A non-smooth regularization of a forward-backward parabolic equation*

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Abstract

In this paper we introduce a model describing diffusion of species by a suitable regularization of a "forward-backward" parabolic equation. In particular, we prove existence and uniqueness of solutions, as well as continuous dependence on data, for a system of partial differential equations and inclusion, which may be interpreted, e.g., as evolving equation for physical quantities such as concentration and chemical potential. The model deals with a constant mobility and it is recovered from a possibly non-convex free-energy density. In particular, we render a general viscous regularization via a maximal monotone graph acting on the time derivative of the concentration and presenting a strong coerciveness property.

Key words: diffusion of species, forward-backward parabolic equation, non-smooth regularization, initial-boundary value problem, well-posedness, hysteresis

AMS (MOS) Subject Classification: 35M13, 35D35, 74N25, 74N30

1 Introduction

The model we are introducing may be applied to different situations, dealing with diffusion of different species (located in some domain $\Omega \subseteq \mathbb{R}^n$ and) described in terms of concentration. In the following, moving from the classical approach in thermodynamics which leads to the well-known Cahn–Hilliard equation, we introduce our point of view and make some comments on its thermodynamical consistency. In particular, we are focusing on diffusion in solids and we have in mind, as a possible final application, hydrogen storage in metals.

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Models for species diffusion. In the simplest setting, a mathematical model describing species diffusion in a solid can be obtained by combining the following three ingredients:

1) the *mass-balance law* for the *concentration* u

$$\dot{u} + \operatorname{div} \mathbf{h} = 0; \tag{1.1}$$

2) a linear *constitutive relation* connecting the *flux of diffusant* $\mathbf{h}$ to the gradient of *chemical-potential* μ through a *mobility constant* m (as it is suggested by the Fick law):

$$\mathbf{h} = -m\nabla\mu, \qquad m > 0; \tag{1.2}$$

3) a possibly nonlinear relation between μ and u:

$$\mu = \psi'(u), \tag{1.3}$$

dictated by the derivative ψ' of a *coarse-grained free energy* ψ.

An important property of the free energy to ensure the well-posedness of the resulting model is convexity. However, in physical situations when diffusion is accompanied by *phase separation*, in general ψ must be assumed to be a non-convex function, a typical choice being the *double-well polynomial potential*:

$$\psi(u) = k(u-1)^2u^2, \qquad k > 0. \tag{1.4}$$

This kind of specifications would make the system (1.1)–(1.3) *backward parabolic* (e.g., for (1.4), in the region where $0 < u < 1$), an undesirable feature both from the physical and from the mathematical standpoint.

In order to make the system well-posed and physically sound, the most popular remedy is the so-called *elliptic regularization* of (1.3):

$$\mu = \psi'(u) - \sigma\Delta u, \qquad \sigma > 0, \tag{1.5}$$

which results into the celebrated *Cahn–Hilliard system* [?]. This mathematical model was originally devised in materials science to describe spinodal decomposition, namely, the process by which a binary alloy undergoes phase separation when its temperature is brought below a critical value. However, the same model finds its application in several other fields, spanning from complex fluids [?] to image processing [?] and [?].

A different type of regularization has been considered by Novick-Cohen and Pego [?] and Plotnikov [?], who replace the elliptic term $-\sigma\Delta u$ with a term of the form $\alpha\dot{u}$, which incorporates viscoelastic relaxation effects. The result is the *viscous regularization*:

$$\mu = \psi'(u) + \alpha\dot{u}, \qquad \alpha > 0. \tag{1.6}$$

The discussion in [?] suggests that in the limit as the viscosity parameter α tends to zero hysteretic effects should be observed.

A combination of energetic and viscous regularization, namely,

$$\mu = \psi'(u) - \sigma\Delta u + \alpha\dot{u}, \tag{1.7}$$

which leads to the so-called *viscous Cahn–Hilliard equation*, was derived by Novick-Cohen in [?] and analytically investigated by Elliott and Garcke in [?] and by Elliott and Stuart in [?], with computations being carried out in [?]. Its vanishing-viscosity limit was studied in [?]. We point out that the regularizing terms in (1.7) with their coefficients σ and α actually correspond to the respective quadratic contributions in the free energy functional for the gradient term and in the dissipation potential for the velocity. More sophisticated generalizations of the Cahn–Hilliard system that still incorporate a viscous contribution have been proposed and investigated in [?, ?]. In the same spirit, a similar viscous regularization has been introduced in the paper [?], which deals with phase separation in binary alloys driven by mechanical effects.

The elliptic regularization as a microforce balance. Derivations of diffusion models of Cahn–Hilliard type have been proposed by Gurtin in [?] and by Podio-Guidugli in [?]. We give here a brief account, instrumental for setting up the mathematical problem we study, and we refer to the quoted papers and to the references cited therein for further details.

Central to the aforementioned derivations are the following ingredients:

- a system of microforces, distinct from Newtonian forces, which obey their own balance laws and whose power expenditure is associated to the evolution of the observable fields of interest — in the present case, the concentration u;
- a collection of constitutive specifications, which relate microforces to the actual evolution of the system;
- a dissipation principle, which sifts away thermodynamically inconsistent constitutive specifications.

Within this common framework, the elliptic regularization (1.5) is a particular instance of the balance statement:

$$\operatorname{div} \boldsymbol{\xi} + \pi + \gamma = 0, \tag{1.8}$$

an instance that arises when we adopt the constitutive specifications:

$$\boldsymbol{\xi} = \sigma \nabla u, \qquad \pi = \mu - \psi(u), \qquad \gamma = 0, \tag{1.9}$$

for the vectorial *microstress* $\boldsymbol{\xi}$ and the two scalar-valued fields, π and γ, respectively, the *internal* and the *external microforce*.

Other constitutive specifications may be taken into consideration of course, provided that they are thermodynamically consistent. In this respect, two distinct options are offered in [?] and [?]. In this paper we opt for the former, where thermodynamical consistency is embodied by the dissipation inequality:

$$\dot{\phi} \le (\mu - \pi)\dot{u} + \boldsymbol{\xi} \cdot \nabla \dot{u} - \mathbf{h} \cdot \nabla \mu, \tag{1.10}$$

with ϕ the *free-energy density*.

A standard argument [?] shows that consistency with (1.10) rules out any constitutive dependence of free energy on the time derivative of u; accordingly, one assumes that the free energy and the concentration fields are related through a specification of the form:

$$\phi = \widehat{\phi}(u, \nabla u). \tag{1.11}$$

A consequence of (1.10) and (1.11) is that, on introducing the *equilibrium parts*

$$\boldsymbol{\xi}^{\mathrm{eq}} := \frac{\partial\widehat{\phi}}{\partial\nabla u}(u,\nabla u) \qquad \text{and} \qquad \pi^{\mathrm{eq}} := \mu - \frac{\partial\widehat{\phi}}{\partial u}(u,\nabla u), \tag{1.12}$$

of, respectively, microstress and internal microforce, the test for consistency of a certain constitutive choice with the dissipation inequality (1.10) reduces to verifying that, for whatever process, the *non-equilibrium parts*

$$\boldsymbol{\xi}^{\mathrm{ne}} := \boldsymbol{\xi} - \boldsymbol{\xi}^{\mathrm{eq}}, \qquad \pi^{\mathrm{ne}} := \pi - \pi^{\mathrm{eq}} \tag{1.13}$$

of microstress and internal microforce satisfy, together with the flux of diffusant $\mathbf{h}$, the *reduced dissipation inequality*:

$$0 \le -\pi^{\mathrm{ne}}\dot{u} + \boldsymbol{\xi}^{\mathrm{ne}}\cdot\nabla\dot{u} - \mathbf{h}\cdot\nabla\mu. \tag{1.14}$$

In particular, the constitutive specifications (1.9) follow from (1.12)–(1.13) on taking

$$\widehat{\phi}(u,\nabla u) = \frac{1}{2}a|\nabla u|^2 + \psi(u),$$

and on choosing

$$\boldsymbol{\xi}^{\mathrm{ne}} = \mathbf{0}, \qquad \pi^{\mathrm{ne}} = 0, \tag{1.15}$$

a choice consistent with (1.14).

The viscous and the "non smooth" regularizations. The viscous variant (1.6) is obtained in a similar fashion: first, we exclude microscopic contact interactions by letting $\boldsymbol{\xi} = \mathbf{0}$, and we set to null the external microforce γ, so that the microforce balance (1.8) reduces to

$$\pi = 0. \tag{1.16}$$

Then, consistent with this choice, we rule out the dependence of free energy on concentration gradient by letting

$$\widehat{\phi}(u,\nabla u) = \psi(u), \tag{1.17}$$

so that the second of (1.12) specializes to

$$\pi^{\mathrm{eq}} = \mu - \psi'(u); \tag{1.18}$$

As to the "non-equilibrium part", the simplest constitutive choice is

$$\pi^{\mathrm{ne}} = -\alpha\dot{u}, \qquad \alpha > 0, \tag{1.19}$$

so that, bearing in mind the second of (1.13), we recover (1.6) from (1.16) and (1.18).

Note that, actually, the above relation could be introduced in terms of a "dissipation functional" Φ: the so-called pseudo-potential of dissipation introduced by Moreau [?], which is a non-negative and convex functional, equal to zero for null dissipation. In particular, letting $\Phi(\dot{u}) = \frac{\alpha}{2}|\dot{u}|^2$, the non-equilibrium part could be introduced as

$$\pi^{\mathrm{ne}} = -\frac{\partial\Phi}{\partial\dot{u}}. \tag{1.20}$$

Note in particular that the assumptions on Φ and the relation (1.20) lead to

$$\pi^{\mathrm{ne}}\dot{u} \le 0, \tag{1.21}$$

which is important to ensure thermodynamical consistency. Indeed, if (1.21) and (1.2) hold, then the reduced dissipation inequality (1.14) is satisfied (recall that $\boldsymbol{\xi}^{\mathrm{ne}} = \mathbf{0}$, which follows from $(1.12)_1$, $(1.13)_1$, and (1.17)).

The choice $\Phi = \frac{1}{2}\alpha\dot{u}^2$, which leads to (1.19), is not the only possible. In fact, to model hysteresis, we make another choice, following a suggestion in [?]. Precisely, we replace (1.19) with the following specification:

$$-(\pi^{\mathrm{ne}} + \alpha\dot{u}) \in \beta(\dot{u}), \tag{1.22}$$

where, as before, $\alpha > 0$ is a constant and $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ is a set-valued mapping whose graph is (maximal) monotone and contains the origin:

$$0 \in \beta(0), \tag{1.23}$$

as in the case $\beta = \partial\zeta$ is the subdifferential of a non-negative convex function ζ, with $\zeta(0) = 0$.

As to the assumption on β, it is easily seen that the constitutive specification (1.22) is consistent with the dissipation inequality (1.14): indeed, owing to the monotonicity of β, we have

$$v_1 \in \beta(w_1) \ \&\ v_2 \in \beta(w_2) \quad \Rightarrow \quad (v_1 - v_2)(w_1 - w_2) \ge 0; \tag{1.24}$$

thus, if the pair $(\pi^{\mathrm{ne}}, \dot{u})$ is compliant with (1.22), then we can take $v_1 = -(\pi^{\mathrm{ne}} + \alpha\dot{u})$ and $w_1 = \dot{u}$ as tests in (1.24); meanwhile, (1.23) entitles us to choose $w_2 = v_2 = 0$; with these choices, (1.24) yields $-(\pi^{\mathrm{ne}} + \alpha\dot{u})\dot{u} \ge 0$, which entails (1.21).

The system we investigate. In view of the aforementioned discussion, in order to assemble the system we study, it now suffices for us to combine the balance law for diffusant, the Fick's law and the constitutive specification of the chemical potential. Specifically, we proceed in two steps:

1) we substitute the expression for the flux of diffusant prescribed by Fick's law (1.2) in the balance equation (1.1) that governs the time evolution of concentration. As a result, we obtain the partial differential equation:

$$\dot{u} - m\Delta\mu = 0; \tag{1.25}$$

2) we combine the second of (1.13) with (1.16) and (1.18) to obtain

$$\pi^{\mathrm{ne}} = \psi'(u) - \mu. \tag{1.26}$$

Then, we substitute (1.26) into the inclusion (1.22) to arrive at

$$\mu \in \psi'(u) + \alpha\dot{u} + \beta(\dot{u}), \tag{1.27}$$

where, with slight abuse of notation, we write $p + \beta(q)$ to denote the set $\{r \in \mathbb{R} : r - p \in \beta(q)\}$. Some heuristic discussions, as in [?] (see also the remark enclosing (1.32) below), suggest that the inclusion (1.27) is able to reproduce phenomenologically the hysteretic behavior observed, e.g., in solid-state hydrogen-storage systems, where the chemical potential during adsorption and desorption is not the same (see also [?]).

Of course, in (1.25) one may want to replace the constant mobility m with a function of concentration: for the standard Cahn–Hilliard system this dependence can be handled, even in the degenerate case [?]. However, we prefer to keep our focus on the main novelty of this paper, that is, the non-smooth dependence of μ on $\dot{u}$, which was suggested, but not treated analitically, in [?].

We finally complete the system (1.25)–(1.27) with a specification of the *initial concentration* $u(0)$ (see (1.39c) below) and a time-dependent specification of the chemical potential on the boundary:

$$\mu(\cdot,t) = \mu_\flat(\cdot,t) \quad \text{on } \Gamma := \partial\Omega. \tag{1.28}$$

Dirichlet-type boundary data involving chemical potential are consistent with applications. An example in the context of the mathematical modeling of hydrogen storage [?, ?, ?, ?] is the following: consider that the domain Ω where diffusion takes place represents a body immersed in a gaseous reservoir at uniform pressure p_g and temperature T_g, both of which may be possibly time dependent. The chemical potential of the diffusant in the reservoir is related to its pressure and temperature through the formula $\mu_g = \mu_0 + RT_g \log(p_g/p_0)$ (see for instance [?, Eq. 5.16]), where μ_0 is the *standard chemical potential*, R is the *Avogadro constant*, and p_0 is the *standard pressure* (typically, $p_0 = 1$bar). If local equilibrium prevails, μ is continuous across the boundary Γ, and hence (1.28) holds with $\mu_\flat(x,t) = \mu_g(t)$. Another example is provided by mechanical theories that describe the behaviour of a permeable elastic solid immersed in an incompressible fluid (see for instance [?]); in these theories, the chemical potential of the fluid is given by $\mu_f = \mu_0 + \nu(p_f - p_0)$, where ν is the molar volume of the solvent and p_f is its pressure.

Of course, boundary conditions other than (1.28) could be considered: for example, the control of the flux on the boundary through the difference of chemical potentials on both sides can be prescribed. However, this would not affect our analysis, as far as the coerciveness of the energy estimate with respect to μ is concerned. Moreover, some care is required to guarantee that μ has the spatial smoothness required for our comparison argument to apply (cf. (3.4) and (3.14)).

Energy and dissipation. The precise assumptions we make on the coarse-grained free energy ψ are stated in (2.1) below. Note in particular that we can allow ψ to be nonconvex (a feature that, as already pointed out, allows for phase separation) and that we can include logarithmic type potentials. On the other hand, we are not able to deal, e.g., with subdifferential of indicator functions of closed intervals.

As to the mapping β, whose choice together with that of the constant α affects the dissipative structure of the system, we assume in (2.1g) below that it is the subdifferential of a non-negative, convex, lower semicontinous potential ζ (without setting any restriction on the growth of ζ) with $\zeta(0) = 0$. Besides the trivial case $\beta = 0$, which leads to the PDE considered in [?], other possible choices are:

(i) $\beta(r) = \beta_0 \operatorname{sign}(r)$, where $\beta_0 > 0$ and the sign graph is defined by

$$\operatorname{sign}(r) = \begin{cases} \{+1\} & \text{if } r > 0, \\ [-1, +1] & \text{if } r = 0, \\ \{-1\} & \text{if } r < 0, \end{cases} \tag{1.29}$$

which, as we shall discuss below, *may induce hysteresis.*

(ii) the subdifferential of the indicator function $I_{[a,b]}$ of a closed bounded interval $[a, b]$ (in our computations we require $0 \in [a, b]$)

$$\beta(r) = \partial I_{[a,b]}(r) = \begin{cases} \{0\} & \text{if } r \in (a, b), \\ [0, +\infty) & \text{if } r = b, \\ (-\infty, 0] & \text{if } r = a, \end{cases} \tag{1.30}$$

forcing *the rate of change of concentration* to be bounded in the interval $[a, b]$.

(iii) the subdifferential of the indicator function of $[0, +\infty)$, namely,

$$\beta(r) = \partial I_{[0,+\infty)}(r) = \begin{cases} \{0\} & \text{if } r \in (0, +\infty), \\ (-\infty, 0] & \text{if } r = 0, \end{cases} \tag{1.31}$$

which is a choice particularly interesting, for it entails that the concentration at a given point *cannot decrease*, that is, $\dot{u}$ in (1.27) has to remain non-negative.

Remarks on hysteresis. For the viscous variant of the Cahn–Hilliard system — namely, the system that arises from (1.6) — it is known that hysteresis (in the sense of irreversibility [**?**]) emerges in the *vanishing-viscosity limit*:

$$\alpha \to 0,$$

provided that ψ is a *non-convex function* [**?**, **?**]. It is not hard to construct examples showing that our constitutive assumptions lead to hysteresis as well, *even if ψ is convex.* In order to provide an illustration, we choose β as in (i) above and we take $\psi(u) = \frac{1}{2}ku^2$, with $k > 0$. As a result, the system (1.25)–(1.27) becomes:

$$\partial_t u = m\Delta\mu, \tag{1.32a}$$

$$\mu \in \alpha\, \partial_t u + \beta_0 \operatorname{sign}(\partial_t u) + ku, \tag{1.32b}$$

where ∂_t denotes the (partial) derivative with respect to t. We supplement (1.32a) with a boundary condition of the form

$$\mu(\cdot, t) = f\left(\frac{t}{\tau}\right) \quad \text{on } \Gamma, \tag{1.33}$$

and we investigate the formal limit when the *characteristic time* τ tends to infinity, which corresponds to the regime of a *slowly-varying* chemical potential imposed at the boundary.

On replacing t with the dimensionless variable

$$s := \frac{t}{\tau},$$

and on considering that the β in (i) is *invariant under time reparametrization*, we can rewrite (1.32) as

$$\tau^{-1}\partial_s u = m\Delta\mu, \tag{1.34a}$$
$$\mu \in \tau^{-1}\alpha\,\partial_s u + \beta_0 \;\mathrm{sign}\,(\partial_s u) + ku. \tag{1.34b}$$

Formally, as $\tau \to +\infty$, the parabolic system (1.34) degenerates into the elliptic system:

$$0 = m\Delta\mu, \tag{1.35a}$$
$$\mu \in \beta_0 \;\mathrm{sign}\,(\partial_s u) + ku. \tag{1.35b}$$

Having stipulated with (1.33) that μ is spatially constant on the boundary, the homogeneous elliptic equation (1.35a) entails that μ is spatially uniform in the bulk:

$$\mu(\cdot, s) = f(s) \quad \text{in } \Omega. \tag{1.36}$$

As a consequence, the concentration field satisfies the following differential inclusion:

$$f(s) \in \beta_0 \;\mathrm{sign}\,(\partial_s u) + ku, \tag{1.37}$$

which is known to exhibit hysteresis (actually, it reproduces the well-known *stop operator*, see e.g. [**?**, **?**]).

The initial boundary value problem. We consider the evolution in a smooth domain Ω in the expanse of time $(0,T)$. We suppose that at time $t = 0$ the concentration field be given by a prescribed function $u_0(x)$, $x \in \Omega$. We also suppose that a time-dependent chemical potential $\mu_\flat(x,t)$ be prescribed on for all $x \in \Gamma$ at all times $t \in [0,T]$.

At each particular time, we harmonically extend $\mu_\flat$ to the interior of Ω (still denoting by $\mu_\flat$ the harmonic extension) and we introduce the characteristic time and lengthscale $T_0 = \alpha$ and $L_0 = \sqrt{\frac{m}{\alpha}}$,[†] as well as the new variables and functions:

$$\widetilde{t} = \frac{t}{T_0}, \qquad \widetilde{x} = \frac{x}{L_0}, \qquad \widetilde{\mu} = \mu - \mu_\flat, \qquad \widetilde{\beta}(r) = \beta(r/T_0). \tag{1.38}$$

We express the system in terms of these new variables and *we henceforth drop tildas*, so as to obtain the following problem:

$$\left.\begin{array}{l} \partial_t u = \Delta\mu, \\ \mu = \partial_t u + \xi + \mu_\flat + \psi'(u), \\ \xi \in \beta(\partial_t u) \end{array}\right\} \quad \text{in } \Omega \times (0,T) \tag{1.39a}$$

with the boundary condition

$$\mu = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.39b}$$

and the initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{1.39c}$$

[†]We assume that all energy densities per unit volume are renormalized to a reference value, and so are dimensionless.

We conclude this section with the outline of the paper. In the next section, we introduce notation, assumptions on the data of the problem and state the main existence and uniqueness result, which is complemented by the continuous dependence of the solution with respect to the data of the problem. In Section 3 we proceed by exploiting the a priori estimates on the solutions of the system we use to prove existence and regularity. In Section 4 we show the continuous dependence estimates on solutions. Finally, in Section 5, we provide a detailed proof of the existence and the uniqueness of the solution. To this aim, we first establish our result for a "regularized" version of the system, obtained by a suitable truncation of the coarse-grained free energy mapping; then, owing to the estimates carried out in Section 3, and by making use of a maximum–principle argument, we establish our result for the original problem.

2 Notation, assumptions, and main results

Before stating the problem we are dealing with and the main existence result, let us make precise our assumptions on the data and the notation we use. In the sequel, Ω is a bounded smooth domain in $\mathbb{R}^3$ with smooth boundary Γ. We introduce the spaces

$$H := L^2(\Omega), \qquad V := H_0^1(\Omega), \qquad W := H^2(\Omega) \cap H_0^1(\Omega).$$

We endow H, V, and W with their usual scalar products and norms, and use a self-explanatory notation, like $\|\cdot\|_V$. For the sake of simplicity, the same symbol will be used both for a space and for any power of it (for instance, we use $\|\cdot\|_V$ to denote the norm on V as well as the norm on $V \times V$). We note that the norms $\|v\|_V$ and $\|\nabla v\|_H$ are equivalent for $v \in V$, thanks to the Poincaré inequality. In addition, let us point out that, after identifying H with its dual, the triplet (V, H, V') is a Hilbert triplet (where V' coincides with the Sobolev space $H^{-1}(\Omega)$). Hence, we use the notation $\langle \cdot, \cdot \rangle$ for the duality pairing between V' and V. Given a final time $T > 0$, we set

$$Q := \Omega \times (0, T), \quad \Sigma := \Gamma \times (0, T).$$

As far as the data of the problem are concerned, we assume that the domain of ψ is an open interval $(a, b) \subseteq \mathbb{R}$, where a and b could be taken equal to $-\infty$ and $+\infty$, respectively, and require that

$$\psi \in C^2(a, b), \tag{2.1a}$$
$$\psi(r) \geq 0 \quad \text{for all } \ r \in (a, b), \tag{2.1b}$$
$$\lim_{r \to a^+} \psi'(r) = -\infty, \qquad \lim_{r \to b^-} \psi'(r) = +\infty, \tag{2.1c}$$
$$\psi''(r) \geq -K_1 \quad \text{for all } \ r \in (a, b), \tag{2.1d}$$

for some positive constant K_1; note that in (2.1c) a^+ has to become $-\infty$ if $a = -\infty$ and b^- reduces to $+\infty$ if $b = +\infty$. For the initial concentration u_0 we suppose that

$$u_0 \in H, \quad \exists\, a_0 > a, \ \ b_0 < b \ \text{ such that } \ a_0 \leq u_0(x) \leq b_0 \ \text{ for a.a. } \ x \in \Omega, \tag{2.1e}$$

whence both u_0 and $\psi'(u_0)$ lie in $L^\infty(\Omega)$. Concerning the known datum $\mu_\flat$, we assume that

$$\mu_\flat \in H^1(0, T; H) \cap L^\infty(Q). \tag{2.1f}$$

Finally, as to the (possibly) multi-valued mapping β, we let (see [**?**])

$$\beta = \partial\zeta, \text{ with } \zeta : \mathbb{R} \to [0, +\infty] \textit{ convex} \text{ and } \textit{lower-semicontinuous}, \text{ such that } \zeta(0) = 0. \tag{2.1g}$$

Let us specify a weak formulation of the problem in the set of the Hilbert triplet (V, H, V'). Let us define the operator $A : V \to V'$, corresponding to the "weak realization" of the Laplace operator $-\Delta$ (combined with homogeneous Dirichlet boundary condition) in the duality between V' and V, by letting

$$\langle Av_1, v_2\rangle := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, \mathrm{d}x, \quad v_1, v_2 \in V, \tag{2.2}$$

and specify its inverse $A^{-1} : V' \to V$, such that for $w, z \in V'$ and $v \in V$

$$\langle Av, A^{-1}w\rangle = \langle w, v\rangle, \quad \langle w, A^{-1}z\rangle = \langle z, A^{-1}w\rangle = \int_\Omega \nabla(A^{-1}w) \cdot \nabla(A^{-1}z). \tag{2.3}$$

We introduce a norm in V', denoted by $\|\cdot\|_*$, which is equivalent to the usual one,

$$\|w\|_*^2 = \langle w, A^{-1}w\rangle, \quad w \in V'. \tag{2.4}$$

Definition 1 (Weak solution). *We say that a triplet (u, ξ, μ) is a weak solution to the problem* (1.39) *if*

$$u \in C^1([0,T]; H), \quad \xi \in C^0([0,T]; H), \quad \mu \in C^0([0,T]; V), \tag{2.5a}$$

$$\psi'(u) \in H^1(0,T; H) \tag{2.5b}$$

and the following equations are satisfied:

$$\partial_t u(t) + A\mu(t) = 0 \quad \textit{in } V', \; \textit{for all } t \in [0,T], \tag{2.6a}$$

$$\mu(t) = (\partial_t u + \xi + \mu_\flat + \psi'(u))(t) \quad \textit{a.e. in } \Omega, \; \textit{for all } t \in [0,T], \tag{2.6b}$$

$$\xi(t) \in \beta(\partial_t u(t)) \quad \textit{a.e. in } \Omega, \; \textit{for all } t \in [0,T], \tag{2.6c}$$

$$u(0) = u_0 \quad \textit{a.e. in } \Omega. \tag{2.6d}$$

Theorem 1 (Existence and uniqueness)**.** *Under the assumptions* (2.1)*, there exists a unique solution to the problem* (1.39)*, in the sense of Definition 1. In addition, it results that*

$$u \in L^\infty(Q) \tag{2.7}$$

and the solution is strong in the following sense: $\mu \in C^0([0,T]; W)$ *and equation* (2.6a) *can be replaced by*

$$\partial_t u(t) - \Delta\mu(t) = 0 \quad \textit{a.e. in } \Omega, \; \textit{for all } t \in [0,T]. \tag{2.8}$$

Moreover, a continuous dependence on the data holds: namely, if (u_1, ξ_1, μ_1), (u_2, ξ_2, μ_2) are two solution triplets corresponding to the initial data u_{01}, u_{02} and bulk data $\mu_{\flat 1}$, $\mu_{\flat 2}$, respectively, then their difference satisfies

$$\|u_1 - u_2\|_{C^1([0,T];H)} + \|\xi_1 - \xi_2\|_{C^0([0,T];H)} + \|\mu_1 - \mu_2\|_{C^0([0,T];W)} \leq R\left(\|\mu_{\flat 1} - \mu_{\flat 2}\|_{C^0([0,T];H)} + \|u_{01} - u_{02}\|_H\right) \tag{2.9}$$

for some constant R depending only on the structural assumptions stated in (2.1)*.*

3 Basic estimates

In this section, for the reader's convenience, before proving Theorem 1, we recover the a priori estimates instrumental to prove in Section 4 a continuous dependence result, from which the uniqueness of the solution follows. The same estimates are used to exploit our fixed point argument enabling us to show the existence of a solution in Section 5. Following the standard convention, we use C as a placeholder for a positive constant depending only on the data of the problem.

Energy estimate. We first show that system (2.6) admits in a natural way a so-called "energy estimate". Indeed, once u and μ are solution components for the problem (1.39), they satisfy (2.5a) so that we are allowed to test (2.6a) by μ and (2.6b) by $\partial_t u$. Then, we combine the resulting equations and integrate by parts in time over $(0,t)$; by exploiting Hölder's and Young's inequalities and using the chain rule and the smoothness of ψ, we find that

$$\begin{aligned}
&\int_\Omega \psi(u(t))\,\mathrm{d}x + \int_0^t\int_\Omega (|\partial_t u|^2 + \xi\,\partial_t u + |\nabla\mu|^2)\,\mathrm{d}x\mathrm{d}s \\
&\le \int_\Omega \psi(u_0)\,\mathrm{d}x + \int_0^t\int_\Omega |\mu_\flat\,\partial_t u|\mathrm{d}x\mathrm{d}s, \\
&\le \int_\Omega \psi(u_0)\,\mathrm{d}x + \frac12\int_0^t\int_\Omega |\mu_\flat|^2\mathrm{d}x\mathrm{d}s + \frac12\int_0^t\int_\Omega |\partial_t u|^2\mathrm{d}x\mathrm{d}s. \qquad (3.1)
\end{aligned}$$

Owing to the monotonicity of β and the fact that $0\in\beta(0)$, we have that

$$\int_0^t\int_\Omega \xi\,\partial_t u\,\mathrm{d}x\mathrm{d}s \ge 0. \qquad (3.2)$$

As (2.5a) holds, we also point out that $u(t) = u_0 + \int_0^t \partial_t u(s)\mathrm{d}s$, whence

$$\|u(t)\|_H^2 \le 2\|u_0\|_H^2 + 2T\int_0^t \|\partial_t u(s)\|_H^2\,\mathrm{d}s,$$

thanks to the Hölder inequality. Thus, by virtue of the Poincaré inequality and the nonnegativity of ψ as well, (3.1) and (2.1) yield the estimate

$$\|\psi(u)\|_{L^\infty(0,T;L^1(\Omega))} + \|u\|_{H^1(0,T;H)} + \|\mu\|_{L^2(0,T;V)} \le C. \qquad (3.3)$$

Note that (3.3) entails in particular that u lies between a and b almost everywhere in Q. Hence, recalling that Ω and Γ are smooth enough and that μ has null trace on Γ, by a comparison in (2.6a) and standard elliptic regularity estimates we obtain, in addition, (2.8) and

$$\|\mu\|_{L^2(0,T;W)} \le C. \qquad (3.4)$$

L$^\infty$ estimate for chemical potential. Let us first introduce $\mu_0\in V$ as the unique solution of the nonlinear elliptic problem

$$A\mu_0 + (I+\beta)^{-1}(\mu_0 - \mu_\flat(0) - \psi'(u_0)) = 0, \qquad (3.5)$$

where I denotes the identity operator. Indeed, note that $\mu_\flat(0) + \psi'(u_0) \in H$ by (2.1f), (2.1e) and (2.1a): then, concerning the sum of the two maximal monotone operators A (when restricted to W with values in H) and

$$v \mapsto (I + \beta)^{-1}(v - \mu_\flat(0) - \psi'(u_0))$$

we can apply [?, Cor. 1.3, p. 48], which ensures that the sum is maximal monotone and surjective thanks to the Lipschitz continuity of the second operator and the coerciveness of A. Moreover, the uniqueness of $\mu_0 \in W$ solving (3.5) follows from the strong monotonicity of A. Clearly, from (2.5) and (2.6) we have that $\mu(0) = \mu_0$ and

$$\partial_t u(0) = \Delta\mu_0 =: u_0' \in H \quad \text{and} \quad \xi(0) = \mu_0 - u_0' - \mu_\flat(0) - \psi'(u_0) =: \xi_0 \in H. \tag{3.6}$$

We emphasize that ξ_0 satisfies $\xi_0 \in \beta(u_0')$ almost everywhere in Ω. Next, in order to show the L^∞ estimate for μ, let us fix $t \in (0, T]$ and, for $n \in \mathbb{N}$, set $\tau_n = t/n$, $t_i^n = i\tau_n$, $i = 0, 1, \dots, n$. For typographical convenience, we henceforth omit the dependence of t_i^n on n. In view of (2.5), equations (2.6a) and (2.6b) hold at the times t_i:

$$\partial_t u(t_i) + A\mu(t_i) = 0 \quad \text{in } V', \tag{3.7}$$

$$\mu(t_i) = \partial_t u(t_i) + \xi(t_i) + (\mu_\flat + \psi'(u))(t_i) \quad \text{in } H, \tag{3.8}$$

for $i = 1, \dots, n$. Test (3.7) by $\mu(t_i) - \mu(t_{i-1})$ and the difference of equalities (3.8) at the steps i and $i-1$ by $\partial_t u(t_i)$. Then, by combining the results it is not difficult to check that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla\mu(t_i)|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla\mu(t_{i-1})|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla\mu(t_i) - \nabla\mu(t_{i-1})|^2 \mathrm{d}x \\ &+ \frac{1}{2}\|\partial_t u(t_i)\|_H^2 - \frac{1}{2}\|\partial_t u(t_{i-1})\|_H^2 + \frac{1}{2}\|\partial_t u(t_i) - \partial_t u(t_{i-1})\|_H^2 \\ &+ (\xi(t_i) - \xi(t_{i-1}), \partial_t u(t_i)) + ((\mu_\flat + \psi'(u))(t_i) - (\mu_\flat + \psi'(u))(t_{i-1}), \partial_t u(t_i)) = 0, \end{aligned} \tag{3.9}$$

where $(\cdot,\cdot)$ denotes the scalar product in H. Now, by the properties of the subdifferential $\beta = \partial\zeta$ (which is a maximal monotone graph) it turns out that the inclusion (see (2.6c) and (2.1g))

$$\xi(t_i) \in \partial\zeta(\partial_t u(t_i))$$

can be rewritten as

$$\partial_t u(t_i) \in \partial\zeta^*(\xi(t_i)) \tag{3.10}$$

almost everywhere in Ω, where $\zeta^*(w) := \sup_{v\in\mathbb{R}}(v\,w - \zeta(v))$, $w \in \mathbb{R}$, is the Legendre-Fenchel transform [?] of ζ, and its subdifferential $\partial\zeta^*$ coincides with β^{-1}, the inverse graph of β. Then, using the definition of subdifferential, it is straightforward to infer that

$$(\xi(t_i) - \xi(t_{i-1}), \partial_t u(t_i)) \geq \int_\Omega \zeta^*(\xi(t_i))\mathrm{d}x - \int_\Omega \zeta^*(\xi(t_{i-1}))\mathrm{d}x.$$

Hence, on performing summation in (3.9) for $i = 1, \dots, n$, we plainly deduce that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla\mu(t)|^2 \mathrm{d}x + \frac{1}{2}\|\partial_t u(t)\|_H^2 + \int_\Omega \zeta^*(\xi(t))\mathrm{d}x \\ &+ \sum_{i=1}^n \tau_n \Big(\frac{(\mu_\flat + \psi'(u))(t_i) - (\mu_\flat + \psi'(u))(t_{i-1})}{\tau_n}, \partial_t u(t_i)\Big) \\ &\leq \frac{1}{2}\int_\Omega |\nabla\mu_0|^2 \mathrm{d}x + \frac{1}{2}\|u_0'\|_H^2 + \int_\Omega \zeta^*(\xi_0)\mathrm{d}x. \end{aligned} \tag{3.11}$$

Owing to (2.1f) and (2.5b), it is a standard matter to infer that

$$\sum_{i=1}^{n} \tau_n \Big(\frac{(\mu_\flat + \psi'(u))(t_i) - (\mu_\flat + \psi'(u))(t_{i-1})}{\tau_n}, \partial_t u(t_i) \Big) \to \int_0^t \int_\Omega \partial_t (\mu_\flat + \psi'(u)) \partial_t u \,\mathrm{d}x\mathrm{d}s$$

as $n \to \infty$, and assumptions (2.1d) and (2.1f) easily yield

$$\int_0^t \int_\Omega \partial_t (\mu_\flat + \psi'(u)) \partial_t u \,\mathrm{d}x\mathrm{d}s \geq -\|\partial_t \mu_\flat\|_{L^2(0,T;H)} \|\partial_t u\|_{L^2(0,T;H)} - K_1 \|\partial_t u\|^2_{L^2(0,T;H)} \geq -C,$$

the last inequality being due to the previous estimate (3.3). As $\xi_0 \in \partial\zeta(u_0')$ and consequently $\zeta^*(\xi_0) + \zeta(u_0') = \xi_0\, u_0'$ almost everywhere in Ω, from (2.1g) it follows that

$$\int_\Omega \zeta^*(\xi_0)\mathrm{d}x \leq \|\xi_0\|_H \|u_0'\|_H - \int_\Omega \zeta(u_0')\mathrm{d}x \leq C. \tag{3.12}$$

On the other hand, it is easy to check that ζ^* is non-negative, whence

$$\int_\Omega \zeta^*(\xi(t))\mathrm{d}x \geq 0.$$

Then, passing to the limit as $n \to \infty$ in (3.11) and exploiting the previous remarks, we find out that

$$\|\mu(t)\|_V^2 + \|\partial_t u(t)\|_H^2 \leq C \quad \text{for all } t \in [0,T]. \tag{3.13}$$

At this point, by comparison in (2.6a) and thanks to well-known elliptic regularity results combined with the Sobolev embedding $W \subset L^\infty(\Omega)$, we recover

$$\|\mu\|_{L^\infty(0,T;W)} + \|\mu\|_{L^\infty(Q)} \leq C. \tag{3.14}$$

In particular, because of the L^∞ boundedness of $\mu_\flat$ postulated in Assumption (2.1f), we infer that there exists a constant M such that

$$\|\mu - \mu_\flat\|_{L^\infty(Q)} \leq M. \tag{3.15}$$

$\mathbf{L}^\infty$ estimate for concentration. We combine (2.6b)–(2.6d) to obtain the Cauchy problem in H:

$$\partial_t u(t) = (I + \beta)^{-1}(\mu(t) - \mu_\flat(t) - \psi'(u(t))), \quad t \in [0,T], \tag{3.16a}$$

$$u(0) = u_0. \tag{3.16b}$$

Now, by assumptions (2.1c) and (2.1e) there exist two constants $k_*, k^* \in (a,b)$ such that

$$\psi'(r) \geq M \quad \text{for all } r \geq k^*, \tag{3.17a}$$

$$\psi'(r) \leq -M \quad \text{for all } r \leq k_*, \tag{3.17b}$$

$$k_* \leq a_0 \leq u_0(x) \leq b_0 \leq k^* \quad \text{for a.a. } x \in \Omega. \tag{3.17c}$$

We test (3.16a) by $(u - k^*)^+ = \max\{u - k^*, 0\}$ and $-(u - k_*)^- = \min\{u - k_*, 0\}$, and we integrate over $(0,t)$. We note that $(I+\beta)^{-1}(r)$ has the same sign of r and the right hand side of (3.16a) is nonpositive if $u \geq k^*$ and nonnegative if $u \leq k_*$, thanks to (3.17) and (3.15). Then, after integration by parts in time, it is a standard matter to infer that

$$k_* \leq u \leq k^* \quad \text{a.e. in } Q, \tag{3.18}$$

which entails (2.7).

4 Continuous dependence on the data

Consider a pair of data $\{u_{0i},\ \mu_{\flat i}\}$, $i = 1, 2$, fulfilling (2.1e), (2.1f) and let (u_i, ξ_i, μ_i), $i = 1, 2$, be the corresponding solutions. Then, the triplet $(\bar{u}, \bar{\xi}, \bar{\mu})$, with $\bar{u} := u_1 - u_2$, $\bar{\xi} := \xi_1 - \xi_2$, $\bar{\mu} := \mu_1 - \mu_2$, satisfies (cf. (2.6) and (2.8))

$$\partial_t \bar{u}(t) - \Delta\bar{\mu}(t) = 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in [0, T], \tag{4.1a}$$

$$\bar{\mu}(t) = (\partial_t \bar{u} + \bar{\xi} + \bar{\mu}_\flat + \psi'(u_1) - \psi'(u_2))(t) \quad \text{a.e. in } \Omega, \text{ for all } t \in [0, T], \tag{4.1b}$$

$$\xi_i(t) \in \beta(\partial_t u_i(t)) \quad \text{a.e. in } \Omega, \text{ for all } t \in [0, T],\ i = 1, 2, \tag{4.1c}$$

$$\bar{u}(0) = \bar{u}_0 \quad \text{a.e. in } \Omega, \tag{4.1d}$$

where $\bar{u}_0 = u_{01} - u_{02}$ and $\bar{\mu}_\flat = \bar{\mu}_{\flat 1} - \bar{\mu}_{\flat 2}$.

In view of the regularities in (2.5a), we can test (4.1a) by $\bar{\mu}(t)$, (4.1b) by $\partial_t\bar{u}(t)$ and add the resulting equations. In particular, by virtue of (4.1c) and by the monotonicity of β, we have that $\int_\Omega \bar{\xi}(t)\partial_t\bar{u}(t) \geq 0$ and consequently

$$\int_\Omega |\nabla\bar{\mu}(t)|^2 \,\mathrm{d}x + \int_\Omega |\partial_t\bar{u}(t)|^2 \,\mathrm{d}x \leq \int_\Omega (|\bar{\mu}_\flat(t)| + |\psi'(u_1(t)) - \psi'(u_2(t))|)|\partial_t\bar{u}(t)| \,\mathrm{d}x. \tag{4.2}$$

Now, since ψ is twice continuously differentiable, its derivative is locally Lipschitz-continuous. Moreover, by the estimate (3.18), it turns out that both solutions u_i stay in a bounded interval J. Consequently, we have that $|\psi'(u_1) - \psi'(u_2)| \leq \|\psi''\|_{L^\infty(J)}|\bar{u}|$ and, by Young's and Poincaré's inequalities, we infer

$$\|\bar{\mu}(t)\|_V^2 + \|\partial_t\bar{u}(t)\|_H^2 \leq C\big(\|\bar{\mu}_\flat(t)\|_H^2 + \|\bar{u}(t)\|_H^2\big). \tag{4.3}$$

Given that $\|\bar{u}(t)\|_H^2 \leq C\left(\|\bar{u}_0\|_H^2 + \int_0^t \|\partial_t\bar{u}(s)\|_H^2 \,\mathrm{d}s\right)$, from (4.3) it follows that

$$\begin{aligned} &\|\bar{\mu}(t)\|_V^2 + \|\partial_t\bar{u}(t)\|_H^2 \\ &\leq C\left(\|\bar{\mu}_\flat(t)\|_H^2 + \|\bar{u}_0(t)\|_H^2 + \int_0^t \|\partial_t\bar{u}(s)\|_H^2 \,\mathrm{d}s\right) \quad \text{for all } t \in [0, T]. \end{aligned}$$

Thus, an application of the Gronwall-Bellmann inequality yields

$$\|\bar{\mu}\|_{C^0([0,T];V)} + \|\partial_t\bar{u}\|_{C^0([0,T];H)} \leq C\left(\|\bar{\mu}_\flat\|_{C^0([0,T];H)} + \|\bar{u}_0\|_H\right). \tag{4.4}$$

Then, the analogous estimates for $A\bar{\mu}$ (and consequently for $\bar{\mu}$ in $C^0([0, T]; W)$) and $\bar{\xi}$ in $C^0([0, T]; H)$ follow from (4.4) by a comparison in (4.1a) and (4.1b), which helps us to conclude the proof of (2.9). Of course, (2.9) implies in particular the uniqueness of the solution to the problem (2.6).

5 Existence of solutions

In this section, we give details on the proof of the existence of the solution to our problem. We use a contracting argument. First let us make a truncation of the function ψ, which allows us to exploit the above a priori bounds on the solutions to (2.6). Let k_* and k^*

be two constants fulfilling (3.17). It is not hard to check that, thanks to the assumption (2.1c), there exist constants K_* and K^* such that

$$(k_*, k^*) \subseteq (K_*, K^*) \tag{5.1}$$

and

$$\psi''(K_*) \geq 0, \qquad \psi''(K^*) \geq 0. \tag{5.2}$$

We introduce the following truncation of ψ:

$$\psi_*(r) = \begin{cases} \psi(r) & \text{if } K_* \leq r \leq K^*, \\ \psi(K^*) + \psi'(K^*)(r - K^*) + \frac{1}{2}\psi''(K^*)(r - K^*)^2 & \text{if } r > K^*, \\ \psi(K_*) + \psi'(K_*)(r - K_*) + \frac{1}{2}\psi''(K_*)(r - K_*)^2 & \text{if } r < K_*, \end{cases} \tag{5.3}$$

and we denote by

$$L := \max_{r \in \mathbb{R}} |\psi_*''(r)| < +\infty \tag{5.4}$$

the Lipschitz constant of its derivative ψ_*'. We note on passing that the truncated function ψ_* satisfies the assumptions (2.1a)–(2.1d) with $(a, b) = (-\infty, +\infty)$. In particular, the bound from below(2.1d) holds for ψ_*'' with the same constant $-K_1$ as for ψ''.

Next, we consider the set

$$S := \{v \in C^0([0, T]; H) : v(0) = u_0\} \tag{5.5}$$

and we introduce the map $\mathcal{F} : S \to S$ which to every $v \in S$ associates $u = \mathcal{F}(v)$ defined by

$$u(t) = \mathcal{F}(v)(t) := u_0 + \int_0^t (I + \beta)^{-1} (\mu(s) - \mu_\flat(s) - \psi_*'(v(s)))\, \mathrm{d}s, \quad t \in [0, T], \tag{5.6}$$

where $\mu(t)$ denotes the unique element of V that solves the nonlinear elliptic equation

$$A\mu(t) + (I + \beta)^{-1}(\mu(t) - \mu_\flat(t) - \psi_*'(v(t)) = 0, \quad t \in [0, T]. \tag{5.7}$$

Before proceeding, let us comment on the existence of a unique $\mu \in C^0([0, T]; V)$ satisfying (5.7) for some v fixed in S. First, we recall (2.1f)) and observe that ψ_*' is Lipschitz continuous, so that the function $t \mapsto \mu_\flat(t) - \psi_*'(v(t))$ is continuous from $[0, T]$ to H. Then, for all $t \in [0, T]$ there exists a unique $\mu(t)$ fulfilling (5.7): this can be shown arguing as for (3.5) and using [**?**, Cor. 1.3, p. 48]. Moreover, as $(I + \beta)^{-1}$ is monotone and Lipschitz continuous, it is not difficult to check that $\mu \in C^0([0, T]; V)$. Once μ is found, the function $u = \mathcal{F}(v) \in S$ is completely determined from (5.6).

Eventually, we shall apply a fixed point argument: indeed, we will see that any fixed point for the operator $\mathcal{F}$ turns out to be a solution to the problem made precise by (2.5)–(2.6). To this aim, we are going to show that some power $\mathcal{F}^j$ ($j \in \mathbb{N}$) is a contraction mapping in S and, as a consequence, it admits a unique fixed point, which results at the end to provide the unique solution to our system.

To this aim, we pick a pair $\{v_i\}_{i=1,2} \subset S$ and set

$$u_i := \mathcal{F}(v_i), \quad \xi_i := \mu_i - \mu_\flat - \psi_*'(v_i) - \partial_t u_i,$$

where μ_i is the solution to (5.7) corresponding to v_i, $i = 1, 2$. Then, it is easy to verify that

$$\partial_t u_i(t) + A\mu_i(t) = 0 \quad \text{in } V', \quad \text{for all } t \in [0, T], \tag{5.8a}$$

$$\mu_i(t) = (\partial_t u_i + \xi_i + \mu_\flat + \psi'(v_i))(t) \quad \text{a.e. in } \Omega, \quad \text{for all } t \in [0, T], , \tag{5.8b}$$

$$\xi_i(t) \in \beta(\partial_t u_i(t)) \quad \text{a.e. in } \Omega, \quad \text{for all } t \in [0, T], \tag{5.8c}$$

$$u_i(0) = u_0 \quad \text{a.e. in } \Omega \tag{5.8d}$$

for $i = 1, 2$. Now, we use the notation $\bar{u}$ for the difference of $u_1 - u_2$, and the same notation for $\bar{\xi}$, $\bar{\mu}$ and $\bar{v}$. We take the difference of (5.8a) for $i = 1, 2$, test it by $A^{-1}(\partial_t \bar{u}(t))$ and, at the same time, we test the difference of (5.8b) by $\partial_t \bar{u}(t)$. Then we combine the obtained equalities and use the properties of A^{-1} stated in (2.3) and (2.4). Hence, we have that

$$\|\partial_t \bar{u}(t)\|_*^2 + \|\partial_t \bar{u}(t)\|_H^2 + \int_\Omega \bar{\xi}(t)\partial_t \bar{u}(t) \leq \int_\Omega |\psi_*'(v_1(t)) - \psi_*'(v_2(t))||\partial_t \bar{u}(t)|$$
$$\leq \frac{1}{2}\|\bar{u}_t(t)\|_H^2 + \frac{1}{2}\int_\Omega |\psi_*'(v_1(t)) - \psi_*'(v_2(t))|^2. \tag{5.9}$$

Due to (5.4), we handle the right hand side observing that

$$\int_\Omega |\psi_*'(v_1(t)) - \psi_*'(v_2(t))|^2 \leq L^2 \int_\Omega |\bar{v}(t)|^2. \tag{5.10}$$

In addition, by the monotonicity of β and (5.8c) we deduce that

$$\int_\Omega \bar{\xi}(t)\partial_t \bar{u}(t) \geq 0.$$

Thus, we easily obtain

$$\|\partial_t \bar{u}(t)\|_H \leq L\|\bar{v}(t)\|_H \tag{5.11}$$

and, as both u_1 and u_2 satisfy the same initial condition (5.8d), we can easily infer that

$$\|\mathcal{F}(v_1(t)) - \mathcal{F}(v_2(t))\|_H \leq L\int_0^t \|v_1(s) - v_2(s)\|_H \mathrm{d}s \quad \text{for all } t \in [0, T]. \tag{5.12}$$

This inequality leads to

$$\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{C^0([0,t];H)} \leq Lt\|v_1 - v_2\|_{C^0([0,t];H)} \quad \text{for all } t \in [0, T]. \tag{5.13}$$

An iteration of the argument, due to (5.12) and (5.13), leads to

$$\|\mathcal{F}^2(v_1) - \mathcal{F}^2(v_2)\|_{C^0([0,t];H)} \leq L\int_0^t \|\mathcal{F}(v_1)(s) - \mathcal{F}(v_2)(s)\|_H \mathrm{d}s$$
$$\leq L^2 \int_0^t s\|v_1 - v_2\|_{C^0([0,s];H)}\mathrm{d}s \leq \frac{(Lt)^2}{2}\|v_1 - v_2\|_{C^0([0,t];H)}.$$

By iterating j times, we find $\|\mathcal{F}^j(v_1) - \mathcal{F}^j(v_2)\|_{C^0([0,t];H)} \leq \frac{(Lt)^j}{j!}\|v_1 - v_2\|_{C^0([0,t];H)}$ for all $t \in [0, T]$, whence, in particular,

$$\|\mathcal{F}^j(v_1) - \mathcal{F}^j(v_2)\|_{C^0([0,T];H)} \leq \frac{(LT)^j}{j!}\|v_1 - v_2\|_{C^0([0,T];H)}. \tag{5.14}$$

Thus, for j large enough $\mathcal{F}^j$ turns out to be a contraction mapping from S into itself, as announced; as a consequence, $\mathcal{F}^j$ has a unique fixed point u^*, which is also the unique fixed point for $\mathcal{F}$. In view of (5.8), this fixed point yields the triplet (u^*, ξ^*, μ^*) that solves the problem (2.6) in which ψ is substituted by ψ_*. Of course, for (u^*, ξ^*, μ^*) we can repeat the estimates carried out in Section 3. In particular — and this is the crucial point — we can derive for μ^* the same estimate as (3.15), namely,

$$\|\mu^* - \mu_\flat\|_{L^\infty(Q)} \leq M, \tag{5.15}$$

with the *same value of the constant* M. In fact, if one checks carefully the estimates, one can see that ψ_* appears in (3.1) with the integral of $\psi_*(u_0) \equiv \psi(u_0)$ and in (3.6) with $\psi_*'(u_0) \equiv \psi'(u_0)$ (cf. (2.1e) and (3.17c)), and especially with the constant K_1 in (2.1d) which, as observed at the beginning of this section, can be the same for ψ and ψ_*. In addition, by its very definition the derivative ψ_*' satisfies

$$\psi_*'(r) \geq M \quad \text{for all } r \geq k^*, \tag{5.16a}$$

$$\psi_*'(r) \leq -M \quad \text{for all } r \leq k_*, \tag{5.16b}$$

as ψ' does in (3.17a)–(3.17b). Thus, a repetition of the argument leading to (3.18) yields

$$k_* \leq u^* \leq k^* \quad \text{a.e. in } Q \tag{5.17}$$

and, since $\psi = \psi_*$ in $[k_*, k^*]$, (5.17) entails

$$\psi_*'(u^*) = \psi'(u^*) \quad \text{a.e. in } Q. \tag{5.18}$$

In other words, (u^*, ξ^*, μ^*) is actually a solution to the original problem (2.6) and fulfills (2.7). Moreover, (u^*, ξ^*, μ^*) is the unique solution of (2.6), owing to the continuous dependence property (2.9) proved in Section 4. Finally, recalling the smoothness of Ω and Γ and the homogeneous boundary condition on Γ, by (2.5), (2.6b) and standard elliptic regularity estimates we obtain (2.8) and the regularity $\mu \in C^0([0,T]; W)$ for (u^*, ξ^*, μ^*). Therefore, Theorem 1 is completely proved.

Remark 1. Note that if we assume $u_0 \in H^1(\Omega)$ and $\mu_\flat \in L^2(0,T; H^1(\Omega))$ besides (2.1e) and (2.1f), then we can recover the additional regularity $u \in H^1(0,T; H^1(\Omega))$ for the solution component u. Indeed, it suffices to take formally the gradient of (3.16a) and test it by $\nabla(\partial_t u)$. Thanks to the Lipschitz continuity (with Lipschitz constant 1) of $(I+\beta)^{-1}$ and of ψ_*' (which can replace ψ' as we have seen) with constant L (cf. (5.4)), by the Young inequality we easily infer that

$$\frac{1}{2}\|\nabla(\partial_t u)(t)\|_H^2 \leq \|\nabla(\mu - \mu_\flat)(t)\|_H^2 + L^2\|\nabla u(t)\|_H^2.$$

Now, pointing out that $\mu - \mu_\flat \in L^2(0,T; H^1(\Omega))$ (cf., e.g., (2.6a)), as

$$\|\nabla u(t)\|_H^2 \leq 2\|\nabla u_0\|_H^2 + 2T\int_0^t \|\nabla(\partial_t u)(s)\|_H^2 \, \mathrm{d}s,$$

we can easily apply the Gronwall lemma and find out that $\partial_t u \in L^2(0,T; H^1(\Omega))$.