

# ALGORITHMS FOR GEOMETRICAL MODELS IN BORROMINI'S SAN CARLINO ALLE QUATTRO FONTANE

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## ABSTRACT

Construction of mathematical models of the vault of Borromini's San Carlino alle Quattro Fontane based on parametric curves and surfaces, including the shape of the vault and rules for its tessellation with crosses and octagonal coffers. Several models of different complexity are optimized and tested measuring their distance from the point cloud of a very accurate 3D survey and the analysis of such measured data is proposed to validate hypothesis of construction procedures by checking symmetries of coffers shape, scale and position in different levels and sectors. Some original algorithms are discussed to produce regular tessellations on a surface with a generic base curve and to construct regular parametric curves section out of simple point cloud data.

## KEY WORDS

Parametric curves, parametric surfaces, tessellations, mathematical models, distance measure, Borromini, symmetry, point clouds, optimization, algorithms.

## INTRODUCTION

Borromini's architectures, despite their appearance, are usually based on elementary geometry at the beginning of the design process, that become progressively more complex during the evolution of the construction defining spaces and shapes. In the case of San Carlo alle Quattro Fontane (San Carlino because of its small size, 1642) the vault surface and its base curve have been extensively studied, with several surveys (see Portoghesi 1967, Sartor 2000, AA.VV. 2007) and sometimes different conclusions. Moreover, the particularly innovative shape of the vault and its decoration has prevented the construction of simple mathematical models which could capture both the elegance and the structural aspects of such masterpiece of Roman baroque architecture.

Starting from a new survey (see Canciani et al., 2013) several models are presented and tested measuring their closeness to the point cloud of the survey or to a set of reference points.

The main goal of the present study is to derive simple geometrical models close to the shapes and structure detected in San Carlino that could help in the analysis of the construction procedures and in the definition of its final symmetries and appearance.

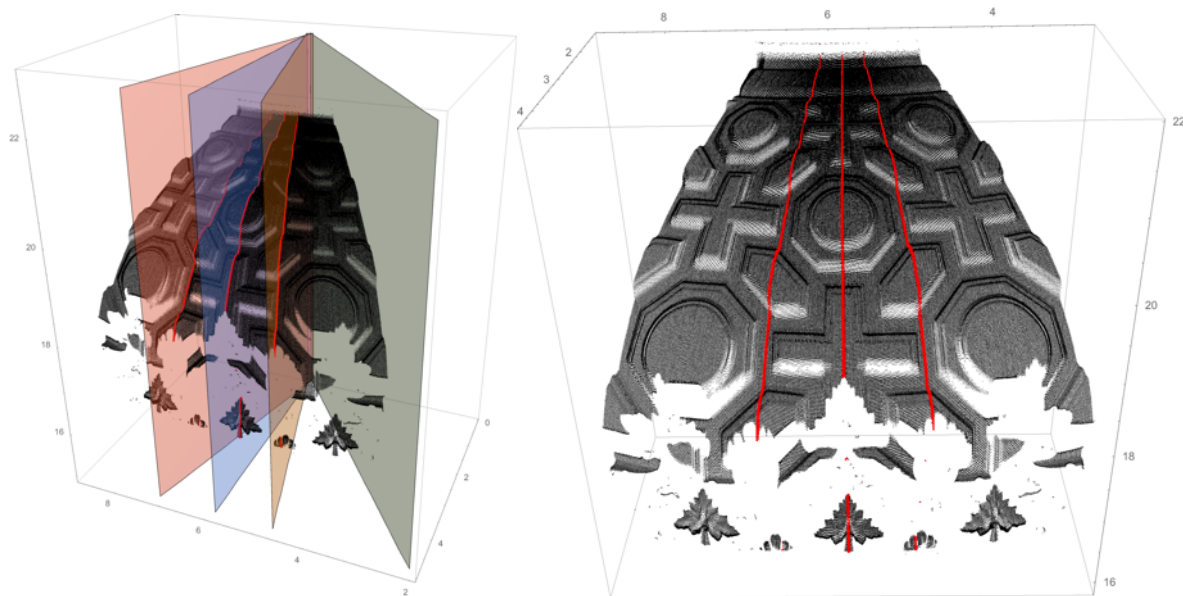
The model itself, as a list of parametric curves and surfaces, could also be used as a starting point for new projects.

Algorithms for the definition of the parametric models are given in an explicit form, in particular regarding the actual shape of the base curve and the geometrical shape of the vault and of its decorations. The results are supported by images and pictures and are tested using some distance function from the point clouds of the survey.

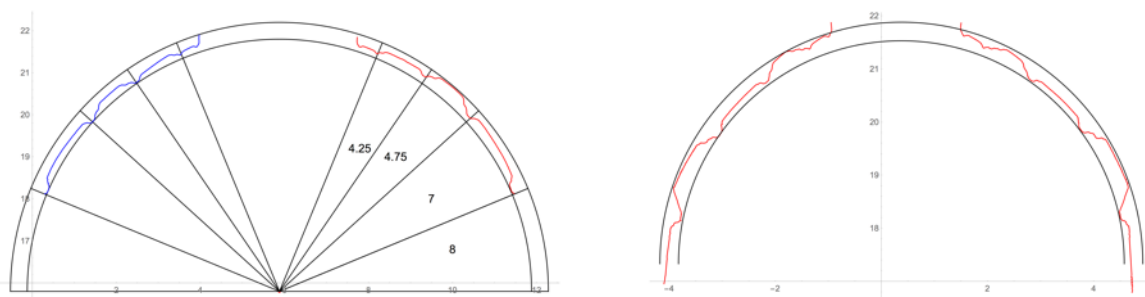
## SECTION CURVES

Starting from a cloud data of 4 million points, we select those points which are sufficiently close to a given plane and we project them on that plane: in this way we get horizontal and vertical sections of the point cloud.

In order to get a curve section we wrote an algorithm to get a parametric planar regular curve which fits the selected points (see Canciani et al., 2013) and automatically makes the selection on the curve of some nodal points with given specific characteristics (symmetry, tangency conditions, corners, ... ).



*Figure 1. Parametric curves fitting the point cloud selection for given vertical planes.*

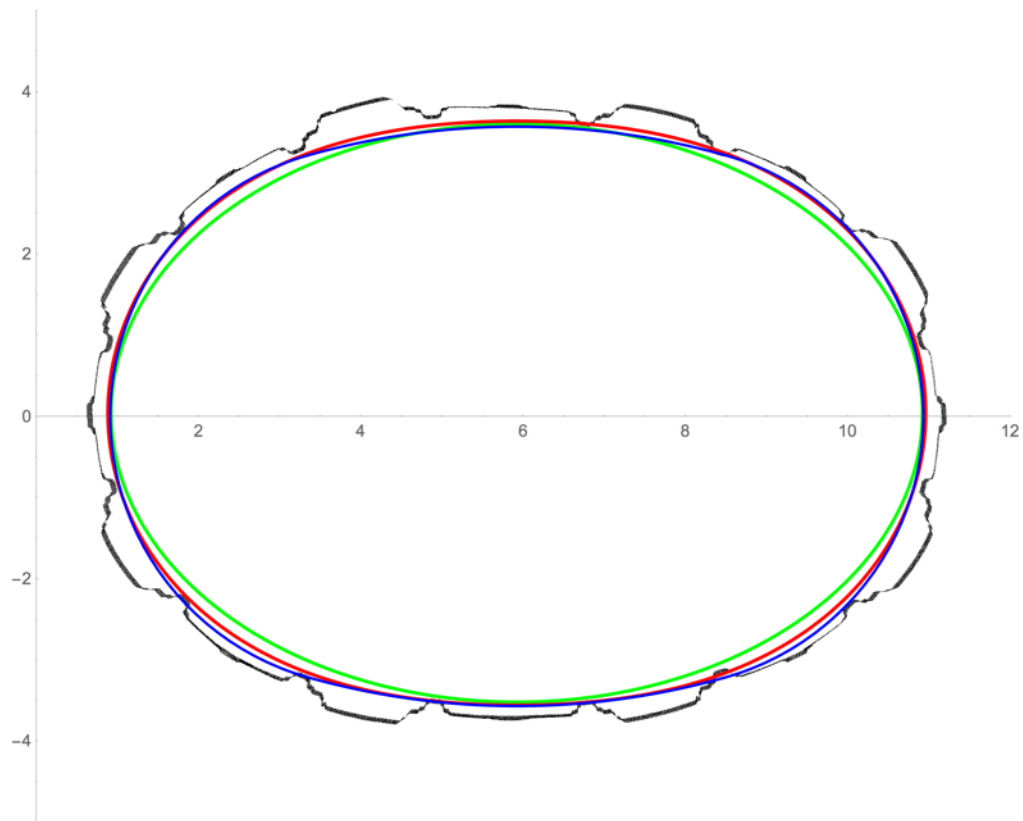


*Figure 2. Parametric curves fitting the point cloud selection. Comparison with circular arcs.*



*Figure 3. Selection of points (in red) close to the horizontal plane  $z=z_0=19.2$ , passing through the lacunars centers of the third level.*

Some oval and the chosen epitrochoid are almost undistinguished (see Figure 3). Note that the point cloud selection, projected back onto the plane of section, is not symmetrical.



*Figure 3. Section with the horizontal plane  $z=z_0=19.2$ . Comparison between an ellipse (green), an oval (red) and an epitrochoid (blue). The red and the blue curves are almost overlapping.*

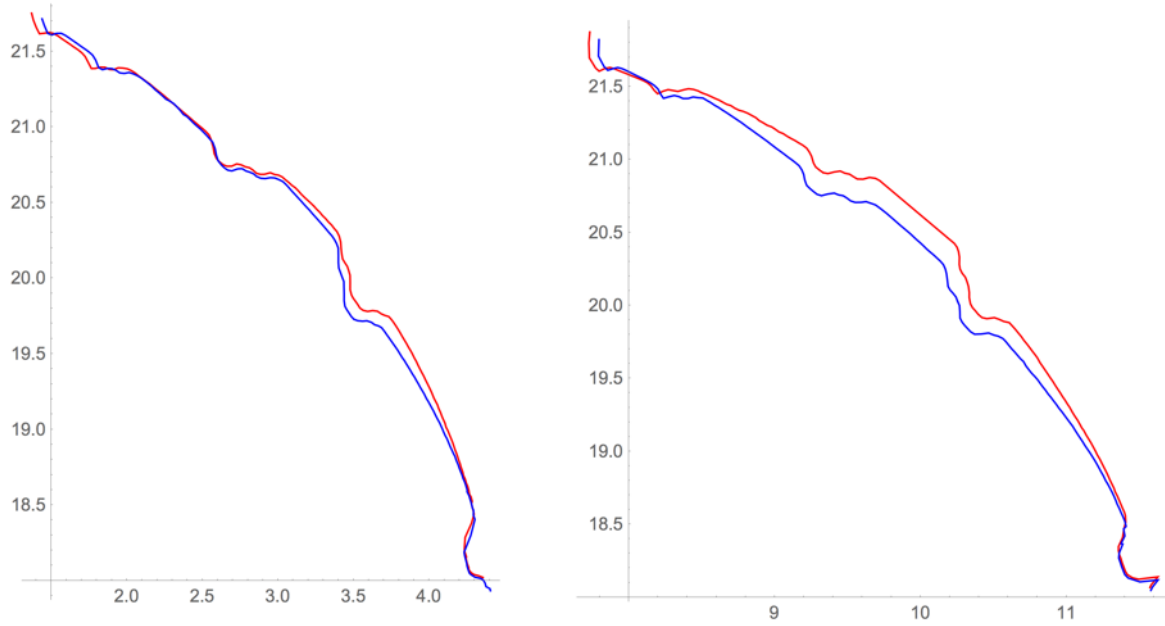


Figure 3. Parametric curves fitting the point cloud selection. Comparison on symmetrical sections of  $(y,z)$ -plane (left) and  $(x,z)$ -plane (right). The differences are greater on the  $(x,z)$ -plane.

## 1 Ovals

Different kind of ovals have been used in Borromini's project and using planar sections of the point cloud is possible to investigate their relations with the construction. Here we will show only some example: sometimes is difficult to choose which kind of oval would best fit the data so we propose to check ovals of a given type with some additional constraint.

In particular let us consider a particular variant of the Serlio's oval of the fourth kind, based on two circles both passing through the center of the other one, by fixing the angular position  $\theta$  of the contact point (where the circles have the same tangent) to be  $\theta=1.29815\dots$  (with successive best approximants  $2/5 \pi$ ,  $5/12 \pi$ ,  $7/17 \pi$ ,  $12/29 \pi$ ) which gives the length ratio of the two circular arcs equal to  $3/5$  (see Figure 4).

In fact, let  $r$  be the radius of the circle centered at  $C_1=(x,0)$  and  $r_2$  be the radius of the circle centered at  $C_2$ , the construction implies  $r = 2x$  and  $r_2 = r + x/\cos \theta$  and the condition to be imposed is

$$\frac{r \cdot 2\theta}{5} = \frac{r_2(\pi - 2\theta)}{3}$$

which gives

$$\frac{r \cdot 2\theta}{5} = \left(r + \frac{x}{\cos \theta}\right) \frac{(\pi - 2\theta)}{3}$$

and finally the equation

$$\cos \theta = -\frac{5(\pi - 2\theta)}{2(5\pi - 16\theta)}$$

solved by the approximate value  $\theta=1.29815$ .

Such a particular value allows to draw 5 lacunars on one circle and 3 lacunars on the other one keeping their horizontal size uniform.

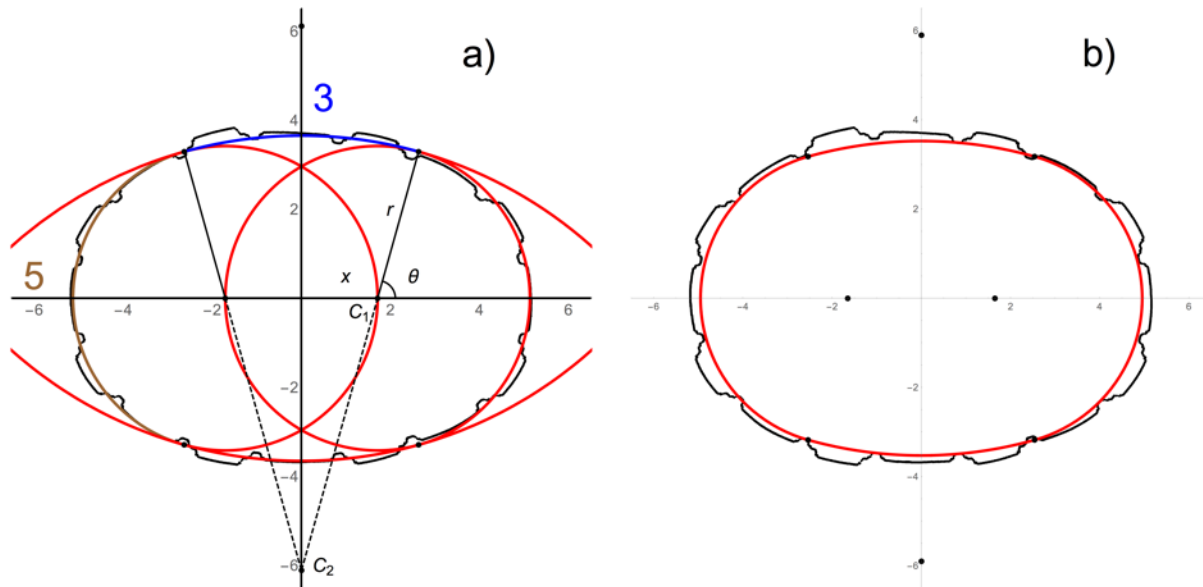


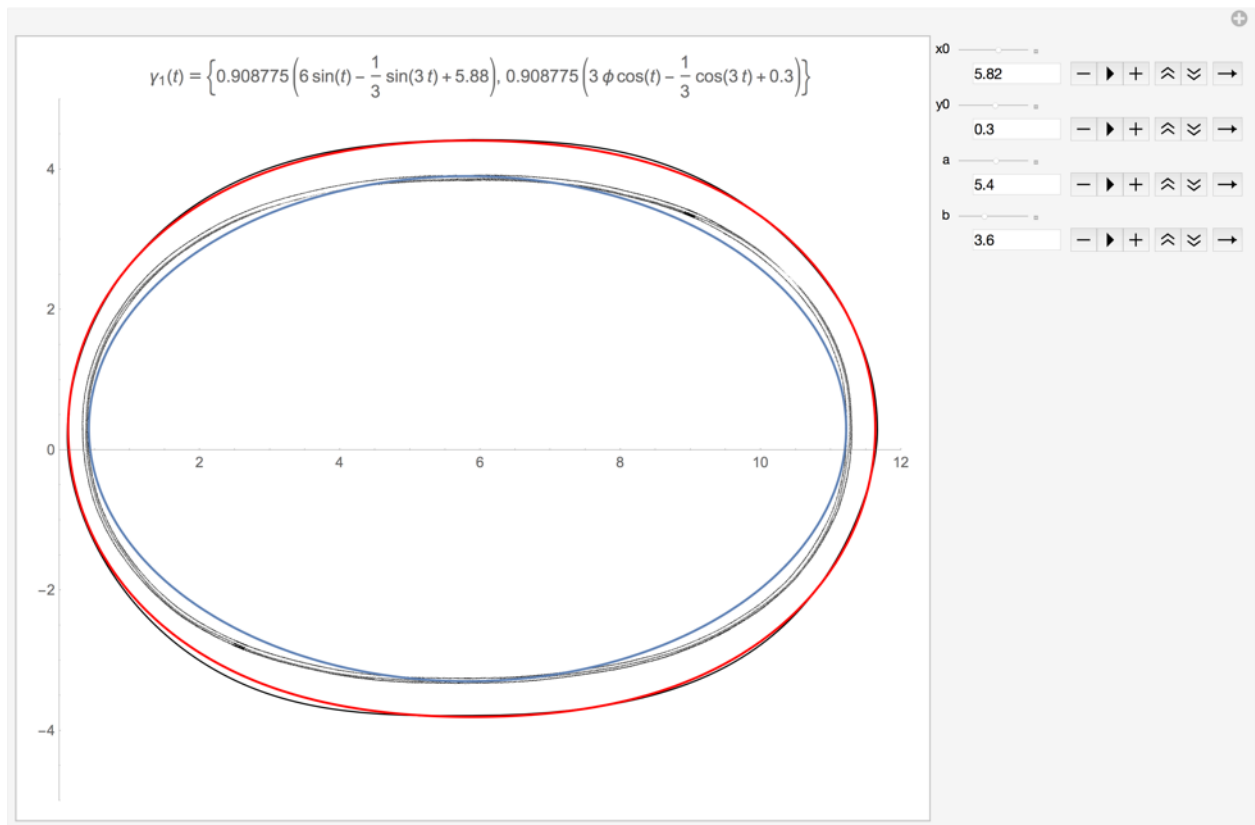
Figure 4. Oval construction on a horizontal section of the point cloud through the lacunars centers of the third level. a)  $\theta=1.29815\dots$  implies the arc ratio of 3/5. b) Same oval passing on the intradox.

## The base curve

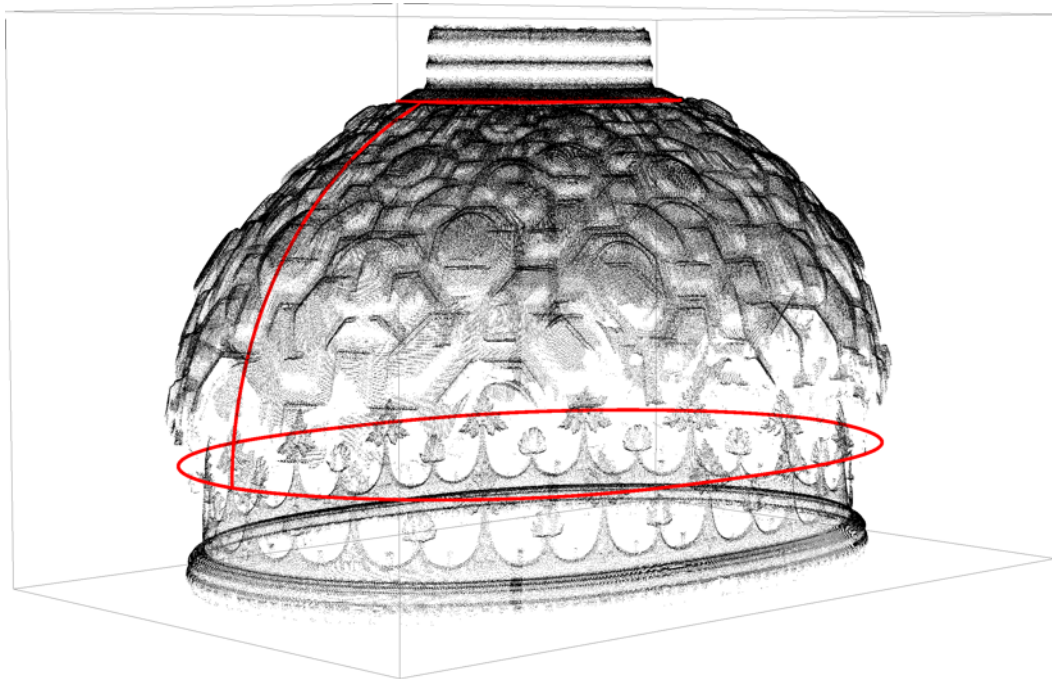
The base curve is found indirectly as a curve at a constant distance from the frame decoration curve: using the software Mathematica (see Figure 9) it is possible to test dynamically the curves of a given shape manipulating the corresponding parameters.

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*Figure 9. The base curve is found indirectly as a curve at a constant distance from the frame decoration curve.*



*Figure 5. The base curve is found indirectly as a curve at a constant distance from the frame decoration curve. In this picture the vertical curve is a circular arc.*

Often the curve at the base of the vault is related, to some extent, to an ellipse or, respecting the original Borromini's design, to an oval.

For instance S. Huerta, in a paper about oval domes (Huerta, 2007), wrote of San Carlino: “the oval which generates the plan changes at the base of the dome. This last oval deviates very much from the usual form of ovals so far. No doubt, Borromini chose this form to provide *tension* in the space. Neither of the ovals corresponds with Serlio’s models” (see Serlio 1566, where Serlio, 1475-1554, defines four kind of ovals).

Ovals and ellipses could be very similar in shape but are very different mathematically: both are smooth curves but the oval, being the union of circular arcs, has the second derivative (and then the curvature) which is only piecewise constant. This discontinuity in the curvature makes the oval structurally more fragile than the ellipse which then seems a good trial precisely in the direction of taking into account the physical *tension* of the structure. But there is a third way: in fact in (Falcolini&Vallicelli, 2011) a curve was proposed which had some feature of an oval but the regularity of an ellipse and seems to fit well some of the horizontal sections even if they corresponds to different kind of ovals.

## The epitrochoid



The epitrochoid, like the ellipse, is a particular epicycloid: it represents the trajectory of a given point at distance  $h$  from the center of a circle of radius  $r$  which rolls around a fixed circle of radius  $R$ , so in particular it would be simple to devise a graphical machine to draw it. The equation of the epitrochoid is:

$$x = (R + r) \cos t - h \cos \frac{R + r}{r} t$$

$$y = (R + r) \sin t - h \sin \frac{R + r}{r} t$$

In the particular case of  $h = 1/3$  (Fig. 1c) two features are reached: regions with almost constant curvature (circular arcs) and points with zero curvature (linear neighborhood).

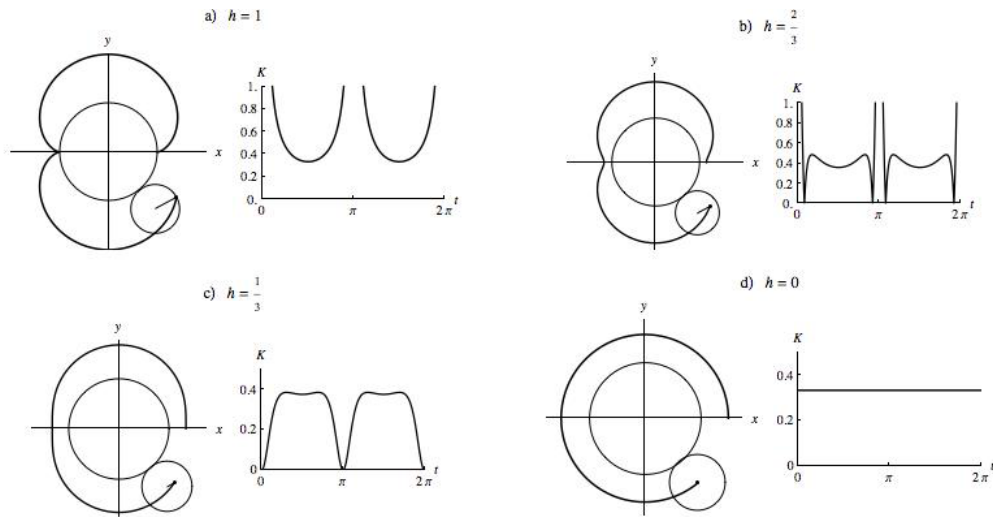


Figure 1: Different epitrochoids for  $h = 1, 2/3, 1/3$  and  $0$  with the corresponding curvature function  $K(t)$ .

Moreover such a curve is as regular as the ellipse and much more regular than an oval made of several circular arcs.

This, implying higher order derivatives, or smooth changes in curvature, affects stresses.

Some additional parameters  $c, d$ , were introduced to stretch the curve along its axis, for a better adaptation to the vault's shape:

$$x = c(R + r) \cos t - h \cos \frac{R + r}{r} t$$

$$y = d(R + r) \sin t - h \sin \frac{R + r}{r} t$$

As it is possible to see in Fig:2, the ellipse (blue) doesn't follow the vault shape along the diagonals, and in the middle is very close to a circular arc. The proposed parametric curve  $\gamma(u)$  is then an epitrochoid with  $R = 2$ ,  $r = 1$ ,  $c = 1.5$ ,  $d = 2$  and  $h = 1/3$  :

$$\gamma(u) = (6 \sin u + \frac{1}{3} \sin 3u, \frac{9}{2} \cos u + \frac{1}{3} \cos 3u)$$

This particular epitrochoid actually in the middle of the vault has curvature very close to 0, since it is locally quite similar to a line, whereas in its lateral sides is very close to a circle.

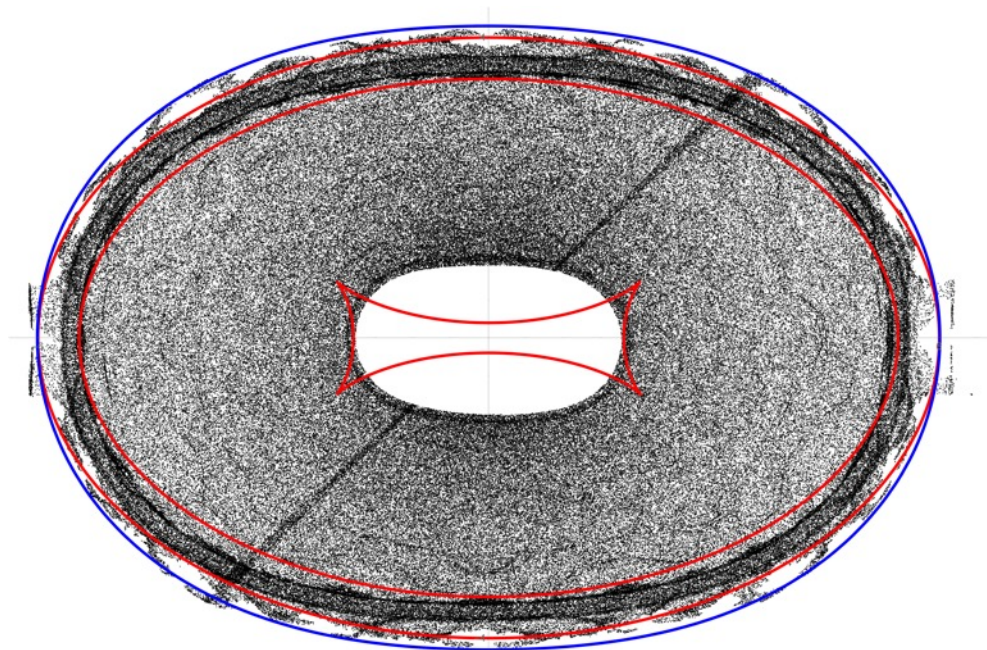


Figure 2: *Comparison between a red ellipse and the blue epitrochoid, superimposed to a vertical projection of the point cloud. The black region correspond to the internal decoration and the base curve is found indirectly as a curve at a constant distance from the frame decoration curve*

## THE SURFACE

The real problem, solved ingeniously by Borromini, is however to construct a structurally stable surface, which would minimize the stresses and, upon it, a tassellation as regular as possible with an underlined sense of vertical perspective.

Our goal is then to describe a possibly simple mathematical model of the actual vault of S. Carlino which could be tested on a three dimensional rendering of a direct survey.

### Simple parametric models

Here we describe three surface models of the vault and try to measure their validity. Keeping fixed the basis of the vault (the stretched epitrochoid (2.3)) a first simple model of the surface can be constructed on it assuming an ellipsoidal shape (see [6]): in parametric coordinates this can be formulated as a parametric surface  $\gamma(u, v)$

$$\gamma(u, v) = \left( \left( 6 \sin u + \frac{1}{3} \sin 3u \right) \sin v, \left( \frac{9}{2} \cos u + \frac{1}{3} \cos 3u \right) \sin v, \frac{19}{3} \cos v \right)$$

where the height of the surface is tentatively chosen to be  $19/3$  in the chosen units as suggested in fig.6; here the drawn ellipses are all similar to the basis of the vault which implies that the height of the vault is equal to its longer semi-axes.

A second model, with similar features, can be constructed by rotating the basis line around the  $y$ -axis (along its minor semi-axes).

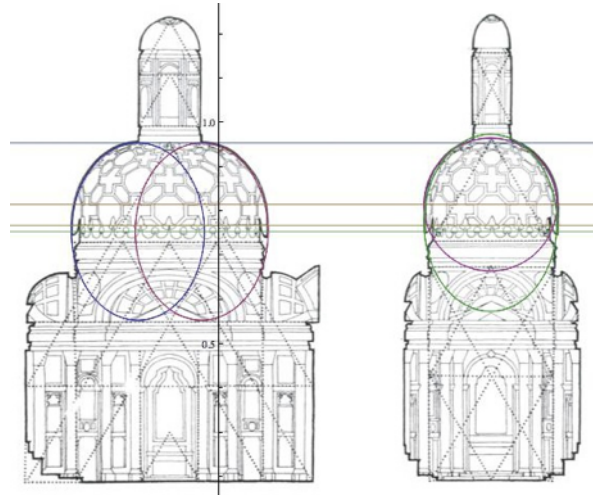


Figure 6: Elliptic and almost circular *sections superimposed to a design from a relief by Portoghesi.*

Finally, in fig.6 the two sections, basically elliptical (ending at the base of the lantern with horizontal tangency) in one direction and more circular in the other, suggests a different model, more similar to a portion of an elliptical torus (suggestive, thinking of S. Costanza) that is the surface of revolution of an ellipse around a coplanar straight line parallel to one of its axis (see fig. 7).

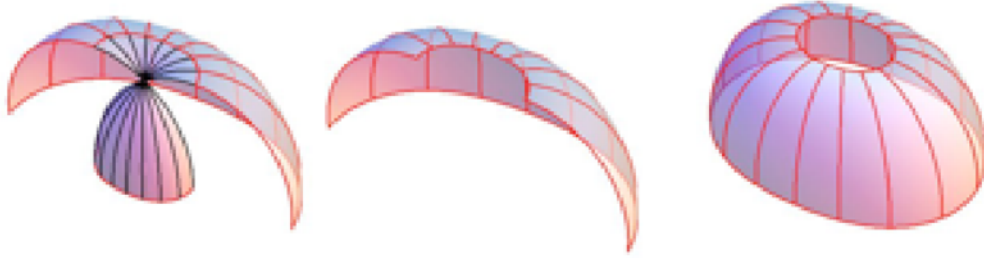


Figure 7: Toric surface defined as an ellipse rotating along the epitrochoid and around a vertical axis which intersects the curve.

In order to fix the corresponding parametric equation of the three models, we have taken a preliminary set of points from a direct relief; with the surface we can then construct also a model of the decorations which we finally want to test with a guided survey and the use of advanced software (like Mathematica and Photomodeler).

We have therefore evaluated the mean distance of a preliminary set of 477 points, taken by a direct survey from the three kind of surfaces with parameters chosen in order to minimize the resulting quantities.

The parametric equations are then:

$$\gamma_1(u, v) = 9.31 \left( \left( 6\sin u - \frac{1}{3}\sin 3u \right) \sin v, \left( \frac{3}{2}\pi \cos u - \frac{1}{3}\cos 3u \right) \sin v, 6.16 \cos v \right)$$

$$\gamma_2(u, v) = 9.27 \left( \left( 6\sin u - \frac{1}{3}\sin 3u \right) \sin v, \left( \frac{3}{2}\pi \cos u - \frac{1}{3}\cos 3u \right), \left( 6\sin u - \frac{1}{3}\sin 3u \right) \cos v \right)$$

$$\gamma_3(u, v) = 6.21 \left( \left( 6\sin u - \frac{1}{3}\sin 3u \right) \left( \frac{1}{2} + \sin v \right), \left( \frac{3}{2}\pi \cos u - \frac{1}{3}\cos 3u \right) \left( \frac{1}{2} + \sin v \right), 8.55 \cos v \right)$$

all translated by the vector (5.94, 0.3, 16.29) to localize the model on the given set of points of the point cloud.

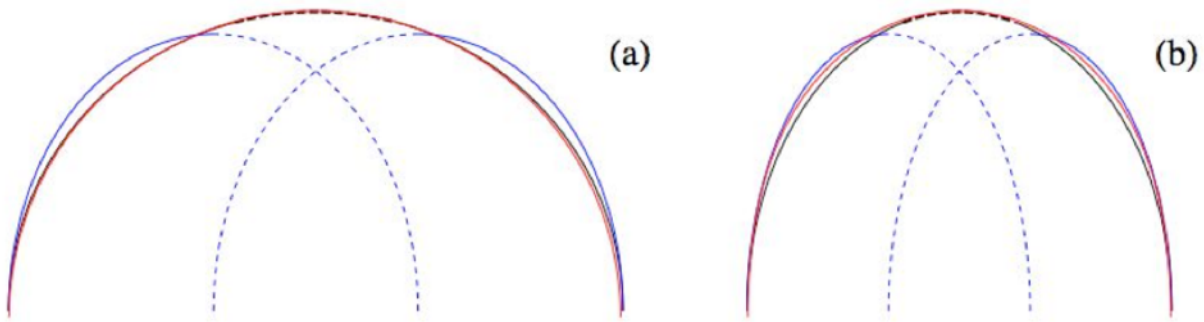


Figure 9: Sections of the three model surfaces ( $\gamma_1(u,v)$  black,  $\gamma_2(u,v)$  red,  $\gamma_3(u,v)$  blue) with the same basis line.

The chosen surfaces are then tested by measuring their mean and maximum distance from the given set of points (see Table 1): the ellipsoidal surface  $\gamma_1(u,v)$  is the closest one whereas the toroidal surface  $\gamma_3(u,v)$  is the worst and the rotated epitrochoid  $\gamma_2(u,v)$  is comparable with  $\gamma_1$ .

Note that the numbers are anyhow very small with respect to the vault size, since they are less than 1% of its major axis.

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	$\langle d(\gamma_1) \rangle$	$\max(d(\gamma_1))$	$\langle d(\gamma_2) \rangle$	$\max(d(\gamma_2))$	$\langle d(\gamma_3) \rangle$	$\max(d(\gamma_3))$
1	5.68	13.89	7.51	11.43	12.32	22.98
2	3.15	7.71	4.95	12.25	16.51	28.40
3	6.87	13.90	7.37	16.90	18.81	33.60
4	6.07	14.67	10.41	22.52	18.87	34.48
5	7.59	14.25	11.69	24.04	15.04	33.91
6	9.01	18.77	7.43	13.89	11.96	24.67
7	6.30	15.53	6.84	14.62	10.66	24.43
8	4.16	11.24	4.93	14.14	18.81	28.87
9	4.93	10.53	6.19	13.46	12.45	28.63
10	4.31	10.49	4.87	12.65	19.94	32.97
11	7.95	22.19	8.69	23.31	15.29	28.98
12	6.85	13.83	10.98	18.24	18.55	33.59
13	6.16	18.27	9.55	14.56	13.61	23.85
14	6.14	12.81	11.12	20.72	19.87	31.76
15	4.90	13.44	6.15	16.19	10.64	24.87
16	5.12	10.01	7.91	13.63	13.46	21.23
Total	5.95	13.85	7.91	16.41	15.43	28.58

Table 1: Mean and maximum distance (in cm.) for each of the 16 horizontal sector (in the  $u$  variable) between points of the relief and the three different surfaces  $\gamma_1(u,v)$ ,  $\gamma_2(u,v)$ ,  $\gamma_3(u,v)$ .

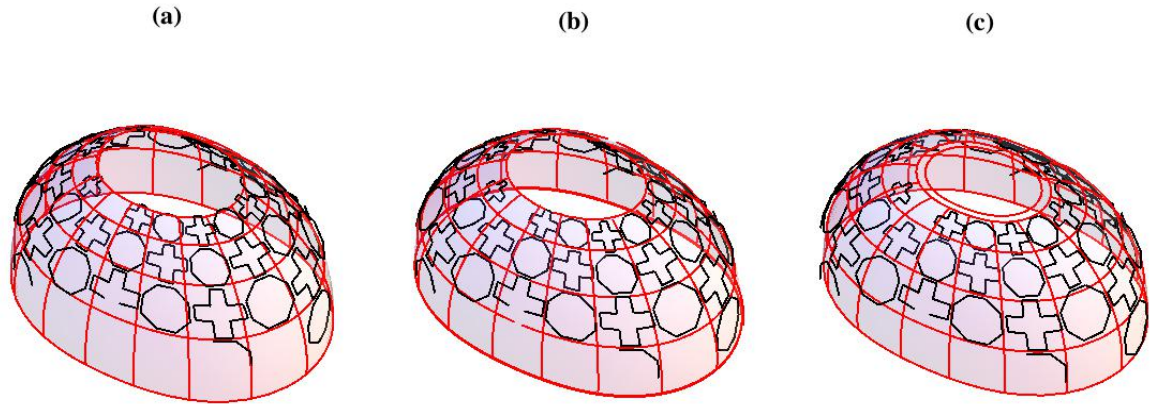


Figure 13: Models of the vault with a decoration sample from the direct survey; (a)  $\gamma_1(u,v)$ ,  $b=0.715$ ; (b)  $\gamma_2(u,v)$ ,  $\cos(v)$  in arithmetic progression; (c)  $\gamma_3(u,v)$ ,  $v = (0, \pi/20, \pi/8, \pi/4, 3/8\pi, \pi/2)$ .

### Edge-Surfaces from four parametric curves

Given four intersecting parametric curves, with parameter values normalized in  $[0,1]$ , it is possible to construct a surface that has the curves as a frame:



$$\begin{aligned} S(u, v) = & (1 - v)S_1(u, 0) + uS_2(1, v) - vS_3(u, 1) + (1 - u)S_4(0, v) \\ & - (1 - u)(1 - v)S_1(0, 0) - u(1 - v)S_2(1, 0) - uvS_3(1, 1) \\ & - (1 - u)vS_4(0, 1) \end{aligned}$$

This mathematical construction algorithm can be used to get a piece of the surface model using regular parametric curve of sections as discussed in the previous paragraphs.

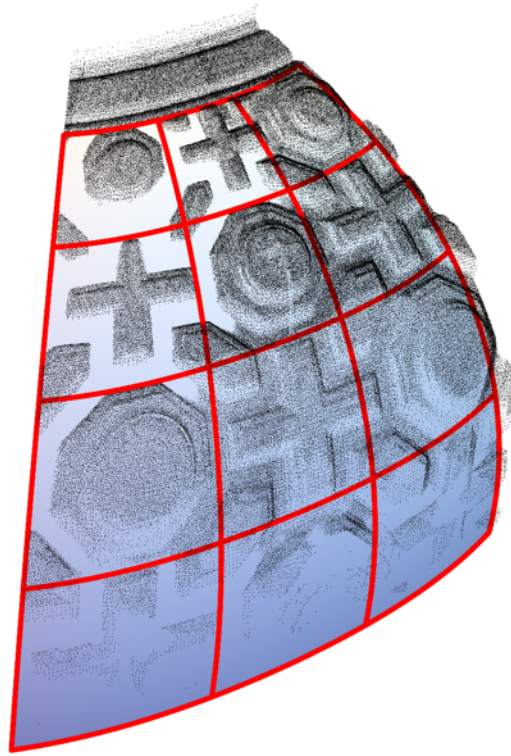
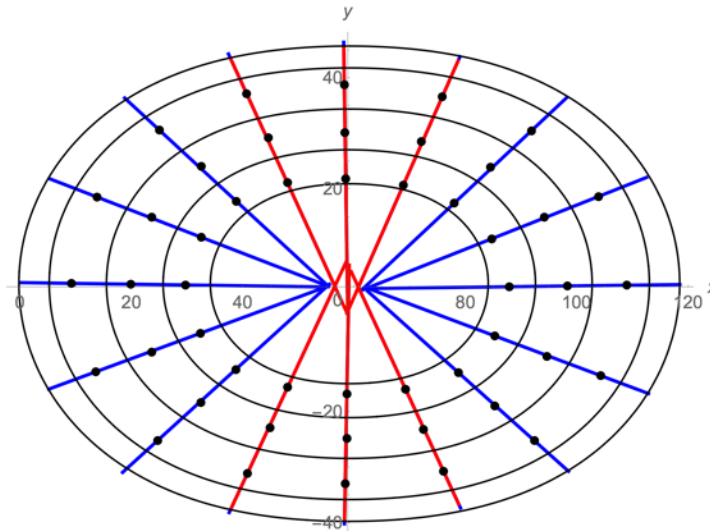


Figure 9: Model of a sector along the large circular arc: the position of the parametric surface based on four given border curves has been optimized.

## Three-dimensional ovals

Starting from the oval of the base curve it is possible to construct a model divided in four sectors corresponding to the four symmetric arcs of the oval. Since also the vertical sections are approximated by circular arcs, the four sectors can be seen as part of a toric surface.

The centers of the lacunars are also oriented along four different centers (see Figure 16).



*Figure 16: The barycenter of the lacunars (black dots) are aligned towards four different centers.*

This suggests the construction using four vertical axis of symmetry passing through such centers, as in Figure 17.

## TESSELLATIONS

### The motif with octagons and crosses



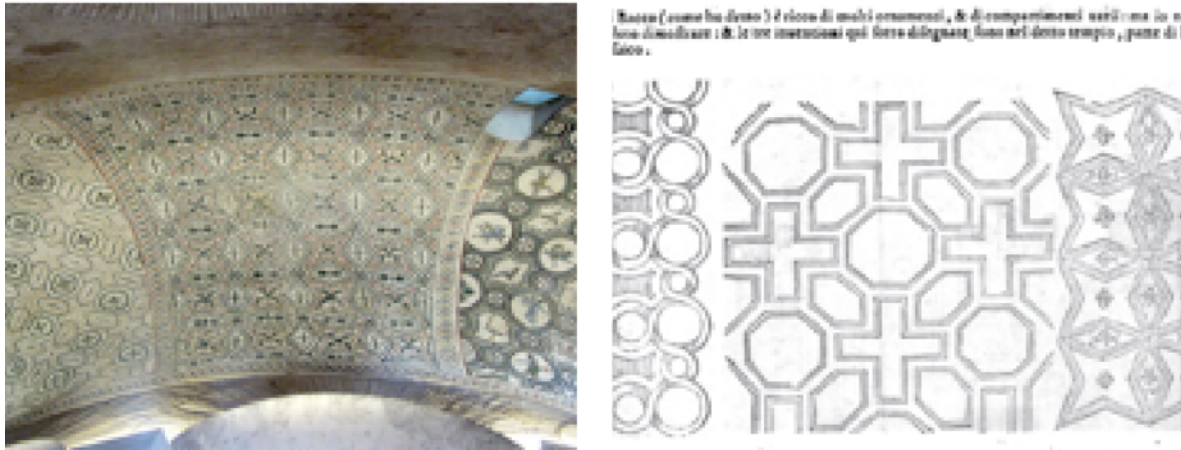
Figure 3: Roman floor mosaics, with the same motif of octagons and crosses, from Aquileia, Piazza Armerina and Pallatina (Spain).

The crosses (symbol of Christianity and of the Order of Padri Trinitari, owner of the church of San Carlino) and octagons tassellation motif of the vault is an ancient roman mosaic floor decoration



(see Fig.3) which appeared also in the toric vault of S. Costanza: Serlio ([15]) reported this geometrical designs in his book (see Fig. 4).

Some connections between mosaics in S.Costanza, Serlio and Borromini are well established



*Figure 4: Ornaments in S. Costanza (Roma), then considered the old Temple of Baccus, as reported in Serlio's Trattato di Architettura, 1540.*

(Blunt, 1979) for S. Carlino and (Bruschi, 1978) for a decoration of an architrave in the Oratorio dei Filippini, let's add some curiosity and mathematical considerations.

The particular oval in the project of S. Carlino is related to the Serlio's oval of the fourth kind, based on two circles both passing through the center of the other one: the entire frame of S. Costanza's mosaics is made of a long line of such circles (Fig. 5). In his Libro IV Serlio [15] also suggest explicitly (with a drawing) a ceiling made by lacunars, again with the same decoration. Moreover, the curvature of the toroidal vault in S. Costanza affects the mosaic design because of the relative size of the rectangular panels which must be distorted and not only curved as in a barrel vault. The effect of such non euclidean iperbolic geometry, even if it had been only a practical solution of a given tessellation (literary) problem, is clearly visible (Fig. 5 and 6) and mathematically is very similar to the problem faced by Borromini: to give the impression, in a non euclidean elliptic geometry, of great regularity as in a planar tessellation.

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**Figure 5:** The decoration over the entrance of S. Costanza. A particular of the frame: note the local displacement between the frame and the internal motif.



Figure 6: S.Costanza: full panel of the toric vault with the cross-octagons and exagonal tassellation.

In this respect, for Borromini's task of rendering a planar tassellation in an ellipsoidal-like vault, it could have been more inspirational the practical work of a 4<sup>th</sup> century mosaic magister rather than a 16<sup>th</sup> century architect.

The 17<sup>th</sup> century has been fundamental for the history of both mathematics and architecture, thank to innovative genius like Cartesio and Borromini, but to settle non-euclidean geometry as a formal theory two more centuries were needed: starting from the work of Gerolamo Saccheri [11] in 1733 up to the work of Lobachevsky and Bolyai in 1829-1833. It's nice however to think that one month after the violent death of Borromini (August 3th, 1667) a baby named Gerolamo was born (september 5<sup>th</sup>, 1667).

## A parametric model

The model depends on few parameters defining a given base curve.

The lacunars seems to be bounded by coordinates curves (“parallels” and “meridians”) in a very regular pattern: alternating crosses and regular octagons on increasing levels seems to rescale towards the top of the vault. Given the  $n^{\text{th}}$  vertical level and the number  $N$  of lacunars for any level we can define position and size of a tassel of the surface by the spherical coordinates of its centre  $(u_0, v_0)$  and the angular “radius” in horizontal ( $u_r$ ) and vertical ( $v_r$ ) direction:

$$\begin{aligned}u_0 &= \frac{2\pi}{N} \\v_0^n &= \frac{2\pi}{N} 2b^n(1+b) \\u_r^n &= u_r^0 = \frac{\pi}{N} \\v_r^n &= \frac{\pi}{4} b^n(1-b)\end{aligned}$$

where  $b$  is the ratio between vertical radii of two successive tassels, that is

$$b = \frac{v_r^{n+1}}{v_r^n}$$

It is possible to check the geometric progression of levels with the help of some graphical tool: in Figure 10 dinamically we could change some parameter, looking for the correct common ratio.

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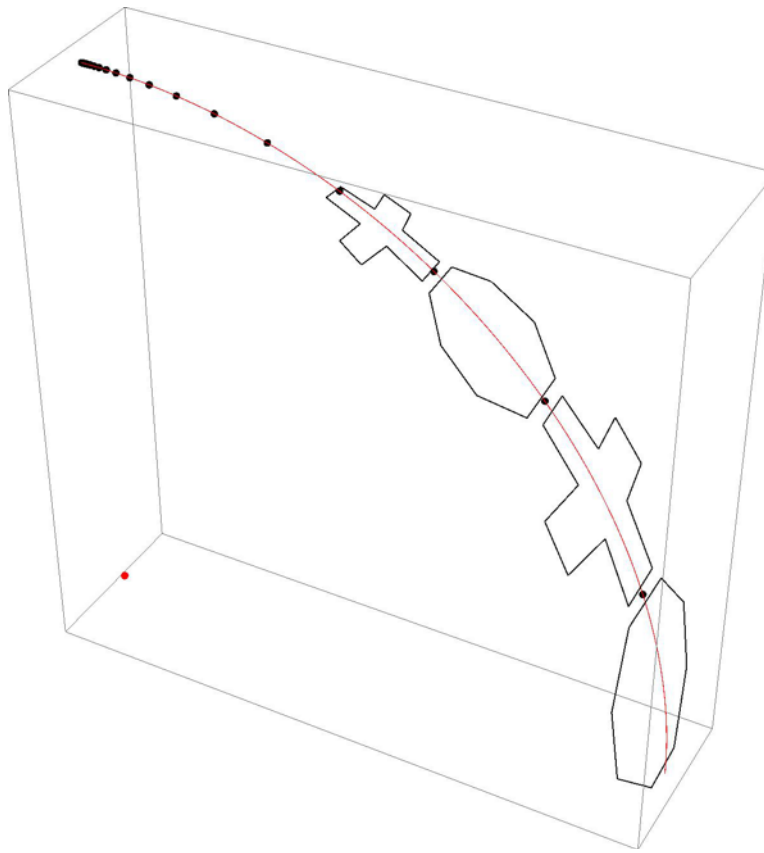


Figure 5: A model of the lacunars based on some reference points at the vertex of octagons and crosses. The black dots indicate the level curves in a geometrical progression.

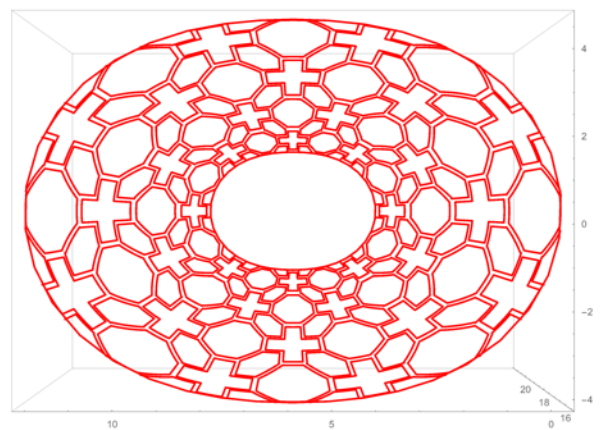
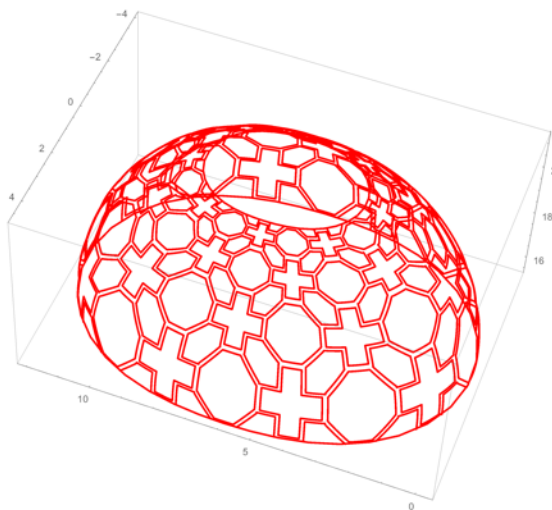


Figure 5: A model of S.Carlino's vault based on an epitrochoidal ellipsoid with  $b=0.71$ .

In order to fix the value of  $b$  and of the vertical size of the first tessell we make (as in Falcolini&Vallicelli, 2011) two hypothesis:

- 1) it should be possible to fit an infinite number of successive tassels of constant ratio which ends up exactly at the pole,
- 2) all tassels have the same size: they should, as much as possible, looks like squares within some accuracy.

These two requirements are at the base of all practical rules to fit domes with regular tessellations which usually tend to the centre of the dome with a perspective sense of infinite height: the general problem is more relevant when the surface is not regular or lack some symmetry.

The first hypothesis implies that the sum of the vertical size of the infinite tassels in angular coordinates would be exactly  $\pi / 2$  , which gives the condition:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty} 2v_r^k = \sum_{k=0}^{\infty} 2v_r^0 b^k = 2v_r^0 \sum_{k=0}^{\infty} b^k = 2v_r^0 \frac{1}{1-b}$$

that is

$$v_r^0 = \frac{\pi}{4} (1-b)$$

The second condition, tassels of similar size on both vertical and horizontal direction, in a spherical approximation of the surface can be fixed by the condition (which depends on the vertical position of the tessell)

$$v_r^n = \frac{\pi}{N} \sin v_0^n$$

or more explicitly

$$\sin\left(\frac{4\pi}{N} b^n (1+b)\right) = \frac{N}{4} b^n (1-b)$$

In our case  $N=16$  and the relation leads to

$$\sin\left(\frac{\pi}{4} b^n (1+b)\right) = 4b^n (1-b)$$

that for  $n=1$  has an approximate numerical solution of  $b=0.712913$  .

For large  $n$  the approximate condition  $\sin(x) \approx x$  holds and for  $N=16$  one gets



$$b = \frac{N^2 - 16\pi}{N^2 + 16\pi} = \frac{16 - \pi}{16 + \pi} = 0.671752 \dots$$

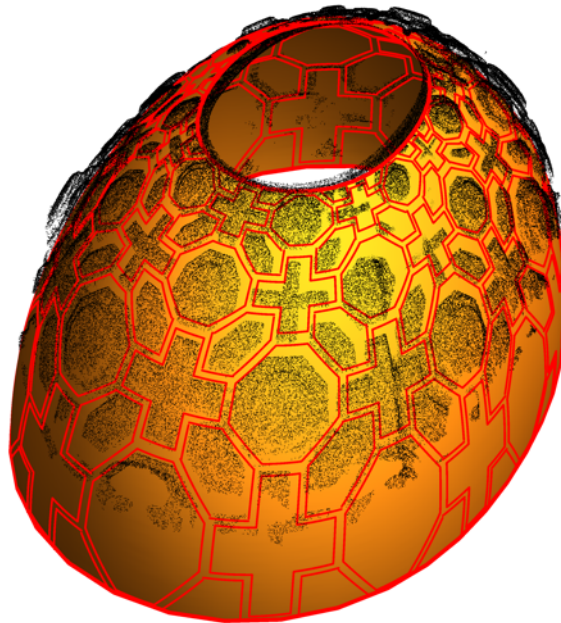
a value which guarantees a regular tassellation for any number of (smaller and smaller) levels.

Since the surface is far from spherical some more detailed analysis is needed .

Here we have shown reasonable values of  $b$  for the models under scrutiny, looking for a graphical control of the prescribed rules for the first levels of the tassellation and keeping fixed the general properties of regularity.

We have written program codes, using Mathematica software (in Appendix), to automatically compute a generic regular tassellation of a given parametric surface evaluating the accuracy of its representation.

An example of a complete model is then superimposed to the points cloud.



*Figure 5 A tessellation model ( $b=0.71$ ) superimposed to a chosen surface and to the points cloud.*

The parametric nature of the automatic algorithm presented here (see the Appendix) to produce different models allows us to apply the same structure of the surface or the decorations but with relevant changes in some of the parameters: on Figure 6a for example is shown the same decoration and the surface is still an ellipsoid built over an epitrochoid but the distance  $h$  of the fixed point from the center of the rolling circle is 2 instead of  $1/3$ ; in Figure 6b the surface is still an ellipsoid on the same epitrochoid of S.Carlino but the number of lacunars per level is 23 instead of 16 on 10 levels instead of 4.

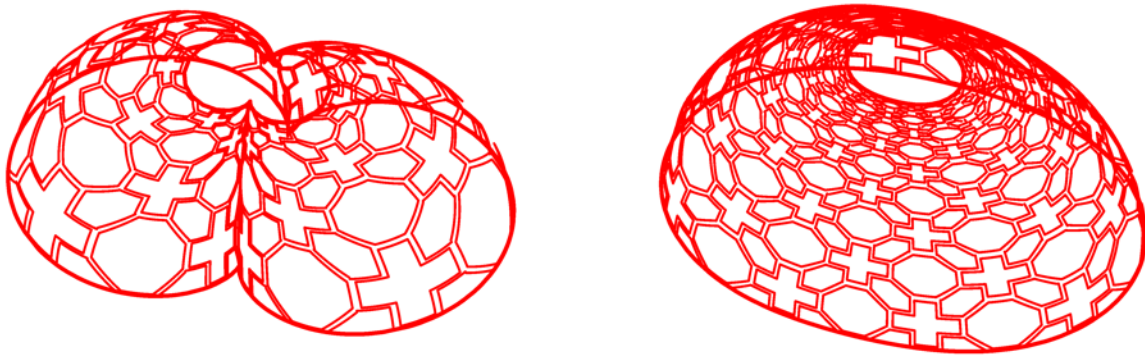


Figure 6: Same tessellation on a different surface with  $h=2$  (left) and on the same surface with  $n=8$  and  $b=.82$  (right).

If we keep fixed the surface and increase the number  $n$  of lacunar levels we see that the uniform shape of the tessels is preserved; in Figure 7 the only difference on the two images is the star-like connection (left) of the 12 vertex points of the cross tessel. Note the "familiar" shape of the curves formed by this simple change in the algorithm.

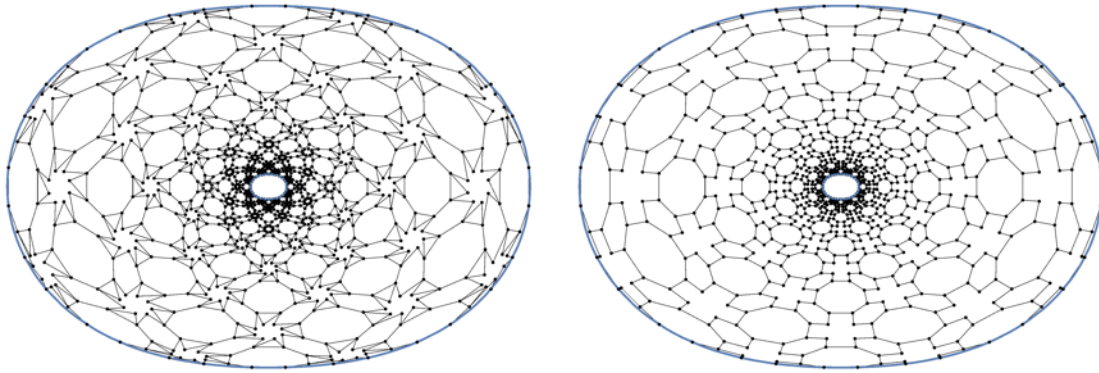


Figure 7: Different tessellations over the same surface with more levels: the only difference on the left is the star-like connection of the 12 vertex points of the cross tessell.

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## KEY TERMS AND DEFINITIONS

**Algorithm:** a self-contained step-by-step set of operations to be performed.

**Epicycloid:** a plane curve produced by tracing the path of a chosen point of a circle which rolls without slipping around a fixed circle.

**Epitrochoid:** a plane curve produced by tracing the path of a point at a given distance from the center of a circle which rolls without slipping around a fixed circle.

**Mathematical model:** a description of a system using mathematical concepts and language.

**Oval:** a symmetrical differentiable curve constructed from two pairs of arcs, with two different radii.

**Parametric curve:** an expression of the coordinates of the generic point of a curve as a function of one variable, called a parameter.

**Parametric surface:** an expression of the coordinates of the generic point of a surface as a function of two variables.

**Point cloud:** a set of three-dimensional data points in some coordinate system.

**Tessellation:** is the tiling of a surface using one or more geometric shapes, called tiles, with no overlaps and no gaps.

**Torus:** a surface of revolution generated by revolving a circle in three-dimensional space about an axis coplanar with the circle.

## APPENDIX

Wolfram Mathematica graphics commands for a model of San Carlino tessellation over a generic parametric surface  $\text{sup}=S(u,v)$  depending on the value of the ratio  $b$ , the number of levels  $M$  and the number  $N$  of lacunars on a given level.

$(u_0,v_0)$  is the center of a tessel and  $ur,v_r$  are the horizontal and vertical radius (in angular coordinates) of the tassel.

```
octagon[sup_][{u0_, v0_}][ur_, vr_] := Show[{
ParametricPlot3D[{sup /. v → v0 + vr, sup /. v → v0 - vr}, {u, u0 - ur/3, u0 + ur/3}],
ParametricPlot3D[{sup /. u → u0 + ur, sup /. u → u0 - ur}, {v, v0 - vr/3, v0 + vr/3}],
ParametricPlot3D[
{sup /. v → vr/ur (u - u0) + v0 + 4/3 vr, sup /. v → -vr/ur (u - u0) + v0 - 4/3 vr}, {u, u0 - ur, u0 + ur/3}],
ParametricPlot3D[
{sup /. v → vr/ur (u - u0) + v0 - 4/3 vr, sup /. v → -vr/ur (u - u0) + v0 + 4/3 vr}, {u, u0 + ur/3, u0 + ur}]]
```

```
cross[sup_][{u0_, v0_}][ur_, vr_] := Show[{
ParametricPlot3D[{sup /. v → v0 + vr, sup /. v → v0 - vr}, {u, u0 - ur/3, u0 + ur/3}],
ParametricPlot3D[{sup /. u → u0 + ur, sup /. u → u0 - ur}, {v, v0 - vr/3, v0 + vr/3}],
ParametricPlot3D[{sup /. v → v0 - vr/3, sup /. v → v0 + vr/3}, {u, u0 - ur, u0 + ur/3}],
ParametricPlot3D[{sup /. u → u0 - ur/3, sup /. u → u0 + ur/3}, {v, v0 - vr, v0 + vr/3}],
ParametricPlot3D[{sup /. v → v0 - vr/3, sup /. v → v0 + vr/3}, {u, u0 + ur/3, u0 + ur}],
ParametricPlot3D[{sup /. u → u0 - ur/3, sup /. u → u0 + ur/3}, {v, v0 + vr/3, v0 + vr}]]
```

```
Show[{
Table[
cross[sup][{h Pi/8, Pi/4 b^n (1 + b)}][Pi/16, Pi/4 b^n (1 - b)],
{n, 0, M, 2}, {h, 1, N, 2}],
Table[
octagon[sup][{(h+1) Pi/8, Pi/4 b^n (1+b)}][Pi/16, Pi/4 b^n (1-b)],
{n, 1, M, 2}, {h, 2, N, 2}],
Table[
octagon [sup][{(h + 1) Pi/8, Pi/4 b^n (1 + b)}][Pi/16, Pi/4 b^n (1 - b)],
{n, 0, M, 2}, {h, 1, N, 2}],
Table[
cross [sup][{h Pi/8, Pi/4 b^n (1 + b)}][Pi/16, Pi/4 b^n (1 - b)],
{n, 1, M, 2}, {h, 2, N, 2}],
ParametricPlot3D[sup /. v → Pi/2, {u, 0, 2 Pi}],
ParametricPlot3D[sup /. v → Pi/2 b^(M + 1), {u, 0, 2 Pi}]]
```