

# Pressure-Field Permeable-Surface Integral Formulations for Sound Scattered by Moving Bodies

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## Abstract

This paper presents two novel integral formulations for the prediction of sound scattered by moving bodies, derived from the Lighthill and the Ffowcs Williams and Hawkings equations for the pressure field. These are expressed in the frequency-domain, over a permeable (fictitious) surface surrounding the scatterer(s), and are numerically evaluated through application of a boundary element technique. The aims of the paper are the assessment of the influence of the nonlinear terms of Lighthill and Ffowcs Williams and Hawkings equations on sound scattering prediction in the presence of nonuniform mean-flow due to scatterer motion, the assessment of the corresponding limits of applicability of the widely-used linear formulations [for solid-wall boundaries](#), and the development of integral formulations capable to predict accurately and efficiently the noise scattered in the far field by moving bodies. The numerical investigation concerns a non-lifting wing in uniform translation impinged by an acoustic disturbance generated by a co-moving source, and includes the comparison of the results obtained through the proposed

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scattering formulations with those provided by a boundary-field velocity-potential approach recently validated for moving-body problems. Its main outcomes reveal that, when the proposed pressure-field formulations include the nonlinear terms through application of a suitable permeable-surface, their predictions match those provided by the boundary-field velocity-potential solver, whereas the fully linear versions of both pressure-field approaches yield underestimated scattered noise predictions, significantly less accurate than those given by the linear version of the velocity-potential formulation.

*Keywords:* Acoustic Scattering, Nonuniform Mean-Flow, Boundary-Field Integral Formulation, Lighthill Equation, Ffowcs Williams and Hawkings Equation

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## 1. Introduction

The prediction of the sound field affected by the presence of an obstacle (scatterer) in the propagation path of an acoustic wave is of paramount importance for civil and military applications. Indeed, the sound-obstacle  
5 interaction causes a re-distribution of the energy content of the impinging wave into reflected and diffracted secondary waves (acoustic scattering). This phenomenon may significantly alter magnitude and directivity of the overall noise with respect to unbounded space propagation. It plays a significant role in several aeroacoustic and hydroacoustic problems where the charac-  
10 teristic size of the moving scatterer is larger than the characteristic acoustic wavelengths (see, for instance, the noise-fuselage/hull interaction effects on noise radiated by propeller driven aircraft/ships [1]-[4]).

Assuming the source of noise as frozen (namely, acoustically independent of the presence of scatterers), the radiated pressure may be evaluated by a prior acoustic analysis as if it were isolated, and hence the acoustic propagation prediction in the presence of obstacles may be conceived as a two-step problem in which the total noise is decomposed into incident and scattered components. This approach avoids the computationally demanding joint solution of sources of sound and scatterers. The examination of the literature shows that the scattered pressure field is usually assumed to be governed by the linear wave equation (in the time domain) or Helmholtz equation (in the frequency domain), with predictions obtained through boundary integral formulations for the velocity potential or the acoustic pressure solved by the Boundary Element Method (BEM) [5]-[13] or the Fast Multipole Method [14]-[16]. Alternative formulations that have been successfully proposed in the last decade are based on the extension of the acoustic analogy to scattering problems and on the boundary integral solution of the corresponding linear version of the Ffowcs Williams and Hawkings Equation (FWHE) [17]-[20]. However, boundary integral approaches for the solution of linear scattering formulations are not capable to take into account the influence of the aerodynamic noise sources arising around moving scatterers because of the presence of nonuniform mean-flow. Indeed, for a wing impinged by a monopole disturbance, [18, 21] prove that the Helmholtz formulations for pressure and velocity potential, and the FWHE-based linear approach provide fully equivalent scattered noise results as long as the scatterer is at rest, whereas relevant discrepancies proportionally increasing with the Mach number are observed when the body moves. These discrepancies are a measure

of the influence of the nonuniform mean-flow related nonlinear terms in the governing equations for pressure and velocity potential (see also [22]) and, at the same time, demonstrate that the nonlinearities differently affect the predictions given by the different scattering formulations considered.

In order to gain a deeper insight on the role of nonlinear terms in the acoustic scattering of moving objects, while demonstrating the perfect equivalence of the mentioned different acoustic models when their complete forms are applied, the present paper proposes two novel pressure-field formulations for the predictions of noise scattered by a body in steady motion (namely, with constant body-air relative velocity in still air). Specifically, under the assumption of subsonic homentropic flows, two boundary integral formulations solving the FWHE and the Lighthill Equation (LE) for scattering problems are introduced, which are expressed on a fictitious, permeable surface surrounding the scatterer, co-moving with it and embedding the corresponding field noise sources. Akin to the FWHE-based approach for porous surfaces commonly used in aeroacoustic radiation problems (see, for instance, [8] and [23], if the permeable surface is large enough, the scattered field outside of the surface is given in terms of surface integrals only, although including also the field sources effects. This avoids the evaluation of the volume integrals that would arise in integral formulations developed in the whole domain external to the solid body surface, and that would imply cumbersome computations making the simulation impracticable for high-frequency, realistic three-dimensional problems.

The effects on acoustic scattering of the field sources arising in moving scatterer, nonuniform mean-flow conditions have been examined in [21]

through a velocity potential boundary-field integral approach (VP). In that formulation, the field sources considered coincide with the whole set of first-  
65 order potential perturbations due to the incident wave extracted from the nonlinear terms, thus allowing the derivation of a frequency-domain formulation depending on the steady-state, nonuniform-mean-flow velocity potential associated to the body motion. Note that, in this context, only a partial component of the first-order potential perturbation contributions from nonlinear  
70 terms is taken into account in the widely-applied Taylor and Taylor-Lorentz scattering formulations (see, for instance, [22] and [24]-[28]). The corresponding numerical investigation shows that the linearized field terms may have a significant impact on the predicted scattered noise, making the inclusion of nonuniform mean-flow effects derived from the nonlinear terms highly recommended for the sound scattering analysis of moving bodies. Although  
attractive and well-posed from a theoretical standpoint, the application of a linearized boundary-field approach similar to that introduced in [21] for the VP formulation is discouraged for the LE and FWHE by the complexity of the Lighthill stress tensor appearing in the corresponding nonlinear terms.  
80 Indeed, a preliminary study has proven that it would yield a not efficient formulation from the numerical point of view, with excessive computational burden required for the evaluation of the volume field terms. This has motivated the development of the permeable surface approaches.

The [novel FWHE and LE sound scattering, porous-surface](#), formulations  
85 presented in this paper extend the linear solution approaches proposed in [18] and [21]. They [are closely related to the formulations for noise radiation discussed in \[29\]](#), and require the knowledge of suitable flow input data to

be provided by an aerodynamic solver on the permeable surface. This is a drawback that makes these formulations less appealing than the VP one which, instead, is capable of determining wherever the unknown field starting from incident wave and body kinematics. Nonetheless, [in this work they are used to](#) investigate on the differences that are observed, for moving scatterers, among the results obtained by potential-field and pressure-field linear formulations [for solid-wall boundaries](#), while defining, at the same time, their limits of applicability.

The paper is structured as follows. Section 2 presents the main features of the novel scattering formulations proposed, whilst Section 3 shows the numerical results concerning the application to the configuration already investigated in [18, 21, 30], consisting of a rectangular, non-lifting wing in uniform translation, impinged by the noise emitted by a monopole co-moving with it. Finally, the conclusions are drawn in Section 4. Mathematical details regarding the derivation of the permeable-surface scattering formulations outlined in Section 2 are provided in Appendix A.

## 2. Permeable-Surface Integral Formulations for Sound Scattering

In this Section, the two permeable-surface, integral scattering formulations based on LE and FWHE are derived (named, respectively, LE-P and FWHE-P). First, the background differential pressure-field problems are stated, and then their solutions are determined through a boundary integral methodology applied to the infinite-size fluid volume bounded by a fictitious (permeable) surface surrounding the scatterer and the region around it where nonuniform mean-flow is significant [\(as mentioned above, these integral so-](#)

lutions are closely related to those discussed in [29] for noise radiation).

### 2.1. The Permeable-Surface Integral Formulation Derived From LE

Let us consider a homentropic, compressible fluid with a solid moving body impinged by a perturbation pressure field generated by an independent source. In a frame of reference,  $\mathcal{R}(\boldsymbol{\xi})$ , fixed to the undisturbed medium, in the presence of an independent source of sound perturbation of intensity  $D_i$  at the point  $\boldsymbol{\xi}_i$ , the combination of the continuity and Navier-Stokes equations yields the following LE for the acoustic disturbance field,  $p'$  [31]

$$\square^2 p' = \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \nabla \cdot \nabla \cdot \mathbf{T} - D_i \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_i) \quad (1)$$

where  $p' = c_0^2(\rho - \rho_0)$ , with  $c_0$  and  $\rho_0$  representing speed of sound and density  
115 of the undisturbed medium, respectively, whereas  $(\rho - \rho_0)$  denotes density perturbation. In addition,  $\mathbf{T} = \rho \mathbf{u} \otimes \mathbf{u} + (p - p') \mathbf{I} + \mathbf{V}$  is the Lighthill tensor, with  $p$ ,  $\mathbf{u}$  and  $\mathbf{V}$  representing, respectively, fluid pressure, fluid velocity and viscous stress tensor.

In order to obtain the integral formulation solution of the problem, it is convenient to recast it into an equivalent infinite-space problem. To this aim, let us introduce, firstly, a fictitious (and hence fully permeable) undeformable surface,  $\mathcal{S}$ , denoted by  $f(\boldsymbol{\xi}, t) = 0$ , that encloses the solid body and rigidly moves with it (a convenient, but not necessary condition), and then the generalized acoustic disturbance function,  $\bar{p}' = p' H(f)$ , with  $H$  denoting the Heaviside function. Noting that

$$\nabla^2 \bar{p}' = H(f) \nabla^2 p' + \nabla H(f) \cdot \nabla p' + \nabla \cdot [p' \nabla H(f)] \quad (2)$$

and that

$$\frac{\partial^2 \bar{p}'}{\partial t^2} = H(f) \frac{\partial^2 p'}{\partial t^2} + \frac{\partial p'}{\partial t} \frac{\partial H(f)}{\partial t} + \frac{\partial}{\partial t} \left[ p' \frac{\partial H(f)}{\partial t} \right] \quad (3)$$

the combination of Eqs. 2 and 3 with Eq. 1 yields

$$\begin{aligned}
-\square^2 \bar{p}' &= \nabla H(f) \cdot \nabla p' + \nabla \cdot [p' \nabla H(f)] - \frac{1}{c_0^2} \frac{\partial p'}{\partial t} \frac{\partial H(f)}{\partial t} - \frac{1}{c_0^2} \frac{\partial}{\partial t} \left[ p' \frac{\partial H(f)}{\partial t} \right] \\
&\quad - H(f) \nabla \cdot \nabla \cdot \mathbf{T} + H(f) D_i \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_i)
\end{aligned} \tag{4}$$

Next, through application of the boundary integral equation approach illustrated in Appendix A for the solution of Eq. 4, it is straightforward to find out that, in a frame of reference,  $\mathcal{R}(\mathbf{x})$ , rigidly connected to  $\mathcal{S}$ , the acoustic disturbance at time  $t_*$  at any receiver (observer) point,  $\mathbf{x}_*$ , external to  $\mathcal{S}$  is expressed by the following integral formulation

$$\begin{aligned}
p'(\mathbf{x}_*, t_*) &= p'_i(\mathbf{x}_*, t_*) - \int_{\mathcal{V}} [(\nabla \cdot \nabla \cdot \mathbf{T}) \hat{G}]_{\vartheta} dV(\mathbf{x}) \\
&\quad + \int_{\mathcal{S}} \left[ \frac{\partial p'}{\partial \hat{n}} \hat{G} - p' \frac{\partial \hat{G}}{\partial \hat{n}} + \frac{\partial p'}{\partial t} \hat{G} \left( \frac{\partial \vartheta}{\partial \hat{n}} + \frac{2}{c_0} \mathbf{M} \cdot \mathbf{n} \right) \right]_{\vartheta} dS(\mathbf{x}) \tag{5} \\
&\quad + \int_{\mathcal{S}} \left[ p' \hat{G} \frac{\partial}{\partial t} [\mathbf{M} \cdot \mathbf{n} (1 - \mathbf{M} \cdot \nabla \vartheta)] \right]_{\vartheta} dS(\mathbf{x})
\end{aligned}$$

where  $p'_i$  is the acoustic disturbance induced by the external source of intensity  $D_i$ ,  $\mathbf{x}$  denotes the integration variable (emission points, sources, in this kind of problems),  $\mathbf{n}$  is the unit outward normal vector on  $\mathcal{S}$ ,  $\mathcal{V}$  denotes the fluid volume around  $\mathcal{S}$  where the field sources from the Lighthill stress tensor are not negligible, whereas  $\frac{\partial(\cdot)}{\partial \hat{n}} = \frac{\partial(\cdot)}{\partial n} - \mathbf{M} \cdot \mathbf{n} \mathbf{M} \cdot \nabla(\cdot)$ , with  $\mathbf{M} = \mathbf{v}/c_0$  and  $\mathbf{v}$  representing the velocity of the rigid space connected to  $\mathcal{S}$ .

In addition, the symbol  $(\dot{\cdot})$  denotes time derivation in the space connected to  $\mathcal{S}$ , while  $[\dots]_{\vartheta}$  indicates evaluation at the retarded emission time  $t = t_* - \vartheta$ , where  $\vartheta$  is the compressibility delay (namely, the time required by an acoustic signal released from a source point  $\mathbf{x}$  to reach the observer at point  $\mathbf{x}_*$  at time  $t_*$ ). This means that the pressure disturbance received



at  $\mathbf{x}_*$  at time  $t_*$  is a combination of the signals radiated by surface and field sources, each emitted at a different time,  $t_* - \vartheta$ , and covering different distances in the air frame (retarded time approach). For given observer time and position,  $(\mathbf{x}_*, t_*)$ , and source position,  $\mathbf{x}$ , the compressibility delay is the root of the equation (see Appendix A)

$$c_0 \vartheta = |\boldsymbol{\xi}(\mathbf{x}_*, t_*) - \boldsymbol{\xi}(\mathbf{x}, t_* - \vartheta)| \quad (6)$$

Furthermore,  $\hat{G}$  indicates the retarded free-space Green function (see Appendix A)

$$\hat{G}(\mathbf{x}_*, \mathbf{x}, t_*) = \frac{-1}{4\pi} \left[ \frac{1}{r(1 - M_r)} \right]_{\vartheta} \quad (7)$$

125 with  $r = |\mathbf{r}| = |\boldsymbol{\xi}(\mathbf{x}_*, t_*) - \boldsymbol{\xi}(\mathbf{x}, t_* - \vartheta)|$ , and  $M_r = \mathbf{M} \cdot (\mathbf{r}/r)$  (component of the moving space Mach number in the direction of radiation).

Observing that the nonlinear field terms related to the Lighthill stress tensor may be of significant value in a domain surrounding the scatterer whose extent depend on the specific problem under examination, it is inferred  
 130 that for  $\mathcal{S}$  large enough to include that domain, the volume integral becomes negligible, and Eq. 5 reduces to a boundary integral representation.

### *The LE-P Permeable-Surface Sound Scattering Formulation*

The acoustic disturbance generated by the interaction of a moving body with an incident, independent pressure perturbation field may conveniently  
 135 be decomposed into the combination of an aerodynamic component arising from the flow perturbation due to the body motion, with the incident field and the field scattered by the body surface (as an effect of the sound-wave/body interference). If the body is in steady motion (as assumed in this

paper in order to highlight the unsteady scattering phenomena), the aerodynamic component,  $p'_{st}$ , is steady and the decomposition reads  $p' = p'_{st} + p'_i + p'_s$ , where  $p'_s$  denotes the scattered field.

Next, for the permeable surface,  $\mathcal{S}$ , large enough to include all the significant field terms, the transformation into frequency domain of Eq. 5 yields the following LE-P permeable-surface, boundary integral representation of the scattered acoustic disturbance

$$\tilde{p}'_s(\mathbf{x}_*, \omega) = \int_{\mathcal{S}} \left[ \hat{G} \frac{\partial \tilde{p}'_s}{\partial \hat{n}} - \tilde{p}'_s \frac{\partial \hat{G}}{\partial \hat{n}} + i\omega \tilde{p}'_s \hat{G} \left( \frac{\partial \vartheta}{\partial \hat{n}} + \frac{2}{c_0} \mathbf{M} \cdot \mathbf{n} \right) \right] e^{-i\omega \vartheta} dS(\mathbf{x}) \quad (8)$$

where  $(\tilde{\cdot})$  denotes Fourier transform and  $\omega$  is the frequency. It can be applied to determine  $\tilde{p}'_s$  everywhere in the external domain, once it is known over  $\mathcal{S}$  along with  $\partial \tilde{p}'_s / \partial n$  through a previous fluid dynamic solution of the flow forced by the incident field  $\tilde{p}'_i$ . Note that, for  $\mathbf{x}_* \in \mathcal{S}$ , Eq. 8 becomes an integral equation that can be used to determine  $\tilde{p}'_s$  on  $\mathcal{S}$  from the knowledge of  $\partial \tilde{p}'_s / \partial n$  only (in this case, a factor 1/2 multiplies  $\tilde{p}'_s(\mathbf{x}_*, \omega)$  in order to deal with regularized integral terms [32]).

It is worth noting that, the Euler equation provides  $\partial \tilde{p}'_s / \partial n$  from  $\partial \tilde{p}'_i / \partial n$  and a linear combination of unsteady velocity perturbation terms. Indeed, for the fluid velocity decomposed similarly to the acoustic disturbance as  $\mathbf{u} = \mathbf{u}_{st} + \mathbf{u}_s + \mathbf{u}_i$ , it is possible to derive the following first-order, unsteady perturbation equation (valid under the assumption that  $\mathbf{u}_p = \mathbf{u}_s + \mathbf{u}_i$  is small with respect to  $\mathbf{u}_{st}$ )

$$\frac{\partial \tilde{p}'_s}{\partial n} = -\frac{\partial \tilde{p}'_i}{\partial n} - i\omega \rho_0 \tilde{\mathbf{u}}_p \cdot \mathbf{n} - \rho_0 [(\tilde{\mathbf{u}}_p \cdot \nabla) \mathbf{u}_{st} - ((\mathbf{v} - \mathbf{u}_{st}) \cdot \nabla) \tilde{\mathbf{u}}_p] \cdot \mathbf{n} \quad (9)$$

Alternatively, the Neumann pressure boundary condition might be obtained through one of the approaches available in the literature for the evaluation

of the gradient of the acoustic pressure [33],[34].

The present LE-P formulation represents the extension to problems appreciably affected by nonuniform mean-flow effects of the sound scattering integral formulations based on the Lighthill equation that are defined over the scatterer solid surface. For the surface  $\mathcal{S}$  suitably extended such to make the volume terms negligible, the nonuniform mean-flow effects are implicitly included in the input provided on  $\mathcal{S}$  by the fluid dynamic solver (either in  $\partial \tilde{p}'_s / \partial n$ , or in  $\tilde{\mathbf{u}}_s$  and  $\mathbf{u}_{st}$  if Eq. 8 is applied in boundary integral equation mode).

The drawback of this formulation is that the knowledge of the incident field is not sufficient for the direct determination of the corresponding scattered field (as it occurs, for instance, in the velocity potential formulation proposed in [21]).

## 2.2. The Permeable-Surface Integral Formulation Derived From FWHE

Considering the same assumptions of Section 2.1, and observing that

$$\nabla \cdot \nabla \cdot [H(f) \mathbf{T}] = H(f) \nabla \cdot \nabla \cdot \mathbf{T} + \nabla \cdot [\mathbf{T} \nabla H(f)] + (\nabla \cdot \mathbf{T}) \cdot \nabla H(f) \quad (10)$$

Eq. 4 can be re-arranged in the following form of the FWHE [18, 23, 35]

$$\begin{aligned} -\square^2 \tilde{p}' &= \nabla \cdot [\mathbf{P} \nabla H(f)] + \nabla \cdot [\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) \nabla H(f)] \\ &\quad - \frac{\partial}{\partial t} [\rho_0 \mathbf{v} \cdot \nabla H(f)] - \frac{\partial}{\partial t} [\rho (\mathbf{u} - \mathbf{v}) \cdot \nabla H(f)] \\ &\quad - \nabla \cdot \nabla \cdot [\mathbf{T} H(f)] + H(f) D_i \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_i) \end{aligned} \quad (11)$$

where  $\mathbf{P} = (p - p_0) \mathbf{I}$  is the compressive stress tensor, with  $p_0$  denoting the undisturbed medium pressure.

Then, considering the boundary integral methodology outlined in Appendix A, in a frame of reference,  $\mathcal{R}(\mathbf{x})$ , rigidly connected to  $\mathcal{S}$ , the solution of Eq. 11 can be derived in the form of the following boundary-field integral formulation for the acoustic disturbance at any receiver (observer) point,  $\mathbf{x}_*$ , external to  $\mathcal{S}$

$$\begin{aligned}
p'(\mathbf{x}_*, t_*) &= p'_i(\mathbf{x}_*, t_*) - \int_0^\infty \int_{\mathcal{V}} \mathbf{T} : (\nabla \nabla G_{SF}) dV(\mathbf{x}) dt \\
&\quad - \rho_0 \int_{\mathcal{S}} \left[ \mathbf{v} \cdot \mathbf{n} \mathbf{v} \cdot \nabla \hat{G} + \frac{\partial}{\partial t} [\mathbf{v} \cdot \mathbf{n} (1 - \mathbf{v} \cdot \nabla \vartheta)] \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
&\quad - \int_{\mathcal{S}} \left[ (\mathbf{P} \mathbf{n}) \cdot \nabla \hat{G} - \left( \frac{\partial \mathbf{P}}{\partial t} \mathbf{n} \right) \cdot \nabla \vartheta \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
&\quad - \int_{\mathcal{S}} \left[ \rho \mathbf{u}^- \cdot \mathbf{n} \mathbf{u}^+ \cdot \nabla \hat{G} + \frac{\partial}{\partial t} [\rho \mathbf{u}^- \cdot \mathbf{n} (1 - \mathbf{u}^+ \cdot \nabla \vartheta)] \hat{G} \right]_{\vartheta} dS(\mathbf{x})
\end{aligned} \tag{12}$$

where  $\mathbf{u}^- = (\mathbf{u} - \mathbf{v})$  and  $\mathbf{u}^+ = (\mathbf{u} + \mathbf{v})$ , whereas  $G_{SF} = \hat{G} \delta(t - t_* + \vartheta)$ .

Akin to Eq. 5, the field contribution is related to the Lighthill stress tensor and, for  $\mathcal{S}$  large enough to include the domain around the scatterer  
170 where it may be of significant value, Eq. 12 reduces to a boundary integral formulation.

#### *The FWHE-P Permeable-Surface Sound Scattering Formulation*

Recalling the same assumptions applied for the derivation of the LE-P scattering formulation (including the application of a surface  $\mathcal{S}$  large enough to make the volume term contribution negligible), considering also the decomposition for the fluid density,  $\rho = \rho_{st} + \rho_s + \rho_i$ , and observing that the perturbation quantities  $p'_p = p'_s + p'_i$  and  $\rho_p = \rho_s + \rho_i$  are related by  $p'_p = c_0^2 \rho_p$ , the following FWHE-P first-order perturbation, frequency-domain, permeable-surface, integral representation of the scattered acoustic disturbance is read-

ily derived from Eq. 12

$$\begin{aligned}
\tilde{p}'_s(\mathbf{x}_*, \omega) = & - \int_{\mathcal{S}} \left[ \tilde{p}'_p \mathbf{n} \cdot \nabla \hat{G} - i\omega \tilde{p}'_p \mathbf{n} \cdot \nabla \vartheta \hat{G} \right] e^{-i\omega\vartheta} dS(\mathbf{x}) \\
& + \int_{\mathcal{S}} \left[ i\omega \left( \frac{\tilde{p}'_p}{c_0^2} \mathbf{v}^- \cdot \mathbf{n} - \rho_{st} \tilde{\mathbf{u}}_p \cdot \mathbf{n} \right) \hat{G} \right] e^{-i\omega\vartheta} dS(\mathbf{x}) \\
& + \int_{\mathcal{S}} \left[ \frac{\tilde{p}'_p}{c_0^2} \mathbf{v}^- \cdot \mathbf{n} \mathbf{v}^+ + \rho_{st} (\mathbf{v}^- \cdot \mathbf{n} \tilde{\mathbf{u}}_p - \tilde{\mathbf{u}}_p \cdot \mathbf{n} \mathbf{v}^+) \right] \cdot \left[ \nabla \hat{G} - i\omega \nabla \vartheta \hat{G} \right] e^{-i\omega\vartheta} dS(\mathbf{x})
\end{aligned} \tag{13}$$

where  $\mathbf{v}^- = \mathbf{v} - \mathbf{u}_{st}$  and  $\mathbf{v}^+ = \mathbf{v} + \mathbf{u}_{st}$ .

Similarly to the LE-P formulation, Eq. 13 yields  $\tilde{p}'_s$  everywhere outside  
175 of  $\mathcal{S}$ , if  $\tilde{p}'_s$  and velocity field perturbations on this surface due to the combination of body motion and incident acoustic disturbance are provided by a previous fluid dynamic analysis. In addition, for  $\mathbf{x}_* \in \mathcal{S}$ , Eq. 13 becomes an integral equation that can be used to determine  $\tilde{p}'_s$  on  $\mathcal{S}$  from the knowledge of the velocity perturbations only (in this case, regularized integral terms are  
180 obtained by application of the formulation presented in [18]).

Therefore, as for the LE-P formulation, the knowledge of the incident field is not a sufficient input to determine the corresponding scattered field and flow data from a previous application of a near-field fluid dynamic solver are required.

185 The present FWHE-P formulation represents the extension to problems appreciably affected by nonuniform mean-flow effects of the scattering formulation based on the acoustic analogy introduced in [18]. Likewise the LE-P formulation, when used in the form of a boundary integral equation and for the surface  $\mathcal{S}$  suitably extended such to make the volume terms negligible, the  
190 nonuniform mean-flow effects are implicitly included in the input provided on  $\mathcal{S}$  by the fluid dynamic solution ( $\tilde{\mathbf{u}}_s$  and  $\mathbf{u}_{st}$ , in this case).

Finally, note that for  $\mathcal{S}$  coinciding with the body surface (and leaving, anyway, volume terms neglected), Eq. 13 reduces to the frequency-domain scattering formulation developed in [18], with the term  $\tilde{\mathbf{u}}_p \cdot \mathbf{n}$  eventually  
195 representing surface porosity effects.

### 2.3. Remarks on FWHE-P and LE-P Formulations

If expressed on a fictitious surface suitably far from a moving scatterer, the integral formulations for sound scattering presented above are capable to capture the nonuniform mean-flow effects included in the field sources  
200 without requiring the evaluation of cumbersome volume integrals.

In this context, it is worth noting that the field contributions in Eqs. 5 and 12 are different, both in terms of intensity of noise sources (directly related to the Lighthill stress tensor or to its double divergence), and in terms of radiation characteristics (of monopole or of quadrupole type): this implies  
205 that their impact on the predicted acoustic disturbance is different, as well as it is different the extension of the porous surface,  $\mathcal{S}$ , that guarantees that the corresponding volume integrals are negligible.

However, regardless the porous boundary surface considered, these pressure-field formulations provide the outer prediction of perturbed noise only once  
210 near-field perturbation acoustic pressure and/or velocity distributions (which implicitly include the effects from nonuniform mean-flow) are known by application of a fluid dynamic solution tool.

Nonetheless, although not applicable for the determination of near-field solutions, these integral formulations remain of interest (i) for the efficient  
215 evaluation of perturbed sound in the far-field, (ii) for the estimation of the relative importance of the field source terms arising in moving scatterer prob-

lems, and hence (iii) for the assessment of the accuracy of the boundary integral formulations expressed on the physical surface, suitably applied for non-moving scatterer problems.

220 Finally, note that, even though the integral scattering formulations are here developed considering an illustrative incident field due to a pressure source, they remain valid for any incident field generated by any kind of flow perturbation.

### 3. Numerical results

225 In this section, the porous-surface formulations presented above are applied for the analysis of a sound scattering problem that consists of a rectangular, non-lifting wing in uniform rectilinear translation, impinged by the incident potential field due to a co-moving pulsating point source located in the mid-span plane [30].

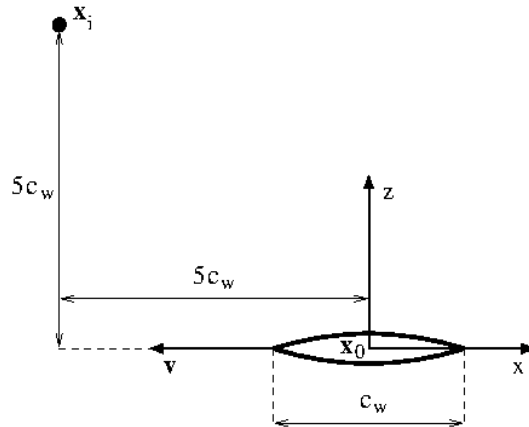


Figure 1: Midplane of the scattering wing. Wing velocity and location of emitting source.

Specifically, considering a coordinate system  $(\mathbf{x}_0, x, y, z)$  connected to the wing, with chordwise  $x$ -axis, spanwise  $y$ -axis and origin,  $\mathbf{x}_0$ , at the centre of the mid-span cross section (see the sketch of Fig 1), for a wing having wing span three times larger than the chord length,  $c_w$ , and the cross section of symmetric, biconvex, parabolic shape with thickness ratio  $t_w/c_w = 0.1$ , the acoustic scattering predictions are presented in terms of the directivity pattern of the ratio between the scattered pressure in the mid-span plane at radial distance  $d/c_w = 52.5$  from  $\mathbf{x}_0$ , and the reference pressure defined as  $p_{ref} = 2 d |\tilde{p}_i(\mathbf{x}_0, k)|/c_w$  [30]. Pressure  $\tilde{p}_i(\mathbf{x}_0, k)$  denotes the incident field at  $\mathbf{x}_0$  induced by a potential velocity source located in front of the wing at  $\mathbf{x}_i = (-5c_w, 0, 5c_w)$ , pulsating at a frequency  $\omega$  such that the wave number is  $k = \omega/c_0$ .

Since scatterer and permeable surface,  $\mathcal{S}$ , surrounding it are both in uniform translation, the retarded free-space Green function appearing in the FWHE-P and LE-P integral formulations presented above is expressed as  $\hat{G} = -1/(4\pi r_\beta)$ , where  $r_\beta = \sqrt{[\mathbf{M} \cdot (\mathbf{x}_* - \mathbf{x})]^2 + \beta^2 |\mathbf{x}_* - \mathbf{x}|^2}$  with  $\beta^2 = (1 - \mathbf{M} \cdot \mathbf{M})$ , whereas the acoustic time delay is given by  $\vartheta = [r_\beta + \mathbf{M} \cdot (\mathbf{x}_* - \mathbf{x})]/(c_0 \beta^2)$  (see [32]). For the numerical solution of the integral formulations a zero-th order BEM scheme is adopted. It consists of dividing the acoustic surface,  $\mathcal{S}$ , into quadrilateral panels, assuming the functions that multiply the kernels of the integral operators to be piecewise constant. When the formulations are applied as boundary integral equations, the satisfaction of the equations is imposed at the center of each panel (collocation method, see also [32]).

First, as a preliminary analysis, the wing is assumed to be at rest, impinged by sound waves with frequencies corresponding to  $k c_w = 3$  and



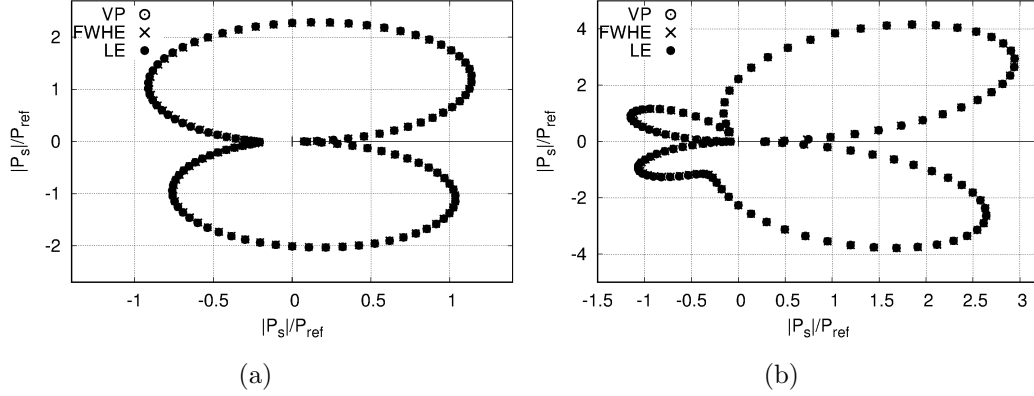


Figure 2: Noise directivity pattern for  $M = 0$ . Linear LE, FWHE and VP formulations.  
 (a):  $k c_w = 3$ ; (b):  $k c_w = 6$ .

255  $k c_w = 6$ . Figure 2 compares the predictions given by the LE and FWHE  
 integral formulations for scattering in Eqs. 8 and 13 written for the surface  $\mathcal{S}$   
 corresponding to the impermeable surface of the scatterer (standard linear,  
 non-porous formulations with fully neglected field terms), with those given  
 by the linear VP formulation proposed in [21]. As expected, for non-moving  
 260 scatterer, in the absence of nonuniform mean-flow effects, the results from  
 these three formulations are perfectly matching.

However, the comparisons from the fully linear formulations expressed  
 over the solid scatterer surface worsen when the scatterer moves. Indeed,  
 for the body Mach number  $M = 0.5$ , Fig. 3 shows significant discrepancies  
 265 among predictions (particularly with respect to those from the VP formu-  
 lation), in spite of the regular, low-curvature, scatterer shape. It is worth  
 noting that, the predictions given by the fully linear VP formulation are in  
 perfect agreement with those obtained in [30] through an equivalent linear  
 velocity potential approach (see also [21]).

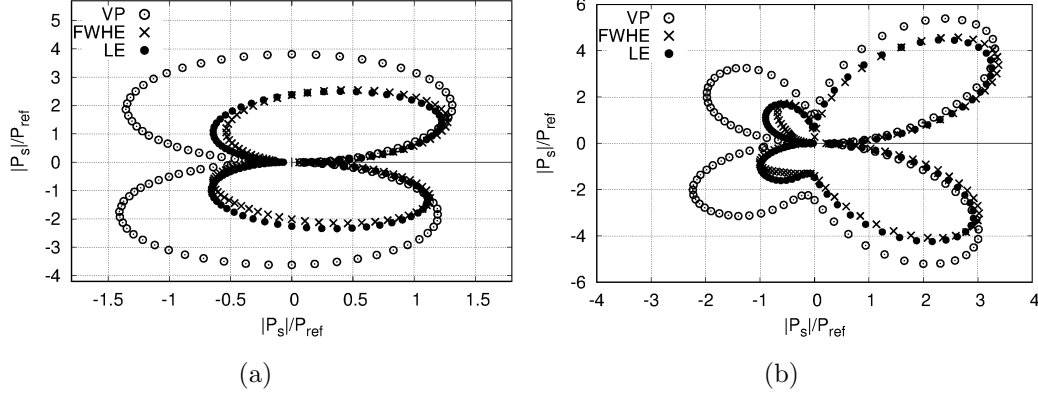


Figure 3: Noise directivity pattern for  $M = 0.5$ . Linear LE, FWHE and VP formulations. (a):  $k c_w = 3$ ; (b):  $k c_w = 6$ .

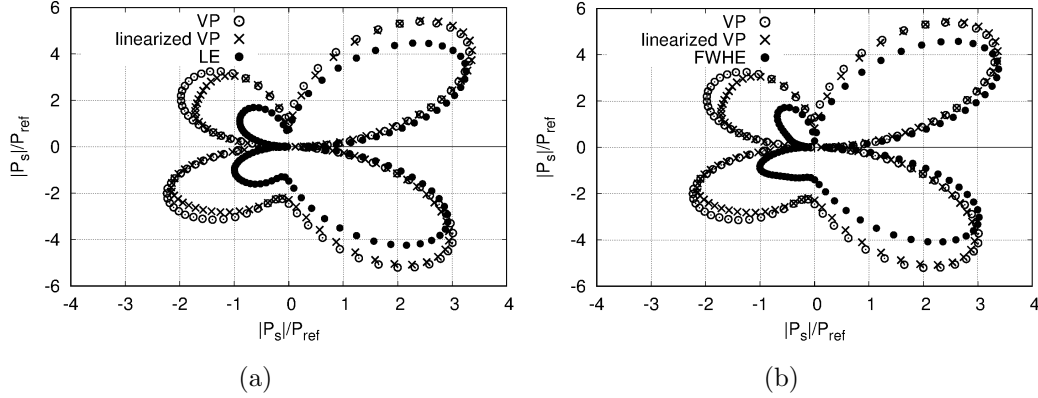


Figure 4: Noise directivity pattern for  $M = 0.5$  and  $k c_w = 6$ . Influence of nonlinear terms. (a): LE; (b): FWHE.

270 Next, for  $M = 0.5$  and  $k c_w = 6$ , Fig. 4 introduces the comparison with  
the directivity patterns predicted by the linearized, boundary-field VP for-  
mulation proposed in [21] that, starting from the non-homogeneous wave  
equation for the velocity potential, includes the contributions from the lin-  
earized field terms related to the nonuniform mean-flow effects, and thus  
275 represents the most complete first-order formulation for noise scattered by

moving bodies. The results from the boundary-field VP formulation are obtained considering the integration field domain extended up to a distance of  $0.7c_w$  from the body surface, discretized into 11500 volume cells, with the wing surface discretized into 2000 body panels (details on the convergence analysis are provided in [21]). The scattering prediction from the linearized VP formulation provides results that are quite close (although clearly different) to those from the linear VP approach, but that significantly differ from the results given by the standard, linear LE and FWHE solutions, particularly in terms of signal amplitudes (the directivity is similar). Observing that the complete VP, LE and FWHE formulations are fully equivalent and must provide identical predictions (at least when the viscous stress tensor is neglected in LE and FWHE), this means that the nonlinear terms of LE and FWHE (neglected in this investigation) strongly affect the scattered field, much more than that appearing in the non-homogeneous wave equation for the velocity potential.

In order to prove this, the developed LE-P and FWHE-P formulations are applied for a permeable cylindrical surface,  $\mathcal{S}$ , that surrounds the wing having elliptical sections with semi-axes  $a$  and  $b$ , closed by domes having the shape of semi-ellipsoids with semi-axes  $a$  and  $b$  and  $e$ , as depicted in Fig. 5.

For both formulations, a preliminary convergence analysis is accomplished by the porous surface configurations described in Table 1, which consider increasing surface size and mesh discretization of growing refinement.

Specifically, using input data from the linearized boundary-field VP formulation [21] with the field integral evaluated over a domain that surrounds the body and is fully enclosed by the porous surface, the LE-P and FWHE-P

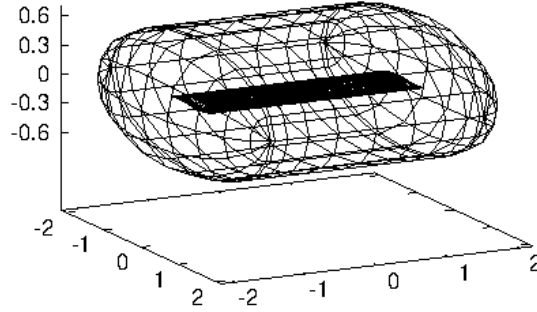


Figure 5: Sketch of the permeable surface surrounding the scatterer.

Table 1: Porous surface configurations.

configuration	panels	surface size
1	1300	$a = 1.2 c_w$ ; $b = 1.00 c_w$ ; $e = 1.00 c_w$
2	5000	$a = 1.5 c_w$ ; $b = 1.25 c_w$ ; $e = 1.25 c_w$
3	20000	$a = 1.8 c_w$ ; $b = 1.50 c_w$ ; $e = 1.50 c_w$
4	45000	$a = 2.1 c_w$ ; $b = 1.75 c_w$ ; $e = 1.75 c_w$

predictions based on the configurations in Table 1 are those illustrated in Fig. 6, whereas the corresponding Root Mean Square (RMS) of the relative error (difference) with respect to results from the linearized VP approach are presented in Table 2.

305 These results demonstrate that the configuration #4 provides predictions close to convergence and, above all, very close to those from the linearized VP approach. This implies that, within the limits of the residual very small numerical errors, this porous surface surrounds the region where the contributions from the Lighthill stress tensor are not negligible and that,

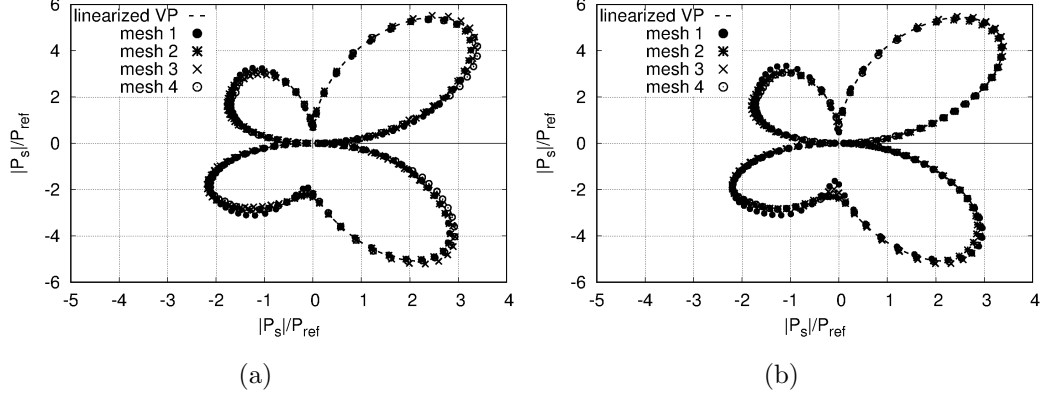


Figure 6: Convergence analysis of the noise directivity pattern for  $M = 0.5$  and  $k c_w = 6$ . (a): LE-P; (b): FWHE-P.

Table 2: RMS of relative error between LE-P/FWHE-P and linearized VP predictions.

configuration	RMS of relative error	
	LE-P	FWHE-P
1	12%	11%
2	6.5%	3.5%
3	4.7%	3.2%
4	3.2%	2.4%

310 correspondingly, LE-P and FWHE-P formulations are capable to capture the nonuniform mean-flow effects and suitably take into account the field sources terms. Incidentally, this proves also the equivalence of the VP, LE and FWHE formulations when not restricted to their linear versions. In addition, Table 2 shows that the FWHE-P solution is closer to the linearized  
315 VP predictions than the LE-P one for any surface configuration: this is related to the different field terms which, therefore, in both cases tend to

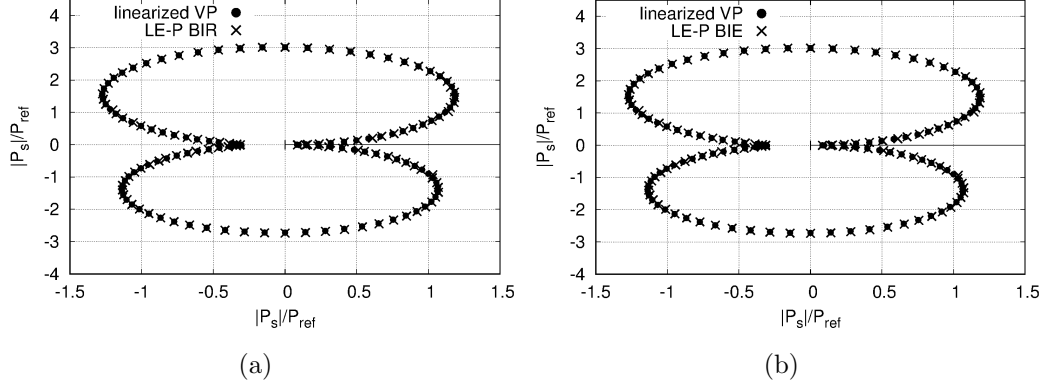


Figure 7: Noise directivity pattern for  $M = 0.3$  and  $k c_w = 3$ . LE-P predictions. (a): BIR; (b): BIE.

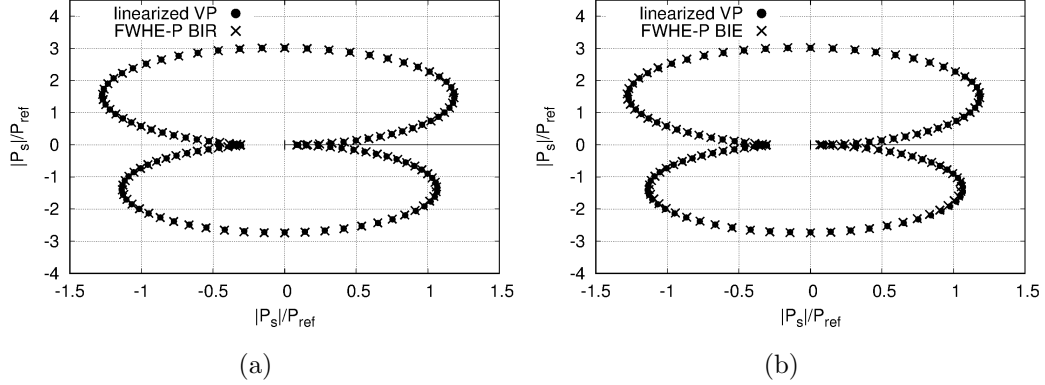


Figure 8: Noise directivity pattern for  $M = 0.3$  and  $k c_w = 3$ . FWHE-P predictions. (a): BIR; (b): BIE.

vanish with the distance from the body, with those in the LE-P more strongly affecting the predictions.

Next, for the sake of completeness, the confirmation of these outcomes  
 320 is examined for different scattering conditions through application of the  
 porous surface of configuration #4. First, Figs. 7, 8 depict the comparisons  
 between the directivity of the scattered acoustic disturbance given by the

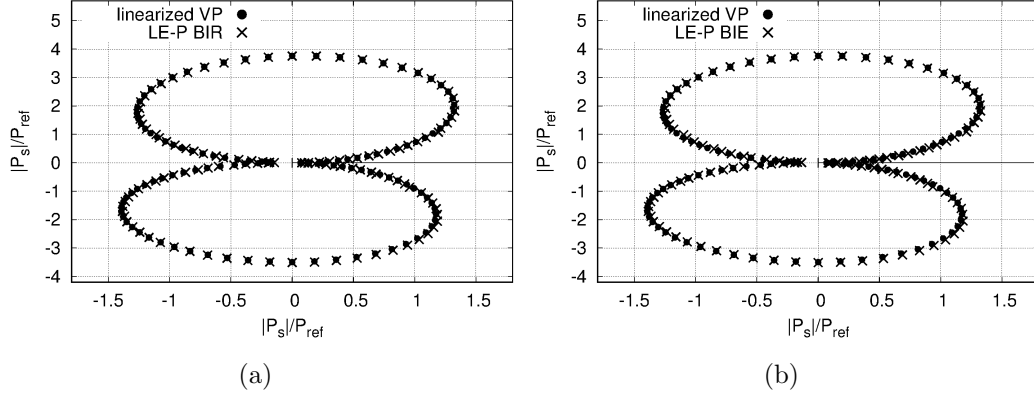


Figure 9: Noise directivity pattern for  $M = 0.5$  and  $k c_w = 3$ . LE-P predictions. (a): BIR; (b): BIE.

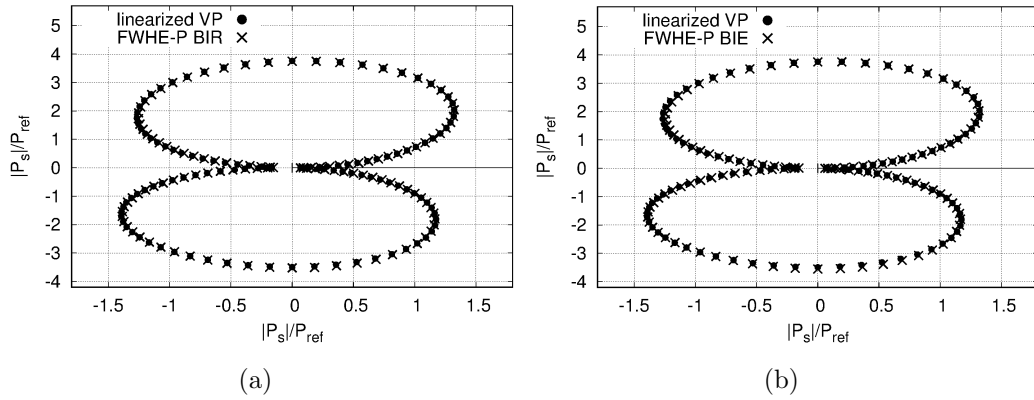


Figure 10: Noise directivity pattern for  $M = 0.5$  and  $k c_w = 3$ . FWHE-P predictions. (a): BIR; (b): BIE.

(reference) linearized VP solver and those computed by the LE-P and FWHE-P approaches, respectively, for  $M = 0.3$  and  $k c_w = 3$ . In details, these figures  
 325 present LE-P and FWHE-P predictions obtained using the formulations in Eqs. 8 and 13 both as boundary integral representations (BIR), and as boundary integral equations (BIE). In the first case, the scattered pressure

outside of  $\mathcal{S}$  is determined by assuming the whole set of pressure and velocity data appearing in the integrals over  $\mathcal{S}$  known from the velocity potential solver presented in [21], whereas in the second case only  $\partial p'_s/\partial n$  in Eq. 8 and  $\mathbf{u}_s$  in Eq. 13 are assumed to be known on  $\mathcal{S}$  from the potential velocity tool, since  $p'_s$  is obtained as solution of the integral equation derived for  $\mathbf{x}_* \in \mathcal{S}$  (note that, also for Eq. 8 in BIE mode  $\mathbf{u}_s$  is a suitable input if Eq. 9 is applied, and that, in any case, the incident fields are assumed to be known a priori). Both pressure-field formulations developed in this work provide results that perfectly match those from the linearized VP solution, for both solution modes (namely, used as BIR and BIE).

Similar analyses are performed for scattering conditions characterized by different body velocity and frequency of the incident disturbance, and specifically  $M = 0.5$  and  $k c_w = 3$ ,  $M = 0.3$  and  $k c_w = 6$ ,  $M = 0.5$  and  $k c_w = 6$ . The corresponding scattered field predictions provided by LE-P and FWHE-P formulations applied both in BIR and BIE modes are presented in Figs. 9 – 14, which show their excellent agreement with those given by the linearized VP approach.

These results confirm the capability of the proposed approaches to provide accurate noise scattering simulations in the far field including nonuniform mean-flow effects, without requiring the computation of cumbersome volume integrals while, at the same time, proving the perfect equivalence of LE, FWHE and VP formulations when same-order approximations are applied.

Finally, it is worth noting that it is verified that identical LE-P BIE solutions are achieved considering known on  $\mathcal{S}$ , through the VP solver, either the distribution of  $\partial p'_s/\partial n$  or that of  $\mathbf{u}_s$ , followed by application of Eq. 9.



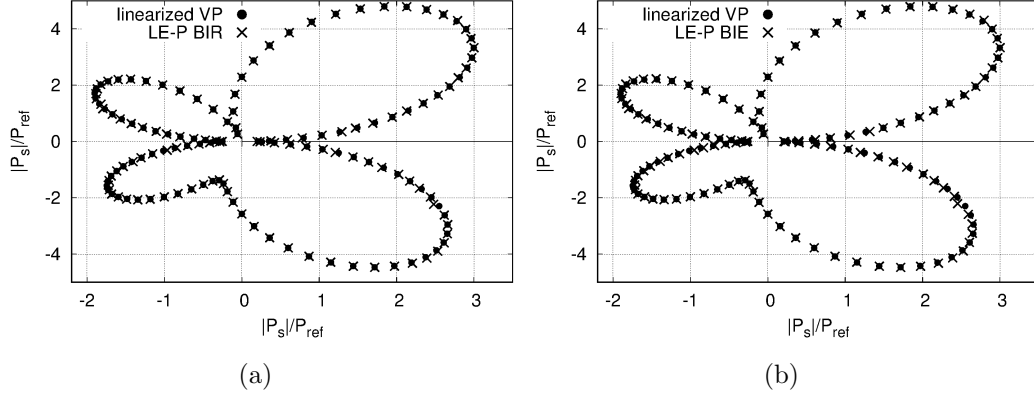


Figure 11: Noise directivity pattern for  $M = 0.3$  and  $k c_w = 6$ . LE-P predictions. (a): BIR; (b): BIE.

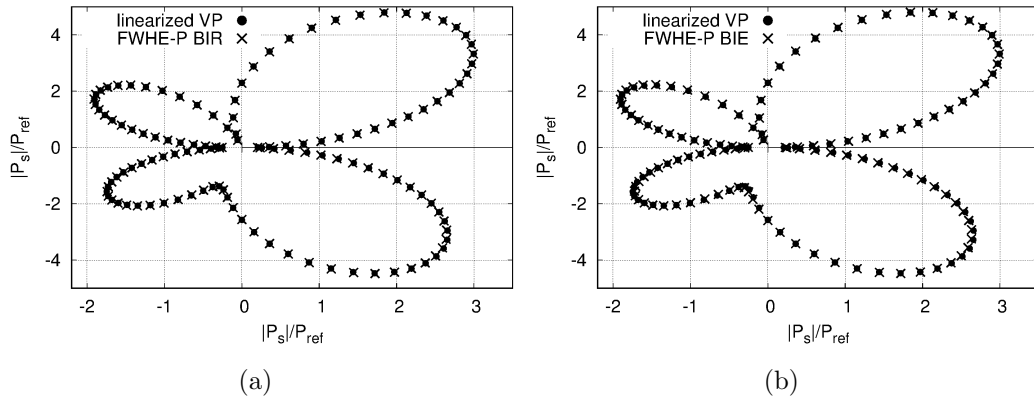


Figure 12: Noise directivity pattern for  $M = 0.3$  and  $k c_w = 6$ . FWHE-P predictions. (a): BIR; (b): BIE.

#### 4. Conclusions

Starting from the Lighthill and Ffowcs Williams and Hawkings equations,  
 355 two novel permeable-surface, pressure-field integral formulations (LE-P and  
 FWHE-P) for the analysis of noise scattered by moving bodies have been pro-  
 posed, which include the acoustic effects of nonuniform mean-flow generated

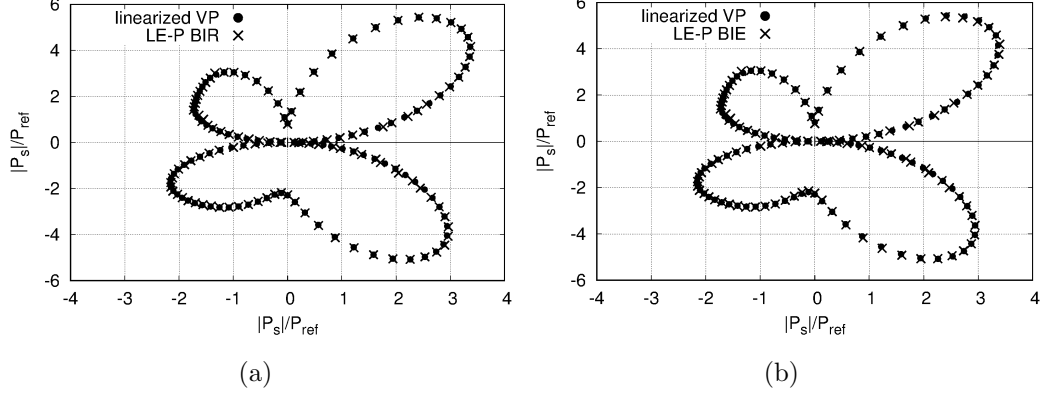


Figure 13: Noise directivity pattern for  $M = 0.5$  and  $k c_w = 6$ . LE-P predictions. (a): BIR; (b): BIE.

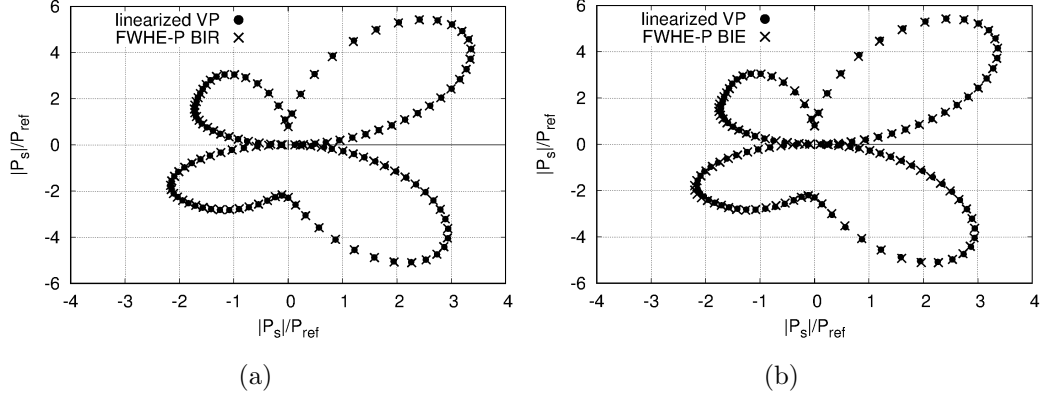


Figure 14: Noise directivity pattern for  $M = 0.5$  and  $k c_w = 6$ . FWHE-P predictions. (a): BIR; (b): BIE.

by the scatterer motion. These have been applied to the problem of noise scattered by a non-lifting wing in uniform translation impinged by the acoustic disturbance generated by a co-moving source. From the comparison of their predictions with those obtained by a validated boundary-field, integral formulation for the velocity potential, the following observations/conclusions are drawn:

- 365
• as long as the scatterer is at rest, the linear LE, FWHE and VP formulations (namely, those derived neglecting the nonlinear terms) provide fully equivalent results when applied on the body surface;
- 370
• large differences among the predictions from the linear LE, FWHE and VP formulations applied on the body surface arise when the scatterer is in motion: this implies that the nonlinear terms, that include the effects of the nonuniform mean-inflow generated by the scatterer, very differently affects these three formulations;
- 375
• the application of the LE-P and FWHE-P formulations on a fictitious surface that surrounds the region where the nonlinear terms are significant yields noise scattering predictions that perfectly match those from the boundary-field VP formulation that takes into account the nonuniform mean-flow through linearized volume integral contributions, and this is achieved by solving LE-P and FWHE-P both in BIR and BIE modes;
- 380
• the observed excellent agreement among these predictions can be interpreted as a sort of cross-validation for the LE, FWHE and VP formulations that, theoretically, are fully equivalent (if the viscous stress tensor is neglected in LE and FWHE, as in this case) and, in addition, demonstrates the capability of the proposed LE-P and FWHE-P scattering formulations to suitably take into account nonuniform mean-flow effects indirectly through their influence on the scattered pressure and/or velocity fields on the surface of integration, without requiring the evaluation of cumbersome volume integrals;
385

- the (nonlinear) nonuniform mean-flow terms affects the scattered field predicted by the LE and FWHE formulations much more than that predicted by the VP one: this implies that when the scatterer moves, the fully linear pressure-field formulations are much less accurate (or better, are much more inaccurate) than that based on the velocity potential;
- LE-P and FWHE-P formulations are capable of predicting far-field noise due to a moving scatterer, but their drawback is that, differently from the VP formulation, require the near-field aerodynamic/aeroacoustic solution from an external solver providing suitable pressure and/or velocity fields data on the porous surface;
- in the numerical investigation, for the sake of simplicity, the scattering body has been arbitrarily considered in steady rectilinear motion, but the discussed outcomes are absolutely general.

Note that, although these observations/conclusions are referred to the specific case of the wing in uniform rectilinear motion examined in the numerical investigation, they remain valid for any moving scatterer producing significant nonuniform mean-flow effects.

## Acknowledgments

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## Appendix A. Integral Formulation for Inhomogeneous Wave Equation Solution in Moving Boundary Problems

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Let us consider a fluid initially at rest fixed to a frame of reference  $\mathcal{R}(\boldsymbol{\xi})$  (fluid frame of reference, FFR), and a closed surface,  $\mathcal{S}$ , denoted by  $f(\boldsymbol{\xi}, t) = 0$ , rigidly moving within it. The most general differential operator governing aerodynamic/aeroacoustic wave propagation of a field  $\psi(\boldsymbol{\xi}, t)$  outside of  $\mathcal{S}$  is of the form (see, for instance, Sections 2.1, 2.2 and [36] for aerodynamic applications)

$$-\square^2 \bar{\psi} = \chi + \mathbf{z} \cdot \nabla H + \nabla \cdot (\mathbf{Z} \nabla H) + k_1 \frac{\partial H}{\partial t} + \frac{\partial}{\partial t} (k_2 \frac{\partial H}{\partial t}) + \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) \quad (\text{A.1})$$

where  $\psi$  is conveniently extended to the whole  $\mathbb{R}^3$  domain by introduction of  $\bar{\psi} = \psi H(f)$ , with  $H$  denoting the Heaviside function, whereas  $k_1, k_2, \mathbf{z}, \mathbf{w}$  and  $\mathbf{Z}$  are scalar, vector and tensor fields representing generic forcing terms to be specified for each particular problem under examination. The definition  
 415 of the differential problem is completed by assuming that both the initial conditions and the boundary conditions at infinity are homogeneous for all the quantities involved.

Next, let us introduce the following fundamental solution for the wave equation,  $G$ , (namely, such that  $-\square^2 G = \delta(\boldsymbol{\xi}_* - \boldsymbol{\xi}) \delta(t_* - t)$ ) and satisfying the conditions  $G = \partial G / \partial t = 0$  at  $t = \infty$  and homogeneous boundary conditions at infinity (free-space Green function)

$$G(\boldsymbol{\xi}, \boldsymbol{\xi}_*, t, t_*) = \frac{-1}{4\pi r} \delta(t - t_* + \frac{r}{c_0}) \quad (\text{A.2})$$

where  $r = |\boldsymbol{\xi}_* - \boldsymbol{\xi}|$  and  $\delta$  denotes the Dirac delta function.

Then, the application of the Green function method yields the following solution of the above problem

$$\begin{aligned}\bar{\psi}(\boldsymbol{\xi}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} G \chi \, dV(\boldsymbol{\xi}) \, dt + \int_0^\infty \int_{\mathbb{R}^3} \mathbf{z} \cdot \nabla H \, G \, dV(\boldsymbol{\xi}) \, dt \\ &+ \int_0^\infty \int_{\mathbb{R}^3} \nabla \cdot (\mathbf{Z} \nabla H) \, G \, dV(\boldsymbol{\xi}) \, dt + \int_0^\infty \int_{\mathbb{R}^3} k_1 \frac{\partial H}{\partial t} \, G \, dV(\boldsymbol{\xi}) \, dt \quad (\text{A.3}) \\ &+ \int_0^\infty \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (k_2 \frac{\partial H}{\partial t}) \, G \, dV(\boldsymbol{\xi}) \, dt + \int_0^\infty \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) \, G \, dV(\boldsymbol{\xi}) \, dt\end{aligned}$$

where  $\boldsymbol{\xi}$  and  $\boldsymbol{\xi}_*$  assume the role of source and observer positions, respectively, whereas  $t$  and  $t_*$  denote emission and receiving time.

Two Dirac delta functions appear in the integrals of Eq. A.3: a space Dirac delta function arising from the derivatives of the Heaviside function,  $H$ , and a time Dirac delta function contained in the expression for  $G$ , Eq. A.2. Since the surface  $\mathcal{S}$  is assumed to be undeformable,  $H$  is time independent in a frame of reference,  $\mathcal{R}(\mathbf{x})$ , fixed to it (surface frame of reference, SFR) and hence, before performing the integrations corresponding to the two Dirac delta functions, it is convenient to re-express the representation for  $\bar{\psi}$  in the SFR.

To this aim: first, note that a transformation,  $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, t)$ , from FFR to SFR is of rigid-body type and, as such, its Jacobian is  $J = 1$  (namely,  $dV(\boldsymbol{\xi}) = dV(\mathbf{x})$ ); then, note that the time derivatives observed in the FFR and in the SFR are related by

$$\left. \frac{\partial(\cdot)}{\partial t} \right|_{\boldsymbol{\xi}} = \left. \frac{\partial(\cdot)}{\partial t} \right|_{\mathbf{x}} - \mathbf{v} \cdot \nabla(\cdot) \quad (\text{A.4})$$

where  $\mathbf{v} = \left. \frac{\partial \boldsymbol{\xi}}{\partial t} \right|_{\mathbf{x}}$  represents the velocity of a point  $\mathbf{x}$  of the SFR with respect to the FFR; finally, observe that, for any  $g(t), h(t)$  and  $q(t)$ , and for  $(\dot{\cdot}) =$

$\partial(\dots)/\partial t|_{\mathbf{x}}$

$$\int_0^\infty q(t) h(t) \delta[g(t)] dt = \sum_k \frac{q(t)h(t)}{|\dot{g}(t)|} \Big|_{t=t_k} = \sum_k \int_0^\infty q(t) \frac{h(t)}{|\dot{g}(t)|} \Big|_{t=t_k} \delta(t-t_k) dt \quad (\text{A.5})$$

where  $t_k$  are the roots of  $g(t) = 0$ . In the present case,  $g = t - t_* + r/c_0$ , where  $r = |\boldsymbol{\xi}_* - \boldsymbol{\xi}| = |\boldsymbol{\xi}(\mathbf{x}_*, t_*) - \boldsymbol{\xi}(\mathbf{x}, t)|$ , and thus  $\dot{g} = 1 + \mathbf{r} \cdot \mathbf{v}/(r c_0)$  (note that, considering in the last term of Eq. A.5 the function  $h(t)/|\dot{g}|$  evaluated at  $t = t_k$  is an arbitrary, but legitimate choice). Limiting, for the sake of simplicity, to the case in which the equation  $g(t) = 0$  has only one root (as occurring for source points in subsonic motion with respect to the FFR), it is denoted as  $\tau = t_* - \vartheta$ , where the delay  $\vartheta$  is the solution of the following equation

$$c_0 \vartheta = |\boldsymbol{\xi}(\mathbf{x}_*, t_*) - \boldsymbol{\xi}(\mathbf{x}, t_* - \vartheta)| \quad (\text{A.6})$$

Combining the above expressions with Eq. A.3, performing integration by parts over the third, fifth and sixth integral on the right hand side and considering the homogeneous boundary conditions at infinity, the following integral solution in the SFR is obtained

$$\begin{aligned} \bar{\psi}(\mathbf{x}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} \chi G_{SF} dV(\mathbf{x}) dt + \int_0^\infty \int_{\mathbb{R}^3} (\mathbf{z} \cdot \nabla H) G_{SF} dV(\mathbf{x}) dt \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} (\mathbf{Z} \cdot \nabla H) \cdot \nabla G_{SF} dV(\mathbf{x}) dt + \int_0^\infty \int_{\mathbb{R}^3} k_1 (\dot{H} - \mathbf{v} \cdot \nabla H) G_{SF} dV(\mathbf{x}) dt \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} k_2 (\dot{H} - \mathbf{v} \cdot \nabla H) (\dot{G}_{SF} - \mathbf{v} \cdot \nabla G_{SF}) dV(\mathbf{x}) dt \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla H (\dot{G}_{SF} - \mathbf{v} \cdot \nabla G_{SF}) dV(\mathbf{x}) dt \end{aligned} \quad (\text{A.7})$$

where  $G_{SF}(\mathbf{x}, \mathbf{x}_*, t_*, t) = \hat{G} \delta(t - t_* + \vartheta)$  represents the free-space Green function expressed in the SFR, with

$$\hat{G}(\mathbf{x}_*, \mathbf{x}, t_*) = \frac{-1}{4\pi} \left[ \frac{1}{r} \left| 1 - \frac{\mathbf{r} \cdot \mathbf{v}}{r c_0} \right|^{-1} \right]_{\vartheta} = \frac{-1}{4\pi} \left[ \frac{1}{r(1 - M_r)} \right]_{\vartheta} \quad (\text{A.8})$$

and  $[\dots]_{\vartheta}$  denoting evaluation at the retarded emission time  $t = t_* - \vartheta$ .

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Next, noting that:

- choosing the function  $f$ , arbitrary but legitimately, such that  $|\nabla f| = 1$ , one obtains

$$\nabla H(f) = \delta(f) \nabla f = \delta(f) \mathbf{n}$$

with  $\mathbf{n}$  denoting the outward unit normal vector of surface  $\mathcal{S}$ ;

- since the surface  $\mathcal{S}$  is rigidly connected with the SFR, the function  $H(f)$  is time independent in this frame of reference, and hence  $\dot{H} = 0$ ;
- since  $\vartheta = \vartheta(\mathbf{x}, \mathbf{x}_*, t_*)$ , one has

$$\nabla G_{SF} = \nabla \hat{G} \delta(t - t_* + \vartheta) + \hat{G} \dot{\delta}(t - t_* + \vartheta) \nabla \vartheta$$

- since  $\hat{G} = \hat{G}(\mathbf{x}, \mathbf{x}_*, t_*)$  one obtains

$$\dot{G}_{SF} = \hat{G} \dot{\delta}(t - t_* + \vartheta)$$

and recalling that for any scalar field,  $h(\mathbf{x}, t)$ , the following relations hold

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- $\int_0^\infty \int_{\mathbb{R}^3} h \delta(f) \delta(t - t_* + \vartheta) dV dt = \int_S [h]_{\vartheta} dS$
- $\int_0^\infty \int_{\mathbb{R}^3} h \delta(f) \dot{\delta}(t - t_* + \vartheta) dV dt = - \int_S [\dot{h}]_{\vartheta} dS$



the integrals appearing in Eq. A.7 can be expressed in the following form

$$\begin{aligned}
\mathcal{I}_1(\mathbf{x}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} (\mathbf{z} \cdot \nabla H) G_{SF} dV(\mathbf{x}) dt = \int_S \left[ \mathbf{z} \cdot \mathbf{n} \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
\mathcal{I}_2(\mathbf{x}_*, t_*) &= - \int_0^\infty \int_{\mathbb{R}^3} (\mathbf{Z} \nabla H) \cdot \nabla G_{SF} dV(\mathbf{x}) dt \\
&= - \int_S \left[ (\mathbf{Z} \mathbf{n}) \cdot \nabla \hat{G} - (\dot{\mathbf{Z}} \mathbf{n}) \cdot \nabla \vartheta \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
\mathcal{I}_3(\mathbf{x}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} k_1 (\dot{H} - \mathbf{v} \cdot \nabla H) G_{SF} dV(\mathbf{x}) dt = - \int_S \left[ k_1 \mathbf{v} \cdot \mathbf{n} \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
\mathcal{I}_4(\mathbf{x}_*, t_*) &= - \int_0^\infty \int_{\mathbb{R}^3} k_2 (\dot{H} - \mathbf{v} \cdot \nabla H) (\dot{G}_{SF} - \mathbf{v} \cdot \nabla G_{SF}) dV(\mathbf{x}) dt \\
&= - \int_S \left[ k_2 \mathbf{v} \cdot \mathbf{n} \mathbf{v} \cdot \nabla \hat{G} + \frac{\partial}{\partial t} \Big|_{\mathbf{x}} (k_2 \mathbf{v} \cdot \mathbf{n} (1 - \mathbf{v} \cdot \nabla \vartheta)) \hat{G} \right]_{\vartheta} dS(\mathbf{x}) \\
\mathcal{I}_5(\mathbf{x}_*, t_*) &= - \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla H (\dot{G}_{SF} - \mathbf{v} \cdot \nabla G_{SF}) dV(\mathbf{x}) dt \\
&= \int_S \left[ \mathbf{w} \cdot \mathbf{n} \mathbf{v} \cdot \nabla \hat{G} + \frac{\partial}{\partial t} \Big|_{\mathbf{x}} (\mathbf{w} \cdot \mathbf{n} (1 - \mathbf{v} \cdot \nabla \vartheta)) \hat{G} \right]_{\vartheta} dS(\mathbf{x})
\end{aligned}$$

and the final integral representation of the solution of Eq. A.1 reads

$$\bar{\psi}(\mathbf{x}_*, t_*) = \sum_{k=1}^5 \mathcal{I}_k + \int_0^\infty \int_{\mathbb{R}^3} \chi G_{SF} dV(\mathbf{x}) dt \quad (\text{A.9})$$

Note that, the definition of  $\hat{G}$  as a retarded function (see Eq. A.8) is a legitimate choice that derives from the application of Eq. A.5, but a fully equivalent integral formulation including additional terms arising from its time derivative could be obtained by considering  $\hat{G}$  not delayed.

#### Appendix A.1. Integral Formulation for the Solution of the LE

Comparing Eqs. 4 and A.1, it is inferred that Eq. A.9 provides the integral form solving the LE, Eq. 5, for:  $\psi = p'$ ,  $\chi = -H [\nabla \cdot \nabla \cdot \mathbf{T} - D_i \delta(\mathbf{x} - \mathbf{x}_i)]$ ,  $\mathbf{z} = \nabla \bar{p}'$ ,  $\mathbf{Z} = p' \mathbf{I}$ ,  $k_1 = -\frac{1}{c_0^2} \frac{\partial p'}{\partial t} \Big|_{\xi} = -\frac{1}{c_0^2} \left( \frac{\partial p'}{\partial t} \Big|_{\mathbf{x}} - \mathbf{v} \cdot \nabla p' \right)$ ,  
 $k_2 = -\frac{p'}{c_0^2}$  and  $\mathbf{w} = 0$ .

It is worth noting that, considering the specific expression of  $\chi$ , the volume integral can be recast in the following form

$$\begin{aligned}
\mathcal{I}_V(\mathbf{x}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} \chi G_{SF} dV(\mathbf{x}) dt = \int_{\mathbb{R}^3} [\chi \hat{G}]_{\vartheta} dV(\mathbf{x}) \\
&= \int_{\mathcal{V}} [(-\nabla \cdot \nabla \cdot \mathbf{T} + D_i \delta(\mathbf{x} - \mathbf{x}_i)) \hat{G}]_{\vartheta} dV(\mathbf{x}) \\
&= - \int_{\mathcal{V}} [(\nabla \cdot \nabla \cdot \mathbf{T}) \hat{G}]_{\vartheta} dV(\mathbf{x}) + [D_i]_{\vartheta} \hat{G}(\mathbf{x}_*, \mathbf{x}_i, t_*) \\
&= p'_i(\mathbf{x}_*, t_*) - \int_{\mathcal{V}} [(\nabla \cdot \nabla \cdot \mathbf{T}) \hat{G}]_{\vartheta} dV(\mathbf{x}) \tag{A.10}
\end{aligned}$$

where  $\mathcal{V}$  denotes the field domain external to the boundary  $\mathcal{S}$  and  $p'_i$  is the pressure field induced by the outer source.

#### *Appendix A.2. Integral Formulation for the Solution of the FWHE*

Akin to the LE solution, the integral solution of the FWHE, Eq. 12, is  
450 given by Eq. A.9 observing that the comparison of Eqs. 11 and A.1 yields:  
 $\psi = p'$ ,  $\chi = -\nabla \cdot \nabla \cdot (H \mathbf{T}) + H D_i \delta(\mathbf{x} - \mathbf{x}_i)$ ,  $\mathbf{z} = 0$ ,  $\mathbf{Z} = \mathbf{P} + \rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v})$ ,  
 $k_1 = 0$ ,  $k_2 = 0$  and  $\mathbf{w} = -\rho_0 \mathbf{v} - \rho (\mathbf{u} - \mathbf{v})$ .

In this case, observing that for any differentiable scalar field,  $\alpha(\mathbf{x})$ , and tensor field,  $\mathbf{A}(\mathbf{x})$ , the following relation holds

$$\alpha (\nabla \cdot \nabla \cdot \mathbf{A}) = \nabla \cdot (\alpha \nabla \cdot \mathbf{A}) - \nabla \cdot (\mathbf{A} \nabla \alpha) + \mathbf{A} : (\nabla \nabla \alpha)$$

where the last term denotes the double dot product (inner product) between the tensors  $\mathbf{A}$  and  $\nabla \nabla \alpha$ , and taking into account the homogeneous boundary conditions at infinity, the volume integral can be recast in the following form

(see also [37])

$$\begin{aligned}
\mathcal{I}_V(\mathbf{x}_*, t_*) &= \int_0^\infty \int_{\mathbb{R}^3} \chi G_{SF} dV(\mathbf{x}) dt \\
&= \int_0^\infty \int_{\mathbb{R}^3} [-\nabla \cdot \nabla \cdot (H\mathbf{T}) + D_i \delta(\mathbf{x} - \mathbf{x}_i)] G_{SF} dV(\mathbf{x}) dt \\
&= p'_i(\mathbf{x}_*, t_*) - \int_0^\infty \int_V (\mathbf{T} : \nabla \nabla G_{SF}) dV(\mathbf{x}) dt \quad (\text{A.11})
\end{aligned}$$

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