies in the interior of f, we obtained a strip planar drawing of (G^+, γ^+), which concludes the proof. □

Claim 2 implies Lemma 6, as proved in the following.

First, the repetition of the above described augmentation leads to a quasi-jagged instance (G^*, γ^*). In fact, whenever the augmentation is performed, the resulting instance is clearly strict, proper, and 2-connected; further, the number of triples (v_m, v_M, g) such that vertices v_m and v_M are visible local minimum and maximum for face g, respectively, and such that both paths connecting v_m and v_M along C_g are not monotone decreases at least by 1, thus eventually the number of such triples is zero, and the instance is quasi-jagged.

Second, we prove that (G^*, γ^*) can be constructed from (G, γ) in $O(kr + n)$ time, that $|G^*| \in O(kr + n)$, and that there are $O(r)$ minima and maxima in (G^*, γ^*). These statements easily descend from the following two arguments.

- The insertion of a monotone path connecting a local minimum v_m with a local maximum v_M for a face g can be easily performed in $O(\gamma(v_M) - \gamma(v_m)) = O(k)$ time, as it consists of introducing $\gamma(v_M) - \gamma(v_m) - 1$ new vertices and $\gamma(v_M) - \gamma(v_m)$ new edges in the graph. Further, whenever the insertion is performed, the number of vertices of the graph increases by $\gamma(v_M) - \gamma(v_m) - 1 \in O(k)$, and the number of distinct pairs (v, g) such that v is a local minimum or maximum for a face g of the graph increases by $O(1)$, given that only vertices v_m and v_M and only the two faces incident to the inserted path can generate new such pairs.
- The number of times the described augmentation is performed is $O(r)$. To prove this claim, it suffices to prove that the number of paths that are inserted in a face g of G is linear in the number of local minima and maxima for g. No two paths P_1 and P_2 are inserted in g connecting a vertex a_m with a vertex a_M and connecting a vertex b_m with a vertex b_M, respectively, such that a_m, b_m, a_M, and b_M appear in this circular order along the boundary of g, as when the second path insertion is performed, the two end-vertices of the path would not be incident to the same face in g. It follows that the graph that has one vertex for each local minimum or maximum for g and one edge between two vertices if a path between them has been inserted in g is planar (in fact outerplanar), hence it has a number of edges that is linear in the number of maxima and minima for g. Thus, the claim follows.

Third, (G^*, γ^*) is an instance of the strip planarity testing problem that is equivalent to (G, γ). This directly comes from repeated applications of Claim 2.

This concludes the proof of Lemma 6.

3.5 From a Quasi-Jagged Instance to a Jagged Instance

In this section we show how to reduce a quasi-jagged instance of the strip planarity testing problem to an equivalent jagged instance.

Lemma 7. *Let (G, γ) be a quasi-jagged instance of the strip planarity testing problem with n vertices, k strips, and r minima and maxima.*

There exists an $O(kr + n)$-time algorithm that constructs an equivalent jagged instance (G^*, γ^*) with $O(kr + n)$ vertices, k strips, and $O(r)$ minima and maxima.

Consider any face f of G that contains some local minimum or maximum which is not a global minimum or maximum for f, respectively. Assume that f contains a local minimum v which is not a global minimum for f. The case in which f contains a local maximum which is not a global maximum for f can be discussed analogously. Denote by u (denote by z) the first global minimum or maximum for f that is encountered when walking along C_f starting at v while keeping f to the left (resp. to the right).

We distinguish two cases, namely the case in which u is a global minimum for f and z is a global maximum for f (*Case 1*), and the case in which u and z are both global maxima for f (*Case 2*). The case in which u is a global maximum for f and z is a global minimum for f, and the case in which u and z are both global minima for f can be discussed symmetrically.

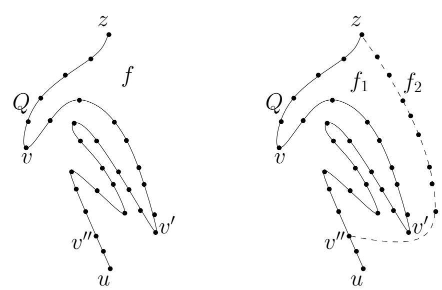


Fig. 11. Augmentation of (G, γ) inside a face f in Case 1. (a) Before the augmentation. (b) After the augmentation.

In Case 1, denote by Q the path connecting u and z in C_f and containing v. Refer to Fig. 11(a). Consider the internal vertex v' of Q that is a local minimum for f and such that $v' = \text{argmin}_{u'} \gamma(u')$ among all the internal vertices u' of Q that are local minima for f. Traverse Q starting from u, until a vertex v'' is found with $\gamma(v'') = \gamma(v')$. Notice that the subpath of Q between u and v'' is monotone. Insert a monotone path connecting v'' and z inside f. See Fig. 11(b). Denote by (G^+, γ^+) the resulting instance of the strip planarity testing problem. We have the following claim:

Claim 3 *Suppose that Case 1 is applied to a quasi-jagged instance (G, γ) to construct an instance (G^+, γ^+). Then, (G^+, γ^+) is strip planar if and only if (G, γ) is strip planar. Also, (G^+, γ^+) is quasi-jagged.*

Proof: We prove that (G^+, γ^+) is strip planar if and only if (G, γ) is strip planar.

One direction of the equivalence is trivial, namely if (G^+, γ^+) is strip planar, then (G, γ) is strip planar, since G is a subgraph of G^+ and $\gamma(x) = \gamma^+(x)$, for every vertex x in G.

We prove the other direction. Consider a strip planar drawing Γ of (G, γ). Observe that, since u and z are consecutive global minimum and maximum for f, they are visible. Since Q is not monotone, by assumption, and since (G, γ) is quasi-jagged, it follows that the path P connecting u and z in C_f and not containing v is monotone. Hence, u and z are the only global minimum and maximum for f, respectively. See Fig. 12.

For every local minimum u' in Q such that $\gamma(u') = \gamma(v')$ (including v'), let $R(u')$ be the bounded region delimited by the two edges incident to u' in Q, and by the horizontal line delimiting the strip corresponding to $\gamma(u')$ from the top; vertically scale $R(u')$ and the part of Γ inside it so that the y-coordinate of u' is larger than the one of v''. Observe that such a modification does not alter the strip planarity of Γ.

If f is an internal face (see Fig. 12(a)), we apply Lemma 1 to construct a curve λ_P following P inside f; otherwise, f is the outer face (see Fig. 12(b)), and we consider a vertical line l lying entirely to the right of Γ. In both cases, we can apply Lemma 2 to construct a curve $\lambda_{v''}$ that moves slightly upward from v'' to a point $p_{v''}$ on either λ_P or l. This can be done since, after the scaling performed for each local minimum u' in Q such that $\gamma(u') = \gamma(v')$, the horizontal segment between v'' and the unique point of λ_P or l with the same y-coordinate lies inside f.

If f is an internal face (see Fig. 12(a)), we construct a y-monotone curve $\mathcal{C}$ between v'' and z inside f that is composed of $\lambda_{v''}$ from v'' to $p_{v''}$, and of the portion of λ_P between $p_{v''}$ and z.

Otherwise, f is the outer face (see Fig. 12(b)). We first apply again Lemma 2 to construct a y-monotone curve λ_z that moves slightly downward from z to a point p_z on l. Observe that, this is possible since z is the unique global maximum for f. Then, we construct a y-monotone curve $\mathcal{C}$ between v'' and z inside f that is composed of $\lambda_{v''}$ from v'' to $p_{v''}$, of the vertical segment from $p_{v''}$ to p_z, and of λ_z from p_z to z.

In both cases, $\mathcal{C}$ is y-monotone and entirely lies inside f; hence, placing each vertex x of the monotone path connecting v'' and z on $\mathcal{C}$ at a suitable y-coordinate, so that x lies in the strip corresponding to $\gamma(x)$, yields a strip planar drawing (G^+, γ^+).

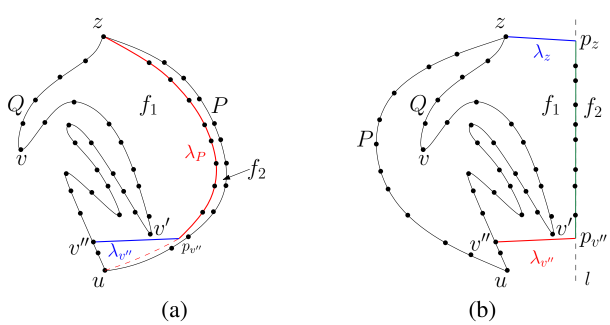


Fig. 12. Inserting a monotone path connecting v'' and z inside f if: (a) f is an internal face, and (b) f is the outer face.

It remains to show that (G^+, γ^+) is quasi-jagged. Clearly, (G^+, γ^+) is strict, proper, and 2-connected. Every face $g \neq f$ of G has not been altered by the augmentation inside f, hence, for any two visible local minimum u_m and local maximum u_M for g, one of the two paths connecting u_m and u_M in g is monotone. Denote by f_1 and f_2 the two faces into which f is split by the insertion of the monotone path connecting v'' and z, where f_1 is the face delimited by such a monotone path and by the subpath of Q

between v'' and z. Face f_2 is delimited by two monotone paths, hence the only pair of visible local minimum and local maximum for f_2 is connected by a monotone path in C_{f_2}. Face f_1, on the other hand, contains a local minimum that is not a local minimum for f, namely v''. However, the existence of a local maximum u'' for f such that v'' and u'' are visible and are not connected by a monotone path in C_{f_1} would imply that u and u'' are a pair of visible local minimum and local maximum for f that is not connected by a monotone path in C_f, which contradicts the fact that (G, γ) is quasi-jagged. □

In Case 2, when both u and z are global maxima for f, there exists a *maximal* path M that is part of C_f, whose end-vertices are two global maxima u_M and v_M for f, that contains v in its interior, and that does not contain any global minimum in its interior (see Fig. 13(a)). Assume, w.l.o.g., that face f is to the right of M when walking along M from u_M to v_M. Possibly $u_M = u$ and/or $v_M = z$. Let u_m (v_m) be the global minimum for f such that u_m and u_M (resp. v_m and v_M) are consecutive global minimum and maximum for f. Possibly, $u_m = v_m$. Denote by P (by Q) the path connecting u_m and u_M (v_m and v_M) along C_f and not containing v. Since M contains a local minimum among its internal vertices, and since (G, γ) is quasi-jagged, it follows that P and Q are monotone.

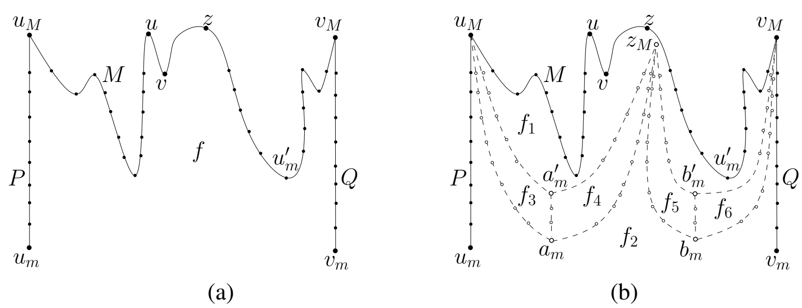


Fig. 13. Augmentation of (G, γ) inside a face f in Case 2. (a) Before the augmentation. (b) After the augmentation.

Insert the plane graph $A(u_M, v_M, f)$ depicted by white circles and dashed lines in Fig. 13(b) inside f. Consider a local minimum $u'_m \in M$ for f such that $u'_m = \text{argmin}_{v'_m} \gamma(v'_m)$ among the local minima v'_m for f in M. Set $\gamma(z_M) = \gamma(u_M)$, set $\gamma(a_m) = \gamma(b_m) = \gamma(u_m)$, and set $\gamma(a'_m) = \gamma(b'_m) = \gamma(u'_m)$. The dashed lines connecting a_m with u_M, a'_m with u_M, a_m with z_M, a'_m with z_M, b_m with z_M, b'_m with z_M, b_m with v_M, b'_m with v_M, a_m with a'_m, and b_m with b'_m represent monotone paths. Denote by (G^+, γ^+) the resulting instance of strip planarity. In the following claim we prove that (G^+, γ^+) is a quasi-jagged instance that is equivalent to (G, γ).

The proof mainly consists of showing that the structure of gadget $A(u_M, v_M, f)$ is "flexible" enough to allow its insertion in any strip planar drawing of (G^+, γ^+) to obtain a strip planar drawing of (G, γ), whatever is the shape of face f in such a drawing. In particular, we have to distinguish two cases based on whether f lies "inside" the region R delimited by P, M, and Q (see Fig. 14) or "outside" it (see Fig. 17). In the first case, we redraw the part of the graph that lies inside R (see Fig. 15) to "make room" for a drawing of $A(u_M, v_M, f)$ (see Fig. 16). In the second case, we again redraw the part of the graph that might interfere with the drawing of $A(u_M, v_M, f)$ (see Fig. 18); however, the drawing of $A(u_M, v_M, f)$ is not straightforward, as it has to "wrap around" R, which can be done by exploiting the existence of the global maximum z_M (see Fig. 19).

Claim 4 *Suppose that Case 2 is applied to a quasi-jagged instance* (G, γ) *to construct an instance* (G^+, γ^+)*. Then,* (G^+, γ^+) *is strip planar if and only if* (G, γ) *is strip planar. Also,* (G^+, γ^+) *is quasi-jagged.*

Proof: One direction of the equivalence is trivial, namely if (G^+, γ^+) is strip planar, then (G, γ) is strip planar, since G is a subgraph of G^+ and $\gamma(v) = \gamma^+(v)$ for every vertex v in G.

We prove the other direction. Consider a strip planar drawing Γ of (G, γ). Slightly perturb the position of the vertices in Γ so that no two of them have the same y-coordinate. Assume w.l.o.g. that f is to the right of P when traversing such a path from u_m to u_M. Denote by l_M (by l_m) the line delimiting the strip corresponding to $\gamma(u_M)$ from below (to $\gamma(u_m)$ from above). Further, denote by l'_m a line above l_m and sufficiently close to l_m so that the horizontal strip delimited by these two lines does not contain any vertex of G.

We distinguish two cases, based on whether the intersection of P with l_M lies left (Case 2A) or right (Case 2B) of the intersection of Q with l_M. Since P and Q are represented in Γ by y-monotone curves that do not intersect each other, in Case 2A (in Case 2B) the intersection of P with l_m lies left (right) of the intersection of Q with l_m. In both cases, we modify Γ while maintaining its strip planarity so that $A(u_M, v_M, f)$ can be planarly drawn in f with y-monotone edges. We first discuss Case 2A.

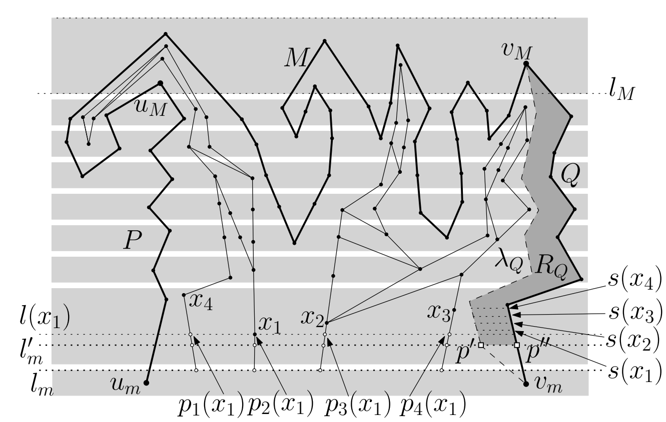


Fig. 14. Illustration for the proof of Claim 4. Paths P, M, and Q are represented by thick lines. The part of the graph that is outside region R is not shown.

We introduce some notation. Refer to Fig. 14. Denote by R the bounded region delimited by P, by M, by Q, and by l_m. Drawing Γ will be only modified in the interior of R. Denote by R' the bounded region delimited by P, by M, by Q, and by l'_m. We define a closed bounded region R_Q inside R' as follows. First, apply Lemma 1 to construct a curve λ_Q that follows Q inside f. Let p' and p'' be the intersection points of l'_m with λ_Q and with Q, respectively. Region R_Q is the one delimited by λ_Q, by Q, and by the horizontal segment between p' and p'' that has v_M on its boundary. By Lemma 1, the interior of R_Q entirely belongs to f. The part of Γ that lies in the interior of R' will be redrawn so that it entirely lies in R_Q.

For each vertex x of G in the interior of R, consider the horizontal line $l(x)$ through x. Let $p_1(x), \dots, p_{f(x)}(x)$ be the left-to-right order of the intersection points of edges of G with $l(x)$, where x is also a point $p_i(x)$ for some $1 \leq i \leq f(x)$. We draw a horizontal segment $s(x)$ inside R_Q, in such a way that: (i) $s(x)$ is contained in the strip corresponding to $\gamma(x)$, (ii) $s(x)$ connects a point in λ_Q with a point in Q, and (iii) if vertices x_1 and x_2 inside R are such that $y(x_1) < y(x_2)$, then $s(x_1)$ lies below $s(x_2)$. For each vertex x of G that lies in the interior of R, insert points $p'_1(x), \dots, p'_{f(x)}(x)$ in this left-to right order on $s(x)$. Also, let $p_1(l'_m), \dots, p_{f(l'_m)}(l'_m)$ be the left-to-right order of the intersection points of edges of G with l'_m. Insert points $p'_1(l'_m), \dots, p'_{f(l'_m)}(l'_m)$ in this left-to right order on the segment between p' and p''.

We now redraw in R_Q the vertices and edges that are inside R' in Γ. Refer to Fig. 15.

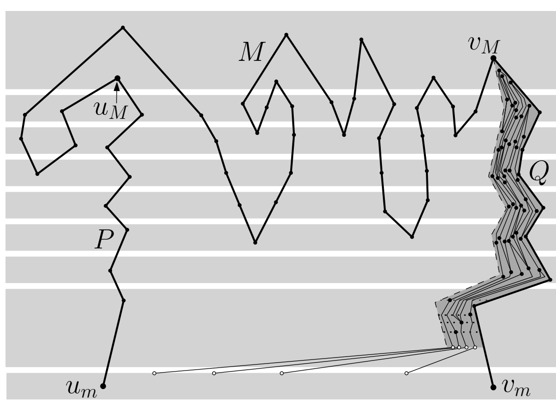
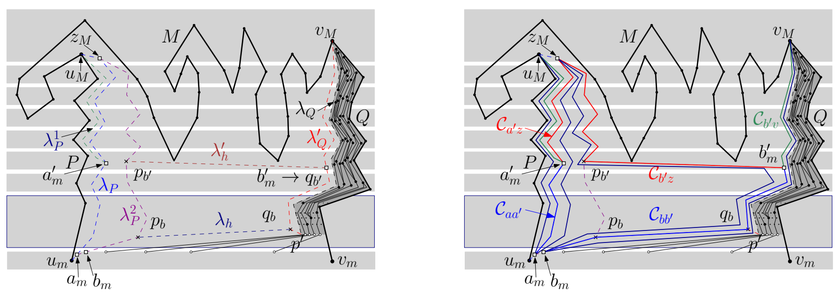


Fig. 15. Redrawing in R_Q the vertices and edges that are inside R in Γ.

For any line segment that is part of an edge of G and that connects two points $p_i(x_1)$ and $p_j(x_2)$, with $x_1 \neq x_2$, (or a point $p_i(l'_m)$ with a point $p_j(x)$) draw a line segment connecting $p'_i(x_1)$ and $p'_j(x_2)$ (resp. connecting $p'_i(l'_m)$ with $p'_j(x)$) inside R_Q. Observe that, if such a line segment exists, then $s(x_1)$ and $s(x_2)$ (resp. $\overline{pp'}$ and $s(x)$) are consecutive horizontal segments in R_Q. Further, the line segments connecting points on two consecutive line segments $s(x_1)$ and $s(x_2)$ (resp. $\overline{pp'}$ and $s(x)$) can be drawn as y-

monotone curves inside R_Q so that they do not cross each other, give that the relative order of the points $p'_i(x)$ on $s(x)$ preserves the order of the points $p_i(x)$ on $l(x)$, for every vertex x of G in the interior of R, and the relative order of the points $p'_i(l'_m)$ on $\overline{pp'}$ preserves the order of the points $p_i(l'_m)$ on l'_m. For each edge e that has non-empty intersection with R, delete from Γ the part e_R of e inside R. If e used to intersect l'_m, denote by $p_i(l_m)$ and $p_i(l'_m)$ the intersection points of e with l_m and l'_m before e_R was removed. Draw a y-monotone curve connecting point $p'_i(l'_m)$ on $\overline{pp'}$ with point $p_i(l_m)$. Such curves can be drawn without introducing crossings, given that the relative order of the points $p'_i(l'_m)$ on $\overline{pp'}$ preserves the order of the points $p_i(l'_m)$ on l'_m.



(a) (b)

Fig. 16. Drawing $A(u_M, v_M, f)$ (vertices are white circles and edges are solid thin lines). (a) Construction of curves λ_P, λ_P^1, λ_P^2, λ_h, λ'_h, and λ'_Q. (b) Assigning paths of $A(u_M, v_M, f)$ to curves.

We now draw $A(u_M, v_M, f)$.

First we draw a set of curves in Γ. Refer to Fig. 16(a). Apply Lemma 1 to construct a curve λ_P between u_m and u_M following P inside f. This splits f into two faces f^1 and f^2 (here we refer to faces as regions of the plane delimited by closed curves, even if such curves do not correspond to paths in the graph), where f^2 is the one having M on its boundary. Place a_m, a'_m and z_M on any three points of λ_P lying in the interior of the strip corresponding to $\gamma(a_m)$, $\gamma(a'_m)$, and $\gamma(z_M)$, respectively. Then, apply Lemma 1 to construct a curve λ_P^1 between a'_m and u_M following λ_P inside f^1, and a curve λ_P^2 between a_m and z_M following λ_P inside f^2. Hence, f_2 is split into two faces; let f^3 be the one having M on its boundary. Place b_m on any point of λ_P^2 lying in the interior of the strip corresponding to $\gamma(b_m)$. Also, apply Lemma 1 to construct a curve λ_Q^1 between p' and v_M following λ_Q inside f^3. Let q_b (let $q_{b'}$) be a point on λ_Q^1 inside the strip corresponding to $\gamma(u_m) + 1$ (to $\gamma(b'_m)$) whose y-coordinate is smaller than the one of every vertex lying inside the same strip. This implies that the horizontal segment between q_b (between $q_{b'}$) and the unique point of λ_P^2 with its same y-coordinate entirely lies inside f^3. Hence, we can apply Lemma 2 to construct inside f^3 a y-monotone curve λ_h that moves slightly downward from q_b to a point p_b on λ_P^2 and a y-monotone curve λ'_h that moves slightly upward from $q_{b'}$ to a point $p_{b'}$ on λ_P^2. Place b'_m on $q_{b'}$.

We now describe an assignment of monotone paths to curves; refer to Fig. 16(b). Assigning a path P to a curve $\mathcal{C}$ means that each vertex v of P will be placed on a point of $\mathcal{C}$ lying inside the strip corresponding to $\gamma(v)$. The path between a_m and a'_m is assigned to a curve $\mathcal{C}_{aa'}$ that is the portion of λ_P between them. The path between a'_m and z_M is assigned to a curve $\mathcal{C}_{a'z}$ that is the portion of λ_P between them. The path between a'_m and u_M is assigned to λ^1_P. The path between a_m and u_M (between a_m and z_M) not containing a'_m is assigned to a curve, obtained by applying Lemma 1, that follows the concatenation of curves $\mathcal{C}_{aa'}$ and λ^1_P (curves $\mathcal{C}_{aa'}$ and $\mathcal{C}_{a'z}$), inside the unique face having these two curves on its boundary. The path between b_m and b'_m is assigned to a curve $\mathcal{C}_{bb'}$ that is obtained by concatenating the portion of λ^2_P from b_m to p_b, curve λ_h, and the portion of λ'_Q between q_b and b'_m. The path between b'_m and z_M is assigned to a curve $\mathcal{C}_{b'z}$ that is obtained by concatenating λ'_h and the portion of λ^2_P between $p_{b'}$ and z_m. The path between b'_m and v_M is assigned to a curve $\mathcal{C}_{b'v}$ that is the portion of λ'_Q between them. The path between b_m and v_M (between b_m and z_M) not containing b'_m is assigned to a curve, obtained by applying Lemma 1, that follows the concatenation of curves $\mathcal{C}_{bb'}$ and $\mathcal{C}_{b'v}$ (curves $\mathcal{C}_{bb'}$ and $\mathcal{C}_{b'z}$), inside the unique face having these two curves on its boundary.

We now discuss Case 2B. We introduce some notation. See Fig. 17. Denote by l'_t the horizontal line passing through the vertex w_M of M with largest y-coordinate, and denote by l_t an horizontal line in the strip corresponding to $\gamma(u_M)$ slightly above l'_t, and close enough to l'_t so that no vertex lies in the interior of the strip delimited by l_t and l'_t. Observe that all the vertices and edges of M, of P, and of Q are entirely below l'_t, except for vertex w_M. Let $s(w_M)$ be the vertical segment connecting w_M with l_t. Denote by l'_p and by l''_p (by l'_q and by l''_q) vertical lines entirely right (left) of M, P, and Q, with l''_p right of l'_p (with l''_q left of l'_q). Let R_A be the region delimited by l_t, by l'_t, by l''_p, and by l''_q. Denote by R_p the bounded region of the plane delimited by l_m, by l''_p, by l_t, by P, by the part of M connecting u_M with w_M, and by $s(w_M)$. Region R_q is defined analogously with l''_q, Q, and v_M in place of l''_p, P, and u_M, respectively. Drawing Γ will be only modified in the interior of $R_p \cup R_q$. In particular, the vertices of G and the intersection points of the edges of G with the lines delimiting $R_p \cup R_q$ will maintain the same position after the modification.

We define some regions inside R_p. Let R'_p be the part of R_p that is left of l'_p, above l'_m and below l'_t; let R''_p be the part of R_p that is above l'_m and below l'_t; let R'''_p be the part of R_p that is right of l'_p, above l'_m and below l'_t (observe that $R''_p = R'_p \cup R'''_p$); finally, let $R_{B,p}$ be the part of R_p below l'_m. Regions R'_q, R''_q, R'''_q, and $R_{B,q}$ are defined analogously with l'_q and l''_q in place of l'_p and l''_p, with left in place of right, and vice versa.

We modify Γ so that no vertex and no part of an edge lies in $R'_p \cup R'_q$. The part of Γ outside $R_p \cup R_q$ is not modified in the process. This modification is similar to the one performed for the proof of Claim 2. Refer to Fig. 18.

We perform a horizontal scaling of the part of Γ that lies inside R''_p (the vertices and edges of P and M stay still). This is done so that every intersection point of an edge with l''_p keeps the same position, and the part of Γ that used to lie inside R''_p is now contained in R'''_p, that is the interior of R'_p contains no vertex and no part of an edge. Some edges of G (those that used to intersect l'_m and l'_t) are now disconnected; e.g., if an edge of G used to intersect l'_m, now such an edge contains a line segment inside R'''_p, which has been scaled, and a line segment inside $R_{B,p}$, which has not been scaled. By construction $R_{B,p}$ does not contain any vertex in its interior. Hence, the line segments

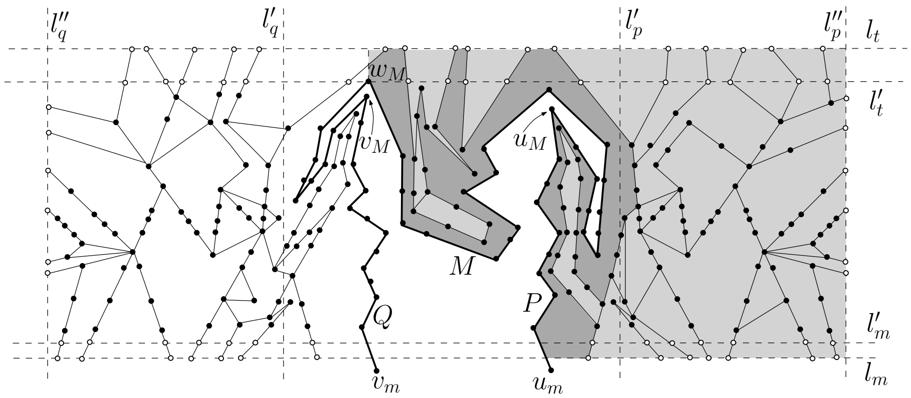


Fig. 17. Drawing Γ inside region $R_p \cup R_q$. Region R_p is colored light and dark gray. In particular, part of face f inside R_p is colored dark gray. Paths P, Q, and M are represented by thick lines. Intersection points of edges with lines l''_p, l''_q, l_m, l'_m, l_t, and l'_t are represented by white circles.

in $R_{B,p}$ form in Γ a planar y-monotone matching between a set A_p of points on l'_m and a set B_p of points on l_m. After the scaling, the position of the points in A_p has been modified, however their relative order on l'_m has not. Thus, we can delete the line segments in $R_{B,p}$ and reconnect the points in B_p with the new positions of the points in A_p on l'_m so that each edge is y-monotone and no two edges intersect.

We also perform a horizontal scaling of the part of Γ inside R''_q (the vertices and edges of Q and M stay still). This is done symmetrically to the scaling inside R''_p. After this scaling, R'_q contains no vertex and no part of an edge.

Finally, the line segments in R_A form in Γ a planar y-monotone matching between points on l'_t and points on l_t. As for the segments in $R_{B,p}$, the scaling does not cause two segments in R_A to cross.

We thus obtain a planar y-monotone drawing Γ' of G in which no vertex and no part of an edge lies in the interior of $R'_p \cup R'_q$. Since no vertex changed its y-coordinate and every edge is y-monotone, Γ' is a strip planar drawing of (G, γ).

We now draw $A(u_M, v_M, f)$.

First we draw a set of curves in Γ'. Refer to Fig. 19(a).

Place z_M on any point of f between l_t and l'_t, and consider the two straight-line segments s_a and s_b connecting z_M to the intersection points between l'_t and l'_p, and between l'_t and l'_q, respectively. Note that s_a and s_b entirely lie inside f. Apply Lemma 1 to construct two y-monotone curves λ_P, between u_m and u_M, and λ_Q, between v_m and v_M, inside f following P and Q, respectively. Let f^1 be the face whose boundary contains both λ_P and λ_Q. Place vertex a_m (vertex b_m) on any point of λ_P (of λ_Q) inside the strip corresponding to $\gamma(a_m) = \gamma(b_m)$. Also, place vertex a'_m (vertex b'_m) on any point of λ_P (of λ_Q) inside the strip corresponding to $\gamma(a'_m) = \gamma(b'_m)$ whose coordinate is smaller than the one of every vertex lying in the same strip. This implies that the horizontal segment between a'_m (between b'_m) and the unique point of l'_p (of l'_q) with its same y-coordinate entirely lies inside f^1. Hence, we can apply Lemma 2

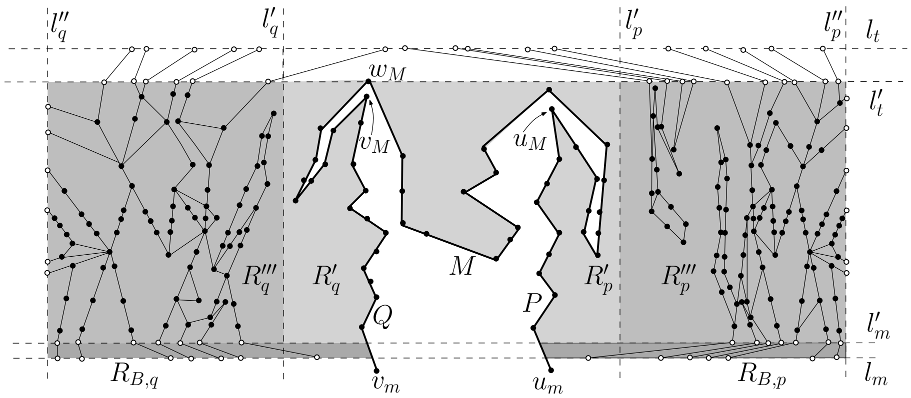


Fig. 18. Drawing Γ' of (G, γ), obtained by scaling the part of Γ that lies inside R''_p and inside R''_q, and by reconnecting points on l_m with points on l'_m and points on l_t with points on l'_t. Regions R'_p and R'_q are light gray, regions R'''_p and R'''_q are medium gray, while regions $R_{B,p}$ and $R_{B,q}$ are dark gray.

to construct inside f^1 a y-monotone curve λ_a that moves slightly upward from a'_m to a point p_a on l'_p and a y-monotone curve λ_b that moves slightly upward from b'_m to a point p_b on l'_q.

We now describe an assignment of monotone paths to curves; refer to Fig. 19(b). Assigning a path P to a curve $\mathcal{C}$ means that each vertex v of P will be placed on a point of $\mathcal{C}$ lying inside the strip corresponding to $\gamma(v)$.

The path between a_m and a'_m is assigned to a curve $\mathcal{C}_{aa'}$ that is the portion of λ_P between them. The path between a'_m and u_M is assigned to a curve $\mathcal{C}_{a'u}$ that is the portion of λ_P between them. The path between a'_m and z_M is assigned to a curve $\mathcal{C}_{a'z}$ that is the curve obtained by concatenating curve λ_a, the portion of l'_p between p_a and the intersection point between l'_p and l'_t, and segment s_a. The path between a_m and u_M (between a_m and z_M) not containing a'_m is assigned to a curve, obtained by applying Lemma 1, that follows the concatenation of curves $\mathcal{C}_{aa'}$ and $\mathcal{C}_{a'u}$ (curves $\mathcal{C}_{aa'}$ and $\mathcal{C}_{a'z}$), inside the unique face having these two curves on its boundary.

The paths connecting b_m, b'_m, v_M, and z_M are assigned analogously, as follows. The path between b_m and b'_m is assigned to a curve $\mathcal{C}_{bb'}$ that is the portion of λ_Q between them. The path between b'_m and v_M is assigned to a curve $\mathcal{C}_{b'v}$ that is the portion of λ_Q between them. The path between b'_m and z_M is assigned to a curve $\mathcal{C}_{b'z}$ that is the curve obtained by concatenating curve λ_b, the portion of l'_q between p_b and the intersection point between l'_q and l'_t, and segment s_b. The path between b_m and v_M (between b_m and z_M) not containing b'_m is assigned to a curve, obtained by applying Lemma 1, that follows the concatenation of curves $\mathcal{C}_{bb'}$ and $\mathcal{C}_{b'v}$ (curves $\mathcal{C}_{bb'}$ and $\mathcal{C}_{b'z}$), inside the unique face having these two curves on its boundary.

This concludes the construction of a strip planar drawing of (G^+, γ^+). It remains to prove that (G^+, γ^+) is quasi-jagged. Clearly, (G^+, γ^+) is strict, proper, and 2-connected.

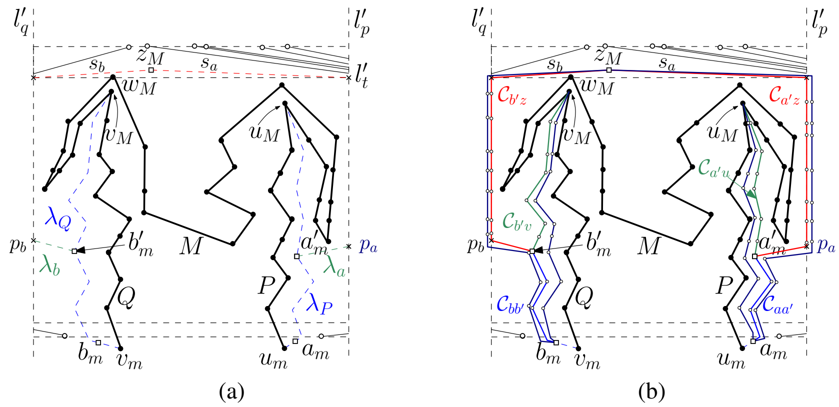


Fig. 19. Augmentation of drawing Γ' of G with a drawing of plane graph $A(u_M, v_M, f)$. (a) Construction of curves λ_Q, λ_P, λ_a, λ_b, s_a, and s_b. (b) Assigning paths of $A(u_M, v_M, f)$ to curves.

Every face $g \neq f$ of G has not been altered by the augmentation inside f, hence, for any two visible local minimum u_m and local maximum u_M for g, one of the two paths connecting u_m and u_M in G is monotone. Denote by $f_1, f_2, \ldots, f_6$ the faces into which f is split by the insertion of $A(u_M, v_M, f)$ (see Fig. 13(b)).

For $i = 3, \ldots, 6$, face f_i is delimited by two monotone paths. Face f_2 contains two local minima, namely a_m and b_m, and one local maximum, namely z_M, that are not incident to f in G. However, u_M and z_M are the only local maxima for f_2 that are visible with a_m; also, a_m and b_m are the only local minima for f_2 that are visible with z_M; further, z_M and v_M are the only local maxima for f_2 that are visible with b_m. For all such pairs of visible local minimum and maximum, there exists a monotone path in C_{f_2} connecting them. Moreover, every pair of visible local minimum and maximum for f_2 which does not include a_m, z_M, or b_m is also a pair of visible local minimum and maximum for f, hence it is connected by the same monotone path in C_{f_2} as in C_f. Finally, consider f_1. As for f_2, each of a'_m, z_M, and b'_m only participates in two pairs of visible local minimum and maximum for f_1, where the second vertex of each pair is one of u_M, a'_m, z_M, b'_m, and v_M. For all such pairs, monotone paths in C_{f_1} exist by construction. Further, every pair of visible local minimum and maximum for f_1 which does not include a'_m, z_M, or b'_m is also a pair of visible local minimum and maximum for f, hence it is connected by the same monotone path in C_{f_1} as in C_f. □

We prove that Claims 3–4 imply Lemma 7. First, the repetition of the augmentation leads to a jagged instance (G^*, γ^*). For an instance (G, γ) and for a face g of G, denote by $n(g, G)$ the number of vertices that are local minima or maxima for g but not global minima or maxima for g. Also, let $n(G) = \sum_g n(g, G)$ over all faces g of G. We claim that, when a face f of G is augmented as in Case 1 or in Case 2 and instance (G, γ) turns into an instance (G^+, γ^+), we have $n(G^+) < n(G)$. The claim implies that $n(G^*) = 0$, hence (G^*, γ^*) is jagged. We prove the claim. For each face $g \neq f$, it holds that $n(g, G) = n(g, G^+)$, given that a vertex u is a local minimum, a local maximum, a global minimum, or a global maximum for g in (G^+, γ^+) if and only if it is in (G, γ).

Suppose that Case 1 is applied, thus splitting f into faces f_1 and f_2, as in Fig. 11(b). Face f_2 is delimited by two monotone paths, hence $n(f_2, G^+) = 0$. Further, every vertex inserted into f is neither a local maximum nor a local minimum for f_1; moreover, vertex v' is a global minimum for f_1, by construction, and it is a local minimum but not a global minimum for f. Hence, $n(f_1, G^+) < n(f, G)$. Suppose that Case 2 is applied, thus splitting f into faces $f_1, \dots, f_6$, as in Fig. 13(b). For $i = 3, \dots, 6$, face f_i is delimited by two monotone paths, hence $n(f_i, G^+) = 0$. Every vertex of $A(u_M, v_M, f)$ incident to f_i, with $i = 1, 2$, is either a global maximum or minimum for f_i, or it is not a local maximum or minimum for f_i at all. Moreover, vertex u'_m is a global minimum for f_1 and it is a local minimum but not a global minimum for f. Hence, $n(f_1, G^+) + n(f_2, G^+) < n(f, G)$.

Second, since each augmentation can be performed in $O(k)$ time by introducing $O(k)$ new vertices and edges, since $O(r)$ augmentations are performed in order to obtain (G^*, γ^*), given that $n(G) \leq r$, and since every augmentation introduces a constant number of minima or maxima, it follows that the number of vertices of G^* is $O(kr+n)$, the number of minima and maxima of (G^*, γ^*) is $O(r)$, and (G^*, γ^*) can be constructed in $O(kr + n)$ time. In particular, an $O(n)$-time preprocessing determines, for all faces g of G, all the pairs (v, g) such that v is a local minimum or maximum for g but it is not a global minimum or maximum for g.

Third, (G^*, γ^*) is an instance of strip planarity equivalent to (G, γ). This comes from repeated applications of Claims 3 and 4, and concludes the proof of Lemma 7.

3.6 Strip Planarity of Jagged Instances

In this section we show that testing whether a jagged instance (G, γ) of the strip planarity testing problem is strip planar is equivalent to testing whether the associated directed graph of (G, γ) is upward planar. Based on this equivalence and on the results of the previous sections, we show that the strip planarity testing problem can be solved in polynomial time for general instances with a prescribed plane embedding.

Recall that the associated directed graph of (G, γ) is the directed plane graph $\overrightarrow{G}$ obtained from (G, γ) by orienting each edge (u, v) in G from u to v if and only if $\gamma(v) = \gamma(u) + 1$. We have the following:

Lemma 8. *A jagged instance (G, γ) of the strip planarity testing problem is strip planar if and only if the associated directed graph $\overrightarrow{G}$ of (G, γ) is upward planar.*

Proof: The necessity is trivial, given that a strip planar drawing of (G, γ) is also an upward planar drawing of $\overrightarrow{G}$, by definition.

We prove the sufficiency. A directed plane graph $\overrightarrow{G}_{st}$ is called *plane st-digraph* if it has exactly one source s and one sink t such that s and t are both incident to the outer face of $\overrightarrow{G}_{st}$. Each face f of a plane st-digraph $\overrightarrow{G}_{st}$ consists of two monotone paths called *left path* and *right path*, where the left path has f to the right when traversing it from its source to its sink. The right path and the left path of the outer face of $\overrightarrow{G}_{st}$ are also called *leftmost path* and *rightmost path* of $\overrightarrow{G}_{st}$, respectively.

Since $\overrightarrow{G}$ is upward planar, $\overrightarrow{G}$ can be augmented [15] to a plane st-digraph $\overrightarrow{G}_{st}$. Also, this can be done by adding only *dummy* edges (u, v) such that u and v are incident to the same face f, and u and v are either both sources or both sinks in C_f (when C_f is oriented according to $\overrightarrow{G}$). Note that, since (G, γ) is jagged, each dummy edge (u, v) is such that $\gamma(u) = \gamma(v)$.

We now compute the *directed dual* $\overrightarrow{G}_{s^*t^*}$ of $\overrightarrow{G}_{st}$. The vertices of $\overrightarrow{G}_{s^*t^*}$ are the faces of $\overrightarrow{G}_{st}$; two special vertices s^* and t^* represent the outer face. There is an edge (f, g) in $\overrightarrow{G}_{s^*t^*}$ if face f shares an edge $(u, v) \neq (s, t)$ with face g, and face f is on the left side of (u, v) when such an edge is traversed from u to v. Graph $\overrightarrow{G}_{s^*t^*}$ is a plane st-digraph [15].

We divide the plane into k horizontal strips of fixed height, each corresponding to one of the strips of (G, γ).

We compute an upward planar drawing of $\overrightarrow{G}_{st}$ in which each vertex lies in the corresponding strip, as follows. First, consider the leftmost path p_l of $\overrightarrow{G}_{st}$, where $p_l = (s = v_1^1, \dots, v_1^{h(1)}, v_2^1, \dots, v_2^{h(2)}, \dots, v_k^1, \dots, v_k^{h(k)} = t)$, with $\gamma(v_i^1) = \dots = \gamma(v_i^{h(i)}) = i$, for $i = 1, \dots, k$. Path p_l is drawn as a y-monotone curve in which each vertex $u \in p_l$ lies inside the strip corresponding to $\gamma(u)$. Then, we add the faces of $\overrightarrow{G}_{st}$ one at a time, in such a way that a face is considered after all its predecessors in $\overrightarrow{G}_{s^*t^*}$ (i.e., the faces can be considered in the order corresponding to any linear extension of the poset represented by $\overrightarrow{G}_{s^*t^*}$). When a face f is considered, its left path has been already drawn as a y-monotone curve λ_l. We apply Lemma 1 to draw the right path of f as a y-monotone curve λ_r that follows λ_l inside the outer face of the current drawing. Then, we place each vertex u belonging to the right path of f on any point of λ_r inside the strip corresponding to $\gamma(u)$. This implies that the rightmost path of the graph in the current drawing is represented by a y-monotone curve.

A strip planar drawing of (G, γ) can be obtained from the upward planar drawing of $\overrightarrow{G}_{st}$ by removing the dummy edges. □

Note how the correctness of the proof of Lemma 8 heavily depends on the fact that (G, γ) is jagged. We thus obtain the main result of this paper:

Theorem 1. *The strip planarity testing problem can be solved in $O(|G|^2)$ time for instances (G, γ) such that G is a plane graph.*

Proof: In the following we denote by $|H|$ the number of vertices of any instance (H, γ) of strip planarity; also, we denote by $r(H)$ the number of minima and maxima of (H, γ), and by $k(H)$ the number of strips of (H, γ). Further, we assume that $k(H) \leq |H|$, since empty strips can be removed without loss of generality.

Let (G, γ) be any instance of strip planarity such that G is a plane graph.

By Lemma 3–8, there exists an $O(|G|^2)$-time algorithm that either decides that (G, γ) does not admit any strip planar drawing or constructs a directed plane graph $\overrightarrow{G^*}$ with $|\overrightarrow{G^*}| \in O(k(G)|G|)$, $s(\overrightarrow{G^*}) \in O(|G| + r(G))$ (where $s(\overrightarrow{G^*})$ is the total number of sources and sinks of $\overrightarrow{G^*}$), and such that (G, γ) is strip planar if and only if $\overrightarrow{G^*}$ is upward planar with respect to its plane embedding.

Finally, by the results of Bertolazzi et al. [7], the upward planarity of $\overrightarrow{G^*}$ can be tested in $O(|\overrightarrow{G^*}| + (s(\overrightarrow{G^*}))^2)$ time. Since $|\overrightarrow{G^*}| \in O(k(G)|G|)$ and since $s(\overrightarrow{G^*}) \in$

$O(|G| + r(G))$, the upward planarity of $\overrightarrow{G^*}$ can be tested in $O(k(G)|G| + (|G| + r(G))^2) \in O(|G|^2)$ time.

This concludes the proof of the theorem. □

3.7 Non-Connected Instances

In this section we show how the problem of testing the strip planarity of non-connected instances can be reduced to the one of testing the strip planarity of connected instances.

The input of the strip planarity testing algorithm might or might not specify the containment relationships between distinct connected components. According to our definition of combinatorial embedding given in Section 2, these relationships are not prescribed. Then a non-connected instance (G, γ) is strip planar if and only if all its connected components are strip planar. Namely, if (G, γ) is strip planar, then all its components are strip planar. Conversely, if all the components of (G, γ) are strip planar, then a strip planar drawing of (G, γ) can be obtained by placing strip planar drawings of the components of (G, γ) "side by side".

Several papers in the graph drawing literature (see, e.g., [2,3,29]), however, assume a definition for "embedding of a non-connected planar graph G" in which the containment relationships between distinct connected components of G are prescribed in advance. More formally, let $G_1, \dots, G_k$ be the connected components of G, let Γ_a and Γ_b be two planar drawings of G, and let $\mathcal{E}_1^a, \dots, \mathcal{E}_k^a$ and $\mathcal{E}_1^b, \dots, \mathcal{E}_k^b$ be the plane embeddings of $G_1, \dots, G_k$ in Γ_a and Γ_b, respectively. Then Γ_a and Γ_b are *equivalent* if: (i) they determine the same clockwise order of the edges incident to each vertex of G, i.e., $\mathcal{E}_i^a = \mathcal{E}_i^b$ for each $1 \leq i \leq k$, and (ii) consider every simple cycle $C \subseteq G_i$ all of whose edges are incident to the same face in $\mathcal{E}_i^a$ (and in $\mathcal{E}_i^b$); then, for every vertex $v \in G_j$ with $j \neq i$, we have that v is inside C in $\mathcal{E}_i^a$ if and only if it is inside C in $\mathcal{E}_i^b$. A *plane embedding* is defined as an equivalence class of planar drawings. This choice for the definition of a plane embedding is somewhat more natural in the sense that, analogously to the connected case, two drawings Γ_a and Γ_b of G have the same embedding if and only if they have faces with the same boundaries. With this definition for "embedding of a non-connected planar graph G", testing the strip planarity of an instance (G, γ) becomes slightly more complicated than just testing the strip planarity of its connected components.

Thus, assume that the input (G, γ) of the strip planarity testing algorithm specifies the containment relationships between distinct connected components. Then the boundary of each face of G is prescribed by the input. Test individually the strip planarity of each connected component of (G, γ). If one of the tests fails, then (G, γ) is not strip planar. Otherwise, construct a strip planar drawing of each connected component of (G, γ). Place the drawings of the connected components containing edges incident to the outer face of G side by side. Repeatedly insert connected components in the internal faces of the currently drawn graph (G', γ) as follows. If a connected component (G_i, γ) of (G, γ) has to be placed inside an internal face f of (G', γ), check whether $\gamma(u_M) \leq \gamma(u_M^f)$ and whether $\gamma(u_m) \geq \gamma(u_m^f)$, where u_M (u_m) is a vertex of (G_i, γ) such that $\gamma(u_M)$ is maximum (resp. $\gamma(u_m)$ is minimum) among the vertices of G_i, and where u_M^f (u_m^f) is a vertex of C_f such that $\gamma(u_M^f)$ is maximum (resp. $\gamma(u_m^f)$ is minimum) among the vertices of C_f. If the test fails, then (G, γ) is not strip planar. Otherwise, using a technique analogous to the one of Claim 2, a strip planar drawing of (G', γ) can be modified so that two consecutive global minimum and maximum for f can be connected by a y-monotone curve $\mathcal{C}$ inside f. Suitably squeezing a strip planar drawing of (G_i, γ) and placing it arbitrarily close to $\mathcal{C}$ provides a strip planar drawing of $(G' \cup G_i, \gamma)$. Repeating such an argument leads either to conclude that (G, γ) is not strip planar, or to construct a strip planar drawing of (G, γ).

4 Reduction

In this section we show that the strip planarity testing problem reduces in polynomial time to the clustered planarity testing problem.

Theorem 2. *Let (G, γ) be an instance of strip planarity. Then, there exists an instance $C(G', T)$ of clustered planarity such that (G, γ) is strip planar if and only if $C(G', T)$ is clustered planar. Further, $C(G', T)$ can be constructed in polynomial time.*

Proof: Denote by k the number of strips of (G, γ). First, we show that, if $k \leq 2$, then the "natural" reduction from strip planarity to clustered planarity, namely the one that transforms each strip into a cluster, is a valid polynomial-time reduction. We now formalize this claim.

If $k = 1$, clustered graph $C(G', T)$ is defined as follows. Graph G' coincides with G and tree T consists of a single internal node μ that is parent of all the vertices of G'. The equivalence between the strip planarity of (G, γ) and the clustered planarity of $C(G', T)$ follows from their equivalence to the planarity of $G = G'$.

If $k = 2$, clustered graph $C(G', T)$ is defined as follows. Graph G' coincides with G and tree T consists of three internal nodes μ, μ_1, and μ_2, where μ is parent of μ_1 and μ_2, and where μ_i is parent of every vertex x of G' such $\gamma(x) = i$, for $i = 1, 2$. From a strip planar drawing Γ of (G, γ), a c-planar drawing Γ' of $C(G', T)$ can be constructed so that the drawings of G and G' coincide, and so that, for $i = 1, 2$, the region $R(\mu_i)$ representing μ_i is a rectangle whose top and bottom sides lie on the top and bottom lines delimiting γ_i, respectively, and whose left (right) side is to the left (right) of all the vertices and edges of G'. Conversely, suppose that $C(G', T)$ is c-planar. Then, it admits a c-planar straight-line drawing Γ' in which μ_1 and μ_2 are represented by convex regions $R(\mu_1)$ and $R(\mu_2)$ (see [4,17]). Thus, $R(\mu_1)$ and $R(\mu_2)$ can be separated by a straight line l; by suitably rotating l and the Cartesian axes, we can assume that l is horizontal and every edge of G' is y-monotone in Γ', with $R(\mu_1)$ below $R(\mu_2)$. Then, define γ_1 (γ_2) as a horizontal strip containing $R(\mu_1)$ (resp. $R(\mu_2)$) and entirely below l (resp. above l). The resulting drawing Γ is a strip planar drawing of (G, γ).

If $k \geq 3$, then the above reduction does not always work (Fig.1 shows an example with $k = 4$). In the following we show how to construct a clustered graph $C(G', T)$ whose c-planarity is equivalent to the strip planarity of (G, γ) if $k \geq 3$. We also assume that G is connected. This is not a loss of generality. Namely, if G is not connected, then (G, γ) is strip planar if and only if each of its connected components $(G_1, \gamma_1), \ldots, (G_m, \gamma_m)$ is strip planar (where $\gamma_i(v) = \gamma(v)$ for every $v \in G_i$ and every $1 \leq i \leq m$). Thus, if an instance $C_i(G'_i, T_i)$ can be constructed in polynomial time equivalent to (G_i, γ_i), for every $1 \leq i \leq m$, then an instance $C(G', T)$ can also be constructed in polynomial time equivalent to (G, γ), where $G' = G'_1 \cup \cdots \cup G'_m$, and where T is a tree consisting of a root having $T_1, \ldots, T_m$ as subtrees.

Further, we assume that (G, γ) is proper. If this is not the case, then the reduction described in Section 3.2 can be applied in order to obtain an equivalent proper instance.

Summarizing, we can suppose w.l.o.g. that (G, γ) is connected, proper, and has $k \geq 3$ strips. We now describe how to construct $C(G', T)$ (see Figs. 20(a)-(b)).

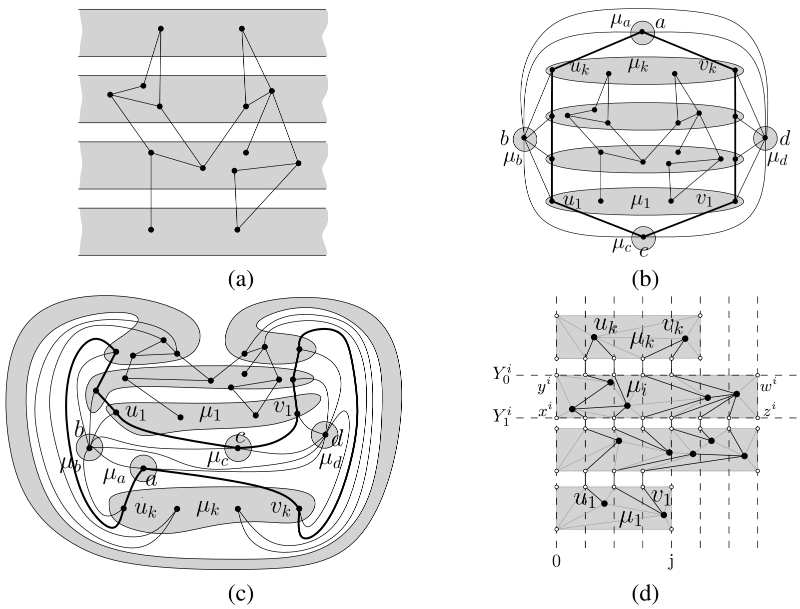


Fig. 20. Illustration for the proof of Theorem 2. (a) Instance (G, γ) of strip planarity; (b) instance $C(G', T)$ of clustered planarity obtained from (G, γ); (c) a c-planar drawing Γ' of $C(G', T)$; (d) the strip planar drawing Γ of (G, γ) obtained from Γ'.

Graph G' is composed of G and of a triconnected plane graph H, which consists of vertices a, b, c, d, $u_1, \ldots, u_k$, $v_1, \ldots, v_k$, and of edges (a, b), (b, c), (c, d), (a, d), (b, d), (a, u_k), (a, v_k), (c, u_1), (c, v_1), $(b, u_1), \ldots, (b, u_k)$, $(d, v_1), \ldots, (d, v_k)$, $(u_1, u_2), \ldots, (u_{k-1}, u_k)$, and $(v_1, v_2), \ldots, (v_{k-1}, v_k)$.

Tree T is constructed as follows. Initialize T with a root cluster μ. Add to T four clusters μ_a, μ_b, μ_c, and μ_d as children of μ, containing vertices a, b, c, and d, respectively. Then, for each $i = 1, \ldots, k$, add a cluster μ_i to T, as a child of μ, that contains vertices u_i, v_i, and each vertex $w \in V(G)$ such that $\gamma(w) = i$.

Clearly, $C(G', T)$ can be constructed in polynomial time. We prove that $C(G', T)$ admits a c-planar drawing if and only if (G, γ) admits a strip planar drawing.

Suppose that $C(G', T)$ admits a c-planar drawing Γ'. We construct a strip planar drawing Γ of (G, γ) as follows.

Since H is triconnected, it has a unique planar embedding [39], hence it has faces delimited by the same sequence of edges in any planar drawing. Since G is connected, by planarity all of its vertices and edges have to be inserted inside a single face of H. By the c-planarity of Γ', the face of H in which G has to be inserted has to contain at least a vertex belonging to each cluster $\mu_1, \dots, \mu_k$. Moreover, since $k \geq 3$, just one of the faces of H has incident vertices belonging to all clusters $\mu_1, \dots, \mu_k$, namely the face f delimited by cycle $C_f = (u_1, \dots, u_k, a, v_k, \dots, v_1, c)$. It follows that all the vertices and edges of G are embedded inside f in Γ'. Moreover, for each $1 \leq i \leq k$, the intersection of the region $R(\mu_i)$ representing cluster μ_i in Γ' with the interior of f is a connected region containing u_i and v_i; in fact, u_i and v_i are separated by path (a, b, c) in the region delimited by cycle C_f different from f. Then, the edges connecting a vertex of μ_i to a vertex of μ_{i+1} cut the boundary of $R(\mu_i)$ consecutively, for every $1 \leq i \leq k-1$; denote by $s^i_1, \dots, s^i_{n_i}$ the clockwise order in which these edges cut the boundary of $R(\mu_i)$, starting at the first edge s^i_1 crossing the boundary of $R(\mu_i)$ after (u_i, u_{i+1}). Analogously, the m_i edges connecting a vertex of μ_i to a vertex of μ_{i-1} cut the boundary of $R(\mu_i)$ consecutively, for every $2 \leq i \leq k$. Observe that, since (G, γ) is proper, it holds that $n_i = m_{i+1}$ for each $i = 1, \dots, k-1$.

We now show how to construct Γ. The outline of such a construction is as follows. We start by performing some modifications on the structure of G. We first subdivide the edges of G connecting vertices in different clusters with two subdivision vertices (denoted by p^i_j and q^{i+1}_j in the following, for $1 \leq i \leq k-1$ and $1 \leq j \leq n_i$); further, we add to G some dummy vertices (denoted by x^i, y^i, z^i, and w^i in the following, for $1 \leq i \leq k$) as well as some dummy edges incident to the dummy vertices and to the subdivision vertices. This modification of G results in a graph (denoted by L in the following) that satisfies the following property: There exists a set of cycles (denoted by $\mathcal{C}^i$ in the following, for $1 \leq i \leq k$) each containing all the vertices of a distinct cluster in its interior (roughly speaking, each of such cycles "simulates" the border of a strip). Then the interior of each cycle $\mathcal{C}^i$ can be triangulated by adding dummy edges. Now the cycles $\mathcal{C}^i$ can be drawn one on top of the other as axis-parallel rectangles and the graph inside $\mathcal{C}^i$ can be drawn with straight-line edges inside such a rectangle, for $1 \leq i \leq k-1$. Interpreting the horizontal sides of the rectangles as part of the strip boundaries and interpreting the dummy vertices as bend points, this results in a strip planar drawing of (G, γ) in which each edge between different strips is a y-monotone curve composed of three straight-line segments, two inside the corresponding strips and one between them. We now make this argument more precise.

We start by constructing the auxiliary graph L. Initialize L as G. We replace each edge s^i_j, where s^i_j connects a vertex u^i_j in μ_i to a vertex u^{i+1}_j in μ_{i+1}, with a path $(u^i_j, p^i_j, q^{i+1}_j, u^{i+1}_j)$. Further, add to L (i) dummy edges (p^i_j, p^i_{j+1}), with $j = 1, \dots, n_i-1$, (ii) dummy edges (q^i_j, q^i_{j+1}), with $j = 1, \dots, m_i - 1$, (iii) dummy edges (p^i_1, y^i), (x^i, y^i), and (x^i, q^i_1), where x^i and y^i are two dummy vertices, and (iv) dummy edges $(p^i_{n_i}, w^i)$, (z^i, w^i), and $(z^i, q^i_{m_i})$, where z^i and w^i are two dummy vertices. Also, add edges (x^1, z^1) and (y^k, w^k) to L.

For each $i = 1, \dots, k$, denote by C_{μ_i} the subgraph of L induced by the vertices inside or on the boundary of cycle $\mathcal{C}^i = (p^i_1, \dots, p^i_n, w^i, z^i, q^i_{m_i}, \dots, q^i_1, x^i, y^i)$.

Consider any set of k horizontal strips $\gamma_1, \dots, \gamma_k$. For each $i = 1, \dots, k$, let $y = Y_0^i$ and $y = Y_1^i$ be the higher and lower horizontal lines delimiting the strip γ_i, respectively.

We first show how to draw each graph C_{μ_i}. Place vertex y^i at point $(0, Y_0^i)$, vertex x^i at point $(0, Y_1^i)$, vertex w^i at point $(\max\{m_i, n_i\} + 1, Y_0^i)$, and vertex z^i at point $(\max\{m_i, n_i\} + 1, Y_1^i)$. Also, place each vertex p_j^i at point (j, Y_0^i), and each vertex q_j^i at point (j, Y_1^i). By construction, $\mathcal{C}^i$ is represented by a convex quadrilateral Q^i. Then, extend Q^i to a straight-line planar drawing Γ_i of C_{μ_i}. Observe that C_{μ_i} can be augmented to an internally-triangulated planar graph with no edge connecting two non-consecutive vertices on the outer face. Hence, Γ_i always exists [9]. Slightly perturbing the position of the internal vertices of C_{μ_i} results in a drawing in which all the edges, except for the ones incident to the outer face, are y-monotone.

Finally, for each $i = 1, \dots, k-1$ and $j = 1, \dots, n_i$, vertices p_j^i and q_j^{i+1} have the same x-coordinate, and hence can be connected with a vertical straight-line segment not intersecting any other edge. Now removing the inserted dummy edges and replacing all dummy vertices p_j^i and q_j^i with bends results in a strip planar drawing Γ of (G, γ).

Suppose now that (G, γ) admits a strip planar drawing Γ. A c-planar drawing Γ' of $C(G', T)$ can be constructed as follows. First, let the drawings of G' in Γ' and of G in Γ coincide. Let X_0 and X_1 be the smallest and the largest x-coordinate of a vertex in Γ, respectively. For each $i = 1, \dots, k$, let $y = Y_0^i$ and $y = Y_1^i$ be the horizontal lines delimiting strip γ_i from above and from below, respectively. Refer to Fig. 21.

Place vertices u_i and v_i at points $(X_0 - 1, \frac{Y_0^i + Y_1^i}{2})$ and $(X_1 + 1, \frac{Y_0^i + Y_1^i}{2})$, respectively, and represent μ_i as a rectangular region $R(\mu_i)$ with corners at $(X_0 - 2, Y_0^i)$, $(X_0 - 2, Y_1^i)$, $(X_1 + 2, Y_0^i)$, and $(X_1 + 2, Y_1^i)$. Then, place vertex a at point $(X_a = \frac{X_0 + X_1}{2}, Y_a = Y_0^k + 1)$, vertex b at point $(X_b = X_0 - 4, Y_b = \frac{Y_0^k + Y_1^1}{2})$, vertex c at point $(X_c = \frac{X_0 + X_1}{2}, Y_c = Y_1^1 - 1)$, and vertex d at point $(X_d = X_1 + 4, Y_d = \frac{Y_0^k + Y_1^1}{2})$.

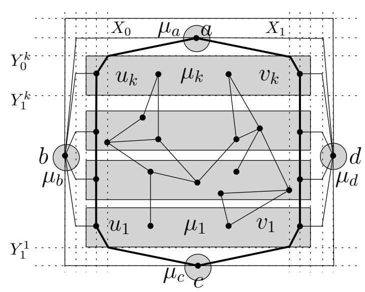


Fig. 21. The c-planar drawing Γ' of $C(G', T)$ obtained from a strip planar drawing Γ of (G, γ).

Draw edges (a, b), (b, c), (c, d), and (d, a) as polygonal lines bending at points $(X_b + 1, Y_a)$, (X_b, Y_c), (X_d, Y_c), and $(X_d - 1, Y_a)$, respectively. Draw edge (b, d) as a polygonal line bending at points $(X_b, Y_a + 1)$ and $(X_d, Y_a + 1)$. For each $i = 1, \dots, k$, draw edges (b, u_i) and (d, v_i) as polygonal lines bending at points $(X_b + 1, \frac{Y_0^i + Y_1^i}{2})$ and $(X_d - 1, \frac{Y_0^i + Y_1^i}{2})$, respectively. Draw edges (a, u_k), (a, v_k), (c, u_1), and (c, v_1) as polygonal lines bending at points (X_0, Y_0^k), (X_1, Y_0^k), (X_0, Y_1^1), and (X_1, Y_1^1), respectively. Draw edges (u_i, u_{i+1}) and (v_i, v_{i+1}) as straight-line segments, for each $1 \leq i \leq k-1$.

Finally, draw cluster μ_a (μ_b, μ_c, μ_d) as a small disk $R(\mu_a)$ (resp. $R(\mu_b)$, $R(\mu_c)$, $R(\mu_d)$) enclosing only vertex a (resp. b, c, d) and not overlapping with any other region.

This results in a c-planar drawing of $C(G', T)$. □

5 Conclusions

In this paper, we introduced the strip planarity testing problem and showed how to solve it in polynomial time if the input graph has a prescribed plane embedding. The main question raised by this paper is whether the strip planarity testing problem can be solved in polynomial time or it is rather $\mathcal{NP}$-hard for graphs without a prescribed plane embedding. The problem is intriguing even if the input graph is a tree.

We also proved the existence of a polynomial-time reduction from the strip planarity testing problem to the clustered planarity testing problem. Fulek proved [22] a stronger result: For every instance (G, γ) of strip planarity, an equivalent instance $C(G, T)$ of clustered planarity can be constructed in polynomial time such that T only contains three clusters. Thus, designing a polynomial-time algorithm for the strip planarity testing problem is a vital step towards deepening our understanding of clustered planarity.

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