Equilibrium selection for multi-portfolio optimization*

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Abstract

We analyze a Nash equilibrium problem arising when trades from different accounts are pooled for execution. We introduce a new general multi-portfolio model and state sufficient conditions for the monotonicity of the underlying Nash equilibrium problem. Monotonicity makes it possible to treat the problem numerically and, for the case of nonunique equilibria, to solve hierarchical problems of equilibrium selection. We also give sufficient conditions for the Nash equilibrium problem formulation to be a potential game. Our computational experience confirms the theoretical insights and substantiates the significance of the equilibrium selection.

Keywords: Game theory, Nash equilibrium problem, hierarchical optimization, variational inequalities, portfolio selection

1. Introduction

Portfolio optimization deals with the allocation of funds among risky assets so that some utility measure is maximized. In the basic mean-variance model of portfolio theory, Markowitz (1952, 1959) suggested a measure which accounts for the trade-off between maximizing the return and minimizing the risk. Based on this, but different from the classical setting where a manager is in charge of a *single* account, we consider the more practical and common situation of *multiple* portfolios to be simultaneously and jointly optimized. This is the case, for example, for financial service providers, which typically accommodate multiple accounts from different clients simultaneously and pool the trades for common execution. In this framework, since

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the transaction costs for each portfolio depend on the trades from all accounts, the multiple portfolios turn out to be linked to each other by the market impact of their transactions.

Transaction costs (see Fabozzi et al., 2010, and the references therein) are incurred when trades are executed. They can be divided into explicit and implicit costs (Kociński, 2017) which are, e.g., the commission applied to the accounts, which is known, and the market (liquidity) impact, respectively. In particular, market impact costs, that originate from price impact and limited liquidity, are the main component of transaction costs when trading volume increases (see Bertsimas and Lo, 1998, for more information). The impact of account trades on the liquidity of the assets in the market is hard to estimate (Lesmond et al., 1999) and the intricate interactions between the multiple portfolios are difficult to treat. Hence, these phenomena are often disregarded, resulting in an actual reduction in realized returns (O'Cinneide et al., 2006; Yang et al., 2013).

Probably due to the rather complicated nature of the problem, the study concerning the so-called multi-portfolio selection is still in its infancy. Some initial attempts have been undertaken mostly by practitioners. The approach that is usually resorted to in practice is to optimize each account ignoring the others; then, once trades are bunched and executed, the actual market impact cost, which is typically larger than the estimated one, is allocated among accounts depending on each account's share of trading activity, i.e., according to a *pro rata* policy. However, such a strategy typically results in a poor performance for all the clients (O'Cinneide et al., 2006).

1.1. Literature review

The first paper concerning multi-portfolio selection dates back to O'Cinneide et al. (2006). There, the complicated interactions between the accounts are taken care of via an optimization model where the sum of the objectives of the individual accounts is maximized. Based on the observation that, as a result of this approach, some accounts (usually those that are smaller in size) turn out to be penalized with respect to others, Savelsbergh et al. (2010) proposed to address the multi-portfolio problem by selecting the set of portfolios that form a Nash equilibrium. This amounts to optimizing each account individually and simultaneously, but such that in each corresponding portfolio selection problem also the aggregate trades from all the other accounts are considered. The resulting solution has the remarkable property that there is no incentive for any account to unilaterally change its trades. Iancu and Trichakis (2014) proposed the accounts manager to jointly optimize the trades *and* the split of market impact costs between the accounts. The main aim of their work is trying to balance the conflicting goals of maximizing the aggregate gains and distributing them equitably across the accounts. For this reason, the multi-portfolio selection is achieved via a utility allocation model: the manager maximizes a welfare function of the allocation of accounts utilities over the set of feasible trades and associated splits of market impact costs.

Following Savelsbergh et al. (2010) and Yang et al. (2013) we rely on the Nash equilibrium approach, since it seems to fit well within a framework in which individual accounts belong to clients who follow their own interest. If the trades executed by the manager do not form a Nash equilibrium, some clients could ask to modify exclusively their trades to improve utility. This undesired and unstable situation cannot happen if the manager relies on the Nash equilibrium approach. We assume the following conditions, which are standard in noncooperative frameworks, to hold: accounts are managed rationally and optimized simultaneously, and each account's information is shared with the others. The Nash Equilibrium Problem (NEP) is widely used to model the resulting multi-agent-like system; for further information, the reader may refer to the classical works of Nash (1951) and Tirole (1988). Whenever the NEP has a unique solution, it provides each account with the best decision to make, since the choices from all the other accounts can be predicted. For other details, including algorithmic developments, see Aussel and Sagratella (2017), Dreves et al. (2011), Facchinei and Sagratella (2011), Sagratella (2016), Sagratella (2017a), Sagratella (2017b) and Sagratella (2019).

In the context of the NEP perspective on multi-portfolio selection, the most advanced results have been obtained by Yang et al. (2013). There, the authors rely upon the restrictive requirement that the problem can be modeled as a potential game, i.e., the special simpler case in which solutions of the NEP can be computed by addressing a single optimization problem. The focus of Yang et al. (2013) is on establishing sufficient conditions for the existence and uniqueness of these solutions, and developing viable solution algorithms.

1.2. Aim and scope

We deal with multi-portfolio NEP models which are neither necessarily potential games, nor uniquely solvable. The resulting more general framework requires modeling strategies and algorithmic choices that stand apart from the existing ones and widen scope and applicability of NEPs in the multi-portfolio context. Our main contributions are:

- a new general multi-portfolio NEP model that improves on the state-of-the-art where restrictive assumptions are called for requiring the underlying NEP to be a simple potential game with a unique solution (see Yang et al. (2013) and the references therein). As a major departure from these results, the NEP is neither assumed to be a potential game, nor uniquely solvable;

- the formulation of a novel selection approach yielding a new hierarchical model in which some (upper-level) criterion is minimized over the (lower-level) set of the NEP (multiple) solutions. Such a strategy allows one to recover, also in the more general framework we consider, a predictable outcome;

- the statement of sufficient conditions for monotonicity of the NEP which is crucial for its explicit numerical treatment. In particular, this makes it possible to rely on a Tikhonov-

like algorithm (Lampariello et al., 2020) and to present some numerical results for our selection method.

We also identify new classes of problems for the general multi-portfolio selection to reduce to a simpler potential game: we show that, under conditions that are weaker than those assumed in Yang et al. (2013), a potential game can still be guaranteed.

2. The game theoretical model

We consider N accounts indexed by $\nu = 1, \ldots, N$. Each account ν's budget $b^\nu \in \mathbb{R}_+$ is to be invested in K assets of a market. The decision variables $y^\nu \in Y_\nu \subseteq R^K$ denote the fractions of b^ν to invest in each asset, where Y_ν is the nonempty convex compact set of feasible portfolios, e.g., the standard simplex. Let $r \in \mathbb{R}^K$ denote random variables where r_k is the return of asset $k \in \{1, \ldots, K\}$ over a single-period investment. Since, unless expectations are assumed homogeneous (e.g., capital asset pricing model), the evaluation of an investment is account dependent, we define $\mu^\nu = \mathbb{E}^\nu(r) \in \mathbb{R}^K$ as expectations of the assets' returns for ν, as well as the positive semidefinite covariance matrix $\Sigma^\nu = \mathbb{E}^\nu((r - \mu^\nu)(r - \mu^\nu)^\top)$. We consider the following measures for portfolio income I_ν and risk R_ν, where we use the portfolio variance as the risk measure:

$$I_\nu(y^\nu) \triangleq b^\nu(\mu^\nu)^\top y^\nu, \tag{1}$$

$$R_\nu(y^\nu) \triangleq \frac{1}{2}(b^\nu)^2(y^\nu)^\top \Sigma^\nu y^\nu. \tag{2}$$

2.1. A model for market impact

It is precisely when trades from multiple accounts are pooled for common execution that individual accounts can suffer the market impact caused by a shortage of liquidity because the joint demand of an asset can be tremendously larger than the individual demand. We focus on liquidity as a main factor for transaction costs, but our model allows one to use also other sources of costs. When each account is managed independently, resulting market impacts due to overall trades are naively ignored, leading to worse solutions (O'Cinneide et al., 2006; Yang et al., 2013).

To take account of the transaction cost effect, we shall consider a market impact matrix $\Omega^\nu \in \mathbb{R}^{K \times K}$ whose entry at position (i, j) is the impact of the liquidity of asset i on the liquidity of asset j. Since these interrelations are hard to estimate (and there does not even exist an univocal definition of liquidity), it is reasonable to assume that Ω^ν is different for each ν.

The cross-asset effects motivate the need for not necessarily diagonal Ω^νs: liquidity and, more generally, trading costs show commonality (liquidity co-movements across assets or markets) (see, e.g., Bertsimas, Hummel and Lo, 1999). This means that each action on a

given security displays effects not only on that security, but also on the others. For example, huge trades in some company's stock can attract more trading in related companies' stocks. As for the specific form of Ω^ν, we assume that it is positive semidefinite, as in Mei et al. (2016). This is a natural assumption, as any eigenvector associated with a negative eigenvalue would correspond to an aggregated trade with unreasonable negative transaction costs. For simplicity, we disregard differences between buy and sell positions.

Adopting a well-established path in the literature (see, e.g., Bertsimas and Lo, 1998, and the references therein), we consider for each ν a linear market impact unitary cost function. Since the trades are pooled and executed together, the market impact unitary cost function depends on the invested capital of the aggregated trades from all accounts, that is

$$c^\nu(y^1, \ldots, y^N) \triangleq \Omega^\nu \sum_{\lambda=1}^{N} b^\lambda (y^\lambda - v^\lambda), \tag{3}$$

where $v^\lambda \in \mathbb{R}^K$ denotes the vector of the current positions for account λ.

The market impact cost function we employ is non-separable among the assets, because the positive semidefinite Ω^ν is not assumed to be symmetric, let alone diagonal. The quantity $b^\lambda(y^\lambda - v^\lambda)$ in (3) denotes the deviation of the amount of λ's budget to be invested in each available asset with respect to the current positions. Then, c^ν is associated with rebalancing from v^λ to y^λ for every λ.

Summarizing, c^ν maps the demand of all accounts to a unitary cost. Thus, the total transaction costs for ν is

$$TC_\nu(y^1, \ldots, y^N) \triangleq b^\nu (y^\nu - v^\nu)^\top c^\nu(y^1, \ldots, y^N) = \underbrace{b^\nu (y^\nu - v^\nu)^\top}_{\text{Invested capital}} \underbrace{\Omega^\nu \sum_{\lambda=1}^{N} b^\lambda (y^\lambda - v^\lambda)}_{\text{Unitary transaction costs}}. \tag{4}$$

Having defined the objective functions to optimize for each account (see (1), (2) and (4)), all the elements are available to present our multi-portfolio selection strategy.

To model the multiple portfolio selection problems it is natural to rely on the NEP consisting of the collection of N (parametric) optimization problems, each borne by an account. As it is standard for NEPs, we denote by y the vector formed by all the decision variables, and by $y^{-\nu}$ the vector composed by all the accounts' decision variables except those of account ν:

$$y \triangleq \begin{pmatrix} y^1 \\ \vdots \\ y^N \end{pmatrix} \in \mathbb{R}^{NK}, \qquad y^{-\nu} \triangleq \begin{pmatrix} y^1 \\ \vdots \\ y^{\nu-1} \\ y^{\nu+1} \\ \vdots \\ y^N \end{pmatrix} \in \mathbb{R}^{(N-1)K}.$$

To emphasize the account ν's decision variables within y, we sometimes write $(y^\nu, y^{-\nu})$ instead of y. Note that this still stands for the vector $y = (y^1, \ldots, y^\nu, \ldots, y^N)$ and that, in particular, the notation $(y^\nu, y^{-\nu})$ does not mean that the block components of y are reordered in such a way that y^ν becomes the first block. For each account, the objective function $\theta_\nu : \mathbb{R}^{NK} \to \mathbb{R}$ depends on variables y^ν as well as on the variables $y^{-\nu}$ from all the other accounts:

$$\begin{aligned}\theta_\nu(y^\nu, y^{-\nu}) &\triangleq -I_\nu(y^\nu) + \rho^\nu R_\nu(y^\nu) + TC_\nu(y^\nu, y^{-\nu}) \\ &= -b^\nu(\mu^\nu)^\top y^\nu + \rho^\nu \tfrac{1}{2}(b^\nu)^2 (y^\nu)^\top \Sigma^\nu y^\nu + b^\nu (y^\nu - v^\nu)^\top \Omega^\nu \textstyle\sum_{\lambda=1}^N b^\lambda (y^\lambda - v^\lambda),\end{aligned}$$

where $\rho^\nu \in \mathbb{R}_+$ is the risk aversion parameter. It is the presence of $TC_\nu(y^\nu, y^{-\nu})$ that makes θ_ν dependent (parametrically) on the other accounts' decision variables $y^{-\nu}$. The objective θ_ν is continuously differentiable, quadratic, and convex with respect to y^ν, since Σ^ν and Ω^ν are positive semidefinite and b^ν, $\rho^\nu \in \mathbb{R}_+$. Summarizing, the NEP we consider consists of the collection of account ν's parametric optimization problems

$$\text{minimize}_{y^\nu}\ \theta_\nu(y^\nu, y^{-\nu}) \quad \text{s.t.} \quad y^\nu \in Y_\nu. \tag{P^ν}$$

For fixed values of $y^{-\nu}$, each parametric problem (P^ν) is convex in y^ν. Denoting $Y \triangleq Y_1 \times \cdots \times Y_N$, the NEP we consider is the problem of finding $y \in Y$ such that

$$\theta_\nu(y^\nu, y^{-\nu}) \le \theta_\nu(w^\nu, y^{-\nu}) \qquad \forall w^\nu \in Y_\nu, \qquad \nu = 1, \ldots, N. \tag{5}$$

Any $y \in Y$ satisfying (5) is an equilibrium or a solution of the NEP. A point is therefore an equilibrium if for no account, given the other accounts' trades, the objective function can be decreased by unilaterally changing trades to any other feasible solution. Accordingly, we indicate with E the (non-parametric) set of equilibria of the NEP:

$$E \triangleq \left\{ y \in Y :\ \theta_\nu(y^\nu, y^{-\nu}) \le \theta_\nu(w^\nu, y^{-\nu})\ \forall w^\nu \in Y_\nu,\ \nu = 1, \ldots, N \right\}.$$

As major deviations from the classical Markowitz (1952) framework that reads, for each account,

$$\text{minimize}_{y^\nu}\ -(\mu^\nu)^\top y^\nu + \frac{1}{2}\rho^\nu (y^\nu)^\top \Sigma^\nu y^\nu \quad \text{s.t.} \quad y^\nu \in Y_\nu, \tag{6}$$

in (P^ν) we consider transaction costs that couple the accounts' problems, as well as different budgets for the accounts. This makes it reasonable to consider amounts to be invested, while, if budgets were all the same, normalizing, one would rather rely on shares (with unitary budgets), as commonly done in the classical Markowitz (1952) model. Moreover, since the NEP is not supposed to be uniquely solvable, one has to discriminate among the equilibria according to some further criterion (see, e.g., Tirole, 1988) in order to recover a solution that is predictable. This leads us to introduce a novel equilibrium selection model where some

upper-level merit function $f : \mathbb{R}^{NK} \to \mathbb{R}$, that is assumed to be $C^{1,1}$ (that is, differentiable with locally Lipschitz continuous gradient) and convex, is minimized over E:

$$\text{minimize } f(y) \quad \text{s.t.} \quad y \in E. \tag{7}$$

The upper-level objective f can be viewed as a common additional criterion shared by all the accounts, or as any other cost function that is chosen by the manager. For example, it can be taken equal to the sum of all accounts' expected portfolio returns or overall objectives. In the rest of our analysis, we do not specify a particular form for f, making our model (7) sufficiently flexible to encompass different alternatives as for the choice of the selection criterion. We stress that (7) is a problem with a hierarchical structure, in that one has to minimize an upper-level objective over the solution set of a lower-level problem. The main difficulty in tackling such problems is that the feasible set is *implicitly* defined as the solution set of another problem which, being in this case a NEP, is particularly hard to solve. For example, one cannot expect standard constraint qualifications to hold. We show that (7) is actually numerically tractable under mild assumptions.

2.2. The variational inequality formulation

The finite-dimensional Variational Inequality (VI) provides a unifying setting to study properties and solution procedures for optimization and equilibrium problems (see the fundamental book by Facchinei and Pang (2003) and, for some more recent developments, Facchinei et al. (2015); Facchinei, Kanzow and Sagratella (2014); Facchinei and Lampariello (2011); Latorre and Sagratella (2016); Shehu et al. (2020)). We rely on VIs as an analytical tool to establish conditions that make the hierarchical problem (7) practically solvable and show that certain stability properties hold. The NEP we consider turns out to be equivalent to the following variational inequality VI(Y, F): find $y \in Y$ such that

$$F(y)^\top (w - y) \geq 0 \quad \forall w \in Y, \tag{8}$$

where

$$F(y) \triangleq \begin{pmatrix} \nabla_{y^1} \theta_1(y) \\ \vdots \\ \nabla_{y^N} \theta_N(y) \end{pmatrix},$$

and $\nabla_{y^\nu} \theta_\nu(y)$ is the gradient of θ_ν (taken with respect to y^ν) at y. We denote by SOL(Y, F) the solution set of VI(Y, F). The explicit expression of $\nabla_{y^\nu} \theta_\nu(y)$ is given by

$$\begin{aligned} \nabla_{y^\nu} \theta_\nu(y^\nu, y^{-\nu}) \;\; = \;\; & -\left(b^\nu \mu^\nu + 2(b^\nu)^2 \operatorname{sym}(\Omega^\nu) v^\nu + b^\nu \Omega^\nu \textstyle\sum_{\lambda \neq \nu} b^\lambda v^\lambda \right) \\ & + (b^\nu)^2 \left(\rho^\nu \Sigma^\nu + 2 \operatorname{sym}(\Omega^\nu) \right) y^\nu + b^\nu \Omega^\nu \textstyle\sum_{\lambda \neq \nu} b^\lambda y^\lambda, \end{aligned}$$

where we denote by $\operatorname{sym}(\Omega^\nu) \triangleq \frac{1}{2}(\Omega^\nu + \Omega^{\nu\top})$ the symmetric part of Ω^ν. Since θ_ν is continuously differentiable and quadratic, F is continuous and affine. In Proposition 2.1 we show that

conditions that are known to be sufficient for a NEP to be equivalently recast as a VI are satisfied by (5).

Proposition 2.1 *The identity $E = SOL(Y, F)$ holds.*

Proof. Since, for every ν, Y^ν is a convex compact set and θ_ν is continuously differentiable and convex in y^ν for every fixed $y^{-\nu}$, the claim follows by applying (Facchinei and Pang, 2003, Proposition 1.4.2). □

The above result allows us to analyze (5) via VI theory. In particular, standard conditions entailing existence of a solution and compactness of the solution set hold true for (8) and, thus, also for (5).

Proposition 2.2 *The equilibrium set E is nonempty and compact.*

Proof. Nonemptiness and compactness of $\mathrm{SOL}(Y, F)$ are due to the continuity of F, and nonemptiness, convexity and compactness of Y (see Facchinei and Pang, 2003, Corollary 2.2.6). The claim follows due to Proposition 2.1. □

VI theory again comes into play to study the sensitivity of E with respect to the current positions parameters v^ν. Let $\delta^\nu \in \mathbb{R}^K$ be perturbations of v^ν for every ν and $\delta \triangleq (\delta^1, \ldots, \delta^N)$, and consider the modified current positions $v^\nu + \delta^\nu$. Correspondingly, we introduce the NEP consisting of the following perturbed problems ν:

$$\text{minimize}_{y^\nu}\ \theta_\nu^\delta(y^\nu, y^{-\nu}) \quad \text{s.t.} \quad y^\nu \in Y_\nu,$$

where

$$\begin{aligned}
\theta_\nu^\delta(y^\nu, y^{-\nu}) \;\triangleq\;& -b^\nu(\mu^\nu)^\top y^\nu + \rho^\nu \tfrac{1}{2}(b^\nu)^2 (y^\nu)^\top \Sigma^\nu y^\nu \\
& + b^\nu [y^\nu - (v^\nu + \delta^\nu)]^\top \Omega^\nu \textstyle\sum_{\lambda=1}^N b^\lambda [y^\lambda - (v^\lambda + \delta^\lambda)] \\
=\;& \theta_\nu(y^\nu, y^{-\nu}) - 2(b^\nu)^2 (y^\nu - v^\nu)^\top \mathrm{sym}(\Omega^\nu)\delta^\nu - b^\nu (y^\nu - v^\nu)^\top \Omega^\nu \textstyle\sum_{\lambda\neq\nu} b^\lambda \delta^\lambda \\
& - \textstyle\sum_{\lambda\neq\nu} b^\lambda (y^\lambda - v^\lambda)^\top \Omega^{\nu\top} b^\nu \delta^\nu + b^\nu (\delta^\nu)^\top \Omega^\nu \textstyle\sum_{\lambda=1}^N b^\lambda \delta^\lambda.
\end{aligned}$$

We consider the problem of finding $y_\delta \in Y$ such that

$$\theta_\nu^\delta(y_\delta^\nu, y_\delta^{-\nu}) \leq \theta_\nu^\delta(w^\nu, y_\delta^{-\nu}) \qquad \forall w^\nu \in Y_\nu, \qquad \nu = 1, \ldots, N. \tag{9}$$

By replicating verbatim the considerations in the proof of Proposition 2.1, (9) can be equivalently recast as the perturbed $\mathrm{VI}(Y, F_\delta)$, with

$$F_\delta(y) \triangleq \begin{pmatrix} \nabla_{y^1}\theta_1^\delta(y) \\ \vdots \\ \nabla_{y^N}\theta_N^\delta(y) \end{pmatrix} = F(y) - \begin{pmatrix} 2(b^1)^2 \mathrm{sym}(\Omega^1)\delta^1 + b^1\Omega^1 \sum_{\lambda\neq 1} b^\lambda \delta^\lambda \\ \vdots \\ 2(b^N)^2 \mathrm{sym}(\Omega^N)\delta^N + b^N\Omega^N \sum_{\lambda\neq N} b^\lambda \delta^\lambda \end{pmatrix}.$$

In the following proposition, relying on the equivalent VI formulation of NEPs, we show that E enjoys some local Lipschitz-like continuity with respect to perturbations of the current positions. Preliminarily, we recall that since F is affine, if Y is polyhedral, $\mathrm{VI}(Y,F)$ is semistable (see Facchinei and Pang, 2003, Corollary 5.5.9), that is, for every open set $U \subseteq \mathbb{R}^{NK}$ containing $\mathrm{SOL}(Y,F)$, there exist positive scalars η and d such that, for every function G satisfying the relation $\max_{y\in Y} \|G(y)-F(y)\| < \eta$, it holds that

$$\mathrm{SOL}(Y,G) \cap U \subseteq \mathrm{SOL}(Y,F) + d\,\omega\,\mathbb{B}(0,1), \tag{10}$$

where $\mathrm{SOL}(Y,G)$ is the solution set of $\mathrm{VI}(Y,G)$, $\omega \triangleq \max_{y\in Y} \|G(y)-F(y)\|$ and $\mathbb{B}(0,1)$ denotes the ball with center at the origin and radius 1. We also say that $\mathrm{SOL}(Y,F)$ is semistable if the VI is so.

Proposition 2.3 *Let Y be polyhedral. Then, positive scalars ε and c exist such that for all perturbations δ of the current positions with $\|\delta\| \le \varepsilon$, and for every equilibrium y of (5) and every equilibrium y_δ of (9), the relation*

$$\|y_\delta - y\| \le c\|\delta\| \tag{11}$$

holds.

Proof. Due to Proposition 2.1, with $\mathrm{SOL}(Y,F)$ also E is semistable. Since a positive scalar ε exists such that if $\|\delta\| \le \varepsilon$, then

$$\|F_\delta(y) - F(y)\| = \left\| \begin{pmatrix} 2(b^1)^2\,\mathrm{sym}(\Omega^1)\delta^1 + b^1\Omega^1 \sum_{\lambda\neq 1} b^\lambda \delta^\lambda \\ \vdots \\ 2(b^N)^2\,\mathrm{sym}(\Omega^N)\delta^N + b^N\Omega^N \sum_{\lambda\neq N} b^\lambda \delta^\lambda \end{pmatrix} \right\| < \eta,$$

by taking $G = F_\delta$ in definition (10), we have

$$\begin{aligned} \mathrm{SOL}(Y,F_\delta) \cap U &\subseteq \mathrm{SOL}(Y,F) + d \left\| \begin{pmatrix} 2(b^1)^2\,\mathrm{sym}(\Omega^1)\delta^1 + b^1\Omega^1 \sum_{\lambda\neq 1} b^\lambda \delta^\lambda \\ \vdots \\ 2(b^N)^2\,\mathrm{sym}(\Omega^N)\delta^N + b^N\Omega^N \sum_{\lambda\neq N} b^\lambda \delta^\lambda \end{pmatrix} \right\| \mathbb{B}(0,1) \\ &\subseteq \mathrm{SOL}(Y,F) + c\|\delta\|\,\mathbb{B}(0,1), \end{aligned} \tag{12}$$

where $\mathrm{SOL}(Y,F_\delta)$ is the solution set of $\mathrm{VI}(Y,F_\delta)$, for some positive scalar c. Claim (11) is a consequence of (12), due to the compactness of Y. □

Other crucial well-known properties of the VI and, thus, equivalently of the NEP, are deter-

mined by the nature of the Jaco-Hessian

$$Q \triangleq \begin{pmatrix} \nabla_{y^1}^\top \nabla_{y^1} \theta_1(y) & \cdots & \nabla_{y^N}^\top \nabla_{y^1} \theta_1(y) \\ \vdots & \ddots & \vdots \\ \nabla_{y^1}^\top \nabla_{y^N} \theta_N(y) & \cdots & \nabla_{y^N}^\top \nabla_{y^N} \theta_N(y) \end{pmatrix}$$

$$= \begin{pmatrix} (b^1)^2[\rho^1 \Sigma^1 + 2\,\mathrm{sym}(\Omega^1)] & b^1 b^2 \Omega^1 & \cdots & b^1 b^N \Omega^1 \\ b^2 b^1 \Omega^2 & (b^2)^2[\rho^2 \Sigma^2 + 2\,\mathrm{sym}(\Omega^2)] & & b^2 b^N \Omega^2 \\ \vdots & & \ddots & \\ b^N b^1 \Omega^N & b^N b^2 \Omega^N & & (b^N)^2[\rho^N \Sigma^N + 2\,\mathrm{sym}(\Omega^N)] \end{pmatrix}.$$

Hereinafter, recalling some basic results from the literature, we focus our attention on two fundamental properties of Q, its symmetry and positive semidefiniteness. When Q is symmetric, (5) reduces to a conceptually simpler potential game (see, e.g., Facchinei et al., 2011; Sagratella, 2017a) whose equilibria correspond to stationary points of the single optimization problem where the potential function

$$p(y) \triangleq \frac{1}{2} y^\top Q y + F(0)^\top y \tag{13}$$

is minimized over Y. Secondly, (see, e.g., Proposition 2.3.2 and the surrounding discussion in Facchinei and Pang, 2003) a positive semidefinite Q corresponds to the monotonicity of F, that is

$$(F(y) - F(w))^\top (y - w) \geq 0, \ \ \forall y, w \in \mathbb{R}^{NK},$$

while a positive definite Q corresponds to the strict monotonicity of F, that is

$$(F(y) - F(w))^\top (y - w) > 0, \ \ \forall y, w \in \mathbb{R}^{NK}.$$

Then, if Q is positive semidefinite (but possibly nonsymmetric), not only the solution set of $\mathrm{VI}(Y, F)$ (and thus E) is convex (see Theorem 2.3.5 in Facchinei and Pang, 2003), but also $\mathrm{VI}(Y, F)$ (and thus (5)) is practically solvable by resorting to effective algorithms available in the literature (see Chapter 12 in Facchinei and Pang, 2003).

If Q is positive definite, $\mathrm{VI}(Y, F)$ has a unique solution so that also (5) has a unique equilibrium (Facchinei and Pang, 2003, Theorem 2.3.3). If Q is symmetric and positive semidefinite, then the potential function p is convex and E corresponds to the set of global minima of p over Y. If Q is symmetric and positive definite, then p is strictly convex, and the unique global minimum of the potential function over Y is the unique equilibrium of (5). To obtain a complete picture of the basic properties of VIs and related NEPs, we refer the reader to Facchinei and Pang (2003).

It is convenient to recast our hierarchical model (7) as the following problem, rewriting the constraint $y \in E$ via its equivalent VI formulation,

$$\mathrm{minimize}_y \, f(y) \quad \text{s.t.} \quad y \in Y, \quad F(y)^\top (w - y) \geq 0 \quad \forall w \in Y. \tag{14}$$

Thus, (7) belongs to the class of Optimization Problems with VI Constraints (OPVIC) (see, e.g., Facchinei, Pang, Scutari and Lampariello, 2014; Scutari et al., 2012, for further information on this class of problems) and we use (7) or (14) interchangeably to refer to our model. If (5) is a convex potential game (as when Q is symmetric and positive semidefinite), then (7) reduces to the simple bilevel (or pure hierarchical) optimization problem (we refer the reader to Dempe et al., 2010; Sabach and Shtern, 2017; Shehu et al., 2019, for further information about this class of problems)

$$\text{minimize}_y \, f(y) \quad \text{s.t.} \quad y \in \arg\min_{w \in Y} p(w), \tag{15}$$

which is clearly a particular instance of (14). Hence, the OPVIC (14) generalizes the framework of simple bilevel problems to the presence of a lower-level variational inequality. As a major departure from the more general bilevel structures (Dempe, 2002; Lampariello and Sagratella, 2017, 2020; Lampariello et al., 2019), we deal only with lower-level problems that are *non-parametric* with respect to the upper-level variables.

3. A model with reference transaction costs

Focusing on the transaction costs matrices Ω^ν, we study the related properties of the Jaco-Hessian matrix Q, which are the key for the practical solution of problem (7) through the OPVIC reformulation (14). Without loss of generality, Ω^ν can be considered, for each account, as the additive perturbation of some matrix $\overline{\Omega}$ by a matrix Δ^ν, i.e.,

$$\Omega^\nu = \overline{\Omega} + \Delta^\nu. \tag{16}$$

One can interpret $\overline{\Omega}$ as a reference estimate that is either commonly shared by all the accounts or provided by some (upper-level) manager. We also introduce the reference transaction costs for account ν,

$$\overline{TC}_\nu(y^1, \ldots, y^N) \triangleq b^\nu (y^\nu - v^\nu)^\top \overline{\Omega} \sum_{\lambda=1}^{N} b^\lambda (y^\lambda - v^\lambda). \tag{17}$$

While neither $\overline{\Omega}$ nor Δ^ν have to be symmetric, it is natural to assume $\overline{\Omega}$ to be positive semidefinite. This guarantees the convexity of $\overline{TC}_\nu$ for account ν, and it avoids a common bias in the perturbations Δ^ν to make Ω^ν positive semidefinite. While the introduction of the expression (16) for Ω^ν does not establish any loss of generality in our model (14), our standing assumption of positive semidefinite matrices Ω^ν now depends on the perturbations Δ^ν. Positive semidefiniteness of Ω^ν follows for any positive semidefinite Δ^ν, but it also results from sufficiently small Δ^ν if $\overline{\Omega}$ is positive definite. In the following we shall quantify this fact.

The positive (semi)definiteness of a not necessarily symmetric perturbation P of some not necessarily symmetric positive definite matrix M may be guaranteed by the following result.

Lemma 3.1 *Let $M \in \mathbb{R}^{n\times n}$ be positive definite and $P \in \mathbb{R}^{n\times n}$. Then, $M + P$ is positive semidefinite whenever*

$$\|P\|_2 \leq \lambda_{\min}(\mathrm{sym}(M)),$$

where $\lambda_{\min}(\mathrm{sym}(M))$ is the smallest eigenvalue of $\mathrm{sym}(M)$. If the latter inequality is strict, then $M + P$ is positive definite.

Proof. For any $x \in \mathbb{R}^n$ we have

$$|x^\top Px| \leq \|P\|_2\|x\|_2^2 \leq \lambda_{\min}(\mathrm{sym}(M))\|x\|_2^2 \leq x^\top \mathrm{sym}(M)x = x^\top Mx$$

which entails

$$0 \leq x^\top Mx - |x^\top Px| \leq x^\top (M + P)x$$

and, thus, $M + P$ is positive (semi)definite. □

Lemma 3.1 entails the following result via the choice $M = \overline{\Omega}$ and $P = \Delta^\nu$.

Proposition 3.2 *Let $\overline{\Omega}$ be positive definite. Then Ω^ν is positive semidefinite whenever*

$$\|\Delta^\nu\|_2 \leq \lambda_{\min}(\mathrm{sym}(\overline{\Omega})). \tag{18}$$

If (18) is strict, then Ω^ν is positive definite.

With regard to the multi-portfolio problem (14), we discuss how different Δ^ν, and thus different structures for the transaction costs matrices, influence the properties of Q and of the (lower-level) NEP and of problem (14). We investigate conditions that make it possible to employ the Tikhonov approach that has been recently proposed by Lampariello et al. (2020) to solve OPVICs. We need to show that the three main conditions (A1)-(A3) from Lampariello et al. (2020) (entailing convergence of the Tikhonov algorithm) hold in the framework we consider. For problem (14) these conditions are:

- ∇f is Lipschitz continuous and monotone plus on Y, i.e., ∇f is monotone on Y and, for every $y, w \in Y$, $(\nabla f(y) - \nabla f(w))^T(y - w) = 0$ implies $\nabla f(y) = \nabla f(w)$;
- F is Lipschitz continuous and monotone on Y;
- Y is nonempty, convex and compact.

Since f is assumed to be $C^{1,1}$ and convex, the first condition is trivially satisfied (see Section 2.3.1 and Corollary 2.3.7 in Facchinei and Pang, 2003), and Y is anyway assumed to satisfy the third condition above. We are left to establish conditions under which F is monotone. Hence, the task consists in studying the definiteness of Q. We also give conditions under which (5) reduces to a convex potential game, and in turn (14) to the simple bilevel problem (15). This side result can help in understanding when specific perturbations make the problem easier to be solved.

3.1. Monotonicity results

To guarantee our assumption of positive semidefinite Ω^νs, in view of Proposition 3.2, we assume $\max_{\nu=1,\dots,N} \|\Delta^\nu\|_2 \leq \lambda_{\min}(\mathrm{sym}(\overline{\Omega}))$. The subsequent developments are aimed at establishing conditions on Ω^ν that entail a positive semidefinite or definite Jaco-Hessian Q and, thus, a monotone or strictly monotone function F. Collecting all Δ^νs in P, the Jaco-Hessian may be written as

$$Q = M + P$$

with

$$\begin{aligned} M &= \begin{pmatrix} (b^1)^2[\rho^1\Sigma^1 + 2\,\mathrm{sym}(\overline{\Omega})] & b^1b^2\overline{\Omega} & \cdots & b^1b^N\overline{\Omega} \\ b^2b^1\overline{\Omega} & (b^2)^2[\rho^2\Sigma^2 + 2\,\mathrm{sym}(\overline{\Omega})] & & b^2b^N\overline{\Omega} \\ \vdots & & \ddots & \\ b^Nb^1\overline{\Omega} & b^Nb^2\overline{\Omega} & & (b^N)^2[\rho^N\Sigma^N + 2\,\mathrm{sym}(\overline{\Omega})] \end{pmatrix} \\ &= \begin{pmatrix} (b^1)^2\rho^1\Sigma^1 & & \\ & \ddots & \\ & & (b^N)^2\rho^N\Sigma^N \end{pmatrix} \\ &\quad + \begin{pmatrix} b^1I & & \\ & \ddots & \\ & & b^NI \end{pmatrix} \left[(ee^\top \otimes \overline{\Omega}) + (I \otimes \overline{\Omega}^\top)\right] \begin{pmatrix} b^1I & & \\ & \ddots & \\ & & b^NI \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} P &= \begin{pmatrix} 2(b^1)^2\,\mathrm{sym}(\Delta^1) & b^1b^2\Delta^1 & \cdots & b^1b^N\Delta^1 \\ b^2b^1\Delta^2 & 2(b^2)^2\,\mathrm{sym}(\Delta^2) & & b^2b^N\Delta^2 \\ \vdots & & \ddots & \\ b^Nb^1\Delta^N & b^Nb^2\Delta^N & & 2(b^N)^2\,\mathrm{sym}(\Delta^N) \end{pmatrix} \\ &= \begin{pmatrix} b^1I & & \\ & \ddots & \\ & & b^NI \end{pmatrix} \left[\begin{pmatrix} \Delta^1 & & \\ & \ddots & \\ & & \Delta^N \end{pmatrix} (ee^\top \otimes I) + \begin{pmatrix} {\Delta^1}^\top & & \\ & \ddots & \\ & & {\Delta^N}^\top \end{pmatrix} \right] \begin{pmatrix} b^1I & & \\ & \ddots & \\ & & b^NI \end{pmatrix}, \end{aligned}$$

where e denotes the all ones vector, and $\otimes$ is the Kronecker product. The resulting M is symmetric if $\overline{\Omega}$ is symmetric. In contrast, P is nonsymmetric even for diagonal perturbation matrices $\Delta^\nu = \mathrm{diag}(\delta^\nu)$ with vectors $\delta^\nu \in \mathbb{R}^K \setminus \mathrm{span}(e)$. Nevertheless, we may derive sufficient conditions for the positive semidefiniteness (definiteness) of such a nonsymmetric matrix $Q = M + P$ and, thus, for the monotonicity (strict monotonicity) of F. Due to the nonsymmetry of P, the positive semidefiniteness of all Δ^νs cannot be proven to be such a sufficient condition.

Theorem 3.3 *Let $\overline{\Omega}$ or Σ^ν for all ν, be positive definite, and let Δ^νs satisfy*

$$\max_{\nu=1,\dots,N} \|\Delta^\nu\|_2 \leq \min\left\{ \lambda_{\min}(\mathrm{sym}(\overline{\Omega})), \frac{\lambda_{\min}(\mathrm{sym}(M))}{(N+1)\max\limits_{\nu=1,\dots,N}(b^\nu)^2} \right\}. \tag{19}$$

Then, Q is positive semidefinite. If the above inequality is strict, then Q is positive definite.

Proof. The symmetric part of M is

$$\begin{aligned}\mathrm{sym}(M) = &\begin{pmatrix} (b^1)^2\rho^1\Sigma^1 & & \\ & \ddots & \\ & & (b^N)^2\rho^N\Sigma^N \end{pmatrix} \\ &+ \begin{pmatrix} b^1 I & & \\ & \ddots & \\ & & b^N I \end{pmatrix} \left[(I + ee^\top) \otimes \mathrm{sym}(\overline{\Omega}) \right] \begin{pmatrix} b^1 I & & \\ & \ddots & \\ & & b^N I \end{pmatrix}.\end{aligned}$$

Since Kronecker products of positive (semi)definite matrices are positive (semi)definite, $\mathrm{sym}(M)$ is positive definite. Hence, in view of Lemma 3.1, Q is positive (semi)definite, whenever $\|P\|_2 < (\leq)\lambda_{\min}(\mathrm{sym}(M))$ holds. The assertions now follow from the estimate

$$\begin{aligned}\|P\|_2 \leq &\left\| \begin{pmatrix} b^1 I & & \\ & \ddots & \\ & & b^N I \end{pmatrix} \right\|_2 \left[\left\| \begin{pmatrix} \Delta^1 & & \\ & \ddots & \\ & & \Delta^N \end{pmatrix} \right\|_2 \left\| ee^\top \right\|_2 \|I\|_2 + \left\| \begin{pmatrix} {\Delta^1}^\top & & \\ & \ddots & \\ & & {\Delta^N}^\top \end{pmatrix} \right\|_2 \right] \\ &\left\| \begin{pmatrix} b^1 I & & \\ & \ddots & \\ & & b^N I \end{pmatrix} \right\|_2 = (N+1)(\max_{\nu=1,\dots,N} b^\nu)^2 \max_{\nu=1,\dots,N} \|\Delta^\nu\|_2\end{aligned}$$

and $(\max_{\nu=1,\dots,N} b^\nu)^2 = \max_{\nu=1,\dots,N}(b^\nu)^2$. □

The matrix $\mathrm{sym}(M)$ is positive definite also if $\overline{\Omega}$ and Σ^ν are just positive semidefinite, but the kernels of the two matrices adding up to $\mathrm{sym}(M)$ only share the zero element. Under this alternative condition that is weaker than the assumptions in Theorem 3.3, the fulfillment of (19) can still be guaranteed.

Alternatively, a lower bound on $\lambda_{\min}(\mathrm{sym}(M))$ yields the following corollary with a stronger, yet more explicit assumption than in Theorem 3.3.

Corollary 3.4 *Let $\overline{\Omega}$ or Σ^ν for all ν, be positive definite, and*

$$\begin{aligned}\max_{\nu=1,\dots,N} \|\Delta^\nu\|_2 \leq \min\Bigg\{ &\frac{\min\limits_{\nu=1,\dots,N}((b^\nu)^2\rho^\nu\lambda_{\min}(\Sigma^\nu)) + \lambda_{\min}(\mathrm{sym}(\overline{\Omega}))\min\limits_{\nu=1,\dots,N}(b^\nu)^2}{(N+1)\max\limits_{\nu=1,\dots,N}(b^\nu)^2}, \\ &\lambda_{\min}(\mathrm{sym}(\overline{\Omega}))\Bigg\}.\end{aligned} \tag{20}$$

Then, Q is positive semidefinite. If the above inequality is strict, then Q is positive definite.

Proof. The symmetric part of M may be written as

$$\operatorname{sym}(M) = \begin{pmatrix} (b^1)^2\rho^1\Sigma^1 + (b^1)^2\operatorname{sym}(\overline{\Omega}) & & \\ & \ddots & \\ & & (b^N)^2\rho^N\Sigma^N + (b^N)^2\operatorname{sym}(\overline{\Omega}) \end{pmatrix} + bb^\top \otimes \operatorname{sym}(\overline{\Omega})$$

which, in view of $\lambda_{\min}(bb^\top) = 0$, yields the lower bound for its smallest eigenvalue

$$\begin{aligned} \lambda_{\min}(\operatorname{sym}(M)) &\geq \min_{\nu=1,\dots,N} \lambda_{\min}((b^\nu)^2\rho^\nu\Sigma^\nu + (b^\nu)^2\operatorname{sym}(\overline{\Omega})) + \lambda_{\min}(bb^\top)\lambda_{\min}(\operatorname{sym}(\overline{\Omega})) \\ &= \min_{\nu=1,\dots,N} \lambda_{\min}((b^\nu)^2\rho^\nu\Sigma^\nu + (b^\nu)^2\operatorname{sym}(\overline{\Omega})) \\ &\geq \min_{\nu=1,\dots,N}((b^\nu)^2\rho^\nu\lambda_{\min}(\Sigma^\nu)) + \lambda_{\min}(\operatorname{sym}(\overline{\Omega})) \min_{\nu=1,\dots,N}(b^\nu)^2. \end{aligned}$$

The assertions now follow from Theorem 3.3. □

Note that in the case

$$\lambda_{\min}(\operatorname{sym}(\overline{\Omega})) \leq \frac{\min_{\nu=1,\dots,N}((b^\nu)^2\rho^\nu\lambda_{\min}(\Sigma^\nu))}{(N+1)\max_{\nu=1,\dots,N}(b^\nu)^2 - \min_{\nu=1,\dots,N}(b^\nu)^2}$$

Corollary 3.4 guarantees the monotonicity of F already under our assumption $\max_{\nu=1,\dots,N}\|\Delta^\nu\|_2 \leq \lambda_{\min}(\operatorname{sym}(\overline{\Omega}))$.

3.2. Potential games

We investigate conditions that make the NEP a structurally easier potential game. Whenever this is the case, the selection problem (7) and its OPVIC reformulation (14) reduce to the simple bilevel instance (15). A straightforward sufficient condition for the symmetry of Q, which is sufficient to obtain a potential game with the potential function (13), is the assumption of identical budgets $b^\nu = \bar{b}$ and the absence of perturbation matrices in the definition of Ω^ν, that is $\Delta^\nu = 0$, for every ν, together with the symmetry of $\overline{\Omega}$. Then, we obtain $\Omega^\nu = \overline{\Omega}$ and Q may be written as

$$Q = \bar{b}^2 \begin{pmatrix} \rho^1\Sigma^1 & & \\ & \ddots & \\ & & \rho^N\Sigma^N \end{pmatrix} + \bar{b}^2(I + ee^\top) \otimes \overline{\Omega}.$$

The symmetry of Q now follows from the symmetry of Σ^ν and $\overline{\Omega}$ together with the fact that Kronecker products of symmetric matrices are symmetric. These assumptions coincide with those from Yang et al. (2013), where a potential game approach for multi-portfolio

problems with transaction costs is studied in detail. Moreover, in this case Q is also positive semidefinite, so that p turns out to be convex: Q is given by the sum of a positive semidefinite matrix and the Kronecker product of two symmetric positive semidefinite matrices.

In the following we show that, under considerably weaker assumptions, a potential game can still be guaranteed, including arbitrary budgets b^ν and arbitrary scaling perturbations Δ^ν, which even lead to a nonsymmetric matrix Q.

Proposition 3.5 *Let, for all ν, $b^\nu > 0$ and $\Delta^\nu = a^\nu \overline{\Omega}$ with $a^\nu > -1$ and $\overline{\Omega}$ symmetric. Then, the NEP is a potential game with potential function*

$$p(y) \triangleq \frac{1}{2} y^\top \widetilde{Q} y + \widetilde{d}^\top y, \tag{21}$$

where

$$\widetilde{Q} \triangleq \begin{pmatrix} \frac{\rho^1 (b^1)^2}{1+a^1}\Sigma^1 + (b^1)^2\overline{\Omega} & & \\ & \ddots & \\ & & \frac{\rho^N (b^N)^2}{1+a^N}\Sigma^N + (b^N)^2\overline{\Omega} \end{pmatrix} + bb^\top \otimes \overline{\Omega},$$

$$\widetilde{d} \triangleq - \begin{pmatrix} b^1 \left(\frac{\mu^1}{1+a^1} + 2b^1\overline{\Omega}v^1 + \overline{\Omega}\sum_{\lambda\neq 1} b^\lambda v^\lambda \right) \\ \vdots \\ b^N \left(\frac{\mu^N}{1+a^N} + 2b^N\overline{\Omega}v^N + \overline{\Omega}\sum_{\lambda\neq N} b^\lambda v^\lambda \right) \end{pmatrix}.$$

Moreover, each Ω^ν and $\widetilde{Q}$ are positive semidefinite and p is convex.

Proof. The assumptions imply $\Omega^\nu = (1+a^\nu)\overline{\Omega}$ with $1 + a^\nu > 0$ for each ν, and the Ω^νs are positive semidefinite. Thus the proof relies on the fact that account ν's solution set is invariant if the objective function is rescaled to

$$\widetilde{\theta}^\nu(y^\nu, y^{-\nu}) \triangleq \frac{1}{1+a^\nu}\,\theta^\nu(y^\nu, y^{-\nu}).$$

The Jaco-Hessian of the equivalent rescaled game then is

$$\begin{pmatrix} \frac{\rho^1 (b^1)^2}{1+a^1}\Sigma^1 & & \\ & \ddots & \\ & & \frac{\rho^N (b^N)^2}{1+a^N}\Sigma^N \end{pmatrix} + \begin{pmatrix} 2(b^1)^2\overline{\Omega} & b^1b^2\overline{\Omega} & \cdots & b^1b^N\overline{\Omega} \\ b^2b^1\overline{\Omega} & 2(b^2)^2\overline{\Omega} & & b^2b^N\overline{\Omega} \\ \vdots & & \ddots & \\ b^Nb^1\overline{\Omega} & b^Nb^2\overline{\Omega} & & 2(b^N)^2\overline{\Omega} \end{pmatrix} = \widetilde{Q}.$$

The symmetry of $\widetilde{Q}$ now follows as above, exploiting in addition the obvious symmetry of the dyadic product $bb^\top$. Hence, observing that

$$\begin{pmatrix} \frac{1}{1+a^1} & & \\ & \ddots & \\ & & \frac{1}{1+a^N} \end{pmatrix} F(0) = \widetilde{d},$$

the game with rescaled objective functions as well as the original game are potential games with potential function p. Finally, $\widetilde{Q}$ is positive semidefinite as the sum of a positive semidefinite matrix and the Kronecker product of two symmetric positive semidefinite matrices. □

The assertion of Proposition 3.5 remains true if for some ν the perturbation is set so that $a^\nu = -1$. In this case transaction costs are not considered and, thus, the corresponding optimization problems are no longer coupled to the other accounts' problems.

If $\widetilde{Q}$ is positive definite, then the potential function is strictly convex and the game has a unique Nash equilibrium. This is the case if all Σ^νs or $\overline{\Omega}$ are positive definite.

4. Problem parameters estimates and computational experience

In this section we explain a way to estimate Ω^ν from real-world data, introduce different selection functions, and use the Tikhonov approach from Lampariello et al. (2020) for numerical equilibrium selection among nonunique solutions. The purpose of our study is to demonstrate that the general multi-portfolio model is solvable for relevant problem sizes and that its solutions can have advantages compared to the case when the coupling effect of transaction costs is naively ignored. Moreover, we illustrate that there are situations where a selection among different equilibria is practically relevant for the outcome of the optimization process. The numerical experiments we consider are based on real financial data sets and involve problems of considerable dimensions compared to the literature on multi-portfolio selection (see, e.g., Iancu and Trichakis, 2014; Savelsbergh et al., 2010; Yang et al., 2013).

4.1. Problem data and a way to estimate the market impact

We consider two different data sets, the first one consisting of daily returns time series of $K = 10$ assets from banking, insurance and financial companies belonging to Euro Stoxx 50 (SX5E) (from 2/1/2019 to 31/12/2019). The second one consists in $K = 29$ assets from the Dow Jones Industrial Average (DJIA) stock markets (from 2/1/2017 to 31/12/2017). We consider $N = 25$ accounts for both test cases. Overall, we face problems involving from $10 \times 25 = 250$ (for the case of SX5E data set) to $29 \times 25 = 725$ (for the case of DJIA data set) variables. All experiments are performed on a Mac mini 8.1, Quad-Core Intel Core i3 3.6 GHz, RAM 8 GB.

We specify how we compute the key parameters in the multi-portfolio model (7). For simplicity, for each income and risk ν we set $\mu^\nu = \overline{\mu}$ and $\Sigma^\nu = \overline{\Sigma}$, where $\overline{\mu}$ and $\overline{\Sigma}$ denote mean and variance of the return of the K assets, respectively. This corresponds to agreeing, for each ν, on the same reference estimates for assets' return expectations and variance. Concerning transaction costs, the market impact is not known a priori, but has to be estimated for each account individually. There are several approaches for measuring transaction costs, see

Frazzini et al. (2018) for a recent study. Our aim is to give a realistic measure of transaction costs, keeping the estimate simple and easy to reproduce.

Since transaction costs are largely driven by liquidity, we focus on estimating the latter. One straightforward way to measure liquidity is via traded volumes of assets. Data for traded volumes of assets from the above stock markets is publicly available, which will make our measure easy to compute and to reproduce. In order to compute the entries Ω^ν_{ij}, let us initially focus on obtaining estimates for the interrelation of traded volumes, α_{ij}. Subsequently, we shall translate this measure to account-dependent transaction costs. Let vol_i^t denote the traded volume of asset i at time t and vol_i the corresponding vector with entries $\mathrm{vol}_i^t, t = 1, \ldots, T$. Moreover, with $\overline{\mathrm{vol}_i}$ we denote the arithmetic mean of vol_i. Then we may measure the interrelation of the traded volume of asset i and the traded volume of asset j by their empirical correlation

$$\alpha_{ij} = \frac{\sum_{t=1}^T (\mathrm{vol}_i^t - \overline{\mathrm{vol}_i})(\mathrm{vol}_j^t - \overline{\mathrm{vol}_j})}{\sqrt{\sum_{t=1}^T (\mathrm{vol}_i^t - \overline{\mathrm{vol}_i})^2}\sqrt{\sum_{t=1}^T (\mathrm{vol}_j^t - \overline{\mathrm{vol}_j})^2}}$$

and obtain the empirical correlation matrix $A \triangleq (\alpha_{ij})_{i,j=1}^K$. To account for different impacts of traded volumes correlations on transaction costs per asset, we pre-multiply A by a positive definite diagonal matrix $\mathrm{diag}(\bar{s})$. In our experiments the resulting matrix $\mathrm{diag}(\bar{s})A$ turns out to be positive semidefinite and we take it as $\overline{\Omega}$ that provides our reference estimate

$$\overline{\Omega} = \mathrm{diag}(\bar{s})A = \begin{pmatrix} \bar{s}_1 a_1^\top \\ \vdots \\ \bar{s}_K a_K^\top \end{pmatrix},$$

where $a_i^\top$ is the row i vector in A. While A and $\bar{s}$ are computable and known for all the accounts, the evaluation about the influence of A on the transactions costs may differ. We model this by setting

$$\Omega^\nu = \mathrm{diag}(\bar{s} + s^\nu)A = \overline{\Omega} + \Delta^\nu,$$

where $\Delta^\nu = \mathrm{diag}(s^\nu)A$. Furthermore, we consider problem instances arising from the following choices: $\bar{s} = (1\text{e-}6)Ke$, s^ν is generated randomly so that all Ω^νs and Q are positive semidefinite; budgets b^νs and risk-aversion parameters ρ^νs are taken randomly from the interval $[0, 2]$. Finally, we consider current positions $v^\nu = 0$ for every ν.

Concerning the structure of problem (7), the feasible set for every account's problem is given by $Y_\nu = \{y^\nu \in \mathbb{R}^K : e^\top y^\nu = 1,\, y^\nu \geq 0\}$ and, without loss of generality, short sales are not allowed. Regarding the upper-level convex selection criterion f in (7), we consider the following function:

$$f_{\mathrm{overall}}(y) \triangleq \sum_{\nu=1}^N \bar{\theta}_\nu(y) \quad \text{with} \quad \bar{\theta}_\nu(y) \triangleq -b^\nu(\overline{\mu})^\top y^\nu + \rho^\nu \frac{1}{2}(b^\nu)^2 (y^\nu)^\top \overline{\Sigma} y^\nu + \overline{TC}_\nu(y^1, \ldots, y^N), \tag{22}$$

that is, the sum of all accounts' overall objectives with reference estimates ($\overline{\mu}$, $\overline{\Sigma}$, $\overline{\Omega}$). The selection function f_{overall} measures the aggregate utility of all the accounts, i.e., the sum of their individual (reference) utilities, which in turn are given by the difference between their (reference) income and risk and transaction costs terms. We add to the selection criterion a negligible regularization term 1e-5 $y^\top y$ so that the resulting sum is strictly convex. Solving (7) allows one to select the equilibrium that is optimal in the sense of the measure f_{overall}.

4.2. A solution method

In order to solve the problems we deal with, we rely on the projected Tikhonov-prox method. The following scheme, when applied to the problems we address, is provably convergent to the solution of problem (14) since all convergence conditions from Theorem 1 in Lampariello et al. (2020) are satisfied. While f is $C^{1,1}$ strictly convex and Y is nonempty, convex and compact by construction, F is monotone since Q is positive semidefinite.

Algorithm 1: projected Tikhonov-prox

Data: $\lambda = 1$, $\gamma = 1$, $y_{0,j} = 1/K$ for every $j = 1, \ldots, NK$, $w_0 = y_0 \in Y$, $i \leftarrow 1$;

for $k = 0, 1, \ldots$ **do**

(S.1) $\tau_k = i$, $\varepsilon_k = 1/(\tau_k)^2$;

(S.2) $y_{k+1} = P_Y(y_k - \gamma[F(y_k) + \frac{1}{\tau_k}\nabla f(y_k) + \lambda(y_k - w_k)])$;

(S.3) **if** $\|y_{k+1} - y_k\| \leq \varepsilon_k$ **then**
$w_{k+1} = y_k$, $i = i + 1$;

(S.4) **else**
$w_{k+1} = w_k$;

end

end

The Tikhonov Algorithm 1 can be seen as a penalty-like approach where the lower problem operator F is "penalized" at the same level of the upper one through the introduction of the Tikhonov (penalty-like) parameter τ_k; also, a proximal term is included for regularization purposes. Given the outer parameters (defined in the scheme of Algorithm 1) τ_k, ε_k and w_k, the core step of the algorithm consists in the iterative (inexact) solution of the strongly monotone "penalized" subproblem $\text{VI}(Y, F + \frac{1}{\tau_k}\nabla f + \lambda(\bullet - w_k))$ until the prescribed accuracy ε_k is reached at step (S.3). As soon as this is the case, the outer parameters are updated: τ_k is increased, ε_k is reduced, and w_{k+1} is taken equal to y_k. The (inexact) solution of the "penalized" subproblem $\text{VI}(Y, F + \frac{1}{\tau_k}\nabla f + \lambda(\bullet - w_k))$ is achieved by simple projection steps (see (S.2) where $P_Y(z)$ denotes the projection of z on the set Y). The sequence of outer iterates $\{w_k\}$ produced by the algorithm converges to the solution of problem (7) (see Lampariello et al., 2020). The execution of the algorithm is stopped if the condition at step (S.3) is met and the measure

$$\frac{1}{\gamma}\|y_{k+1} - y_k\| + \frac{1}{\tau_k}\|\nabla f(y_k)\| + \lambda\|y_k - w_k\| \tag{23}$$

is smaller than 1e-4. The first addendum in (23) accounts for the solution of the penalized subproblem $\mathrm{VI}(Y, F+\frac{1}{\tau_k}\nabla f+\lambda(\bullet-w_k))$ (consistently, it appears in the condition at step (S.3)). The second term is related to the upper-level objective function weighted by the reciprocal of the Tikhonov penalty-like parameter. As typical in Tikhonov approaches, being $\|\nabla f\|$ bounded, this addendum eventually vanishes. Finally, the last term measures the distance between two successive outer iterations. We also recall from Lampariello et al. (2020) that

$$V(y_k) \triangleq \|P_Y(y_k - F(y_k)) - y_k\| \leq \frac{1}{\gamma}\|y_{k+1} - y_k\| + \frac{1}{\tau_k}\|\nabla f(y_k)\| + \lambda\|y_k - w_k\|. \tag{24}$$

V is the natural residual map for the lower-level $\mathrm{VI}(Y, F)$ (see Facchinei and Pang, 2003). It is a continuous nonnegative measure such that $V(y) = 0$ if and only if y is a solution of $\mathrm{VI}(Y, F)$; $V(y)$ also gives an upper-bound for the distance between y and $\mathrm{SOL}(Y, F)$, since Y is polyhedral and F is affine (see Facchinei and Pang, 2003, Chapter 6 and, in particular, Proposition 6.3.3). Requiring the measure in (23) to be smaller than 1e-4 entails $V(y_k) \leq$ 1e-4: since, up to a constant, $V(y_k)$ is a majorant of the distance between y^k and $\mathrm{SOL}(Y, F)$, this means, having satisfied the condition at step (S.3), that w_{k+1} is "close" to $\mathrm{SOL}(Y, F)$.

4.3. Results and comparison with the independent approach

As a benchmark we consider the independent approach. This model ignores the aggregate effect from other accounts, and reduces to the collection of the following optimization problem for each ν:

$$\text{minimize}_{y^\nu} \; -b^\nu(\mu^\nu)^\top y^\nu + \rho^\nu\frac{1}{2}(b^\nu)^2(y^\nu)^\top\Sigma^\nu y^\nu + (b^\nu)^2(y^\nu - v^\nu)^\top\Omega^\nu(y^\nu - v^\nu) \quad \text{s.t.} \quad y^\nu \in Y_\nu. \tag{25}$$

In this simplified model, to evaluate an account's transaction costs, only the trades from that single account are considered, and the real market impact (see (4)) that is yielded also by the trades from all the other accounts in the common pool is neglected. Since problems (25) are not linked to each other, they end up to be N independent optimization problems that, apart from the presence of an additional transaction costs term and (in this case inessential) non-unitary budgets, correspond to the Markowitz (1952) model (6). In our computational experience, we solve problems (25) via the Tikhonov-like algorithm. Let y^ν_{indep} be a solution of (25) and $y_{\text{indep}} \triangleq (y^1_{\text{indep}}, \ldots, y^N_{\text{indep}})$. If a manager relies on the independent approach, the resulting trades y^ν_{indep} do not form, in general, an equilibrium for (5), i.e., $y_{\text{indep}} \notin E$. Since accounts are pooled for common execution, TC_νs that are actually incurred by each account deviate from the anticipated ones $(b^\nu)^2(y^\nu - v^\nu)^\top\Omega^\nu(y^\nu - v^\nu)$ from (25). As a result, the realized utility for each ν, that is given by $-\theta_\nu(y_{\text{indep}})$, might differ largely from the negative of the optimal value of (25).

We demonstrate that our equilibrium selection model improves on the independent approach, also in the general hierarchical framework we consider. In Table 1 and 2, we report the

Approaches	Income	Risk	TCs	utility
Independent	1.96e-3 (1.77e-6)	3.00e-5 (5.57e-10)	4.47e-4 (1.95e-7)	1.49e-3 (1.18e-6)
Eq. selection	1.86e-3 (1.66e-6)	2.87e-5 (6.17e-10)	2.06e-4 (4.50e-8)	1.62e-3 (1.31e-6)

Table 2: Mean (variance) of accounts' objective values for the Dow Jones case

results obtained when the manager resorts to either the independent or the equilibrium selection approach. For both SX5E and Dow Jones instances, we detail income, risk, transaction costs and utility obtained by the accounts. We indicate mean and variance of the corresponding values over the accounts. With the equilibrium selection model (7), one obtains larger realized utility means.

Approaches	Income	Risk	TCs	utility
Independent	1.60e-3 (4.83e-7)	4.38e-5 (1.20e-9)	3.46e-4 (8.80e-8)	1.21e-3 (3.06e-7)
Eq. selection	1.53e-3 (4.31e-7)	5.23e-5 (2.21e-9)	1.54e-4 (1.24e-8)	1.32e-3 (3.44e-7)

Table 1: Mean (variance) of accounts' objective values for the SX5E case

In Fig. 1 we detail, for each ν, the utility improvement entailed by our model (with $f = f_{\text{overall}}$ as selection criterion) with respect to the independent approach (25), namely

$$-\frac{\theta_\nu(y_{\text{overall}}) - \theta_\nu(y_{\text{indep}})}{|\theta_\nu(y_{\text{indep}})|},$$

where y_{overall} denotes the optimal solution for problem (7) with $f = f_{\text{overall}}$. Also, in 'Total', we report the quantity

$$-\frac{\sum_{\nu=1}^{N} (\theta_\nu(y_{\text{overall}}) - \theta_\nu(y_{\text{indep}}))}{|\sum_{\nu=1}^{N} \theta_\nu(y_{\text{indep}})|}.$$

Properly taking transaction costs into account results in an improvement of utility.

4.4. Results with different choices for the selection function f in (7)

Standard sufficient conditions for E to be a singleton do not hold for the instance of (5) we consider in our numerical experience: in this case, Q is not positive definite but just positive semidefinite. However, the question on the presence of multiple lower-level equilibria remains open since it is difficult to prove that a NEP has actually multiple solutions, especially when hundreds of decision variables are involved. Even if one has to rely on the selection approach (7) not only when the presence of multiple lower-level equilibria is certain, but also, as in this case, if uniqueness cannot be proven easily for (5), we show that E is not a singleton for the problem at hand. By considering an upper-level objective function that is different from f_{overall} and obtaining a different solution, we observe a posteriori that our approach makes a

selection among multiple lower-level equilibria. We introduce in (7) the upper-level objective function

$$f_{\text{sparse}}(y) \triangleq z^\top \sum_{\nu=1}^{N} y^\nu,$$

where $z \in \mathbb{R}^K$ is a randomly generated vector of positive weights, and we add to the function a negligible regularization term 1e-5 $y^\top y$ so that the resulting sum is strictly convex. Since $y^\nu \geq 0$ for every ν, function $f_{\text{sparse}}(y)$ is the weighted ℓ_1-norm of $\sum_{\nu=1}^{N} y^\nu$, where z_i denotes the weight associated to asset i. The use of the weighted ℓ_1-norm as a sparsity-promoting functional traces back to at least a decade. One can take different weights to selectively penalize different components in the solution: since f_{sparse} is minimized, large weights z_i are used to discourage nonzero $\sum_{\nu=1}^{N} y_i^\nu$, while small weights are used to encourage nonzero $\sum_{\nu=1}^{N} y_i^\nu$, where $\sum_{\nu=1}^{N} y_i^\nu$ is the sum of all accounts' budget fractions to be invested in asset i. Solving the hierarchical problem (7) with $f = f_{\text{sparse}}$ makes it possible to pick among the equilibria of the lower-level NEP the sparsest one, i.e., the equilibrium that involves the smallest number of assets to be traded, according to the weights z_i. Building sparse portfolios is attractive because accounts incur monitoring costs for each nonzero position and to reduce the complexity of management. For these reasons but also with the aim of mitigating parameter uncertainty and estimation errors, sparse portfolio selection is an active research topic (see, e.g., Bertsimas, Darnell and Soucy (1999) and, for some more recent developments, Cesarone et al. (2013); Li (2015)).

In Fig. 2 the results obtained whenever $f = f_{\text{sparse}}$ are compared to those for $f = f_{\text{overall}}$. We report, for each ν, the quantity

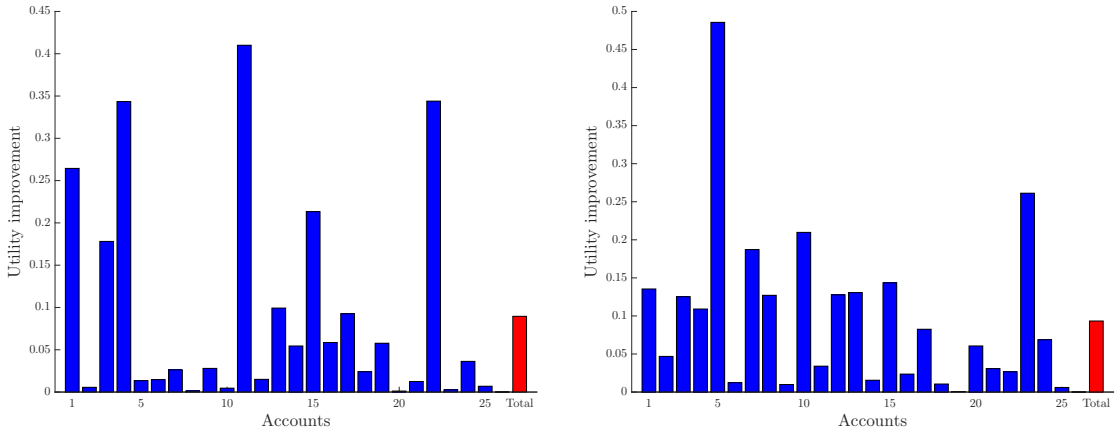
$$-\frac{\theta_\nu(y_{\text{sparse}}) - \theta_\nu(y_{\text{overall}})}{|\theta_\nu(y_{\text{overall}})|},$$

where y_{sparse} denotes the optimal solution for (7) with $f = f_{\text{sparse}}$. In 'Total', we report the quantity

$$-\frac{\sum_{\nu=1}^{N} (\theta_\nu(y_{\text{sparse}}) - \theta_\nu(y_{\text{overall}}))}{|\sum_{\nu=1}^{N} \theta_\nu(y_{\text{overall}})|}.$$

We indicate in Table 3 the distance, in terms of ℓ_1 and ℓ_∞ norms and objective functions values, between y_{overall} and y_{sparse}, and their ℓ_0 norm. The results in Fig. 2 and the value of the distance measures in Table 3 show that, changing the upper-level objective f from f_{overall} to f_{sparse}, we obtain different solutions for (7). Hence, we obtain numerical evidence that addressing (7) yields a selection over the lower-level equilibrium set. As expected, minimizing f_{sparse} over E, we obtain an equilibrium that is sparser than the one that is computed minimizing f_{overall} over E (see the ℓ_0-norm values in Table 3). We also notice that, differently from f_{sparse}, f_{overall} seems to be rather flat over E.

Finally, in Table 4, we report the CPU time and the number of iterations needed by Algorithm 1 to reach termination for the independent approach and for the equilibrium selection



(a) SX5E stock market assets

(b) Dow Jones stock market assets

Figure 1: Accounts' utility improvement of the equilibrium selection with $f = f_{\text{overall}}$ w.r.t. the independent approach

Approaches	SX5E	Dow Jones
$\|y_{\text{overall}} - y_{\text{sparse}}\|_1$	2.968	3.661
$\|y_{\text{overall}} - y_{\text{sparse}}\|_\infty$	0.354	0.285
$f_{\text{overall}}(y_{\text{sparse}}) - f_{\text{overall}}(y_{\text{overall}})$	1.279e-4	3.454e-5
$f_{\text{sparse}}(y_{\text{overall}}) - f_{\text{sparse}}(y_{\text{sparse}})$	3.625e-2	1.923e-2
$\|y_{\text{overall}}\|_0$	53	44
$\|y_{\text{sparse}}\|_0$	48	35

Table 3: Solutions distances

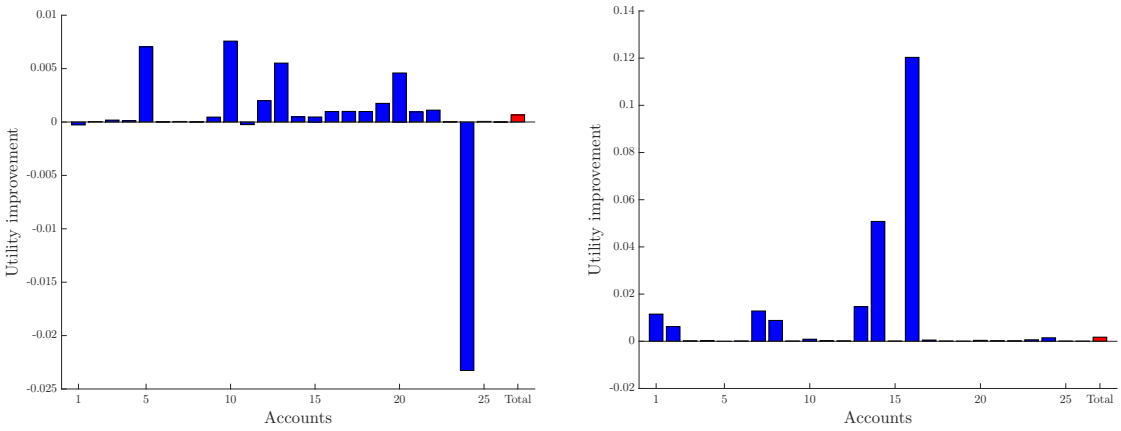
(considering both the cases of $f = f_{\text{overall}}$ and $f = f_{\text{sparse}}$). When solving (7), Algorithm 1 proves to achieve convergence in at most 377 seconds.

5. Conclusions

We formulate a new general portfolio selection model involving several accounts whose trades are coupled due to market impact via transaction costs. The resulting NEP is not necessarily assumed to be a potential game, nor to lead to a unique solution. We consider

Approaches	SX5E	Dow Jones
Independent	44 sec – 34438 iter	128 sec – 19547 iter
Eq. selection with f_{overall}	26 sec – 19008 iter	130 sec – 20124 iter
Eq. selection with f_{sparse}	46 sec – 35376 iter	377 sec – 49738 iter

Table 4: CPU times – number of iterations



(a) SX5E stock market assets

(b) Dow Jones stock market assets

Figure 2: Accounts' utility improvement of the equilibrium selection with $f = f_{\text{sparse}}$ w.r.t. the equilibrium selection with $f = f_{\text{overall}}$

different merit functions that yield a selection over the set of equilibria arising from the NEP. We obtain a hierarchical program with a lower-level NEP. We analyze the main theoretical properties of this model and establish practical sufficient conditions that make it possible to treat the problem numerically via a recently proposed Tikhonov approach. The experimental study confirms the theoretical insights. Our model has proven to be easily solvable when dealing with test problems that involve up to several hundreds of decision variables, and our model's outcomes improve on the benchmark method (the independent approach) that is usually relied on by practitioners.

A fundamentally different structure is generated if some upper level manager controls parameters of the lower-level accounts and wishes to minimize some objective depending on these parameters. In the setting of multi-portfolio optimization such parameters may be, for example, the budgets for the accounts or correction terms for their transaction costs. As this structure is no longer purely hierarchical, but rather a multi-follower game, the analysis of the present paper does not apply, and we leave the investigation of such models to future research.

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