Almost periodic invariant tori for the NLS on the circle

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Abstract

In this paper we study the existence and linear stability of almost periodic solutions for a NLS equation on the circle with external parameters. Starting from the seminal result of Bourgain in [15] on the quintic NLS, we propose a novel approach allowing to prove in a unified framework the persistence of finite and infinite dimensional invariant tori, which are the support of the desired solutions. The persistence result is given through a rather abstract “counter-term theorem” à la Herman, directly in the original elliptic variables without passing to action-angle ones. Our framework allows us to find “many more” almost periodic solutions with respect to the existing literature and consider also non-translation invariant PDEs.

Keywords: Almost periodic solutions; Nonlinear Schrödinger equation; KAM for PDEs

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1. Introduction and main results

A classical result in KAM theory for PDEs is the existence and stability of quasi-periodic solutions for semilinear Hamiltonian PDEs on the circle with an elliptic fixed point. The first pioneering works on this topic (see [34], [47]) were direct generalizations of the corresponding results on elliptic tori for finite dimensional Hamiltonian systems and dealt with the existence and linear stability of finite dimensional invariant tori which are the support of a quasi-periodic solution. In order to avoid resonances and simplify the inherent small divisor problems, one can work with an \( n \)-parameter family of PDEs and show that for most values of the parameters there exist invariant tori of dimension \( n \). Of course one would prefer to have information on a fixed PDE. Following the strategy of finite dimensional dynamical systems, this is typically done by showing that, due to the presence of the non-linearity, the initial data modulate the frequency and hence can be used as parameters. There is a vast literature on quasi-periodic solutions for PDEs we mention just a few [21,14,24,31,5].

It must be noted that quasi-periodic solutions by construction are not typical (w.r.t. any reasonable measure on the phase space), this is true already in finite dimension where lower dimensional tori clearly have measure zero. A natural step forward is to look for \textit{almost-periodic solutions}, i.e. solutions which are limit (in the uniform topology in time) of quasi-periodic solutions. A very naif approach would be to construct the desired solution by just constructing quasi-periodic solutions supported on invariant tori of dimension \( n \) and then taking the limit \( n \to \infty \). Unfortunately the classical KAM procedure (of say [42], [34]) is not uniform in the dimension \( n \), and by taking the limit one just falls on the elliptic fixed point.

A refined version of this very natural idea was in fact used by Pöschel in [44], to prove the existence of almost-periodic solutions with very high regularity, i.e. with Fourier coefficients \( u_j \) which decay in a super-exponential way as \( j \to \infty \). His idea was to construct a sequence of invariant tori of growing dimension using at each step the invariant torus of the previous one as an unperturbed solution: the KAM procedure being not uniform in the dimension \( n \), the \( n+1 \)'th and \( n \)'th tori are extremely close, this leading to very regular solutions.

The model of Pöschel [44] is an NLS equation on the circle with a multiplicative potential and smoothing non-linearity. The potential is not fixed a priori, but gives an infinite set of \textit{external parameters}, which are used to tune the frequencies and control the small divisors. Of course this means that the existence of almost-periodic solutions is proved for \textit{most} choices of the potential. The fact such potential can be used to modulate the frequencies, comes from spectral results and from the fact that the non-linearity is smoothing.

Recently Geng and Xu in [30] proved existence of analytic almost periodic solutions for the NLS equation with external parameters given by Fourier multipliers, without assuming any smoothing assumptions on the nonlinearity. Their approach generalizes the one of [44] by applying the ideas of Töplitz-Lipschitz functions which give a better control on the asymptotics of the frequencies.

A different approach was proposed by Bourgain in [15] to study a translation invariant NLS equation with a Fourier multiplier providing external parameters in \( \mathbb{L}^\infty \). The main result is to prove -for most values of the parameters-existence and linear stability of almost-periodic solutions with \textit{Gevrey} regularity. The idea is to construct a converging sequence of \( \infty \)-dimensional approximately invariant manifolds and prove that the limit is the support of the desired almost-periodic solution. The fact that one does not restrict to neighborhoods of finite dimensional tori allows a better control of the small-divisors and hence the construction of more general \textit{i.e. less regular solutions}.

Bourgain points out that a weak point of the construction through finite dimensional tori comes from using the action-angle coordinates, i.e. the symplectic coordinates adapted to the finite dimensional approximately invariant tori. This is due to the fact that action-angle coordinates in the \( \infty \)-dimensional context are not, in general, well defined; then the idea is to work directly in the Fourier basis and exploit the properties of functions analytic in a neighborhood of zero. We mention also [19], where the authors discuss non-linear stability of the invariant manifolds studied in [15].

In the present paper we take the same point of view as Bourgain. On the other hand our analysis extends Bourgain’s result in many ways.

We introduce a new functional setting which allows us to construct both finite and infinite dimensional tori by a KAM algorithm which is \textit{uniform} in the dimension. In particular our only restriction on the actions is that they live in a ball around the origin, whereas those of [15] belong to a set of zero measure. Moreover we discuss whether (and with respect to which topology) our invariant manifolds are embedding of an \( \infty \)-torus and also whether one can define action angle variables close to it. Finally we are able to consider also non translation invariant NLS equations.

The theory of quasi-periodic solutions for PDEs is now rather developed and considers a much wider class of equa-
tions: without external parameters, on more general domains (see [7]) and with unbounded non-linearities. Regarding this last point in recent years a new approach, based on pseudo-differential calculus, has been developed, see [6,16]. We believe that this approach should allow some substantially new achievements also for almost-periodic solutions, see [20].

Of course proving the existence of full-dimensional tori in a neighborhood of zero is a first step in the study of close-to-integrable PDEs. As far as we know all the literature in this direction is restricted to proving the existence of finite dimensional tori close to finite gap solutions, see [11].

Before describing further our results, let us briefly discuss our model.

Following Bourgain in [15] we consider families of NLS equations on the circle with external parameters of the form:

$$i u_t + u_{xx} - V * u + f(x, |u|^2) u = 0. \tag{1.1}$$

Here $i = \sqrt{-1}$, $u = \sum_{j \in \mathbb{Z}} u_j e^{ijx}$, $V *$ is a Fourier multiplier

$$V * u = \sum_{j \in \mathbb{Z}} V_j u_j e^{ijx}, \quad (V_j)_{j \in \mathbb{Z}} \in \ell^\infty \tag{1.2}$$

and $f(x, y)$ is $2\pi$ periodic and real analytic in $x$ and is real analytic in $y$ in a neighborhood of $y = 0$. We shall assume that $f(x, 0) = 0$. By analyticity, for some $a, R > 0$ we have

$$f(x, y) = \sum_{d=1}^\infty f^{(d)}(x)y^d, \quad |f|_{a, R} := \sum_{d=1}^\infty |f^{(d)}(x)|_{T^d} R^d < \infty, \tag{1.3}$$

where, given a real analytic function $g(x) = \sum_{j \in \mathbb{Z}} g_j e^{ijx}$, we set $|g|_{T^a} := \sup_{j \in \mathbb{Z}} |g_j| e^{ajl}$. It is well known that (1.1) is a Hamiltonian system with Hamiltonian

$$H_{\text{NLS}}(u) := \int_T |u_x|^2 + V * |u|^2 + F(x, |u(x)|^2) dx, \quad F(x, y) := \int_0^y f(x, s) ds, \tag{1.4}$$

w.r.t. the symplectic form $\Omega(u, v) = \text{Im} \int_T u \bar{v}$ induced by the Hermitian product on $L^2(T, \mathbb{C})$.

Note that Equation (1.1) with $f(x, |u|^2) = |u|^4$ is the model considered in [15].

Passing to the Fourier side, i.e. setting

$$u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx}, \quad (u_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}),$$

$H_{\text{NLS}}$ in (1.4) is an infinite dimensional Hamiltonian System consisting in an infinite chain of harmonic oscillators, with linear frequencies $\lambda_j = j^2 + V_j$, coupled by a non-linearity.

If we ignore the non-linearity all the bounded solutions are of the form

$$u_{\text{Lin}}(x, t) = \sum_{j \in \mathbb{Z}} u_j(0) e^{i(jx + \lambda_j t)}, \quad \lambda_j = j^2 + V_j$$

hence –for most values of $V$– they are periodic, quasi-periodic or almost-periodic1 accordingly whether one, finitely many or infinitely many modes $u_j(0)$ are excited.

It is natural to ask if these solutions persist when the nonlinearity is plugged in. In this direction, for the translation invariant NLS, Bourgain in [15] proves that2:

Given any $s > 0$, $0 < \theta < 1$, for “most choices” of $V = (V_j)_{j \in \mathbb{Z}} \in \ell^\infty$ there exist almost periodic solutions $u(x, t)$ such that:

---

1 We recall that an almost-periodic function is the uniform limit of quasi-periodic ones.
2 Actually Bourgain proves (1.5) with $\theta = 1/2$ the extension for $0 < \theta < 1$ was given later in [19].
\[
\frac{r}{2} |(j)|^{-2} e^{-s(j)\theta} \leq |u_j(t)| \leq r |(j)|^{-2} e^{-s(j)\theta} \quad \forall j \in \mathbb{Z}, \quad \langle j \rangle := \max \{|j|, 1\},
\]

for some small enough \( r > 0 \) and for all times.

Informally speaking in this paper we remove the lower bound in (1.5) and generalize the result to non translation invariant equation (1.1).

In order to formulate our result in a more precise way, let us introduce some functional setting. We start by passing to the Fourier side and identifying \( u(x) \) with the sequence of Fourier coefficients \( (u_j)_{j \in \mathbb{Z}} \). We work on \( \ell^2(\mathbb{C}) \) with the standard real symplectic structure coming from the Hermitian product.\(^3\) Then (1.4) becomes

\[
H_{\text{NLS}}(u) := \sum_{j \in \mathbb{Z}} (j^2 + V_j) |u_j|^2 + P, \quad P := \frac{1}{T} \int_T F(x, |\sum_j u_j e^{ixj}|^2) dx.
\]

Fixing once and for all \( 0 < \theta < 1 \), for \( s > 0, a \geq 0, p > 1 \), we consider the following scale of Banach spaces of sequences on \( \mathbb{C} \)

\[
\mathfrak{w}^\infty_{p,s,a} := \left\{ v := (v_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}) : |v|_{p,s,a} := \sup_{j \in \mathbb{Z}} |v_j| \langle j \rangle^p e^{|a| + s(j)\theta} < \infty \right\}.
\]

We endow \( \mathfrak{w}^\infty_{p,s,a} \subset \ell^2 \) with the symplectic structure inherited from \( \ell^2 \).

Our aim is to prove the existence of a symplectic change of variables, well defined and analytic in some open ball \( B_r(\mathfrak{w}^\infty_{p,s,a}) \subset \mathfrak{w}^\infty_{p,s,a} \) centered at the origin and with radius \( r > 0 \), which conjugates \( H_{\text{NLS}} \) to a Hamiltonian which has an invariant torus which supports an almost-periodic solution of Diophantine frequency. Following [15], we fix the hypercube

\[
Q := \left\{ \omega = (\omega_j)_{j \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}, \quad \sup_j |\omega_j - j^2| \leq 1/2 \right\},
\]

and define infinite diophantine frequencies \( \omega \) as follows.

**Definition 1.1.** Given \( 0 < \gamma < 1 \), we denote by \( \mathcal{D}_\gamma \) the set of \( \gamma \)-Diophantine frequencies

\[
\mathcal{D}_\gamma := \left\{ \omega \in Q : |\omega \cdot \ell| > \gamma \prod_{n \in \mathbb{Z}} \frac{1}{(1 + |\epsilon_n|^2 n^2)} \right\}, \quad \forall \ell \in \mathbb{Z}^\mathbb{Z}, \quad |\ell| := \sum_i |\ell_i| < \infty.
\]

The strategy in this definition is to give very strong non-resonance conditions, while still obtaining a positive measure set of frequencies, see section 3.1 for a more detailed discussion.

Given a sequence

\[
I = (I_j)_{j \in \mathbb{Z}}, \quad I_j \geq 0, \quad \sqrt{I} := \left( \sqrt{I_j} \right)_{j \in \mathbb{Z}} \in \mathfrak{w}^\infty_{p,s,a},
\]

we consider the torus

\[
\mathcal{T}_I := \{ u \in \mathfrak{w}^\infty_{p,s,a} : |u_j|^2 = I_j, \quad \forall j \in \mathbb{Z} \}.
\]

We say that \( \mathcal{T}_I \) is a KAM torus of frequency \( \omega \) for the Hamiltonian \( N \) if

\[
N = \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2 + P,
\]

where the Hamiltonian vector field \( X_P \) vanishes on the torus \( \mathcal{T}_I \). Indeed under the hypotheses above \( \mathcal{T}_I \) is an invariant torus for the dynamics of \( N \). Moreover the restricted dynamics is linear with frequency \( \omega \), namely

\[
\langle u, v \rangle = 2 \text{Re}(u, v), \quad \omega(u, v) = 2 \text{Im}(u, v).
\]

---

\(^3\) We recall that given a complex Hilbert space \( H \) with a Hermitian product \( (\cdot, \cdot) \), its realification is a real symplectic Hilbert space with scalar product and symplectic form given by

\[
\langle u, v \rangle = 2 \text{Re}(u, v), \quad \omega(u, v) = 2 \text{Im}(u, v).
\]
\[ u_j(t) = u_j(0)e^{i\omega_j t}, \quad |u_j(0)|^2 = I_j, \quad j \in \mathbb{Z}. \] (1.12)

We are now ready to state our result.

**Theorem 1.** For any \( p > 1, s > 0, 0 \leq a < a, \gamma > 0 \) there exists \( \varepsilon_* = \varepsilon_*(p, s, a - a) > 0 \) such that for all \( r > 0 \) satisfying
\[ \frac{|f|_{a, R}}{\gamma R^r} \leq \varepsilon_* \] (1.13)

the following holds. For all \( \omega \in \mathbb{D}_\gamma \) and \( \sqrt{T} \in \tilde{B}_r(\mathbb{w}_p^\omega) \) as in (1.10) there exists a potential \( V \in \ell^\infty \) and a symplectic analytic change of variables \( \Phi : \tilde{B}_r(\mathbb{w}_p^\omega) \rightarrow \tilde{B}_r(\mathbb{w}_p^\omega) \) such that \( T_I \) is a KAM torus of frequency \( \omega \) for \( H_{NLS} \circ \Phi \). Finally \( V \) depends on \( \omega \) in a Lipschitz way.

Of course the change of variables \( \Phi \) is invertible in the sense that there exists \( \Psi : \tilde{B}_r(\mathbb{w}_p^\omega) \rightarrow \tilde{B}_r(\mathbb{w}_p^\omega) \) such that \( \Psi \circ \Phi u = \Phi \circ \Psi u = u \), for all \( u \) in some smaller ball. We denote as is habitual \( \Phi^{-1} := \Psi \).

In order to have a well defined change of variables \( \Phi \) from the phase space \( \mathbb{w}_p^\omega \) in itself we need \( s > 0, \theta > 0 \) and \( p > 1 \). The condition \( p > 1 \) is needed in order to have the algebra property, see Proposition 2.1, while \( s > 0 \) and \( \theta > 0 \) is needed in order to bound the solution of the homological equation, see Lemma 3.1. In particular \( \varepsilon_* \) shrinks to zero as \( \theta \) approaches zero (and one) in our estimates. As a consequence we construct solutions that are at least Gevrey. Of course, this does not prevent to find more regular solutions (for example analytic). Indeed finding very regular almost-periodic solutions is simpler since they are very close to finite dimensional tori. On the other hand, no results concerning almost periodic solutions were known in Sobolev category so far. However, in the forthcoming work [13], by taking full advantage of our strategy we make a step forward in this direction, by constructing (non maximal) infinite dimensional Sobolev tori.

If \( I_j > 0 \) for infinitely many \( j \in \mathbb{Z} \) then \( \Phi(T_I) \) supports truly almost-periodic solutions.\(^4\) Anyway, if \( \rho := \inf_{j \in \mathbb{Z}} I_j(j)^{2p} e^{2a|j| + 2x(j)^6} > 0 \) then we are in the framework (1.5). Moreover, as we show in Appendix B, in this setting we can introduce action-angle variables around the torus \( T_I \). In this last case, the action-angle map is a diffeomorphism from a neighborhood of \( T_I \) into \( |J - I|_{2p, 2s, 2a} < \rho/2 | \times \mathbb{T}^Z \), where \( \mathbb{T} := \mathbb{R}/2\pi \mathbb{Z} \), and \( \mathbb{T}^Z \) is a differential manifold modeled on \( \ell^\infty \), hence \( \Phi(T_I) \) is an embedded invariant torus.

In order to compare our result with the one of [15] we remark that the condition (1.5) places strong restriction on the action space. Indeed the set of \( u \in \tilde{B}_r(\mathbb{w}_p^\omega) \) such that \( |u_j|^2 \) satisfies (1.5) has zero measure with respect to the natural product measure in a weighted \( \ell^\infty \) space. We remark (see Appendix B) that the set
\[
\left\{ u \in \mathbb{w}_p^\omega : \inf_j \frac{\inf_{|j| < s} \text{inf}_{|j| < s} |u_j|^2}{|u_j|^2} \right\}
\] (1.14)
is topologically generic in the sense of Baire, but still has measure zero. Our result instead holds for all actions \( \sqrt{T} \in \tilde{B}_r(\mathbb{w}_p^\omega) \).

Another interesting advantage of this result is that it is independent from the dimension of the invariant torus (i.e. how many actions are non-zero), and one can prove within the same unified scheme both the persistence of quasi-periodic (if only a finite number of \( I_j \neq 0 \)) and almost-periodic solutions which is independent of the dimension. Of course if we follow Theorem 1 directly, it seems that one needs to modulate infinitely many parameters and require an infinite dimensional diophantine condition even to construct finite dimensional tori. In fact this is not true: as expected we only need to modulate as many parameters as the “non-zero actions”.

Let us fix any non empty set \( S \subseteq \mathbb{Z} \) and denote correspondingly \( S^c \) its complementary. Now we wish to prove the persistence of a KAM torus \( |u_j|^2 = I_j \), with \( I_j = 0 \) for \( j \in S^c \). As in (1.8) we set

\(^4\) Indeed, the map \( t \mapsto u(t) \) defined in (1.12) is almost periodic from \( \mathbb{R} \) to the phase space \( \mathbb{w}_p^\omega \) with \( p' < p \).
\[ Q_S := \left\{ \alpha = (\alpha_j)_{j \in S} \in \mathbb{R}^S, \quad \sup \limits_j |\alpha_j - j^2| \leq 1/2 \right\}. \] (1.15)

The main point of the next theorem is that we can fix the Fourier modes of \( V_j, \ j \in S^c \) in any way (in order to highlight this, we shall denote them by \( W_j \)). In order to keep the proof as simple as possible we shall avoid technical issues related to double eigenvalues by assuming that the NLS Hamiltonian (1.6) is translation invariant, namely that \( f(x, |u|^2) \) does not depend on \( x \). We refer the reader to [24] and [39] for a discussion of double eigenvalues.

**Theorem 2.** Consider a translation invariant NLS Hamiltonian as in (1.6). For any \( p > 1, s > 0, 0 \leq a < \alpha, \gamma > 0 \) there exists \( \varepsilon_* = \varepsilon_0(p, s, \alpha - a) > 0 \) such that:

- for all \( r > 0 \) satisfying (1.13), for every \( \sqrt[I]{T} \in B_r(\omega^\infty_{p,s,a}) \) with \( I_j = 0 \) for \( j \in S^c \) and for any \( |W_j| \leq 1/4 \) with \( j \in S^c \), such that if \( 0 \in S^c \) then \( W_0 \neq 0 \), the following holds.
- There exist a positive measure Cantor-like set \( \mathcal{C} \subset Q_S \), such that for all \( \alpha \in \mathcal{C} \) there exists a potential \( V \in \mathcal{C}_\infty \) and a change of variables \( \Phi : B_{2r}(\omega^\infty_{p,s,a}) \to B_{4r}(\omega^\infty_{p,s,a}) \) such that \( T_1 \) is an elliptic KAM torus of frequency \( \alpha \) for \( H_{\text{NLS}} \circ \Phi \) and \( V_j = W_j \) for \( j \in S^c \). Finally \( V \) depends on \( \alpha \) in a Lipschitz way.

The set \( \mathcal{C} \) is explicitly described in (7.12). In section 7 we also show how this set is related to “second order Melnikov conditions”, see Lemma 7.2. In Remark 7.2 we discuss the relation between our result and known results about quasi-periodic solutions (finite \( S \)) from a technical point of view.

The general strategy of our paper is to rephrase Theorems 1, 2 as a counterterm problem and look for the change of variables \( \Phi \) by performing an iterative KAM scheme.

1.1. A normal form theorem à la Herman

In finite dimension, the first generalization of KAM theory to degenerate systems is due to J. Moser in [38]. The main point is the introduction in the perturbed system of some extra parameters (or counter-terms), in order to compensate its eventual degeneracies, such as absence of twist properties or Hamiltonian symmetries that usually make the general KAM-scheme work.

Starting from the 80’s many authors exploited this idea to derive KAM-type results through the elimination of parameters see for instance [32,45,18,10,25,23,36,17] and, for quasi-periodic solutions in PDEs [3,2,1,4].

It is then natural to extend this approach also to deal with infinitely many parameters and derive the existence of an almost periodic torus in two separated steps:

- prove a normal form which does not rely on any non-degeneracy condition (but containing the hard analysis)
- show that the counter-terms can be eliminated by using internal or external parameters and convenient non-degeneracy assumptions (twist condition, symmetries...) satisfied by the system under analysis, through the application of the usual implicit function theorem: if the extra corrections vanish, then the perturbed systems under normal form possess an invariant almost-periodic torus.

Theorems 1, 2 will then be proved through a direct application to the NLS Hamiltonian of normal form theorems in the spirit of “Herman’s twist theorem” for degenerate Hamiltonians [33], and the elimination of the counter terms through the (\( V_j \)).

While in finite dimension, this kind of results can be stated in a very synthetic and somehow more conceptual way (see for instance, [25,29,37] and references therein), in infinite dimensions one must be more precise regarding quantitative aspects of the functional spaces involved.

In order to motivate our strategy let us briefly describe the classical approach in the finite dimensional case to prove the persistence of an invariant torus \( T_j \). As is to be expected, the simplest scenario is the Lagrangian case, i.e. \( I_j > 0 \) for all \( j = 1, \ldots, d \) (where \( d \) is the dimension of the configuration space). In this context, the simplest approach is to write \( H \) in action-angle variables centered at \( I \), i.e. set

\[
\chi : (y, \theta) \to u(I, y, \theta), \quad u_j = \sqrt{I_j} + y_j e^{i\theta_j}, \quad |y_j| < I_j.
\] (1.16)
so that $H(y, \theta)$ is analytic in $y$ and we can Taylor expand it at $y = 0$. Then the KAM scheme corresponds to looking for a change of variables - analytic for $|y|$ small enough - which cancels the homogeneous terms up to the order one in $y$ in the Hamiltonian (namely those terms, depending on $\theta$, which prevent $T_I$ to be invariant and quasi-periodic).

Then it is very natural to make the ansatz that the change of variables above should be affine in $y$, see [43] for example. Since the action angle coordinates introduce a singularity at $y = -I$, the KAM schemes in action-angle variables produce a change of variables defined and analytic in some neighborhood of $T_I$ (essentially an annulus). In the case of lower dimensional tori, one adapts the scheme above see for instance [41]. Note that in this context also the dynamics normal to the torus comes into play. To bound the resulting small divisors, one requires the so called “Melnikov conditions” which control interactions between tangential and normal modes.

It is worthwhile to mention that, in the case of a dynamical system with an elliptic fixed point at zero, one can also work directly in the natural elliptic variables $(p, q)$ (or in complex notation $u = p + iq$), see [17,46,22] and conjugate a Hamiltonian with external parameters to normal form by a change of variables which is analytic in a neighborhood of zero, just like in Theorem 1.

We stress that in this scheme, by its very nature, there is no need to specify whether $T_I$ is maximal or not.

Trying to reproduce the approach described above in the infinite dimensional setting is not at all straightforward, except in the case of finite dimensional elliptic tori.

First of all, introducing infinite dimensional action-angles variables is much more delicate: whether the map $(\theta, y) \mapsto u$ defines a diffeomorphisms in the neighborhood of $\mathbb{T}^\infty \times \mathbb{R}^\infty$ is strongly related to the chosen topology endowing the spaces involved. In Appendix B we show that, if we work in a weighted $\ell^\infty$ space, then they can be defined at least if $I$ satisfies some appropriate conditions. Nonetheless, even in this simplest setting we are not able to perform the KAM scheme in action-angle variables. The main reason relying on the fact that we are not able to prove the analyticity of the solution of the homological equation $L_\omega F = G$, if $G$ is defined only in an annulus around $T_I$.

It is then preferable to work directly in the elliptic variables $u$, where we can control the solution of the homological equation, provided that $G$ is analytic in a ball at $u = 0$, see Lemma 3.1.

The whole problem then amounts to understanding which terms need to be canceled in order to have an invariant almost-periodic torus. The key difference with respect to the action-angle approach is that these terms must be analytic in a ball centered at the origin. In order to guarantee these two conditions at the same time, we perform a decomposition of the Hamiltonian following the idea of Bourgain in [15], but constructing a more natural functional setting, that we are going to describe.

We shall consider Hamiltonians of the form

$$\sum_{j \in \mathbb{Z}} \omega_j |u_j|^2 + P, \quad \omega = (\omega_j)_{j \in \mathbb{Z}} \in \mathbb{D},$$

where $P$ is a regular Hamiltonian, namely the Cauchy majorant of $P$ is an analytic function from some ball $\tilde{B}_r(w^\infty_{p,s,a})$ to $\mathbb{R}$, whose Hamiltonian vector field $X_P$ is again a bounded analytic function from $\tilde{B}_r(w^\infty_{p,s,a})$ to $w^\infty_{p,s,a}$, (see Definition 2.2). We denote the space above by $\mathcal{H}_{r,s,\eta}$ and we endow it with the norm $| \cdot |_{r,s,\eta}$ (here the parameter $\eta$ is technical, used to control terms in the Hamiltonian which do not preserve momentum). In order to control the dependence of a Hamiltonian on $\omega \in \mathbb{D}$, we introduce weighted Lipschitz norms which we denote by $\| \cdot \|_{r,s,\eta}$ (see (2.24)).

By the very definition of a KAM torus, we wish to decompose regular Hamiltonians as a sum of regular terms with an increasing “order of zero” at $T_I$, $\sqrt{T} \in \tilde{B}_{r'}(w^\infty_{p,s,a})$ with $0 < r' < r$. To this purpose we consider the direct sum decomposition

$$\mathcal{H}_{r,s,\eta} = \mathcal{H}_{r,s,\eta}^{(-2)} \oplus \mathcal{H}_{r,s,\eta}^{(0)} \oplus \mathcal{H}_{r,s,\eta}^{(2)}, \quad H = H^{(-2)} + H^{(0)} + H^{(2)}, \quad (1.17)$$

so that $H \in \mathcal{H}_{r,s,\eta}^{(0)}$ vanishes at $T_I$ (however $X_H$ is tangent but not necessarily null), while $\mathcal{H}_{r,s,\eta}^{(2)}$ has a zero of order at least 2 at $T_I$ (this means that the corresponding vector field vanishes at $T_I$). A crucial property for the convergence of the KAM scheme is the behavior of such decomposition with respect to Poisson brackets, namely $\{ F, G^{(2)} \}^{(-2)} = 0$ and, if $F^{(-2)} = 0$, also $\{ F, G^{(2)} \}^{(0)} = 0$ (see Lemma 4.5).

---

5 Actually (1.17) holds at any even order $d \geq -2$, namely $\mathcal{H}_{r,s,\eta} = \mathcal{H}_{r,s,\eta}^{(-2)} \oplus \cdots \oplus \mathcal{H}_{r,s,\eta}^{(d)} \oplus \mathcal{H}_{r,s,\eta}^{(d+2)}$, see Proposition 4.1.
Let us roughly discuss the decomposition (1.17). The main idea is to make a power series expansion centered at $I$ without introducing a singularity. Start from a regular Hamiltonian $H(u)$ expanded in Taylor series at $u = 0$ and rewrite every monomial $u^a \bar{u}^b$ as $|u|^{2m} u^a \bar{u}^b$ with $\alpha, \beta$ with distinct support. Then define an auxiliary Hamiltonian $\tilde{H}(u, w)$ (here $w = (w_j)_{j \in \mathbb{Z}}$ are auxiliary “action” variable) by the substitution $|u|^{2m} u^a \bar{u}^b \sim w^m u^a \bar{u}^b$ (see (4.16) below). Since we are considering functions on a product space, it turns out that $\tilde{H}(u, w)$ is analytic in both $u$ and $w$. In particular we can Taylor expand with respect to $w$ at the point $w = I$, being $I$ in the domain of analyticity. Then we set $H^{(-2)} := H(u, I)$, $H^{(0)} := \tilde{H}(u, I) |u|^{2} - I$ and $H^{(\geq 2)}$ is what is left. As an example the Hamiltonian

$$H = |u|^2 |u_2|^4 \text{Re}(u_1 \bar{u}_3) = (|u_1|^2 - I_1)(|u_2|^2 - I_2 + I_2)^2 \text{Re}(u_1 \bar{u}_3),$$

has auxiliary Hamiltonian $\tilde{H}(u, w) = w_1 w_2^2 \text{Re}(u_1 \bar{u}_3)$ and decomposes as

$$H^{(-2)} := I_1 I_2^2 \text{Re}(u_1 \bar{u}_3), \quad H^{(0)} := \left[I_2^2 (|u_1|^2 - I_1) + 2I_1 I_2 (|u_2|^2 - I_2)\right] \text{Re}(u_1 \bar{u}_3)$$

$$H^{(\geq 2)} := \left(|u_1|^2 (|u_2|^2 - I_2) + 2I_2 (|u_1|^2 - I_1)\right) \text{Re}(u_1 \bar{u}_3).$$

The above decomposition is, at a formal level, the same introduced by Bourgain in [15]; the main novelty here is that we introduce a suitable Banach space, namely $\mathcal{H}_{r,s,\eta}$, such that the decomposition holds in the same space with no loss of regularity, see (1.17), while the analogous is not true for the norm used in [15], where a loss of regularity in the Grevy parameter $s$ is necessary. An important point is that all our construction works independently of the “dimension” of $T_I$, namely it never requires conditions of the form $I_j \neq 0$.

In view of the above decomposition we give the following

**Definition 1.2 (Normal forms).** We say that a Hamiltonian $N \in D_\omega + \mathcal{H}_{r,s,\eta}$ is in normal form at $T_I$ if $N - D_\omega \in \mathcal{H}_{r,s,\eta}^{(\geq 2)}$ and denote the (affine) subspace of normal forms by $\mathcal{N}_{r,s,\eta}(\omega, I) \equiv \mathcal{N}_{r,s,\eta}$.

Of course if $N$ is in normal form at $T_I$ then the torus is invariant and the $N$-flow is linear on $T_I$, with frequency $\omega$. In our finite dimensional example we could pass the Hamiltonian to action-angle variables, see (1.16)

$$\tilde{H} = H \circ \chi = (I_1 + y_1)(I_2 + y_2)^2 \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3).$$

Then the terms canceled in a classical KAM scheme would be the first two terms in the Taylor expansion at $y = 0$, namely

$$\tilde{H}_0 = I_1^3 I_2^2 \sqrt{I_3} \cos(\theta_1 - \theta_3), \quad \tilde{H}_1 = \left(\frac{3}{2} I_2 \sqrt{I_3} y_1 + 2I_1 \sqrt{I_3} y_2 + I_1 I_2 \frac{2}{\sqrt{I_3}} y_3\right) I_2 \sqrt{I_1} \cos(\theta_1 - \theta_3).$$

Direct computations show that

$$H^{(-2)} \circ \chi = I_1 I_2^2 \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3),$$

$$H^{(0)} \circ \chi = \left(I_2 y_1 + 2I_1 I_2 y_2\right) \sqrt{(I_1 + y_1)(I_3 + y_3)} \cos(\theta_1 - \theta_3)$$

contain $\tilde{H}_0$ and $\tilde{H}_1$ as well as other terms at least quadratic in $y$, and obviously $H^{(\geq 2)} \circ \chi$ is at least quadratic in $y$. In this way we are canceling more terms than are strictly necessary, but in doing so we avoid introducing the singularity $I = 0$.

In typical KAM schemes one can assume that $\tilde{H} - \tilde{H}_0 - \tilde{H}_1$ is negligible, since it is at least quadratic in $y$, by restricting to a sufficiently small neighborhood of $T_I$. In our setting we need to control $H^{(\geq 2)}$ in a ball centered at zero (and containing $T_I$), where clearly it is not small.

Now we fix parameters

$$r_0, s_0, \eta_0 > 0 \quad \text{and take} \quad 0 < r < \frac{r_0}{2}, \quad 0 < r \leq \frac{r_0}{2\sqrt{2}}, \quad 0 < \sigma < \min\left\{\frac{\eta_0}{2}, 1\right\}. \quad (1.18)$$

We then have the following normal form theorem.
Theorem 3 (Twisted conjugacy à la Herman). Consider $r_0, s_0, \eta_0, \rho, r, \sigma$ as in (1.18). There exists $\tilde{\epsilon}, \tilde{C} > 0$, depending only on $\rho/r_0$ and $\sigma$ such that the following holds. Let $\sqrt{I} \in \tilde{B}_r(w^\infty_{p,s_0^0+\sigma,a})$, $N_0 \in \mathcal{N}_{r_0,s_0,\eta_0}(I, \omega)$ and $H \in D_\omega + \mathcal{H}_{r_0,s_0,\eta_0}$. If

$$(1 + \Theta)^2 \epsilon \leq \tilde{\epsilon}, \quad \text{where} \quad \epsilon := \gamma^{-1}\|H - N_0\|_{r_0,s_0,\eta_0}, \quad \Theta = \gamma^{-1}\|D_\omega - N_0\|_{r_0,s_0,\eta_0}$$

(1.19)

then there exist a symplectic diffeomorphism $\Psi: \tilde{B}_{r_0-\rho}(w^\infty_{p,s_0+\sigma,a}) \to \tilde{B}_{r_0}(w^\infty_{p,s_0+\sigma,a})$, close to the identity, a unique correction (counter term) $\Lambda = \sum \lambda_j (|u_j|^2 - I_j)$, Lipschitz depending on $\omega \in D_\gamma$, with:

$$\|\lambda\|_\infty \leq \tilde{C}\gamma (1 + \Theta)\epsilon$$

and a Hamiltonian $N \in \mathcal{N}_{r_0-\rho,s_0+\sigma,\eta_0}(I, \omega)$, such that

$$(\Lambda + H) \circ \Psi = N.$$ \hspace{1cm} (1.20)

Remark 1.1. The quantities $\tilde{\epsilon}$ and $\tilde{C}$ can be explicitly evaluated; see (6.21) below.

The name “twisted conjugacy”, comes from the fact that there exists a conjugacy between $H$ and $N$ which is “twisted” by the correction in frequencies $\Lambda$ (geometrically, the Hamiltonian action on $H$ is “twisted” by the presence of the counter-term: $H = N \circ \Psi^{-1} - \Lambda$, cf. equation (1.20)).

The proof of Theorem 3 is based on an iterative scheme that kills out the obstructing terms, namely terms belonging to $\mathcal{H}_{r_0,\sigma,\eta}$ and $\mathcal{H}_{r_0,\sigma,\eta}^{(0)}$, by solving homological equations of the form

$$L_\omega F^{(d)}(I) = G^{(d)}, \quad G^{(d)} \in \mathcal{H}_{r_0,\sigma,\eta}^{(d)}, \quad d = -2, 0.$$ \hspace{1cm} (2.1)

The control of the solution $L_\omega^{-1} G^{(d)}$ relies strongly on the fact that $G^{(d)}$ is analytic in a ball around 0, see Lemma 3.1.

Stability of tori In [19] the authors prove, with respect to Bourgain’s work, the long time stability of the almost periodic tori. As can be expected, this result can be recovered also in the slightly more general setting of Theorem 3, namely without requiring any smallness condition on the non linear part of the normal form $N_0$. In section 5.1 we consider the Hamiltonian flow of a normal form $N$ and we show that initial data $\delta$-close to the invariant torus, stay $2\delta$-close to the torus for polynomially long times.

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2. Functional setting

Consider the scale of Banach spaces $w^\infty_{p,s,a}$ defined in (1.7). For any $p \leq p', a \leq a', s \leq s'$ we have

$$w^\infty_{p',s',a'} \subset w^\infty_{p,s,a}$$ \hspace{1cm} (2.1)

and

$$|v|_{p,s,a} \leq |v|_{p',s',a'}, \quad \forall v \in w^\infty_{p',s',a'}.$$ \hspace{1cm} (2.2)

Note that

$$u, v \in \ell^\infty(\mathbb{C}), \quad |u_j| \leq |v_j| \quad \forall j \in \mathbb{Z} \quad \Rightarrow \quad |u|_{p,s,a} \leq |v|_{p,s,a}.$$
We endow $\mathcal{H}^\infty_{p,s,a}$ with the standard real symplectic structure coming from the Hermitian product on $\ell^2(\mathbb{C})$. For convenience and to keep track of the complex structure, we write everything in complex notation, that is
\[
i \sum_j d\eta_j \wedge d\bar{\eta}_j, \quad X_H^{(j)} = i\frac{\partial}{\partial \bar{\eta}_j} H.
\]

2.1. Spaces of Hamiltonians

Remark 2.1. Here and in the following we shall always assume that $p > 1$ and $s > 0$.

Multi-index notation. In the following we denote, with abuse of notation, by $\mathbb{N}^Z$ the set of multi-indexes $\alpha, \beta$ etc. such that $|\alpha| := \sum_{j \in \mathbb{Z}} \alpha_j$ is finite. As usual $\alpha! := \prod_{j \in \mathbb{Z}, \alpha_j \neq 0} \alpha_j$. Moreover $\alpha \leq \beta$ means $\alpha_j \leq \beta_j$ for every $j \in \mathbb{Z}$, then $(\alpha^\beta) := \frac{\beta^!}{\alpha! \beta! - \alpha! \alpha!}$. Finally, take $j_1 < j_2 < \ldots < j_n$ such that $\alpha_j \neq 0$ if and only if $j = j_i$ for some $1 \leq i \leq n$, as usual we set $\partial^\alpha f := \partial^\alpha u_{j_1} \ldots \partial^\alpha u_{j_n} f$; analogously for $\partial^\beta f$.

Definition 2.1 (majorant analytic Hamiltonians). For $r > 0$, let $\mathcal{A}_r(\mathcal{H}^\infty_{p,s,a})$ be the space of Hamiltonians
\[
H : \tilde{\mathcal{B}}_r(\mathcal{H}^\infty_{p,s,a}) \to \mathbb{R}
\]
such that there exists a pointwise absolutely convergent power series expansion
\[
H(u) = \sum_{\alpha, \beta \in \mathbb{N}^Z} H_{\alpha, \beta} u^\alpha \bar{u}^\beta, \quad u^\alpha := \prod_{j \in \mathbb{Z}} u_j^\alpha
\]
with the following properties:

(i) Reality condition:
\[
H_{\alpha, \beta} = \overline{H_{\beta, \alpha}} ;
\]

(ii) Mass conservation:
\[
H_{\alpha, \beta} = 0 \quad \text{if} \quad |\alpha| \neq |\beta|,
\]

namely the Hamiltonian Poisson commutes with the mass $\sum_{j \in \mathbb{Z}} |u_j|^2$.

Finally, given $H$ as above, we define its majorant $\tilde{H} : \tilde{\mathcal{B}}_r(\mathcal{H}^\infty_{p,s,a}) \to \mathbb{R}$ as
\[
\tilde{H}(u) := \sum_{\alpha, \beta \in \mathbb{N}^Z} |H_{\alpha, \beta}| u^\alpha \bar{u}^\beta
\]
and, for $\eta \geq 0$ its $\eta$-majorant:
\[
\tilde{H}_\eta(u) := \sum_{\alpha, \beta \in \mathbb{N}^Z} |H_{\alpha, \beta}| e^{\eta|\pi(\alpha - \beta)|} u^\alpha \bar{u}^\beta \in \mathcal{A}_r(\mathcal{H}^\infty_{p,s,a}),
\]

where given $\alpha \in \mathbb{N}^Z$
\[
\pi(\alpha) := \sum_{j \in \mathbb{Z}} j \alpha_j.
\]

Note that $\pi(\alpha - \beta)$ is the eigenvalue of the adjoint action of the momentum Hamiltonian $P = \sum_{j \in \mathbb{Z}} j |u_j|^2$ on the monomial $u^\alpha \bar{u}^\beta$. The exponential weight $e^{\eta|\pi(\alpha - \beta)|}$ is added in order to ensure that the monomials which do not preserve momentum have an exponentially small coefficient.

---

6 As usual given a vector $k \in \mathbb{Z}^Z$, $|k| := \sum_{j \in \mathbb{Z}} |kj|$.
Definition 2.2 (η-regular Hamiltonians). We say that a Hamiltonian $H \in \mathcal{A}_r(w_{p,s,a}^{\infty})$ is η-regular\(^7\) if $H_\eta(u) \in \mathcal{A}_r(w_{p,s,a}^{\infty})$ and its Hamiltonian vector field is bounded, i.e.

$$|H_{r,s,\eta}^{(a,p)}| := \frac{1}{r} \left( \sup_{|u|_{p,s,a} \leq r} |X_{H_{\eta}}|_{p,s,a} \right) < \infty.$$ 

We denote such space by $\mathcal{H}_{r,s,\eta} = \mathcal{H}_{r,s,\eta}^{(a,p)}$.

Note that $|\cdot|_{r,s,\eta}$ is a seminorm on $\mathcal{H}_{r,s,\eta}$ and a norm on its subspace

$$\mathcal{H}_{r,s,\eta}^{(a,p)} := \{ H \in \mathcal{H}_{r,s,\eta} \text{ with } H(0) = 0 \},$$

endowing $\mathcal{H}_{r,s,\eta}^{(a,p)}$ with a Banach space structure. Analogously we set

$$\mathcal{A}_{r}^{0}(w_{p,s,a}^{\infty}) := \{ H \in \mathcal{A}_{r}(w_{p,s,a}^{\infty}) \text{ with } H(0) = 0 \}$$

Lemma 2.1. For $H \in \mathcal{A}_{r}(w_{p,s,a}^{\infty})$, the norm in Definition 2.2 can be expressed as

$$|H_{r,s,\eta}^{(a,p)}| = \sup_{j} \sum_{\alpha, \beta \in \mathbb{N}^{2}} |H_{\alpha, \beta} \xi^{j} \eta^{\alpha(\alpha - \beta)}(\alpha - \beta)| \leq r \sum_{j} \sup_{|u|_{p,s,a} \leq r} \left| u_{\alpha, \beta}^{(a,p)} \right|_{p,s,a}$$

where $u_{0, j} = u_{0}(r)$ is defined as

$$u_{0, j}(r) := r^{j} - p e^{-a j} s^{j} |j|^{\theta}.$$ 

Note that

$$|u|_{p,s,a} \leq r \quad \implies \quad |u_{j}| \leq u_{0, j}, \quad \forall j \in \mathbb{Z}.$$ \hspace{1cm} (2.13)

Then a formal power series as in (2.7) such that (2.11) is finite, totally converges in the closed ball $\bar{B}_{r}(w_{p,s,a}^{\infty})$ with estimate

$$\sum_{\alpha, \beta} \sup_{|u|_{p,s,a} \leq r} \left| H_{\alpha, \beta} \xi^{j} \eta^{\alpha(\alpha - \beta)}(\alpha - \beta)\right| u_{\alpha, \beta}^{(a,p)} \leq |H(0)| + r^{\alpha} |H_{\eta}^{(\alpha, p)}|.$$ \hspace{1cm} (2.14)

Therefore $H(u)$ and $H_{\eta}(u)$ are analytic in the open ball\(^8\) $B_{r}(w_{p,s,a}^{\infty})$ and

$$|H_{r,s,\eta}^{(a,p)}| = \frac{1}{r} \left( \sup_{|u|_{p,s,a} \leq r} |X_{H_{\eta}}(u)|_{p,s,a} \right) = \frac{1}{r} \left| X_{H_{\eta}}(u_{0}(r)) \right|_{p,s,a}.$$ 

where $X_{H_{\eta}}$ is the vector field associated to the η-majorant Hamiltonian defined in (2.7).

The proof is postponed to Appendix A.

Remark 2.2. The norm defined in (2.11) in $\mathcal{H}_{r,s,\eta}^{(a,p)}$ is weaker than the one introduced by Bourgain in [15] (see formula (1.14)), which is a weighted $L^{\infty}$-norm on the coefficients of the Taylor expansion of $H$ at the origin. Note that both norms are not intrinsic since they depend on such Taylor expansion.

\(^7\) When $\eta = 0$ we simply say that $H$ is regular.

\(^8\) In particular by (2.14) $X_{H_{\eta}}$ exists on the open ball $B_{r}(w_{p,s,a}^{\infty})$ and can be continuously extended on the closed ball $\bar{B}_{r}(w_{p,s,a}^{\infty})$, so that the expression $H_{\eta}(u_{0}(r))$ makes sense.
By the reality condition\(^9\) (2.4) we get
\[
|H_{r,s,\eta}(\alpha, p)| = \sup_{j} \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}|^2 \geq |\alpha + \beta - 2e_j e^\eta|\pi(\alpha - \beta)|
\]
\[
= \frac{1}{2} \sup_{j} \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}|^2 |(\alpha + \beta + j)u_0|^2 e^\eta|\pi(\alpha - \beta)|
\]
\[
= \frac{1}{2} \sup_{j} \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}|^2 |(\alpha + \beta + j)\xi^{(j)}(\alpha, \beta)|
\]
(2.15)

where
\[
\xi^{(j)}(\alpha, \beta) := u_0 |\alpha + \beta - 2e_j e^\eta|\pi(\alpha - \beta)|.
\]
(2.16)

**Notations:** Since we will always keep the parameters \(a, p\) fixed, we shall drop them from our notations. Hence we will set
\[
\omega := \omega_{p,s,a}^\infty, \quad \omega := \omega_{p,s,a}, \quad \omega := \omega_{r,s,\eta}.
\]
(2.17)

**Remark 2.3.** We note that if \(H\) preserves momentum, i.e.
\[
H_{\alpha, \beta} = 0 \quad \text{if} \quad \pi(\alpha - \beta) \neq 0
\]
then \(|H|_{r,s,\eta} = |H|_{r,s,0}\), namely it does not depend on \(\eta\).

**Proposition 2.1.** Let \(f, R, a\) as in (1.3) and \(P\) as in (1.6). Let \(p > 1, r > 0, a, s, \eta \geq 0\) with \(a + \eta < a\). Set
\[
C_{a lg} := C_{a lg}(p) := 2^{p+1} \left(1 + 2 \sum_{n \geq 1} n^{-p}\right), \quad C(p, s, t) := \sup_{j} e^{-t|j|^{t+j}(j)p}
\]
(2.18)

and assume that \((C_{a lg})^2 \leq R\). Then
\[
|P|_{r,s,\eta} \leq C(p, s, a - a - \eta) \frac{(C_{a lg})^2}{R} |f|_{a, R} < \infty.
\]
(2.19)

2.2. *Inhomogenous weighted Lipschitz norm*

In the following, we shall keep track of the Lipschitz dependence of the Hamiltonians on the frequency \(\omega\). The frequencies will live in the set
\[
\mathcal{Q} := \left\{ \omega = (\omega_j)_{j \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}, \sup_j |\omega_j - j^2| \leq 1/2 \right\}
\]
(2.20)

which is isomorphic to \([-1/2, 1/2]^\mathbb{Z}\) (endowed with the sup-norm) via the map
\[
\omega : \xi \mapsto \omega(\xi), \quad \text{where} \quad \omega_j(\xi) = j^2 + \xi_j
\]
(2.21)

\(\mathcal{Q}\) is endowed with the probability measure \(\mu\) induced\(^10\) by the product measure on \([-1/2, 1/2]^\mathbb{Z}\).

\(^9\) Indeed
\[
\sum |H_{\alpha, \beta}|^2 |u_0|^2 |\alpha - \beta - 2e_j e^\eta| \pi(\alpha - \beta)| = \sum |H_{\alpha, \beta}|^2 |u_0|^2 |\alpha - \beta - 2e_j e^\eta| \pi(\alpha - \beta)|.
\]

\(^10\) Denoting by \(\mu\) the measure in \(\mathcal{Q}\) and by \(\nu\) the product measure on \([-1/2, 1/2]^\mathbb{Z}\), then \(\mu(A) = \nu(\omega(\mathcal{Q})\mathcal{A})\) for all sets \(A \subset \mathcal{Q}\) such that \(\omega(\mathcal{Q})\mathcal{A}\) is \(\nu\)-measurable.
Let \( \mathcal{O} \subset \mathcal{Q} \) be a closed bounded set of positive Lebesgue measure and assume that \( H = H(\omega) \in \mathcal{H}_{r,s,\eta} \) is \( \eta \)-regular uniformly with respect to \( \omega \in \mathcal{O} \), we define its Lipschitz semi-norm as

\[
|H|_{r,s,\eta}^{\text{Lip},\mathcal{O}} = \sup_{\omega,\omega' \in \mathcal{O}, \omega \neq \omega'} \frac{|H(\omega) - H(\omega')|}{|\omega - \omega'|} = \sup_{\omega,\omega' \in \mathcal{O}, \omega \neq \omega'} \Delta_{\omega,\omega'} H_{r,s,\eta},
\]

where, as usual \( |v|_\infty := \sup_{j \in \mathbb{Z}} |v_j| \) and

\[
\Delta_{\omega,\omega'} H := \frac{H(\omega) - H(\omega')}{|\omega - \omega'|}. \tag{2.23}
\]

Set

\[
\mathcal{H}_{r,s,\eta}^{\mathcal{O}} := \left\{ H(\omega) \in \mathcal{H}_{r,s,\eta}, \ \omega \in \mathcal{O}, \ \text{with} \ \sup_{\omega \in \mathcal{O}} |H(\omega)|_{r,s,\eta} < \infty, \ |H|_{r,s,\eta}^{\text{Lip},\mathcal{O}} < \infty \right\}
\]

and

\[
\mathcal{H}_{r,s,\eta}^{\mathcal{O},0} := \left\{ H \in \mathcal{H}_{r,s,\eta}^{\mathcal{O}} \text{ with } H(0) = 0 \right\}.
\]

For any \( \mu \geq 0 \)

\[
\|H\|_{r,s,\eta}^{\mu,\mathcal{O}} := \sup_{\omega \in \mathcal{O}} |H(\omega)|_{r,s,\eta} + \mu |H|_{r,s,\eta}^{\text{Lip},\mathcal{O}}
= \sup_{\omega \in \mathcal{O}} |H(\omega)|_{r,s,\eta} + \mu \sup_{\omega,\omega' \in \mathcal{O}, \omega \neq \omega'} \Delta_{\omega,\omega'} H_{r,s,\eta} < \infty,
\]

(2.24)

is a weighted Lipschitz semi-norm on \( \mathcal{H}_{r,s,\eta}^{\mathcal{O}} \) and a norm on \( \mathcal{H}_{r,s,\eta}^{\mathcal{O},0} \). It is immediate to verify that \( \mathcal{H}_{r,s,\eta}^{\mathcal{O},0} \) is a Banach space endowed with the above norm.

**Remark 2.4.** In the following, for brevity, we will often write \( \|\cdot\|_{r,s,\eta} \) instead of \( \|\cdot\|_{r,s,\eta}^{\mu,\mathcal{O}} \).

**Definition 2.3.** Given any subset \( S \subset \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z} \) let us define

\[
\Pi^{S} H := \sum_{(\alpha, \beta) \in S} H_{\alpha, \beta} u^\alpha \bar{u}^\beta.
\]

Then

\[
\|\Pi^{S} H\|_{r,s,\eta} \leq \|H\|_{r,s,\eta} \tag{2.25}
\]

**Lemma 2.2.** Given \( 0 < r_1 < r \) and \( N \in \mathbb{N} \)

\[
\|\Pi^{(|\alpha| = |\beta| > N)} H\|_{r_1,s,\eta} \leq (r_1/r)^{2N}\|\Pi^{(|\alpha| = |\beta| > N)} H\|_{r_1,s,\eta} \leq (r_1/r)^{2N}\|H\|_{r_1,s,\eta}.
\]

(2.26)

**Proof.** By (2.15) and noting that \( u_0(r_1) = (r_1/r) u_0(r) \) (recall (2.12)). \( \square \)

We set

\[
\Pi^{S} H := \sum_{(\alpha, \beta) \in \mathbb{N}^\mathbb{Z} \times \mathbb{N}^\mathbb{Z}, \alpha \neq \beta} H_{\alpha, \beta} u^\alpha \bar{u}^\beta, \quad \Pi^{K} H := H - \Pi^{S} H = \sum_{\alpha \in \mathbb{N}^\mathbb{Z}} H_{\alpha} u^\alpha, \quad \Pi^{K} H := H - \Pi^{S} H = \sum_{\alpha \in \mathbb{N}^\mathbb{Z}, \alpha \neq \beta} H_{\alpha} u^\alpha.
\]

(2.27)

and correspondingly we define the following subspaces of \( \mathcal{H}_{r,s,\eta} \):

\[
\mathcal{H}_{r,s,\eta}^{S} := \{ H \in \mathcal{H}_{r,s,\eta} : \Pi^{S} H = H \}, \quad \mathcal{H}_{r,s,\eta}^{K} := \{ H \in \mathcal{H}_{r,s,\eta} : \Pi^{K} H = H \}.
\]

(2.28)

By (2.25) we get

\[
\|\Pi^{S} H\|_{r,s,\eta} \leq \|H\|_{r,s,\eta}.
\]

(2.29)
2.3. Poisson structure and Hamiltonian flows

**Proposition 2.2.** For any \( F, G \in \mathcal{H}_{r+s,\eta} \), with \( \rho > 0 \), we have \( \{ F, G \} \in \mathcal{H}^0_{r+s,\eta} \) and

\[
\| \{ F, G \} \|_{r+s,\eta} \leq 8 \max \left\{ 1, \frac{r}{\rho} \right\} \| F \|_{r+s,\eta} \| G \|_{r+s,\eta}.
\] (2.30)

Moreover the Leibniz formula holds:

\[
\delta^\alpha_u \delta^\beta_u (f \cdot g) = \sum_{\gamma \leq \alpha, \delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \delta^\alpha_u \partial^\gamma_u \partial^\delta_u f \cdot \delta^\alpha_u \partial^\gamma_u \partial^\delta_u g \quad \text{on } \tilde{B}_r.
\] (2.31)

and

\[
\delta^\alpha_u \delta^\beta_u \{ f, g \} = \sum_{\gamma \leq \alpha, \delta \leq \beta} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \left\{ \delta^\alpha_u \partial^\gamma_u \partial^\delta_u f \cdot \partial^\gamma_u \delta^\delta_u g \right\} \quad \text{on } \tilde{B}_r.
\] (2.32)

Analogously for any \( F, G \in \mathcal{H}^0_{r+s,\eta} \)

\[
\| \{ F, G \} \|_{r+s,\eta} \leq 8 \max \left\{ 1, \frac{r}{\rho} \right\} \| F \|_{r+s,\eta} \| G \|_{r+s,\eta}.
\] (2.33)

The proof is given in Appendix A.

**Proposition 2.3 (Hamiltonian flow).** Let \( S \in \mathcal{H}^0_{r+s,\eta} \) with

\[
\| S \|_{r+s,\eta} \leq \| \rho \| = \frac{16e(r + \rho)}{16e(r + \rho)}.
\] (2.34)

Then, for every \( \omega \in \mathcal{O} \) the time 1-Hamiltonian flow \( \Phi^1_{S(\omega)} : \tilde{B}_r(\omega_s) \to \tilde{B}_{r+s}(\omega_s) \) is well defined, analytic in \( B_r(\omega_s) \), symplectic with

\[
\sup_{u \in B_r(\omega_s)} \left| \Phi^1_{S(\omega)}(u) - u \right|_{s} \leq (r + \rho) \| S \|_{r+s,\eta} \leq \frac{\rho}{16e}.
\] (2.35)

For any \( H \in \mathcal{H}^0_{r+s,\eta} \) we have that \( H \circ \Phi^1_{S} = e^{(S,1)} H \in \mathcal{H}^0_{r+s,\eta} \), \( e^{(S,1)} H - H \in \mathcal{H}^0_{r+s,\eta} \) and

\[
\left\| e^{(S,1)} H \right\|_{r+s,\eta} \leq 2 \left\| H \right\|_{r+s,\eta},
\] (2.36)

\[
\left\| \left( e^{(S,1)} - 1 \right) H \right\|_{r+s,\eta} \leq \left\| S \right\|_{r+s,\eta} \left\| H \right\|_{r+s,\eta},
\] (2.37)

\[
\left\| \left( e^{(S,1)} - 1 \right) - (S,1) \right\|_{r+s,\eta} \leq \frac{1}{2} \left\| S \right\|_{r+s,\eta}^2 \left\| H \right\|_{r+s,\eta}.
\] (2.38)

More generally for any \( h \in \mathbb{N} \) and any sequence \( (c_k)_{k \in \mathbb{N}} \) with \( |c_k| \leq 1/k! \), we have

\[
\left\| \sum_{k \geq h} c_k \text{ad}^k_{\tilde{S}} (H) \right\|_{r+s,\eta} \leq 2 \left\| H \right\|_{r+s,\eta} \left( \left\| S \right\|_{r+s,\eta} / 2 \delta \right)^h,
\] (2.39)

where \( \text{ad}_S (\cdot) := \{ S, \cdot \} \).

The proof is completely analogous to the one of Lemma 2.1 of [12] and it is based on (2.33) and on the Lie series expansion for \( e^{(S,1)} \).
2.4. Monotonicity

The following properties of monotonicity are fundamental in bounding solutions of the linearized problem.

**Proposition 2.4.** The following inequalities hold:

1. **Monotonicity.** The norm $\| \cdot \|_{r,s,\eta}$ is increasing\(^{11}\) in $r$, $\mu$, $\eta$ and\(^{12}\) $O$.

2. **Variation w.r.t. the parameter $s$.** For any $0 < \sigma < \eta$ and $H_{r,s,\eta}^0$ we have

$$\| H \|_{r,s,\eta - \sigma} \leq \| H \|_{r,s,\eta}. \quad (2.40)$$

Note that item (2) corresponds to monotonicity with respect to $s$ whenever $H$ preserves momentum.

**Proof.** In order to prove Proposition 2.4 we first need the following Lemma, which we shall use also in the proof of Lemma 3.1 below. Its proof directly follows from (2.15)-(2.16) and (2.24).

**Lemma 2.3.** Let $H^{(1)} \in H_{r_1,s_1,\eta_1}^0$ and $H^{(2)} \in H_{r_2,s_2,\eta_2}^0$, be such that, for all $\alpha, \beta \in \mathbb{N}^Z$ and $j \in \mathbb{Z}$ with $|\alpha| = |\beta|$ and $\alpha_j + \beta_j \neq 0$, one has for all $\omega \in O$

$$|H_{\alpha,\beta}^{(1)}(\omega)|c_{r_1,s_1,\eta_1}^{(j)}(\alpha, \beta) \leq |H_{\alpha,\beta}^{(2)}(\omega)|c_{r_2,s_2,\eta_2}^{(j)}(\alpha, \beta) \quad (2.41)$$

and for all $\omega \neq \omega' \in O$

$$|\Delta_{\omega,\omega'} H_{\alpha,\beta}^{(1)}|c_{r_1,s_1,\eta_1}^{(j)}(\alpha, \beta) \leq |\Delta_{\omega,\omega'} H_{\alpha,\beta}^{(2)}|c_{r_2,s_2,\eta_2}^{(j)}(\alpha, \beta), \quad (2.42)$$

where the coefficients $c_{r,s,\eta}(\alpha, \beta)$ are defined in (2.16). Then

$$\| H^{(1)} \|_{r_1,s_1,\eta_1} \leq \| H^{(2)} \|_{r_2,s_2,\eta_2}. \quad (2.43)$$

Let us come back to the proof of Proposition 2.4. While point (1) is immediate from (2.15)-(2.16), (2.24) and Lemma 2.3 (being $c_{r_1,s_1,\eta_1}(\alpha, \beta)$ not decreasing with respect to $r$ and $\eta$ since\(^{13}\) $|\alpha| = |\beta|$), point (2) is more delicate. We need to show that, for all $\alpha, \beta \in \mathbb{N}^Z$ and $j \in \mathbb{Z}$ with $|\alpha| = |\beta|$ and $\alpha_j + \beta_j \neq 0,$

$$c_{r,s,\eta}(\alpha, \beta) \leq \exp\left[ -\sigma \left( \sum_i (i)^{\theta(\alpha_i + \beta_i)} - 2(j)^{\theta} + |\pi(\alpha - \beta)| \right) \right] \leq 1$$

or, equivalently, that

$$\sum_i (i)^{\theta(\alpha_i + \beta_i)} - 2(j)^{\theta} + |\pi(\alpha - \beta)| \geq 0. \quad (2.44)$$

The proof of the non trivial estimate (2.44) is contained in [12] (see formula (3.20)) and it is based on an idea proposed by Bourgain in [15] (see also [19]).

\(^{11}\) Not strictly.

\(^{12}\) Namely if $O' \subseteq O$ then $\| \cdot \|_{r,s,\eta} \leq \| \cdot \|_{r,s,\eta'}.$

\(^{13}\) Note that in the case $\alpha = \beta = 0$ we have $(\alpha_j + \beta_j) c_{r,s,\eta}(\alpha, \beta) = 0.$ Otherwise, since $|\alpha| = |\beta|$, the exponent $\alpha + \beta - 2\epsilon_j$ in the definition of $c_{r,s,\eta}(\alpha, \beta)$ in (2.16) is non negative.
3. Small divisors and homological equation

3.1. Small divisors

We start by recalling that the set of Diophantine frequencies, of Definition 1.1 are typical in $[-1/2,1/2]^2$, namely there exists a positive constant $C$ such that

$$\text{meas}([-1/2,1/2]^2 \setminus D_{\gamma}) \leq C \gamma,$$

(3.1)

where $\text{meas}$ is the product measure on $[-1/2,1/2]^2$. The proof of (3.1) is contained in [12] (see Lemma 4.1). As we mentioned in the introduction, the general idea is to impose the “strongest possible” Diophantine condition. Even if condition (1.9) can be slightly improved/modified, we are not able to find a stronger one which significantly improves the bounds on the Homological equation Lemma 3.1. It is interesting to compare condition (1.9) with the ones used in constructing Birkhoff Normal Form for PDEs, [26,9,8,27,28]. In particular we note that our condition is stronger than the ones discussed in [26] Proposition A.1.

3.2. Homological equation

The proof of the following classical Lemma (core of any small divisors problem), relies on some notation and results introduced by Bourgain in [15] and extended later on by Cong-Li-Shi-Yuan in [19] (see Lemma 3.2 below). We shall send the reader to [12] for the detailed proof of such results.

Recalling the definitions of $\Pi^R$ and $\mathcal{H}^R_{r,s,0}$ given in (2.27) and (2.28), we introduce the following operator on the space of formal power series:

$$\text{L}_{\omega} G := \sum i(\omega \cdot (\alpha - \beta)) G_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}.$$  

(3.2)

The operator $\text{L}_{\omega}$ is nothing but the action of the Poisson bracket $\{ \sum j \omega j | u j |^2 \cdot \}$ and it is invertible on the subspace of formal power series such that $F = \Pi^R F$ with inverse

$$\text{L}_{-\omega}^{-1} F = G := \sum \frac{F_{\alpha, \beta}}{i(\omega \cdot (\alpha - \beta))} u^{\alpha} \bar{u}^{\beta}.$$  

(3.3)

**Lemma 3.1 (Straightening the torus).** Let $0 < \sigma < \min\{\eta, 1\}$, $r > 0$ and let $D_{\gamma} \ni \omega \mapsto F(\omega) \in \mathcal{H}^R_{r,s,\eta}$ be a Lipschitz family of Hamiltonians. Then, defining $G$ as in (3.3), the following bound holds

$$\|G\|_{\mathcal{H}^R_{r,s,\eta}} \leq \gamma^{-1} \epsilon \sigma^{-3/2} \|F\|_{\mathcal{H}^R_{r,s,\eta}}$$

for suitable $C \geq 1$.

**Proof.** We first claim that

$$\sup_{\omega \in D_{\gamma}} |G|_{r,s+\sigma,\eta-\sigma} \leq \frac{1}{3\gamma} \epsilon \sigma^{-3} \sup_{\omega \in D_{\gamma}} |F|_{r,s,\eta}.$$  

(3.4)

Note that by Lemma 2.1 estimate (3.4) ensures that the formal power series $G$ is actually totally convergent, together with $G_{0,0}.$

Let us prove (3.4). In order to apply Lemma 2.3 and get the stated bound, we shall start by proving that, for all $j \in \mathbb{Z}$ and $\alpha, \beta$ with $|\alpha| = |\beta|, \alpha \neq \beta$ and $\alpha_j + \beta_j \neq 0$ one has

$$\frac{|F_{\alpha, \beta}(\omega)|}{|\omega \cdot (\alpha - \beta)|} c_{r,s+\sigma,\eta-\sigma}^{(j)}(\alpha, \beta) \leq \frac{1}{3\gamma} \epsilon \sigma^{-3/2} |F_{\alpha, \beta}(\omega)| c_{r,s,\eta}^{(j)}(\alpha, \beta)$$

for a suitable $C \geq 1$ large enough. This is equivalent to proving

\[ \text{for example taking } |\omega \cdot \ell| > \gamma \prod_{n \in \mathbb{Z}} \frac{1}{|n + (\alpha, \beta) \ell|} \text{ or } |\omega \cdot \ell| > \gamma \prod_{n \in \mathbb{Z}} \frac{1}{|n + (\alpha, \beta) \ell|} \text{ with } c > 1. \]
\[
e^{-\sigma(\sum_i (i)^\theta (\alpha_i + \beta_i) - 2(j)^\theta + |\pi|)} \leq \frac{1}{3\gamma} e^{c\sigma^{-3}}. \tag{3.5}
\]

In order to prove (3.5) we consider two cases. The first case is when

\[
\sum_i (\alpha_i - \beta_i)^2 \geq 10|\alpha - \beta|. \tag{3.6}
\]

Then, denoting \(\omega_j = j^2 + \xi_j\) with \(|\xi_j| \leq \frac{1}{2}\),

\[
|\omega \cdot (\alpha - \beta)| \geq 10|\alpha - \beta| - \frac{1}{2}|\alpha - \beta| \geq 9|\alpha - \beta| \geq 9. \tag{3.7}
\]

Then by (2.43) the left hand side of (3.5) is bounded by 1/9 and (3.5) follows (since \(\gamma < 1\) and \(C > 0\)). Otherwise, in order to control small divisors, we shall make use of the following result (see [15] and Lemma 4.2 of [12]).

**Lemma 3.2.** Consider \(\alpha, \beta \in \mathbb{N}^Z\) with \(1 \leq |\alpha| = |\beta|\) and \(\alpha \neq \beta\). If

\[
\sum_i (\alpha_i - \beta_i)^2 \leq 10 \sum_i |\alpha_i - \beta_i| \tag{3.8}
\]

then for all \(j\) such that \(\alpha_j + \beta_j \neq 0\) one has

\[
\sum_i |\alpha_i - \beta_i|(i)^\theta/2 \leq \frac{13}{1 - \theta} \left(\sum_i (\alpha_i + \beta_i)(i)^\theta - 2(j)^\theta + |\pi|\right), \tag{3.9}
\]

where \(\pi = \sum_i i(\alpha_i - \beta_i)\) is defined in (2.8).

Then if (3.8) holds, by Lemma 3.2 and using that \(\omega \in \mathcal{D}_\nu\), the following chain of inequalities holds

\[
e^{-\sigma(\sum_i (i)^\theta (\alpha_i + \beta_i) - 2(j)^\theta + |\pi|)} \leq \gamma^{-1} e^{-c\omega \cdot \sum_i |\alpha_i - \beta_i|^{\theta/2}} \prod_i \left(1 + |\alpha_i - \beta_i|^2(i)^2\right) \leq \gamma^{-1} \exp \sum_i \left[-\frac{\sigma(1 - \theta)}{13} |\alpha_i - \beta_i|(i)^\theta + \ln \left(1 + |\alpha_i - \beta_i|^2(i)^2\right)\right] = \gamma^{-1} \exp \sum_i f_i(|\alpha_i - \beta_i|, \sigma) \tag{3.10}
\]

where, for \(0 < \sigma \leq 1\), \(i \in \mathbb{Z}\) and \(x \geq 0\), we set

\[
f_i(x, \sigma) := -\frac{\sigma(1 - \theta)}{13} x(i)^\theta + \ln \left(1 + x^2(i)^2\right). \tag{3.11}
\]

Now we exploit the following estimate whose proof is given [12] (see Lemma 4.5)

\[
\sum_i f_i(|\ell_i|, \sigma) \leq 21i_\gamma(\sigma) \ln i_\gamma(\sigma), \quad \text{with} \quad i_\gamma(\sigma) := \left(\frac{312}{\sigma(1 - \theta)} \ln \frac{156}{\sigma(1 - \theta)} \right)^{\frac{1}{\theta}} \tag{3.12}
\]

for every \(\ell \in \mathbb{Z}^Z\) with \(|\ell| < \infty\). By (3.10) we get that (3.5) holds for \(C\) large enough; then (3.4) follows.

Let us now estimate the Lipschitz semi-norm. By Leibniz’s rule we have (recall (2.23))

\[
\Delta_{\omega,\omega'} F_{\omega,\omega'} = \sum_{i \omega, (\alpha - \beta)} F_{\alpha, \beta}(\omega) \Delta_{\omega,\omega'} \left(\frac{1}{i \omega \cdot (\alpha - \beta)}\right) u^\alpha \bar{u}^\beta =: G_1 + G_2. \tag{3.13}
\]
Arguing as in the estimate of $G$ in (3.4) we get
\[ \sup_{\omega, \omega' \in \mathcal{D}_\gamma, \omega \neq \omega'} |G_1|_{r,s + \sigma, \eta - \sigma} \leq \frac{1}{3\gamma^2} e^{C\alpha \cdot \beta} \sup_{\omega, \omega' \in \mathcal{D}_\gamma, \omega \neq \omega'} |\Delta_{\omega, \omega'} F|_{r,s, \eta}. \] (3.14)

Regarding the term $G_2$ we claim that
\[ \sup_{\omega, \omega' \in \mathcal{D}_\gamma, \omega \neq \omega'} |G_2|_{r,s + \sigma, \eta - \sigma} \leq \frac{1}{3\gamma^2} e^{C\alpha \cdot \beta} \sup_{\omega, \omega' \in \mathcal{D}_\gamma, \omega \neq \omega'} |F|_{r,s, \eta}, \] (3.15)
taking $C \geq 1$ large enough. Set for brevity
\[ \Delta_{\omega, \omega'}(\alpha, \beta) := \left| \Delta_{\omega, \omega'} \left( \frac{1}{|\omega \cdot (\alpha - \beta)|} \right) \right|. \]

By (3.13) the claim (3.15) follows by Lemma 2.3 if we prove that, for all $j \in \mathbb{Z}$ and $\alpha, \beta$ with $|\alpha| = |\beta|$ and $\alpha + \beta \neq 0$
\[ \Delta_{\omega, \omega'}(\alpha, \beta) c_r, s + \sigma, \eta - \sigma(\alpha, \beta) \leq \frac{1}{3\gamma^2} e^{C\alpha \cdot \beta} c_r, s, \eta(\alpha, \beta) \]
or, equivalently, taking the logarithm
\[ -\sigma \left( \sum_i (i^\theta (\alpha_i + \beta_i) - 2j^\theta + |\pi|) \right) + \ln \Delta_{\omega, \omega'}(\alpha, \beta) \leq \ln \frac{1}{3\gamma^2} e^{C\alpha \cdot \beta}. \] (3.16)

We have that
\[ \Delta_{\omega, \omega'}(\alpha, \beta) = \frac{1}{|\omega \cdot (\alpha - \beta)|} \leq \frac{1}{|\omega \cdot (\alpha - \beta)|} \frac{(\omega' \cdot (\alpha - \beta))}{|\omega' \cdot (\alpha - \beta)|} \frac{1}{|\omega' \cdot (\alpha - \beta)|}. \] (3.17)

As above we have two cases. In the first case, namely when (3.6) holds, by (3.7) and (3.17) we get
\[ \Delta_{\omega, \omega'}(\alpha, \beta) \leq 1. \]

Then, by (2.44), (3.16) holds in this first case since its left hand side is negative while its right hand side is positive.

In the second case, when (3.8) holds, by (3.9) and (3.17) we get, for $\omega, \omega' \in \mathcal{D}_\gamma$,
\[ \Delta_{\omega, \omega'}(\alpha, \beta) \leq |\alpha - \beta| \gamma^{-2} \prod_{i \in \mathbb{Z}} \left( 1 + |\alpha_i - \beta_i|^2 (i^2) \right)^2 \leq \gamma^{-2} \prod_{i \in \mathbb{Z}} (1 + |\alpha_i - \beta_i|^2 (i^2))^3. \] (3.18)

Then by (3.9)
\[ -\sigma \left( \sum_i (i^\theta (\alpha_i + \beta_i) - 2j^\theta + |\pi|) \right) + \ln \Delta_{\omega, \omega'}(\alpha, \beta) \]
\[ \leq \ln \gamma^{-2} + 3 \sum_i \left[ -\frac{\sigma(1 - \theta)}{3\gamma^2} |\alpha_i - \beta_i|^2 (i^2) + \ln \left( 1 + |\alpha_i - \beta_i|^2 (i^2) \right) \right] \]
\[ \leq \ln \gamma^{-2} + 3 \sum_i f_i(|\alpha_i - \beta_i|, \sigma/3) \]
\[ \leq \ln \gamma^{-2} + 63i_2(\sigma/3) \ln i_2(\sigma/3). \]
Then (3.16) and, hence, (3.15) follow also in the second case taking $C$ large enough.

Recollecting, recalling (2.24), (2.13) and using (3.4), (3.14) and (3.15), we get

$$\|G\|_{r,s+\sigma,\eta-\sigma} \leq \sup_{\omega \in D_{\omega}} |G|_{r,s+\sigma,\eta-\sigma} + \gamma \sup_{\omega,\omega' \in D_{\omega}} |G|_{r,s+\sigma,\eta-\sigma} + \gamma \sup_{\omega,\omega' \in D_{\omega}} |G|_{r,s+\sigma,\eta-\sigma}$$

$$\leq \frac{1}{\gamma} e^{C_{\beta_{\gamma}}^{-1}} \|F\|_{r,s,\eta}.$$

The proof is completed. \(\Box\)

4. Projections on the torus

We now fix a torus $T_l$ associated to an action $I$, as explained in the introduction, then given a regular Hamiltonian we define a degree decomposition with increasing order of zero at $T_l$. The general ideas of this section were sketched in [15], here we adapt them to our norm and give detailed explanations of the projections and their properties.

We fix $r > 0$ and (recall (1.10))

$$(I_j)_{j \in \mathbb{Z}} , \quad I_j \geq 0, \quad \forall j \in \mathbb{Z} , \quad \text{with} \quad |\sqrt{I}|_s \leq \kappa r, \quad \kappa < 1. \quad (4.1)$$

Then we define the torus

$$T_I = \{ u \in \mathcal{H}_{r,s,\eta}^\infty : |u|^2 = 1 \}$$

where for brevity we set

$$\left( |u|^2 \right)_j \equiv |u_j|^2, \quad j \in \mathbb{Z}. \quad (4.2)$$

Note that by construction $T_I$ is contained in the interior of $\tilde{B}_r(\nu_{s'})$ for all $s' \leq s$. In the following we will use the quantity

$$c_\kappa := \left\{ \begin{array}{ll} \frac{1}{\ln \kappa^{-2}} & \text{if } \frac{1}{2} < \kappa^2 < 1, \\ 2\kappa^2 & \text{if } 0 < \kappa^2 \leq \frac{1}{2}. \end{array} \right. \quad (4.3)$$

The main result of the section is condensed in the following

**Proposition 4.1.** Let $I$ be fixed as in (4.1). For every $d = 2q - 2, \; q \in \mathbb{N}$ and $I$ fixed as in (4.1), there exist linear continuous “projection” operators $\Pi^d : \mathcal{H}_{r,s,\eta}^\infty \rightarrow \mathcal{H}_{r,s,\eta}^\infty$ such that the following holds:

(i) $\Pi^d \Pi^d = \Pi^d$ and $\Pi^d \Pi^{d'} = \Pi^d \Pi^{d'} = 0$ for every even $d' \neq d, \; d' \geq -2$.

(ii) For every $\kappa_\ast \leq 1$

$$\|\Pi^d H\|_{r,s,\eta} \leq \kappa_{\ast}^{-2} \left( 1 + \frac{\kappa^2}{\kappa_\ast^2} \right)^q c_\kappa^d \|H\|_{r,s,\eta}, \quad (4.4)$$

with $c_\kappa$ defined in (4.3).

(iii) For any $q \geq 0$,

$$\Pi^2q^{-2} H = 0, \; \forall q \leq q \quad \Longrightarrow \quad \partial_\alpha^q \partial_\beta^q H \equiv 0 \quad \text{on} \quad T_I, \quad \text{for} \quad 0 \leq |\alpha| + |\beta| \leq q. \quad (4.5)$$

(iv) Setting for brevity

$$\Pi^{<d} := \sum_{j=-2}^{d-2} \Pi^j,$$
if $\Pi^{<d_1} F = \Pi^{<d_2} G = 0$, then
\[ \Pi^{<d_1+d_2} \{ F, G \} = 0. \tag{4.6} \]

**Remark 4.1.** (i) The “projection” operators $\Pi^d$ are explicitly defined below in formula (4.22).
(ii) For $u$ such that $|u|^2$ belongs to the interior of $I + \tilde{B}_{(1-\kappa^2)r^2}(\omega_{2\rho}, 2\pi, 2a)$ we have
\[ H(u) = \sum_{q \in \mathbb{N}} \Pi^{2q-2} H(u). \tag{4.7} \]
Moreover for $\kappa^2 < 1/2$ (4.7) actually holds in the whole ball $\tilde{B}_r$.
(iii) The vice versa of (4.5) does not hold, in general, if $I_j = 0$ for some $j \in \mathbb{Z}$. On the other hand if all the $I_j$ are positive the vice versa of (4.5) holds.

**Notation:** for any even integer $d \geq -2$ we define
\[ \Pi^{\geq d} := \text{Id} - \Pi^{<d}, \]
moreover for $H \in \mathcal{H}_{r,s,\eta}^O$ we set
\[ H^{(d)} := \Pi^d H, \quad H^{\geq d} := \Pi^{\geq d} H := \sum_{j=-2}^{d} H^{(j)}, \quad H^{\geq d} := \Pi^{\geq d} H, \tag{4.8} \]
and denote
\[ \mathcal{H}^{(d)}_{r,s,\eta} = \left\{ H \in \mathcal{H}_{r,s,\eta} : \Pi^d H = H \right\}. \tag{4.9} \]
In the proof of Proposition 4.1, in particular points (iii)-(iv), we will use the following result, which is interesting in itself.

**Proposition 4.2 (Bourgain’s representation).** Let $I$ and $\kappa$ be as in (4.1). Let $H \in \mathcal{H}_{r,s,\eta}^O$ and $q \geq 0$. Then the following representation formula holds
\[ \Pi^{\geq 2q-2} H(u) = \sum_{\delta \in \mathbb{N}^2 : |\delta| = q} (|u|^2 - 1)^\delta \tilde{H}_\delta(u) \quad \text{for } |u|_s < r, \tag{4.10} \]
where $\tilde{H}_\delta(u)$ are analytic in $\tilde{B}_r(\omega_s)$ and can be written in totally convergent power series in every ball $|u|_s \leq \kappa_s r$ with $\kappa_s < 1$. Moreover
\[ \| \Pi^{\geq 2q-2} H \|_{r,s,\eta} \leq \frac{1}{\kappa^2_s} \left( \frac{\kappa^2_s + \kappa^2}{\kappa^2_s} c_{\kappa_s} \right)^q \| H \|_{r,s,\eta}, \tag{4.11} \]
where $c_{\kappa_s}$ was defined in (4.3).

We prove first Proposition 4.1 and then Proposition 4.2.

**Proof of Proposition 4.1.** We start constructing the projections $\Pi^d$.
Assume that a formal power series
\[ H(u) = \sum_{|\alpha| = |\beta|} \tilde{H}_{\alpha, \beta} u^\alpha \bar{u}^\beta \]

---

15 $|u|^2$ is defined in (4.2).
Let \( |H|_{r,s,\eta} < \infty \). Since by Lemma 2.1 the above series totally converges in \( |u|_s \leq r \), we can rearrange its terms in the unique way

\[
H(u) = \sum_{m,\alpha,\beta} H_{m,\alpha,\beta} |u|^{m-\alpha-\beta} \hat{u}^\beta, \quad \text{with} \quad H_{m,\alpha,\beta} = \hat{H}_{m+\alpha,m+\beta}, \quad m, \alpha, \beta \in \mathbb{N}^Z
\]  

(4.12)

and

\[
\sum_{m,\alpha,\beta} := \sum_{m,\alpha,\beta \in \mathbb{N}^Z \atop |\alpha| = |\beta|, \alpha_j \beta_j = 0, \forall j \in \mathbb{Z}}.
\]  

(4.13)

Note that the map

\[
(m, \alpha, \beta) \rightarrow (\alpha, \beta), \quad \alpha := m + \alpha, \quad \beta := m + \beta
\]

with inverse

\[
\alpha := \alpha - m, \quad \beta := \beta - m, \quad m_j := \min\{\alpha_j, \beta_j\}, \quad \forall j \in \mathbb{Z},
\]

is a bijection between the sets of indexes

\[
\{m, \alpha, \beta \in \mathbb{N}^Z : |\alpha| = |\beta|, \alpha_j \beta_j = 0, \forall j \in \mathbb{Z}\} \quad \text{and} \quad \{\alpha, \beta \in \mathbb{N}^Z : |\alpha| = |\beta|\}.
\]

Note that by (4.15) we get

\[
|H|_{r,s,\eta} = \frac{1}{2} \sup_{j} \sum_{m,\alpha,\beta} |H_{m,\alpha,\beta}| (2m + \alpha + \beta) u_0^{2m + \alpha + \beta - 2e_j \varphi_j |\alpha - \beta|),
\]

(4.14)

where \( u_0 = u_0(r) \) is defined in (4.12).

Given

\[
w = (w_j)_{j \in \mathbb{Z}} \in \mathbb{W}_{2,p,2s,2a}^\infty, \quad |w|_{2p,2s,2a} \leq r^2,
\]

(4.15)

and noting that \( |w_j| \leq u_0^2 \) for every \( j \in \mathbb{Z} \), we define (recall (4.13))

\[
H(u, w) := \sum_{m,\alpha,\beta} H_{m,\alpha,\beta} \hat{w}^m \hat{u}^\alpha \hat{\beta}.
\]

(4.16)

Reasoning as in (2.14) we get that the series in (4.16) totally converges in \( \{|w|_{2p,2s,2a} \leq r^2\} \times \{|u|_s \leq r\} \) with estimate

\[
\sum_{m,\alpha,\beta} \sup_{|w|_{2p,2s,2a} \leq r^2} \sup_{|u|_s \leq r} |H_{m,\alpha,\beta}| |w|^m |u|^\alpha |\hat{\beta}| = \sum_{m,\alpha,\beta} |H_{m,\alpha,\beta}| u_0^{2m+\alpha+\beta} \leq |H_{0,0,0}| + r^2 |H|_{r,s,\eta}.
\]

(4.17)

Then \( H(u, w) \) is analytic in the interior of

\[
\bar{B}_r(\mathbb{W}_{2p,2s,2a}^\infty) \times \bar{B}_r(w_0) \ni (w, u).
\]

(4.18)

Note that

\[
H(u) = H(u, |u|^2).
\]

(4.19)

Recalling (4.1), and defining the multi-index \( k^2 u_0^2 \), with \( (k^2 u_0^2)_j := k^2 u_0^2 \) for every \( j \in \mathbb{Z} \), we have

\[
I \leq k^2 u_0^2.
\]

(4.20)

Let \( H \in \mathcal{H}_{r,s,\eta} \). For every \( u \) in the interior of \( \bar{B}_r(w_0) \), by analyticity we can Taylor expand the function \( w \mapsto H(u, w) \) at \( w = I \) obtaining

\[
\text{...}
\]

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\[ H(u, w) = \sum_{q \in \mathbb{N}} \frac{1}{q!} D^q_u H(u, I)[w - I, \ldots, w - I], \]

for \( w - I \) in the interior of \( \bar{B}_{(1-\kappa^2)^2}(w_{2p,2s,2a}) \). For every \( u \) in the interior of \( \bar{B}_r(w_s) \),

\[ D^q_u H(u, I) : w_{2p,2s,2a} \times \ldots \times w_{2p,2s,2a} \to \mathbb{C} \]

is the bounded \( q \)-linear symmetric operator of the \( q \)’th derivative evaluated at\(^{16} \) \( w = I \). For \( q \in \mathbb{N}_+ \) and \( u \) in the interior of \( \bar{B}_r(w_s) \) we set

\[ \Pi^{-2} H(u) = H^{(-2)}(u) := H(u, I) \quad \text{and} \quad \Pi^{2q-2} H(u) = H^{(2q-2)}(u) := \frac{1}{q!} D^q_u H(u, I)[|u|^2 - I, \ldots, |u|^2 - I]. \]  

(4.22)

Then, recalling (4.19), we get

\[ H(u) = H(u, |u|^2) = \sum_{q \in \mathbb{N}} H^{(2q-2)}(u) \]  

(4.23)

for \( |u|^2 \) in the interior of \( I + \bar{B}_{(1-\kappa^2)^2}(w_{2p,2s,2a}) \).

We now rewrite \( H^{(2q-2)} \) in order to prove that it is not only analytic but also regular. Since the norm in (2.11), equivalently (4.14), is not intrinsic but is defined in terms of the coefficients of the Taylor expansion at zero, we now expand \( H^{(2q-2)}(u) \) at \( u = 0 \). Since for any \( u \) in the interior of \( \bar{B}_r(w_s) \)

\[ \frac{1}{q!} D^q_u H(u, I)[\xi, \ldots, \xi] = \sum_{\delta} \frac{1}{\delta!} \delta^q_u H(u, I) \xi^\delta = \sum_{\delta} \sum_{m \geq \delta} \binom{m}{\delta} H_{m,\alpha,\beta} I^{m-\delta} u^\alpha \overline{u}^\beta \xi^\delta \]

by the definition in (4.22) we get

\[ H^{(2q-2)}(u) = \sum_{\delta} \left( |u|^2 - I \right)^\delta \sum_{m \geq \delta} \sum_{\gamma \leq \delta} \binom{m}{\delta} \binom{\delta}{\gamma} H_{m,\alpha,\beta} I^{m-\gamma} u^\alpha \overline{u}^\beta \]  

(4.24)

Since

\[ \left( |u|^2 - I \right)^\delta = \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} (-1)^{\delta-\gamma} |I^{\delta-\gamma} |u|^{2\gamma} \]

we get

\[ H^{(2q-2)}(u) = \sum_{\delta} \sum_{\gamma \leq \delta} \sum_{m \geq \delta} \binom{m}{\delta} \binom{\delta}{\gamma} (-1)^{\delta-\gamma} I^{m-\gamma} H_{m,\alpha,\beta} |u|^{2\gamma} u^\alpha \overline{u}^\beta \]  

(4.25)

First we show that the series in (4.25) totally converges in the closed ball \( |u|_{p,s,a} \leq r \). Indeed for \( \kappa \leq 1 \) we get

\[ \sum_{\delta} \sum_{\gamma \leq \delta} \sum_{m \geq \delta} \sup_{|u|_{p,s,a} \leq \kappa r} \binom{m}{\delta} \binom{\delta}{\gamma} I^{m-\gamma} H_{m,\alpha,\beta} |u|^{2\gamma} u^\alpha \overline{u}^\beta. \]

Note that

\[ \frac{1}{q!} D^q_u H(u, I)[\xi, \ldots, \xi] = \sum_{\delta} \frac{1}{\delta!} \delta^q_u H(u, I) \xi^\delta. \]  

(4.21)
\[ (2.12) \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \sum_{m \geq \delta}^* \binom{m}{\delta}^\gamma \frac{(\delta)}{\gamma} \frac{I^{m-\gamma} |H_{m,\alpha,\beta}| (\kappa u_0)^{2\gamma + \alpha + \beta}}{m^\delta} \]

\[ \leq \frac{1}{2} \sup_{j} \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \sum_{m \geq \delta}^* \binom{m}{\delta} \frac{(\kappa u_0)^{2\gamma + \alpha + \beta}}{m^\delta} \]

\[ \leq (\kappa r)^{\frac{1}{2}} \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \sum_{m \geq \delta}^* \binom{m}{\delta} \frac{(\kappa u_0)^{2\gamma + \alpha + \beta - 2e_j e^\nu \pi(\alpha - \beta)}}{m^\delta} \]

\[ =: C_{q,\kappa} . \]  \hspace{1cm} (4.26)

Noting that

\[ I^{m-\gamma} \leq \kappa^{2|m-\gamma|} u_0^{2m-2\gamma} \leq \kappa^{2|m-2\gamma|} u_0^{2m-2\gamma} \]

we have

\[ C_{q,\kappa} \leq \frac{r^2}{2} \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \sum_{m \geq \delta}^* \binom{m}{\delta} \frac{(\kappa u_0)^{2\gamma |\delta|}}{m^\delta} \frac{(\kappa u_0)^{2|m| |H_{m,\alpha,\beta}| (2m_j + \alpha_j + \beta_j)u_0^{2m+\alpha+\beta-2e_j e^\nu \pi(\alpha-\beta)}}{m^\delta} \]

\[ \leq \left( 1 + \frac{\kappa^2}{k^2} \right) \frac{r^2}{2} \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \sum_{m \geq \delta}^* \binom{m}{\delta} \frac{(\kappa u_0)^{2|m| |H_{m,\alpha,\beta}| (2m_j + \alpha_j + \beta_j)u_0^{2m+\alpha+\beta-2e_j e^\nu \pi(\alpha-\beta)}}{m^\delta} \]

noting that for \(|\delta|=q\)

\[ \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} (\kappa u_0)^{2|\gamma|} = \left( 1 + \frac{\kappa^2}{k^2} \right)^q \]  \hspace{1cm} (4.27)

Now we need the following technical result, whose proof is postponed to the Appendix.

\[ \textbf{Lemma 4.1.} \text{ Let } 0 < \kappa < 1. \text{ For } q \in \mathbb{N} \text{ and } |m| \geq q \text{ we get}^{17} \]

\[ \kappa^{2|m|} \sum_{|\delta|=q} \binom{m}{\delta} \leq c_k^q , \]  \hspace{1cm} (4.28)

where \(c_k\) was defined in (4.3).

Then by (4.28) and exchanging the order of summation between \(\delta\) and \(m\) we get

\[ C_{q,\kappa} \leq \left( 1 + \frac{\kappa^2}{k^2} \right)^q c_k^q \frac{r^2}{2} \sup_{j} \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \binom{m}{\delta} (-1)^{|\delta-\gamma|} I^{m-\gamma} \frac{H_{m,\alpha,\beta}}{m^\delta} \]

\[ \leq \left( 1 + \frac{\kappa^2}{k^2} \right)^q c_k^q r^2 |H|_{r,s,q} . \]  \hspace{1cm} (4.29)

This implies that the series in (4.25) totally converges in the closed ball \(|u|_{p,s,a} \leq r\) (taking \(\kappa=1\)).

Then we can exchange the order of summation obtaining the representation formula

\[ H^{(2q-2)}(u) = \sum_{|\gamma| \leq q, \alpha, \beta} \sum_{\delta \geq \gamma, |\delta|=q} \sum_{m \geq \delta} \binom{m}{\delta} (-1)^{|\delta-\gamma|} I^{m-\gamma} H_{m,\alpha,\beta} |u|^2 u^\alpha u^\beta , \quad \forall |u|_{p,s,a} \leq r , \]

\[ \hspace{1cm} (4.30) \]

\[^{17}\text{ We set } \binom{m}{\delta} := \prod_{\ell \in \mathbb{Z}} \binom{m}{\delta}. \]
which is in the form (4.12). Then, by (4.14), we can evaluate the norm of $H^{(2q-2)}$ obtaining
\[
|H^{(2q-2)}|_{k_r,s,\eta} \leq \frac{1}{2} \sup_{m} \sum_{j \geq m} \left( \sum_{\delta \geq \gamma,|\delta|=q} (m,\delta) \left( \frac{\delta}{\gamma} \right) I^{m-\gamma} H_{m,\alpha,\beta} \right) (2y_j + \alpha_j + \beta_j) (\kappa_u u_0)^{2\gamma + \alpha + \beta - 2e_j} c_\kappa \|\pi(\alpha - \beta)\|
\]
using that $m \geq \gamma$ and (4.26). Then by (4.29)
\[
|H^{(2q-2)}|_{k_r,s,\eta} \leq \kappa^{-2}_u \left( 1 + \frac{\kappa^2}{\kappa^2} \right) c_\kappa^q |H|_{r,s,\eta}.
\]
Since the Lipschitz estimate is analogous we get (4.4).

We now prove that $\Pi^d \Pi^d H = \Pi^d H$ and $\Pi^d \Pi^d H = \Pi^d \Pi^d H = 0$. Setting $\tilde{H}(u) := \Pi^{2q-2} H(u) = H^{(2q-2)}(u)$ and recalling (4.16), by (4.24) we get as in (4.16) with $H \rightsquigarrow H$,
\[
\tilde{H}(u, w) = \sum_{|\delta| = q} (w-I)^\delta \sum_{m \geq \gamma,|\alpha|,|\beta|} \left( m,\delta \right) H_{m,\alpha,\beta} I^{m-\gamma} u^\alpha \bar{u}^\beta.
\]
Noting that $\frac{1}{\delta!} \sum_{|\delta| = q} (w-I)^\delta$ evaluated at $w = I$ is equal to 1 if $|\delta'| = q$ and vanishes if $|\delta'| \neq q$ we conclude by the definition of the projections in (4.22) (recall also (4.21)).

Now we note that (4.5) directly follows by the representation formula (4.10) with $q = 1 + q$.

Finally we use again (4.10) in order to prove (4.6). By linearity it suffices to work on monomials. Namely let us assume that $F = (|u|^2 - I)^{\delta_1} |u|^{2k(1)} u^\alpha \bar{u}^\beta$ and $G = (|u|^2 - I)^{\delta_2} |u|^{2k(2)} u^\alpha \bar{u}^\beta$ with $|\delta(i)| \geq 1 + \frac{1}{2} d_1$, for $i = 1, 2$. Then by Leibniz rule we have
\[
H := \{F, G\} = \left\{ (|u|^2 - I)^{\delta_1}, |u|^{2k(1)} u^\alpha \bar{u}^\beta \right\} (|u|^2 - I)^{\delta_2} |u|^{2k(1)} u^\alpha \bar{u}^\beta +
\]
\[
+ \left\{ |u|^{2k(1)} u^\alpha \bar{u}^\beta, (|u|^2 - I)^{\delta_2} \right\} (|u|^2 - I)^{\delta_1} |u|^{2k(2)} u^\alpha \bar{u}^\beta +
\]
\[
+ \left\{ |u|^{2k(1)} u^\alpha \bar{u}^\beta, |u|^{2k(2)} u^\alpha \bar{u}^\beta \right\} (|u|^2 - I)^{\delta_1} (|u|^2 - I)^{\delta_2}.
\]
We can write $H$ in the form (4.41) with $q = |\delta(1)| + |\delta(2)| - 1 \geq 1 + \frac{1}{2} (d_1 + d_2)$. Then $H^{d_1+d_2} = 0$, proving (4.6).

The proof of Proposition 4.1 is now completed. \(\Box\)

**Proof of Proposition 4.2.** First we note that the case $q = 0$ is trivial since $\Pi^{2q-2} = \text{Id}$ and formula (4.10) reduces to (4.12). Then we consider now $q \geq 1$.

Let us introduce the auxiliary function, defined for $(u, w)$ in the interior of $\tilde{B}_r(\mathcal{W}_s) \times \tilde{B}_r(\mathcal{W}_s)$,
\[
F(u, w) := H(u, w) - \sum_{i=0}^{q-1} \frac{1}{i!} D^{\delta}_w H(u, I) [w - I, \ldots, w - I] \\
= \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!} D^{\delta}_w H(u, I + t (|u|^2 - I)) [w - I, \ldots, w - I] dt
\]
by Taylor’s formula, where $I_w(t) := I + t (w - I)$. Note that by (4.22)
\[
F(u, |u|^2) = H(u, |u|^2) - \sum_{i=0}^{q-1} H^{(2i-2)}(u) = \Pi^{2q-2} H(u).
\]
So, for \( q \geq 1 \), we get the representation formula

\[
H^{2q-2}(u) = \int_0^1 \frac{1}{(q-1)!} D_q^H(u) H(u, t(|u|^2 - I)) \sum_{\nu=1}^q \left( \left[ |u|^2 - I \right]_\nu \right) dt ,
\]

now

\[
\frac{1}{(q-1)!} D_q^H(u, w)[h, \ldots, h] = q \sum_{|\delta|=q} \frac{1}{\delta!} \partial_w^\delta H(u, w) h^\delta \quad (4.12)
\]

in every summation, the order of summation in \( m \) and \( k \) and used that

\[
\int_0^1 (1-t)^{|m|-|k|-1} dt = \frac{|k|!(|m| - |k| - 1)!}{|m|!}.
\]

The capability to exchange the order of summation in \( (4.31) \) and in the rest of the proof follows by the absolute convergence of the series, which we will prove below.

We now set for \( \delta, k, \alpha, \beta \in \mathbb{N}^2 \)

\[
\tilde{H}_{\delta, k, \alpha, \beta} := q \sum_{m \geq \delta + k} \binom{m - \delta}{k} H_{m, \alpha, \beta} I^{m-(\delta+k)} \frac{|k|!(|m| - |k| - 1)!}{|m|!}
\]

and we claim that the above series absolutely converges. We also claim that

\[
\tilde{H}_{\delta}(u) := \sum_{k, \alpha, \beta} \tilde{H}_{\delta, k, \alpha, \beta} |u|^{2k} u^\alpha \bar{u}^\beta \quad \text{totally converges for } |u|_{s,r} \leq \kappa_s r
\]

for every \( \kappa_s < 1 \). Then \( (4.10) \) follows by \( (4.33) \), \( (4.34) \) and \( (4.31) \). Developing \( (|u|^2 - I)^\beta \) as \( \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} |u|^{2\gamma} (-I)^{\delta-\gamma} \)

and exchanging again the order of summation we get

\[
H^{2q-2}(u) = \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} |u|^{2\gamma} (-I)^{\delta-\gamma} \sum_{k, \alpha, \beta} \tilde{H}_{\delta, k, \alpha, \beta} |u|^{2k} u^\alpha \bar{u}^\beta
\]

\[
= \sum_{m, \alpha, \beta} \left( \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} (-I)^{\delta-\gamma} \tilde{H}_{\delta, m, \gamma, \alpha, \beta} \right) |u|^{2m} u^\alpha \bar{u}^\beta
\]

\[
= \sum_{k, \alpha, \beta} \left( \sum_{|\delta|=q} \sum_{\gamma \leq \delta} \binom{\delta}{\gamma} (-I)^{\delta-\gamma} \tilde{H}_{\delta, k, \alpha, \beta} \right) |u|^{2\gamma + 2k} u^\alpha \bar{u}^\beta.
\]

We now claim that the last series totally converges on every ball \( |u|_{s,r} \leq \kappa_s r < r \), in particular that
\[
\sum_{k, \alpha, \beta} \left( \sum_{|\delta|=q \leq \delta} \left( \begin{array}{c} m \\ \delta \end{array} \right) \right) |I^\delta | \tilde{H}_{8, k, \alpha, \beta}| (\kappa^*_s u_0)^{2^{\gamma+2k+\alpha+\beta}} \leq (4.33)
\]

\[
\sum_{k, \alpha, \beta} \left( \sum_{|\delta|=q \leq \delta} \left( \begin{array}{c} m \\ \delta \end{array} \right) \right) q \sum_{m \geq \delta+k} \left( \begin{array}{c} m - \delta \\ k \end{array} \right) |H_{m, \alpha, \beta} | I^{m-\gamma-k} \frac{|k|!(|m| - |k| - 1)!}{|m|!} (\kappa^*_s u_0)^{2^{\gamma+2k+\alpha+\beta}} \leq (4.20)
\]

\[
q \sum_{m, \alpha, \beta} |H_{m, \alpha, \beta} | u_0^{2m+\alpha+\beta} \kappa^*_s^{2|m|} \sum_{|\delta|=q} \left( \begin{array}{c} m \\ \delta \end{array} \right) \sum_{y \leq \delta} \left( \begin{array}{c} \delta \\ y \end{array} \right) \sum_{k \leq m - \delta} \left( \begin{array}{c} \delta \\ k \end{array} \right) \left( \frac{\kappa^*_s}{\kappa_s} \right)|H_{m, \alpha, \beta} | u_0^{2 \delta} |k|!(|m| - |k| - 1)! \leq (4.20)
\]

Before proving (4.36) we stress that it shows that the series in (4.31) and (4.35) absolutely converge and that \( \tilde{H}_{8, k, \alpha, \beta} \) in (4.33) is well defined since the series absolutely converges. Moreover since the first line of (4.36) is bounded we get (4.34).

Note that the first two inequalities in (4.36) are straightforward and we have to prove only the last one, which follows by the auxiliary estimate

\[
A_{q, k, \kappa_s} := \frac{q}{2} \sup_j \sum_{m, \alpha, \beta} e^{\eta|\sigma(\alpha-\beta)|} |H_{m, \alpha, \beta} | (2m_j + \alpha_j + \beta_j) u_0^{2m+\alpha+\beta-2\epsilon j} \kappa^*_s^{2|m|}
\]

\[
\sum_{|\delta|=q} \left( \begin{array}{c} m \\ \delta \end{array} \right) \sum_{|\delta|=q \leq \delta} \left( \begin{array}{c} \delta \\ y \end{array} \right) \sum_{k \leq m - \delta} \left( \begin{array}{c} \delta \\ k \end{array} \right) \left( \frac{\kappa^*_s}{\kappa_s} \right)|I^{\delta} | \frac{|k|!(|m| - |k| - 1)!}{|m|!} \leq \frac{1}{2} \left( \frac{\kappa^*_s}{\kappa_s} \right)^q |H_{r, s, \eta} |
\]

Indeed in the series in (4.36) the index \( m \) cannot be zero, since \( |m| \geq |\delta| = q \geq 1 \), and, therefore, the term \( 2m_j + \alpha_j + \beta_j \) in (4.37) is always greater or equal than 2 for some \( j \). Then (4.37) implies (4.36).

Let us now prove (4.37). For \( |\delta| = q \) we have

\[
\sum_{k \leq m - \delta} \left( \begin{array}{c} m - \delta \\ k \end{array} \right) \left( \frac{\kappa^*_s}{\kappa_s} \right)|I^{\delta} | \frac{|k|!(|m| - |k| - 1)!}{|m|!} = \int_0^1 \left( \sum_{k \leq m - \delta} \left( \begin{array}{c} m - \delta \\ k \end{array} \right) \left( \frac{\kappa^*_s}{\kappa_s} \right) \frac{k}{(1 - t)^{m-\delta-k}} \right) (1 - t)^{|\delta| - 1} dt
\]

\[
= \int_0^1 \left( \frac{\kappa^*_s}{\kappa_s} \right) \frac{m-|\delta|}{(1 - t)^{|\delta| - 1}} dt = \kappa^*_s^{2q-2|m|} \int_0^1 (\kappa^*_s + t(\kappa^*_s - \kappa_s)^{2q-1}(1 - t)^q - 1) dt.
\]

Moreover integrating by parts we get

\[
0 \leq \int_0^1 (\kappa^*_s + t(\kappa^*_s - \kappa_s)^{2q-2|m|} - 1) dt
\]

\[
= \int_0^1 (\kappa^*_s + t(\kappa^*_s - \kappa_s)^{2q-2|m|} - 1) dt
\]

Since for \( |m| \geq q \)

\[
0 \leq (|m| - q) \int_0^1 (\kappa^*_s + t(\kappa^*_s - \kappa_s)^{2q-2|m|} - 1) dt
\]
Since (4.27) and (4.28) we get

\[
\int_0^1 \left( \kappa_*^2 - t \left( \kappa_*^2 - \kappa^2 \right) \right) |m| - q \, dt \leq \frac{1}{q} \left( \kappa_*^2 |m| - 2q \right) .
\]

Then in conclusion we get

\[
\sum_{\kappa \leq m - \delta} \left( \frac{m - \delta}{k} \right) \left( \frac{\kappa_*^2}{\kappa} \right)^{2|m|} \frac{|k|! |m|! |k| - 1)!}{|m|!} \leq \frac{1}{q} \left( \frac{\kappa_*^2}{\kappa} \right)^{2|m| - 2q} .
\]

(4.38)

Recalling the definition of \( A_{q, \kappa, \kappa_*} \) we get (4.37)

\[
A_{q, \kappa, \kappa_*} \leq \frac{1}{2} \left( \frac{\kappa_*}{\kappa} \right)^{2q} \sup_j \sum_{m, \alpha, \beta} e^{n \pi |(\alpha - \beta)|} |H_{m, \alpha, \beta}| (2m_j + \alpha_j + \beta_j) u_0^{2m + \alpha + \alpha - 2e_j \kappa_*} \left( \frac{\kappa_*^2}{\kappa} \right)^{2|m|} \sum_{|\delta| = q, \delta \leq m} \left( \frac{m}{\delta} \right) |\gamma| \right)
\]

(4.27)

\[
= \frac{1}{2} \left( 1 + \frac{\kappa_*^2}{\kappa} \right)^{q} \sup_j \sum_{m, \alpha, \beta} e^{n \pi |(\alpha - \beta)|} |H_{m, \alpha, \beta}| (2m_j + \alpha_j + \beta_j) u_0^{2m + \alpha + \alpha - 2e_j \kappa_*} \sum_{|\delta| = q, \delta \leq m} \left( \frac{m}{\delta} \right) .
\]

(4.39)

Then (4.37) follows by (4.28) and (4.14).

Let us now prove (4.11). By (4.14) and (4.35) we get

\[
\left| H_{2q - 2}^{\gamma - \eta} \right|_{k, s, \eta} \leq \frac{1}{2} \sup_j \sum_{m, \alpha, \beta} \left( \sum_{|\delta| = q, \gamma \leq \delta} \sum_{|\gamma| \leq m} \left( \frac{\delta}{\gamma} \right) I_{\delta - \gamma} \left( \tilde{H}_{m - \gamma, \alpha, \beta} \right) (2m_j + \alpha_j + \beta_j) \left( \kappa_* u_0 \right)^{2m + \alpha + \alpha - 2e_j \kappa_*} e^{n \pi |(\alpha - \beta)|} \right)
\]

(4.33)

\[
\leq \frac{q}{2} \sup_j \sum_{k, \alpha, \beta} \sum_{|\delta| = q, \gamma \leq \delta} \sum_{|\gamma| \leq m + \delta + k} \left( \frac{2k_j + 2\gamma_j + \alpha_j + \beta_j}{} \right) \left( \kappa_* u_0 \right)^{2(k + \gamma_j) + \alpha + \alpha - 2e_j \kappa_*} e^{n \pi |(\alpha - \beta)|} \left( \frac{m}{\delta} \left( \frac{m - \delta}{k} \right) \right) |H_{m, \alpha, \beta}| \left( \frac{m}{|m|} \right) \left( \frac{|m| - |k| - 1)! |m|! \right)
\]

(4.20)

\[
\leq \frac{q}{2} \sup_j \sum_{k, \alpha, \beta} \sum_{|\delta| = q, \gamma \leq \delta} \sum_{|\gamma| \leq m + \delta + k} \left( \frac{2k_j + 2\gamma_j + \alpha_j + \beta_j}{} \right) \left( \frac{\kappa_*}{\kappa} \right)^{2(|\gamma| + 1) |\gamma|} \left( \frac{|m| - |k| - 1)! |m|! \right)
\]

Since all the terms are positive we can exchange the order of summation (of \( k \) and \( m \)) obtaining
\[
\left| H^{\leq 2q-\delta} \right|_{r,s,\eta,\kappa}
\leq \frac{q}{2} \sup_{j} \left( \sum_{m,\alpha,\beta : || \alpha || = q} \sum_{\gamma \leq \delta} \sum_{k \leq m - \delta} (2k_j + 2\gamma_j + \alpha_j + \beta_j) \left( \frac{\kappa_s}{\kappa} \right)^{2(|\gamma| + |\alpha|)} k_s^{(|\alpha| + |\beta|) - 2} 2^{m \gamma_j} \kappa \sum_{u_0}^{2m + \alpha + \beta - 2e_j \eta} e_j |\pi(\alpha - \beta)| \right)
\]

Using \( \kappa_s^{(|\alpha| + |\beta|) - 2} \leq \kappa_s^{-2} \). Since the Lipschitz estimate is analogous we get (4.11). \( \square \)

**Lemma 4.2.** Assume that \( H \in \mathcal{H}_{r,s,\eta} \) can be written as a totally convergent series for \( |u|_s \leq r' \) with \( kr < r' < r \) (\( \kappa \) introduced in (4.1)), of the form

\[
H(u) = \sum_{|\alpha| = |\beta| = q} (|u|^2 - I)^{\delta} \sum_{k,\alpha,\beta} H_{k,\alpha,\beta} |u|^{2k} u^\alpha \bar{u}^\beta, \quad \text{for } |u|_s \leq r',
\]

then

\[
H^{\leq 2q-4} = 0.
\]

**Proof.** Recalling (4.16) we have

\[
H(u, w) = \sum_{|\delta| = q} (w - I)^{\delta} \sum_{k,\alpha,\beta} H_{k,\alpha,\beta} w^k u^\alpha \bar{u}^\beta.
\]

Then

\[
D^{q'} H(u, I) = 0, \quad \forall q' < q.
\]

We conclude by (4.22). \( \square \)

**Remark 4.2.** Proposition 4.2 shows a connection between the order at the origin \( u = 0 \) and on the torus \( T_r \). Indeed if we consider a polynomial function

\[
H(u) = \sum_{|\alpha| = |\beta| \leq N} H_{\alpha,\beta} u^\alpha \bar{u}^\beta
\]

then one has

\[
\Pi_0^{\geq 2N} H = 0.
\]

Conversely if we consider a Hamiltonian of the form

\[
H(u) = \sum_{|\alpha| = |\beta| > N} H_{\alpha,\beta} u^\alpha \bar{u}^\beta,
\]

then, given a parameter \( \kappa_0 \) with

\[
\kappa_s < \kappa_0 < \min\{1, \kappa_s/k\},
\]

by (4.11) (applied with \( \kappa_s \to \kappa_0, \kappa \to \kappa_1 := \kappa_0 \kappa_s \) and \( r \to r_1 := \kappa_s r/\kappa_0 \) so that (4.1) reads \( \sqrt{T} |p, r, a, s, \eta \leq \kappa r = \kappa_1 r_1 \) and by (2.26) we get

\[
\| \Pi_0^{\geq 2q-4} H \|_{r,s,\eta,\kappa} \leq \frac{1}{\kappa_0} \left( \frac{\kappa_s^2 + \kappa_0^2}{\kappa_s^2} c_{\kappa_0} \right)^q \left( \frac{\kappa_s}{\kappa_0} \right)^{2N} \| H \|_{r,s,\eta}. \tag{4.42}
\]
One can easily verify that the projections $\Pi^d$ commute with the projections $\Pi^K$ and $\Pi^C$ defined in (2.27) so that we may define

$$\mathcal{H}^{d,R}_{r,s,\eta} := \{ H \in \mathcal{H}_{r,s,\eta} : \Pi^{d,R} H := \Pi^d \Pi^{R} H = H \}.$$  \hfill (4.43)

Similarly for $\mathcal{H}^{\geq d,R}_{r,s,\eta}$, $\mathcal{H}^{\geq d,K}_{r,s,\eta}$ etc. Note that $\mathcal{H}^{d,K}_{r,s,\eta} = \mathcal{H}^{d,K}_{r,s,\eta} \oplus \mathcal{H}^{d,R}_{r,s,\eta}$.

A particularly important space is $\mathcal{H}^{0,K}_{r,s,\eta}$ which, as can be seen in the following Lemma, is identified isometrically with $\ell_\infty$.

Given $\lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \ell_\infty$ possibly depending in a Lipschitz way on the parameter $\omega \in \mathcal{O}$ we set$^{18}$

$$\| \lambda \|_{\infty, \mu}^O := \sup_{\omega} |\lambda|_{\ell_\infty} + \mu \sup_{\omega \neq \omega'} |\Delta_{\omega, \omega'} \lambda|_{\ell_\infty},$$  \hfill (4.44)

and as usual set $\| \lambda \|_\infty = \| \lambda \|_{\infty, \gamma}^{\gamma, \gamma}$.

Lemma 4.3 (Norms in $\mathcal{H}^{0,K}_{r,s,\eta}$). Every $H \in \mathcal{H}^{0,K}_{r,s,\eta}$ has the form

$$H = \sum_{j \in \mathbb{Z}} \lambda_j (|u_j|^2 - I_j), \quad \lambda = (\lambda_j) \in \ell_\infty : \| H \|_{r,s,\eta} = \| \lambda \|_\infty.$$  \hfill (4.45)

Proof. We start by noting that for $H$ as in (4.12)

$$\Pi^{0,K} H = \sum_m H_{m,0,0} \sum_{i \in \mathbb{Z}} m! I^{m-\ell} (|u_i|^2 - I_i),$$

so $\Pi^{0,K} H = H$ means that $H_{m,\alpha,\beta} = 0$ if $(m, \alpha, \beta) \neq (0, 0, 0)$ or $(e_j, 0, 0)$; moreover all the $H_{e_j,0,0}$ are free real parameters and finally $H_{0,0,0} = -\sum_j H_{e_j,0,0} I_j$. So we set $\lambda_j = H_{e_j,0,0}$ and obtain the representation in (4.45). Regarding the estimate we have

$$|H|_{r,s,\eta} = \frac{1}{2} \sup_j \sum_m |H_{m,\alpha,\beta}| (2m + \alpha + \beta) u_0^{2m + \alpha - 2e_j} e^{\eta \pi (\alpha - \beta)} = \frac{1}{2} \sup_j \sum_m |\lambda_j| (2\delta(i,j)) u_0^{2e_j - 2e_j} = \sup_j |\lambda_j| = \| \lambda \|_{\ell_\infty}.$$  

The Lipschitz estimate is analogous. \hfill \Box

Notation. Lemma 4.3 shows that, for every $r, s, \eta$ the space $\mathcal{H}^{0,K}_{r,s,\eta}$ is isometrically identified by (4.45) with $\ell_\infty$, namely

for $H$ as in (4.45) we write $H \in \ell_\infty$ and $\| H \|_\infty := \| \lambda \|_\infty$. \hfill (4.46)

Such Hamiltonians will be referred to as counterterms.

Lemma 4.4. If $H \in \mathcal{H}_{r,s,\eta}$ then $\| \Pi^{-2} H \|_{r,s,\eta} \leq \| H \|_{r,s,\eta}$. Moreover if $\kappa \leq 1/\sqrt{2}$ we have

$$\| \Pi^0 H \|_{r,s,\eta}, \quad \| \Pi^{0,K} H \|_{\infty} \leq 3 \| H \|_{r,s,\eta}, \quad \| \Pi^{\leq 2} H \|_{r,s,\eta} \leq 4 \| H \|_{r,s,\eta}, \quad \| \Pi^{\geq 2} H \|_{r,s,\eta} \leq 5 \| H \|_{r,s,\eta}.$$  \hfill (4.47)

Proof. By (4.4) with $\kappa_* = 1$, (4.3) and (2.29). \hfill \Box

$^{18}$ Recall (2.23).
We shall consider also the Hamiltonian
\[
D_\omega := \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2 \quad \text{with} \quad \omega \in \mathbb{D}_y.
\] (4.48)

Note that such Hamiltonian belongs to \(\mathcal{A}_r^0(\omega_x)\) for all \(r, s\), see (2.10), but is not in \(\mathcal{H}_{r,s,N}\). We define
\[
\Pi^{-2,K} D_\omega = \sum_{j \in \mathbb{Z}} \omega_j I_j, \quad \Pi^0 K D_\omega = \sum_{j \in \mathbb{Z}} \omega_j (|u_j|^2 - I_j), \quad \Pi^{-2,R} D_\omega = \Pi^0, R D_\omega = \Pi^{\leq 2} D_\omega = 0,
\]
thus extending the projections to the affine space \(D_\omega + \mathcal{H}_{r,s,N}\). We conclude this section with the following Lemma which can be deduced directly from Proposition 4.1.

**Lemma 4.5.** Given \(F, G \in \mathcal{H}_{r,s,N}\) such that \(\Pi^{\leq 0} G = 0\) then
\[
\Pi^{-2} \{F, G\} = 0,
\] (4.49)
similarly if also \(\Pi^{-2} F = 0\) then
\[
\Pi^0 \{F, G\} = 0,
\] (4.50)
finally
\[
\Pi^d \{D_\omega, F\} = \{D_\omega, F^{(d)}\}.
\] (4.51)

5. Normal forms, invariant tori and nearby dynamics

Here we discuss the relation between the projections of the previous section and the dynamics on the torus \(T_L\). The first trivial remark is that if
\[
H = D_\omega + P, \quad P \in \mathcal{H}_{r,s,N}^0,
\]
then \(T_L\) is an invariant manifold for the dynamics. Indeed, by Proposition 4.2 one has
\[
H = \sum_j (|u_j|^2 - I_j) \hat{H}_j(u),
\]
and a direct computation ensures that \(d_t(|u_j|^2) = 0\) on \(T_L\). Recalling the Definition 1.2, of course if \(N\) is in normal form at \(T_L\) then the torus is invariant and the \(N\)-flow is linear on \(T_L\), with frequency \(\omega\). Indeed, for all \(j \in \mathbb{Z}\) one has
\[
u_j(t) = u_j(0)e^{i\omega jt}, \quad |u_j(0)| = \sqrt{I_j},
\]
is an almost-periodic solution for the \(N\)-flow.

In the appendix we show that if
\[
\rho := \inf_{j \in \mathbb{Z}} I_j (\langle j \rangle)^2 \rho^{2\langle j \rangle + 2s(j)}} > 0
\]
then one can introduce action-angle variables around the torus \(T_L\). The action-angle map is a diffeomorphism from a neighborhood of \(T_L\) into \(|I - I| < \rho / 2\} \times \mathbb{T}_N^2\), where \(\mathbb{T}_N^2\) is a differential manifold modeled on \(\ell_1\). We shall call such an object a maximal torus.

Now for \(N \in \mathcal{N}_{r,s,N}(\omega, I)\) we discuss conditions which entail orbital stability at \(T_L\), i.e. that the trajectory of initial data \(\delta\)-close to the torus stays \(\delta\)-close to the torus for times of order \(\delta^{-d}\). To make this statement precise let us fix \(r, s\) and \(\sqrt{I} \in \mathcal{B}_{cr}(w_x)\). Then we define the annulus\(^19\)
\[
A_\delta := \{u \in \mathcal{B}_r(w_x) : \sqrt{||u|^2 - I||} < \delta r \} \quad \text{for} \quad 0 < \delta < \sqrt{1 - \kappa^2}.
\] (5.1)

\(^19\) Where \(\sqrt{||u|^2 - I||}\) is defined componentwise \((\sqrt{||u|^2 - I||})_j := \sqrt{|u_j|^2 - I_j}|.\
Now in order to measure the variation of the actions we can, in the case of a maximal torus, pass to action angle variables and compute the action component of the Hamiltonian vector field on $A_δ$. In order to give a result which holds also in the case of non-maximal tori we use the following

**Lemma 5.1.** Fix $r, s > 0$, $0 < κ < 1$, $d ≥ 2$, $C ≥ 1$ and $\sqrt{I} \in B_{κr}(w_s)$. Then there exist $δ_0 > 0$, $T_0 > 1$ such that if

$$N = Dω + P, \sup_{u \in A_{2δ}} |X_p|_s < \infty, \frac{1}{\sqrt{2r}} \sup_{u \in A_{2δ}} \left| \{P, |u_j|^2\} \right|_{2p, 2s, 2a} \leq Cδ^d, \quad \forall 0 < δ ≤ δ_0,$$

then for any initial datum $u(0) \in A_δ$ the $N$-flow is well defined and remains in $A_{2δ}$ for all times $|t| ≤ T_0δ^{−d}$.

**Proof.** The first two conditions ensure that the Hamilton equations of $N$ are at least locally well-posed in $w_s$. The last condition is an energy estimate. Indeed $\frac{d}{dt}|u_j|^2 = \{N, |u_j|^2\} = \{P, |u_j|^2\}$ hence as long as the flow stays in $A_{2δ}$

$$\frac{d}{dt} \left| |u_j|^2 - I \right|_{2p, 2s, 2a} \leq \sup_{u \in A_{2δ}} \left| \{P, |u_j|^2\} \right|_{2p, 2s, 2a} \leq r^2δ^{d+2}C,$$

so the proof follows by standard contraction arguments. □

We now prove that a Hamiltonian in $\mathcal{H}_r^{\geq 1}$ when restricted to $A_δ$ leaves the actions approximately invariant.

**Lemma 5.2.** Given $H \in \mathcal{H}_r^{\geq 1}$ and $κ < κ^* < 1$, if $0 < δ < \sqrt{κ^2 − κ^2}$ one has

$$\frac{1}{δ^2} \sup_{u \in A_δ} \left| \{H, |u_j|^2\} \right|_{2p, 2s, 2a} \leq 2δ^d c^d_{κ^2} κ^{-d-2} |H|_{r,s,0},$$

moreover

$$\frac{1}{δ^r} \sup_{u \in A_δ} \left| X_H \right|_s \leq δ^{d−1} c^d_{κ^2} κ^{-d+1} |H|_{r,s,0}.$$

**Proof.** First we note that by (2.12) and (4.1) $\sqrt{T_j} \leq κu_{0j}(r) = κu_{0j}$ and

$$u ∈ A_δ \quad ⇒ \quad \sqrt{|u_j|^2 − I_j} ≤ δu_{0j}, \quad |u_j| ≤ κ^2u_{0j}. \tag{5.2}$$

Setting $2q − 2 = d$, and using the representation formula (4.10) and the Leibniz rule, we get for every $j ∈ \mathbb{Z}$

$$\{H(u), |u_j|^2\} = \sum_{|h|=q} (|u|^2 − I)^h \{\tilde{H}_h(u), |u_j|^2\}.$$

Then for $u ∈ A_δ$ we have

$$\frac{1}{δ^2} \sup_{u \in A_δ} \left| \{H, |u_j|^2\} \right|_{2p, 2s, 2a} \leq \frac{1}{δ^2} \sup_{j} \left| \sum_{|h|=q} (|u_j|^2 − I)^h \{\tilde{H}_h(u), |u_j|^2\} \right| \leq \frac{1}{δ^2} \sup_{j} \left| \sum_{|h|=q} \tilde{H}_h(k,α,β)(|u|^2 − I)^h |u|^{2k} u^α \bar{u}^β (2k_j + α_j + β_j) \right| \leq \delta^d \sum_{j} \sum_{|h|=q} \sum_{κ^2} \tilde{H}_h(k,α,β)|u_0|^{2h+2k} |u|^{|α|+|β|} |2k_j + α_j + β_j| \leq q^d \sum_{j} \sum_{|h|=q} \sum_{κ^2} \left( \begin{array}{c} m-h \hline k \end{array} \right) |H_m,α,β| \left( |m| − |h| − |k| \right)! \left( |m| − |h| − |k| − 1 \right)! \left( |m| − |h| − |k| − 2 \right)! \left( |m| − |h| − |k| − 3 \right)! \left( |m| − |h| − |k| − 4 \right)! \left( |m| − |h| − |k| − 5 \right)! \left( |m| − |h| − |k| − 6 \right)!$$
We prove the following.

**Proposition 5.1.** Fix $r, s, \eta > 0$, $0 < \kappa < 1$, $d \geq 2$, and $\sqrt{r} \in B_{k_r}(\omega_s)$ For any $N \in N_{r,s,\eta,\nu}(\omega, I)$, there exists $d > 0$, $T_0 > 1$ such that for $0 < \delta < d$, calling $\Phi_t^\nu(u)$ the flow of $N$ at time $t$ and with initial datum $u$, one has

$$u \in A_{\delta} \Rightarrow \Phi_t^\nu(u) \in A_{2\delta}, \quad \forall |t| \leq T_0 \delta^{-d}.$$

**Proof.** The point is that, for any $d \geq 2$ and given any normal form $N \in N_{r,s,\eta,\nu}$, one can prove the existence of a symplectic change of variables $\Phi_t : A_{\delta} \rightarrow A_{2\delta}$ (for all sufficiently thin annular domains $A_{\delta}$, namely $\delta$ small) that conjugates $N$ to $N_{q} = D_\omega + P_d$, where $P_d$ satisfies the hypotheses of Lemma 5.1. Note that $\Phi_d$ is NOT defined on the whole ball $B_{k_r}(\omega_s)$ unless we assume further restriction on $\kappa$ (rapidly decreasing to 0 as $d$ increases).

Let us briefly discuss how to prove such a result.

We start with a normal form Hamiltonian

$$N = D_\omega + P, \quad P = P_{\geq 2},$$

such that $P \in \mathcal{H}_{r,s,\eta}$. Assume that $0 < \sigma < \eta$ and fix

$$0 < \rho \leq \frac{1}{4} (1 - \kappa) r.$$

We claim that there exists $S \in \mathcal{H}_{r-\rho,s+\sigma,\eta-\sigma}$ such that

$$\Pi_{d/2, \mathcal{R}} \sum_{k=0}^{d/2} \frac{a d^k}{k!} N = 0.$$

To prove our claim we decompose $S = \sum_{h=1}^{d/2} S^{2h}$ where we fix $\sigma_h = \frac{2h}{d} \sigma$, $\rho_h = \frac{2h}{d} \rho$ and assume $S^{2h} \in \mathcal{H}_{r-\rho_h,s+\sigma_h,\eta-\sigma_h}$. The functions $S^{2h}$ are defined recursively as

$$\{D_\omega, S^{2h}\} = \Pi_{2h, \mathcal{R}} \left( \sum_{k=0}^{h-1} \frac{a d^k}{k!} \{S^{<2h}, D_\omega\} + \sum_{k=0}^{h-1} \frac{a d^k}{k!} P \right), \quad S^{<2h} := \sum_{j=1}^{h-1} S^{2j}.$$
By Lemma 3.1 and Proposition 2.2 we get the recursive estimate
\[ |\{D_\omega, S^{2h}\}|_{r-\rho_h,s+\sigma_h-1,\eta-\sigma_h-1} \leq C_h |P|_{r,s,\eta}^{1}, \quad |S^{2h}|_{r-\rho_h,s+\sigma_h,\eta-\sigma_h} \leq C_h |P|_{r,s,\eta}^{1}, \]
which implies
\[ |\{D_\omega, S\}|_{r-\rho,s+\sigma,\eta-\sigma} \leq C \max\{|P|_{r,s,\eta}^{1}, |P|_{r,s,\eta}^{d}\}, \]
where the constants \( C_h, C \) depend on \( r/\rho, \sigma, \theta \) and \( d \); moreover the dependence on \( d \) is superexponentially large. Note that \( S \) belongs to \( \mathcal{R}_{r-\rho,s+\sigma,\eta-\sigma} \); but it is not necessarily small in \( \mathcal{B}_{r-\rho}(\omega_s) \). Of course, by Lemma 5.2 and since \( S \in \mathcal{H}^{2,\infty} \), if we take a sufficiently thin annulus \( A_\delta \), we fall under the hypotheses of Lemma 5.1. In particular, recalling (5.3), we apply Lemma 5.2 with
\[ r \sim r-2\rho, \quad \kappa \sim \frac{kr}{r-2\rho}, \quad \delta \sim \frac{2\delta r}{r-2\rho}, \quad \kappa_2 \sim \frac{r-2\rho}{r}. \]
Then assuming \( \delta \) small enough not only the time one flow of \( S \) is well defined and generates a symplectic change of variables \( \Phi_d : A_\delta \rightarrow A_{2\delta} \) but the Lie series expansion \( e^{[S,\cdot]}N \) is totally convergent. Indeed one has
\[ \frac{ad^k}{k!} N = \frac{ad^{k-1}}{k!} \{S, D_\omega\} + \frac{ad^k}{k!} P \]
and by standard computations (see for instance Lemma 2.1 of [12]) one has
\[ |L_k|_{r-2\rho,s+\sigma,\eta-\sigma} :|\frac{ad^{k-1}}{k!} \{S, D_\omega\}|_{r-2\rho,s+\sigma,\eta-\sigma} \leq \left( \frac{8e|S|_{r-\rho,s+\sigma,\eta-\sigma}}{\rho} \right)^{k-1} |\{S, D_\omega\}|_{r-\rho,s+\sigma,\eta-\sigma} \]
hence by Lemma 5.2
\[ \sup_{u \in A_{2\delta}} |X_L_k| \leq r c_k^2 \kappa_2^k \delta^2 \left( \frac{8e c_k^2 \kappa_2^k \delta^2 |S|_{r-\rho,s+\sigma,\eta-\sigma}}{\rho} \right)^{k-1} |\{S, D_\omega\}|_{r-\rho,s+\sigma,\eta-\sigma}. \]
The same kind of bound holds for \( \frac{ad^k}{k!} P \). Then for all \( u \in A_{2\delta} \)
\[ N \circ \Phi_d = e^{[S,\cdot]} N = \prod_{k=0}^{d/2} \frac{ad^k}{k!} (\sum_{k=0}^{d/2} \frac{ad^k}{k!} N) + \prod_{k=d/2+1}^{\infty} \sum_{k=d/2+1}^{\infty} \frac{ad^k}{k!} N \]
where the first two terms are finite sums and hence analytic in \( \mathcal{B}_{r-\rho}(\omega_s) \) while the series on the right hand side is totally convergent. Then one may apply Lemmata 5.2 and 5.1 and the stability estimates follow.

6. Proof of Theorems 1 and 3

Remark 6.1. In this Section we shall fix \( \gamma \) and always assume that \( \omega \in D_\gamma \) and that Hamiltonian functions depend on \( \omega \) in a Lipschitz way. Hence, for ease of notations, we shall denote the norm (2.24) with \( \mu = \gamma \) and \( \mathcal{O} = D_\gamma \) as
\[ |||r,s,\eta||| :|||r,s,\eta|||_{\gamma, D_\gamma}. \] (6.1)
We start this section by proving Theorem 1 through a direct application to the NLS Hamiltonian of the normal form Theorem 3, and the elimination of the counter terms via the \((V_j)\).

Proof of Theorem 1. Recalling the notations of Theorem 1 we fix
\[ \rho_0 := 2\sqrt{2}r, \quad \rho := \rho_0 - 2r, \quad \eta_0 := \frac{a-a}{2}, \quad \sigma := \frac{1}{2} \min\{s, \eta_0, 2\}, \quad s_0 := s - \sigma, \] (6.2)
and choose \( I = (I_j)_{j \in \mathbb{Z}} \) such that \( \sqrt{T} \in \mathcal{B}_r(\omega_s) \). For \( \omega \in D_\gamma \), let us write the Hamiltonian \( H_{NLS} \) as
\[ H_{NLS} = \sum_{j \in \mathbb{Z}} (j^2 + V_j) |u_j|^2 + P = D_\omega + \Lambda + P', \] (6.3)
where we set
\[ \Lambda = \sum_{j \in \mathbb{Z}} \lambda_j (|u_j|^2 - I_j), \quad \lambda_j = j^2 - \omega_j + V_j, \quad P' = P + \sum_{j \in \mathbb{Z}} \lambda_j I_j. \] (6.4)

Of course \( P' \) satisfies (2.19). By the definition of \( V_j \) and the fact that \( D_r \subset Q \) (recall definitions (1.2) and (1.8)) we have that \( \Lambda \in \ell^\infty \).

In order to apply Theorem 3 to \( H = N_0 + P' \) where \( N_0 = D_\omega \) we fix
\[ \varepsilon_0 := \frac{\bar{\epsilon}}{8 C_{\text{alg}}(p) C(p, s_0, \eta_0)} \] (6.5)
where \( s_0, \eta_0 \) are defined in (6.2) and \( C_{\text{alg}}(p), C(p, s_0, \eta_0) \) are the ones in Proposition 2.1. With this choice the smallness conditions (1.19) are satisfied, since in this case \( \Theta = 0 \). Finally, by setting
\[ V_j(\omega) = \Lambda_j(\omega) + \omega_j - j^2, \]
we obtain
\[ H_{NLS} \circ \Psi = N. \]
Finally note that \( V_j(\omega) \) is Lipschitz in \( \omega \) since \( \Lambda(\omega) \) is so. \( \square \)

The proof of Theorem 3 follows a quadratic KAM scheme, which consists in constructing the solutions of equation (1.20) iteratively, by linearizing the problem at each step and solving the equation as a linear one (homological equation) plus a remainder, which is proved to converge to 0. More precisely we apply the following iterative procedure.

Fix \( r_0, s_0, \eta_0, \rho, r, \sigma \) as in (1.18) and let \( \{\rho_n\}_{n \in \mathbb{N}}, \{\sigma_n\}_{n \in \mathbb{N}} \) be the summable sequences:
\[ \rho_n = \frac{\rho}{4} 2^{-n}, \quad \sigma_0 = \frac{\sigma}{8}, \quad \sigma_n = \frac{9\sigma}{4\pi^2 n^2} \quad \forall n \geq 1. \] (6.6)

Let us define recursively
\[ r_{n+1} = r_n - 2\rho_n \rightarrow r_\infty := r_0 - \rho \quad (\text{decreasing}) \]
\[ s_{n+1} = s_n + 2\sigma_n \rightarrow s_\infty := s_0 + \sigma \quad (\text{increasing}) \]
\[ \eta_{n+1} = \eta_n - 2\sigma_n \rightarrow \eta_\infty := \eta_0 - \sigma \quad (\text{decreasing}). \] (6.7)

Note that for every \( r' \geq r_\infty, s' \leq s_\infty \)
\[ \sqrt{I} \in \mathcal{B}_r(w_{s_\infty}) \begin{array}{c} (1.18) \end{array} \sqrt{I} \in \mathcal{B}_{r'}(w_{s_\infty}) \subset \mathcal{B}_{r/s_\infty} (w_{s_\infty}) \subset \mathcal{B}_{r/s'} (w_{s'}) \] (6.8)
and that the projections \( \Pi^d \) are well defined on every space \( \mathcal{H}_{r'/s', \eta'} \). Moreover by (4.47)
\[ H \in \mathcal{H}_{r'/s', \eta'} \quad \text{with} \quad r' \geq r_\infty, \quad s' \leq s_\infty, \quad \eta' \geq 0, \quad \Rightarrow \quad \|\Pi^0 K H\|_\infty \leq 3 \|H\|_{r', s', \eta'}. \] (6.9)

Let
\[ H_0 := D_\omega + G_0 + \Lambda_0, \quad G_0 \in \mathcal{H}_{r_0, s_0, \eta_0}, \quad \Lambda_0 \in \ell^\infty, \] (6.10)
(recall (4.46)) where the counterterms \( \Lambda_0 \) are free parameters. We define
\[ \varepsilon_0 := \gamma^{-1} \left( \left\| G_0^{(0, K)} \right\|_\infty + \left\| G_0^{(0, R)} \right\|_{r_0, s_0, \eta_0} + \left\| G_0^{(2)} \right\|_{r_0, s_0, \eta_0} \right), \quad \Theta_0 := \gamma^{-1} \left\| G_0^{\leq 2} \right\|_{r_0, s_0, \eta_0} + \varepsilon_0. \] (6.11)

**Lemma 6.1 (Iterative step).** Let \( 0 \leq \rho, \sigma, r, s, \eta, \rho_n, \sigma_n, r_n, s_n, \eta_n \) as in (6.6)-(6.7), \( H_0, G_0, \Lambda_0 \) as in (6.10) and \( \varepsilon_0, \Theta_0 \) as in (6.11). Let \( \sqrt{I} \in \mathcal{B}_r(w_{s_\infty}) \). There exists a constant \( \mathcal{C} > 1 \) large enough such that if

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20 Recall also that \( 0 < \theta < 1 \) was fixed one and for all.
\[
\varepsilon_0 \leq (1 + \Theta_0)^{-3} K^{-2}, \quad K := \mathcal{C} \left( \frac{r_0}{\rho} \right)^4 \sup_n 2^{4n} e^{2\gamma^2 \sigma_{n+\theta}} e^{-\chi^2 (2-\chi)}, \quad C' := 2 \left( \frac{4\pi^2}{9\sigma} \right)^{\frac{3}{2}} C
\]  
(6.12)

(C defined in Lemma 3.1) then we can iteratively construct a sequence of generating functions \( S_i = S_i^{(-2)} + S_i^{(0)} \in \mathcal{H}_{\tau_1 - \rho, \tau_1 + \eta_i - \sigma_i} \) with \( S_i^{(-2)} \in \mathcal{H}_{\tau_1, \tau_1 + \eta_i + \sigma_i} \) and a sequence of counterterms \( \Lambda_i \in \ell^\infty \) such that the following holds, for \( n \geq 0 \).

(1) For all \( i = 0, \ldots, n - 1 \) and any \( s_0 + \eta_0 \geq s' \geq s_{i+1} \) the time-1 hamiltonian flow \( \Phi_{S_i} \) generated by \( S_i \) satisfies

\[
\sup_{u \in \mathcal{B}_{s_{i+1}}(s')} \left| \Phi_{S_i}(u) - u \right|_{s'} \leq \rho 2^{-2i-7} 
\]  
(6.13)

Moreover

\[
\Psi_n := \Phi_{S_0} \circ \cdots \circ \Phi_{S_{n-1}} 
\]  
(6.14)

is a well defined, analytic map \( \tilde{B}_{s_n}(s') \to \tilde{B}_{s_0}(s') \) for all \( s_n \leq s' \leq s_0 + \eta_0 \) with the bound

\[
\sup_{u \in \mathcal{B}_{s_n}(s')} |\Psi_n(u) - \Psi_{n-1}(u)| \leq \rho 2^{-2n+2}. \]  
(6.15)

(2) We set \( L_0 := 0 \) and for \( i = 1, \ldots, n \)

\[
L_i + \text{Id} := e^{\{S_i, \cdot\}}(L_{i-1} + \text{Id}), \quad \Lambda_{i+1} := \Lambda_i - \tilde{\Lambda}_{i-1}, \quad H_i := e^{\{S_i, \cdot\}} H_{i-1}
\]

where \( \Lambda_{i+1} \) are free parameters and \( L_i : \ell^\infty \to \mathcal{H}_{\tau_1, \tau_1 + \eta_i} \) are linear operators. We have

\[
H_i = D_\omega + G_i + (\text{Id} + L_i) \Lambda_i, \quad G_i, \in \mathcal{H}_{\tau_1, \tau_1 + \eta_i}. \]  
(6.16)

Setting for \( i = 0, \ldots, n \)

\[
\varepsilon_i := \gamma^{-1} \left( \left\| G_i^{(0, R)} \right\|_{\tau_1, \tau_1 + \eta_i} + \left\| G_i^{(-2)} \right\|_{\tau_1, \tau_1 + \eta_i} \right), \quad \Theta_i := \gamma^{-1} \left\| G_i^{(-2)} \right\|_{\tau_1, \tau_1 + \eta_i} + \varepsilon_i 
\]  
(6.17)

we have

\[
\varepsilon_i \leq \varepsilon_0 e^{\chi^2 \rho^2}, \quad \chi := 3/2, \quad \Theta_i \leq \Theta_0 \sum_{j=0}^{i} 2^{-j} \]  
(6.18)

\[
\left\| (L_i - L_{i-1}) h \right\|_{\tau_1, \tau_1 + \eta_i} \leq K \varepsilon_0 (1 + \Theta_0) 2^{-i} \left\| h \right\|_{\infty}, \quad \left\| L_i h \right\|_{\tau_1, \tau_1 + \eta_i} \leq K (1 + \Theta_0) \varepsilon_0 \sum_{j=1}^{i} 2^{-j} \left\| h \right\|_{\infty}, \]  
(6.19)

for all \( h \in \ell^\infty \). Finally the counter-terms satisfy the bound

\[
\left\| \tilde{\Lambda}_{i-1} \right\|_{\infty} \leq \gamma K \varepsilon_{i-1} (1 + \Theta_0), \quad i = 1, \ldots, n. \]  
(6.20)

By (6.15), (6.20) and (6.19) we get

Corollary 6.1. \( \Psi := \lim_{n \to \infty} (\Psi_n)_n \) is well defined as a map from \( \tilde{B}_{s_0}(s) \to \tilde{B}_{s_0}(s) \). Moreover the sequence \( \tilde{\Lambda}_n \) is summable. Finally the sequence \( \mathcal{L}_n \) converges to an operator \( \mathcal{L} : \ell^\infty \to \mathcal{H} \) in \( \ell^\infty \). Let us now prove Theorem 3 by applying Lemma 6.1.

**Proof of Theorem 3.** Take

\[
\bar{\varepsilon} := 2^{-15} K^{-2}, \quad \bar{C} := 2^{7} K, \]  
(6.21)

with \( K \) defined in (6.12). Recalling that \( H - D_\omega \in \mathcal{H}_{\rho_0, \rho_0 - \rho_0} \) let us set

\[
H_0 := D_\omega + G_0 + \Lambda_0, \quad G_0 := H - D_\omega := (N_0 - D_\omega) + G \]

\[21\] Namely \( S_i^{(-2)} \) is more regular since \( \mathcal{H}_{\tau_1, \tau_1 + \eta_i - \sigma_i} \subset \mathcal{H}_{\tau_1, \rho_1, \rho_1 + 1, \eta_i + 1}. \]
where \( \Lambda_0 \in \ell^\infty \) are free parameters and \( \| G \|_{r_0, s_0, \eta_0} = \| H - N_0 \|_{r_0, s_0, \eta_0} = \gamma \epsilon \) is small. Since

\[ G_0^{\leq 0} = (N_0 - D_\omega + G)^{\leq 0} = G^{\leq 0}, \]

by (1.18) and Lemma 4.4 we have that

\[ \| G_0^{(-2)} \|_{r_0, s_0, \eta_0} \leq \| G \|_{r_0, s_0, \eta_0}, \quad \| G_0^{(0, K)} \|, \quad \| G_0^{(0, R)} \|_{r_0, s_0, \eta_0} \leq 3 \| G \|_{r_0, s_0, \eta_0} \]

hence

\[ \varepsilon_0 := \gamma^{-1} \left( \| G_0^{(0, K)} \| + \| G_0^{(0, R)} \|_{r_0, s_0, \eta_0} + \| G_0^{(-2)} \|_{r_0, s_0, \eta_0} \right) \leq 7 \epsilon. \] (6.22)

Moreover, since \( G_0^{\geq 2} = (N_0 - D_\omega)^{\geq 2} + G^{\geq 2} \), we also have that

\[ \Theta_0 := \gamma^{-1} \| G_0^{\geq 2} \|_{r_0, s_0, \eta_0} + \varepsilon_0 \leq 4 \gamma^{-1} \left( \| N_0 - D_\omega \|_{r_0, s_0, \eta_0} + \| G \|_{r_0, s_0, \eta_0} \right) + \varepsilon_0 \leq 4 \Theta + 11 \epsilon. \]

By (6.21) the hypothesis (6.12) of Lemma 6.1 is satisfied. Then by Lemma 6.1 and by Corollary 6.1 we pass to the limit in (6.16) and obtain

\[ H_0 \circ \Phi = D_\omega + G_\infty + (\text{Id} + \mathcal{L})(\Lambda_0 - \sum_{i=0}^\infty \tilde{\Lambda}_i) =: N. \]

Now by formula (6.18) we have \( N^{\leq 2} = D_\omega \) provided that we fix

\[ \Lambda_0 = \Lambda = \sum_{i=0}^\infty \tilde{\Lambda}_i. \]

This concludes the proof. \( \square \)

**Proof of the iterative Lemma 6.1.** Throughout the Lemma we do not keep track of constant terms in the Hamiltonians \( H_i \). Of course such terms do not contribute to the seminorm \( \| \cdot \|_{r_i, s_i, \eta_i} \), moreover, by Proposition 2.3, \( H_i(0) = H_0(0) \) for all \( i \).

In the following by \( a \lesssim b \) we mean that there exists a positive constant \( c \) (depending only on \( \theta \), which is fixed) such that \( a \leq cb \).

Moreover, we will repeatedly make use of Proposition 2.2, Lemma 3.1 and Proposition 4.1, this last one always and tacitly with \( \kappa_a = 1 \).

**Initialization.** Proving that Lemma 6.1 holds at \( n = 0 \) is essentially tautological. Indeed item (1) is empty, while item (2) follows directly from the definitions: (6.16) and (6.17) coincide resp. with (6.10) and (6.11) while the bound (6.18) is trivial. Now, assuming that Lemma 6.1 holds up to \( n \geq 0 \), we verify that it holds also for \( n + 1 \).

**Proving the \( n + 1 \) step.** We first observe that, since

\[ (\sqrt{T_i})_{j \in \mathbb{Z}} \in \bar{B}_r(w_{s_0 + \sigma}) \subset \bar{B}_{r_j}(w_{s_j}), \quad \forall n \]

by Proposition 4.1, the projections \( \Pi^{0, K} \) and \( \Pi^{0, R} \) are well defined and continuous throughout the iteration. Assume that

\[ H_n := D_\omega + (\text{Id} + \mathcal{L}_n)\Lambda_n + G_n \]

satisfies (6.18) and (6.19) with \( i = n \). We fix the generating function \( S_n = S_n^{(-2)} + S_n^{(0)} \) and the counter-term \( \tilde{\Lambda}_n \in \ell^\infty \) as the unique solutions of the homological equation

\[ \Pi^{\leq 0}\left( (\text{Id} + \mathcal{L}_n)\tilde{\Lambda}_n + G_n + \left\{ S_n, D_\omega + G_n^{\geq 2} \right\} \right) = G_n^{(-2, K)}. \] (6.23)

Solving this equation amounts to canceling the non-quadratic terms which prevent the torus \( T_I \) to be invariant for the Hamiltonian \( e^{(S_n)_{-1}}H_n \) (recall Lemma 4.5).

Let us project (6.23) on the three subspaces \( \ell^\infty, \mathcal{H}^{0, R}, \mathcal{H}^{-2} \); by Lemma 4.5 the equation (6.23) splits into the following triangular system

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Let us first solve equation (6.24) “modulo $\tilde{\Lambda}_n$”; secondly we determine the counter term $\tilde{\Lambda}_n$ in equation (6.25) and, eventually, solve equation (6.26). In what follows we repeatedly apply Lemma 3.1, Proposition 2.2 and Proposition 4.1 in order to solve the (system of) homological equation (6.23) and bound appropriately the solutions to obtain (6.13)-(6.20) for $i = n$.

**Existence of $S_n$, $\tilde{\Lambda}_n$ and corresponding bounds.** Let us start with (6.24), which gives

$$S_n^{(-2)} = L_\omega^{-1} \left( G^{(-2),R} + \Pi^{-2,R} \mathcal{L}_n \tilde{\Lambda}_n \right).$$

(6.27)

By substituting $S_n^{(-2)}$ in (6.25), we get

$$\left( \text{Id} + \Pi^{0,K} \mathcal{L}_n \right) \tilde{\Lambda}_n + \Pi^{0,K} \left\{ L_\omega^{-1} \left( G^{(-2)} + \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n \right), G_n^{2+} \right\} = -G_n^{(0,K)},$$

(6.28)

namely

$$(\text{Id} + M_n) \tilde{\Lambda}_n = -G_n^{(0,K)} - \Pi^{0,K} \left\{ L_\omega^{-1} G_n^{(-2)}, G_n^{2+} \right\},$$

(6.29)

where $M_n : \ell^\infty \to \ell^\infty$ is the operator defined by:

$$h \mapsto M_n h := \Pi^{0,K} \mathcal{L}_n h + \Pi^{0,K} \left\{ \Pi^{-2} L_\omega^{-1} \mathcal{L}_n h, G_n^{2+} \right\}.$$

(6.30)

The following Lemma will be proved in Appendix A.3.

**Lemma 6.2.** $\| M_n h \|_{\ell^\infty} \leq \frac{1}{2} \| h \|_{\ell^\infty}$.

Since the operator norm of $M_n$ is smaller than $\frac{1}{2}$, then Id$+M_n$ is invertible by Neumann series and its inverse is bounded by 2, in operator norm. We thus conclude that

$$\tilde{\Lambda}_n = -\left( \text{Id} + M_n \right)^{-1} \left( G_n^{(0,K)} + \Pi^{0,K} \left\{ L_\omega^{-1} G_n^{(-2)}, G_n^{2+} \right\} \right).$$

(6.31)

By (6.17), (4.47), (2.33) and Lemma (3.1) we get

$$\left\| \tilde{\Lambda}_n \right\|_{\ell^\infty} \leq 2 \left\| G_n^{(0,K)} \right\|_{\ell^\infty} + 2 \left\| \Pi^{0,K} \left\{ L_\omega^{-1} G_n^{(-2)}, G_n^{2+} \right\} \right\|_{\ell^\infty} \leq 2 \gamma_\epsilon + 6 \left\| L_\omega^{-1} G_n^{(-2)}, G_n^{2+} \right\|_{\ell^\infty} \leq 2 \gamma_\epsilon + 6 \left\| L_\omega^{-1} G_n^{(-2)}, G_n^{2+} \right\|_{\ell^\infty}$$

(6.32)

where the last inequality follows by (6.12) and noting that $\eta_\alpha \geq \eta_0 + \sigma \geq \sigma = 8\sigma_0$. This proves (6.20) for $i = n + 1$. We can thus bound the solution $S_n^{(-2)}$ determined in (6.27) as

$$S_n^{(-2)} \left\|_{\ell^\infty} \right. \leq \gamma^{-1} e^{C_{\eta_0}^{-1/3}/\epsilon_{\eta_0}} \left( \left\| G_n^{(-2)} \right\|_{\ell^\infty} + \left\| \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n \right\|_{\ell^\infty} \right) \leq e_\eta e^{C_{\eta_0}^{-1/3}/\epsilon_{\eta_0}} \left( 1 + \kappa^2 (1 + \Theta_0)^2 \epsilon_0 \right) \leq 2 e_\eta e^{C_{\eta_0}^{-1/3}/\epsilon_{\eta_0}}$$

(6.33)

by (6.17), (6.19), (6.32), (6.12). Equation (6.26) determines $S_n^{(0)}$

$$S_n^{(0)} = -L_\omega^{-1} \left( G_n^{(0,R)} + \Pi^{0,R} \mathcal{L}_n \tilde{\Lambda}_n + \Pi^{0,R} \left\{ S_n^{(-2)}, G_n^{2+} \right\} \right).$$

22 Note that at the first step we have $M_0 \equiv 0$, since $\mathcal{L}_0 = 0$ and equation (6.25) determines $\tilde{\Lambda}_0$ trivially.
which, by Lemma 3.1, (6.17), (4.47), (6.19), (6.32), (6.33), (2.33) and (6.12) satisfies
\[
\|S_n^{(0)}\|_{r_n - \rho_n, s_{n+1}, \eta_{n+1}} \leq e^{C r_n^{-3/\rho}} n \left( 1 + K^2 (1 + \Theta_0)^2 \varepsilon_0 + \frac{r_n}{\rho_n} e^{C r_n^{-3/\rho}} \Theta_0 \right)
\]
\[
\lesssim e^{2 C r_n^{-3/\rho}} n \varepsilon_0 (1 + \Theta_0).
\]
Recalling (6.33), (6.6), (6.7) and using \(\varepsilon_n \leq \varepsilon_0 e^{-\chi^{n+1}}\) (by the inductive hypothesis (6.18)) we get
\[
\|S_n\|_{r_n - \rho_n, s_{n+1}, \eta_{n+1}} \leq C r_0 \frac{2^{-n-10} e^{C r_n^{-3/\rho}} e^{-\chi^{n}} n \varepsilon_0 (1 + \Theta_0)}{\rho_n} \leq C 2^{-n-10} \varepsilon_0 (1 + \Theta_0) \sqrt{K},
\]
where \(C > 1\) is a universal constant (only depending on the fix quantity \(\theta\)).

**The maps \(\Phi_S\) and \(\Psi_n\).** For any \(s_0 + \eta_0 \geq s' \geq s_{n+1}\), fixing \(\eta' = s_0 + \eta_0 - s'\), by the monotonicity entailed in (2.40) and by (6.35), we have that
\[
\|S_n\|_{r_n - \rho_n, s', \eta'} \leq \|S_n\|_{r_n - \rho_n, s_{n+1}, \eta_{n+1}} \leq C 2^{-n-10} \varepsilon_0 (1 + \Theta_0) \sqrt{K},
\]
where \(C\) is the constant in (6.35). We wish to apply Proposition 2.3 with \(\rho \sim \rho_n\) and \(r \sim r_n - \rho_n\). Indeed
\[
\|S_n\|_{r_n - \rho_n, s', \eta'} \leq \frac{2^{-n-10}}{r_0} \frac{\rho_n}{16e(r_n - \rho_n)} 2^{n-1} \leq \frac{\rho_n}{16e(r_n - \rho_n)} 2^{n-1}
\]
by (6.12). In turn, the bound (6.37) implies (2.34); then (6.13) for \(i = n\) follows by (2.35) and (6.6) (1.18).

By (6.14) and the estimate (6.37) we have
\[
\Psi_{n+1} := \Psi_n \circ \Phi_{S_0} = \Phi_{S_0} \circ \cdots \circ \Phi_{S_{n-1}} : \tilde{B}_{r_0 + \rho_n} (w_{s'}) \rightarrow \tilde{B}_{r_0 - \rho_0} (w_{s'})
\]
(recall that \(r_{n+1} + \rho_n = r_n - \rho_n\) for all \(s_{n+1} \leq s' \leq s_0 + \eta_0\). Note that \(\Phi_{S_{n-1}} \circ \Phi_{S_1} \) is well defined for any \(i = 1, \ldots, n\) since \(\tilde{B}_{r_{i-1} - \rho_{i-1}} (w_{s'}) \subset \tilde{B}_{r_{i-1} - 2 \rho_{i-1}} (w_{s'})\).

Again Proposition 2.3 with \(s' = s_{n+1}\) implies that \(H_{n+1} = e^{\{S_{n+1}\}} H_n\) is well defined and \(H_{n+1}\)-majorant analytic (where \(\eta_{n+1} = \eta_n - 2 \sigma_n\), recall definition (6.6)).

We now prove (6.15). In formula (6.13) in Lemma 6.1, we have proved that for any \(n \geq 0\)
\[
\Psi_n = \Phi_{S_0} \circ \cdots \circ \Phi_{S_{n-1}}, \quad \Psi_n : \tilde{B}_{r_n + \rho_n - \rho_0} (w_{s'}) \rightarrow \tilde{B}_{r_0} (w_{s'}).
\]
Setting \(u_t = (1 - t)u + t \Phi_{S_0} (u), \ t \in [0, 1]\) for \(u \in \tilde{B}_{r_n - \rho_0} (w_{s'})\), we have
\[
\Psi_n (u) - \Psi_n (u) = \Psi_n (\Phi_{S_0} (u)) - \Psi_n (u) = \int_0^1 \Psi_n (u_t) \left[ \Phi_{S_0} (u) - u \right] dt.
\]

In the following, in order not to burden the notations we will only indicate indexes of norms which undergo some variation as
\[
|\cdot|_{r_n} := \sup_{u \in \tilde{B}_{r_n} (w_{s'})} |\cdot|_{s'}, |\cdot|_{op, r_n} := \sup_{u \in \tilde{B}_{r_n} (w_{s'})} |\cdot|_{op}, \Phi_n := \Phi_{S_n}.
\]
So,
\[
|\Psi_{n+1} - \Psi_n|_{r_{n+1}} \leq \int_0^1 \left\| \Psi'_n (u_t) \right\|_{op, r_n - \rho_n} dt \left| \Phi_n - \text{Id} \right|_{r_{n+1}} \leq \int_0^1 \left\| \Psi'_n \right\|_{op, r_n - \rho_n} dt \left| \Phi_n - \text{Id} \right|_{r_{n+1}}.
\]
By the chain rule and Cauchy estimates, we have that
\[ |\Psi_n|_{H_{\rho_n}} = \sup_{u \in H_{\rho_n}} |\Phi'_{n-1}(u) \cdots \cdot \Phi'(u)| \leq L \rho_n^{-1} \left( |\Phi_{n-1} - \Phi'(u)|_{\rho_n} + 1 \right) \leq L (2^{-j - 5} + 1) \leq 2. \]

So,

\[ |\Psi_{n+1}(u) - \Psi_n(u)|_{\rho_{n+1}} \leq 2 |\Phi_n - \text{Id}|_{\rho_{n+1}} \leq \rho^{2-2n-6}. \]

**Bound on** \( L_{n+1} - L_n \). By (6.19) for \( i = n \) and (6.12) we have

\[ \|(L_n + \text{Id})h\|_{\rho_{n+1}, \eta_{n+1}} \leq \|(L_n + \text{Id})h\|_{\rho_0} \leq 2 \|

Since by construction

\[ (L_{n+1} - L_n) = \left( e^{S_{n+1}} - \text{Id} \right) \circ (L_n + \text{Id}). \]

by (2.37), (6.35) and (6.6) we get

\[ \|(L_{n+1} - L_n)h\|_{\rho_0} \leq 16e^{\rho_0} \rho_0 \|

proving the first bound in (6.19) for \( i = n + 1 \), taking \( C \) large enough in (6.12).

The second bound in (6.19) follows directly from the first one.

**Bounds on** \( G_{n+1} \) and \( G_{n+1}^{\geq 2} \). By construction

\[ G_{n+1} = e^{S_{n+1}} H_n - [D_0 + (L_{n+1}) \Lambda_{n+1}]. \]

Since \( S_n \) solves the Homological equation (6.23), we have that

\[ G_{n+1} = G_{n+1}^{\leq 2, K} + G_{n+1}^{\geq 2} + \Pi \left( L_{n+1} \Lambda_n + \left\{ S_n, G_{n+1}^{\geq 2} \right\} \right) + G_{n+1}^{*, \leq 0} \]

Note that \( G_{n+1}^{*, \leq 0} \) is quadratic in \( S_n \sim G_{n+1}^{0, \leq 0} \).

In order to prove (6.18) for \( i = n + 1 \) we just need to apply Proposition 2.3 and repeatedly use Lemmas 2.2-4.1. In the following formula only the radius of analyticity changes, hence, for brevity, we omit to write \( s_{n+1}, \eta_{n+1} \) in the indexes of the norms. We have

\[ \|G_{n+1}^{*, \leq 0}\|_{\rho_{n+1}} \leq \frac{r_0}{\rho_n} \left[ \|S_n\|_{\rho_{n+1}} \|G_{n+1}^{0, \leq 0}\|_{\rho_{n+1}} + \frac{r_0}{\rho_n} \|S_n\|_{\rho_{n+1}} \|G_{n+1}^{0, \leq 0}\|_{\rho_{n+1}} + \frac{r_0}{\rho_n} \|S_n\|_{\rho_{n+1}} \|G_{n+1}^{0, \leq 0}\|_{\rho_{n+1}} + \frac{r_0}{\rho_n} \|S_n\|_{\rho_{n+1}} \|G_{n+1}^{0, \leq 0}\|_{\rho_{n+1}} \right]. \]

Hence by (6.17), (6.18), (6.32), (6.33) and (6.35) we have

\[ \gamma_1 \frac{r_0}{\rho_n} \left[ (1 + \Theta_0)^2 e^{2\rho_0 \rho_n} e ^{2\rho_0 \rho_n} e^{-\chi_0} \right]. \]

Since the same estimate holds for \( \|G_{n+1}^{0, K}\|_{\rho_{n+1}} \) and \( \|G_{n+1}^{0, R}\|_{\rho_{n+1}} \), we get

\[ \varepsilon_{n+1} \leq \left( \frac{r_0}{\rho_n} \right)^3 K(1 + \Theta_0)^2 e^{2\rho_0 \rho_n} e ^{2\rho_0 \rho_n} e^{-\chi_0} \leq K(1 + \Theta_0)^2 e^{2\rho_0 \rho_n} e^{-\chi_0}. \]
Then the first estimate in (6.18) for \( i = n + 1 \) follows by (6.12). With similar calculations we estimate \( \| G_{\mu+1}^{\ge 2} \|_{r_{n+1}} \), obtaining

\[
\Theta_{n+1} - \Theta_n \lesssim \varepsilon_2^2 (1 + \Theta_0)^2 e^{-\chi^n} + \varepsilon_0 (1 + \Theta_0) \Theta_0 2^{2n} e^{-\chi^n} e^{C n^6/\theta} + 2^4 e^{2C n^6/\theta} e^{-2\chi^n} \varepsilon_0^2 (1 + \Theta_0)^3.
\]

Then, taking \( C \) large enough in (6.12) we get

\[
\Theta_{n+1} - \Theta_n \leq \Theta_0 2^{-n},
\]

proving the second estimate in (6.18) for \( i = n + 1 \). \( \square \)

7. Lower dimensional tori

As discussed in the previous section an advantage of our counterterm method is that it is uniform in the dimension of the torus, namely in the number of non-zero actions. Of course one should expect that in constructing a non-maximal torus one only needs to modulate the frequencies relative to the non-zero actions. In this section we make this statement precise and we discuss an application to the NLS.

Let \( S \) be any subset of \( \mathbb{Z} \) and denote \( v := (v_j)_{j \in S} := (u_j)_{j \in S} \) and \( z := (z_j)_{j \in S^c} := (u_j)_{j \in S^c} \). With abuse of notation we will write \( u = (v, z) \). Analogously we will write \( V = (U, W, \lambda = (v, \mu), \omega = (\alpha, \Omega) \). In particular we consider \( I = (J, 0) \) with \( I_j = J_j > 0 \) for \( j \in S \).

Similarly to section 4, given \( H \in \mathcal{H}_{r,s,\eta} \), we expand in Taylor series as

\[
H = \sum_{m, a, \beta, a, b \in \mathbb{N}^S \atop \alpha|\beta| = \theta, \delta \leq m} H_{m, a, \beta, a, b} \left| v \right|^{2m} \bar{v}^\beta \bar{z}^a \bar{z}^b
\]

and define appropriate projection operators depending on the variables we are considering. More precisely:

- on the variables \( v \) supported in \( S \), we use the projections defined in section 4
- on the variables \( z \) supported in \( S^c \), we use the projections on the terms of fixed homogeneous degree at \( z = 0 \).

This gives rise to two degree indices; nevertheless we will not distinguish them but instead define just one degree as follows

\[
H^{(d)} = \sum_{m, a, \beta, a, b \in \mathbb{N}^S \atop \alpha|\beta| = \theta, \delta \leq m} H_{m, a, \beta, a, b} \left( \begin{array}{c} m \\ \delta \end{array} \right) J^{m-\delta} \left| v \right|^2 - J^\delta \bar{v}^\beta \bar{z}^a \bar{z}^b. \tag{7.1}
\]

In this way, if \( S = \mathbb{Z} \), projections coincide with the ones of section 4, while if \( S = \emptyset \), \( H^{(d)} \) represents the usual homogeneous degree at \( z = 0 \).

Note that in this definition the degree \( d \) is always \( \geq -2 \) but now it is no more necessarily even. In particular, for the projections onto \( \mathcal{H}_{r,s,\eta}^{0,K} \), following the previous definitions, we shall set for any \( H \in \mathcal{H}_{r,s,\eta}^{0,K} \)

\[
\Pi^{0,K} H = \Pi^{0,K} S H + \Pi^{0,K} S^c H := \sum_{m \neq 0} \sum_{\ell \in S} J^{m-\ell} m \left( \left| v_\ell \right|^2 - J_\ell \right) + \sum_{j \in S^c} H_0, 0, 0, \epsilon, \mu_j \left| z_j \right|^2. \tag{7.2}
\]

Of course, all the properties enjoyed by \( \Pi^{0,K} \) and, in general by \( \Pi^{(d)} \) entailed in Propositions 4.1 and 4.2 hold also in this case. The only difference is that now \( q \in \mathbb{N}/2 \) and \( c_k \sim \max \{1, c_k\} \).

We point out that in this frame, the space \( \ell_\infty^\infty \) of counter-terms (i.e. Hamiltonians \( \Lambda \) such that \( \Pi^{0,K} \Lambda = \Lambda \)) now consists of elements of the form

\[
\Lambda = \sum_{j \in S} v_j \left( \left| v_j \right|^2 - J_j \right) + \sum_{j \in S^c} \mu_j \left| z_j \right|^2, \quad v = (v_j)_{j \in S}, \mu = (\mu_j)_{j \in S^c} \in \ell_\infty^\infty. \tag{7.3}
\]

\( ^{23} \) Consisting in a reordering of the indexes \( j \).
Definition 7.1 (Diophantine condition). We say that a vector $\omega \in \mathbb{Q}$ belongs to $D_{\gamma,S}$ if it satisfies

$$|\omega \cdot \ell| \geq \gamma \prod_{n \in \mathbb{Z}} \left( \frac{1}{1 + |\ell_n|^2(n)} \right), \quad \forall \ell \in \mathbb{Z}^Z : \ell \neq 0, \quad |\ell| := \sum_i |\ell_i| < \infty, \quad \sum_j |\ell_j| \leq 2.$$  (7.4)

Note that $D_{\gamma} \subset D_{\gamma,S}$ (recall (1.9)). We also call $D_{\gamma,S}^0$ the set of $\omega \in \mathbb{Q}$ satisfying (7.4) only for $\ell$ with zero momentum, namely $\pi(\ell) = 0$. Clearly $D_{\gamma,S}^0 \subset D_{\gamma,S}$.

Theorem 7.1 (à la Herman–Féjoz). Given any $S \subset \mathbb{Z}$, if $I = (J, 0)$ then Theorem 3 holds word by word with $\omega \in D_{\gamma,S}$, $\bar{\epsilon} \sim (1 + \Theta)^{-2}\epsilon$ and $\bar{C} \sim (1 + \Theta)\bar{C}$.

Remark 7.1. The main point in this result is that if some of the actions are zero (say all those supported on $S^c$) then one may impose weaker diophantine conditions, namely $\omega \in D_{\gamma,S}$, instead of $\omega \in D_{\gamma}$. 

Proof. We start with a Hamiltonian $H_0 = D_\omega + \Lambda + G_0$ with $\Lambda \in \ell^\infty$. We need the corresponding of the iterative Lemma 6.1, to construct a change of variables and a counterterm such that

$$\Pi_{D_{H_0}}^{\leq 0} e^{(S;\cdot)}(D_\omega + \Lambda + G) = D_\omega.$$  

As in Lemma 6.1, at the $n$th step we have an expression of the form

$$H_n = D_\omega + (\text{Id} + \mathcal{L}_n)\Lambda_n + G_n$$

with $G_n \in \mathcal{H}_{\mathcal{S}_n, \mathcal{S}_n, \Lambda_n}$. The generating function $S_n$ and the counterterm $\Lambda_n$ are fixed as the unique solutions of the Homological equation

$$\Pi_{D_{H_0}}^{\leq 0} \left( \left\{ S_n, D_\omega + G_n^{\mathcal{S},1} \right\} + (\text{Id} + \mathcal{L}_n)\Lambda_n + G_n \right) = G_n^{(-2, \mathcal{K})}.$$  

As before this equation can be written componentwise as a triangular system and solved consequently. We have

$$\begin{align*}
S_n^{(-2)} & \quad + \Pi^{-2,\mathcal{R}} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(-2,\mathcal{R})} = 0, \\
S_n^{(-1)} & \quad + \Pi^{-1} \left( S_n^{(-2)}, G_n^{\mathcal{S},1} \right) + \Pi^{-1} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(-1)} = 0, \\
\Pi^{0,\mathcal{K}} \left( S_n^{(-2)} + S_n^{(-1)}, G_n^{\mathcal{S},1} \right) + \tilde{\Lambda}_n + \Pi^{0,\mathcal{K}} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(0,\mathcal{K})} = 0, \\
S_n^{(0,\mathcal{R})} & \quad + \Pi^{0,\mathcal{R}} \left( S_n^{(-2)} + S_n^{(-1)}, G_n^{\mathcal{S},1} \right) + \Pi^{0,\mathcal{R}} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(0,\mathcal{R})} = 0. 
\end{align*}$$  

Now we solve

$$\begin{align*}
S_n^{(-2)} & = L_\omega^{-1} \left( \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(-2)} \right), \\
S_n^{(-1)} & = L_\omega^{-1} \left( \Pi^{-1} \left( L_\omega^{-1} \left( \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(-2)} \right), G_n^{\mathcal{S},1} \right) + \Pi^{-1} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(-1)} \right), \\
S_n^{(0,\mathcal{R})} & = L_\omega^{-1} \left( \Pi^{0,\mathcal{R}} \left( S_n^{(-2)} + S_n^{(-1)}, G_n^{\mathcal{S},1} \right) + \Pi^{0,\mathcal{R}} \mathcal{L}_n \tilde{\Lambda}_n + G_n^{(0,\mathcal{R})} \right). 
\end{align*}$$  

Then we solve for $\tilde{\Lambda}_n$

$$\begin{align*}
\Pi^{0,\mathcal{K}} \left( L_\omega^{-1} \left( \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n + \Pi^{-1} \left( L_\omega^{-1} \Pi^{-2} \mathcal{L}_n \tilde{\Lambda}_n, G_n^{\mathcal{S},1} \right), G_n^{\mathcal{S},1} \right) + \tilde{\Lambda}_n + \Pi^{0,\mathcal{K}} \mathcal{L}_n \tilde{\Lambda}_n = \\
\Pi^{0,\mathcal{K}} \left( L_\omega^{-1} \left( \Pi^{-1} \left( L_\omega^{-2} \mathcal{L}_n h + \Pi^{-1} \left( L_\omega^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{\mathcal{S},1} \right), G_n^{\mathcal{S},1} \right) + \Pi^{0,\mathcal{K}} \mathcal{L}_n h \right) \right). 
\end{align*}$$

As in the previous case this amounts to showing that the operator $M_n : \ell^\infty \rightarrow \ell^\infty$ defined as

$$M_n h = \Pi^{0,\mathcal{K}} \left( L_\omega^{-1} \left( \Pi^{-2} \mathcal{L}_n h + \Pi^{-1} \left( L_\omega^{-1} \Pi^{-2} \mathcal{L}_n h, G_n^{\mathcal{S},1} \right), G_n^{\mathcal{S},1} \right) + \Pi^{0,\mathcal{K}} \mathcal{L}_n h \right)$$

satisfies an estimate of the type
\[ \|M_n h\|_\infty \leq \frac{1}{2} \|h\|_\infty. \]

The proof of this last bound follows just like the corresponding Lemma 6.2. Then
\[ \tilde{\alpha}_n = (\text{Id} + M_n)^{-1}(\Pi^{0,\mathcal{X}}\{ L_{\omega}^{-1}\left( G_{n}^{\omega(-2)} + \Pi^{-1}\{ L_{\omega}^{-1} G_{n}^{\omega(-2)} , G_{n}^{\omega(1)} \} \right) \}), \]

is fixed. Now we substitute in the equations (7.9), compute \( \mathcal{C}^{\omega(-2)} \) and \( \mathcal{C}^{\omega(-1)} \) and finally \( \mathcal{C}^0,\mathcal{X} \). The estimates follow exactly as in Section 6. \( \Box \)

Let us discuss the consequences of Theorem 7.1 on the NLS equation (1.1). Rewrite the NLS Hamiltonian (1.6) in the form
\[ H_{\text{NLS}} = \sum_{j \in \mathcal{S}} (j^2 + U_j) |v_j|^2 + \sum_{j \in \mathcal{S}'} (j^2 + W_j) |\alpha_j|^2 + P \]  \hspace{1cm} (7.10)

where \( U = (U_j)_{j \in \mathcal{S}} = (V_j)_{j \in \mathcal{S}} \) are free parameters while \( W = (W_j)_{j \in \mathcal{S}'} = (V_j)_{j \in \mathcal{S}'} \) are fixed.

In order to keep the proof as simple as possible, we shall avoid technical issues related to double eigenvalues by assuming that \( P \) preserves momentum, namely that \( f(x, |u|^2) \) in (1.1) does not depend directly on \( x \). Regarding the parameters \( W_j \) the only assumption is that if \( 0 \in \mathcal{S}' \) then one has \( W_0 \neq 0 \). We reformulate Theorem 2 in a more precise way as follows.

**Theorem 7.2.** Fix \( \gamma > 0 \) and consider the momentum preserving Hamiltonian \( H_{\text{NLS}} \) in (1.6)-(7.10). Assume that \( r \) satisfies (1.13) and take \( J = (J_j)_{j \in \mathcal{S}}, J > 0 \), such that \( (\sqrt{J}, 0) \in B_{r}(\omega_{s+\theta}) \). There exists a Lipschitz map
\[ \Omega : \mathcal{Q}_S \to \mathcal{Q}_{S'}, \quad \|\Omega_j - j\|^2_{j \in \mathcal{S}_j} \|\mathcal{Q}_S \leq C \gamma \epsilon, \]  \hspace{1cm} (7.11)

for some \( C > 1 \) such that the following holds.

For all \( \alpha \) belonging to
\[ C := \{ \alpha \in \mathcal{Q}_S : \omega(\alpha) = (\alpha, \Omega(\alpha)) \in B_{y_0,r}, \mathcal{S} \} \]  \hspace{1cm} (7.12)

there exist \( U = U(\alpha, J) \in \ell^\mathcal{S} \) and a symplectic change of variables \( \Psi : B_{r}(\omega_{s}) \to B_{r}(\omega_{s}) \) such that
\[ H_{\text{NLS}} \circ \Psi = \sum_{j \in \mathcal{S}} \alpha_j |v_j|^2 + \sum_{j \in \mathcal{S}'} \Omega_j(\alpha)|\alpha_j|^2 + R, \quad R \in \mathcal{H}^{\mathcal{S}}. \]

**Proof.** In order to apply Theorem 7.1, we recall the estimates (2.19) on \( P \). For \( \alpha \in B_{y_0,r}, \mathcal{S} \), let us write the Hamiltonian \( H_{\text{NLS}} \) as
\[ H_{\text{NLS}} = \sum_{j \in \mathcal{S}} (j^2 + U_j) |v_j|^2 + \sum_{j \in \mathcal{S}'} (j^2 + W_j) |\alpha_j|^2 + P = D_{\alpha,\Omega} + \Lambda + P', \]

where
\[ \Lambda = \sum_{j \in \mathcal{S}} v_j \left( |u_j|^2 - J_j \right) + \sum_{i \in \mathcal{S}'} \mu_j |\alpha_j|, \quad v_j = j^2 - \alpha_j + U_j, \quad \mu_j = j^2 - \Omega_j + W_j, \quad P' = P + \sum_{j \in \mathcal{S}} v_j J_j. \]

Of course \( P' \) satisfies (2.19) by definition of the norm and, by construction, \( \Lambda \in \ell^\mathcal{S} \). By the definition of \( r_0 \) one can apply Theorem 7.1 to \( H = N_0 + P' \) where \( N_0 = D_{\alpha,\Omega} \). In order to prove that \( H_{\text{NLS}} \circ \Phi = N \) we require that \( H_{\text{NLS}} = H + \Lambda(J, \alpha, \Omega) \) fixed in Theorem 7.1. This amounts to requiring
\[ \begin{cases} \Omega_j + \mu_j (J, \alpha, \Omega) = j^2 + W_j \\ \alpha_j + v_j (J, \alpha, \Omega) = j^2 + U_j. \end{cases} \]  \hspace{1cm} (7.13)

In order to solve the equations above, the only key point is to extend the map \( \mu : B_{y_0,r}^{\mathcal{S}} \to \ell^\mathcal{S} \) to the whole square \( \mathcal{S} \) preserving the weighted Lipschitz norm \( \| \cdot \| \). This in fact is guaranteed by Kirtzbraum theorem on metric spaces (see for instance [40]).
At this point by direct application of the contraction Lemma, we solve $\Omega = \Omega(\alpha)$ which by construction is a Lipschitz map and satisfies (7.11). Finally we set $U_j = \alpha_j + v_j(\alpha, \Omega(\alpha)) - j^2$. \hfill $\Box$

Of course in order for Theorem 7.2 to be non empty we need measure estimates, namely we need to show the following result, whose proof is postponed to Appendix C.

**Lemma 7.1** (Measure estimates). Let $\Omega$ be a map satisfying (7.11). Then the set $C$ defined in (7.12) has positive relative measure in $Q_S$.

The following result states that the frequencies $(\alpha, \Omega(\alpha))$ with $\alpha \in C$ satisfied the so called Melnikov non-resonance conditions in the following way

**Lemma 7.2.** Let $M_\gamma$ the set of $\alpha \in Q_S$ such that

$$|\alpha \cdot h + \sigma \Omega_j(\alpha) + \sigma' \Omega_k(\alpha)| > \gamma \prod_{n \in S} \frac{1}{(1 + |h_n|^{\beta}(n))^{\delta}},$$

for every $h \in Z, j, k \in S^\gamma, \alpha, \sigma', \alpha' = \pm 1, 0$ satisfying $\pi(h) + \alpha j + \alpha' k = 0$. Then $C \subseteq M_\gamma$.

The proof is postponed to Appendix C.

**Remark 7.2.** If the set $S = \{-m, \ldots, m\}$ then $\alpha \in M_\gamma$ implies the Melnikov conditions

$$|\alpha \cdot h + \sigma \Omega_j(\alpha) + \sigma' \Omega_k(\alpha)| > \frac{\gamma}{(|m|!)^{12}|h|^{\frac{12m}{\infty}}},$$

which are the usual ones imposed in the literature on finite dimensional KAM tori (with a non optimal exponent $12m$). We could take this comparison further and assume that all the actions $I_j$ with $j \in S$ are equidistributed. Then of course we would see a dependence between the size of the perturbation and the dimension $m$ as in [1] or in [35]. Note that even if these constants are surely non optimal, in any case the size of the perturbation shrinks to zero more than factorially with the dimension of the torus. The fact that our Theorem 7.2 is uniform in the dimension is due to the appropriate distribution of the actions $I_j$, the use of suitable weighted norms and Diophantine condition.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Appendix A. Technicalities**

**A.1. Proof of Lemma 2.1**

Recalling (2.13) we get

$$\frac{1}{r^2} \sup_{|\alpha|=|\beta|=0} \left| H_{\alpha, \beta} \left( e^{\eta|\pi(\alpha - \beta)|} \right) u_{\alpha + \beta} \right| \leq \frac{1}{r^2} \sum_{|\alpha|=|\beta|=0} \left| H_{\alpha, \beta} \left( e^{\eta|\pi(\alpha - \beta)|} \right) u_{\alpha + \beta} \right| \leq \sup_{j \in Z} \left| H_{\alpha, \beta} \left( e^{\eta|\pi(\alpha - \beta)|} \right) u_{\alpha + \beta} \right| \leq \sup_{j \in Z} \left| H_{\alpha, \beta} \left( e^{\eta|\pi(\alpha - \beta)|} \right) u_{\alpha + \beta} \right|.$$
Then (2.14) follows by the mass conservation. As a consequence $H_x(u)$ and, a fortiori, $H(u)$ are analytic functions on the open ball $\{ |u| < r \}$ and continuous on the closed ball $\{ |u| \leq r \}$. Analogously for the hamiltonian vector fields $X_{H_x}$ and $X_H$. Indeed it is easily seen that

$$X^{(j)}_{H_x}(u) = i \sum_{\alpha, \beta \in \mathbb{N}^\mathbb{Z}} |H_{\alpha, \beta}||\beta_j e^{\xi |(\alpha - \beta)|}u^{\alpha \beta - \delta_j}$$

and, therefore, for every $|u|_{p, r, a} \leq r$

$$|X^{(j)}_{H_x}(u)| \leq \sum_{\alpha, \beta \in \mathbb{N}^\mathbb{Z}} |H_{\alpha, \beta}||\beta_j e^{\xi |(\alpha - \beta)|}u^{\alpha \beta - \delta_j}$$

(2.13) follows

$$\sum_{\alpha, \beta \in \mathbb{N}^\mathbb{Z}} |H_{\alpha, \beta}||\beta_j e^{\xi |(\alpha - \beta)|}u^{\alpha + \beta - \delta_j} = |X^{(j)}_{H_x}(u_0)|,$$

proving the first equality in (2.15). Then

$$\frac{1}{r} |X_{H_x}(u_0(r))|_{r, a, p} = \sup \sum_{\alpha, \beta \in \mathbb{N}^\mathbb{Z}} |H_{\alpha, \beta}||\beta_j u_0^{\alpha + \beta - 2\delta_j} e^{\xi |(\alpha - \beta)|},$$

concluding the proof of the lemma. \qed

A.2. Proof of Lemma 4.1

We have

$$\sum_{|\delta| = q \atop \delta \leq m} \binom{m}{\delta} = \sum_{|\delta| = q \atop \delta \leq m} \prod_{i} \binom{m_i}{\delta_i} \leq \sum_{|\delta| = q \atop \delta \leq m} \prod_{i} m_i^{\delta_i} = \sum_{|\delta| = q \atop \delta \leq m} \frac{m^q}{\delta!} \leq \frac{|m|^q}{q!} \leq \left( \frac{|e|m|}{q} \right)^q$$

and also

$$\sum_{|\delta| = q \atop \delta \leq m} \prod_{i} \binom{m_i}{\delta_i} = \prod_{i} \binom{m_i}{\delta_i} = 2^{|m|},$$

so that

$$\sum_{|\delta| = q \atop \delta \leq m} \binom{m}{\delta} \leq \min \left\{ \left( \frac{|e|m|}{q} \right)^q, 2^{|m|} \right\}.$$

Then, for $|m| \geq q$, we get

$$\kappa^{2|m|} \sum_{|\delta| = q \atop \delta \leq m} \binom{m}{\delta} \leq \min \left\{ \max_{|m| \geq q} \left( \frac{|e|m|}{q} \right)^q \kappa^{2|m|}, \max_{|m| \geq q} \left( 2\kappa^2 \right)^{|m|} \right\} \leq c_q^q \kappa^q \kappa^{2|m|}.$$

A.3. Proof of Lemma 6.2

We treat the two summands of $M_n$ separately, we recall that by (4.47)

$$\| \Pi^{0, K} L_n h \|_\infty \leq 3 \| L_n h \|_{L_3^{\infty}, s_n} \leq 3 \kappa (1 + \Theta) e_0 \sum_{j=1}^n 2^{-j} \| h \|_\infty < \frac{1}{4} \| h \|_\infty,$$

24 Use that $\binom{0}{q} \leq \frac{q^q}{q!}$. Moreover $|x|^q = \left( \sum_{1 \leq i \leq n} x_i \right)^q = \sum_{v \in \mathbb{N}^n, |v| = q} q^{|v|} = q^q$. Note that, being $\delta \leq m$, the support of $\delta$ is contained in the support of $m$. Finally we use that $q^q \leq e^q q!$.\]
provided that

\[ 3K(1 + \Theta)e_0 < 1/4. \]

As for the second summand we have, again by (4.47), Lemma 3.1 and (2.40)

\[
\| \Pi_{0,K} \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \| \leq 3 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \leq 48 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \leq 48 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \leq 48 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \leq 48 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \leq 4 \| \{ (\Pi^{-2} L_{w}^{-1} L_{n} h, G_{n}^{2}) \} \|_{\infty} \]

(2.33)

noting that \( \eta_{n} \geq \eta_{0}/2 \) by (6.7) and (1.18). \( \square \)

Appendix B. On action-angle coordinates

**Lemma B.1.** The set

\[ U := \left\{ u \in w_{p,s,a}^{\infty} : \inf_{j} (j)^{p} e^{a|j|+s(j)^{\theta}} |u_{j}| > 0 \right\} \]  

(B.1)

is open and dense in \( w_{p,s,a}^{\infty} \).

Moreover, the map to action angles

\[ \Phi : \mathbb{T}^{Z} \times w_{2a,2a,2p}^{\infty}(\mathbb{R}+) \to w_{p,s,a}^{\infty}(\mathbb{C}), \quad (\theta, J) \mapsto u = \Phi(J, \theta) := \left( \sqrt[|j|]{e^{a|j|}} \right)_{j \in \mathbb{Z}} \]

where \( \mathbb{T}^{Z} = (\mathbb{R}/2\pi \mathbb{Z})^{Z} \) is endowed with the norm:

\[ |\theta - \theta'|_{\infty} = \sup_{j} |\theta_{j} - \theta'_{j}|_{\text{mod}2\pi} \]

is locally well defined in some ball \( \hat{B}_{\epsilon}(\hat{u}, w_{p,s,a}^{\infty}) \) for all \( \hat{u} \in U \).

**Proof.** Let us fix a \( \hat{u} \in U \), then by definition

\[ \inf_{j} (j)^{p} e^{a|j|+s(j)^{\theta}} |\hat{u}_{j}| = \varphi(\hat{u}) > 0, \]

moreover for any \( r < \varphi(\hat{u}) \) the ball \( \hat{B}_{r}(\hat{u}) \subset U \), indeed

\[ \inf_{j} (j)^{p} e^{a|j|+s(j)^{\theta}} |u_{j}| \geq \inf_{j} (j)^{p} e^{a|j|+s(j)^{\theta}} |\hat{u}_{j}| - \sup_{j} (j)^{p} e^{a|j|+s(j)^{\theta}} |\hat{u}_{j} - u_{j}| \geq \varphi(\hat{u}) - r > 0. \]

In order to prove the continuity of the action/angle map we start by recalling that we may identify isometrically \( w_{p,s,a}^{\infty}(\mathbb{C}) \) with \( \ell^{\infty}(\mathbb{C}) \) via the map

\[ i_{p,s,a} : w_{p,s,a}^{\infty}(\mathbb{C}) \leftrightarrow \ell^{\infty}(\mathbb{C}) ; \quad i_{p,s,a} u = \left( (j)^{p} e^{a|j|+s(j)^{\theta}} u_{j} \right)_{j \in \mathbb{Z}}, \]

and the same map \( i_{2a,2a,2p} \) identifies \( w_{2a,2a,2p}^{\infty}(\mathbb{R}+) \) with \( \ell^{\infty}(\mathbb{R}+) \). Finally the diagram

\[
\begin{array}{ccc}
\mathbb{T}^{Z} \times w_{2a,2a,2p}^{\infty}(\mathbb{R}+) & \xrightarrow{\Phi} & w_{a,s,p}^{\infty}(\mathbb{C}) \\
\downarrow i_{2a,2a,2p} & & \downarrow i_{a,s,p} \\
\mathbb{T}^{Z} \times \ell^{\infty}(\mathbb{R}+) & \xrightarrow{\Phi} & \ell^{\infty}(\mathbb{C})
\end{array}
\]
is commutative so we just need to prove the statement for $\ell^\infty(\mathbb{C})$ where it is trivial. Indeed fix $\hat{u} = \Phi(\hat{J}, \hat{\theta}) \in \ell^\infty$ and consider the preimage through $\Phi$ of the ball $\tilde{B}_r(\hat{u}, \ell^\infty(\mathbb{C}))$, i.e.

$$\left\{ (J, \theta) \in \mathbb{T}^2 \times \ell^\infty(\mathbb{R}_+) : \sup_j |\sqrt{J} e^{i\theta} - \sqrt{\tilde{J}} e^{i\hat{\theta}}| \leq r \right\}$$

then provided that we assume that $3r|\hat{u}|^\infty < \pi(\hat{u})$ the ellipses

$$\tilde{B}_1(r) := \left\{ (J, \theta) : |J - \hat{J}|^\infty < 3r|\hat{u}|^\infty, \ |\theta - \hat{\theta}|^\infty < \arctan(\frac{r}{\pi(\hat{u})}) \right\}$$

$$\tilde{B}_2(r) := \left\{ (I, \theta) : |J - \hat{J}|^\infty < r \frac{\pi(\hat{u})}{2}, \ |\theta - \hat{\theta}|^\infty < \arctan(\frac{r}{|\hat{u}|^\infty}) \right\}$$

satisfy

$$\Phi(\tilde{B}_2(r)) \subseteq \tilde{B}_r(\hat{u}, \ell^\infty) \subseteq \Phi(\tilde{B}_1(r)).$$

This implies that $\Phi$ is a homeomorphism. □

Appendix C. Measure estimates

C.1. Proof of Lemma 7.1

For $\ell \neq 0$ with $|\ell| < \infty$, $\sum_{j \in \mathcal{S}^c} |\ell_j| \leq 2$ and $\pi(\ell) = 0$, we define the resonant set

$$\mathcal{R}_\ell := \left\{ \alpha \in \mathcal{Q}_\mathcal{S} : |\omega(\alpha) \cdot \ell| \leq \gamma \prod_{n \in \mathbb{Z}} \frac{1}{(1 + |\ell_n|^2(n^2)^2)} \right\}.$$ 

We first note that if $\ell$ is supported only on $\mathcal{S}^c$ (i.e. $\ell_j = 0$ for all $j \in \mathcal{S}$) then $\mathcal{R}_\ell$ is empty. Indeed in this case $\ell = \sigma \mathbf{e}_j + \sigma' \mathbf{e}_k$ for some $j, k \in \mathcal{S}^c$, $|j| \geq |k|$, $\sigma = \pm 1$, $\sigma' = \pm 1, 0$, $\sigma j + \sigma' k = 0$. Then $\sigma \sigma' = -1$, otherwise, since the momentum is zero, we get $j = k$ and $\ell = 0$. Consider first the case $\sigma \sigma' = 1$; then $k = -j$ and

$$|\omega(\alpha) \cdot \ell| = |\Omega_j + \Omega_k| \geq |j^2 + k^2| - (|W_j| + |W_k|) - O(\epsilon) \geq 2j^2 - 1 - O(\epsilon),$$

which, if $\ell \neq 0$, is bigger than $1/2$ and, therefore, $\mathcal{R}_\ell$ is empty whenever $\gamma < 1/2$. Otherwise, when $\ell = 0$ we get

$$|\omega(\alpha) \cdot \ell| \geq 2|W_0| - O(\epsilon) \geq |W_0|,$$

then $\mathcal{R}_\ell$ is empty provided $\gamma$ is small enough with respect to $|W_0|$, that we are assuming to be different from zero (since we are in the case $0 \in \mathcal{S}^c$).

It remains only the case $\sigma \sigma' = 0$, namely $\sigma' = 0$. Then $|\omega(\alpha) \cdot \ell| = |\Omega_j|$ and we conclude as above.

On the other hand, if $\ell_j \neq 0$ for some $j \in \mathcal{S}$, then we can bound from below the Lipschitz variation in the direction $s$. Indeed (recalling that $|\Omega_j(\alpha)|_{\text{lip}} \sim \epsilon$) one has

$$|\Delta_{\alpha_j} \omega(\alpha) \cdot \ell| \geq |\ell_s| - 2 \sup_{j \in \mathcal{S}^c} |\Omega_j(\alpha)|_{\text{lip}} \geq |\ell_s| - O(\epsilon) \geq \frac{1}{2}$$

if $\epsilon$ is small enough. Then following the proof of Lemma 4.1 of [12] verbatim one gets

$$\text{meas}(\mathcal{Q}_\mathcal{S} \setminus C) \leq \sum_{\ell \neq 0} \text{meas}(\mathcal{R}_\ell) \leq \gamma \sum_{\ell \neq 0} \prod_{n \in \mathbb{Z}} \frac{1}{(1 + |\ell_n|^2(n^2)^2)} \sim O(\gamma).$$
C.2. Proof of Lemma 7.2

Let us consider \( \alpha \in \mathcal{C} \) and estimate

\[
|\alpha \cdot h + \sigma \Omega_j(\alpha) + \sigma' \Omega_k(\alpha)| = |\omega(\alpha) \cdot \ell|, \quad \ell = (h, \sigma e_j + \sigma' e_k).
\]

We start by remarking that, since \( \alpha \in Q_S \) and \( \Omega_j(\alpha) - W_j - j^2 \sim \gamma \epsilon \), one has

\[
|\alpha \cdot h + \sigma \Omega_j(\alpha) + \sigma' \Omega_k(\alpha)| \geq j^2 + \sigma |k|^2 - 2 \sum_{i \in S} i^2 |h_i|
\]

thus unless \( \sigma \sigma' = -1 \) and \( j = -k \) we can deduce that if \( |j|, |k| \geq C \sum_{i \in S} i^2 |h_i| \) then the left hand side above is \( \geq \frac{1}{2} \gamma \epsilon \) and hence the conditions defining \( M_\gamma \) are trivially met. On the other hand if \( |j|, |k| \leq C \sum_{i \in S} i^2 |h_i| \) then

\[
|\alpha \cdot h + \sigma \Omega_j(\alpha) + \sigma' \Omega_k(\alpha)| = |\omega(\alpha) \cdot \ell| \geq \frac{1}{(1 + j^2)(1 + k^2)} \prod_{n \in S} \frac{1}{(1 + |h_n|^2(n)^2)} \geq \frac{\gamma}{(1 + \sum_{i \in S} i^2 |h_i|)^2} \prod_{n \in S} \frac{1}{(1 + |h_n|^2(n)^2)} \geq \frac{\gamma}{(1 + |h_n|^6(n)^6)}.
\]

Finally if \( \sigma \sigma' = -1, j = -k \) we use momentum conservation to deduce

\[
2 |j| \leq |\pi(h)| \leq \sum_{i \in S} i^2 |h_i|.
\]

This concludes the proof. \( \Box \)

References

35. X. Li, S. Liu, The relation between the size of perturbations and the dimension of tori in an infinite-dimensional KAM theorem of Pöschel, Nonlinear Anal. 197 (2020) 111754.