

GENUS TWO CURVES ON ABELIAN SURFACES

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ABSTRACT. This paper deals with singularities of genus 2 curves on a general (d_1, d_2) -polarized abelian surface (S, L) . In analogy with Chen's results concerning rational curves on K3 surfaces [Ch1, Ch2], it is natural to ask whether all such curves are nodal. We prove that this holds true if and only if d_2 is not divisible by 4. In the cases where d_2 is a multiple of 4, we exhibit genus 2 curves in $|L|$ that have a triple, 4-tuple or 6-tuple point. We show that these are the only possible types of unnodal singularities of a genus 2 curve in $|L|$. Furthermore, with no assumption on d_1 and d_2 , we prove the existence of at least one nodal genus 2 curve in $|L|$. As a corollary, we obtain nonemptiness of all Severi varieties on general abelian surfaces and hence generalize [KLM, Thm. 1.1] to nonprimitive polarizations.

1. INTRODUCTION

The minimal geometric genus of any curve lying on a general abelian surface is 2 and there are finitely many curves of such genus in a fixed linear system. The role of genus two curves on abelian surfaces is thus analogous to that of rational curves on K3 surfaces, but until now it has not been investigated as extensively. Their enumeration is now well understood. Their count in the primitive case was carried out by Göttsche [Go], Debarre [De] and Lange-Sernesi [LS1], and used in [De] in order to compute the Euler characteristic of generalized Kummer varieties. Only recently, Bryan, Oberdieck, Pandharipande and Yin [BOPY] handled the nonprimitive case, thus obtaining a formula parallel to the full Yau-Zaslow conjecture for rational curves on K3 surfaces (cf. [KMPS]).

Singularities of rational curves on K3 surfaces have received plenty of attention. Mumford [MM, Appendix] first proved the existence of a nodal rational curve in the primitive linear system $|L|$ on a general genus g polarized K3 surface (S, L) ; as a byproduct, he obtained nonemptiness of the Severi variety $|L|_\delta$ parametrizing δ -nodal curves in $|L|$ for any integer $0 \leq \delta \leq g$. His results were then generalized by Chen [Ch1, Ch2] to nonprimitive linear systems. In the primitive case, Chen managed to deal with all rational curves in $|L|$ showing that they are all nodal; the analogue for nonprimitive linear systems is still an open problem.

Singularities of genus 2 curves on abelian surfaces are not as well understood, even though they are necessarily ordinary (cf. [LS2, Prop. 2.2]). The natural question whether any genus 2 curve on a general (d_1, d_2) -polarized abelian surface is nodal [LC, Pb. 2.7] has negative answer if one does not make any assumption on d_1 and d_2 . Indeed, multiplication by 2 on a principally polarized abelian surface (A, L) identifies the six Weierstrass points of its theta divisor, whose image is thus a genus 2 curve with a 6-tuple point lying in (a translate) of the linear system $|L^{\otimes 4}|$ (cf. Example 2). Since this is a polarization of type $(4, 4)$, all genus 2 curves may still be expected to be nodal in primitive linear systems (or even in linear systems not divisible by 4, cf. [LC, Conj. 2.10]). Our main result is that this expectation does not hold in its full generality and detects all the cases where it fails, thus completely answering the question.

Theorem 1.1. *Let (S, L) be a general abelian surface with a polarization of type (d_1, d_2) . Then any genus 2 curve in the linear system $|L|$ is nodal if and only if 4 does not divide d_2 .*

When d_2 is a multiple of 4, we exhibit genus 2 curves in $|L|$ that have an unnodal singularity and, more precisely, a triple, a 4-tuple or a 6-tuple point (cf. Examples 1 and 2). We also show that these are the only types of unnodal singularities that a genus 2 curve on a general abelian surface may acquire (cf. Remark 1). To our knowledge, the best bound on the order of such a singularity in the literature was $\frac{1}{2}(1 + \sqrt{8d_1d_2 - 7})$ by Lange-Sernesi, cf. [LS2, Prop. 2.2]. The existence of unnodal genus 2 curves in all primitive linear systems of type $(1, 4k)$ is quite striking and highlights a major difference with the $K3$ case.

When 4 divides d_2 , the above theorem does not exclude the existence in $|L|$ of some nodal genus 2 curves. This is indeed proved in the following:

Theorem 1.2. *Let (S, L) be a general (d_1, d_2) -polarized abelian surface. Then the linear system $|L|$ contains a nodal curve of genus 2.*

Given a nodal genus 2 curve as above, standard deformation theory enables one to smooth any of its nodes independently remaining inside the linear system $|L|$. As a consequence, Theorem 1.2 yields nonemptiness of all Severi varieties on general abelian surfaces:

Corollary 1.3. *Let (S, L) be a general (d_1, d_2) -polarized abelian surface. Then, for any $0 \leq \delta \leq d_1d_2 - 1$ the Severi variety $|L|_\delta$ is nonempty and smooth of dimension equal to $d_1d_2 - 1 - \delta$.*

This generalizes [KLM, Thm. 1.1] to nonprimitive linear systems. Note that, since S has trivial canonical bundle, the regularity of $|L|_\delta$ stated in Corollary 1.3 follows for free from its nonemptiness by the proofs of Propositions 1.1 and 1.2 in [LS2]. We mention that the irreducible components of the Severi varieties on a general primitively polarized abelian surface have been determined very recently by Zahariuc in [Za1].

We now spend some words on the proofs of Theorems 1.1 and 1.2. In contrast to the methods proposed in [Ch1, Ch2, KLM], we need neither to degenerate S to a singular surface nor to specialize it to an abelian surface with large Neron-Severi group. Instead, we exploit the universal property of Jacobians in order to translate the if part of Theorem 1.1 and Theorem 1.2 into the following statement concerning Brill-Noether theory on a general curve of genus 2:

Theorem 1.4. *Let $[C] \in \mathcal{M}_2$ be a general genus 2 curve and fix any integer $d \geq 4$. If C admits a g_d^2 totally ramified at three points P_1, P_2, P_3 , then d is even and P_1, P_2, P_3 are Weierstrass points.*

We refer to Section §2 for the details of this reduction, that we mention here only briefly. The key fact is that any genus 2 curve \bar{C} on a (d_1, d_2) -polarized abelian surface S with normalization C arises as image of a composition

$$(1) \quad C \xrightarrow{u} J(C) \xrightarrow{\lambda} S,$$

where u is the Abel-Jacobi map and λ is an isogeny. Three points $P_1, P_2, P_3 \in C = u(C)$ identified by λ necessarily differ by elements in its kernel. Since the order of any such element is divisible by d_2 , the three divisors $d_2P_1, d_2P_2, d_2P_3 \in C_d$ are linearly equivalent and thus define (for $d_2 \geq 4$) a $g_{d_2}^2$ on C totally ramified at three points. Theorem 1.4 excludes the existence of such a linear series for C general and odd values of d_2 , thus implying our main results in these cases. If instead d_2 is even, a $g_{d_2}^2$ totally ramified at three points does exist: as soon as P_1, P_2, P_3 are Weierstrass points of C , the divisors

$2P_1, 2P_2, 2P_3$ are linearly equivalent and thus the same holds true for d_2P_1, d_2P_2, d_2P_3 . Conversely, by Theorem 1.4, any $g_{d_2}^2$ with three points of total ramification on C is of this type. This characterization is used in Section §2 both to prove the if part of Theorem 1.1 and Theorem 1.2 for $d_2 \equiv 2 \pmod{4}$, and to provide examples of genus 2 curves with a triple, 4-tuple or 6-tuple point (cf. Examples 1 and 2) when $d_2 \equiv 0 \pmod{4}$ implying the only if part of Theorem 1.1. These examples are based on the construction of suitable isogenies λ as in (1) or, equivalently by taking their kernels, suitable isotropic (with respect to the commutator pairing) subgroups of the group $J(C)[d_1d_2]$ of d_1d_2 -torsion points of $J(C)$.

Section §3 is devoted to the proof of Theorem 1.4. This is done in two steps. First, we degenerate C to the transversal union of two elliptic curves meeting at a point and reduce Theorem 1.4 into a statement of Brill-Noether theory with ramification on a general elliptic curve (cf. Proposition 3.1). This reduction seriously involves the theory of limit linear series on curves of compact type, for which we refer to the original papers by Eisenbud and Harris [EH1, EH2, EH3]. Proposition 3.1 is then proved by an infinitesimal study of a *generalized Severi variety* (cf. [CH, Za2]).

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2. POLARIZED ISOGENIES AND PROOF OF THE MAIN THEOREMS

In this section we review some known facts concerning polarized isogenies and genus 2 curves on complex abelian surfaces and reduce the proof of Theorems 1.1 and 1.2 to a statement concerning Brill-Noether theory with prescribed ramification on a general curve of genus 2.

2.1. Polarized isogenies and genus 2 curves. Let S be an abelian surface defined over \mathbb{C} and consider a genus 2 curve $\overline{C} \subset S$ such that the line bundle $L := \mathcal{O}_S(\overline{C})$ is a polarization of type (d_1, d_2) . The normalization map $\nu : C \rightarrow \overline{C} \subset S$ then factors as

$$C \xrightarrow{u} J(C) \xrightarrow{\lambda} S,$$

where u is the Abel-Jacobi map (that is is an embedding only defined up to translation) and λ is an isogeny. We set $A := J(C)$.

By the Push-Pull formula, the above isogeny λ has degree d_1d_2 and thus $\lambda^*L \simeq L_1^{\otimes d_1d_2}$, where L_1 is a principal polarization on A . We write $A = V/\Lambda$, where V is a 2-dimensional \mathbb{C} -vector space and Λ is a rank 4 lattice. Chosen a symplectic basis $\lambda_1, \lambda_2, \mu_1, \mu_2$ of Λ , we denote by $\mathfrak{e}'_1 := \lambda_1/(d_1d_2), \mathfrak{e}'_2 := \lambda_2/(d_1d_2), \mathfrak{f}'_1 := \mu_1/(d_1d_2), \mathfrak{f}'_2 := \mu_2/(d_1d_2)$ the standard generators of the group $A[d_1d_2]$ of d_1d_2 -torsion points of A . By definition, the commutator pairing on $A[d_1d_2]$ is the nondegenerate multiplicative alternating form

$$e_{d_1d_2} : A[d_1d_2] \times A[d_1d_2] \rightarrow \mathbb{C}^*$$

that takes value 1 on all pairs of standard generators with the only two following exceptions:

$$(2) \quad e_{d_1d_2}(\mathfrak{e}'_1, \mathfrak{f}'_1) = e_{d_1d_2}(\mathfrak{e}'_2, \mathfrak{f}'_2) = e^{\frac{2\pi i}{d_1d_2}}.$$

For a fixed principally polarized abelian surface A with a fixed theta divisor Θ , by [Mu, §23] there is a bijection between the following two sets:

- (*) polarized isogenies $\lambda : A \rightarrow S$ onto abelian surfaces S such that $\lambda(\Theta) \in |L|$ for some polarization L on S of type (d_1, d_2) ;
- (**) isotropic subgroups G of $A[d_1d_2]$ of cardinality d_1d_2 such that $G^\perp/G \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$.

Indeed, the kernel G of any isogeny λ in (*) is a subgroup of $A[d_1d_2]$ in (**); furthermore, G^\perp/G is isomorphic to the kernel $K(L)$ of the isogeny defined by L :

$$\phi_L : S \rightarrow \hat{S}, \quad \phi_L(x) = t_x^*L \otimes L^\vee,$$

where t_x denotes the translation by x on S . Viceversa, given a subgroup G in (**), the quotient map $\lambda : A \rightarrow A/G$ is an isogeny as in (*).

Given λ as in (*), let

$$\hat{\lambda} : \hat{S} \rightarrow \hat{A}$$

be its dual isogeny and denote by \hat{L} the dual polarization of L . Again by [Mu, §23] the kernel \hat{G} of $\hat{\lambda}$ is a maximal isotropic subgroup of $K(\hat{L}) \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$. On the other hand, \hat{G} is the character group of G (cf. [BL, Prop. 2.4.3]) and thus $\hat{G} \simeq G$. In particular, the order of any element of $G = \ker \lambda$ divides d_2 .

2.2. Reduction of Theorem 1.1 to Theorem 1.4 for odd values of d_2 . Since isogenies are étale, all singularities of the image $\lambda(\Theta)$ of a theta divisor under an isogeny λ as in (*) are ordinary (cf. [LS2, Prop. 2.2]). The only pathology that might prevent $\lambda(\Theta)$ from being nodal is thus the existence of three points $x, y, z \in \Theta$ such that $\lambda(x) = \lambda(y) = \lambda(z)$. Since the order of any element in the kernel of λ divides d_2 , such a triple of points x, y, z would be identified by the multiplication map

$$m_{d_2} : A \rightarrow A;$$

equivalently, if $A = J(C)$, the three divisors d_2x, d_2y, d_2z on C would be linearly equivalent. For $d_2 \geq 4$ this implies the existence of a $g_{d_2}^2$ on C totally ramified at x, y, z . In the cases $d_2 = 2$ and $d_2 = 3$ the same conclusion holds up to replacing d_2 with a multiple of it. Theorem 1.1 for odd values of d_2 then follows from Theorem 1.4.

2.3. Theorem 1.1 for even values of d_2 . Theorem 1.4 also implies that the image of a theta divisor under an isogeny λ as in (*) for even values of d_2 may have non-nodal singularities only at the image of its Weierstrass points. In order to analyze this possibility, we denote by $\mathfrak{e}_1 := \lambda_1/2, \mathfrak{e}_2 := \lambda_2/2, \mathfrak{f}_1 := \mu_1/2, \mathfrak{f}_2 := \mu_2/2$ the standard generators of the group $A[2]$ of 2-torsion points of A . Note that

$$(3) \quad \mathfrak{e}_i = \frac{d_1d_2}{2}\mathfrak{e}'_i, \quad \mathfrak{f}_i = \frac{d_1d_2}{2}\mathfrak{f}'_i \text{ for } i \in \{1, 2\}.$$

As the Abel-Jacobi map $u : C \hookrightarrow J(C) = A$ is defined up to translation, we may assume its image to coincide with a symmetric theta divisor Θ so that (the image under u of) the six Weierstrass points of C are exactly the 2-torsion points of A contained in Θ , namely, without loss of generality by [BL, Ex. 10.2.7], the points:

$$(4) \quad \mathfrak{e}_1, \mathfrak{f}_1, \mathfrak{e}_1 + \mathfrak{f}_1, \mathfrak{e}_2, \mathfrak{f}_2, \mathfrak{e}_2 + \mathfrak{f}_2.$$

Theorem 1.1 for $d_2 \equiv 2 \pmod{4}$ then follows from the following Lemma.

Lemma 2.1. *Let G be an isotropic subgroup of $A[d_1d_2]$ such that $|G| = d_1d_2$, $G^\perp/G \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$ and at least three among the six 2-torsion points in (4) lie in the same G -orbit of $A[d_1d_2]$. Then, one necessarily has $d_2 \equiv 0 \pmod{4}$.*

Proof. Up to exchanging the ϵ_i 's with the f_i 's and up to relabelling the indices i , we may assume that one of the three points in the same G -orbit of $A[d_1d_2]$ is ϵ_1 . It follows that G contains at least two elements g_1, g_2 in the following set:

$$(5) \quad \{\epsilon_1 + f_1, f_1, \epsilon_2 + \epsilon_1, f_2 + \epsilon_1, \epsilon_2 + f_2 + \epsilon_1\}.$$

Using (2) and (3) one obtains $e_{d_1d_2}(g_1, g_2) = e^{\frac{2d_1d_2\pi i}{4}}$ (up to exchanging g_1 and g_2), and thus

$$(6) \quad d_1d_2 \equiv 0 \pmod{4}$$

since G is isotropic.

In order to exclude the case $d_2 \equiv 2 \pmod{4}$ (that would also imply $d_1 \equiv 2 \pmod{4}$ by (6)), we proceed by contradiction. Both the elements $\frac{2g_1}{d_1d_2}$ and $\frac{2g_2}{d_1d_2}$ have order d_1d_2 and $e_{d_1d_2}(\frac{2g_1}{d_1d_2}, \frac{2g_2}{d_1d_2}) = e^{\frac{2\pi i}{d_1d_2}}$; therefore, there exists an automorphism φ of $A[d_1d_2]$ preserving the alternating form $e_{d_1d_2}$ such that $\varphi(\frac{2g_1}{d_1d_2}) = \epsilon'_1$ and $\varphi(\frac{2g_2}{d_1d_2}) = \epsilon'_2$. In particular, we may assume $g_1 = \epsilon_1$ and $g_2 = f_1$. We consider the group

$$K := \langle \epsilon_1, f_1 \rangle,$$

and its orthogonal $K^\perp = \langle 2\epsilon'_1, 2f'_1, \epsilon'_2, f'_2 \rangle$. By (3), one gets

$$(7) \quad K^\perp/K \simeq \mathbb{Z}_{\frac{d_1d_2}{4}}^{\oplus 2} \oplus \mathbb{Z}_{d_1d_2}^{\oplus 2} \simeq \mathbb{Z}_{\frac{d_1d_2}{4}}^{\oplus 4} \oplus \mathbb{Z}_4^{\oplus 2} \text{ with } \frac{d_1d_2}{4} \text{ odd,}$$

where the second isomorphism follows from the assumption $d_2 \equiv 2 \pmod{4}$. The inclusions $K < G < G^\perp < K^\perp$ imply that

$$K^\perp/K > G^\perp/K \text{ and } G^\perp/G \simeq (G^\perp/K)/(G/K);$$

in particular G^\perp/G is a quotient of a subgroup of K^\perp/K . However, our assumption yields

$$G^\perp/G \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2} \simeq \mathbb{Z}_2^{\oplus 4} \oplus \mathbb{Z}_{\frac{d_1}{2}}^{\oplus 2} \oplus \mathbb{Z}_{\frac{d_2}{2}}^{\oplus 2} \text{ with } \frac{d_1}{2}, \frac{d_2}{2} \text{ odd.}$$

As a consequence, $\mathbb{Z}_2^{\oplus 4}$ is a quotient of a subgroup of K^\perp/K and thus of $\mathbb{Z}_4^{\oplus 2}$ by (7). This is a contradiction because the only quotient of a subgroup of $\mathbb{Z}_4^{\oplus 2}$ having cardinality 16 is $\mathbb{Z}_4^{\oplus 2}$ itself. \square

By the following example, as soon as $d_2 \equiv 0 \pmod{4}$, there do exist isotropic subgroups G of $A[d_1d_2]$ as in Lemma 2.1. As a consequence, a general polarized abelian surface of type (d_1, d_2) contains an unnodal genus 2 curve and the only if part of Theorem 1.1 follows.

Example 1. We fix positive integers d_1, d_2, a, b such that $d_1|d_2$ and the relation $ab = d_1^2d_2$ holds. We consider the following subgroup of $A[d_1d_2]$

$$G := \langle a\epsilon'_1, bf'_1, d_2\epsilon'_2 \rangle.$$

One has $|G| = \frac{d_1d_2}{a} \cdot \frac{d_1d_2}{b} \cdot d_1 = d_1d_2$ and

$$G^\perp = \left\langle \frac{d_1d_2}{b}\epsilon'_1, \frac{d_1d_2}{a}f'_1, \epsilon'_2, d_1f'_2 \right\rangle,$$

and hence $G^\perp/G \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$. In particular, the group G corresponds to a polarized isogeny from the principally polarized abelian surface A to a (d_1, d_2) -polarized abelian surface (S, L) . If both a and b divide $d_1d_2/2$ (and thus $ab = d_1^2d_2$ divides $(d_1d_2)^2/4$, or equivalently, $d_2 \equiv 0 \pmod{4}$), then G contains the 2-torsion points $\epsilon_1, f_1, \epsilon_1 + f_1$ and we find a genus 2 curve in $|L|$ with a singularity that is (at least) a triple point. Note that

for any values of d_1, d_2 such that $d_1|d_2$ and $d_2 \equiv 0 \pmod{4}$, the integers $a = d_1d_2/2$ and $b = 2d_1$ satisfy all the above conditions. If moreover d_1 is even, then G contains also the point e_2 and the image of the theta divisor acquires a 4-tuple point.

To our knowledge the only example of an unnodal genus 2 curve on an abelian surface from the published literature until now was the image of the theta divisor under the multiplication by two on a principally polarized abelian surface. This example has played interesting roles in various works concerning curve singularities (cf. [DS, Ex. 4.14]) and Seshadri constants (cf. [St, Pf. of Prop. 2], [Ba, Rmk. 6.3], [KSS, Ex. 4.2]). We recall and generalize this example:

Example 2. Assume we have an isogeny λ as in (*) identifying all the six Weierstrass points of Θ . The group $A[2]$ of 2-torsion points of A is necessarily contained in the kernel of such a λ , that hence factors through the multiplication by 2

$$m_2 : A \rightarrow A.$$

In fact, the image $m_2(\Theta)$ has only one singularity at the image of the six Weierstrass points, that is thus a 6-tuple point. Furthermore, $m_2(\Theta) \in |L_1^{\otimes 4}|$ where L_1 is a principal polarization on A . As soon as $d_1 \equiv 0 \pmod{4}$, one constructs a genus 2 curve with a sextuple point on a general (d_1, d_2) -polarized abelian surface (S, L) by composing m_2 with an isogeny $\lambda' : A \rightarrow S$ such that $\lambda'(\Theta) \in |L'|$ where L' is a polarization on S satisfying $L'^{\otimes 4} \simeq L$.

Remark 1. Theorem 1.4 along with the fact that any smooth genus 2 curve has exactly 6 Weierstrass points yields that 6 is the maximal order of any singularity of a genus 2 curve on a general abelian surface. Examples 1 and 2 exhibit genus 2 curves with a triple, a 4-tuple or a 6-tuple point. It is natural to ask whether one can construct a genus 2 curve with a 5-tuple point. Such a curve would correspond to an isotropic subgroup G containing exactly 4 points in the set (5). However, any subgroup of $A[2]$ generated by 4 elements in the set (5) contains e_1, e_2, f_1, f_2 and thus coincides with $A[2]$. As a consequence, if one requires G to contain four points in (5), then G contains the whole $A[2]$ and one falls in Example 2 thus obtaining a 6-tuple and not a 5-tuple point.

Remark 2. While looking for a proof of Theorem 1.1, we realized that the proof of [DL, Proposition 3.1] contains a gap since it somehow assumes that an isogeny between two principally polarized abelian surfaces $\lambda : A \rightarrow B$ never identifies three or more points on the theta divisor of A . In [DL] the abelian varieties are defined over an algebraically closed field \mathbb{K} of arbitrary characteristic and the kernel of λ is a maximal isotropic subgroup of $A[p]$ for some prime integer $p \neq \text{char } \mathbb{K}$. Theorem 1.4 repairs the mentioned gap for $\mathbb{K} = \mathbb{C}$.

2.4. Reduction of Theorem 1.2 to Theorem 1.4. We conclude this section by proving the following lemma, to which Theorem 1.2 reduces thanks to Theorem 1.4.

Lemma 2.2. *For any positive integers d_1, d_2 with $d_1|d_2$, there exists an isotropic subgroup G of $A[d_1d_2]$ as in (**) such that any G -orbit of $A[d_1d_2]$ contains at most two points in (4).*

Proof. The group

$$G := \langle d_1 e'_1, d_2 e'_2 \rangle$$

is clearly isotropic. By (3), G contains e_1 if d_2 is even and e_2 if d_1 is even; in no case G contains other elements of order 2. As a consequence, the only set of points contained in (4) and lying in the same G -orbit are $\{f_1, e_1 + f_1\}$ for even values of d_2 , and $\{f_2, e_2 + f_2\}$ if d_1 is even. One easily checks that

$$G^\perp := \langle e'_1, d_2 f'_1, e'_2, d_1 f'_2 \rangle$$

so that $G^\perp/G \simeq \mathbb{Z}_{d_1}^{\oplus 2} \oplus \mathbb{Z}_{d_2}^{\oplus 2}$.

□

3. PROOF OF THEOREM 1.4

We proceed by degeneration to a curve C_0 having two irreducible smooth elliptic components E_1 and E_2 meeting at a point P .

Let $\pi : \mathcal{C} \rightarrow B$ be a flat family of curves over a *local* one-dimensional base B (that is, $B = \text{Spec}R$ for some discrete valuation ring R) with special fiber C_0 and generic fiber C_b being a smooth irreducible curve of genus 2; also assume that the total space \mathcal{C} is smooth. A relative linear series of type g_d^2 on \mathcal{C} is a pair $\mathfrak{l} = (\mathcal{A}, \mathcal{V})$ such that \mathcal{A} is a line bundle on \mathcal{C} flat over B and \mathcal{V} is a rank-3 subbundle of $\pi_*\mathcal{A}$. We assume the existence of a linear series $l_b = (A_b, V_b)$ of type g_d^2 on the generic fiber C_b of π totally ramified at three points. Possibly after finitely many sequences of base changes and blow-ups at the nodes of the special fiber, we obtain a family $\pi' : \mathcal{C}' \rightarrow B'$ such that:

- (i) the generic fiber of π' is again C_b ;
- (ii) the special fiber C'_0 of π' is obtained from C_0 by inserting a chain of $h \geq 0$ rational curves at the node P ;
- (iii) l_b is the restriction of a relative linear series $\mathfrak{l} = (\mathcal{A}, \mathcal{V})$ on \mathcal{C}' ;
- (iv) there are three sections $\sigma_1, \sigma_2, \sigma_3$ of π' such that l_b is totally ramified at the points $\sigma_1(b), \sigma_2(b), \sigma_3(b)$;
- (v) the points $x_1 := \sigma_1(0), x_2 := \sigma_2(0), x_3 := \sigma_3(0)$ lie in the smooth locus of C'_0 (but are allowed to coincide).

We label the rational components inserted at P with $\gamma_1, \dots, \gamma_h$ and set $\gamma_0 := E_1, \gamma_{h+1} := E_2$ and $P_i := \gamma_{i-1} \cap \gamma_i$ for $1 \leq i \leq h+1$. The restriction of \mathfrak{l} to C'_0 is a (crude) limit linear series [EH2], whose *aspect* on γ_i (cf. [EH2, Def. p. 348]) is denoted by $l_i = (A_i, V_i)$. If $P_j \in \gamma_i$, let $\underline{\alpha}^i(P_j) = (\alpha_0^i(P_j), \alpha_1^i(P_j), \alpha_2^i(P_j))$ denote the ramification sequence of l_i at P_j . We recall the following compatibility conditions [EH2, p. 346]:

$$(8) \quad \alpha_j^{i-1}(P_i) + \alpha_{2-j}^i(P_i) \geq d - 2 \text{ for } 1 \leq i \leq h+1 \text{ and } 0 \leq j \leq 2.$$

Furthermore, any two points Q, Q' on the same component γ_i satisfy:

$$(9) \quad \alpha_j^i(Q) + \alpha_{2-j}^i(Q') \leq d - 2 \text{ for } 0 \leq i \leq h+1 \text{ and } 0 \leq j \leq 2.$$

Since $E_2 = \gamma_{h+1}$ is elliptic, then $\alpha_1^{h+1}(P_{h+1}) \leq d - 3$ and thus $\alpha_1^h(P_{h+1}) \geq 1$ by (8). Inequality (9) then yields $\alpha_1^h(P_h) \leq d - 3$. By the same argument, we obtain that

$$(10) \quad \alpha_1^{i-1}(P_i) \geq 1 \text{ for } 1 \leq i \leq h+1.$$

Analogously, using the fact that $E_1 = \gamma_0$ is elliptic, one proves that

$$(11) \quad \alpha_1^i(P_i) \geq 1 \text{ for } 1 \leq i \leq h+1.$$

In particular, $\gamma_0 = E_1$ has at least a cusp at P_1 and $\gamma_{h+1} = E_2$ has at least a cusp at P_{h+1} .

If x_1 lies on the component γ_i , then $\alpha_2^i(x_1) = d - 2$ and thus (9) yields $\alpha_0^i(P_i) = 0$ as soon as $i \neq 0$ and $\alpha_0^i(P_{i+1}) = 0$ for $i \neq h+1$. By (8), we get that both $\alpha_2^{i-1}(P_i) \geq d - 2$ if $i \neq 0$ and $\alpha_2^{i+1}(P_{i+1}) \geq d - 2$ if $i \neq h+1$. Inductively, we obtain

$$(12) \quad \alpha_2^j(P_{j+1}) \geq d - 2 \text{ for } 0 \leq j \leq i - 1 \text{ (if } i \neq 0),$$

$$(13) \quad \alpha_2^j(P_j) \geq d - 2 \text{ for } i + 1 \leq j \leq h + 1 \text{ (if } i \neq h + 1).$$

We get the same conclusion if $x_2 \in \gamma_i$ or $x_3 \in \gamma_i$. In particular, l_0 has total ramification at P_1 as soon as at least one among the points x_1, x_2, x_3 does not lie on $\gamma_0 = E_1$. Analogously, if at least one among x_1, x_2, x_3 lies outside of $\gamma_{h+1} = E_2$ we obtain that P_{h+1} is a total ramification point for l_{h+1} .

By abuse of notation, we set $\underline{\alpha}^i(x_1)$ to be the 0-sequence if x_1 does not lie on γ_i , and the same for x_2 and x_3 . In the case where x_1, x_2, x_3 are distinct the additivity of the Brill-Noether number (cf. [EH2, Lem. 3.6]) then yields:

$$\begin{aligned} -4 &= \rho(2, 2, d, (0, \dots, 0, d-2), (0, \dots, 0, d-2), (0, \dots, 0, d-2)) \\ &\geq \rho(1, 2, d, \underline{\alpha}^0(P_1), \underline{\alpha}^0(x_1), \underline{\alpha}^0(x_2), \underline{\alpha}^0(x_3)) \\ &\quad + \sum_{i=1}^h \rho(0, 2, d, \underline{\alpha}^i(P_i), \underline{\alpha}^i(P_{i+1}), \underline{\alpha}^i(x_1), \underline{\alpha}^i(x_2), \underline{\alpha}^i(x_3)) \\ &\quad + \rho(1, 2, d, \underline{\alpha}^{h+1}(P_{h+1}), \underline{\alpha}^{h+1}(x_1), \underline{\alpha}^{h+1}(x_2), \underline{\alpha}^{h+1}(x_3)). \end{aligned}$$

If $x_2 = x_1$ and $x_3 \neq x_1$, the above inequality still holds up to deleting all the $\underline{\alpha}^i(x_2)$. The cases where x_3 coincides with x_1 and/or x_2 can be treated similarly. We recall that:

- the adjusted Brill-Noether number of any linear series on \mathbb{P}^1 with respect to any collection of points is nonnegative (cf. [EH3, Thm. 1.1]);
- the adjusted Brill-Noether number of any linear series on an elliptic curve with respect to any point is nonnegative (cf. [EH3, Thm. 1.1]);
- the adjusted Brill-Noether number of any g_d^2 on an elliptic curve with respect to any two points is ≥ -2 (cf. [F, Prop. 4.1]).

Concerning the position of the points x_1, x_2, x_3 , we can thus conclude (up to relabelling them) that either

- (a) x_1 lies on E_1 , x_2 lies on E_2 and x_3 lies on γ_i for some $1 \leq i \leq h$, or
- (b) x_1 and x_2 are distinct and lie on the same elliptic component.

In case (a), one has $\underline{\alpha}^i(x_3) \geq (0, 0, d-2)$ and inequalities (10), (11), (12), (13) imply both $\underline{\alpha}^i(P_i) \geq (0, 1, d-2)$ and $\underline{\alpha}^i(P_{i+1}) \geq (0, 1, d-2)$; this contradicts the Plücker Formula [EH1, Prop. 1.1] according to which the total ramification of any g_d^r on \mathbb{P}^1 equals $(r+1)d - r(r+1)$.

Thus we necessarily fall in case (b). Without loss of generality, we assume that $x_1, x_2 \in E_1 = \gamma_0$. If $x_3 = x_1$ or $x_3 = x_2$, the ramification weight of l_0 at x_3 is $\geq 2(d-2)$ since it equals the sum of the weights of the ramification points of C_b tending to x_3 (cf., e.g., [HM, p. 263]). On the other hand, x_3 cannot be a base point and thus $\underline{\alpha}^0(x_3) = (0, d-2, d-2)$ and this is a contradiction because E_1 is elliptic. We conclude that l_0 is totally ramified at three distinct points, namely, x_1, x_2, x_3 if $x_3 \in E_1$ and x_1, x_2, P_1 if $x_3 \notin E_1$; in both cases, l_0 also has a cusp at P_1 . We apply the next proposition; in the former case this implies the relation $2x_1 \sim 2x_2 \sim 2x_3$ on E_1 , while in the latter case we obtain $2x_1 \sim 2x_2 \sim 2P_1$ on E_1 .

Let $\pi'_0 : J_{\pi'} \rightarrow B'$ be the relative generalized jacobian of the family π' , whose generic fiber is the jacobian $J(C_b)$ and whose special fiber is the generalized jacobian $J(X'_0)$ parametrizing isomorphism classes of line bundles having degree 0 on every irreducible component of X'_0 . Hence, one has $J(X'_0) \simeq \text{Pic}^0(E_1) \times \text{Pic}^0(E_2)$ and π'_0 is a family of smooth principally polarized abelian surfaces. The relative degree-0 line bundle $\mathcal{O}_{C'}(\sigma_2 - \sigma_1)$ defines a *torsion* section of π'_0 (since $d\sigma_1(b) \sim d\sigma_2(b)$ by (iv)) intersecting the special fiber $J(X'_0)$ in the 2-torsion point $(\mathcal{O}_{E_1}(x_2 - x_1), \mathcal{O}_{E_2})$. By [Mi, Pf.

of Prop. VII.3.2 and Cor. VII.3.3] (that works for families of abelian varieties of arbitrary dimension), the group of torsion sections of π'_0 injects in the torsion subgroup of any fiber of π'_0 and thus we conclude that $\mathcal{O}_{C'}(\sigma_2 - \sigma_1)$ is 2-torsion. In particular, on the generic fiber C_b of π' the divisors $2\sigma_1(b)$ and $2\sigma_2(b)$ are linearly equivalent, that is, $\sigma_1(b)$ and $\sigma_2(b)$ are Weierstrass points.

We claim that $\sigma_3(b)$ is a Weierstrass point, as well (this is clear in the case where $2x_1 \sim 2x_2 \sim 2x_3$ but needs some work when $2x_1 \sim 2x_2 \sim 2P_1$). Let ι_b be the hyperelliptic involution on C_b and set $\sigma_4(b) := \iota_b(\sigma_3(b))$. By contradiction, we assume $\sigma_4(b) \neq \sigma_3(b)$. As d is even, then $d\sigma_3(b) \sim d\sigma_1(b) \sim \frac{d}{2}(\sigma_3(b) + \sigma_4(b))$ and thus $d\sigma_4(b) \sim d\sigma_3(b) \sim d\sigma_1(b)$. As a consequence, the linear series $l'_b := (\mathcal{O}_{C_b}(d\sigma_1(b)), \langle d\sigma_1(b), d\sigma_3(b), d\sigma_4(b) \rangle)$ is a g_d^2 on C_b totally ramified at $\sigma_1(b), \sigma_3(b), \sigma_4(b)$. The first part of the proof applied to l'_b thus yields that at least two points among $\sigma_1(b), \sigma_3(b), \sigma_4(b)$ are Weierstrass points of C_b and thus a contradiction.

Proposition 3.1. *Fix an integer $d \geq 3$. If a general elliptic curve possesses a g_d^2 totally ramified at three points P_1, P_2, P_3 and with a cusp, then d is even, the g_d^2 is not birational and the relation $2P_1 \sim 2P_2 \sim 2P_3$ holds.*

Proof. We first show that, if a g_d^2 on a general elliptic curve E totally ramified at three points P_1, P_2, P_3 is not birational, then d is even and $2P_1 \sim 2P_2 \sim 2P_3$. We consider the Stein factorization of the map $f : E \rightarrow \mathbb{P}^2$ defined by the g_d^2 (which is base point free since it admits three points of total ramification):

$$\begin{array}{ccc} E & & \\ p \downarrow & \searrow f & \\ \mathbb{P}^1 & \xrightarrow{q} & R \subset \mathbb{P}^2, \end{array}$$

where p is a cover of degree $k \geq 2$, q is birational and R is a singular plane curve of degree d/k . Since f is totally ramified at P_1, P_2, P_3 , the same holds for p . The Riemann-Hurwitz formula thus implies $k \leq 3$. The case $k = 3$ can be excluded for general E because, by Riemann's Theorem along with the fact that all triples of points on \mathbb{P}^1 are projectively equivalent, there is a unique genus 1 triple cover of \mathbb{P}^1 totally ramified at three points.

It remains to show that a birational g_d^2 on a general elliptic curve E totally ramified at three points P_1, P_2, P_3 admits no cusps. Let X be the degree d plane curve image of E under the map defined by the g_d^2 . We note that the three lines L_1, L_2, L_3 cutting the divisors dP_1, dP_2, dP_3 cannot belong to a pencil of lines through a fixed point of \mathbb{P}^2 since otherwise this pencil would cut a g_d^1 on E totally ramified at three points, thus contradicting the Riemann-Hurwitz formula for $d \geq 4$ and the generality of E for $d = 3$, as above. The curve X defines a point in the *generalized Severi variety*

$$V_{d,1}(L_1 + L_2 + L_3, (d, d, d))$$

parametrizing reduced and irreducible plane curves of geometric genus 1 and degree d having contact order d at three *unassigned* points in the smooth locus of $L_1 + L_2 + L_3$. More strongly, since all triples of lines with no common points are projectively equivalent, the image of any elliptic curve under any birational g_d^2 totally ramified at three points is represented by a point in $V_{d,1}(L_1 + L_2 + L_3, (d, d, d))$. Conversely, the normalization map of any member of $V_{d,1}(L_1 + L_2 + L_3, (d, d, d))$ defines a g_d^2 on an elliptic curve totally ramified at three points.

We next note that an elliptic curve E has a finite number of g_d^2 totally ramified at three points up to automorphisms; indeed, the relation $dP_1 \sim dP_2 \sim dP_3$ on E yields that the line bundles $\mathcal{O}(P_i - P_j)$ are d -torsion elements of $\text{Pic}^0(E) \simeq E$. To prove the desired statement that a birational g_d^2 on a general elliptic curve E totally ramified at three points admits no cusps, it is therefore enough to show that a general element X in any irreducible component V of $V_{d,1}(L_1 + L_2 + L_3, (d, d, d))$ is *immersed* (that is, the differential of its normalization map is everywhere injective).

Generalized Severi varieties were introduced by Caporaso-Harris in [CH] (cf. [Za2] for recent results on the topic). Our situation is slightly different since we fix the ramification profile at three lines instead of one; however, the local computations in [CH, §2.2] are proved for fixed contact order with any smooth curve and thus apply also in our case where the points of contact lie in the smooth locus of $L_1 + L_2 + L_3$.

We recall the main deformation theoretic arguments in [CH], adapting them to our setting¹. Let X be a general element of any irreducible component V of $V_{d,1}(L_1 + L_2 + L_3, (d, d, d))$ and let $f : E \rightarrow X \subset \mathbb{P}^2$ denote the normalization map. Then $f^*(L_1 + L_2 + L_3) = dP_1 + dP_2 + dP_3$ for some points $P_1, P_2, P_3 \in E$. We consider the normal sheaf N_f , its torsion subsheaf H_f supported at the vanishing divisor Z of the differential df of f , and the quotient $\overline{N}_f := N_f/H_f$. We have the following commutative diagram (cf. [Se, (3.51)]):

$$(14) \quad \begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \downarrow & \\ & & & & & H_f & \\ & & & & & \downarrow & \\ 0 & \longrightarrow & T_E \simeq \mathcal{O}_E & \xrightarrow{df} & f^*T_{\mathbb{P}^2} & \longrightarrow & N_f \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & T_E(Z) \simeq \mathcal{O}_E(Z) & \longrightarrow & f^*T_{\mathbb{P}^2} & \longrightarrow & \overline{N}_f \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

As in [CH, p. 363], for $1 \leq i \leq 3$ let l_i be the order of vanishing of the differential of f at P_i and define the two following divisors on E :

$$D := \sum_{i=1}^3 (d-1)P_i,$$

$$D_0 := \sum_{i=1}^3 l_i P_i;$$

¹In the notation of Caporaso-Harris the line L is here replaced by $L_1 + L_2 + L_3$ and we have $\alpha = 0$, that is, we are not imposing contact order at any fixed points of $L_1 + L_2 + L_3$, and

$$\beta = (\underbrace{0, \dots, 0}_{d-1}, d),$$

that is, we are imposing contact order d at three unassigned points of $L_1 + L_2 + L_3$.

note that the difference $D_1 := D - D_0$ is effective in our case². Furthermore, the divisor $Z - D_0$ is effective by the definition of Z and l_i .

By [CH, Lemma 2.3 and Lemma 2.6] (cf. also [AC, p.26]), the tangent space $T_{[X]}V$ of V at the point $[X]$ injects in $H^0(E, \overline{N}_f(-D_1))$. From (14) we get:

$$(15) \quad \deg \overline{N}_f(-D_1) = 3d - \deg Z - \deg D + \deg D_0 = 3 - \deg(Z - D_0) \leq 3,$$

and thus

$$(16) \quad \dim V \leq \dim T_{[X]}(V) \leq h^0(E, \overline{N}_f(-D_1)) \leq 3.$$

On the other hand, 3 equals the expected dimension of V because $3 = 3d - 3(d - 1)$, where $3d$ is the dimension of the Severi variety of degree d genus 1 plane curves and $3(d - 1)$ comes from the ramification imposed at three unassigned points (cf. [CH, §2.1]). Hence, $\dim V = 3$ and all inequalities in (15) and (16) are equalities. In particular, V is smooth at $[X]$ and $T_{[X]}V$ can be identified with $H^0(E, \overline{N}_f(-D_1))$. Having degree 3, the line bundle $\overline{N}_f(-D_1)$ is very ample and thus possesses a section vanishing at any point of E with order exactly 1; as in [CH, Proof of Prop. 2.2 p. 364], this implies that X is immersed and concludes the proof. \square

Remark 3. Proposition 3.1 is sharp in the following sense. Take three points P_1, P_2, P_3 on an elliptic curve E satisfying $2P_1 \sim 2P_2 \sim 2P_3$ and let $d \geq 4$ be an even integer. Then the map f defined by the linear series $\langle dP_1, dP_2, dP_3 \rangle$ factors as follows:

$$\begin{array}{ccc} E & & \\ \downarrow p & \searrow f & \\ \mathbb{P}^1 & \xrightarrow{q} & R \subset \mathbb{P}^2, \end{array}$$

where p is the double cover branched at P_1, P_2, P_3 and at a further point P_0 , and q is the (unique up to projectivities) map from \mathbb{P}^1 defined by the linear series $\langle \frac{d}{2}x_1, \frac{d}{2}x_2, \frac{d}{2}x_3 \rangle$ with $x_i := p(P_i)$. Plücker's Formula yields that q has no ramification outside of x_1, x_2, x_3 . Therefore, one computes that the ramification sequence of the g_d^2 on E at P_i is $(0, 1, d - 2)$ for $1 \leq i \leq 3$ and $(0, 1, 2)$ for $i = 0$. In particular, the g_d^2 has cusps at all the four points P_0, P_1, P_2, P_3 .

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