

Explicit estimates on the measure of primary KAM tori*

(Draft)

L. Biasco & L. Chierchia
 Dipartimento di Matematica e Fisica
 Università degli Studi Roma Tre
 Largo San L. Murialdo 1 - 00146 Roma, Italy
 biasco@mat.uniroma3.it, luigi@mat.uniroma3.it

October 1, 2018

Abstract

From KAM Theory it follows that the measure of phase points which do not lie on Diophantine, Lagrangian, “primary” tori in a nearly-integrable, real-analytic Hamiltonian system is $O(\sqrt{\varepsilon})$, if ε is the size of the perturbation. In this paper we discuss how the constant in front of $\sqrt{\varepsilon}$ depends on the unperturbed system and in particular on the phase-space domain.

Contents

1	Introduction	2
2	Notations and set up	4
3	Assumptions	5
4	The local frequency map	5
5	The classical analytic KAM Theorem	7
6	Proof of KAM Theorem 1	7
6.1	Local reduction	8
6.2	A KAM local normal form après [22]	8
6.3	Applying the KAM normal form to $\mathbb{H} _{B_{\hat{r}}(p_i) \times \mathbb{T}^n}$	10
6.4	Kolmogorov’s tori: The sets $\mathcal{T}_\alpha^{(i)}$ and \mathcal{T}_α	12
6.5	Properties of the torus embedding	12
6.6	Measure estimates	17

*This research was partially supported by the European Research Council under FP7 “Hamiltonian PDEs and small divisor problems: a dynamical systems approach” and by MIUR (Italy): PRIN-“Variational and perturbative aspects of nonlinear differential problems”.

1 Introduction

According to classical KAM theory, the majority of the Lagrangian, non-resonant invariant tori of a “general” completely integrable Hamiltonian system persists under the effect of a small enough perturbations ([17], [1], [19]; see, also, [2, §6.3] for a review and [15] for a divulgative exposition).

Indeed, in bounded regions of the phase space $\mathbb{R}^n \times \mathbb{T}^n$ (action-angle) such tori – which are also called “primari” tori – form a set of positive Liouville (Lebesgue) measure, whose complement has a measure proportional to $\sqrt{\varepsilon}$, if ε measures the size of the perturbing function, [20] and [21].

The square root behavior, in such measure estimates, is optimal in the sense that, in general, at simple resonances, for $\varepsilon \neq 0$, there appear regions of size proportional to $\sqrt{\varepsilon}$ free of primary invariant tori as trivially shows the example of the simple pendulum with gravity¹ ε .

It is therefore natural to look for explicit evaluations of the constant in front of $\sqrt{\varepsilon}$ in the KAM measure estimates of the complement of invariant primary tori.

In [20] and [21] such constant, which depends on analytic properties of the integrable limit, is left implicit, and, somewhat surprisingly, to the best of our knowledge, there are no explicit evaluations of it in the vast literature on classical KAM theory. On the other hand, KAM is a constructive technique and discussions about “KAM constants” are clearly relevant, as also testified by the large literature on them; compare, e.g., [12], [16], [9], [5], [13], [18], [6], [14], [7], [10], [11], [8].

We also point out that an explicit dependence upon the domain in the above measure estimate is crucial in investigating the more complicate problem of the existence and abundance of *secondary tori*, i.e., of those tori which arise by effect of the perturbation around simple resonances. In [2] it is conjectured that “in a generic system with three or more degrees of freedom the measure of the non-torus set has order ε ”, while in [4] it is given a sketchy proof² that *the union of primary and secondary tori leave out (for general mechanical systems) a region of measure $\varepsilon |\log \varepsilon|^a$ (for a suitable $a > 0$)*. To achieve such result one needs to control simultaneously a large number of regions around simple resonances and to apply KAM measure estimates taking into account different (local) phase-space domains, including neighborhoods of separatrices: To carry out such strategy simple explicit measure estimates – such as the above – are necessary.

In this paper we compute explicitly the constant in front of $\sqrt{\varepsilon}$ up to a constant depending only on n (the number of degrees of freedom) and $\tau > n - 1$ (the uniform “Diophantine exponent”).

More precisely, we consider a real-analytic, nearly-integrable Hamiltonian

$$H : (p, q) \in D \times \mathbb{T}^n \mapsto H(p, q) = h(p) + f(p, q) \in \mathbb{R}$$

where D is an *arbitrary* bounded domain in \mathbb{R}^n and \mathbb{T}^n is the standard flat n -torus (with periods 2π); f is a small perturbation function and the integrable limit h is Kolmogorov non-degenerate on D , i.e., its hessian is invertible on D .

The main result – Theorem 1 below – will be formulated in terms of a few (five) parameters, which we now describe briefly (precise definitions will be given in § 2 below):

¹Just look at the phase portrait of the simple pendulum $\frac{1}{2}p^2 + \varepsilon \cos q$, $(p, q) \in \mathbb{R} \times \mathbb{T}$, and observe that the region enclosed by the separatrix $\frac{1}{2}p^2 + \varepsilon \cos q = \varepsilon$ has measure $4\sqrt{2} \cdot \sqrt{\varepsilon}$.

²A complete proof will appear elsewhere.

- The Hamiltonian H is assumed to be real-analytic on $D \times \mathbb{T}^n$: therefore there exists $r_0 > 0$ and $0 < s \leq 1$ such that H is holomorphic on a complex r_0 -neighborhood of D and a s -complex neighborhood of \mathbb{T}^n .
- The smallness of the perturbation f will be measured by $\epsilon := \frac{\varepsilon}{Mr_0^2}$ where: $\varepsilon = \|f\|_{r_0, s}$ denotes the sup-norm on the above complex neighborhood of f and $M := \|h_{pp}\|_{r_0}$ the sup norm of the Hessian matrix of h .
- The “torsion” associated to h will be measured by $\mu := \frac{\inf_D |\det h_{pp}|}{M^n}$; note that $0 < \mu \leq 1$ (compare (5) below).
- Last “independent parameter” will be the number $\lambda := LM$, where L denotes a suitable uniform Lipschitz constant of the local complex inverse of the “frequency map” $p \mapsto \omega = h_p(p)$ (compare (9)); indeed one can show that $1 \leq \lambda \leq 2 \cdot n! \mu^{-1}$ (see (14) below).

Notice that the parameters ϵ , μ and λ are *dimensionless parameters* (i.e., do not have physical dimensions).

Then, fixed $\nu := \tau + 1 > n$, we will show that there exist a positive constants $c < 1$ depending only on n and ν such that if the perturbation is so small that

$$\epsilon \leq c \frac{\mu^6}{\lambda^2} s^{4\nu} ,$$

then one can construct a family \mathcal{T}_α of H -invariant primary tori. Such tori live in $D_{r_0} \times \mathbb{T}^n$ (where D_{r_0} is a real r_0 -neighborhood of D) and the H -flow on them is analytically conjugated to the Kronecker flow $x \in \mathbb{T}^n \mapsto x + \omega t$ for a frequency $\omega \in \mathbb{R}^n$ which is (α, τ) -Diophantine³ with $\tau = \nu - 1$ and α proportional to $\sqrt{\epsilon}$:

$$\alpha := \frac{\lambda}{\hat{c} \mu s^{3\nu}} (Mr_0) \sqrt{\epsilon} ,$$

where $\hat{c} < 1$ is a suitable constant depending only on n and ν .

The upshot is, then, the following measure estimate (where “meas” denotes outer Lebesgue measure):

$$\text{meas} \left((D \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha \right) \leq C \sqrt{\epsilon}$$

where the constant C is given by

$$C := \kappa \left(\max \{ \mu^2 r_0, \text{diam } D \} \right)^n \cdot \frac{\lambda^{n+2}}{\mu^3 s^{3\nu}} ,$$

where $\kappa > 0$ is a suitable constant depending only on n and ν .

Remarks (i) In fact, we shall prove a stronger statement, which is non trivial even in case of D of measure zero or even finite (compare (19) below).

(ii) Of course, more refined estimates are possible if one adds extra hypotheses on the domain D (e.g., smooth boundary) and it would be interesting to give bounds which take into account geometrical properties of D .

³I.e., $|\omega \cdot k| \geq \alpha/|k|_1^\tau$ for all $k \in \mathbb{Z}^n \setminus \{0\}$; compare (1) below.

(iii) We do not compute explicitly the dependence upon n (and ν): Indeed it is well known that in such generality explicit bounds on c tend to be quite “pessimistic”, however, in concrete example, such as a forced pendulum, the standard map or particular three body problems computer-assisted (rigorous) upper bounds on ε are in excellent agreement with experimental data (see, e.g., [5], [13], [6], [8] and references therein).

2 Notations and set up

Given $r > 0$, p_0 a point of \mathbb{R}^n or \mathbb{C}^n and D a subset of \mathbb{R}^n or \mathbb{C}^n , we denote:

$$\begin{aligned} \mathbb{B}_r(p_0) &:= \{p \in \mathbb{R}^n \mid |p - p_0| < r\}, & (p_0 \in \mathbb{R}^n), \\ \mathbb{B}_r(p_0) &:= \{p \in \mathbb{C}^n \mid |p - p_0| < r\}, & (p_0 \in \mathbb{C}^n), \\ \mathbb{B}_r(D) &:= \bigcup_{p_0 \in D} \mathbb{B}_r(p_0), & (D \subseteq \mathbb{R}^n), \\ \mathbb{B}_r(D) &:= \bigcup_{p_0 \in D} \mathbb{B}_r(p_0), & (D \subseteq \mathbb{C}^n), \end{aligned}$$

where in \mathbb{R}^n and \mathbb{C}^n , $|x| = |(x_1, \dots, x_n)|$ will denote the sup-norm $\max_i |x_i|$.

For a matrix (or a tensor) A , $\|A\|$ denotes the standard operator norm $\sup_{|x|=1} |Ax|$.

The standard flat n -torus $\mathbb{R}^n/(2\pi\mathbb{Z}^n)$ is denoted by \mathbb{T}^n and, for $s > 0$, \mathbb{T}_s^n denotes its complex neighbourhood of points q with norm of the imaginary part $|\operatorname{Im} q| = |(\operatorname{Im} q_1, \dots, \operatorname{Im} q_n)| < s$:

$$\mathbb{T}_s^n := \{y \in \mathbb{C}^n \mid |\operatorname{Im} y| < s\} / (2\pi\mathbb{Z}^n).$$

If D is an arbitrary bounded set in \mathbb{R}^n and h , respectively, f , a real-analytic function (with values in \mathbb{R}^m or in matrix spaces) with bounded holomorphic extension on $\mathbb{B}_r(D)$ for some $r > 0$, respectively, on $\mathbb{B}_r(D) \times \mathbb{T}_s^n$ for some $r, s > 0$, we define its analytic sup-norm as, respectively,

$$\|h\|_{D,r} := \sup_{y \in \mathbb{B}_r(D)} |h(y)|, \quad \|f\|_{D,r,s} := \sup_{(p,q) \in \mathbb{B}_r(D) \times \mathbb{T}_s^n} |f(p,q)|.$$

The Lipschitz semi-norm of a function $f : \Omega \rightarrow \mathbb{R}^m$, will be denoted by

$$|f|_{\operatorname{Lip}, \Omega} := \sup_{\omega_1, \omega_2 \in \Omega, \omega_1 \neq \omega_2} \frac{|f(\omega_1) - f(\omega_2)|}{|\omega_1 - \omega_2|}.$$

If D is an open set and $H : D \times \mathbb{T}^n \rightarrow \mathbb{R}$ is a C^2 function, ϕ_H^t denotes its Hamiltonian flow, namely, $(p(t), q(t)) = \phi_H^t(p, q)$ solves the standard Hamilton equations⁴

$$\begin{cases} \dot{p}(t) := \frac{dp}{dt}(t) = -\partial_q H(p(t), q(t)) \\ \dot{q}(t) := \frac{dq}{dt}(t) = \partial_p H(p(t), q(t)) \end{cases}, \quad (p(0), q(0)) = (p, q).$$

For example, if $H(p, q) = h(p)$, then the flow ϕ_h^t is linear with frequency $\omega := \partial_p h(p)$, namely, $\phi_h^t(p, q) = (p, q + \omega t)$.

⁴Equivalently, ϕ_H^t denotes the Hamiltonian associated to the standard symplectic form $dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq_i$.

Given $\alpha, \tau > 0$, a vector $\omega \in \mathbb{R}^n$ is said to be (α, τ) -Diophantine if

$$|\omega \cdot k| := \left| \sum_{j=1}^n \omega_j k_j \right| \geq \frac{\alpha}{|k|_1^\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}, \quad (1)$$

where $|k|_1 := \sum |k_j|$ denotes the 1-norm. It is well known that, fixed $\tau > n - 1$, almost all (in the sense of Lebesgue measure) $\omega \in \mathbb{R}^n$ are (α, τ) -Diophantine for some $\alpha > 0$. Indeed such statement follows immediately observing that⁵

$$\text{meas} \{ \omega \in B_R(0) \mid \omega \text{ is not } (\alpha, \tau)\text{-Diophantine} \} \leq c R^{n-1} \alpha,$$

with a constant c depending only on n and τ .

Finally, given a $2n$ -vector (y, x) , π_1 and π_2 denote, respectively, the projectionst on the first and second n components:

$$\pi_1(y, x) = y \quad \text{and} \quad \pi_2(y, x) = x. \quad (2)$$

3 Assumptions

Fix $n \geq 2$ and $\tau > n - 1$. Let D be any non-empty, bounded subset of \mathbb{R}^n . Let

$$H := h + f$$

with h and f real-analytic functions with holomorphic extensions on, respectively, $\mathbb{B}_{r_0}(D)$ and $\mathbb{B}_{r_0}(D) \times \mathbb{T}_s^n$ for some $r_0 > 0$ and $0 < s \leq 1$, and having finite norms:

$$M := \|h_{pp}\|_{D, r_0}, \quad \varepsilon := \|f\|_{D, r_0, s}. \quad (3)$$

Assume that the frequency map $p \in D \rightarrow \omega = h_p$ is a local diffeomorphism, namely, assume:

$$d := \inf_D |\det h_{pp}| > 0. \quad (4)$$

4 The local frequency map

Under assumption (4) the frequency map is a *local* real-analytic diffeomorphism in the neighbourhood of any point of D . More precisely, the following lemma holds. Define⁶

$$\mu := \frac{d}{M^n} \leq 1. \quad (5)$$

Lemma 1 *Let*

$$c_0 = \frac{1}{8n \cdot n!^2}, \quad \hat{c}_0 = \frac{1}{4n \cdot n!}. \quad (6)$$

⁵“meas” stands for Lebesgue measure, or, in general, for outer Lebesgue measure.

⁶Since any eigenvalue of h_{pp} is bounded in absolute value by $\|h_{pp}\| \leq M$, $d \leq \sup_D |\det h_{pp}| \leq M^n$.

and define

$$r_\star := \hat{c}_0 \mu r_0, \quad \rho_\star := c_0 \mu^2 M r_0. \quad (7)$$

Then, for every $p_0 \in \mathbb{D}$ the frequency map $p \rightarrow \omega = h_p$ has a real-analytic inverse map, $\omega \rightarrow \mathbf{p}(\omega; p_0)$, defined in a neighborhood of $\omega_0 := h_{p_0}$

$$\mathbf{p} = h_p^{-1} : \omega \in \mathbb{B}_{\rho_\star}(\omega_0) \mapsto \mathbf{p}(\omega; p_0) \in \mathbb{B}_{r_\star}(p_0), \quad (8)$$

with uniform Lipschitz constant⁷

$$\mathbf{L} := \sup_{p_0 \in \mathbb{D}} |\mathbf{p}(\cdot; p_0)|_{\text{Lip}, \mathbb{B}_{\rho_\star}(\omega_0)} = \sup_{p_0 \in \mathbb{D}} \sup_{\mathbb{B}_{\rho_\star}(\omega_0)} \|\mathbf{p}_\omega(\cdot; p_0)\| \quad (9)$$

satisfying

$$\mathbf{L} \leq \frac{M^{-1}}{2n \hat{c}_0 \mu}, \quad (10)$$

and

$$\sup_{p_0 \in \mathbb{D}} \sup_{\mathbb{B}_{\frac{3}{4}\rho_\star}(\omega_0)} \|\mathbf{p}_\omega(\cdot; p_0)\| \leq \frac{\mathbf{L}}{\frac{c_0}{4} \mu^2 M r_0}. \quad (11)$$

Proof Writing out the inverse of the matrix h_{pp} (Cramer's rule), by Leibniz formula for the ji -minor of h_{pp} , one has, uniformly on⁸ D :

$$|(h_{pp}^{-1})_{ij}| \leq \frac{1}{d} (n-1)! M^{n-1},$$

which implies

$$\sup_{\mathbb{D}} \|(h_{pp})^{-1}\| \leq \frac{n!}{\mu} M^{-1}. \quad (12)$$

Let $T := h_{pp}^{-1}(p_0)$. Then, by standard Cauchy estimates⁹, it follows that for any $p \in \mathbb{C}^n$ such that $|p - p_0| \leq r_\star$ one has

$$\|I - Th_{pp}(p)\| \leq \|T\| \|h_{pp}(p) - h_{pp}(p_0)\| \leq \|T\| n \frac{M}{r_0 - r_\star} r_\star \stackrel{(12)}{\leq} \frac{n \cdot n!}{\mu} \frac{r_\star}{r_0 - r_\star} \stackrel{(6)}{\leq} \frac{1}{2}.$$

Thus, by the standard Inverse Function Theorem (see Appendix A, Eqn's (74), (75), (76)) and Cauchy estimates, relations (8), (9), (10) and (11) follow immediately with the constants in (6). \blacksquare

For later use, we point out that¹⁰

$$\lambda := \mathbf{L} M \geq 1, \quad (13)$$

and that, by (10),

$$\lambda \leq \frac{1}{2n\hat{c}_0} \frac{1}{\mu} = 2 \cdot n! \frac{1}{\mu}. \quad (14)$$

⁷ Notice that on convex domains the Lipschitz semi-norm of a differentiable function coincides with the sup-norm of its Jacobian.

⁸ Note that $\sup_{i,j} \sup_D |h_{p_i p_j}| \leq M := \sup_D \sup_i \sum_j |h_{p_i p_j}|$.

⁹ If $f : \mathbb{B}_r(D) \rightarrow \mathbb{C}^m$ is holomorphic, ∂^α is a partial derivative of order $k = \alpha_1 + \dots + \alpha_n$ and $0 < r' < r$, then

$$\sup_{\mathbb{B}_{r'}(D)} |\partial^\alpha f| \leq \frac{\sup_{\mathbb{B}_r(D)} |f|}{(r - r')^k}.$$

¹⁰ Indeed: $1 = \|I\| = \|h_{pp}(p)h_{pp}^{-1}(p)\| = \|h_{pp}(p)\mathbf{p}_\omega(h_p(p))\| \leq \|h_{pp}(p)\| \|\mathbf{p}_\omega(h_p(p))\| \leq \mathbf{L} M$.

5 The classical analytic KAM Theorem

Theorem 1 *Let the assumptions in § 3 hold and let λ , μ and c_0 be as in § 4 and let $\nu = \tau + 1$. There exist positive constants $c_\star < 1/(8 \cdot n!)$ and κ , depending only on n and τ , such that, if*

$$c := \frac{c_\star^2}{2^{17} \cdot n^2 (n!)^6}, \quad \hat{c} := \frac{c_\star}{2^4}, \quad (15)$$

and if ε is such that

$$\varepsilon := \frac{\varepsilon}{Mr_0^2} \leq c \frac{\mu^6}{\lambda^2} s^{4\nu}, \quad (16)$$

then the following holds. Define

$$\alpha := \frac{\lambda}{\hat{c} \mu s^{3\nu}} (Mr_0) \sqrt{\varepsilon}, \quad \hat{r} := \frac{c_0}{2} \mu^2 r_0, \quad r_\varepsilon := \frac{\lambda}{c_\star} \sqrt{\varepsilon} r_0. \quad (17)$$

Then, there exists a positive measure set $\mathcal{T}_\alpha \subseteq B_{2\hat{r}}(\mathbb{D}) \times \mathbb{T}^n$ formed by “primary” Kolmogorov’s tori; more precisely, for any point $(p, q) \in \mathcal{T}_\alpha$, $\phi_{\mathbb{H}}^t(p, q)$ covers densely an \mathbb{H} -invariant, analytic, Lagrangian torus, with \mathbb{H} -flow analytically conjugated to a linear flow with (α, τ) -Diophantine frequencies $\omega = h_p(p_0)$, for a suitable $p_0 \in \mathbb{D}$; each of such tori is a graph over \mathbb{T}^n r_ε -close to the unperturbed trivial graph $\{(p, \theta) = (p_0, \theta) \mid \theta \in \mathbb{T}^n\}$.

Finally, the Lebesgue outer measure of $(\mathbb{D} \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha$ is bounded by:

$$\text{meas}((\mathbb{D} \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha) \leq C \sqrt{\varepsilon} \quad (18)$$

with

$$C := \kappa \left(\max\{\mu^2 r_0, \text{diam } \mathbb{D}\} \right)^n \cdot \frac{\lambda^{n+2}}{\mu^3 s^{3\nu}};$$

indeed, there exist N and, for $1 \leq i \leq N$, $p_i \in \mathbb{D}$, such that $\mathbb{D} \subseteq \bigcup_{i=1}^N B_{\hat{r}}(p_i)$ and

$$\text{meas} \left(\left(\bigcup_{i=1}^N B_{\hat{r}}(p_i) \times \mathbb{T}^n \right) \setminus \mathcal{T}_\alpha \right) \leq C \sqrt{\varepsilon}. \quad (19)$$

6 Remarks

- (i) The constant c_\star is, essentially, the “smallness” constant appearing in a local KAM normal form (see Theorem 2 below). The constant κ is given in (73).
- (ii) Estimate (19) implies at once (18) and notice that (19) is meaningful also in the case of sets \mathbb{D} of measure zero (such as a singleton).

6 Proof of KAM Theorem 1

The proof of Theorem 1 is divided in six steps.

6.1 Local reduction

The first step consists in covering D with N balls centered at points of D (with an explicit upper bound on N), thus reducing the Theorem to the special case in which the domain is a ball. Indeed, the following simple result holds.

Lemma 2 (Covering Lemma) *Let $E \subseteq \mathbb{R}^n$ be a non-empty set of finite diameter. Then, for any $r > 0$ there exists an integer N , with¹¹*

$$1 \leq N \leq \left(\left\lceil \frac{\text{diam } E}{r} \right\rceil + 1 \right)^n, \quad (20)$$

and N points $p_i \in E$ such that

$$E \subseteq \bigcup_{i=1}^N B_r(p_i). \quad (21)$$

Proof Let $\delta := \text{diam } E$ and let $z_i = \inf\{x_i \mid x \in E\}$. Then $E \subseteq K := z + [0, \delta]^n$. Let $0 < r' < r$ close enough to r so that $\lceil \delta/r' \rceil = \lceil \delta/r \rceil + 1 =: M$. Then, one can cover K with M^n closed, contiguous cubes K_j , $1 \leq j \leq M^n$, with edge of length r' . Let j_i be the indices such that $K_{j_i} \cap E \neq \emptyset$ and pick a $p_i \in K_{j_i} \cap E$; let $1 \leq N \leq M^n$ be the number of such cubes. Observe that, since we have chosen the sup-norm in \mathbb{R}^n , one has $K_{j_i} \subseteq B_r(p_i)$ and, therefore, (21) follows with N as in (20). \blacksquare

We now apply the Lemma with $E = D$ and $r = \hat{r}$ defined by¹²

$$\hat{r} := \frac{\rho_\star}{2M} = \frac{c_0}{2} \mu^2 r_0 \leq \frac{r_0}{128} < \frac{r_0}{2}. \quad (22)$$

Thus, Lemma 2 yields that:

For suitable N points $p_i \in D$, one has

$$D \subseteq \bigcup_{i=1}^N B_{\hat{r}}(p_i), \quad 1 \leq N \leq \left(\left\lceil \frac{\text{diam } D}{\frac{c_0}{2} \mu^2 r_0} \right\rceil + 1 \right)^n. \quad (23)$$

Notice that, by (22), H is holomorphic and bounded on $\mathbb{B}_{r_0/2}(B_{\hat{r}}(p_i)) \times \mathbb{T}_s^n$, for every $i \leq N$.

Next we shall prove a “local” version of Theorem 1

6.2 A KAM local normal form après [22]

Fix one of the balls $B_{\hat{r}}(p_i)$ in the covering (23). We first prove Theorem 1 with D and r_0 replaced, respectively, by

$$D_i := B_{\hat{r}}(p_i) \quad \text{and} \quad \frac{r_0}{2}.$$

We shall use a “KAM normal form with parameters”; more specifically, we shall use Theorem B of [22], whose statement we recall here for convenience of the reader.

¹¹ $[x]$ denotes the integer-part (or “floor”) function $\max\{n \in \mathbb{Z} \mid n \leq x\}$, while $\lceil x \rceil$ denote the “ceiling function” $\min\{n \in \mathbb{Z} \mid n \geq x\}$.

¹²Recall (7), (5) and (6).

Let $r, \alpha, \mathbf{h} > 0$, $0 < s \leq 1$, $\tau > n - 1$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with piecewise smooth boundary; let

$$\Omega_\alpha := \{\omega \in \Omega, \text{ s.t. } \text{dist}(\omega, \partial\Omega) \geq \alpha \text{ and } \omega \text{ is } (\alpha, \tau)\text{-Diophantine}\};$$

let $\omega \rightarrow e(\omega)$ and $(I, \theta, \omega) \rightarrow P(I, \theta, \omega)$ be real-analytic functions with holomorphic extension on, respectively, $\mathbb{B}_\mathbf{h}(\Omega_\alpha)$ and $\mathbb{B}_r(0) \times \mathbb{T}_s^n \times \mathbb{B}_\mathbf{h}(\Omega_\alpha)$. Finally, if $a > 0$, we define the ‘‘action rescaling map’’:

$$R_a(I, \theta) := (I/a, \theta). \quad (24)$$

Consider the Hamiltonian function, parametrized by ω ,

$$H(I, \theta, \omega) := N(I, \omega) + P(I, \theta, \omega), \quad \text{where } N := e(\omega) + \omega \cdot I,$$

with respect to the standard symplectic form $dI \wedge d\theta$; in particular, the integrable flow ϕ_N^t is given by $\phi_N^t(I, \theta) = (I, \theta + \omega t)$.

In [22] the following result is proven.

Theorem 2 *Under the above definitions and assumptions, there exist constants*

$$0 < c_\star < \frac{\hat{c}_\star}{4 \cdot n!} < \frac{1}{8 \cdot n!}, \quad (25)$$

depending only on n and τ , such that if

$$|P|_{r,s,\mathbf{h}} := \sup_{\mathbb{B}_r(0) \times \mathbb{T}_s^n \times \mathbb{B}_\mathbf{h}(\Omega_\alpha)} |P| \leq c_\star \alpha r s^\nu, \quad \alpha s^\nu \leq \mathbf{h}, \quad (\nu := \tau + 1), \quad (26)$$

then, there exist a Lipschitz homeomorphism $\varphi : \Omega \hookrightarrow \Omega$ and a family of torus embeddings

$$\Phi : \mathbb{T}^n \times \Omega \rightarrow \mathbb{B}_r(0) \times \mathbb{T}^n \subseteq \mathbb{R}^n \times \mathbb{T}^n$$

such the following holds. For every $\omega \in \Omega_\alpha$, $\Phi(\mathbb{T}^n, \omega)$ is an invariant torus for $H|_{\varphi(\omega)} := H(I, \theta, \varphi(\omega))$ and

$$(\phi_{H|_{\varphi(\omega)}}^t \circ \Phi)(\theta, \omega) = \Phi(\theta + \omega t, \omega).$$

Moreover, for each $\omega \in \Omega$, $\theta \rightarrow \Phi(\theta, \omega)$ is real-analytic on $\mathbb{T}_{s/2}^n$ and if

$$(\theta, \omega) \rightarrow \Phi_0(\theta, \omega) := (0, \theta)$$

denotes the trivial torus embedding, one has, uniformly on, respectively, $\mathbb{T}_{s/2}^n \times \Omega$ and Ω , the following estimates:

$$|R_r(\Phi - \Phi_0)|, \quad \alpha s^\nu |R_r(\Phi - \Phi_0)|_{\text{Lip}, \Omega} \leq \frac{|P|}{\hat{c}_\star \alpha r s^\nu}, \quad (27)$$

$$|\varphi - \text{id}|, \quad \alpha s^\nu |\varphi - \text{id}|_{\text{Lip}, \Omega} \leq \frac{|P|}{\hat{c}_\star r}. \quad (28)$$

Remarks

- (i) The constants c_\star and \hat{c}_\star can be taken equal to, respectively, $\gamma/2$ and $1/c$ where γ and c are the constants appearing in Theorem A of [22], where are not explicitly evaluated.

In [3] an infinite dimensional KAM Theorem¹³ (implying Theorem 2) is proved, substituting (26) with the stronger (for s small) condition $|P|_{r,s,h} \leq \kappa \alpha r$, where $\kappa = \kappa(s, n, \tau) := \kappa_\star^{-c\kappa_\star}$ with $\kappa_\star = (n + \tau) \ln((n + \tau)/s)$ and where $c > 0$ is an absolute constant.

The numerical relations between c_\star and \hat{c}_\star in (25) are assumed for later convenience (and, obviously, are compatible with [22]).

- (ii) For simplicity – and because it would play no rôle – in (27) we reported slightly weaker estimates with respect to those appearing in Theorem A, where in place of R_r there appears the rescaling $W(I, \theta) := (I/r, \theta/s)$ (which means that the estimates on the angle components in [22] are better by a factor $s < 1$ than those in (27)).
- (iii) Actually, the above Theorem is a synthesis of Theorem A and Theorem B in [22]. In particular, the final measure estimate in Theorem B is not reported since the constant (and its dependence upon Ω) is left implicit.
- (iv) We point out that from the estimates (27) *does not follows that Φ is Lipschitz close to the trivial embedding Φ_0* . Indeed, if π_2 denotes the projection over the θ -component, taking into account that $\alpha = O(\sqrt{\epsilon})$ (compare (17)), from (26) and (27) it follows that

$$|\pi_2 \circ \Phi - \pi_2 \circ \Phi_0|_{\text{Lip}} \leq \frac{c_\star}{\hat{c}_\star} \cdot \frac{1}{\alpha s^\nu} = O\left(\frac{1}{\sqrt{\epsilon}}\right).$$

To overcome this fact one needs suitable asymmetric rescalings of action and angle variables.

- (v) As well known ([21], [12]), maps constructed via KAM methods are smooth in the sense of Whitney, or, what is the same, have C^∞ extensions: Indeed, φ and Φ are $C^\infty(\Omega)$.

6.3 Applying the KAM normal form to $\mathbb{H}|_{\mathbb{B}_{\hat{r}}(p_i) \times \mathbb{T}^n}$

We now apply Theorem 2 to \mathbb{H} restricted to

$$D_i \times \mathbb{T}^n := \mathbb{B}_{\hat{r}}(p_i) \times \mathbb{T}^n$$

where \hat{r} is defined in (22) and p_i is one of the points introduced in Lemma 2. Recall that, by (22), $\hat{r} < r_0/2$ so that \mathbb{H} has holomorphic extension to $\mathbb{B}_{r_0/2}(D_i) \times \mathbb{T}_s^n$.

Let

$$\Omega^{(i)} := h_p(D_i) = h_p(\mathbb{B}_{\hat{r}}(p_i)), \quad \mathbf{h} := \frac{\rho_\star}{4} = \frac{M\hat{r}}{2}, \quad (29)$$

and notice that, by (22),

$$\Omega^{(i)} \subseteq \mathbb{B}_{M\hat{r}}(h_p(p_i)) \subseteq \mathbb{B}_{\rho_\star/2}(h_p(p_i)) = \mathbb{B}_{2\mathbf{h}}(h_p(p_i)) \implies \mathbb{B}_{\mathbf{h}}(\Omega^{(i)}) \subseteq \mathbb{B}_{\frac{3}{4}\rho_\star}(h_p(p_i)),$$

¹³More precisely Theorem 5.1, case (H3) on page 755 and Remark 5.3 on page 758.

which, by Lemma 1, shows that h_p has an inverse¹⁴ $\mathbf{p} = h_p^{-1}$ with holomorphic extension

$$\mathbf{p} = \mathbf{p}(\cdot; p_i) : \mathbb{B}_{\mathbf{h}}(\Omega^{(i)}) \rightarrow \mathbb{B}_{r_\star}(p_i). \quad (30)$$

Following [22], we introduce $\omega \in \mathbb{B}_{\mathbf{h}}(\Omega^{(i)})$ as parameter, and let¹⁵

$$p = \mathbf{p}(\omega) + I, \quad \text{with} \quad \omega \in \mathbb{B}_{\mathbf{h}}(\Omega^{(i)}), \quad |I| < r := \sqrt{\frac{\varepsilon}{\mathbf{M}}} = \sqrt{\varepsilon} \cdot r_0 < \frac{r_0}{4}, \quad (31)$$

and define

$$\begin{cases} N(I, \omega) := e(\omega) + \omega \cdot I := h(\mathbf{p}(\omega)) + \omega \cdot I \\ P(I, \theta, \omega) := \int_0^1 (1-t) h_{pp}(\mathbf{p}(\omega) + tI) I \cdot I dt + f(\mathbf{p}(\omega) + I, \theta), \end{cases} \quad (32)$$

so that

$$\mathbf{H}(\mathbf{p}(\omega) + I, \theta) = h(\mathbf{p}(\omega) + I) + f(\mathbf{p}(\omega) + I, \theta) = N(I, \omega) + P(I, \theta, \omega) =: H(I, \theta, \omega).$$

By (30), (31), (7), (6), one has that if $\omega \in \mathbb{B}_{\mathbf{h}}(\Omega^{(i)})$ and $I \in \mathbb{B}_r(0)$, then

$$|\mathbf{p}(\omega) + I - p_i| \leq |\mathbf{p}(\omega) - p_i| + |I| < r_\star + r = \hat{c}_0 \mu r_0 + \sqrt{\varepsilon} r_0 < \frac{r_0}{4} + \frac{r_0}{4} = \frac{r_0}{2}, \quad (33)$$

so that $\mathbf{p}(\omega) + I \in \mathbb{B}_{r_0/2}(p_i)$. Thus, by (3), (31) and (32), H is real-analytic with holomorphic extension to $\mathbb{B}_r(0) \times \mathbb{T}_s^n \times \mathbb{B}_{\mathbf{h}}(\Omega^{(i)})$ with

$$|P|_{r,s,\mathbf{h}} \leq 2\varepsilon. \quad (34)$$

Thus, if α is as in (17) and r as in (31), then,

$$\frac{|P|_{r,s,\mathbf{h}}}{c_\star \alpha r s^\nu} \leq \frac{2\varepsilon}{c_\star \alpha r s^\nu} = \frac{2\hat{c}}{c_\star} \frac{\mu s^{2\nu}}{\lambda} \stackrel{(15)}{=} \frac{1}{8} \frac{\mu s^{2\nu}}{\lambda} \leq \frac{1}{8}, \quad (35)$$

and the first condition in (26) is satisfied. Observe that,

$$\alpha s^\nu = \frac{\lambda}{\hat{c} \mu s^{2\nu}} (\mathbf{M} r_0) \sqrt{\varepsilon} \stackrel{(16)}{\leq} \frac{\sqrt{c}}{\hat{c}} \mu^2 \mathbf{M} r_0,$$

and, if \mathbf{h} is as in (29), one has, in view of (15),

$$\frac{\alpha s^\nu}{\mathbf{h}} \stackrel{(16)}{\leq} \frac{4\sqrt{c}}{\hat{c} c_0} \leq 1.$$

Thus, also the second condition in (26) is satisfied and we can apply Theorem 2, obtaining the family of torus embedding¹⁶

$$\Phi : \mathbb{T}^n \times \Omega^{(i)} \rightarrow \mathbb{B}_r(0) \times \mathbb{T}^n \quad (r = \sqrt{\varepsilon} r_0)$$

as described in Theorem 2.

¹⁴Recall that $\mathbf{p} = \mathbf{p}(\cdot; p_i)$ and therefore depends upon i ; however for ease of notation we do not indicate explicitly the dependence upon i .

¹⁵In general, I is complex. Notice that by (16), (13), (5) (and the assumption $s \leq 1$), $\varepsilon < 1/16$ since $c < 1/16$.

¹⁶Obviously, also Φ depends on i but, as above (compare footnote 14), for ease of notation we do not indicate explicitly the dependence upon i .

6.4 Kolmogorv's tori: The sets $\mathcal{T}_\alpha^{(i)}$ and \mathcal{T}_α

The tori we obtained in the preceding section live in the “local” phase space $\{(I, \theta) \mid (I, \theta) \in B_r(0) \times \mathbb{T}^n\}$. To translate the invariant tori into the original phase space $\{(p, q) \mid (p, q) \in B_{r_0/2}(D_i) \times \mathbb{T}^n\}$, we define the ω -family of torus embeddings¹⁷

$$\theta \mapsto \Psi(\theta, \omega) := (\mathbf{p} \circ \varphi(\omega), 0) + \Phi(\theta, \omega), \quad \omega \in \Omega^{(i)}, \quad (36)$$

which, as function of θ , is real-analytic on $\mathbb{T}_{s/2}^n$. Then, from § 6.3 it follows that for $\omega \in \Omega_\alpha^{(i)}$ the torus $\Psi(\mathbb{T}^n, \omega)$ is invariant for the flow of \mathbb{H} and, furthermore:

$$(\phi_{\mathbb{H}}^t \circ \Psi)(\theta, \omega) = \Psi(\theta + \omega t, \omega).$$

We therefore obtain the following family of “Kolmogorov’s tori” (recall, (22), that $\hat{r} < r_0/2$):

$$\mathcal{T}_\alpha^{(i)} := \Psi(\mathbb{T}^n \times \Omega_\alpha^{(i)}) \subseteq B_{r_0/2}(D_i) \times \mathbb{T}^n, \quad \mathcal{T}_\alpha := \bigcup_{i=1}^N \mathcal{T}_\alpha^{(i)} \subseteq B_{r_0}(\mathcal{D}) \times \mathbb{T}^n,$$

Below, we shall show that actually \mathcal{T}_α lives in a smaller neighborhood of \mathcal{D} .

Analytic quantitative properties of the torus embedding Ψ , and hence of the family of Kolmogorov’s tori, will be described in detailed in the following section.

6.5 Properties of the torus embedding

Lemma 3 *Let Ψ be defined as in (36) and let Ψ_0 denote the “trivial embedding”*

$$\Psi_0 : (\theta, \omega) \in \mathbb{T}_s^n \times \mathbb{B}_{\mathbf{h}}(\Omega^{(i)}) \mapsto (\mathbf{p}(\omega), \theta) \in \mathbb{B}_{r_\star}(p_i) \times \mathbb{T}_s^n. \quad (37)$$

Then¹⁸,

$$\sup_{\Omega^{(i)}} \sup_{\mathbb{T}^n} |\pi_1(\Psi - \Psi_0)| \leq r_\epsilon \stackrel{(17)}{:=} \frac{\lambda \sqrt{\epsilon}}{c_\star} r_0, \quad (38)$$

and, hence,

$$\mathcal{T}_\alpha^{(i)} \subseteq B_{r_\epsilon}(D_i) \times \mathbb{T}^n = B_{\hat{r}+r_\epsilon}(p_i) \times \mathbb{T}^n, \quad (39)$$

$$\mathcal{T}_\alpha \subseteq \bigcup_{i=1}^N B_{\hat{r}+r_\epsilon}(p_i) \times \mathbb{T}^n \subseteq B_{2\hat{r}}(\mathcal{D}) \times \mathbb{T}^n. \quad (40)$$

Proof By the definitions (36) and (37), one has

$$\Psi = \Psi_0 + (\mathbf{p} \circ \varphi - \mathbf{p}, 0) + \Phi - \Phi_0. \quad (41)$$

Notice that, since $\mathbf{h} = \rho_\star/4$, from (9) and (11), one has

$$\sup_{\mathbb{B}_{\mathbf{h}}(\Omega^{(i)})} \|\mathbf{p}_\omega\| \leq L, \quad \sup_{\mathbb{B}_{\mathbf{h}}(\Omega^{(i)})} \|\mathbf{p}_{\omega\omega}\| \leq \frac{4}{c_0} \frac{L}{\mu^2 M r_0}. \quad (42)$$

¹⁷Recall (33) and footnote 16.

¹⁸Recall, (2), that π_1 denotes projection onto the first n components.

Thus, (42), (28), (34), one gets

$$\sup_{\Omega^{(i)}} |\mathbf{p} \circ \varphi - \mathbf{p}| \leq L \frac{2\varepsilon}{\hat{c}_\star r} \stackrel{(31)}{=} \frac{2}{\hat{c}_\star} \lambda r . \quad (43)$$

Then, by (27), (35), (25) and (15), one gets (38).

Since (recall (37))

$$\Psi_0(\mathbb{T}^n \times \Omega_\alpha^{(i)}) = \mathbf{p}(\Omega_\alpha^{(i)}) \times \mathbb{T}^n \subseteq D_i \times \mathbb{T}^n = B_{\hat{r}}(p_i) \times \mathbb{T}^n ,$$

(39) follows from (38); (40) follows since $r_\varepsilon \leq \hat{r}$. \blacksquare

To control Lipschitz norm we introduce suitable partial rescalings. Let

$$1 < \beta := \frac{\lambda}{\check{c}\mu s^\nu} , \quad \text{with} \quad \check{c} := \frac{n\hat{c}_0\hat{c}_\star}{16} = \frac{\hat{c}_\star}{2^6 \cdot n!} . \quad (44)$$

Define, for any $a > 0$, the following rescaling

$$S_a : (\theta, x) \in \mathbb{T}^n \times \mathbb{R}^n \mapsto (\theta, \omega) := (\theta, ax) .$$

Now, recall (24) and let

$$\begin{cases} \tilde{\Phi} := R_{\beta r} \circ \Phi \circ S_\alpha \\ \tilde{\Phi}_0 := R_{\beta r} \circ \Phi_0 \circ S_\alpha \end{cases} , \quad \begin{cases} \tilde{\Psi} := R_{\beta r} \circ \Psi \circ S_\alpha \\ \tilde{\Psi}_0 := R_{\beta r} \circ \Psi_0 \circ S_\alpha \end{cases} , \quad (45)$$

which are defined on the domain $\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}$. The rescaled version of (41), then becomes:

$$\tilde{\Psi} - \tilde{\Psi}_0 = \left(\frac{1}{\beta r} (\mathbf{p} \circ \varphi - \mathbf{p}) \circ S_\alpha, 0 \right) + \tilde{\Phi} - \tilde{\Phi}_0 . \quad (46)$$

Finally, let

$$\Psi_* := (\tilde{\Psi} - \tilde{\Psi}_0) \circ \tilde{\Psi}_0^{-1} , \quad (47)$$

which is defined on $\frac{1}{\beta r}D_i \times \mathbb{T}_{s/2}^n$.

The above rescaled embeddings may, now, be shown to be close, in Lipschitz norm, to the unperturbed rescaled embeddings:

Lemma 4 *The following bounds hold:*

$$\sup_{\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}} |\tilde{\Phi} - \tilde{\Phi}_0| \leq \frac{s^\nu}{8} , \quad \sup_{\mathbb{T}_{s/2}^n} |\tilde{\Phi} - \tilde{\Phi}_0|_{\text{Lip}, \frac{1}{\alpha}\Omega^{(i)}} \leq \frac{s^\nu}{8} , \quad (48)$$

$$\sup_{\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}} |\tilde{\Psi} - \tilde{\Psi}_0| \leq \frac{s^\nu}{4} , \quad \sup_{\mathbb{T}_{s/2}^n} |\tilde{\Psi} - \tilde{\Psi}_0|_{\text{Lip}, \frac{1}{\alpha}\Omega^{(i)}} \leq \frac{1}{4} , \quad (49)$$

$$\sup_{\frac{1}{\beta r}D_i \times \mathbb{T}_{s/2}^n} |\Psi_*| \leq \frac{s^\nu}{4} , \quad \sup_{\mathbb{T}_{s/2}^n} |\Psi_*|_{\text{Lip}, \frac{1}{\beta r}D_i} \leq \frac{1}{4} , \quad (50)$$

$$\sup_{\frac{1}{\beta r}D_i \times \mathbb{T}^n} \|\partial_\theta \Psi_*\| \leq \frac{s^{\nu-1}}{2} . \quad (51)$$

Proof Since $\beta > 1$, by (27) and (35) we have that

$$\sup_{\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}} |\tilde{\Phi} - \tilde{\Phi}_0| \leq \sup_{\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}} |R_\beta^{-1}(\tilde{\Phi} - \tilde{\Phi}_0)| = \sup_{\mathbb{T}_{s/2}^n \times \Omega^{(i)}} |R_r(\Phi - \Phi_0)| \leq \frac{2\hat{c}}{\hat{c}_*} \frac{\mu s^{2\nu}}{\lambda} \leq \frac{s^\nu}{8}, \quad (52)$$

last inequality holding because of the definition of \hat{c} in (15).

Analogously, (by (27), (35) and the definition of \hat{c}) we have that¹⁹

$$\sup_{\mathbb{T}_{s/2}^n} |\tilde{\Phi} - \tilde{\Phi}_0|_{\text{Lip}, \frac{1}{\alpha}\Omega^{(i)}} \leq \sup_{\mathbb{T}_{s/2}^n} |R_\beta^{-1}(\tilde{\Phi} - \tilde{\Phi}_0)|_{\text{Lip}, \frac{1}{\alpha}\Omega^{(i)}} = \alpha \sup_{\mathbb{T}_{s/2}^n} |R_r(\Phi - \Phi_0)|_{\text{Lip}, \Omega^{(i)}} \leq \frac{2\hat{c}}{\hat{c}_*} \frac{\mu s^\nu}{\lambda} \leq \frac{s^\nu}{8},$$

which, together with (52), proves (48).

Now, by (43), (31), (44), we get

$$\sup_{\mathbb{T}_{s/2}^n \times \frac{1}{\alpha}\Omega^{(i)}} \left| \left(\frac{1}{\beta r} (\mathbf{p} \circ \varphi - \mathbf{p}) \circ S_\alpha, 0 \right) \right| = \frac{1}{\beta r} \sup_{\Omega}^{(i)} |\mathbf{p} \circ \varphi - \mathbf{p}| \leq \frac{2\lambda}{\hat{c}_* \beta} \stackrel{(44)}{=} \frac{\mu s^\nu}{2^5 n!} < \frac{s^\nu}{8}. \quad (53)$$

The first estimate in (49) now follows at once in view of (46), the first inequality in (48) and (53).

In order to prove the second estimate in (49), in view of (46), we need to estimate the Lipschitz semi-norm of $(\mathbf{p} \circ \varphi - \mathbf{p})$. Fix $\omega, \omega' \in \Omega^{(i)}$ and set $p = \mathbf{p}(\omega)$ and $p' = \mathbf{p}(\omega')$. Let also $\gamma(t) := (p' - p)t + p$, for $t \in [0, 1]$ and²⁰ $\tilde{\gamma} := h_p \circ \gamma$. Then²¹,

$$\begin{aligned} & \left| \mathbf{p}(\varphi(\omega')) - \mathbf{p}(\omega') - \mathbf{p}(\varphi(\omega)) + \mathbf{p}(\omega) \right| \\ &= \left| \int_0^1 \left(\mathbf{p}_\omega(\varphi(\tilde{\gamma}(t))) \varphi_\omega(\tilde{\gamma}(t)) - \mathbf{p}_\omega(\tilde{\gamma}(t)) \right) h_{pp}(\gamma(t)) dt \right| (p' - p) \\ &\leq M |p' - p| \int_0^1 \left| \mathbf{p}_\omega(\varphi(\tilde{\gamma}(t))) \varphi_\omega(\tilde{\gamma}(t)) - \mathbf{p}_\omega(\tilde{\gamma}(t)) \right| dt \\ &= M |p' - p| \int_0^1 \left| \mathbf{p}_\omega(\varphi(\tilde{\gamma}(t))) (\partial_\omega(\varphi - \text{id})|_{\tilde{\gamma}(t)}) + \mathbf{p}_\omega(\varphi(\tilde{\gamma}(t))) - \mathbf{p}_\omega(\tilde{\gamma}(t)) \right| dt \\ &\stackrel{(42)}{\leq} \lambda |\omega' - \omega| \left(L |\varphi - \text{id}|_{\text{Lip}} + \frac{4L}{c_0 \mu^2 M r_0} |\varphi - \text{id}| \right) \\ &\stackrel{(28), (34)}{\leq} \lambda |\omega' - \omega| \left(L \frac{2\varepsilon}{\hat{c}_* \alpha r s^\nu} + \frac{4L}{c_0 \mu^2 M r_0} \frac{2\varepsilon}{\hat{c}_* r} \right) \\ &\stackrel{(16), (31)}{\leq} |\omega' - \omega| \frac{2\lambda^2 r}{\hat{c}_*} \left(\frac{1}{\alpha s^\nu} + \frac{4}{c_0 \mu^2 M r_0} \right). \end{aligned}$$

Now, observe that, by (17), (16), (15) and (6) (which implies that $c < \hat{c}^2 c_0^2 / 4$), one has

$$\frac{4}{c_0 \mu^2 M r_0} < \frac{1}{\alpha s^\nu}.$$

¹⁹ If f is a Lipschitz map defined on $\Omega^{(i)}$, then $f \circ S_a$ is Lipschitz on $\frac{1}{a}\Omega^{(i)}$ and $|f \circ S_a|_{\text{Lip}, \frac{1}{a}\Omega^{(i)}} = a |f|_{\text{Lip}, \Omega^{(i)}}$.

²⁰ The introduction of the lifted curve $\tilde{\gamma} \subseteq \Omega^{(i)}$ to join ω and ω' is due to the fact that, in general, $\Omega^{(i)}$ is not convex.

²¹ By Remark (v) in 7.2, φ is differentiable; the differentiability of φ almost everywhere also follows, independently, from Rademacher's Theorem. Notice also that, if f is a function differentiable (a.e.) on $\Omega^{(i)}$, then $\sup_{\Omega}^{(i)} |\nabla f| \leq |f|_{\text{Lip}, \Omega^{(i)}}$ (a.e.), the equality holding if $\Omega^{(i)}$ is convex.

Thus,

$$|\mathbf{p} \circ \varphi - \mathbf{p}|_{\text{Lip}, \Omega^{(i)}} < \frac{4\lambda^2 r}{\hat{c}_* \alpha s^\nu},$$

and, therefore (recalling footnote 19),

$$\begin{aligned} \frac{1}{\beta r} |(\mathbf{p} \circ \varphi - \mathbf{p}) \circ S_\alpha|_{\text{Lip}, \frac{1}{\alpha} \Omega^{(i)}} &= \frac{\alpha}{\beta r} |\mathbf{p} \circ \varphi - \mathbf{p}|_{\text{Lip}, \Omega^{(i)}} \\ &< \frac{4\lambda^2}{\beta \hat{c}_* s^\nu} \\ &\stackrel{(14)}{\leq} \frac{2\lambda}{n \hat{c}_0 \mu \hat{c}_* s^\nu} \frac{1}{\beta} \\ &\stackrel{(44), (15)}{=} \frac{1}{8}, \end{aligned}$$

which, together with the second estimate in (48), in view of (46), yields also the second bound in (49). To estimate Ψ_* (defined in (47)), observe that

$$\tilde{\Psi}_0^{-1}(y, \theta) = (\theta, \alpha^{-1} h_p(\beta r y)), \quad \tilde{\Psi}_0^{-1} : \frac{1}{\beta r} D_i \times \mathbb{T}_{s/2}^n \xrightarrow{\text{onto}} \mathbb{T}_{s/2}^n \times \frac{1}{\alpha} \Omega^{(i)}.$$

Thus, the first estimate in (50) follows immediately from the first bound in (49). As for the Lipschitz semi-norm of Ψ_* , by (44), (17), (31), (15) and (25) we have, for all $\theta \in \mathbb{T}_{s/2}^n$, that²²

$$|\tilde{\Psi}_0^{-1}|_{\text{Lip}, \frac{1}{\beta r} D_i} = \frac{1}{\alpha} |h_p(\beta r \cdot)|_{\text{Lip}, \frac{1}{\beta r} D_i} = \frac{\beta r}{\alpha} |h_p|_{\text{Lip}, D_i} \leq \frac{\beta r M}{\alpha} = \frac{\hat{c} s^{2\nu}}{\check{c}} = \frac{c_*}{\hat{c}_*} 4 \cdot n! s^{2\nu} < s^{2\nu}.$$

Thus, in view of the second estimate in (49), we have, for all $\theta \in \mathbb{T}_{s/2}^n$,

$$|\Psi_*|_{\text{Lip}, \frac{1}{\beta r} D_i} \leq |\tilde{\Psi} - \tilde{\Psi}_0|_{\text{Lip}, \frac{1}{\alpha} \Omega^{(i)}} |\tilde{\Psi}_0^{-1}|_{\text{Lip}, \frac{1}{\beta r} D_i} < \frac{s^{3\nu}}{4}.$$

By (50) and Cauchy estimates we get (51). \blacksquare

We shall also need the following

Lemma 5 *Let*²³

$$\rho := \frac{\beta s^\nu}{4} r. \quad (54)$$

Then,

$$\tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} D_i \times \mathbb{T}^n \right) := \tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} B_{\hat{r}}(p_i) \times \mathbb{T}^n \right) \supseteq \frac{1}{\beta r} B_{\hat{r}-\rho}(p_i) \times \mathbb{T}^n. \quad (55)$$

²² $|h_p(\beta r \cdot)|_{\text{Lip}, \frac{1}{\beta r} D_i}$ denotes the Lipschitz norm of the function $y \rightarrow h_p(\beta r y)$ on the rescaled domain $(\beta r)^{-1} D_i$.

²³Recall that $r = \sqrt{\epsilon} r_0$ is defined in (31).

Proof Since²⁴

$$\frac{1}{\beta r} B_{\hat{r}-\rho}(p_i) = B_{\frac{\hat{r}-\rho}{\beta r}}\left(\frac{p_i}{\beta r}\right) \stackrel{(54)}{=} B_{\frac{\hat{r}}{\beta r} - \frac{s^\nu}{4}}\left(\frac{p_i}{\beta r}\right),$$

one sees that (55) will hold if, for any given $(y_0, \theta_0) \in \frac{1}{\beta r} B_{\hat{r}-\rho}(p_i) \times \mathbb{T}^n$, there exists a point²⁵

$$(y_1, \theta_1) \in \overline{B_{s^\nu/4}(0)} \times \mathbb{T}^n$$

such that

$$(y_0, \theta_0) = \tilde{\Psi} \circ \tilde{\Psi}_0^{-1}(y_0 + y_1, \theta_0 + \theta_1) \stackrel{(47)}{=} (y_0 + y_1, \theta_0 + \theta_1) + \Psi_*(y_0 + y_1, \theta_0 + \theta_1).$$

Such relation is, in turn, equivalent to the fixed point equation

$$(y_1, \theta_1) = -\Psi_*(y_0 + y_1, \theta_0 + \theta_1). \quad (56)$$

We shall solve (56) in two steps: (i), we prove that there exists a unique function $y_*(\theta)$ such that²⁶

$$y_*(\theta) = -\pi_1 \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta), \quad \forall \theta \in \mathbb{T}^n, \quad (57)$$

and, (ii), we show that the map

$$\theta \in \mathbb{T}^n \mapsto \theta + \pi_2 \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta) \in \mathbb{T}^n, \quad (58)$$

is onto, guaranteeing that there exists a $\theta_1 \in \mathbb{T}^n$ so that $\theta_1 + \pi_2 \Psi_*(y_0 + y_*(\theta_1), \theta_0 + \theta_1) = 0$. These two facts will show that $y_1 := y_*(\theta_1)$ and θ_1 are solutions of (56), proving the claim.

Proof of (i): Let

$$X := \{\theta \mapsto y(\theta) \in C(\mathbb{T}^n, \mathbb{R}^n) \mid \sup_{\mathbb{T}^n} |y| \leq s^\nu/4\}, \quad F(y)(\theta) := -\pi_1 \Psi_*(y_0 + y(\theta), \theta_0 + \theta).$$

Then, by the first inequality in (50), $F : X \rightarrow X$, and the second inequality in (50) shows that F is a contraction from X into X . Hence, there exists a unique fixed point $y_* \in X$ satisfying (57). Furthermore, since Ψ_* is real-analytic, so is y_* and, in particular, its Jacobian $\partial_\theta y_*$ satisfies the equation

$$\left(\text{id} + \pi_1 \partial_y \Psi_*(y_0 + y(\theta), \theta_0 + \theta) \right) \partial_\theta y_* = -\pi_1 \partial_\theta \Psi_*(y_0 + y(\theta), \theta_0 + \theta),$$

which, by Neumann series, by the second inequality²⁷ in (50) and by (51), yields

$$\sup_{\mathbb{T}^n} \|\partial_\theta y_*\| \leq \frac{1}{1 - \frac{1}{4}} \frac{s^{\nu-1}}{2} = \frac{2}{3} s^{\nu-1}. \quad (59)$$

²⁴Notice that it is $\hat{r} - \rho > 0$: Indeed, by (44) and (31), we see that $\rho = \lambda \sqrt{\epsilon r_0} / (4\check{c}\mu)$, so that (recalling the definition of \hat{r} in (22)) $\rho < \hat{r}$ is seen to be equivalent to $\epsilon < 4\check{c}^2 c_0^2 \mu^6 / \lambda^2 \stackrel{(44),(6)}{=} \check{c}_*^2 \mu^6 / (2^{16} n^2 n!^6 \lambda^2)$, which is guaranteed by (16), observing that, by (15), $c < \check{c}_*^2 / (2^{16} n^2 n!^6)$.

²⁵As standard, the overline denotes closure and observe that $y_0 + y_1 \in \frac{1}{\beta r} B_{\hat{r}}(p_i)$.

²⁶Recall, (2), that π_i denotes projection: $\pi_1(y, \theta) = y$ and $\pi_2(y, \theta) = \theta$.

²⁷Recall the second remark in footnote 7.

Proof of (ii): Observe that from the standard Contraction Lemma it follows easily that²⁸:

If g is a $C^1(\mathbb{T}^n, \mathbb{T}^n)$ map such that $\lambda := \sup_{\mathbb{T}^n} \|\partial_\theta g\| < 1$, then, the map $G : \theta \in \mathbb{T}^n \mapsto \theta + g(\theta) \in \mathbb{T}^n$ has a unique inverse $\tilde{G} : \theta \in \mathbb{T}^n \mapsto \theta + \tilde{g}(\theta) \in \mathbb{T}^n$ with $\tilde{g} \in C^1(\mathbb{T}^n, \mathbb{T}^n)$ and $\sup_{\mathbb{T}^n} \|\partial_\theta \tilde{g}\| \leq \lambda/(1-\lambda)$.

Now, recalling (58), to finish the proof is enough to check that the Jacobian of the map

$$\theta \mapsto \pi_2 \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta)$$

has (operator) norm strictly smaller than one. But, by the second inequality in (50), (59) and (51), one has, for any $\theta \in \mathbb{T}^n$,

$$\begin{aligned} \|\partial_\theta \pi_2 \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta)\| &\leq \|\partial_y \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta)\| \|\partial_\theta y_*(\theta)\| + \|\partial_\theta \Psi_*(y_0 + y_*(\theta), \theta_0 + \theta)\| \\ &\leq \frac{1}{4} \cdot \frac{2}{3} s^{\nu-1} + \frac{s^{\nu-1}}{2} = \frac{2}{3} s^{\nu-1} < 1. \quad \blacksquare \end{aligned}$$

6.6 Measure estimates

We first provide measure estimates on $(D_i \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha^{(i)} = (D_i \times \mathbb{T}^n) \setminus \Psi(\mathbb{T}^n \times \Omega_\alpha^{(i)})$.

Clearly²⁹,

$$(D_i \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha^{(i)} \subseteq ((D_i \times \mathbb{T}^n) \setminus \Psi(\mathbb{T}^n \times \Omega^{(i)})) \dot{\cup} \Psi(\mathbb{T}^n \times (\Omega^{(i)} \setminus \Omega_\alpha^{(i)})). \quad (60)$$

Now, the following estimates hold.

Lemma 6 (Measure estimates) *Recall (6), (15) and (44) and define the following constants:*

$$\kappa_1 := (2\pi)^n \frac{nc_0^{n-1}}{2\hat{c}}, \quad \kappa_2 := \left(\frac{5\pi}{2}\right)^n; \quad (61)$$

$$\kappa'_3 := 2n^{\frac{n-1}{2}} \left(\sum_{k \neq 0} \frac{1}{|k|_1^{\tau+1}} \right) \frac{c_0^{n-1}}{\hat{c}}, \quad \kappa''_3 := \frac{2nc_0^{n-1}}{\hat{c}}, \quad \kappa_3 := \kappa'_3 + \kappa''_3. \quad (62)$$

Then, one has

$$\text{meas}((D_i \times \mathbb{T}^n) \setminus \Psi(\mathbb{T}^n \times \Omega^{(i)})) \leq \kappa_1 \mu^{2n-3} r_0^n \sqrt{\epsilon}, \quad (63)$$

$$\text{meas}(\Psi(\mathbb{T}^n \times \Omega^{(i)} \setminus \Omega_\alpha^{(i)})) \leq \kappa_2 L^n \text{meas}(\Omega^{(i)} \setminus \Omega_\alpha^{(i)}), \quad (64)$$

$$\text{meas}(\Omega^{(i)} \setminus \Omega_\alpha^{(i)}) \leq \kappa_3 \frac{\mu^{2n-3} \lambda^2}{s^{3\nu}} (Mr_0)^n \sqrt{\epsilon}. \quad (65)$$

Proof Observe that by (45)

$$R_{\beta r} \circ \Psi = \tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \circ R_{\beta r} \circ \Psi_0. \quad (66)$$

Thus, since $\Psi_0(\mathbb{T}^n \times \Omega^{(i)}) = D_i \times \mathbb{T}^n$, we have

$$R_{\beta r} \circ \Psi(\mathbb{T}^n \times \Omega^{(i)}) = \tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} D_i \times \mathbb{T}^n \right). \quad (67)$$

²⁸Indeed, $G \circ \tilde{G} = \text{id}$ is equivalent to the fixed point equation $\tilde{g} = -g \circ (\text{id} + \tilde{g})$ and if we let X denote $C(\mathbb{T}^n, \mathbb{T}^n)$ endowed with the standard metric $d(h_1, h_2) := \sup_{\mathbb{T}^n} d_{\mathbb{T}^n}(h_1(\theta), h_2(\theta))$ (where $d_{\mathbb{T}^n}$ denotes the standard flat metric on \mathbb{T}^n), one sees immediately that the assumption implies that the map $h \in X \mapsto -g \circ (\text{id} + h) \in X$ is a contraction from X to X , whose unique fixed point yields \tilde{g} . Furthermore, since g is C^1 , so is \tilde{g} and the inequality on the Jacobian of \tilde{g} follows by Neumann series after having differentiated the identity $\tilde{g} = -g \circ (\text{id} + \tilde{g})$.

²⁹The dot over union denotes ‘‘disjoint union’’.

Therefore³⁰,

$$\begin{aligned}
\text{meas} \left((D_i \times \mathbb{T}^n) \setminus \Psi(\mathbb{T}^n \times \Omega^{(i)}) \right) &= (\beta r)^n \text{meas} \left(R_{\beta r}((D_i \times \mathbb{T}^n)) \setminus R_{\beta r} \circ \Psi(\mathbb{T}^n \times \Omega^{(i)}) \right) \\
&\stackrel{(67)}{=} (\beta r)^n \text{meas} \left(\left(\frac{1}{\beta r} D_i \times \mathbb{T}^n \right) \setminus \tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} D_i \times \mathbb{T}^n \right) \right) \\
&\stackrel{(55)}{\leq} (\beta r)^n \text{meas} \left(\left(\frac{1}{\beta r} D_i \times \mathbb{T}^n \right) \setminus \left(\frac{1}{\beta r} B_{\hat{r}-\rho}(p_i) \times \mathbb{T}^n \right) \right) \\
&= \text{meas} \left((D_i \setminus B_{\hat{r}-\rho}(p_i)) \times \mathbb{T}^n \right) \\
&= (2\pi)^n \left((2\hat{r})^n - (2(\hat{r}-\rho))^n \right) \\
&\leq (2\pi)^n n 2^n \hat{r}^{n-1} \rho \\
&\stackrel{(61)}{=} \kappa_1 \lambda \mu^{2n-3} r_0^n \sqrt{\epsilon},
\end{aligned}$$

proving (63).

To prove (64), observe that if $A \subseteq \Omega^{(i)}$, from (66) and the identity (recall (37))

$$R_{\beta r} \circ \Psi_0(\mathbb{T}^n \times A) = \frac{1}{\beta r} \mathbf{p}(A) \times \mathbb{T}^n,$$

there follows, by (45),

$$\Psi(\mathbb{T}^n \times A) = R_{\beta r}^{-1} \circ \tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} \mathbf{p}(A) \times \mathbb{T}^n \right). \quad (68)$$

Observe also that, since³¹

$$\tilde{\Psi} \circ \tilde{\Psi}_0^{-1} = \text{id} + \Psi_*,$$

from (50) there follows

$$|\tilde{\Psi} \circ \tilde{\Psi}_0^{-1}|_{\text{Lip}, \frac{1}{\beta r} D_i \times \mathbb{T}^n} \leq 5/4.$$

Now, for every measurable set $A \subseteq \Omega^{(i)}$, one has³²

$$\begin{aligned}
\text{meas}(\Psi(\mathbb{T}^n \times A)) &\stackrel{(68)}{=} (\beta r)^n \text{meas} \left(\tilde{\Psi} \circ \tilde{\Psi}_0^{-1} \left(\frac{1}{\beta r} \mathbf{p}(A) \times \mathbb{T}^n \right) \right) \\
&\leq (2\pi)^n \left(|\tilde{\Psi} \circ \tilde{\Psi}_0^{-1}|_{\text{Lip}, \frac{1}{\beta r} D_i \times \mathbb{T}^n} \right)^n \mathbf{L}^n \text{meas}(A) \\
&\leq \kappa_2 \mathbf{L}^n \text{meas}(A),
\end{aligned}$$

and (64) follows.

To prove (65), observe that

$$\Omega^{(i)} \setminus \Omega_\alpha^{(i)} \subseteq \{ \omega \in \Omega^{(i)} \mid \omega \text{ is not } (\alpha, \tau)\text{-Diophantine} \} \cup \Omega^{(i)}(\alpha) \quad (69)$$

³⁰Recall that $D_i = B_{\hat{r}}(p_i)$; in the last inequality use that for every $0 < x < 1$ and for every integer $n > 1$, one has $1 - (1-x)^n < nx$ and for the last equality recall (22), (54), (44), (31).

³¹Recall the definition of Ψ_* in (47).

³²In the first inequality, we use (twice) the following fact: If $A \subseteq \mathbb{R}^n$ is a measurable set and $f : A \rightarrow \mathbb{R}^n$ is a Lipschitz map, then $\text{meas} f(A) \leq |f|_{\text{Lip}, A}^n \text{meas} A$.

where

$$\Omega^{(i)}(\alpha) := \left\{ \omega \in \Omega^{(i)}, \text{ s.t. } \text{dist}(\omega, \partial\Omega^{(i)}) < \alpha \right\}.$$

Let us begin with estimating the measure of the first set in the r.h.s. of (69) keeping track of constants. Notice that, if $\hat{\Omega}^{(i)}$ denotes the euclidean ball of center $h_p(p_i)$ and radius $\sqrt{n}M\hat{r}$, then

$$\Omega^{(i)} = h_p(B_{\hat{r}}(p_i)) \subseteq \hat{\Omega}^{(i)}.$$

Thus, denoting by $\|\cdot\|$ the euclidean norm in \mathbb{R}^n , we have

$$\begin{aligned} \text{meas} \{ \omega \in \Omega^{(i)} \mid \omega \text{ not } (\alpha, \tau)\text{-Dioph.} \} &\leq \text{meas} \left\{ \omega \in \hat{\Omega}^{(i)} \mid \exists k \in \mathbb{Z}^n, k \neq 0 : |\omega \cdot k| < \frac{\alpha}{|k|_1^\tau} \right\} \\ &\leq 2^n \sum_{k \neq 0} (\sqrt{n}M\hat{r})^{n-1} \frac{\alpha}{|k|_1^{\tau+1}} \\ (22)_{i(17)} &\stackrel{=}{=} \kappa'_3 \frac{M^n \lambda \mu^{(2n-3)}}{s^{3\nu}} r_0^n \sqrt{\epsilon}. \end{aligned} \quad (70)$$

As for the measure of the second set in (69), we observe that either $\hat{r} \leq L\alpha$ or $\hat{r} > L\alpha$. In the first case we have

$$\text{meas}(\Omega^{(i)}(\alpha)) \leq \text{meas}(\Omega^{(i)}) \leq M^n \text{meas}(B_{\hat{r}}(p_i)) = 2^n M^n \hat{r}^n \leq 2^n M^n L \hat{r}^{n-1} \alpha.$$

In the second case, let

$$\check{r} := \hat{r} - L\alpha > 0.$$

We claim that

$$h_p(B_{\hat{r}}(p_i) \setminus B_{\hat{r}}(p_i)) \supseteq \Omega^{(i)}(\alpha). \quad (71)$$

Indeed, by contradiction, assume that there exist $\omega = h_p(p) \in \Omega^{(i)}(\alpha)$, $\omega_* = h_p(p_*) \in \partial\Omega^{(i)}$ (namely $|p_* - p_i| = \hat{r}$) with $|p - p_i| < \check{r}$, and $|\omega - \omega_*| < \alpha$. Then

$$L\alpha = \hat{r} - \check{r} < |p_* - p| \leq L|\omega_* - \omega| < L\alpha,$$

proving (71). Thus³³,

$$\begin{aligned} \text{meas}(\Omega^{(i)}(\alpha)) &\leq \text{meas} \left(h_p(B_{\hat{r}}(p_i) \setminus B_{\hat{r}}(p_i)) \right) \leq M^n \text{meas}(B_{\hat{r}}(p_i) \setminus B_{\hat{r}}(p_i)) = 2^n M^n (\hat{r}^n - \check{r}^n) \\ &< (2M)^n n \hat{r}^{n-1} L\alpha \end{aligned}$$

Thus, in either case, by (22) and (17), we have

$$\text{meas} \Omega^{(i)}(\alpha) \leq (2M)^n n \hat{r}^{n-1} L\alpha = \kappa''_3 \frac{M^n \lambda^2 \mu^{(2n-3)}}{s^{3\nu}} r_0^n \sqrt{\epsilon} \quad (72)$$

By (69), (70) and (72), we have

$$\text{meas}(\Omega^{(i)} \setminus \Omega_\alpha^{(i)}) \leq \kappa_3 \frac{M^n \lambda^2 \mu^{(2n-3)}}{s^{3\nu}} r_0^n \sqrt{\epsilon}.$$

³³Recall footnote 30.

Lemma 6 is proved. \blacksquare

From (60) and Lemma 6 there follows

$$\text{meas} \left((D_i \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha^{(i)} \right) \leq (\kappa_1 + \kappa_2 \kappa_3) \mu^{2n} r_0^n \frac{\lambda^{n+2}}{\mu^3 s^{3\nu}} \sqrt{\epsilon}.$$

Now, since it is

$$\begin{aligned} (\mathbb{D} \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha &= (\mathbb{D} \times \mathbb{T}^n) \setminus \bigcup_{i=1}^N \mathcal{T}_\alpha^{(i)} \\ &\subseteq \bigcup_{i=1}^N (D_i \times \mathbb{T}^n) \setminus \bigcup_{i=1}^N \mathcal{T}_\alpha^{(i)} \\ &\subseteq \bigcup_{i=1}^N (D_i \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha^{(i)} \\ &= \bigcup_{i=1}^N (\mathbb{B}_{\hat{r}}(p_i) \times \mathbb{T}^n) \setminus \mathcal{T}_\alpha^{(i)}. \end{aligned}$$

Now, in view of (23), one obtains (19) with

$$\kappa := \frac{2^{2n}}{c_0^n} (\kappa_1 + \kappa_2 \kappa_3), \quad (73)$$

and (18) follows at once. \blacksquare

A The standard quantitative Inverse Function Theorem

The following is a standard Inverse Function Theorem in \mathbb{C}^n ; the bar over sets denotes closure.

Proposition *Let $f : \overline{\mathbb{B}_r(p_0)} \rightarrow \mathbb{C}^n$ be a holomorphic function with invertible Jacobian $f_p(p_0)$ and with r such that*

$$\sup_{\overline{\mathbb{B}_r(p_0)}} \|I - f_p^{-1}(p_0) f_p(p)\| \leq \delta < 1. \quad (74)$$

Then, there exists a unique holomorphic inverse g of f such that

$$g : \overline{\mathbb{B}_\rho(\omega_0)} \rightarrow \overline{\mathbb{B}_r(p_0)}, \quad \text{with} \quad \rho := (1 - \delta) \frac{r}{\|f_p^{-1}(p_0)\|}, \quad \omega_0 := f(p_0). \quad (75)$$

Furthermore,

$$\sup_{\overline{\mathbb{B}_\rho(\omega_0)}} \|g_\omega\| \leq \frac{1}{1 - \delta} \|f_p^{-1}(p_0)\|. \quad (76)$$

If f is real-analytic, so is g .

The elementary proof follows by checking that the map $h \mapsto \Phi(h) := h + f_p^{-1}(p_0)(f \circ h - \text{id})$ is a contraction on the space of continuous functions from $\overline{\mathbb{B}_\rho(\omega_0)}$ in $\overline{\mathbb{B}_r(p_0)}$. Then, by the Contraction Lemma, $g = \lim \Phi^n(p_0)$, which also shows, by Weierstrass Theorem on the uniform limit of holomorphic functions, that g is holomorphic (and real-analytic, if so is f). The bound (76) is a general fact following from Neumann series: Indeed, if A and B are $(n \times n)$ matrices and $\|I - AB\| \leq \delta < 1$, then, by Neumann series, AB is invertible and so are A and B , furthermore $\|B^{-1}\| \leq \|(AB)^{-1}\| \|A\| \leq (1 - \delta)^{-1} \|A\|$. ■

References

- [1] V. I. Arnold, Proof of a theorem of A. N. Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian. (Russian) *Uspehi Mat. Nauk* 18 1963 no. 5 (113), 13–40
- [2] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics, volume 3 of Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, third edition, 2006. [Dynamical systems. III], Translated from the Russian original by E. Khukhro.
- [3] M. Berti and L. Biasco, Branching of Cantor Manifolds of Elliptic Tori and Applications to PDEs. *Commun. Math. Phys.* 305, 741–796 (2011)
- [4] L. Biasco, and L. Chierchia. On the measure of Lagrangian invariant tori in nearly-integrable mechanical systems. *Rend. Lincei Mat. Appl.* **26** (2015), 1–10
- [5] A. Celletti; L. Chierchia, Construction of analytic KAM surfaces and effective stability bounds. *Comm. Math. Phys.* 118 (1988), no. 1, 119–161
- [6] A. Celletti; L. Chierchia, A constructive theory of Lagrangian tori and computer-assisted applications. *Dynamics reported*, 60–129, *Dynam. Report. Expositions Dynam. Systems (N.S.)*, 4, Springer, Berlin, 1995
- [7] A. Celletti; L. Chierchia, On the stability of realistic three-body problems. *Comm. Math. Phys.* 186 (1997), no. 2, 413–449
- [8] A. Celletti; L. Chierchia, KAM stability and celestial mechanics. *Mem. Amer. Math. Soc.* 187 (2007), no. 878, viii+134 pp.
- [9] A. Celletti; A. Giorgilli, On the numerical optimization of KAM estimates by classical perturbation theory. *Z. Angew. Math. Phys.* 39 (1988), no. 5, 743–747
- [10] A. Celletti; A. Giorgilli, Improved estimates on the existence of invariant tori for Hamiltonian systems. (English summary) *Nonlinearity* 13 (2000), no. 2, 397–412
- [11] A. Celletti; A. Giorgilli; U. Locatelli, Improved estimates on the existence of invariant tori for Hamiltonian systems. *Nonlinearity* 13 (2000), no. 2, 397–412
- [12] Chierchia, L., Gallavotti, G. Smooth prime integrals for quasi-integrable Hamiltonian systems. *Nuovo Cimento B* (11) 67 (1982), no. 2, 277–295
- [13] R. de la Llave; D. Rana, Accurate strategies for small divisor problems. *Bull. Amer. Math. Soc. (N.S.)* 22 (1990), no. 1, 85–90
- [14] A. Delshams; P. Gutiérrez, Effective stability and KAM theory. *J. Differential Equations* 128 (1996), no. 2, 415–490
- [15] H. S. Dumas, *The KAM Story*, World Scientific, 2014
- [16] A. Giorgilli; L. Galgani, Rigorous estimates for the series expansions of Hamiltonian perturbation theory. *Celestial Mech.* 37 (1985), no. 2, 95–112

- [17] A. N. Kolmogorov, On conservation of conditionally periodic motions for a small change in Hamilton's function. (Russian) Dokl. Akad. Nauk SSSR (N.S.) 98, (1954). 527–530
- [18] A. Morbidelli; A. Giorgilli, Quantitative perturbation theory by successive elimination of harmonics. (English summary) Celestial Mech. Dynam. Astronom. 55 (1993), no. 2, 131–159
- [19] J.K. Moser, On invariant curves of area-preserving mappings of an annulus. Nachr. Akad. Wiss. Gttingen Math.-Phys. Kl. II 1962 1962 1–20
- [20] A. I. Neishtadt, Estimates in the Kolmogorov theorem on conservation of conditionally periodic motions, J. Appl. Math. Mech. 45 (1981), no. 6, 766–772
- [21] J. Pöschel, Integrability of Hamiltonian systems on Cantor sets, Comm. Pure Appl. Math., v. 35 (1982), no. 1, 653–695
- [22] J. Pöschel, A lecture on the classical KAM theorem. Smooth ergodic theory and its applications (Seattle, WA, 1999), 707-732, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001