UNIVERSITÀ DEGLI STUDI ROMA TRE

DOTTORATO DI RICERCA IN MATEMATICA

XXXVIII CICLO

Dynamics and Topology of Infinite-Dimensional Hamiltonian Systems: a Geometric Approach to Complete Integrability

Tesi di

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Anno Accademico 2025/2026

Ringraziamenti

Sono profondamente grato alle persone che hanno guidato lo sviluppo di questa tesi.

Ringrazio Michela Procesi per l'impegno dedicato alla realizzazione di questo lavoro, per la totale disponibilità, il sincero interesse per il problema e per aver alimentato costantemente il mio entusiasmo. Le sono immensamente riconoscente per i suoi insegnamenti e per aver sempre stimolato la mia curiosità con le sue domande e le sue idee. E soprattutto grazie perché, anche nelle occasioni in cui il mio contributo alla discussione era pari a zero, ha sempre avuto la capacità di farmi sentire parte attiva dei suoi ragionamenti. La ringrazio inoltre per i bei momenti di convivialità condivisi, dalle mozzarelle a Sabaudia ai gelati a Pizzo.

Grazie a Jean-Pierre Marco per le discussioni sempre ricche di spunti e per avermi insegnato quanto sia importante e bello guardare le cose da punti di vista imprevedibili. Le sue idee sono state una fonte primaria di ispirazione e hanno contribuito a definire il carattere *dinamico* di questo lavoro. Lo ringrazio anche per avermi fatto divertire con la sua ironia e i suoi scherzi, come quello del certificato di presenza per la segreteria amministrativa, che per rendergli giustizia devo riportare qui integralmente:

> *To whom it may concern,*
>
> *I hereby certify that Lorenzo Baroni does absolutely nothing, he visits clubs all nights long and spends his time then eating sandwiches which he purchases in highly expensive bakeries from Paris. He wants to produce an article from ChatGPT; I have to warn you that it may be completely wrong, in particular if he pretends to learn the behavior of orbits related to resonant modules.*
>
> *Yours sincerely,*
>
> *Jean-Pierre Marco.*

Un grazie infinito va a Jessica Elisa Massetti che per prima mi ha trasmesso il fascino per i sistemi dinamici al suo corso di Teoria KAM, e mi ha introdotto a persone che si collocano sicuramente tra le rare singolarità umane nel mondo accademico, e forse anche più in generale. La ringrazio per il supporto costante e perché ciò che ha dato valore a una buona parte di questi tre anni di dottorato dipende direttamente da lei, o comunque da qualcuno o qualcosa che, volente o nolente, orbita attorno a Jessica. In lei ho trovato un'amica preziosa, che ti invita a ragionare sinceramente e che ama il discorso trasparente. Grazie perché la stima e la fiducia che mi ha sempre dato mi sono servite nei momenti più difficili. Infine, grazie per le infinite chiacchierate e per avermi insegnato che per i voli intercontinentali in aeroporto ci si va tre ore prima e si prende il seggiolino comodo perché il viaggio è lungo. Grazie.

Desidero ringraziare Claudio Procesi per i suoi utili suggerimenti e per aver condiviso la sua profonda conoscenza dell'algebra tramite lunghi scambi di mail o passeggiando verso la trattoria ad Arpino. Il suo contributo è stato prezioso per la comprensione del meccanismo che sta alla base della struttura topologica dei sottoinsiemi minimali nei flussi di Kronecker sul toro infinito.

Grazie anche a tutto il gruppo dei *Dinamici a Roma Tre*, dai più grandi ai più piccini, che mi ha sempre fatto sentire accolto con calore e in cui ho avuto la fortuna di imparare tanta matematica in un ambiente stimolante e molto umano. Un ringraziamento speciale va a Roberto Feola e Livia Corsi, con cui posso dire di aver condiviso molto. Ringrazio infine Shulamit perché ho trovato un'amica.

E poi ci sono gli amici del primo anno, Simone, Rocco, Mario e Martina (in ordine di altezza), che proprio non avrei mai scommesso di incontrare. E ci mettiamo anche Ilaria, che ormai sta anche nel gruppo. Tre anni sono tanti e, che tu voglia o meno, le cose succedono. E a noi normalmente ce ne sono successe tante ma, per quanto possibile, ci siamo sempre divisi il peso delle sofferenze e provato a trasmettere una totale presenza. In tutto questo tempo passato insieme sento di essermi tanto arricchito, e li ringrazio perché mi hanno regalato tanti bei momenti, che avranno valore per tutta una vita.

Grazie anche a tutti i dottorandi e le dottorande, più o meno giovani, sempre presenti. Un grazie speciale va a Federico e Wei per aver avuto la pazienza e la gentilezza di spiegarmi tutte le cose troppo algebricose che incontravo nel mio lavoro.

Quando un giorno guarderò indietro al mio dottorato, un pensiero immediato e spontaneo andrà ai mesi a Parigi. Ringrazio prima di tutto Tomas Lazaro per avermi regalato un sorriso ogni mattina appena mettevo piede nello studio, per tutto il tempo trascorso insieme e per i pranzi *al trotto* al *Jardin des Plantes*. Grazie a Laurent Niederman per la sua gentilezza, le interessanti discussioni e per tutti i momenti di convivialità. Poi ringrazio Mar e Cesar, i primi amici di Parigi (già conosciuti a Marsiglia) con cui subito è scattata quella complicità che ha portato a un legame sincero. Mi preme assolutamente ringraziare anche tutto il pazzo gruppo di amici sudamericani Odylo, Pietro, Gabriel, Gerson (a.k.a. Luis) e Joaquin. Infine ringrazio Matteo per le chiacchierate e le serate passate a giocare a ping-pong.

Vorrei ringraziare anche delle persone con cui ho passato dei momenti speciali durante i viaggi per conferenze.

Ringrazio Diego e Stefano per l'indimenticabile gita fuori porta nelle campagne di Allumiere e per le serate al bar di Arpino. A tal proposito, ringrazio Thierry Paul per avermi sempre coinvolto nei workshop.

Un grazie speciale va anche a tutte le persone con cui ho condiviso bei momenti nel mio ultimo viaggio in Cina. Ringrazio Jose, anche se non ci ha più accompagnato in aeroporto, e ringrazio Junhao, Qi e Deyu per l'ospitalità disarmante, le bevute, la bellissima giornata che ci hanno regalato e per tutte le discussioni interessanti che abbiamo avuto.

Ringrazio infinitamente mio fratello, i miei genitori e la mia famiglia, perché è in loro che ritrovo le coordinate quando mi sento nel vuoto. E poi li ringrazio perché, non so per quale motivo, mi hanno sempre fatto credere di poter riuscire in qualsiasi cosa mi fossi cimentato.

Ringrazio Rebecca perché mi ha fatto alzare dal letto quando ero sopraffatto dallo sconforto e volevo rinunciare a fare gli esami orali per l'ammissione ai dottorati. Subito dolce, e dura un attimo dopo, aveva capito, o forse lo sapeva e basta, che per farmi muovere doveva risvegliarmi quel senso del rigore e quella tenacia propria delle cose antiche per ricordarmi che il tempo, la fatica e le cose importanti non si buttano via così con leggerezza, con un capriccio. Grazie per avermi sempre incoraggiato e manifestato stima, e infine grazie per aver sopportato anche tutti gli effetti negativi di questo percorso.

Grazie ai valutatori Benoit Grebert e Alberto Abbondandolo per il tempo dedicato alla lettura e alla valutazione di questa tesi e per gli interessanti spunti su cui ragionare.

Ringrazio i commissari Luigi Chierchia, Alfonso Sorrentino e Paolo Giulietti.

Contents

Notations

Throughout this work, we adopt the following conventions:

$\mathbb{N}_0, \mathbb{N}$ — $\mathbb{N}_0$ denotes the set of natural numbers $\{0, 1, 2, \dots\}$, while $\mathbb{N}$ denotes the set of positive natural numbers $\{1, 2, \dots\}$.

$\mathbb{T}$ — $\mathbb{T}$ denotes the one dimensional torus $\mathbb{R}/2\pi\mathbb{Z}$.

$X^{(Y)}$ — Given $X, Y \subseteq \mathbb{Z}$, we denote the set of sequences with finite support as:

$$X^{(Y)} := \left\{ (\lambda_k)_{k\in Y} \ : \ \lambda_k \in X \ \forall k \in Y, \ |\lambda| := \sum_{k\in Y} |\lambda_k| < \infty \right\}.$$

z^λ — Multi-index notation for a list $z = (z_k)_{k\in Y} \in \mathbb{C}^Y$ and $\lambda \in X^{(Y)}$:

$$z^\lambda := \prod_{k\in Y} z_k^{\lambda_k}.$$

$|z|^n$ — For any $z \in \mathbb{C}^Y$, $n \in \mathbb{N}$, the list: $|z| := (|z_k|^n)_{k\in Y}$.

e_j — For $j \in \mathbb{N}$, the standard basis vector $e_j = ((e_j)_k)_{k\in\mathbb{N}}$ where $(e_j)_k = \delta_{jk}$.

$\mathscr{U}_r(X)$ — Given a normed space $(X, |\cdot|_X)$, the open ball of radius r centered at the origin is $\mathscr{U}_r(X) := \{x \in X : |x|_X < r\}$. Its closure is denoted by $\overline{\mathscr{U}}_r(X)$.

$\mathscr{U}_{r,x_0}(X)$ — For $X \subset \mathbb{C}^\mathbb{Z}$ and $x_0 \in \mathbb{C}^\mathbb{Z}$, the open ball centered at x_0 is:

$$\mathscr{U}_{r,x_0}(X) := x_0 + \mathscr{U}_r(X) = \{x \in \mathbb{C}^\mathbb{Z} : |x - x_0|_X < r\}.$$

The closure is denoted by $\overline{\mathscr{U}}_{r,x_0}(X)$.

Introduction

The primary objective of this thesis is to generalize ideas and techniques from the theory of finite-dimensional Hamiltonian systems to the infinite-dimensional setting. We focus specifically on concepts related to Liouville-Arnold integrability, with particular emphasis on systems on infinite lattices and PDEs on the circle. Although several results concerning the complete integrability of infinite-dimensional Hamiltonian systems have been established (see for e.g. [ZF71] for seminal results on the KdV, [AS81] for the IST method, or [KP03] for a modern approach), none of them provides a decomposition of the phase space as rigid as the one provided by the Liouville-Arnold theorem in the finite-dimensional case. This is largely due to the choice of the topology for the phase space, a crucial issue that will be discussed in detail later on.

In this context, the present work aims to fill this gap. To the best of our knowledge, the results presented in this thesis provide the first instance of analytic invariant foliations by infinite tori for infinite-dimensional Hamiltonian systems, alongside the first construction of action-angle coordinates that are analytic in an open neighborhood of the phase space (see Theorems 2.6.1 and 3.1.4). These results establish a more faithful analogue of the Liouville-Arnold theorem in the infinite-dimensional setting.

Specifically, this work addresses the following questions:

- **Foliation into infinite-dimensional tori (Theorem 1.3.2):** *When do the level sets of a family of analytic Hamiltonians induce a regular foliation of an open subset of the phase space into infinite-dimensional invariant tori?*

- **Action-Angle variables (Theorem 2.5.12):** *Assuming the family of Hamiltonians is in involution, what additional analytical regularity assumptions are required to construct a set of action-angle coordinates adapted to the foliation?*

- **Liouville-Arnold integrability (Theorem 2.6.1):** *When the integrals of motion induce both a regular foliation and a set of action-angle coordinates, is it possible to establish an infinite-dimensional analogue of the Liouville-Arnold theorem for a given analytic Hamiltonian commuting with the family?*

- **Applications to Hamiltonian PDEs (Theorem 3.2.7):** *Are there Hamiltonian PDEs for which a complete family of first integrals in involution is already known and to which our results can be applied to prove Liouville-Arnold integrability?*

The simplification provided by the Liouville-Arnold theorem shifts the analysis of the dynamical properties of an integrable system towards the characterization of *Kronecker flows*, whose properties in finite dimension are well understood through the lens of arithmetic relations between frequencies (see Properties A and B, page 9). However, extending this analysis to the infinite-dimensional torus introduces a significantly higher level of complexity. As suggested by recent results in [SV24], the behaviour of linear flows in infinite dimensions can be far more involved than in the finite-dimensional case, exhibiting phenomena that lack a direct finite-dimensional counterpart, such as the existence of orbits whose closure is not a torus (contradicting Property B, page 9).

In this setting, the choice of the topology is not merely a technical detail but a pivotal matter. Indeed, key dynamical properties, such as the continuity, density, and ergodicity of the flow, depend heavily on the specific topological framework. Our study is based on a topological setting designed to allow for direct applications to Hamiltonian systems in Hilbert spaces, as well as to spaces modelled on ℓ^∞ but endowed with the weak* topology. In particular, we tackle the following issues:

- **Generalization of Kronecker flows (see Chapter 4):** *To what extent can the dynamical properties of finite-dimensional Kronecker flows be extended to infinite tori, and how does the frequency module determine the topology of the closure of the orbits and their homeomorphism classes?*

- **Exotic orbits in Hamiltonian PDEs (Theorem BO):** *Are we able to exhibit examples of Hamiltonian PDEs admitting almost-periodic solutions whose orbit closures exhibit a more complex topological structure than the usual invariant tori encountered in finite-dimensional integrable Hamiltonian systems?*

The rest of this introduction is dedicated to providing a historical and theoretical context for these topics, tracing the evolution of integrable systems from their classical roots to the modern infinite-dimensional framework.

1 Historical Perspectives on Integrable Systems

In the immediate aftermath of the publication of Newton's *Philosophiae Naturalis Principia Mathematica*, which introduced the fundamental principle of dynamics $F = ma$, and despite a calculus not yet established on rigorous foundations, the scientific community began to tackle the challenge of finding exact solutions for non-trivial mechanical problems. While Newton himself analytically solved the Kepler problem on planetary motion, it became clear that such cases were exceptional. Beyond this example, only a handful of problems could be treated exactly, i.e., avoiding intermediate approximations, additional modelling hypotheses, or numerical methods, all of which inherently introduce approximations.

Thus, despite the publication of Lagrange's *Mécanique Analytique* and the establishment of the deterministic mechanistic paradigm, by the early 19th century, the examples of dynamical models with solutions known in analytical form were few: the aforementioned Kepler problem, the systems of harmonic oscillators, the rigid body with a fixed point in the absence of external forces (Euler's top), and the Lagrange top. Their solvability was guaranteed by the existence of a sufficient number of analytical conserved quantities, which allowed for the reduction of the problem to the computation of integrals, a process known as reduction to quadratures.

Throughout the first half of the 19th century, Hamilton, Jacobi, and Liouville refined the notion of *integrability* for what are now termed *Hamiltonian systems*. These systems are of great relevance in physics, as they describe the evolution of mechanical systems obeying Newton's laws. In this context, the Liouville theorem, later generalized by Arnold, represent the cornerstones of the theory, leading to what is known today as complete integrability *à la Liouville-Arnold*, which constitutes the primary focus of this thesis.

In such integrable systems, the dynamics is characterized by the superposition of periodic motions. This specific type of evolution is said to be *quasi-periodic* and can be modelled as a linear flow on a compact torus, that is, a *Kronecker flow*. The existence of these tori ensures that the trajectories remain confined and predictable, representing the highest degree of order within Hamiltonian mechanics.

However, H. Poincaré in his *Les méthodes nouvelles de la mécanique céleste* called the task to study perturbations of integrable systems the *Problème général de la dynamique*. The scientist was motivated by the celestial mechanics, where perturbed integrable systems play a very important role[1]. Poincaré proved that the three-body problem lacks a general analytical solution in terms of algebraic expressions and integrals. He further showed that many scenarios exhibit a sensitive dependence on initial conditions: by studying motions near periodic orbits and employing first-order approximations, Poincaré observed that the generic situation involves an exponential divergence from regular motion. He thus provided the first description of a dynamics that, nearly a century later, would be termed deterministic chaos.

This led to a conceptual crossroads: if mechanical systems generally tend toward chaotic behaviour and integrable systems are a rarity, are the latter merely mathematical curiosities? The question then becomes: to what extent can an integrable system be deformed before its integrable structure breaks down? The theory initiated by the seminal work of Kolmogorov (see [Kol54]) and later extended by Arnold and Moser (see [Arn63] and [Mos61, Mos66], respectively), now widely known as *KAM theory*, establishes that most of the recurrent motions exhibited by an integrable system persist under small perturbations. This result validates the use of perturbative methods and demonstrates that the dynamics is not necessarily lost to chaos even under perturbation, provided the underlying integrable structure is sufficiently robust. This, however, also implies the challenge of identifying integrable systems to serve as starting points for such analysis, beyond the classical examples already established. It suggests that the study of integrable systems is far from being a closed chapter; rather, it remains a fertile field of research.

This perspective remains equally relevant when moving from finite-dimensional mechanics to the domain of field theories and continuum mechanics. Indeed, many physical systems with an infinite number of degrees of freedom admit a Hamiltonian formulation. During the sixties, a significant breakthrough occurred with the realization that some of these models possess an infinite number of independent constants of motion in involution (see, for e.g., [ZS] and [Gar71]). This suggests that the dynamics is constrained to specific invariant subsets of the phase space which are homeomorphic to tori of possibly infinite dimension. In this setting, the resulting motion is characterized by an *almost-periodic* behaviour, that is, a generalization of the quasi-periodicity observed in finite dimension to cases involving an infinite number of combined frequencies. These structures appear in continuous models, such as the *Korteweg–de Vries equation* for shallow water waves, as well as in discrete systems like the *Toda lattice* [Tod89]. The study of such systems has been intensively pursued using techniques ranging from algebraic geometry to Riemann-Hilbert problems, depending largely on the topology of the manifold of independent variables (see, for e.g., [AS81]).

While KAM theory does not generally guarantee the persistence of quasi-periodic solutions in the infinite-dimensional setting, the study of the complete integrability and the search for almost-periodic solutions is an active and challenging topic in the study of global solutions for evolution Hamiltonian PDEs.

2 Finite-Dimensional Completely Integrable Hamiltonian Systems: the Liouville-Arnold Theorem

In the seminal work [Lio55], Liouville identified that the common denominator of most of the systems that can be reduced to quadratures, such as the aforementioned Kepler problem, the harmonic oscillators, the Euler's top and the Lagrange top, was the existence of a sufficient number of constants of motion *independent* and *in involution*. He further proved that, under such conditions, the equations of motion can be solved through elementary formulas. This result was significantly advanced by Arnold [Arn89], who unveiled the underlying geometric architecture of such systems. The synthesis of these two contributions

[1] For example, the solar system, regarded as a system of 8 interacting planets rotating around the Sun, is a small perturbation of the Kaplerian system. The latter is integrable.

is now known as the *Liouville-Arnold theorem*, which constitutes the foundational framework of the present thesis. Indeed, the present work is motivated by the aim of extending this classical result to Hamiltonian systems with infinitely many degrees of freedom.

Let us recall the basic definitions and statements of the theory of Hamiltonian mechanics of finitely many degrees of freedom. Throughout this discussion, we shall work within the real-analytic setting. Nevertheless, we emphasize that all definitions and results remain applicable to the $\mathscr{C}^\infty$ (smooth) regularity class.

A symplectic manifold is a real analytic manifold M of even dimension $2N$, with $N \in \mathbb{N}$, endowed with a closed, non-degenerate differential 2-form α. According to the Darboux theorem, any symplectic manifold is locally indistinguishable from the standard symplectic manifold $(\mathbb{R}^{2N}_{(p,q)}, \omega)$, where ω is the standard symplectic form given by

$$\omega := \sum_{k=1}^{N} dp_k \wedge dq_k, \tag{1}$$

meaning that there always exists a system of *Darboux coordinates* such that α takes the form (1).

Given a real-analytic function H on M, which we shall refer to it as *Hamiltonian function*, there is an associated *Hamiltonian vector field* V_H defined as follows through the symplectic structure:

$$\omega(\cdot, V_H) = dH[\,\cdot\,]. \tag{2}$$

The existence and uniqueness of this vector field are guaranteed by the non-degeneracy and skew-symmetry of the symplectic form.

We now introduce another structure intimately connected to the symplectic form. Given a real-analytic manifold M, let $\mathcal{A}(M)$ denote the space of real valued analytic function. Then, a Poisson bracket on M is a bilinear, skew-symmetric map

$$\{\cdot,\cdot\} : \mathcal{A}(M) \times \mathcal{A}(M) \to \mathcal{A}(M)$$

that obeys both the Leibniz rule and the Jacobi identity.

Remark 1. *Crucially, the non-degeneracy of the symplectic form allows for the definition of an associated Poisson bracket:*

$$\{f, g\} := \alpha(V_f, V_g), \qquad \textit{for all } f, g \in \mathcal{A}(M).$$

Explicitly, by setting a Darboux coordinate system $(p_k, q_k)_{k=1}^N$, one has

$$\{f, g\} = \sum_{k=1}^{N} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} \right).$$

Notice that, by virtue of (2), the Lie derivative of a real analytic function f in the direction of the *Hamiltonian flow* ϕ_H induced by a Hamiltonian vector field V_H can be equivalently expressed as

$$\frac{d}{dt}(f \circ \phi_H^t) := df[V_H] \circ \phi_H^t = \alpha(V_H, V_f) \circ \phi_H^t = \{H, f\} \circ \phi_H^t. \tag{3}$$

The Hamilton's equations for a N-degrees of freedom *Hamiltonian system* with Hamiltonian H, defined on a simplectic manifold of dimension $2N$, describe the flow ϕ_H^t induced by the Hamiltonian vector field V_H. In a Darboux coordinate system $(p_k, q_k)_{k=1}^N$, these equations take the form:

$$\begin{cases} \dot{p}_k = \{H, p_k\} = -\dfrac{\partial H}{\partial q_k} \\ \dot{q}_k = \{H, q_k\} = \dfrac{\partial H}{\partial p_k} \end{cases}, \qquad k = 1, \dots, N.$$

These equations are invariant under *symplectomorphisms*, namely coordinate transformations that preserve the standard symplectic form ω, or, equivalently, the Poisson bracket $\{\cdot,\cdot\}$. While physicists typically refer to the variables q as *positions* and p as *conjugate momenta*, there is no fundamental mathematical necessity to distinguish between them, as they can be interchanged via symplectomorphisms.

Generally, this system of first-order ordinary differential equations is non-linear and can be arbitrarily complex. However, if the Hamiltonian depends solely on the conjugate momenta, the Hamilton's equations reduce to:

$$\dot{p}_k = 0, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad k = 1, \ldots, N.$$

In this case, the system can be integrated directly, leading to the following solution:

$$\phi_H^t(p,q) = (p, q + \lambda(p)t),$$

where the frequencies $\lambda_k(p) = \partial_{p_k} H$ are clearly constant. Moreover, if the motion is constrained to compact level sets, the q coordinates are identified as angles, and this dynamics is equivalent to a linear flow on an N-dimensional torus.

An illustrative example is provided by the system of *uncoupled harmonic oscillators*, which is described by the Hamiltonian H_{HO} given by

$$H_{HO}(p,q) = \sum_{k=1}^{N} \frac{\lambda_k}{2}(p_k^2 + q_k^2).$$

Although this Hamiltonian depends on both positions and momenta, it can be reduced to the previously discussed integrable form through the symplectomorphism that defines the so called *symplectic polar coordinates* (I,θ):

$$p_k = \sqrt{2I_k}\cos\theta_k, \quad q_k = \sqrt{2I_k}\sin\theta_k.$$

In these new variables (I,θ), called *action-angle* coordinates, the Hamiltonian takes the form $H(I) = \sum_{k=1}^{N} \lambda_k I_k$. Since H now depends exclusively on the actions I_k, the equations of motion become linear: $\dot{I}_k = 0$ and $\dot{\theta}_k = \lambda_k$. Consequently, the dynamics is a linear flow on the N-dimensional torus $\mathbb{T}^N := \mathbb{R}^N/(2\pi\mathbb{Z})^N$, with constant frequencies given by λ_k, $k = 1, \ldots, N$.

In order to produce symplectomorphisms, one often employs the *generating function approach*. In this framework, a generating function $S(I,q)$ of the coordinates q and the new actions I defines the transformation $(p,q) \to (I,\theta)$ through the relations:

$$p = \frac{\partial S}{\partial q}, \quad \theta = \frac{\partial S}{\partial I},$$

whenever they are invertible. To verify that this transformation is indeed a symplectomorphism, we check the invariance of the standard symplectic form:

$$\begin{aligned}
\sum_{k=1}^{N} dI_k \wedge d\theta_k &= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\partial^2 S}{\partial q_\ell \partial I_k} dI_k \wedge dq_\ell = \sum_{k=1}^{N}\sum_{\ell=1}^{N} \frac{\partial p_\ell}{\partial I_k} dI_k \wedge dq_\ell \\
&= \sum_{\ell=1}^{N} \left(\sum_{k=1}^{N} \frac{\partial p_\ell}{\partial I_k} dI_k \right) \wedge dq_\ell = \sum_{\ell=1}^{N} \left(dp_\ell - \sum_{k=1}^{N} \frac{\partial p_\ell}{\partial q_k} dq_k \right) \wedge dq_\ell \\
&= \sum_{\ell=1}^{N} dp_\ell \wedge dq_\ell - \sum_{\ell=1}^{N}\sum_{k=1}^{N} \frac{\partial^2 S}{\partial q_k \partial q_\ell} dq_k \wedge dq_\ell \\
&= \sum_{\ell=1}^{N} dp_\ell \wedge dq_\ell.
\end{aligned}$$

The last step holds due to the symmetry of the second derivatives of S and the skew-symmetry of the wedge product, which causes the corresponding term to vanish. This approach becomes extremely powerful when the Hamiltonian system possesses a sufficient number of analytic constant of motions *independent* and in *involution*, and it plays a crucial role in the Liouville-Arnold theorem.

Definition 2. *Given a symplectic manifold M of dimension $2N$, a* ***momentum map*** *is an analytic mapping $F : M \to \mathbb{R}^N$ whose components $(F^{(m)})_{m=1}^N$ are independent and in involution:*

- *(Independence) The differential dF is surjective on a dense open set;*
- *(In involution) $\{F^{(m)}, F^{(n)}\} = 0$ for all $m, n = 1, \dots, N$.*

Consider an N-degree-of-freedom Hamiltonian system (M, α, H). We say that the system admits a momentum map if there exists a map F as defined above such that the Hamiltonian H Poisson-commutes with each component $F^{(m)}$, namely $\{F^{(m)}, H\} = 0$ for all $m = 1, \dots, N$. In this setting, the components of F are called ***first integrals*** *or, equivalently,* ***constants of motion***. *This nomenclature is justified by the property described in equation* (3).

A Hamiltonian system admitting a momentum map is said to be *completely integrable*. The following celebrated theorem provides the motivation for this terminology.

Theorem 3 (Liouville-Arnold integrability)**.** *Consider a N-degrees of freedom Hamiltonian system with Hamiltonian H and assume that it admits a momentum map F. Let $c \in \mathbb{R}^N$ be such that the level set $F^{-1}(c)$ is non-empty, compact and connected. Then, the following hold:*

(i) $F^{-1}(c)$ is an N-dimensional Lagrangian torus[2] *that is invariant under the Hamiltonian flow ϕ_H^t.*

(ii) There exists an open neighborhood $U \subset \mathbb{R}^N$ of c such that $F^{-1}(U)$ is analytically foliated by the level sets of F, all of which are N-dimensional invariant Lagrangian tori. Specifically, there exists a bi-analytic diffeomorphism χ such that the following diagram commutes:

$$\begin{tikzcd} F^{-1}(U) \arrow[r, "\chi"] \arrow[d, "F"'] & U \times \mathbb{T}^N \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

where p_1 denotes the projection onto the first factor.

(iii) There exist action-angle coordinates (I, θ) adapted to this foliation and such that

$$\alpha = \sum_{k=1}^{N} dI_k \wedge d\theta_k \quad \text{and} \quad H = H(I).$$

As a straightforward consequence, Hamilton's equations simplify to

$$\dot{I}_k = -\frac{\partial H}{\partial \theta_k} = 0, \quad \dot{\theta}_k = \frac{\partial H}{\partial I_k}(I(0)),$$

meaning that the Hamiltonian dynamics is conjugate to a linear flow on the torus $\mathbb{T}^N$.

[2]A Lagrangian submanifold $L \subset M$ of a symplectic manifold is a submanifold such that $\alpha|_L \equiv 0$ and $\dim(L) = \frac{1}{2}\dim(M)$. Namely, any Lagrangian submanifold has *maximal* dimension, which is exactly half the dimension of the manifold.

The proof of Theorem 3 also provides a prescription for constructing this powerful set of action-angle coordinates via the generating function method, provided that a momentum map and a parameterization of its compact fibers are available. In particular, the fact that the level sets are Lagrangian submanifolds allows for the construction of a generating function by integrating the *Liouville* 1*-form* $\sum_{k=1}^{N} p_k dq_k$ along paths lying on the invariant tori. The canonical transformation induced by this generating function maps the original system into a framework where the Hamiltonian depends exclusively on the new momenta, effectively reducing the motion to a linear flow. Explicitly, the *Arnold formula* defines the action variables for a Hamiltonian system with momentum map F as:

$$I_k(c) := \frac{1}{2\pi} \oint_{\sigma_k} p\, dq, \tag{4}$$

where $\{\sigma_k\}_{k=1}^{n}$ is a basis of cycles for the fundamental group of the invariant torus $F^{-1}(c)$.

It is therefore evident that the problem of integrating Hamilton's equations is reduced to the task of identifying a momentum map for the system and characterizing its level sets. To date, no general method exists that is applicable in every context for this purpose. Later, we shall discuss a modern approach widely used in the infinite-dimensional setting, known as the *Lax pair method*.

The significance of the Liouville-Arnold theorem lies in the fact that it provides a decomposition of the phase space into simple invariant objects, each supporting an elementary motion. Indeed, the Hamiltonian dynamics is conjugate to a linear flow on a torus; such a motion, often referred to as a *Kronecker flow*, represents a radical simplification of the problem. Indeed, this reduction allows one to translate the complexity of the evolution into the study of the arithmetic relations between the frequencies $\lambda(I) = \nabla H(I)$, as *Kronecker flows* on finite-dimensional tori satisfy the following properties:

A The (non-resonance) condition that the frequencies are rationally independent is equivalent to topological transitivity, to minimality, and to unique ergodicity (see [Laz93] for a modern exposition).

B Any Kronecker flow is topologically conjugate by an automorphism of the torus to another Kronecker flow having the property that the non-zero frequencies are rationally independent (see, for e.g., [FM10]). Consequently, by Property A, the closure of any orbit is still a torus.

Remark 4. *It is worth remarking on the quotient space structure of the level sets. This structure arises from the fact that the Hamiltonian vector fields* $(V_{F(m)})_{m=1}^{N}$ *associated with the integrals induce an abelian, differentiable, and transitive action of the group* $\mathbb{R}^N$ *on the compact level sets via the joint flow:*

$$\mathbb{R}^N \times F^{-1}(c) \to F^{-1}(c), \quad (t_1, \dots, t_N, (p,q)) \mapsto \Phi^{t_N}_{V_{F(N)}} \circ \cdots \circ \Phi^{t_1}_{V_{F(1)}}(p,q).$$

Consequently, the level sets of the momentum map are diffeomorphic to the quotient of the acting group by the stabilizer subgroup.

The above remark highlights the algebraic nature of this notion of integrability. Indeed, the existence of a momentum map effectively reduces the study of the solutions to that of the orbits of a specific group action. Since the action is abelian and transitive, the stabilizer is necessarily a discrete subgroup (specifically a lattice) which, under the compactness assumption, ensures that the resulting quotient space possesses the topology of a torus. From this perspective, the fact that the Hamiltonian dynamics is conjugate to a linear flow emerges as a direct consequence of compactness and the symmetries encoded by the integrals of motion, which constrain the evolution to such a rigid geometric configuration.

3 Infinite-Dimensional Integrable Systems

In the infinite-dimensional setting, the most effective method for producing an infinite family of first integrals relies on the existence of a *Lax pair* (see the original contribution in [Lax68]). This approach consists in representing the equations of motion in the form

$$\frac{d}{dt}L(t) = [B(t), L(t)],$$

where L and B are operators, self-adjoint and anti-self-adjoint, respectively, acting on an appropriate auxiliary Hilbert space. A fundamental property of this formulation is that the flow induced by the system is isospectral, namely its spectrum remains invariant over time (actually, more is true: the $L(t)$'s are unitary equivalent). Consequently, the spectrum of L provides a natural set of conserved quantities. This framework characterizes a restricted but fundamental class of infinite dimensional integrable Hamiltonian systems. Notable examples of Hamiltonian PDEs admitting a Lax pair include the cubic nonlinear Schrödinger (NLS) in dimension $(1+1)$, the Benjamin-Ono, and the Korteweg-de Vries (KdV) equations, alongside the Camassa-Holm and Degasperis-Procesi equations, and the Szegő equation (see [ZS], [BK79] and [Nak79], [Lax68], [CH93], [DHH02], [GG10]). Beyond the continuous setting, this approach also extends to infinite lattice systems, such as the Toda lattice (see [Fla74]).

For instance, in the case of the Korteweg-de Vries equation in $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$:[3]

$$u_t + u_{xxx} - 6uu_x = 0, \quad x \in \mathbb{R}/\mathbb{Z},$$

a celebrated example of Hamiltonian PDE, the operator L is the second-order differential operator $L = -\partial_x^2 + u$, whose spectrum, known to be composed of bands and gaps, yields an infinite family of integrals of motion. Typically these constants of motion are encoded in the lengths of the spectral gaps, which, together with the mean value of the potential u, completely characterize the spectrum (see [GT84]). A closer inspection of the isospectral sets shows that, for any finite N, they are compact analytic submanifolds homeomorphic to N-dimensional tori, where N corresponds to the number of open gaps. While this topological identification also holds for $N = \infty$ (see [MT76]), such level sets cannot be analytic submanifolds of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ since an infinite-dimensional compact subset of a Banach space can never be a submanifold (see e.g. Appendix C of [PT87]). If we vary the widths of the open gaps $(g_1, \dots, g_N) \in U \subset \mathbb{R}^N$, we obtain a set of functions isomorphic to $U \times \mathbb{T}^N$. This subset is invariant for the dynamics, and the restriction of the KdV system is a finite ($2N$-)dimensional integrable system. In [MT76], it was shown that the squared gap lengths g_j^2 are real analytic functions of u that Poisson commute with each other, representing the meeting point between the Lax pair structure and the Hamiltonian framework.

From a Hamiltonian perspective, the ultimate goal is to determine whether these spectral invariants can be used to construct a bi-analytic symplectomorphism, often referred to as a Birkhoff map, ensuring that, in the new coordinate frame, the Hamiltonian depends exclusively on the conjugate momenta. The strategy can be understood through the lens of algebraic geometry, a perspective that is not explicitly developed in this thesis. We limit ourselves to briefly give the basic idea for the case of the KdV. In [FM76], Flaschka and McLaughlin obtained an explicit formula for the action-angle variables $(I_k, \theta_k)_{k=1}^N$ in a neighbourhood of any invariant torus of dimension N by interpreting the spectral data as coordinates on a Riemann surface whose genus is determined by the number of open spectral gaps. The action variables are then recovered as integrals of the Liouville 1-form along the cycles surrounding these gaps as described by the Arnold formula (4). Consequently, the Hamiltonian dynamics is linearized on the associated Jacobian variety, which serves as the algebraic counterpart to the Liouville tori.

[3]The Hamiltonian structure of the KdV equation is a classic result in the theory of integrable systems; for a brief review, we refer the reader to Chapter 3. For a more detailed and modern overview, see [KP03] and references therein.

In [Kap91] it was proved that in the neighborhood of any finite-dimensional torus associated with an isospectral set with N open gaps, there exists a local fibration into tori of the same dimension N. While this provides a beautiful geometric structure to the results of Flaschka and McLaughlin, the analysis still remains restricted to finite-dimensional invariant submanifolds. Consequently, the codimension of these tori within the Hilbert space $L^2(\mathbb{R}/\mathbb{Z},\mathbb{R})$ is infinite, suggesting that such a description does not capture the full complexity of the infinite-dimensional setting.

Topological obstruction. One might think of repeating the construction of Flaschka and McLaughlin for all N and then taking the limit $N \to \infty$ to cover the whole phase space. However, as pointed out in [HK14], the limit is singular on the dense subset of $L^2(\mathbb{R}/\mathbb{Z},\mathbb{C})$ given by

$$\Game := \{u : g_n(u) = 0 \text{ for some } n \in \mathbb{N}\}.$$

This is actually a consequence of the topological obstruction imposed by the Hilbert norm, which prevents the possibility of foliating an open subset of the phase space with infinite-dimensional tori. To see this, let us identify $L^2(\mathbb{R}/\mathbb{Z},\mathbb{R})$ with $\ell^2(\mathbb{N},\mathbb{C})$ via the Fourier transform $u \mapsto (u_k)_{k\in\mathbb{N}}$. Given $I = (I_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N},\mathbb{R})$ such that $I_k > 0$ for all $k \in \mathbb{N}$, consider the torus[4]

$$\mathcal{T}_I := \{u \in \ell^2(\mathbb{N},\mathbb{C}) : |u_k| = \sqrt{I_k} \text{ for all } k \in \mathbb{N}\}.$$

Finally, note that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that every $u \in \mathcal{T}_I$ is ϵ-close in ℓ^2 to the N-dimensional truncated torus

$$\mathcal{T}_I^N := \{u \in \ell^2(\mathbb{N},\mathbb{C}) : |u_k| = \sqrt{I_k} \text{ for all } k \le N, \text{ and } u_k = 0 \text{ for all } k > N\}$$

leading to the conclusion that any neighborhood of an infinite-dimensional torus contains tori of finite dimension. It is worth stressing that, since each $\mathcal{T}_I^N$ is compact, this approximation property implies that $\mathcal{T}_I$ is a compact subset of the Hilbert space. Consequently, it cannot be an analytic submanifold, as any infinite-dimensional compact subset of a Banach space fails to satisfy the definition of a submanifold (see, for e.g., Appendix C of [PT87]).

The solution for overcoming the topological obstruction, as developed in [BKM96] (see also [KP03]), relies on the introduction of the so-called *Birkhoff coordinates*

$$x_k = \sqrt{2I_k}\cos(\theta_k), \quad y_k = \sqrt{2I_k}\sin(\theta_k),$$

which represent the Cartesian version of the action-angle coordinates. The Birkhoff coordinates are well-defined and analytic on the entire phase space. In these variables, the symplectic form takes the standard form $\omega = \sum_{k\in\mathbb{N}} dx_k \wedge dy_k$, and the Hamiltonian flow is linearized on the invariant tori.

Nevertheless, the foliated structure provided by the Liouville-Arnold theorem is still far from being achieved. This leads to a fundamental question:

is it possible to recover a version of the Liouville-Arnold theorem by weakening the compactness requirement, namely by choosing a different topological setting?

The results presented in this thesis (see Theorems 1.3.2, 2.5.12 and 2.6.1) give a positive answer to this question by proposing a new construction for analytic invariant foliations and action-angle coordinates in a more suitable functional framework.

[4] Note that $\mathcal{T}_I$ is indeed an infinite product of circles, as it is parametrized by $(\theta_k)_{k\in\mathbb{N}}$ where $u_k = \sqrt{I_k}e^{i\theta_k}$.

4 Structure of the Thesis and Summary of Main Results

This section serves as a conceptual roadmap for the research problem and the primary findings of this thesis. The dissertation is organized into four chapters: the first three establish a progressive analytical and Hamiltonian framework, whereas Chapter 4, which investigates Kronecker flows on infinite-dimensional tori, stands as a self-contained study that can be consulted independently of the previous chapters.

Throughout this text, we adopt the following notation: $\mathbb{N}_0$ denotes the set of natural numbers, while $\mathbb{N}$ denotes the set of positive integers $\{1, 2, \dots\}$. The one-dimensional torus is denoted by $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

4.1 Analytic Foliations by Infinite Tori

The first part of this work addresses the challenge of the topological obstruction to the foliation by infinite tori inherent to the L^2 setting. To overcome this issue, we shift the functional framework to a phase space constructed as a generalization of the cotangent bundle of a finite-dimensional torus $T^*\mathbb{T}^n \cong \mathbb{R}^n \times \mathbb{T}^n$. Specifically, we consider as phase space the product $\ell^\infty_w \times \mathbb{T}^\infty$, where ℓ^∞_w is the Banach space of sequences:

$$\ell^\infty_w := \left\{ y = (y_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|y\|_{\ell^\infty_w} := \sup_{k\in\mathbb{N}} |y_k| w_k < \infty \right\},$$

and $w = (w_k)_{k\in\mathbb{N}}$ is a *weight sequence* of positive real numbers. The infinite-dimensional torus

$$\mathbb{T}^\infty := \ell^\infty / 2\pi\mathbb{Z}^{\mathbb{N}}$$

is then endowed with the metric induced by the ℓ^∞-norm.

While the robustness of these foliations is a general property independent of the specific weights w, a more delicate analysis is required when introducing a symplectic structure, as discussed in Subsection 2.3.1. Roughly speaking, to ensure that

$$\sum_{k\in\mathbb{N}} dy_k \wedge d\varphi_k \tag{5}$$

gives well-defined (weak)-symplectic form, the weight sequence must satisfy the summability condition $\mathsf{c}_w := \sum_{j\in\mathbb{N}} w_j^{-1} < \infty$. Geometrically, this requirement ensures that ℓ^∞_w embeds (compactly) into $\ell^1(\mathbb{N}, \mathbb{R})$, which in turn embeds isometrically into the space of continuous functionals acting on ℓ^∞ (the tangent space at any point of $\mathbb{T}^\infty$); in particular, this assumption implies that $\ell^\infty_w \times \mathbb{T}^\infty$ is a sub-bundle of the weak*-cotangent bundle of the infinite torus, which is an infinite-dimensional weak-symplectic manifold when endowed with the weak-symplectic form (5).[5]

Since all the results only require the structure given by the Poisson bracket, we shall see that the condition $\mathsf{c}_w < \infty$ is not strictly required for our analysis in our functional setting.

In this framework, we consider a class of analytic functions representable as Taylor-Fourier series, which are majorant-analytic in the sense of the functional setting presented in Section 1.2.1. The chosen norm for such space of functions allows for effective Cauchy estimates, which are essential for the analytical steps of our construction. In this framework, we prove that the trivial foliation of the phase space induced by the projection onto the first factor, namely

$$p_1 : \ell^\infty_w \times \mathbb{T}^\infty \to \ell^\infty_w, \qquad p_1(y, \varphi) = y,$$

is robust under sufficiently small analytic perturbations. Our first main result (see Theorem 1.3.2 for the precise statement) can be informally stated as follows:

[5] In contrast with the strong-symplectic case, a weak-symplectic form ω on an infinite-dimensional manifold M is not required to be non-degenerate but only weakly-non-degenerate, meaning that the map $\omega^\# : TM \to T^*M$ defined by $\omega^\#(V) = \omega(\cdot, V)$, for all $V \in TM$, is injective but not surjective.

Theorem. *Let $\Gamma : \ell^\infty_w \times \mathbb{T}^\infty \to \ell^\infty_w$ be a sufficiently small real analytic perturbation of the projection onto the first factor. Then its level sets are infinite tori analytically diffeomorphic to $\mathbb{T}^\infty$ and form a real analytic foliation in a neighbourhood of $\{0\} \times \mathbb{T}^\infty$.*

We prove this theorem essentially by constructing a majorant analytic map A such that

$$\Gamma(\gamma - A(\gamma, \varphi), \varphi) = \gamma \qquad \forall \varphi \in \mathbb{T}^\infty. \tag{6}$$

In fact, the statement of Theorem 1.3.2 remains valid for more general functions of the form

$$\Gamma(y, \varphi) = f(y) + P(y, \varphi),$$

where f is analytic with an invertible differential. Indeed, by expanding f in a Taylor series about a point y_0, one can locally reduce Γ to the form $f(y_0) + df(y_0)[y - y_0] + \mathcal{O}(\|y - y_0\|^2_{\ell^\infty_w}) + P(y, \varphi)$. This generalization is consistent with the fact that the constant term $f(y_0)$ merely shifts the codomain of the map Γ. Explicitly, this generalized problem can be solved following the same procedure as in (6), provided one sets $\tilde{\gamma} = y_0 + df(y_0)^{-1}[\gamma - f(y_0)]$ and requires the smallness condition on the redefined perturbation $\tilde{P}(y, \varphi) = df(y_0)^{-1}(\mathcal{O}(\|y - y_0\|^2_{\ell^\infty_w}) + P(y, \varphi))$.

Notably, this particular setting will allow us to recover the invariant tori for a completely integrable Hamiltonian system no longer through a purely algebraic construction, but by describing them analytically as graphs of majorant analytic functions. This establishes the necessary geometric structure for the infinite-dimensional Liouville-Arnold theorem developed in Chapter 2.

4.2 Symplectic Straightening and the Liouville-Arnold Theorem

The second part of this thesis, developed in Chapter 2, extends the previous analytical construction to a Hamiltonian setting. In the classical finite-dimensional framework, the Liouville-Arnold theorem ensures the existence of a symplectomorphism that straightens the foliation induced by a momentum map; however, in our infinite-dimensional setting, the transition from a merely analytic foliation to a Lagrangian one is far from trivial.

To bridge this gap, we first establish a consistent theory of analytic vector fields and Poisson structures on the Banach manifold $\ell^\infty_w \times \mathbb{T}^\infty$. The primary objective is to achieve the *symplectic straightening* of a foliation induced by a momentum map belonging to a specific regularity class.

Notation 1. *For simplicity of exposition, throughout this introduction we shall denote by $\mathcal{X}$ the class of analytic maps $\Gamma : \ell^\infty_w \times \mathbb{T}^\infty \to \ell^\infty_w$ whose Hamiltonian vector fields associated with its component functions are analytic maps from the phase space to its tangent space, and such that the adjoint of its differential is bounded. For more details, see Definition 2.3.8, Remark 2.3.9 and Proposition 2.3.10.*

Notably, this condition mirrors the "summability assumption" on vector fields adopted by Bambusi and Stolovitch in [BS20], as well as the boundedness requirement on the adjoint of the differential of normally analytic germs considered by Kuksin and Perelman in their work on the infinite-dimensional Vey theorem [KP10].

Within this framework, we prove that for any momentum map $\Gamma \in \mathcal{X}$ satisfying a suitable smallness assumption, one can construct a change of coordinates that is not only a bi-analytic diffeomorphism but also a *symplectomorphism*. This map straightens the perturbed foliation back to the trivial one while preserving the underlying Poisson structure. This result, proved in Subsection 2.5.2 (Theorem 2.5.12), can be heuristically outlined as follows:

Let $\Gamma \in \mathcal{X}$ be a sufficiently small perturbation of the projection onto the first factor. If the components of Γ Poisson-commute, there exists a symplectomorphism that "straightens" the leaves of the foliation *given by the level sets of* Γ *and defines a set of action-angle coordinates in a neighbourhood of* $\{0\} \times \mathbb{T}^\infty$.

This symplectic straightening provides the necessary geometric foundation for a local version of the Liouville-Arnold theorem in infinite dimension. Indeed, once the momentum map is reduced to its canonical form, the integration of any Hamiltonian H that commutes with the components of Γ becomes straightforward. In these straightened coordinates, the complex dynamics of the original system is conjugate to a Kronecker flow. This result, presented here in an informal version, constitutes, to the best of our knowledge, the first instance of a local Liouville-Arnold theorem in an infinite-dimensional setting (see Theorem 2.6.1 for the correct statement and proof).

Theorem. *Consider a majorant analytic Hamiltonian* $H : \ell^\infty_w \times \mathbb{T}^\infty \to \mathbb{R}$ *that commutes with the components of a momentum map* $\Gamma \in \mathcal{X}$ *which is a sufficiently small perturbation of the projection onto the first factor. Then, the system is completely integrable in a neighborhood of* $\{0\} \times \mathbb{T}^\infty$. *Specifically:*

(i) the Hamiltonian H *generates a well-defined flow;*

(ii) the phase space is foliated by invariant analytic tori, which coincide with the level sets of the momentum map;

(iii) there exists a symplectomorphism that gives action-angle coordinates such that the dynamics is reduced to a linear flow on $\mathbb{T}^\infty$.

These results find a natural application in the context of Hamiltonian systems on infinite lattices and PDEs, which we shall now briefly discuss.

4.3 Application to Hamiltonian sytems on infinite lattices and PDEs

A fundamental question in the study of Hamiltonian systems on infinite lattices and Hamiltonian PDEs is under what conditions a family of first integrals ensures the integrability of the system. In the finite-dimensional analytic setting, a powerful, local counterpart to the Arnold-Liouville theorem is provided by Vey's Theorem on the convergence of the Birkhoff normal form.

The extension of these ideas to the infinite-dimensional setting has been the subject of significant research, notably in the works of Kuksin and Perelman [KP10] and Bambusi and Stolovitch [BS20]. In both these frameworks, as well as in the setting adopted in this thesis, a key requirement is the existence of a family of first integrals possessing sufficient regularity to induce the necessary coordinate transformations.

Before stating our result, let us give some definitions. Given a sequence of positive real numbers $(\eta_j)_{j\in\mathbb{N}}$, we consider as a phase space the complex weighted Banach space $\ell^\infty_\eta(\mathbb{C})$ defined by

$$\ell^\infty_\eta(\mathbb{C}) := \left\{ u \in \mathbb{C}^\mathbb{N} : \|u\|_{\ell^\infty_\eta(\mathbb{C})} := \sup_{j\in\mathbb{N}} \eta_j |u_j| < \infty \right\}.$$

Note that $\ell^\infty_\eta(\mathbb{C})$ can be embedded into $\ell^2(\mathbb{C})$ if the weight η satisfies the condition that $\mathsf{c}_\eta := \sum_{j\in\mathbb{N}} \eta_j^{-2}$ is finite. From this embedding, one can inherit the standard symplectic form of $\ell^2(\mathbb{C})$, namely

$$\omega := \mathrm{i} \sum_{j\in\mathbb{N}} du_j \wedge d\bar{u}_j,$$

by pulling it back onto $\ell^\infty_\eta(\mathbb{C})$. Finally, let $\xi = (\xi_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that

$$0 < \inf_{k\in\mathbb{N}} \eta_k^2 \xi_k \le \sup_{k\in\mathbb{N}} \eta_k^2 \xi_k < \infty.$$

It then follows that the set

$$\mathcal{T}_\xi := \{u \in \ell^\infty_\eta(\mathbb{C}) : |u_k|^2 = \xi_k \text{ for all } k \in \mathbb{N}\}$$

is analytically diffeomorphic to $\mathbb{T}^\infty$ via the map $\mathbb{T}^\infty \ni \varphi \mapsto (\sqrt{\xi_k} e^{\mathrm{i}\varphi_k})_{k\in\mathbb{N}}$.

We are now ready to informally state our theorem on the complete integrability for systems on infinite lattices (see Theorem 3.1.4 for the precise statement). This result should be compared with [BS20] and [KP10].

Theorem. *Consider a majorant real analytic Hamiltonian $H : \ell^\infty_\eta(\mathbb{C}) \to \mathbb{R}$ that commutes with the components of a majorant analytic momentum map $\Gamma^{(m)}(u) = |u_m|^2 + P^{(m)}(u)$, where the perturbation P is such that the quantity*

$$\sup_{k\in\mathbb{N}} \eta_k \sum_{m=1}^{\infty} \left\| \frac{\partial P^{(m)}}{\partial u_k} \right\|_R$$

is sufficiently small. Here, $\|\cdot\|_R$ denotes a majorant analytic norm which will be properly defined in Section 3. Then, for a suitable range of values ξ, the system is completely integrable in a neighborhood of the torus $\mathcal{T}_\xi$. Specifically:

(i) the Hamiltonian H generates a well-defined flow in a neighborhood of the torus;

(ii) the phase space is locally foliated by invariant analytic manifolds, each diffeomorphic to the infinite torus $\mathbb{T}^\infty$;

(iii) there exists a symplectomorphism providing action-angle coordinates (x, θ) that reduce the dynamics to a linear flow on $\mathbb{T}^\infty$.

In Section 3.2, we apply Theorem 2.6.1 to show that the Korteweg-de Vries (KdV) equation can be viewed as a completely integrable infinite-dimensional Hamiltonian system in the sense of Liouville-Arnold. This statement is proved in Theorem 3.2.7. By the nature of our results, this integrability is established in a small annulus surrounding the origin. Unlike the global results obtained through the Birkhoff map in $L^2(\mathbb{T})$, our approach provides a local decomposition of the phase space that mirrors the geometry provided by the Liouville-Arnold theorem.

It is important to emphasize that a foliation into infinite-dimensional invariant tori for the KdV equation could also be obtained by exploiting the fact that the Birkhoff map is the identity plus a 1-smoothing operator [BKM96], combined with standard Sobolev embeddings. In contrast, our approach proceeds by verifying whether the spectral gaps of the Lax operator associated with the KdV system satisfies the specific regularity assumptions required by our general theorems.

4.4 Kronecker flow on the infinite torus

Initial investigations on Kronecker flow on infinite tori can be traced back to Jessen [Jes34], who established a specific instance of Birkhoff's Ergodic Theorem. Further analysis of the ergodic properties of Kronecker flows on infinite tori, equipped with the product topology, was conducted in [Koz21] and [SV24]. In the latter, Sakbaev and Volovich [SV24] demonstrated the emergence of a novel class of trajectories that are absent in the finite-dimensional case, namely, orbits whose closure fails to be a torus. These findings indicate that the typical dynamical behavior of finite-dimensional flows does not necessarily apply to the case of Kronecker flows on infinite tori.

Let us give the basic definitions. In the following discussion, the infinite torus is endowed with the product topology and it will be denoted by $\mathbb{T}^\mathbb{N}$. Some comments on the concrete realizations of $\mathbb{T}^\mathbb{N}$ are given in Appendix D.

Given $\omega = (\omega_j)_{j\in J} \in \mathbb{R}^J$, we define the Kronecker flow associated with the "frequency vector" ω as the topological dynamical system $(\mathbb{T}^J, \Phi_\omega)$ induced by the continuous $\mathbb{R}$-action

$$\Phi_\omega : \mathbb{R} \times \mathbb{T}^J \to \mathbb{T}^J \ , \quad (t, \theta) \mapsto \Phi^t_\omega(\theta) := (\Theta_j + \omega_j t + 2\pi\mathbb{Z})_{j\in J},$$

where, given $\theta \in \mathbb{T}^J$, $\Theta := (\Theta_j)_{j\in J}$ denotes any list of real numbers such that $\theta = (\Theta_j + 2\pi\mathbb{Z})_{j\in J}$.[6] The real numbers ω_j are called "frequencies".

1. *The Main Results.* In Section 4.3, we show that for non-resonant Kronecker flows on $\mathbb{T}^\mathbb{N}$, the same properties of the finite-dimensional analogue hold. More precisely, the first purpose of Chapter 4 is to give a proof of the following assertion.

Theorem A. *For a Kronecker flow* $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$*, the following statements are equivalent:*

(a) it is non-resonant;

(b) it is uniquely ergodic;

(c) it is minimal;

(d) it is topologically transitive.

This theorem should be compared with [Jes34], where the author essentially proves that the average of any integrable function along the orbits of a non-resonant Kronecker flow on the infinite torus equals the average over the entire torus with respect to an appropriate invariant integral. The ergodic properties of Kronecker flows on infinite-dimensional tori are also studied in [Koz21] and [SV24].

It is worth stressing that non-resonant Kronecker flows are in a certain sense "typical". This was first observed by Bourgain [Bou05] who proved that, for all $\delta > 0$, the set

$$\Omega_\delta := \left\{ \omega \in [-1,1]^\mathbb{N} \ : \ \sum_{j\in\mathbb{N}} \omega_j \nu_j \geq \delta \prod_{j\in\mathbb{N}} (1 + \nu_j^2 j^4)^{-1}, \ \forall \nu \in \mathbb{Z}^{(\mathbb{N})} \right\}$$

has measure $\mathfrak{m}(\Omega_\delta) = 1 - O(\delta)$, where $\mathfrak{m}$ is the product Lebesgue measure on $[-1,1]^\mathbb{N}$.

Next we consider the resonant case. In the infinite-dimensional setting, Property B does not hold in general: in the above mentioned paper [SV24], Sakbaev and Volovich give examples of linear flows on the infinite-dimensional torus where the closures of the orbits are not tori, even though these orbits are accumulated by periodic ones. However, a legitimate question to ask is whether a Kronecker flow on $\mathbb{T}^\mathbb{N}$ is topologically conjugate to another Kronecker flow whose non-vanishing frequencies are rationally independent. In Section 4.2 we present an algorithm to reduce resonant Kronecker flows on the infinite torus in the presence of finitely many independent resonances. More precisely, in Proposition 4.2.23 we show that for a Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ where the resonance module $\mathscr{R}_\omega$ has finite dimension $1 \leq d < \infty$, the system is topologically conjugate to another Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_{\tilde\omega})$ characterized by a frequency vector $\tilde\omega = (0_d, \bar\omega)$ where 0_d is the null vector in $\mathbb{R}^d$ and $\bar\omega$ is non-resonant. Aiming to fill the gap left by the assumption of a finite-dimensional resonance module, we establish the following more general result, which connects an algebraic property of the frequency vector with the $\mathscr{C}^0$-conjugacy class to which the resonant Kronecker flow belongs (see Section 4.4.1 for the proof).

[6] Given an element g of an additive abelian group G and a subgroup H of G, we denote by $g + H$ the equivalence class of g with respect to the relation $g \sim g'$ if and only if $g - g' \in H$, that is, $g + H := \{g' \in G \ : \ \exists h \in H \text{ such that } g = g' + h\} \in G/H$.

Theorem B. *Let $\omega \in \mathbb{R}^{\mathbb{N}}$ and let $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ be the corresponding Kronecker flow. Then, $\mathscr{M}_\omega$ is a free abelian group if and only if $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ is topologically conjugate to another Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ whose non-vanishing frequencies are rationally independent. If this is the case, $\mathscr{M}_\omega$ is isomorphic to $\mathbb{Z}^{(supp(\tilde{\omega}))}$.*

In particular, if the frequency module is free, $\mathbb{T}^{\mathbb{N}}$ is fibered by invariant, minimal embedded tori such that the projection of the flow on each fiber is conjugate to a non-resonant Kronecker flow. This implies that the closure of any orbit is still a torus, of possibly infinite dimension.

Note that Property B of the finite-dimensional case follows from the fact that, as already pointed out in Remark 4.2.11, if the frequency vector has finite support, then $\mathscr{M}_\omega$ is finitely generated, and hence, by the structure theorem for finitely generated abelian groups without torsion, it is free.

Furthermore, in the presence of finitely many resonances, Proposition 4.2.23 provides a conjugacy to a flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ whose non-vanishing frequencies are rationally independent. By Theorem B, this ensures that $\mathscr{M}_\omega$ must be a free abelian group, even though it is necessarily no longer finitely generated.

Nevertheless, in the general case, linear dynamics on an infinite torus can be much more complex. In Theorem 4.4.7 of Section 4.4.2, we construct a special class of Kronecker flows with the property that the closure of any orbit is a solenoid (see Definition 4.4.6), which is the total spaces of a particular fiber bundle over the circle with a Cantor set as fiber. By Lemma 4.4.24, the frequency module of any Kronecker flow in such class is isomorphic to a non-free subgroup of $(\mathbb{Q}, +)$.

Next, we show that the closure of any orbit induced by a Kronecker flow on the infinite torus is homeomorphic to the Pontryagin dual of the associated frequency module (Proposition 4.4.23). At the end of Section 4.4.3, we use this result, together with Theorem 4.4.7 and classical results from Pontryagin's theory of locally compact abelian groups, to prove the following statement.

Theorem C. *Let I be a countable set of indices and $(G_i)_{i\in I}$ a family of subgroups of $(\mathbb{Q}, +)$. Then, there exists a Kronecker flow whose frequency module is isomorphic to $\bigoplus_{i\in I} G_i$. Moreover, the closure of any orbit is homeomorphic to a product $\prod_{i\in I} X_i$, where each factor X_i is a circle if G_i is free, and a solenoid otherwise.*

This is an extension of the recent result in [SV24] concerning the existence of Kronecker flows on the infinite torus inducing orbits whose closure is not a torus (see Remark 4.4.26), as we exhibit a well-defined class and we provide a more refined understanding of the topological structure.

In Section 4.4.4 we apply these results to the Benjamin-Ono equation, namely, we prove the following

Theorem BO. *Consider the Benjamin-Ono equation for a real-valued function $u(t, x)$ that is 2π-periodic in space:*

$$u_t = \mathrm{H}(u_{xx}) - 2uu_x, \quad x \in \mathbb{T},\ t \in \mathbb{R},$$

where $\mathrm{H}(\cdot)$ denotes the Hilbert transform defined in (4.2). *Let*

$$\Psi : L^2_0(\mathbb{T}) \ni u \mapsto z \in h^{1/2}$$

be its Birkhoff map [GK21]. Given $z \in h^{1/2}$, if there exists $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $|z_j|^2 \in \beta\mathbb{Q} \setminus \{0\}$ for all $j \in \mathbb{N}$, then the closure of the orbit with initial data $u = \Psi^{-1}(z)$ under the Benjamin-Ono flow is homeomorphic to the product of a circle and a solenoid.

It would be of interest to determine whether this family of almost-periodic solutions corresponds to specific physical features, such as a novel form of turbulence.

We conclude this work by discussing a general classification problem for Kronecker flows on the infinite torus. First, we prove the following statement.

Theorem D. *Given $\omega, \tilde{\omega} \in \mathbb{R}^{\mathbb{N}}$, let $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ and $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ be two Kronecker flows on the infinite torus. The closure of any orbit of Φ_ω is homeomorphic to the closure of any orbit of $\Phi_{\tilde{\omega}}$ if and only if the associated frequency modules $\mathscr{M}_\omega$ and $\mathscr{M}_{\tilde{\omega}}$ are isomorphic as abelian groups.*

Finally, we show how to associate to each countable abelian group G without torsion a Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ in such a way that G is isomorphic to $\mathscr{M}_\omega$. Therefore, by Remark 4.2.10, Theorem D implies that completely classifying Kronecker flows on $\mathbb{T}^{\mathbb{N}}$ up to homeomorphism of the closure their orbits is a highly complex issue, as it is equivalent to the classification problem for countable abelian groups without torsion. To date, a complete classification is known only for those groups having rank[7] 1, namely, additive subgroups of the rationals (see [Bae37], [BZ51] and [Fuc15]) which corresponds to the class of Kronecker flows (constructed in Section 4.4.3) whose orbits are dense in a product of circles and solenoids, in the sense of Theorem C.

[7] See Appendix E for the definition of rank and other classical results from the theory of abelian groups.

Chapter 1

Foliations by Infinite Tori

The main concern of this chapter is the study of the robustness of foliations of Banach manifolds by infinite tori under real analytic perturbations. To give an idea of what we are going to do, let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and consider the cotangent bundle of the n-dimensional torus $\mathbb{T}^n$, $n \in \mathbb{N}$, which is naturally isomorphic to $\mathbb{R}^n \times \mathbb{T}^n$. The projection onto the first factor

$$\Gamma_0 : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \ , \quad \Gamma_0(y, \varphi) = y \ , \quad y \in \mathbb{R}^n \ , \ \varphi \in \mathbb{T}^n \ ,$$

gives rise to a real analytic foliation with leaves $\Gamma_0^{-1}(\gamma) \cong \mathbb{T}^n$, $\forall \gamma \in \mathbb{R}^n$. A real analytic perturbation of the map Γ_0 is a function Γ_P of the form

$$\Gamma_P(y, \varphi) = y + P(y, \varphi) \ ,$$

where P is real analytic on an open neighbourhood $\mathscr{U}_r(\mathbb{R}^n) \times \mathbb{T}^n$ of $\{0\} \times \mathbb{T}^n$ and "small" with respect to some suitable norm. Then one can ask the following question:

Do the level sets of Γ_P give rise to a real analytic foliation whose leaves are n-dimensional tori for an open set of "sufficiently small" values of Γ_P?

In this chapter (see Theorem 1.3.2) we give a positive answer to an infinite-dimensional version of this question. Let us briefly illustrate our procedure in this finite-dimensional context. The idea is the following: since the perturbation is small, we look for a real analytic map

$$A : \mathscr{U}_{r-\rho}(\mathbb{R}^n) \times \mathbb{T}^n \to \mathscr{U}_\rho(\mathbb{R}^n)$$

such that, for all $\gamma \in \mathscr{U}_{r-\rho}(\mathbb{R}^n)$, the equation for the level set $\Gamma_P(y, \varphi) = \gamma$ can be solved by parametrizing y as $y = \gamma - A(\gamma, \varphi)$, namely

$$\Gamma_P(\gamma - A(\gamma, \varphi), \varphi) = \gamma \qquad \forall \varphi \in \mathbb{T}^n \ .$$

By using the definition of Γ_P, the above equation is equivalent to

$$A(\gamma, \varphi) = P(\gamma - A(\gamma, \varphi), \varphi) \ ,$$

which has a (unique) solution if the map $A \mapsto P \circ \Phi_{-A}$, where

$$\Phi_{-A} : \mathscr{U}_{r-\rho}(\mathbb{R}^n) \times \mathbb{T}^n \to \mathscr{U}_r(\mathbb{R}^n) \times \mathbb{T}^n \ , \quad \Phi_{-A}(\gamma, \varphi) = (\gamma - A(\gamma, \varphi), \varphi) \ ,$$

is a contraction on a closed ball about the origin of the space of real analytic functions from an open neighbourhood of $\{0\} \times \mathbb{T}^n$ in the cotangent bundle of the torus to $\mathbb{R}^n$. It is then clear that we shall need to understand the behaviour of the chosen norm under the operation of composition and to give suitable estimates on the derivatives of the involved maps in order to produce a contraction. If this is the case, then for all $\gamma \in \mathscr{U}_{r-\rho}(\mathbb{R}^n)$, the level set $\Gamma_P^{-1}(\gamma)$ is identified with the graph of the real analytic function[1]

$$\mathbb{T}^n \ni \varphi \mapsto \gamma - A(\gamma, \varphi) \in \mathscr{U}_r(\mathbb{R}^n) \, ,$$

namely it is analytically diffeomorphic to $\mathbb{T}^n$. Since such a function depends analytically on the parameter γ, we get that the neighbourhood of $\{0\} \times \mathbb{T}^n$ given by the image of $\mathscr{U}_{r-\rho}(\mathbb{R}^n) \times \mathbb{T}^n$ under the map Φ_{-A} is analytically foliated by the level sets of Γ_P, which are all analytically diffeomorphic to $\mathbb{T}^n$.

In order to apply this general scheme in an infinite-dimensional context, we must first specify the setting. Unlike the finite-dimensional case, the choice of the topology for the product space, as well as the selection of the functional space, is crucial. In particular, we shall prefer a topology that allows us to perform differential calculus and to deal with functions in normed spaces in which it is possible to provide suitable Cauchy estimates for the differentials. Let us start by giving some basic definitions to introduce our infinite-dimensional setting.

1. Infinite torus. We denote by $\mathbb{T}^\infty$ the infinite torus, i.e., the countably infinite direct product of circles

$$\mathbb{T}^\infty := \prod_{j \in \mathbb{N}} \mathbb{T}_j \, , \quad \mathbb{T}_j = \mathbb{T} \ \ \forall j \in \mathbb{N} \, ,$$

endowed with the topology induced by the uniform distance

$$d_{\mathbb{T}^\infty}(\varphi, \varphi') := \sup_{j \in \mathbb{N}} d(\varphi_j, \varphi'_j) \, , \qquad \forall \, \varphi = (\varphi_j)_{j \in \mathbb{N}} \, , \ \varphi' = (\varphi'_j)_{j \in \mathbb{N}} \in \mathbb{T}^\infty,$$

where d is the usual distance on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We shall see in the next section that such a metric space has a natural structure of Banach manifold modelled on ℓ^∞.

We emphasize that there are other possible infinite-dimensional extensions that one could consider. For example, if one equips the infinite product of circles with the product topology, there would be no way to introduce a topological manifold structure on the resulting infinite torus. Roughly speaking, the cylinders form a basis for the product topology, and non-contractible loops can be found in each cylinder. This contrasts with the local simple connectedness required of any topological manifold.

2. ℓ^∞-weighted space. Given a sequence of positive real numbers $w = (w_j)_{j \in \mathbb{N}}$, we define the real weighted (by the weight w) Banach space ℓ^∞_w based on $\ell^\infty := \ell^\infty(\mathbb{N}, \mathbb{R})$ by setting

$$\ell^\infty_w := \left\{ y \in \mathbb{R}^{\mathbb{N}} \ : \ |y|_{\ell^\infty_w} := \sup_{j \in \mathbb{N}} w_j |y_j| < \infty \right\} .$$

Clearly, $\ell^\infty_w = \ell^\infty$ if $w_j = 1$ for all $j \in \mathbb{N}$. In the following, we shall denote by $\ell^\infty_w(\mathbb{C})$ the complex analogue of ℓ^∞_w.

Since we shall need to consider holomorphic extensions of real analytic functions on ℓ^∞, given $s > 0$, let us define $\ell^\infty_s(\mathbb{C})$ as the open subset of the complex Banach space $\ell^\infty(\mathbb{C}) := \ell^\infty(\mathbb{N}, \mathbb{C})$ given by

$$\ell^\infty_s(\mathbb{C}) := \left\{ \varphi = (\varphi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ : \ |\varphi|_{\ell^\infty(\mathbb{C})} := \sup_{j \in \mathbb{N}} |\varphi_j| < \infty, \ \sup_{j \in \mathbb{N}} |\mathrm{Im}\, \varphi_j| < s \right\} .$$

This chapter is devoted to the formulation and proof (see Theorem 1.3.2) of a theorem that can be informally stated as follows:

[1] Geometrically, it represents a one-form on $\mathbb{T}^n$.

If $\Gamma : \ell^\infty_w \times \mathbb{T}^\infty \to \ell^\infty_w$ is a sufficiently small real analytic perturbation of the projection onto the first factor, its level sets are infinite tori and form a real analytic foliation.

In the next sections we describe the Banach manifold structure of the infinite torus and introduce a class of real analytic functions on $\ell^\infty_w \times \mathbb{T}^\infty$ for which we provide Cauchy estimates (see Lemma 1.2.13). In Proposition 1.2.19, we give estimates for the composition of functions in this class with close-to-identity analytic maps $\ell^\infty_w \times \mathbb{T}^\infty \to \ell^\infty_w \times \mathbb{T}^\infty$, and we apply these results to construct a contraction, as explained before.

For the sake of clarity, we stress that all the results presented in this chapter hold for all choices of the weight w.

1.1 Quotient Space and Banach Manifold Structures of $\mathbb{T}^\infty$

Here we explore different ways to view the infinite torus defined in the previous section as a product space. In particular, we will examine how $\mathbb{T}^\infty$ can be understood as:

- a **quotient space**, offering a way to understand $\mathbb{T}^\infty$ through equivalence classes and the identification of certain elements;
- a **Banach manifold**, providing a smooth and geometrically rich framework that ties $\mathbb{T}^\infty$ to the world of functional analysis and geometry.

Each of these views not only broadens our understanding of the infinite torus but also opens up different techniques and tools that can be applied to study more effectively analytic functions defined on it.

Definition 1.1.1. *Let us denote by Λ the submodule $\Lambda := (2\pi\mathbb{Z})^{\mathbb{N}} \cap \ell^\infty$. Hereafter, π_∞ denotes the surjective map $\pi_\infty : \ell^\infty \to \ell^\infty/\Lambda$ defined as*

$$\pi_\infty(\hat\varphi) = [\hat\varphi] := \{\hat\varphi + M : M = (M_j)_{j\in\mathbb{N}} \in \Lambda\}, \quad \text{for all } \hat\varphi = (\hat\varphi_j)_{j\in\mathbb{N}} \in \ell^\infty.$$

We endow ℓ^∞/Λ with the metric $\mathtt{d}$ given by

$$\mathtt{d}([\hat\varphi],[\hat\varphi']) := \inf_{\hat\varphi\in[\hat\varphi],\hat\varphi'\in[\hat\varphi']} |\hat\varphi - \hat\varphi'|_{\ell^\infty} = \min_{M\in\Lambda} \sup_{j\in\mathbb{N}} |\hat\varphi_j - \hat\varphi'_j + M_j|.$$

Remark 1.1.2. *$\pi_\infty : \ell^\infty \to \ell^\infty/\Lambda$ is the universal covering of ℓ^∞/Λ.*

The following proposition gives a characterization of the infinite torus as a quotient space. In particular, $\mathbb{T}^\infty$ is isometric to the quotient of ℓ^∞ with respect to the submodule of all bounded sequences of real numbers that are integer multiples of 2π.

Proposition 1.1.3. *The metric spaces ℓ^∞/Λ and $\mathbb{T}^\infty$, endowed with the metrics $\mathtt{d}$ and $d_{\mathbb{T}^\infty}$, respectively, are isometric.*

Proof. It is straightforward to see that the map

$$\Phi : \ell^\infty/\Lambda \to \mathbb{T}^\infty, \quad \Phi([\hat\varphi]) = (\varphi_1, \ldots, \varphi_j, \ldots), \tag{1.1}$$

where $\varphi_j := \{\hat\varphi_j + m : m \in 2\pi\mathbb{Z}\}$, is a bijection. To prove that Φ is an isometry, we show that for any $\hat\varphi, \hat\varphi' \in \ell^\infty$,

$$\min_{M\in\Lambda} \sup_{j\in\mathbb{N}} |\hat\varphi_j - \hat\varphi'_j + M_j| = \sup_{j\in\mathbb{N}} \min_{m\in 2\pi\mathbb{Z}} |\hat\varphi_j - \hat\varphi'_j + m|.$$

For any fixed $j \in \mathbb{N}$, the set $\{|\hat{\varphi}_j - \hat{\varphi}'_j + m| : m \in 2\pi\mathbb{Z}\}$ is discrete and bounded from below; hence, there exists $m_j \in 2\pi\mathbb{Z}$ such that

$$|\hat{\varphi}_j - \hat{\varphi}'_j + m_j| = \min_{m \in 2\pi\mathbb{Z}} |\hat{\varphi}_j - \hat{\varphi}'_j + m|.$$

It follows that

$$\min_{M \in \Lambda} \sup_{j \in \mathbb{N}} |\hat{\varphi}_j - \hat{\varphi}'_j + M_j| \leq \sup_{j \in \mathbb{N}} |\hat{\varphi}_j - \hat{\varphi}'_j + m_j| = \sup_{j \in \mathbb{N}} \min_{m \in 2\pi\mathbb{Z}} |\hat{\varphi}_j - \hat{\varphi}'_j + m|.$$

Conversely, for any $M \in \Lambda$, the following inequality holds for each $j \in \mathbb{N}$:

$$|\hat{\varphi}_j - \hat{\varphi}'_j + M_j| \geq \min_{m \in 2\pi\mathbb{Z}} |\hat{\varphi}_j - \hat{\varphi}'_j + m|.$$

Taking the supremum over $j \in \mathbb{N}$ and subsequently the minimum over $M \in \Lambda$, we obtain the reverse inequality, which establishes that Φ is an isometry. ■

We now turn to the issue of showing that $\mathbb{T}^\infty$ can be endowed with a natural structure of analytic Banach manifold modelled on ℓ^∞. For the basic theory of differentiable and analytic functions on Banach spaces and manifolds, we refer to [PT87], Appendices A, B, and C, and references therein.

For all $\sigma = (\sigma_j)_{j \in \mathbb{N}} \in \{0, 1, 2\}^{\mathbb{N}}$, let φ^σ denote

$$\varphi^\sigma := (\{2\sigma_j \pi/3 + m : m \in 2\pi\mathbb{Z}\})_{j \in \mathbb{N}} \in \mathbb{T}^\infty ,$$

and let $\mathscr{U}_\sigma$ be the open set

$$\mathscr{U}_\sigma := \{\varphi \in \mathbb{T}^\infty : d_{\mathbb{T}^\infty}(\varphi, \varphi^\sigma) < 2\pi/3\} .$$

Note that $(\mathscr{U}_\sigma)_{\sigma \in \{0,1,2\}^{\mathbb{N}}}$ is an open covering of $\mathbb{T}^\infty$. Let

$$\Phi_\sigma : \mathscr{U}_\sigma \to \prod_{j \in \mathbb{N}} \Big(-\frac{2\pi}{3}(1 + \sigma_j), \frac{2\pi}{3}(1 + \sigma_j) \Big) \subset \ell^\infty$$

be the map that associates to any $\varphi = (\varphi_j)_{j \in \mathbb{N}} \in \mathscr{U}_\sigma$ the list $(\hat{\varphi}_j)_{j \in \mathbb{N}}$, where, for all $j \in \mathbb{N}$, $\hat{\varphi}_j$ is the unique representative of φ_j in

$$\Big(-\frac{2\pi}{3}(1 + \sigma_j), \frac{2\pi}{3}(1 + \sigma_j) \Big) \subset \mathbb{R} .$$

Then $(\mathscr{U}_\sigma, \Phi_\sigma)_{\sigma \in \{0,1,2\}^{\mathbb{N}}}$ is an analytic atlas for $\mathbb{T}^\infty$. Indeed, the transition functions

$$\Psi_{\sigma,\sigma'} : \Phi_\sigma(\mathscr{U}_\sigma \cap \mathscr{U}_{\sigma'}) \to \Phi_{\sigma'}(\mathscr{U}_\sigma \cap \mathscr{U}_{\sigma'})$$

are all equal to the identity map $\mathrm{Id} : \ell^\infty \to \ell^\infty$, which is clearly analytic.

The same argument holds verbatim if one replaces $\mathbb{T}^\infty$ with ℓ^∞/Λ: write $[\hat{\varphi}^\sigma]$, where $\hat{\varphi}^\sigma = (2\sigma_j/3)_{j \in \mathbb{N}}$, instead of φ^σ and $\mathtt{d}$ in place of $d_{\mathbb{T}^\infty}$. It follows that

$$(\Phi^{-1}(\mathscr{U}_\sigma), \Phi_\sigma \circ \Phi)_{\sigma \in \{0,1,2\}^{\mathbb{N}}} ,$$

where Φ is defined in (1.1), is an analytic atlas for ℓ^∞/Λ. Then we have the following:

Remark 1.1.4. *The infinite torus $\mathbb{T}^\infty$ and ℓ^∞/Λ are also analytically diffeomorphic as Banach manifolds.*

Remark 1.1.5. *Noting that the covering map π_∞ is analytic, the latter remark together with Remark 1.1.2 allows us to identify real analytic functions on $\mathbb{T}^\infty$ with real analytic, multi-periodic functions defined on the universal covering ℓ^∞ of ℓ^∞/Λ.*

1.2 Functional Setting and Preliminaries

Throughout this chapter, we fix a weight w, namely a sequence of positive real numbers $w = (w_j)_{j\in\mathbb{N}}$, once and for all. In this section, we introduce a class of real analytic functions on $\ell^\infty_w \times \mathbb{T}^\infty$ and discuss Cauchy estimates and their behaviour under the operation of composition.

1.2.1 Space of functions

We restrict to analytic functions on infinite tori which are represented by trigonometric series on the universal covering ℓ^∞ of $\mathbb{T}^\infty$, in the sense explained in the previous section. In the following, we shall adopt the notation φ not to denote elements of $\mathbb{T}^\infty$, but instead to refer to elements of the universal covering ℓ^∞.

Definition 1.2.1 (The space of formal series $\mathcal{A}$)**.** *For any $r, s > 0$, we denote by $\mathcal{A}_{r,s}$ the space of formal power-trigonometric series f*

$$f(y,\varphi) = \sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}} \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} f_{\nu\alpha} y^\alpha e^{\mathrm{i}\nu\cdot\varphi} \ , \quad y \in \mathscr{U}_r(\ell^\infty_w)\ , \varphi \in \ell^\infty \ , \tag{1.2}$$

with $f_{\nu\alpha} \in \mathbb{C}$, $f_{\nu\alpha} = \bar{f}_{-\nu\alpha}$ for all $(\nu,\alpha) \in \mathbb{Z}^{(\mathbb{N})} \times \mathbb{N}_0^{(\mathbb{N})}$, such that

$$|f|_{r,s} := \sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}} \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |f_{\nu\alpha}| v^\alpha_{w,r} e^{s|\nu|} < \infty \ ,$$

where $\forall r > 0$, $v_{w,r} \in \ell^\infty_w$ denotes the sequence defined by $(v_{w,r})_j = r w_j^{-1}$ for all $j \in \mathbb{N}$. We denote by $\mathcal{A}$ the space of formal power-trigonometric series defined by

$$\mathcal{A} := \bigcup_{r,s>0} \mathcal{A}_{r,s} \ .$$

We shall now show that the formal series in $\mathcal{A}$ are in fact totally convergent and define functions that can be uniformly approximated by analytic functions of finitely many variables. To state it more precisely, let us give the definition of exhaustion of a set of indices by finite subsets.

Definition 1.2.2. *An exhaustion of a countable set of indices X is a nested sequence of subsets S_n, $n \in \mathbb{N}$, of X (i.e., $S_1 \subset S_2 \subset S_3 \subset \ldots$) such that $X = \cup_{n\in\mathbb{N}} S_n$. An exhaustion by finite subsets is an exhaustion such that each S_n is a finite set.*

We have the following proposition.

Proposition 1.2.3. *If $f \in \mathcal{A}_{r,s}$ for some $r, s > 0$, then it is absolutely convergent and defines a function*

$$\mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathbb{R} \ , \qquad (y,\varphi) \mapsto f(y,\varphi) \ ,$$

whose complex extension $f_\mathbb{C}$, obtained by considering $(y,\varphi) \in \mathscr{U}_r(\ell^\infty_w(\mathbb{C})) \times \ell^\infty_s(\mathbb{C})$ in formula (1.2)*, is totally convergent. In particular, for all exhaustions $(S_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^{(\mathbb{N})} \times \mathbb{N}_0^{(\mathbb{N})}$ by finite subsets, one has*

$$\lim_{n\to\infty} \sup_{(y,\varphi)\in\mathscr{U}_r(\ell^\infty_w(\mathbb{C}))\times\ell^\infty_s(\mathbb{C})} \left| f_\mathbb{C}(y,\varphi) - \sum_{(\nu,\alpha)\in S_n} f_{\nu\alpha} y^\alpha \, e^{\mathrm{i}\nu\cdot\varphi} \right| = 0 \ . \tag{1.3}$$

Proof. Consider the complex extension of f obtained by formula (1.2) assuming that (y,φ) extend to variables defined in $\mathscr{U}_r(\ell^\infty_w(\mathbb{C}))\times\ell^\infty_s(\mathbb{C})$. Clearly, it is totally convergent because

$$\sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}\sup_{(y,\varphi)\in\mathscr{U}_r(\ell^\infty_w(\mathbb{C}))\times\ell^\infty_s(\mathbb{C})}|f_{\nu\alpha}y^\alpha e^{\mathrm{i}\nu\cdot\varphi}|=|f|_{r,s}<\infty\ .$$

In particular, f is absolutely convergent and hence the attained values do not depend on the order of summation. Namely, f defines a function

$$\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty\to\mathbb{R}\ ,\qquad (y,\varphi)\mapsto f(y,\varphi)\ ,$$

whose complex extension $f_\mathbb{C}$ is represented by a totally convergent power-trigonometric series. Formula (1.3) follows by the total convergence. ∎

Definition 1.2.4 (Majorant)**.** *Given $f\in\mathcal{A}$, we define its majorant f_M as the formal series*

$$f_M(y,\varphi):=\sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}|f_{\nu\alpha}|y^\alpha e^{\mathrm{i}\nu\cdot\varphi}\ .$$

Proposition 1.2.5 (Majorant analyticity)**.** *If $f\in\mathcal{A}_{r,s}$ for some $r,s>0$, then the function*

$$\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty\to\mathbb{R}\ ,\qquad (y,\varphi)\mapsto f(y,\varphi)\ ,$$

is majorant real analytic.

Proof. Let $(S_n)_{n\in\mathbb{N}}$ be an exhaustion of $\mathbb{Z}^{(\mathbb{N})}\times\mathbb{N}_0^{(\mathbb{N})}$ by finite subsets. Define the sequence of functions $(f_{M,n})_{n\in\mathbb{N}}\subset\mathcal{A}_{r,s}$ of finitely many variables as

$$f_{M,n}(y,\varphi):=\sum_{(\nu,\alpha)\in S_n}|f_{\nu\alpha}|y^\alpha e^{\mathrm{i}\nu\cdot\varphi}.$$

Now, being each $f_{M,n}$ a trigonometric polynomial with polynomial functions as coefficients, it extends to a complex valued analytic function by considering $(y,\varphi)\in\mathscr{U}_r(\ell^\infty_w(\mathbb{C}))\times\ell^\infty_s(\mathbb{C})$ in formula (1.2). Moreover, by Proposition 1.2.3, the sequence $(f_{M,n})_{n\in\mathbb{N}}$ converges uniformly in $\mathscr{U}_r(\ell^\infty_w(\mathbb{C}))\times\ell^\infty_s(\mathbb{C})$ to f_M (more precisely, it converges to its complex extension obtained by considering each y_j and each φ_j as complex variables as before). The conclusion follows by the fact that the uniform limit of a sequence of analytic functions on Banach spaces is an analytic function. Of course, the limit is real on reals. ∎

Proposition 1.2.6 (Algebra)**.** *For all $r,s>0$, $\mathcal{A}_{r,s}$ are Banach algebras with respect to the product of series. More precisely, for all $f,g\in\mathcal{A}_{r,s}$, the following algebra estimate hold:*

$$|fg|_{r,s}\le|f|_{r,s}|g|_{r,s}\ .$$

Proof. Formally, one has

$$(fg)(y,\varphi)=\sum_{\mu\zeta}\Bigg(\sum_{\substack{\nu+\nu'=\mu\\ \alpha+\alpha'=\zeta}}f_{\nu\alpha}g_{\nu'\alpha'}\Bigg)y^\zeta e^{\mathrm{i}\mu\cdot\varphi}.$$

Then,

$$|fg|_{r,s}=\sum_{\mu\zeta}\Bigg|\sum_{\substack{\nu+\nu'=\mu\\ \alpha+\alpha'=\zeta}}f_{\nu\alpha}g_{\nu'\alpha'}\Bigg|v^\zeta_{w,r}e^{s|\mu|}\le\sum_{\nu,\alpha}|f_{\nu\alpha}|v^\alpha_{w,r}e^{s|\nu|}\sum_{\nu',\alpha'}|g_{\nu'\alpha'}|v^{\alpha'}_{w,r}e^{s|\nu'|}=|f|_{r,s}|g|_{r,s}\ .$$

Hence, the claim follows. ∎

1.2.2 Taylor-Fourier series interpretation and comments on the $|\cdot|_{r,s}$-norm

Now we prove that the series in $\mathcal{A}$ are indeed the Taylor-Fourier series of the associated function. The theory of analytic functions depending on infinitely many angles, in a slightly different setting, has been developed, for e.g., in [MP21] and [CGP24]. The Taylor part follows from the fact that any $f \in \mathcal{A}$ defines a real analytic function, as proved in Proposition 1.2.5. More precise Cauchy estimates are given in the next subsection. Let us focus on the periodic variables.

Definition 1.2.7. *Given $f \in \mathcal{A}_{r,s}$, with $r, s > 0$, and $\nu \in \mathbb{Z}^{(\mathbb{N})}$, we denote by f_ν the majorant analytic function*

$$f_\nu : \mathscr{U}_r(\ell^\infty_w) \to \mathbb{R}\,, \qquad f_\nu(y) = \sum_\alpha f_{\nu\alpha} y^\alpha$$

and by $f_{\nu,M}$ its majorant $f_{\nu,M}(y) = \sum_\alpha |f_{\nu\alpha}| y^\alpha$.

Remark 1.2.8. *Note that $f(y,\varphi) = \sum_\nu f_\nu(y) e^{\mathrm{i}\nu\cdot\varphi}$ for all $\varphi \in \ell^\infty$, and*

$$|f|_{r,s} = \sum_\nu \sup_{y \in \mathscr{U}_r(\ell^\infty_w)} f_{\nu,M}(y)\; e^{s|\nu|}\,.$$

The next lemma, together with the Cauchy estimates for the n^{th}-order derivatives with respect to each of the y_k-variables (see Lemma 1.2.13 and Proposition 1.2.14), implies that the formal power-trigonometric series $f \in \mathcal{A}$ can be interpreted as the Taylor-Fourier series of the function associated with f. In particular, it shows that, for all $r, s > 0$ and every $f \in \mathcal{A}_{r,s}$, the majorant real analytic function f_0 (see Definition 1.2.7) is such that, for all $y \in \mathscr{U}_r(\ell^\infty_w)$, the real number

$$f_0(y) = \sum_\alpha f_{0\alpha} y^\alpha$$

can be interpreted as the average of f on the infinite torus $\{y\} \times \mathbb{T}^\infty$, namely[2]

$$f_0(y) = \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f(y,\varphi)\, d\varphi_1 \dots d\varphi_N\,. \tag{1.4}$$

More precisely, we prove the following:

Lemma 1.2.9 (Fourier series). *Consider $f \in \mathcal{A}_{r,s}$, with $r, s > 0$. Then, for all $\nu \in \mathbb{Z}^{(\mathbb{N})}$, the equality*

$$f_\nu(y) = \int_{\mathbb{T}^\infty} f(y,\varphi) e^{-\mathrm{i}\nu\cdot\varphi}\, d\varphi := \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f(y,\varphi) e^{-\mathrm{i}\nu\cdot\varphi}\, d\varphi_1 \dots d\varphi_N \tag{1.5}$$

holds pointwise for any $y \in \mathscr{U}_r(\ell^\infty_w)$.

Proof. First note that, since for all $y \in \mathscr{U}_r(\ell^\infty_w)$ the series $\sum_\nu f_\nu(y) e^{\mathrm{i}\nu\cdot\varphi}$ converges uniformly for $\varphi \in \ell^\infty$, one has

$$\int_{\mathbb{T}^N} f(y,\varphi)\, d\varphi_1 \dots d\varphi_N = \sum_\nu f_\nu(y) \int_{\mathbb{T}^N} e^{\mathrm{i}\nu\cdot\varphi} d\varphi_1 \dots d\varphi_N$$

for all $N \in \mathbb{N}$. Moreover, for any $N \in \mathbb{N}$, let S_N denote the subset of $\mathbb{Z}^{(\mathbb{N})}$ given by

$$S_N := \{\nu \in \mathbb{Z}^{(\mathbb{N})} \;:\; \nu \neq 0\,,\; \nu_j = 0 \quad \forall j > N\}\,.$$

[2] Note that the functional in (1.4) is the integral with respect to the product measure on $\mathbb{T}^\infty$ as described in Chapter 4.

Thus, for all $N \in \mathbb{N}$ and any $\nu \in S_N$, one has

$$\frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{\mathrm{i}\nu\cdot\varphi}\, d\varphi_1 \dots d\varphi_N = 0 \; .$$

Hence,

$$\begin{aligned}
&\frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f(y,\varphi)\, d\varphi_1 \dots d\varphi_N \\
&\qquad = \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} \Bigg(f_0(y) + \sum_{\nu \in S_N} f_\nu(y) e^{\mathrm{i}\nu\cdot\varphi} + \sum_{0\neq\nu\notin S_N} f_\nu(y) e^{\mathrm{i}\nu\cdot\varphi} \Bigg) d\varphi_1 \dots d\varphi_N \\
&\qquad = f_0(y) + \sum_{0\neq\nu\notin S_N} f_\nu(y) \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{\mathrm{i}\nu\cdot\varphi}\, d\varphi_1 \dots d\varphi_N \; .
\end{aligned}$$

The latter integral can only take the values 0 or 1. Moreover, by Remark 1.2.8, the tail of the series with $0 \neq \nu \notin S_N$ goes to zero as $N \to \infty$. This proves (1.5) for $\nu = 0$.

Now, take $\nu' \in \mathbb{Z}^{(\mathbb{N})} \setminus \{0\}$ and set

$$f(y,\varphi,\nu') := f(y,\varphi) e^{-\mathrm{i}\nu'\cdot\varphi} = \sum_\nu f_\nu(y) e^{\mathrm{i}(\nu-\nu')\cdot\varphi} = \sum_\nu f_{\nu+\nu'}(y) e^{\mathrm{i}\nu\cdot\varphi} \; .$$

By applying (1.5) with $\nu = 0$ to $f(y,\varphi,\nu')$ and using the fact that ν' is arbitrary, the equality (1.5) follows for $\nu \neq 0$ as well. ■

We conclude this subsection with some comments on the $|\cdot|_{r,s}$-norm; in particular, we discuss the presence of the weight $e^{s|\nu|}$. In the finite-dimensional case, a formal, real trigonometric series

$$f(\varphi) = \sum_{\nu \in \mathbb{Z}^N} f_\nu e^{\mathrm{i}\nu\cdot\varphi} \; , \quad \varphi \in \mathbb{R}^N \; ,$$

that admits an absolutely and uniformly convergent complex extension obtained by extending each variable φ_k to a complex strip $|\mathrm{Im}\,\varphi_k| < s$, for some $s > 0$, defines a real analytic multi-periodic function. Moreover, the formal series coincides with the Fourier series of the function, whose Fourier coefficients are controlled by $e^{-|\nu|s}$. This decay for the Fourier coefficients is sufficient for proving Cauchy estimates on the derivatives. This is essentially due to the following facts:

(a) $\sum_{\nu\in\mathbb{Z}^N} e^{-|\nu|\sigma} < \infty$ for all $\sigma > 0$;

(b) $\lim_{n\to\infty} \sup_{|\mathrm{Im}\,\varphi_k|<s} \left| f(\varphi) - \sum_{\nu\in S_n} f_\nu e^{\mathrm{i}\nu\cdot\varphi} \right| = 0$ for all exhaustions $(S_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^N$ by finite subsets.

In the case of functions depending on infinitely many angles, the exponential decay of the Fourier coefficients does not ensure even the pointwise convergence of the Fourier series. The main point is that the bound in (a) is false whenever the sum is taken over $\nu \in \mathbb{Z}^{(\mathbb{N})}$.

Note that, in both the finite and infinite-dimensional settings, the Fourier coefficients decay exponentially if either the *sup-norm* or the *weighted Fourier-norm*, defined respectively below, is finite:

$$|f|_s^{\sup} := \sup_{|\mathrm{Im}(\varphi_k)|<s} \left| \sum_\nu f_\nu e^{\mathrm{i}\nu\cdot\varphi} \right| , \qquad |f|_s := \sum_\nu |f_\nu| e^{|\nu|s} \; . \tag{1.6}$$

Furthermore, we stress that these norms are equivalent in the finite-dimensional case. More precisely, in [BBP13], the authors proved that, for multi-periodic functions of m variables, the bounds

$$|f|_s^{\text{sup}} \le |f|_s \le 2^m |f|_s^{\text{sup}} \tag{1.7}$$

hold for all $s > 0$, whenever one of the due norms is finite. This is why the sup-norm and the weighted Fourier norm are interchangeably used to characterize the class of multi-periodic analytic functions of finitely many variables as Banach spaces.

Let us prove that the norms defined in (1.6) are not equivalent in infinite dimension. We first prove that the bounds in (1.7) are optimal.

Proposition 1.2.10. *Let $m \in \mathbb{N}$. Then for any $c_1 < 1$ and for all $c_2 < 2^m$ there exist $s_1, s_2 > 0$ such that the trigonometric polynomial*

$$p^{(m)}(\varphi_1, \dots, \varphi_m) = \prod_{j=1}^{m} \cos(\varphi_j)$$

satisfies

$$|p^{(m)}|_{s_1}^{sup} > c_1 |p^{(m)}|_{s_1} , \qquad |p^{(m)}|_{s_2} > c_2 |p^{(m)}|_{s_2}^{sup} .$$

Proof. Given $s > 0$, the sup-norm of the function $\varphi \mapsto \cos(\varphi)$ is $\cosh(s)$. Indeed, writing $\varphi = x + \mathrm{i}y$, with $x, y \in \mathbb{R}$, one has

$$\begin{aligned}
\sup_{|\mathrm{Im}(\varphi)|<s} |\cos(\varphi)| &= \sup_{|\mathrm{Im}(\varphi)|<s} |\cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)| \\
&= \sup_{|\mathrm{Im}(\varphi)|<s} \sqrt{\cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)} \\
&= \sup_{|\mathrm{Im}(\varphi)|<s} \sqrt{\cosh^2(y) - \sin^2(x)} = \cosh(s) .
\end{aligned}$$

Then, since $p^{(m)}$ is a product of functions that depend on only one variable, the supremum is attained when the functions attain their supremum separately, namely we have that

$$|p^{(m)}|_s^{\text{sup}} = (\cosh(s))^m .$$

Moreover, the weighted Fourier-norm of $p^{(m)}$ is equal to e^{ms}. Then we have

$$\frac{|p^{(m)}|_s}{|p^{(m)}|_s^{\text{sup}}} = 2^m (1 + e^{-2s})^{-m} . \tag{1.8}$$

The claim follows by the fact that, for all $s > 0$, the above ratio is larger than 1, it goes to 1 as $s \to 0$, and it tends to 2^m for $s \to \infty$. ∎

Now we use equation (1.8) of the above proposition to prove that the sup-norm and the weighted Fourier-norm are not equivalent in infinite dimension.

Proposition 1.2.11. *With the same notation of Proposition 1.2.10, for all $M > 0$, there exists $m \in \mathbb{N}$ such that*

$$|p^{(m)}|_s > M |p^{(m)}|_s^{sup} , \qquad \text{for all } s \ge 1 .$$

Proof. Given $m \in \mathbb{N}$, let $s \geq 1$. Then, by equation (1.8), one has

$$\frac{|p^{(m)}|_s}{|p^{(m)}|_s^{\sup}} = 2^m(1+e^{-2s})^{-m} \geq 2^m(1+e^{-2})^{-m} .$$

Since $2(1+e^{-2})^{-1}$ is larger than 1, there exists $m \in \mathbb{N}$ such that $2^m(1+e^{-2})^{-m} > M$. This concludes the proof. ∎

Note that, regardless of the dimension, the boundedness of the sup-norm implies the absolute convergence of the formal trigonometric series, and hence it defines a multi-periodic function, but we are not able to prove the uniform convergence nor to provide a counterexample. Therefore, we cannot assert that such a multi-periodic function is the uniform limit of trigonometric polynomials and hence analytic. This reflects the difficulty (or, possibly, the impossibility) of performing Cauchy estimates with respect to the sup-norm in our infinite-dimensional setting.

Let us explain why the weighted Fourier-norm allows us to overcome this problem. It is worth stressing that one can give another definition of majorant (suggested by C. Procesi, private communication). Namely, given a formal, real trigonometric series $f(\varphi) = \sum_\nu f_\nu e^{\mathrm{i}\nu\cdot\varphi}$, we define the $\{\}$-majorant of f as

$$f_{\{M\}}(\varphi) = \sum_\nu |f_\nu| e^{\mathrm{i}\{\nu\}\cdot\varphi} ,$$

where, for all $\nu = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$, $\{\nu\} := (|\nu_1|, |\nu_2|, \dots, |\nu_n|, \dots)$. Now, define $z = (z_k)_{k\in\mathbb{N}}$, where $z_k := e^{\mathrm{i}\varphi_k}$, and consider the formal power series

$$g_f(z) := \sum_\nu |f_\nu| z^{\{\nu\}} .$$

This definition is remarkable because it allows us to associate with each formal trigonometric series f a formal power series g_f such that the $|\cdot|_s$ norm of f is nothing but the usual sup-norm of g_f in the ball $\mathscr{U}_{e^s}(\ell^\infty(\mathbb{C}))$, namely

$$|f|_s = \sum_\nu |f_\nu| e^{|\nu|s} = \sup_{z\in\mathscr{U}_{e^s}(\ell^\infty(\mathbb{C}))} \left| \sum_\nu |f_\nu| z^{\{\nu\}} \right| .$$

Therefore, the Cauchy estimates for a function associated with f follow from the Cauchy estimates for the function with Taylor series given by g_f in the ball $\mathscr{U}_{e^s}(\ell^\infty)$.

On the other hand, defining in $\mathbb{Z}^{(\mathbb{N})}$ the $\star$-*norm* as

$$|\nu|_\star := \sum_{j\in\mathbb{N}} h_j|\nu_j|, \quad h_j \in \mathbb{R}_+ \ \forall\, j \in \mathbb{N}, \quad h_{j+1} \geq h_j, \quad \lim_{j\to+\infty} h_j = +\infty,$$

one has $\sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}} e^{-|\nu|_\star\sigma} < \infty$ for all $\sigma > 0$. The $\star$-norm has been used in [CGP24] to define a space of functions on the infinite torus which can be analytically extended to holomorphic functions on the thickened infinite torus:

$$\mathbb{T}^\infty_{s,h} := \left\{ \varphi \in \mathbb{C}^\mathbb{N} : \operatorname{Re}(\varphi) \in \mathbb{T}^\infty, \ \sup_{j\in\mathbb{N}} \left| h_j^{-1} \operatorname{Im}(\varphi_j) \right| < s \right\},$$

which is a complex Banach manifold. Many of the difficulties of our setting come from the fact that we make a much weaker analyticity assumption: a holomorphic function on $\mathbb{T}^\infty_{s,h}$ has the property that the width of the strip of analyticity with respect to the $k^{\mathtt{th}}$ periodic variable goes to infinity as $k \to \infty$.

1.2.3 Cauchy estimates

By virtue of Proposition 1.2.5, the elements of $\mathcal{A}$ define analytic functions on their domain of convergence. Throughout the remainder of this text, we shall therefore refer to them as either power-trigonometric series or functions, depending on the context, without further distinction. In this subsection we prove Cauchy estimates for the differential of functions in the class $\mathcal{A}$.

Definition 1.2.12. *Given $r, s > 0$ and $f \in \mathcal{A}_{r,s}$, for all $n \geq 0$ and any $k_1, \ldots, k_n \in \mathbb{N}$, we define the n^{th} partial derivative of f by setting*

$$\frac{\partial^n f}{\partial y_{k_1} \ldots \partial y_{k_n}}(y, \varphi) := \sum_{\nu,\alpha} f_{\nu\alpha} \alpha_{k_1} \prod_{j=2}^{n} \left(\alpha_{k_j} - \sum_{\ell=1}^{j-1} \delta_{k_j k_{j-\ell}} \right) y^{\alpha - \sum_{j=1}^n e_{k_j}} e^{\mathrm{i}\nu\cdot\varphi} ,$$

$$\frac{\partial^n f}{\partial \varphi_{k_1} \ldots \partial \varphi_{k_n}}(y, \varphi) := \sum_{\nu,\alpha} f_{\nu\alpha}\, \mathrm{i}^n \prod_{j=1}^{n} \nu_{k_j} y^{\alpha} e^{\mathrm{i}\nu\cdot\varphi} .$$

Lemma 1.2.13 (Cauchy estimates)**.** *For all $r, s > 0$ and any $f \in \mathcal{A}_{r,s}$, the following estimates*

$$\sum_{k_1,\ldots,k_n \in \mathbb{N}} \prod_{j=1}^{n} w_{k_j}^{-1} \left| \frac{\partial^n f}{\partial y_{k_1} \ldots \partial y_{k_n}} \right|_{r-\rho,s} \leq \left(\frac{r}{r-\rho} \right)^n \frac{n^n e^{-n}}{\rho^n} |f|_{r,s} , \tag{1.9}$$

$$\sum_{k_1,\ldots,k_n \in \mathbb{N}} \left| \frac{\partial^n f}{\partial \varphi_{k_1} \ldots \partial \varphi_{k_n}} \right|_{r,s-\sigma} \leq \frac{n^n e^{-n}}{\sigma^n} |f|_{r,s} \tag{1.10}$$

hold for all $\rho \in (0, r)$ and $\sigma \in (0, s)$.

Proof. The left-hand side of (1.9) is equal to

$$\sum_{k_1,\ldots,k_n \in \mathbb{N}} \prod_{j=1}^{n} w_{k_j}^{-1} \sum_{\nu,\alpha} |f_{\nu\alpha}| \alpha_{k_1} \prod_{m=2}^{n} \left(\alpha_{k_m} - \sum_{\ell=1}^{m-1} \delta_{k_m k_{m-j}} \right) v_{w,r-\rho}^{\alpha - \sum_{j=1}^n e_{k_j}} e^{s|\nu|},$$

which is bounded by

$$\begin{aligned} \left(\frac{1}{r-\rho} \right)^n \sum_{\nu,\alpha} |f_{\nu\alpha}| |\alpha|^n v_{w,r-\rho}^{\alpha} e^{s|\nu|} &\leq \left(\frac{1}{r-\rho} \right)^n \sup_{x \geq 0} \left[x^n \left(\frac{r-\rho}{r} \right)^x \right] |f|_{r,s} \\ &\leq \left(\frac{r}{r-\rho} \right)^n \frac{n^n e^{-n}}{\rho^n} |f|_{r,s}. \end{aligned}$$

Let us perform the second estimate. The left-hand side of (1.10) is

$$\sum_{\nu,\alpha} |f_{\nu\alpha}| |\nu|^n v_{w,r}^{\alpha} e^{(s-\sigma)|\nu|} = \sum_{\nu,\alpha} |f_{\nu\alpha}| v_{w,r}^{\alpha} e^{s|\nu|} |\nu|^n e^{-\sigma|\nu|}$$

and is bounded by

$$\sup_{x \geq 0} \left(x^n e^{-\sigma x} \right) \sum_{\nu,\alpha} |f_{\nu\alpha}| v_{w,r}^{\alpha} e^{s|\nu|} \leq \frac{n^n e^{-n}}{\sigma^n} |f|_{r,s} .$$

This concludes the proof. ∎

The functions defined in Definition 1.2.12 are indeed the partial derivatives, as proved in the following proposition.

Proposition 1.2.14. *Given $r, s > 0$, consider $f \in \mathcal{A}_{r,s}$. Then,*

(a) the n^{th} Gateaux derivative of f in the direction of $\mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(n)}$, with $\mathfrak{h}^{(j)} = (h^{(j)}, 0) \in \ell^\infty_w \times \ell^\infty$ for all $j = 1, \dots, n$, is

$$\begin{aligned} \partial_y^n f(y,\varphi)[h^{(1)}, \dots, h^{(n)}] &:= \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_n}\bigg|_{t_n=0} f\Big(y + \sum_{j=1}^n t_j h^{(j)}, \varphi\Big) \\ &= \sum_{k_1,\dots,k_n \in \mathbb{N}} \prod_{j=1}^n h^{(j)}_{k_j} \frac{\partial^n f}{\partial y_{k_1} \dots \partial y_{k_n}}(y,\varphi)\,; \end{aligned} \tag{1.11}$$

(b) the n^{th} Gateaux derivative of f in the direction of $\mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(n)}$, with $\mathfrak{h}^{(j)} = (0, h^{(j)}) \in \ell^\infty_w \times \ell^\infty$ for all $j = 1, \dots, n$, is

$$\begin{aligned} \partial_\varphi^n f(y,\varphi)[h^{(1)}, \dots, h^{(n)}] &:= \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_n}\bigg|_{t_n=0} f\Big(y, \varphi + \sum_{j=1}^n t_j h^{(j)}\Big) \\ &= \sum_{k_1,\dots,k_n \in \mathbb{N}} \prod_{j=1}^n h^{(j)}_{k_j} \frac{\partial^n f}{\partial \varphi_{k_1} \dots \partial \varphi_{k_n}}(y,\varphi)\,. \end{aligned}$$

Proof. Formula (1.11) holds pointwise in (y, φ) for all monomials of the form $f(y,\varphi) = y^\alpha e^{\mathrm{i}\nu\cdot\varphi}$, $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, $\nu \in \mathbb{Z}^{(\mathbb{N})}$ and both the series

$$\sum_{\nu,\alpha} f_{\nu\alpha} \Big(y + \sum_{j=1}^n t_j h^{(j)}\Big)^\alpha e^{\mathrm{i}\nu\varphi}$$

$$\sum_{\nu,\alpha} f_{\nu\alpha} \sum_{k_1,\dots,k_n \in \mathbb{N}} h^{(1)}_{k_1} \alpha_{k_1} \prod_{j=2}^n h^{(j)}_{k_j} \Big(\alpha_{k_j} - \sum_{\ell=1}^{j-1} \delta_{k_j k_{j-\ell}}\Big) \Big(y + \sum_{j=1}^n t_j h^{(j)}\Big)^{\alpha - \sum_{j=1}^n e_{k_j}} e^{\mathrm{i}\nu\cdot\varphi}$$

converge for all $(y,\varphi) \in \mathscr{U}_{r-\rho}(\ell^\infty_w) \times \ell^\infty$ absolutely and uniformly in a neighbourhood of $t_1 = \cdots = t_n = 0$. More precisely, let $\epsilon_1, \dots, \epsilon_n > 0$ be such that $y + \sum_{j=1}^n t_j h^{(j)} \in \mathscr{U}_{r-\rho}(\ell^\infty_w)$ for all $t_1, \dots, t_n$ with $|t_j| \le \epsilon_j$. Then one has

$$\sup_{|t_j| \le \epsilon_j} \sum_{\nu,\alpha} \left| f_{\nu\alpha} \Big(y + \sum_{j=1}^n t_j h^{(j)}\Big)^\alpha e^{\mathrm{i}\nu\varphi} \right| \le \sup_{y \in \mathscr{U}_{r-\rho}(\ell^\infty_w)} \sum_{\nu\alpha} |f_{\nu\alpha} y^\alpha| \le |f|_{r,s}$$

and

$$
\begin{aligned}
&\sup_{|t_j|\le\epsilon_j}\sum_{\nu,\alpha}\left|f_{\nu\alpha}\sum_{k_1,\dots,k_n\in\mathbb{N}}h^{(1)}_{k_1}\alpha_{k_1}\prod_{j=2}^{n}h^{(j)}_{k_j}\left(\alpha_{k_j}-\sum_{\ell=1}^{j-1}\delta_{k_jk_{j-\ell}}\right)\left(y+\sum_{j=1}^{n}t_jh^{(j)}\right)^{\alpha-\sum_{j=1}^{n}e_{k_j}}e^{\mathrm{i}\nu\cdot\varphi}\right|\\
&\le\sup_{y\in\mathscr{U}_{r-\rho}(\ell^\infty_w)}\sum_{\nu,\alpha}|f_{\nu\alpha}|\sum_{k_1,\dots,k_n\in\mathbb{N}}|h^{(1)}_{k_1}|\alpha_{k_1}\prod_{j=2}^{n}|h^{(j)}_{k_j}|\left(\alpha_{k_j}-\sum_{\ell=1}^{j-1}\delta_{k_jk_{j-\ell}}\right)y^{\alpha-\sum_{j=1}^{n}e_{k_j}}\\
&\le\prod_{j=1}^{n}|h^{(j)}|_{\ell^\infty_w}\sup_{y\in\mathscr{U}_{r-\rho}(\ell^\infty_w)}\sum_{k_1,\dots,k_n\in\mathbb{N}}\prod_{j=1}^{n}w^{-1}_{k_j}\sum_{\nu,\alpha}|f_{\nu\alpha}|\alpha_{k_1}\prod_{j=2}^{n}\left(\alpha_{k_j}-\sum_{\ell=1}^{j-1}\delta_{k_jk_{j-\ell}}\right)y^{\alpha-\sum_{j=1}^{n}e_{k_j}}\\
&\le\prod_{j=1}^{n}|h^{(j)}|_{\ell^\infty_w}\sum_{k_1,\dots,k_n\in\mathbb{N}}\prod_{j=1}^{n}w^{-1}_{k_j}\left|\frac{\partial^n f}{\partial y_{k_1}\dots\partial y_{k_n}}\right|_{r-\rho,s}\\
&\overset{\text{Lemma 1.2.13}}{\le}\prod_{j=1}^{n}|h^{(j)}|_{\ell^\infty_w}\left(\frac{r}{r-\rho}\right)^n\frac{n^ne^{-n}}{\rho^n}|f|_{r,s}\,.
\end{aligned}
$$

An analogous argument holds for the n^{th} Gateaux derivative of $f\in\mathcal{A}_{r,s}$ in the direction of $\mathfrak{h}^{(1)},\dots,\mathfrak{h}^{(n)}$, with $\mathfrak{h}^{(j)}=(0,h^{(j)})\in\ell^\infty_w\times\ell^\infty$ for all $j=1,\dots,n$, that is

$$
\begin{aligned}
\partial^n_\varphi f(y,\varphi)[h^{(1)},\dots,h^{(n)}]&:=\left.\frac{d}{dt_1}\right|_{t_1=0}\cdots\left.\frac{d}{dt_n}\right|_{t_n=0}f\left(y,\varphi+\sum_{j=1}^{n}t_jh^{(j)}\right)\\
&=\sum_{k_1,\dots,k_n\in\mathbb{N}}\prod_{j=1}^{n}h^{(j)}_{k_j}\frac{\partial^n f}{\partial\varphi_{k_1}\dots\partial\varphi_{k_n}}(y,\varphi),
\end{aligned}
$$

which proves the claim. ■

Remark 1.2.15. *The maps $\partial^n_y f(y,\varphi)$ and $\partial^n_\varphi f(y,\varphi)$ correspond with the restriction of the n^{th}-differential of f at (y,φ), namely $d^nf(y,\varphi)$, on the subspaces $\ell^\infty_w\times\{0\}$ and $\{0\}\times\ell^\infty$, respectively.*

1.2.4 Pullback of functions by close to identity maps

We now define suitable norms that we shall use to provide estimates for the pullback of functions by close to identity maps.

Definition 1.2.16. *Given $r,s>0$, we denote by $\mathcal{A}^{\infty,w}_{r,s}$ the space of functions*

$$N:\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty\to\ell^\infty_w,\quad N(y,\varphi)=(N^{(m)}(y,\varphi))_{m\in\mathbb{N}},$$

such that, for any $m\in\mathbb{N}$, $N^{(m)}\in\mathcal{A}_{r,s}$, with

$$|N|^{\infty,w}_{r,s}:=\sup_{m\in\mathbb{N}}w_m|N^{(m)}|_{r,s}<\infty.$$

Accordingly, $\mathcal{A}^\infty_{r,s}$ denotes the space of functions

$$\Omega:\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty\to\ell^\infty,\quad \Omega(y,\varphi)=(\Omega^{(m)}(y,\varphi))_{m\in\mathbb{N}},$$

such that, for any $m\in\mathbb{N}$, $\Omega^{(m)}\in\mathcal{A}_{r,s}$, with

$$|\Omega|^\infty_{r,s}:=\sup_{m\in\mathbb{N}}|\Omega^{(m)}|_{r,s}<\infty.$$

Remark 1.2.17. *The spaces $\mathcal{A}^{\infty,w}_{r,s}$ and $\mathcal{A}^{\infty}_{r,s}$ are Banach spaces when endowed with the norms $|\cdot|^{\infty,w}_{r,s}$ and $|\cdot|^{\infty}_{r,s}$ respectively.*

Given $r, s > 0$, let $\mathcal{C}_{r,s}(\mathcal{A})$ be the affine space passing through the identity and directed by $\mathcal{A}^{\infty,w}_{r,s} \times \mathcal{A}^{\infty}_{r,s}$, namely

$$\mathcal{C}_{r,s}(\mathcal{A}) := \left\{ \Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty \ : \ \Phi = \mathrm{id} + (N, \Omega) \ , \ N \in \mathcal{A}^{\infty,w}_{r,s}, \ \Omega \in \mathcal{A}^{\infty}_{r,s} \right\} .$$

For any $\rho, \sigma > 0$, let $\mathcal{C}^{\rho;\sigma}_{r,s}(\mathcal{A})$ be the subset of those maps $\Phi = \mathrm{id} + (N, \Omega) \in \mathcal{C}_{r,s}(\mathcal{A})$ such that

$$|N|^{\infty,w}_{r,s} \le \rho \qquad \text{and} \qquad |\Omega|^{\infty}_{r,s} \le \sigma \ .$$

In this case, we say that Φ is close to identity. Accordingly, we denote by $\mathcal{C}(\mathcal{A})$ the space of functions

$$\mathcal{C}(\mathcal{A}) := \bigcup_{r,s>0} \mathcal{C}_{r,s}(\mathcal{A}) \ .$$

Notation 1.2.18. *In the following, given two functions*

$$\Phi : U \to V \ , \qquad f : V \to Z \ ,$$

we denote by $\Phi^* f$ the pullback of f by Φ, namely the function $f \circ \Phi : U \to Z$.

Proposition 1.2.19 (Estimate on the pullback of functions)**.** *Given $r_0, s_0 > 0$, let $\rho \in [0, r_0)$ and $\sigma \in [0, s_0)$. Consider a function $\Psi = id + (N, \Omega) \in \mathcal{C}^{\rho,\sigma/2}_{r_0-\rho, s_0-\sigma}(\mathcal{A})$. Then, for all $r \in (\rho, r_0]$, $s \in (\sigma, s_0]$, and every $f \in \mathcal{A}_{r,s}$, the pullback of f by Ψ belongs to $\mathcal{A}_{r-\rho,s-\sigma}$ with estimate*

$$|\Psi^* f|_{r-\rho,s-\sigma} \le 2|f|_{r,s}.$$

Proof. Let $r \in (\rho, r_0]$, $s \in (\sigma, s_0]$, and $f \in \mathcal{A}_{r,s}$. First, we formally manipulate the series

$$f(\Psi(y, \varphi)) = \sum_{\nu,\alpha} f_{\nu\alpha}(y + N(y, \varphi))^\alpha e^{\mathrm{i}\nu\cdot(\varphi+\Omega(y,\varphi))} \tag{1.12}$$

with the aim of obtaining a representation of the form

$$\sum_{\lambda,\beta} (f \circ \Psi)_{\lambda\beta} y^\beta e^{\mathrm{i}\lambda\cdot\varphi}.$$

Given $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, let $\mathrm{supp}(\alpha)$ denote the support of α. For all $\mu \in \mathbb{Z}^{(\mathbb{N})}$, $\zeta \in \mathbb{N}_0^{(\mathbb{N})}$, we denote by $\Lambda_{\mu,\zeta}(\alpha)$ the set of lists of multi-indices $(\overline{\mu}, \overline{\zeta})_\alpha := ((\mu^{(k,j_k)}, \zeta^{(k,j_k)})_{j_k=1,\dots,\alpha_k})_{k\in\mathrm{supp}(\alpha)}$ defined by

$$\Lambda_{\mu,\zeta}(\alpha) := \left\{ (\overline{\mu}, \overline{\zeta})_\alpha \ : \ \sum_{k\in\mathrm{supp}(\alpha)} \sum_{j_k=1}^{\alpha_k} \mu^{(k,j_k)} = \mu, \ \sum_{k\in\mathrm{supp}(\alpha)} \sum_{j_k=1}^{\alpha_k} \zeta^{(k,j_k)} = \zeta \right\},$$

where, for any $k \in \mathrm{supp}(\alpha)$ and any $j_k = 1, \dots, \alpha_k$, $\mu^{(k,j_k)} \in \mathbb{Z}^{(\mathbb{N})}$ and $\zeta^{(k,j_k)} \in \mathbb{N}_0^{(\mathbb{N})}$. Let us define

$$\begin{aligned} a_{\mu\zeta\alpha} &:= \sum_{(\overline{\mu},\overline{\zeta})_\alpha\in\Lambda_{\mu,\zeta}(\alpha)} \prod_{k\in\mathrm{supp}(\alpha)} \prod_{j_k=1}^{\alpha_k} \hat{N}^{(k)}_{\mu^{(k,j_k)}\zeta^{(k,j_k)}}, \quad \text{where} \quad \hat{N}(y, \varphi) := y + N(y, \varphi); \\ b_{\mu'\zeta'\nu} &:= \Bigg(\delta_{\mu',0}\delta_{\zeta',0} + \mathrm{i}\nu \cdot \Omega_{\mu'\zeta'} + \frac{1}{2!}\mathrm{i}\nu \cdot \sum_{\mu'',\zeta''} \Omega_{\mu''\zeta''} \times \mathrm{i}\nu \cdot \sum_{\substack{\mu'''=\mu'-\mu'' \\ \zeta'''=\zeta'-\zeta''}} \Omega_{\mu'''\zeta'''} + \cdots \Bigg). \end{aligned} \tag{1.13}$$

Then,

$$
\begin{aligned}
(1.12) &= \sum_{\nu,\alpha} f_{\nu\alpha} e^{\mathrm{i}\nu\cdot\varphi}\Big(y+\sum_{\mu,\zeta} N_{\mu\zeta} y^{\zeta} e^{\mathrm{i}\mu\cdot\varphi}\Big)^{\alpha} \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\mathrm{i}\nu\cdot\sum_{\mu',\zeta'}\Omega_{\mu'\zeta'} y^{\zeta'} e^{\mathrm{i}\mu'\cdot\varphi}\Big)^{n}\\
&= \sum_{\nu,\alpha} f_{\nu\alpha}\sum_{\mu,\zeta} a_{\mu\zeta\alpha}\sum_{\mu',\zeta'} b_{\mu'\zeta'\nu} y^{\zeta+\zeta'} e^{\mathrm{i}(\nu+\mu+\mu')\cdot\varphi}\\
&= \sum_{\lambda,\beta}\Big(\sum_{\substack{\nu+\mu+\mu'=\lambda\\ \zeta+\zeta'=\beta}} f_{\nu\alpha} a_{\mu\zeta\alpha} b_{\mu'\zeta'\nu}\Big) y^{\beta} e^{\mathrm{i}\lambda\cdot\varphi}.
\end{aligned}
$$

Now we perform estimates on the latter series.

$$
\begin{aligned}
&\sum_{\lambda,\beta}\Big|\sum_{\substack{\nu+\mu+\mu'=\lambda\\ \zeta+\zeta'=\beta}} f_{\nu\alpha} a_{\mu\zeta\alpha} b_{\mu'\zeta'\nu}\Big| v_{w,r-\rho}^{\beta} e^{(s-\sigma)|\lambda|} \le \sum_{\nu,\mu,\mu'}\sum_{\zeta,\zeta'} |f_{\nu\alpha}||a_{\mu\zeta\alpha}||b_{\mu'\zeta'\nu}| v_{w,r-\rho}^{\zeta+\zeta'} e^{(s-\sigma)|\nu+\mu+\mu'|}\\
&\le \sum_{\nu,\alpha} |f_{\nu\alpha}| e^{(s-\sigma)|\nu|}\sum_{\mu,\zeta} |a_{\mu\zeta\alpha}| v_{w,r-\rho}^{\zeta} e^{(s-\sigma)|\mu|}\sum_{\mu',\zeta'} |b_{\mu'\zeta'\nu}| v_{w,r-\rho}^{\zeta'} e^{(s-\sigma)|\mu'|}
\end{aligned}
\tag{1.14}
$$

By implementing again triangular inequalities and recalling (1.13) we obtain

$$
\begin{aligned}
(1.14) \le{}& \sum_{\nu,\alpha} |f_{\nu\alpha}| e^{(s-\sigma)|\nu|}\sum_{\mu,\zeta}\sum_{(\overline{\mu},\overline{\zeta})_{\alpha}\in\Lambda_{\mu,\zeta}(\alpha)}\prod_{k\in\mathrm{supp}(\alpha)}\prod_{j=1}^{\alpha_k} |\hat N^{(k)}_{\mu^{(k,j_k)}\zeta^{(k,j_k)}}| v_{w,r-\rho}^{\zeta} e^{(s-\sigma)|\mu|}\\
&\times\Big(1+\Big|\mathrm{i}\nu\cdot\sum_{\mu',\zeta'}\Omega_{\mu'\zeta'} v_{w,r-\rho}^{\zeta'} e^{(s-\sigma)|\mu'|}\Big|\\
&+\frac{1}{2!}\Big|\mathrm{i}\nu\cdot\sum_{\mu'',\zeta''}\Omega_{\mu''\zeta''} v_{w,r-\rho}^{\zeta''} e^{(s-\sigma)|\mu''|}\mathrm{i}\nu\cdot\sum_{\mu''',\zeta'''}\Omega_{\mu'''\zeta'''} v_{w,r-\rho}^{\zeta'''} e^{(s-\sigma)|\mu'''|}\Big|+\dots\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{\nu\alpha} |f_{\nu\alpha}|\Big(\sum_{\mu,\zeta} |\hat N_{\mu\zeta}| v_{w,r-\rho}^{\zeta} e^{(s-\sigma)|\mu|}\Big)^{\alpha} e^{(s-\sigma)|\nu|}\sum_{n=0}^{\infty}\frac{1}{n!}\Big|\mathrm{i}\nu\cdot\sum_{\mu'\zeta'}\Omega_{\mu'\zeta'} v_{w,r-\rho}^{\zeta'} e^{(s-\sigma)|\mu'|}\Big|^{n}\\
&\le \sum_{\nu\alpha} |f_{\nu\alpha}|\Big(\sum_{\mu,\zeta} |\hat N_{\mu\zeta}| v_{w,r-\rho}^{\zeta} e^{(s-\sigma)|\mu|}\Big)^{\alpha} e^{(s-\sigma)|\nu|}\sum_{n=0}^{\infty}\frac{|\nu|^n}{n!}\Big(\sup_{m\in\mathbb{N}}\sum_{\mu'\zeta'} |\Omega^{(m)}_{\mu'\zeta'}| v_{w,r-\rho}^{\zeta'} e^{(s-\sigma)|\mu'|}\Big)^{n}\\
&\le \sum_{\nu\alpha} |f_{\nu\alpha}|\Big((r-\rho+|N|^{\infty,w}_{r-\rho,s-\sigma}) v_{w,1}\Big)^{\alpha} e^{(s-\sigma)|\nu|}\sum_{n=0}^{\infty}\frac{|\nu|^n}{n!}(|\Omega|^{\infty}_{r-\rho,s-\sigma})^{n}\\
&\le \sum_{n=0}^{\infty}\sum_{\nu\alpha} |f_{\nu\alpha}||\nu|^n v_{w,r}^{\alpha} e^{(s-\sigma)|\nu|}\frac{1}{n!}(|\Omega|^{\infty}_{r_0-\rho,s_0-\sigma})^{n}.
\end{aligned}
\tag{1.15}
$$

Therefore, by (1.10), we have

$$
\begin{aligned}
(1.15) &\le \sum_{n=0}^{\infty}\frac{n^n e^{-n}}{\sigma^n}\frac{1}{n!}(|\Omega|^{\infty}_{r_0-\rho,s_0-\sigma})^{n}|f|_{r,s} \le \sum_{n=0}^{\infty}\frac{n^n e^{-n}}{n!}\Big(\frac{|\Omega|^{\infty}_{r_0-\rho,s_0-\sigma}}{\sigma}\Big)^{n}|f|_{r,s}\\
&\le \sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^{n}|f|_{r,s} = 2|f|_{r,s},
\end{aligned}
$$

which ends the proof. ■

Now we prove a formula for the derivative of the pullback of functions by $\Phi = \mathrm{id} + (N,\Omega) \in \mathcal{C}(\mathcal{A})$. In order to do this we shall need to require a smallness condition on N to ensure that the image of Φ is contained in the domain of real analyticity of the given function with respect to the y variable.

Proposition 1.2.20. *Given $r_0, s_0 > 0$, let $\rho \in [0, r_0)$. Consider $\Phi = id + (N,\Omega) \in \mathcal{C}_{r_0,s_0}(\mathcal{A})$ such that $|N|^{\infty,w}_{r_0,s_0} < \rho$. Let us denote by $\hat{N}$ and $\hat{\Omega}$ the functions $\hat{N}(y,\varphi) = y + N(y,\varphi)$ and $\hat{\Omega}(y,\varphi) = \varphi + \Omega(y,\varphi)$. Then, for all $r \in (\rho, r_0+\rho]$, $s > 0$, and every $f \in \mathcal{A}_{r,s}$, the pullback of f by Φ satisfies*

$$\begin{aligned} (i) \quad & \frac{\partial \Phi^* f}{\partial y_k}(y,\varphi) = \sum_{\nu\alpha} f_{\nu\alpha} \frac{\partial (\hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}})}{\partial y_k}(y,\varphi) \\ (ii) \quad & \frac{\partial \Phi^* f}{\partial \varphi_k}(y,\varphi) = \sum_{\nu\alpha} f_{\nu\alpha} \frac{\partial (\hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}})}{\partial \varphi_k}(y,\varphi) \end{aligned} \tag{1.16}$$

for all $(y,\varphi) \in \mathscr{U}_{r-\rho}(\ell^\infty_w) \times \ell^\infty$.

Proof. Given $r \in (\rho, r_0+\rho]$, $s > 0$, and $f \in \mathcal{A}_{r,s}$, the series

$$\sum_{\nu\alpha} f_{\nu\alpha} \hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}}$$

converges to $\Phi^* f$. Now, given $(y,\varphi) \in \mathscr{U}_{r-\rho}(\ell^\infty_w) \times \ell^\infty$, let $\epsilon > 0$ be such that $\overline{\mathscr{U}}_{\epsilon,y}(\ell^\infty_w) \subset \mathscr{U}_{r-\rho}(\ell^\infty_w)$. Let us show that the series in (1.16) are totally convergent in $\mathscr{U}_{\epsilon,y}(\ell^\infty_w) \times \ell^\infty$. Let $r' > 0$ be such that $\rho < r' < r$, and $\mathscr{U}_{\epsilon,y}(\ell^\infty_w) \subset \mathscr{U}_{r'-\rho}(\ell^\infty_w)$. Then we have

$$\sum_{\nu\alpha} |f_{\nu\alpha}| \sup_{(y,\varphi)\in\mathscr{U}_{\epsilon,y}(\ell^\infty_w)\times\ell^\infty} \left| \frac{\partial (\hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}})}{\partial y_k}(y,\varphi) \right| \le \sum_{\nu\alpha} |f_{\nu\alpha}| \sup_{(y,\varphi)\in\mathscr{U}_{r'-\rho}(\ell^\infty_w)\times\ell^\infty} \left| \frac{\partial (\hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}})}{\partial y_k}(y,\varphi) \right| .$$

The right-hand side of the above inequality is bounded by

$$\begin{aligned} & \sum_{\nu\alpha} |f_{\nu\alpha}| \sup_{(y,\varphi)\in\mathscr{U}_{r'-\rho}(\ell^\infty_w)\times\ell^\infty} \sum_{m\in\mathbb{N}} \alpha_m \left| \frac{\partial \hat{N}^{(m)}}{\partial y_k}(y,\varphi) \right| |\hat{N}(y,\varphi)|^{\alpha - e_m} \\ & + \sum_{\nu\alpha} |f_{\nu\alpha}| \sup_{(y,\varphi)\in\mathscr{U}_{r'-\rho}(\ell^\infty_w)\times\ell^\infty} \sum_{m\in\mathbb{N}} |\nu_m| \left| \frac{\partial \Omega^{(m)}}{\partial y_k}(y,\varphi) \right| |\hat{N}(y,\varphi)|^{\alpha} \\ \le & \sum_{\nu\alpha} |f_{\nu\alpha}| \sum_{m\in\mathbb{N}} \left(\alpha_m \left| \frac{\partial \hat{N}^{(m)}}{\partial y_k} \right|_{r'-\rho,s_0} \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell - \delta_{\ell m}}_{r-\rho,s_0} + |\nu_m| \left| \frac{\partial \Omega^{(m)}}{\partial y_k} \right|_{r'-\rho,s_0} \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell}_{r-\rho,s_0} \right). \end{aligned} \tag{1.17}$$

By Lemma 1.2.13, and by the fact that $N \in \mathcal{A}^{\infty,w}_{r_0,s_0}$ and $\Omega \in \mathcal{A}^\infty_{r_0,s_0}$, there exists a constant c such that

$$\begin{aligned} (1.17) & \le c \left(\sum_{\nu\alpha} |f_{\nu\alpha}| |\alpha| \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell}_{r-\rho,s_0} + \sum_{\nu\alpha} |f_{\nu\alpha}| |\nu| \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell}_{r-\rho,s_0} \right) \\ & \le c \left(\sum_{\nu\alpha} |f_{\nu\alpha}| |\alpha| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) v_{w,1} \Big)^\alpha + \sum_{\nu\alpha} |f_{\nu\alpha}| |\nu| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) v_{w,1} \Big)^\alpha \right) . \end{aligned}$$

Since $|N|^{\infty,w}_{r_0,s_0} < \rho$, we have that $(r-\rho+|N|^{\infty,w}_{r_0,s_0})/r < 1$, and hence

$$c' := \sup_{|\alpha|} |\alpha| \left(\frac{r-\rho+|N|^{\infty,w}_{r_0,s_0}}{r} \right)^{|\alpha|} < \infty .$$

Therefore

$$\sum_{\nu\alpha} |f_{\nu\alpha}||\alpha| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0})v_{w,1}\Big)^{\alpha} \le c' \sum_{\nu\alpha} |f_{\nu\alpha}| v^{\alpha}_{w,r} \le c'|f|_{r,s} < \infty \, .$$

Moreover, for all $\sigma \in (0,s)$, one has

$$\begin{aligned} \sum_{\nu\alpha} |f_{\nu\alpha}||\nu| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0})v_{w,1}\Big)^{\alpha} &\le \sum_{\nu\alpha} |f_{\nu\alpha}||\nu| v^{\alpha}_{w,r} e^{(s-\sigma)|\nu|} \\ &\le \Big(\sup_{|\nu|} |\nu| e^{-\sigma|\nu|}\Big) |f|_{r,s} < \infty \, , \end{aligned}$$

which gives formula (i) of (1.16). Analogous estimates allow to prove that formula (ii) of (1.16) holds as well. ■

Remark 1.2.21. *For the sake of clarity, we now prove that functions belonging to either $\mathcal{A}^{\infty,w}_{r,s}$ or $\mathcal{A}^{\infty}_{r,s}$ are majorant real analytic, meaning that, if one defines the majorant F_M of a given $F \in \mathcal{A}^{\infty,w}_{r,s}$ ($\mathcal{A}^{\infty}_{r,s}$) by setting*

$$F_M(y,\varphi) = (F^{(m)}_M(y,\varphi))_{m\in\mathbb{N}} \, ,$$

then F_M admits an extension to an analytic function on $\mathscr{U}_r(\ell^\infty_w(\mathbb{C})) \times \ell^\infty_s(\mathbb{C})$ taking values in $\ell^\infty_w(\mathbb{C})$ ($\ell^\infty(\mathbb{C})$). The proof relies on Theorem 1.2.24 and on the fact that the map $\ell^\infty_w(\mathbb{C}) \ni (y_j)_{j\in\mathbb{N}} \mapsto (w_j y_j)_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{C})$ is globally bi-analytic. As a consequence, it will turn out that any $\Phi \in \mathcal{C}(\mathcal{A})$ is analytic (being a sum of analytic functions).

Definition 1.2.22. *Let X, Y be Banach spaces and $\mathscr{U} \subseteq X$ an open subset. A countable family of functions $(f^{(m)})_{m\in I}$, $f^{(m)} : \mathscr{U} \to Y$, where I is a countable set of indices, is uniformly bounded if there exists $C < \infty$ such that*

$$\sup_{m\in I} \sup_{x\in\mathscr{U}} |f^{(m)}(x)|_Y \le C.$$

Remark 1.2.23. *Let $F \in \mathcal{A}^{\infty,w}_{r,s}$, $F(y,\varphi) = (F^{(m)}(y,\varphi))_{m\in\mathbb{N}}$. For any $m \in \mathbb{N}$, consider the complex extension $F^{(m)}_{M,\mathbb{C}}$ of $F^{(m)}_M$ on the complex domain $\mathscr{U}_r(\ell^\infty_w(\mathbb{C})) \times \ell^\infty_s(\mathbb{C})$ as in Remark 1.2.5. Then $(f^{(m)})_{m\in\mathbb{N}}$, where*

$$f^{(m)} : \mathscr{U} \to \mathbb{C} \, , \quad f^{(m)} := w_m F^{(m)}_{M,\mathbb{C}} \quad \forall m \in \mathbb{N} \, ,$$

and $\mathscr{U}$ denotes the open subset $\mathscr{U}_r(\ell^\infty_w(\mathbb{C})) \times \ell^\infty_s(\mathbb{C})$ of the complex Banach space $\ell^\infty_w(\mathbb{C}) \times \ell^\infty(\mathbb{C})$, defines a uniformly bounded family of analytic functions.

Theorem 1.2.24. *Let X be a complex Banach space and $\mathscr{U} \subseteq X$ an open subset. Let $(f^{(m)})_{m\in I}$ be a uniformly bounded countable family of analytic functions $f^{(m)} : \mathscr{U} \to \mathbb{C}$. Then, the function f defined by $f(x) := (f^{(m)}(x))_{m\in I}$ defines an analytic function*

$$f : \mathscr{U} \to \ell^\infty(I, \mathbb{C}) \, .$$

Proof. For any $n \in \mathbb{N}$, let $L^n_S(X, \ell^\infty(\mathbb{C}))$ denote the space of all bounded n-linear symmetric maps from X^n, i.e., the direct product of n copies of X, taking values in $\ell^\infty(\mathbb{C})$, and let us define the map

$$d^n f : \mathscr{U} \to L^n_S(X, \ell^\infty(\mathbb{C})), \qquad x \mapsto d^n f(x) := (d^n f^{(m)}(x))_{m\in I},$$

where, for any $m \in I$, $d^n f^{(m)}$ denotes the n^{th}-differential of $f^{(m)}$. Namely, for any $n \in \mathbb{N}$ and for all $(h^{(j)})^n_{j=1} \in X^n$ and all $x \in \mathscr{U}$,

$$d^n f(x)[h^{(1)}, \dots, h^{(n)}] = (d^n f^{(m)}(x)[h^{(1)}, \dots, h^{(n)}])_{m\in I}.$$

Note that, for all $x \in \mathscr{U}$, by the n-linearity and the symmetry property of each $d^n f^{(m)}(x)$, $d^n f(x)$ is indeed an n-linear and symmetric map. Moreover, given $x \in \mathscr{U}$ and $h^{(1)}, \dots, h^{(n)} \in X$, let $r > 0$ be such that $\overline{\mathscr{U}}_{r,x}(X) \subset \mathscr{U}$ and let $(\rho_j)_{j=1}^n$ be a list of real numbers such that, for all $j \in \{1, \dots, n\}$, $0 < \rho_j < r/(n|h^{(j)}|_X)$. Then, for all $m \in I$, since each $f^{(m)}$ is analytic (and hence weak-analytic), by the Cauchy's integral formula applied to the complex valued functions of n complex variables

$$\prod_{j=1}^n \mathscr{U}_{r/(n|h^{(j)}|_X)} \ni (z_1, \dots, z_n) \mapsto f^{(m)}\left(x + \sum_{j=1}^n \xi_j h^{(j)}\right) \in \mathbb{C},$$

we have

$$d^n f^{(m)}(x)[h^{(1)}, \dots, h^{(n)}] = \frac{1}{(2\pi \mathrm{i})^n} \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_n|=\rho_n} \frac{f^{(m)}(x + \sum_{j=1}^n \xi_j h^{(j)})}{\prod_{j=1}^n \xi_j^2} d\xi_1 \dots d\xi_n,$$

and the integrals do not depend on $(\rho_j)_{j=1}^n$ as long as, $\forall j = 1, \dots, n$, $0 < \rho_j < r/(n|h^{(j)}|_X)$. Therefore, one immediately has the estimate

$$|d^n f^{(m)}(x)[h^{(1)}, \dots, h^{(n)}]| \le C \prod_{j=1}^n \frac{1}{2\pi} \int_{|\xi_j|=\rho_j} \frac{|d\xi_j|}{|\xi_j|^2} = C \prod_{j=1}^n \frac{1}{\rho_j},$$

where $C := \sup_{m \in I} \sup_{x \in \mathscr{U}} |f^{(m)}(x)|$, from which, by taking the limits $\rho_j \to r/(n|h^{(j)}|_X)$, one derives the boundedness of $d^n f(x)$, namely,

$$|d^n f(x)[h^{(1)}, \dots, h^{(n)}]|_{\ell^\infty(\mathbb{C})} \le C \frac{n^n}{r^n} \prod_{j=1}^n |h^{(j)}|_X.$$

Now we show that, for all $n \in \mathbb{N}$, $d^n f$ is indeed the n^{th}-differential of f. Of course, we set $d^0 f := f$. Now, given $x \in \mathscr{U}$ and $(h^{(j)})_{j=1}^n \in X^n$, let r and $(\rho_j)_{j=1}^n$ be as before. Then, one has

$$\begin{aligned} &d^{n-1} f^{(m)}(x + h^{(n)})[h^{(1)}, \dots, h^{(n-1)}] \\ &\quad = \frac{1}{(2\pi \mathrm{i})^{n-1}} \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_{n-1}|=\rho_{n-1}} \frac{f^{(m)}(x + \sum_{j=1}^{n-1} \xi_j h^{(j)} + h^{(n)})}{\prod_{j=1}^{n-1} \xi_j^2} d\xi_1 \dots d\xi_{n-1}. \end{aligned} \tag{1.18}$$

By implementing again Cauchy's integral formula applied to the analytic function

$$\mathscr{U}_{r/(n|h^{(n)}|_X)} \ni z \mapsto f^{(m)}\left(x + \sum_{j=1}^{n-1} \xi_j h^{(j)} + z h^{(n)}\right) \in \mathbb{C},$$

one gets

$$(1.18) = \frac{1}{(2\pi \mathrm{i})^n} \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_n|=\rho_n} \frac{f^{(m)}(x + \sum_{j=1}^n \xi_j h^{(j)})}{\prod_{j=1}^{n-1} \xi_j^2 (\xi_n - 1)} d\xi_1 \dots d\xi_n.$$

Furthermore, the expansion

$$\frac{1}{\xi_n - 1} = \frac{1}{\xi_n} + \frac{1}{\xi_n^2} + \frac{1}{\xi_n^2(\xi_n - 1)},$$

and further applications of Cauchy's integral formula, yield

$$\begin{aligned} (1.18) &= d^{n-1} f^{(m)}(x)[h^{(1)}, \dots, h^{(n-1)}] + d^n f^{(m)}(x)[h^{(1)}, \dots, h^{(n)}] \\ &\quad + \frac{1}{(2\pi \mathrm{i})^n} \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_n|=\rho_n} \frac{f^{(m)}(x + \sum_{j=1}^n \xi_j h^{(j)})}{\prod_{j=1}^n \xi_j^2 (\xi_n - 1)} d\xi_1 \dots d\xi_n. \end{aligned}$$

Therefore,

$$\begin{aligned}&\left|(d^{n-1}_{x+h^{(n)}}f^{(m)}-d^{n-1}f^{(m)}(x))[h^{(1)},\dots,h^{(n-1)}]-d^nf^{(m)}(x)[h^{(1)},\dots,h^{(n)}]\right|\\&\quad\le C\frac{1}{\rho_n(\rho_n-1)}\prod_{j=1}^{n-1}\frac{1}{\rho_j},\end{aligned}$$

from which, by taking the limits $\rho_j\to r/(n|h^{(j)}|_X)$ one gets

$$\begin{aligned}&\sup_{h^{(1)},\dots,h^{(n-1)}\in\mathscr{U}_1(X)}|(d^{n-1}f(x+h^{(n)})-d^{n-1}f(x))[h^{(1)},\dots,h^{(n-1)}]-d^nf(x)[h^{(1)},\dots,h^{(n)}]|_{\ell^\infty(\mathbb{C})}\\&\quad\le C\frac{n^n}{r^n}\frac{|h^{(n)}|_X^2}{(r/n-1)}=o(|h^{(n)}|_X).\end{aligned}$$

We conclude the proof by showing that, for any $x\in\mathscr{U}$, there exists $r_0>0$ such that the Taylor series

$$\sum_{n=0}^{\infty}\frac{1}{n!}d^nf(x)[h,\dots,h]$$

converges to $f(x+h)$ uniformly for $h\in\mathscr{U}_{r_0}(X)$. The thesis follows by Pöschel-Trubowitz, Appendix A, Theorem 1. First of all, let $r>0$ be such that $\overline{\mathscr{U}}_{r,x}(X)\subset\mathscr{U}$, $h\in X$, and take any $\rho\in(1,r/|h|_X)$. Then, Cauchy's integral formula gives

$$d^nf^{(m)}(x)[h,\dots,h]=\frac{n!}{2\pi\mathrm{i}}\int_{|\xi|=\rho}\frac{f^{(m)}(x+\xi h)}{\xi^{n+1}}d\xi.$$

This formula, together with the expansion

$$\frac{1}{(\xi-1)}=\sum_{n=0}^{N}\frac{1}{\xi^{n+1}}+\frac{1}{\xi^{N+1}(\xi-1)}\ ,$$

yields

$$f^{(m)}(x+h)-\sum_{n=0}^{N}\frac{1}{n!}d^nf^{(m)}(x)[h,\dots,h]=\frac{1}{2\pi\mathrm{i}}\int_{|\xi|=\rho}\frac{f^{(m)}(x+\xi h)}{\xi^{N+1}(\xi-1)}d\xi.$$

Therefore,

$$\left|f(x+h)-\sum_{n=0}^{N}\frac{1}{n!}d^nf(x)[h,\dots,h]\right|_{\ell^\infty(\mathbb{C})}\tag{1.19}$$

is bounded by

$$C\frac{|h|_X^{N+1}}{r^N(r-|h|_X)}.$$

From this estimate, we conclude that

$$\lim_{N\to\infty}(1.19)=0$$

uniformly for $h\in\mathscr{U}_{r_0}(X)$, for any $r_0<r$. ■

Remark 1.2.25. *Note that the proof of Theorem 1.2.24 implies that, for all $r,s>0$ and for all $n\in\mathbb{N}$, the n^{th} differential of any F belonging to either $\mathcal{A}^{\infty,w}_{r,s}$ or $\mathcal{A}^\infty_{r,s}$, at $(y,\varphi)\in\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty$, is given by*

$$d^nF(y,\varphi)[h^{(1)},\dots,h^{(n)}]=(d^nF^{(m)}(y,\varphi)[h^{(1)},\dots,h^{(n)}])_{m\in\mathbb{N}}.$$

Analogous relations hold for ∂^n_yF and $\partial^n_\varphi F$.

1.2.5 Changes of coordinates

As announced in the introduction to the chapter, in the next section we shall prove a theorem about foliations. It follows that we need to introduce the notion of change of coordinates.

Definition 1.2.26 (Changes of coordinates)**.** *A function $\Phi \in \mathcal{C}_{r_0,s_0}(\mathcal{A})$ is a change of coordinates if there exist $\rho \in [0, r_0/2)$ and $\sigma \in [0, s_0/2)$ such that:*

(a) for all $r \le r_0$,

$$\Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho}(\ell^\infty_w) \times \ell^\infty$$

with the property that, for all $r \in (\rho, r_0 + \rho]$, $s \in (\sigma, s_0 + \sigma]$, and every $f \in \mathcal{A}_{r,s}$, the pullback $\Phi^* f$ belongs to $\mathcal{A}_{r-\rho,s-\sigma}$ with

$$|\Phi^*|^{\rho,\sigma}_{r,s} := \sup_{f \neq 0} \frac{|\Phi^* f|_{r-\rho,s-\sigma}}{|f|_{r,s}} < \infty;$$

(b) there exists $\Psi \in \mathcal{C}_{r_0-\rho,s_0-\sigma}(\mathcal{A})$ such that, for all $r \le r_0 - \rho$,

$$\Psi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho}(\ell^\infty_w) \times \ell^\infty$$

and, for all $r \in (\rho, r_0]$, $s \in (\sigma, s_0]$, and every $f \in \mathcal{A}_{r,s}$, $\Psi^* f \in \mathcal{A}_{r-\rho,s-\sigma}$ with

$$|\Psi^*|^{\rho,\sigma}_{r,s} := \sup_{f \neq 0} \frac{|\Psi^* f|_{r-\rho,s-\sigma}}{|f|_{r,s}} < \infty,$$

which is the inverse of Φ in the following sense:

$$\Phi \circ \Psi = id_{\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty}, \quad \forall r \le r_0 - \rho, \qquad \Psi \circ \Phi = id_{\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty}, \quad \forall r \le r_0 - 2\rho.$$

Note that, given $r_0, s_0 > 0$, $\rho \in [0, r_0)$ and $\sigma \in [0, s_0)$, any invertible map $\Phi \in \mathcal{C}^{\rho,\sigma/2}_{r_0,s_0}(\mathcal{A})$ with inverse $\Psi \in \mathcal{C}^{\rho,\sigma/2}_{r_0-\rho,s_0-\sigma}(\mathcal{A})$ is a (close to identity) change of coordinates, as follows by Proposition 1.2.19.

Remark 1.2.27. *Any change of coordinates Φ is a bi-analytic diffeomorphism because, in particular, both Φ and its inverse Ψ belong to $\mathcal{C}(\mathcal{A})$. Hence, by Remark 1.2.21, Φ is an analytic map with an analytic inverse.*

Note that, given $r_0 > 0$ and $\rho \in [0, r_0)$, any map

$$\Phi : \mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad \Phi = \mathrm{id} + (N, \Omega),$$

satisfying condition (a) of the above definition is such that $N \in \mathcal{A}^{\infty,w}_{r_0-\rho,s_0-\sigma}$. Indeed, one has the bound:

$$\begin{aligned}
\sup_{k\in\mathbb{N}} w_k |N^{(k)}|_{r_0-\rho,s_0-\sigma} &= \sup_{k\in\mathbb{N}} w_k |\Phi^* y_k - \mathrm{id}^* y_k|_{r_0-\rho,s_0-\sigma} \\
&\le r_0 \sup_{k\in\mathbb{N}} \frac{|\Phi^* y_k|_{r_0-\rho,s_0-\sigma}}{w_k r_0} + r_0 - \rho \\
&= r_0 \left(1 + \sup_{k\in\mathbb{N}} \frac{|\Phi^* y_k|_{r_0-\rho,s_0-\sigma}}{|y_k|_{r_0,s_0}} \right) - \rho \\
&\le r_0 (1 + |\Phi^*|^{\rho,\sigma}_{r_0,s_0}) - \rho.
\end{aligned}$$

The same argument does not apply to Ω because the maps $(y, \varphi) \mapsto \varphi_k$, $k \in \mathbb{N}$, are not periodic and, hence, do not belong to the class $\mathcal{A}$. One is tempted to follow the same reasoning using the family of functions $(y, \varphi) \mapsto e^{i\varphi_k}$, $k \in \mathbb{N}$, and subsequently attempt to provide an estimate for the logarithm. It can be shown that this approach works if Ω depends only on finitely many angles; however, we are neither able to extend the result to the general case nor to provide a counterexample. For this reason, we decided to consider the condition $\Omega \in \mathcal{A}^{\infty}_{r_0-\rho, s_0-\sigma}$ as an assumption.

1.3 Robustness of Foliations by Infinite Tori

In this section, we prove the main theorem of the chapter, which asserts the robustness of foliations by infinite tori under real analytic perturbations. The result is established for an open neighborhood of $\{0\} \times \mathbb{T}^\infty$ within the Banach manifold $\ell^\infty_w \times \mathbb{T}^\infty$. Before stating the theorem, let us introduce a notation.

Notation 1.3.1. *Given $P \in \mathcal{A}^{\infty,w}_{r,s}$, we denote by Γ_P the map*

$$\Gamma_P : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \to \ell^\infty_w, \qquad \Gamma_P(y, \pi_\infty(\varphi)) = y + P(y, \varphi), \quad \forall y \in \mathscr{U}_r(\ell^\infty_w),\ \varphi \in \ell^\infty.$$

Accordingly, Γ_0 denotes the projection onto the first factor.

Our goal is to show that if P is sufficiently small, namely if Γ_P is a perturbation of the projection onto the first factor, then its level sets are infinite tori and give rise to a foliation of a neighborhood of $\{0\} \times \mathbb{T}^\infty$. More precisely, we have the following:

Theorem 1.3.2 (Foliation by infinite tori)**.** *Given $r_0 > 0$, if $P \in \mathscr{U}_{r_0/6}(\mathcal{A}^{\infty,w}_{r_0,s_0})$ for some $s_0 > 0$, then for all $r \leq r_0 - 3|P|^{\infty,w}_{r_0,s_0}$ one has that:*

(a) for all $\gamma \in \mathscr{U}_r(\ell^\infty_w)$, the level set $\Gamma_P^{-1}(\gamma)$ is the graph of a majorant analytic function from $\mathbb{T}^\infty$ to ℓ^∞_w and hence it is analytically diffeomorphic to $\mathbb{T}^\infty$;

(b) $\mathscr{N}_0 := \Gamma_P^{-1}(\mathscr{U}_r(\ell^\infty_w))$ is a neighborhood of $\{0\} \times \mathbb{T}^\infty$, and there is a bi-analytic map χ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{N}_0 \arrow[r, "\chi"] \arrow[d, "\Gamma_P"'] & \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \arrow[dl, "\Gamma_0"] \\ \mathscr{U}_r(\ell^\infty_w) & \end{tikzcd}$$

Remark 1.3.3. *It is worth noting that Theorem 1.3.2 extends to a broader class of maps of the form*

$$\Gamma(y, \varphi) = f(y) + P(y, \varphi),$$

provided that f is analytic and possesses an invertible differential. In fact, by performing a local Taylor expansion of f around a point y_0, the map Γ can be recast as $f(y_0) + df(y_0)[y - y_0] + \mathcal{O}(\|y - y_0\|^2_{\ell^\infty_w}) + P(y, \varphi)$. Therefore, the statement still holds provided one sets $\tilde{\gamma} = y_0 + df(y_0)^{-1}[\gamma - f(y_0)]$ and requires the smallness condition on the redefined perturbation $\tilde{P}(y, \varphi) = df(y_0)^{-1}(\mathcal{O}(\|y - y_0\|^2_{\ell^\infty_w}) + P(y, \varphi))$.

Remark 1.3.4. *A less quantitative version of Theorem 1.3.2 (with no explicit upper bound on r) would have a simpler proof based on the implicit function theorem in the analytic Banach manifolds setting. Nevertheless, this quantitative version requires the more delicate analysis.*

The approach for proving Theorem 1.3.2 has already been outlined in the introduction to this chapter. However, since we prefer to work with functions defined on the universal cover of $\mathbb{T}^\infty$, we need to provide a more detailed explanation of the strategy.

Definition 1.3.5. *We denote by q the projection $q : \ell^\infty_w \times \ell^\infty \to \ell^\infty_w \times \mathbb{T}^\infty$ defined by*

$$q(y, \varphi) = (y, \pi_\infty(\varphi)).$$

Definition 1.3.6. *Given a function $\Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty$ such that $\Phi - id$ is 2π-periodic in each φ_k-variable, we denote by $\tilde{\Phi}$ the map that makes the following diagram commute:*

$$\begin{tikzcd}
\mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \arrow[r, "\Phi"] \arrow[d, "q"'] & \ell^\infty_w \times \ell^\infty \arrow[d, "q"] \\
\mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \arrow[r, "\tilde{\Phi}"'] & \ell^\infty_w \times \mathbb{T}^\infty
\end{tikzcd}$$

Remark 1.3.7. *If Φ is a change of coordinates with inverse Ψ, then $\tilde{\Phi}$ is a bi-analytic diffeomorphism with inverse $\tilde{\Psi}$. This follows from the fact that, if Φ has inverse Ψ, then $\tilde{\Phi}$ is also invertible with inverse $\tilde{\Psi}$; moreover, any change of coordinates is a bi-analytic diffeomorphism (see Remark 1.2.27), and π_∞ is analytic.*

Notation 1.3.8. *Given $G \in \mathcal{A}^{\infty,w}_{r,s}$, we denote by Φ_G the map*

$$\Phi_G(y, \varphi) = (y + G(y, \varphi), \varphi), \qquad \forall y \in \mathscr{U}_r(\ell^\infty_w),\ \varphi \in \ell^\infty.$$

Clearly, $\Phi_G \in \mathcal{C}_{r,s}(\mathcal{A})$.

The procedure for proving Theorem 1.3.2 can be summarized as follows:

- In Corollary 1.3.11, we show that if the perturbation P satisfies a suitable smallness condition such as the one in Theorem 1.3.2, there exists $r > 0$ and a family of multi-periodic, majorant analytic maps $(A(\gamma, \cdot) : \ell^\infty \to \ell^\infty_w)_{\gamma \in \mathscr{U}_r(\ell^\infty_w)}$ such that, for any $\gamma \in \mathscr{U}_r(\ell^\infty_w)$, the level set associated with the value γ of the function

  $$\mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty \ni (y, \varphi) \mapsto y + P(y, \varphi) \in \ell^\infty_w$$

  is the graph of the majorant analytic function $\ell^\infty \ni \varphi \mapsto \gamma - A(\gamma, \varphi) \in \mathscr{U}_{r_0}(\ell^\infty_w)$. Clearly, the functions $A(\gamma, \cdot)$ must satisfy the condition

  $$A(\gamma, \varphi) = P(\gamma - A(\gamma, \varphi), \varphi), \quad \text{for all } \gamma \in \mathscr{U}_r(\ell^\infty_w),\ \varphi \in \ell^\infty.$$

  It is then clear that a good strategy consists in formalizing our question as a fixed point problem for the nonlinear operator $G \mapsto \Phi^*_G P$ acting on a closed ball of the Banach space $\mathcal{A}^{\infty,w}_{r,s_0}$, where $\Phi^*_G P(y, \varphi) := (\Phi^*_G P^{(m)}(y, \varphi))_{m \in \mathbb{N}}$. In order to show that the operator $\Phi^*_{(\cdot)} P$ acts as a contraction, we show that it is differentiable in the sense of Fréchet and provide a suitable estimate for the differential. These technical results are proved in Proposition 1.3.9 and Lemma 1.3.10. Therefore, Corollary 1.3.11 follows directly from Lemma 1.3.10. This proves claim (a) of Theorem 1.3.2.

- Note that the function

  $$\mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \ni \pi_\infty(\varphi) \mapsto \gamma - A(\gamma, \varphi) \in \ell^\infty_w$$

depends analytically on γ because A is the fixed point of a contraction acting on a closed ball of a Banach space of analytic functions. Thus, one expects that the level sets $(\Gamma_P^{-1}(\gamma))_\gamma$ form an analytic foliation by maximal tori for all $\gamma \in \mathscr{U}_r(\ell_w^\infty)$. We prove that this intuition is correct in the following way: first, in Theorem 1.3.12, we show that Φ_P is a (close to identity) change of coordinates with inverse Φ_{-A} such that $\Phi_{-A}(\mathscr{U}_r(\ell_w^\infty) \times \ell^\infty)$ is a neighborhood of $\{0\} \times \ell^\infty$. Following Remark 1.3.7, it follows that $\tilde{\Phi}_P$ is a bi-analytic diffeomorphism from an open neighborhood of $\{0\} \times \mathbb{T}^\infty$ onto its image, with inverse $\tilde{\Phi}_{-A}$. Combining these results, we can provide the proof of Theorem 1.3.2.

Given $r, s > 0$, let $F \in \mathcal{A}_{r,s}^{\infty,w}$ and consider the action of the nonlinear operator

$$\mathscr{F} : G \mapsto \Phi_G^* F$$

on the open ball of radius $\rho < r$ about the origin of the space of analytic functions $\mathcal{A}_{r-\rho,s}^{\infty,w}$. Note that the action is well defined because

$$\sup_{m\in\mathbb{N}} w_m \sup_{\mathscr{U}_{r-\rho}(\ell_w^\infty)\times\ell^\infty} |y_m + G^{(m)}(y,\varphi)| \le r - \rho + |G|_{r-\rho,s}^\infty < r,$$

i.e., the image of $\mathscr{U}_{r-\rho}(\ell_w^\infty) \times \ell^\infty$ under the action of Φ_G is contained in the domain of F. Let us start by investigating the differentiability properties of this action.

Proposition 1.3.9. *Given $r, s > 0$, let $F \in \mathcal{A}_{r,s}^{\infty,w}$. Then, for any $\rho \in (0, r)$, the map*

$$\mathscr{F} : \mathscr{U}_\rho(\mathcal{A}_{r-\rho,s}^{\infty,w}) \ni G \mapsto \Phi_G^* F \in \mathcal{A}_{r-\rho,s}^{\infty,w}$$

is Fréchet differentiable. Moreover, the following estimate holds:

$$|d\mathscr{F}(G)|_{op} \le \frac{r}{r-\rho_G}\frac{2}{e\rho_G}|F|_{r,s}^{\infty,w}, \tag{1.20}$$

where $\rho_G := \min\{\rho - |G|_{r-\rho,s}^{\infty,w}, r/2\}$.

Proof. Let us show that the Fréchet differential $d\mathscr{F}(G)$ of $\mathscr{F}$ at $G \in \mathscr{U}_\rho(\mathcal{A}_{r-\rho,s}^{\infty,w})$ is the linear map that associates to each $H \in \mathcal{A}_{r-\rho,s}^{\infty,w}$ the function in $\mathcal{A}_{r-\rho,s}^{\infty,w}$ given by

$$(\Phi_G^*\, \partial_y F)[H] : (y,\varphi) \mapsto \partial_y F(y + G(y,\varphi), \varphi)[H(y,\varphi)].$$

Let $H \in \mathcal{A}_{r-\rho,s}^{\infty,w}$ be such that $G + H \in \mathscr{U}_\rho(\mathcal{A}_{r-\rho,s}^{\infty,w})$. We must provide a suitable estimate for the following quantity:

$$|\, \mathscr{F}(G+H) - \mathscr{F}(G) - (\Phi_G^*\, \partial_y F)[H]\, |_{r-\rho,s}^{\infty,w}. \tag{1.21}$$

Note that, for $|t| \le 1$, the equality

$$\begin{aligned}
&(\mathscr{F}^{(m)}(G+tH))(y,\varphi) - (\mathscr{F}^{(m)}(G))(y,\varphi) - t\partial_y F^{(m)}(y+G(y,\varphi),\varphi)[H(y,\varphi)] \\
&= \sum_{\nu,\alpha} F_{\nu\alpha}^{(m)} \Bigg((y+G(y,\varphi)+tH(y,\varphi))^\alpha - (y+G(y,\varphi))^\alpha \\
&\quad - t\sum_{k\in\mathbb{N}} H^{(k)}(y,\varphi)\alpha_k (y+G(y,\varphi))^{\alpha-e_k} \Bigg) e^{\mathrm{i}\nu\cdot\varphi}
\end{aligned}$$

holds pointwise for all $(y,\varphi) \in \mathscr{U}_{r-\rho}(\ell_w^\infty) \times \ell^\infty$. Defining the analytic function

$$g_\alpha : t \mapsto g_\alpha(t) := (y + G(y,\varphi) + tH(y,\varphi))^\alpha,$$

one can write the above formula as

$$\sum_{\nu,\alpha} F^{(m)}_{\nu\alpha}\left(\int_0^t g''_\alpha(\tau)(t-\tau)d\tau\right)e^{\mathrm{i}\nu\cdot\varphi},$$

which is equivalent to

$$\sum_{\nu,\alpha} F^{(m)}_{\nu\alpha}\left(\int_0^t \sum_{k_1,k_2\in\mathbb{N}} (H^{(k_1)}H^{(k_2)})(y,\varphi)\frac{\partial^2 y^\alpha}{\partial y_{k_1}\partial y_{k_2}}\circ\Phi_{G+\tau H}(y,\varphi)(t-\tau)d\tau\right)e^{\mathrm{i}\nu\cdot\varphi},$$

where, by abuse of notation, y^α denotes the majorant analytic function $(y,\varphi)\mapsto y^\alpha$. By Proposition 1.2.6 and the triangle inequality, the $|\cdot|_{r-\rho,s}$ norm of the above expression is bounded by

$$(|H|^{\infty,w}_{r-\rho,s})^2\sum_{\nu,\alpha}|F^{(m)}_{\nu\alpha}|\left(\int_0^t\sum_{k_1,k_2\in\mathbb{N}} w_{k_1}^{-1}w_{k_2}^{-1}\left|\frac{\partial^2 y^\alpha}{\partial y_{k_1}\partial y_{k_2}}\circ\Phi_{G+\tau H}\right|_{r-\rho,s}(t-\tau)d\tau\right)e^{s|\nu|}. \tag{1.22}$$

Evaluating at $t=1$, Proposition 1.2.19 yields

$$(1.22)\le 2(|H|^{\infty,w}_{r-\rho,s})^2\sum_{\nu,\alpha}|F^{(m)}_{\nu\alpha}|\left(\int_0^1\sum_{k_1,k_2\in\mathbb{N}} w_{k_1}^{-1}w_{k_2}^{-1}\left|\frac{\partial^2 y^\alpha}{\partial y_{k_1}\partial y_{k_2}}\right|_{r-\rho',s}(1-\tau)d\tau\right)e^{s|\nu|},$$

for any ρ' such that $0<\rho'\le\rho-|G|^{\infty,w}_{r-\rho,s}-|H|^{\infty,w}_{r-\rho,s}$. Furthermore, by Lemma 1.2.13, one can estimate the above expression by

$$\left(\frac{r}{r-\rho'}\right)^2\frac{4e^{-2}}{(\rho')^2}(|H|^{\infty,w}_{r-\rho,s})^2|F^{(m)}|_{r,s}.$$

Hence,

$$(|H|^{\infty,w}_{r-\rho,s})^{-1}\times(1.21)\le\left(\left(\frac{r}{r-\rho'}\right)^2\frac{4e^{-2}}{(\rho')^2}|F|^{\infty,w}_{r,s}\right)|H|^{\infty,w}_{r-\rho,s},$$

which tends to 0 as $|H|^{\infty,w}_{r-\rho,s}\to 0$.

It remains to show that $\mathcal{A}^{\infty,w}_{r-\rho,s}\ni H\mapsto(\Phi_G^*\,\partial_y F)[H]\in\mathcal{A}^{\infty,w}_{r-\rho,s}$ is a bounded operator. Proposition 1.2.6 and the triangle inequality imply:

$$\begin{aligned}|\,(\Phi_G^*\,\partial_y F)[H]\,|^{\infty,w}_{r-\rho,s}&\le\sup_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}}|H^{(k)}|_{r-\rho,s}\left|\frac{\partial F^{(m)}}{\partial y_k}\circ\Phi_G\right|_{r-\rho,s}\\&\le|H|^{\infty,w}_{r-\rho,s}\sup_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}}w_k^{-1}\left|\frac{\partial F^{(m)}}{\partial y_k}\circ\Phi_G\right|_{r-\rho,s}.\end{aligned}\tag{1.23}$$

Invoking again Proposition 1.2.19 and Lemma 1.2.13, we obtain:

$$(1.23)\le 2|H|^{\infty,w}_{r-\rho,s}\sup_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}}w_k^{-1}\left|\frac{\partial F^{(m)}}{\partial y_k}\right|_{r-\rho',s}\le 2|H|^{\infty,w}_{r-\rho,s}\frac{r}{r-\rho'}\frac{1}{e\rho'}|F|^{\infty,w}_{r,s},$$

where $0<\rho'\le\rho-|G|^{\infty,w}_{r-\rho,s}$. The function

$$(0,r)\ni\rho'\mapsto\frac{1}{(r-\rho')\rho'}$$

is strictly convex and attains its minimum at $\rho' = r/2$. Hence, $\mathscr{F}$ is Fréchet differentiable and its differential satisfies:

$$|d\mathscr{F}(G)|_{\text{op}} = \sup_{H\in\mathscr{U}_1} |\,(\Phi_G^*\,\partial_y F)[H]\,|_{r-\rho,s}^{\infty,w} \leq \frac{r}{r-\rho_G}\frac{2}{e\rho_G}|F|_{r,s}^{\infty,w},$$

where $\rho_G := \min\{\rho - |G|_{r-\rho,s}^{\infty,w}, r/2\}$. This concludes the proof. ■

Lemma 1.3.10 (First Fixed Point Lemma)**.** *Given $r,s > 0$, let $F \in \mathscr{U}_{r/4}(\mathcal{A}_{r,s}^{\infty,w})$ and define $\rho := 3|F|_{r,s}^{\infty,w}$. Consider the map*

$$\mathscr{F} : \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w}) \ni G \mapsto \Phi_G^* F \in \mathcal{A}_{r-\rho,s}^{\infty,w}$$

with Φ_G as in 1.3.8. Then, there exists a unique $\hat{G} \in \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})$ such that

$$\hat{G} = \Phi_{\hat{G}}^* F.$$

Proof. The strategy is the following: first, we show that

$$\sup_{G\in\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})} |\Phi_G^* F|_{r-\rho,s}^{\infty,w} \leq 2\rho/3\ ;$$

then, we prove that $\mathscr{F}$ acts as a contraction on $\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})$ and the thesis will follow by Remark 1.2.17 and by the fact that $\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w}) \subset \mathcal{A}_{r-\rho,s}^{\infty,w}$ is closed.

The first point follows as an immediate consequence of Proposition 1.2.19 by setting, with the same notations of Proposition 1.2.19, $\Psi = \Phi_G$, i.e., $N = G$, $\Omega \equiv 0$. Indeed,

$$|G|_{r-\rho,s}^{\infty,w} \leq 2\rho/3 \leq \rho.$$

Then, for all $G \in \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})$, the sequence $(G_n)_{n\in\mathbb{N}}$ defined by

$$G_{n+1} := \mathscr{F}(G_n), \qquad G_1 := G \tag{1.24}$$

is contained in $\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})$.

Since $\mathscr{F}$ is Fréchet differentiable[3] (see Proposition 1.3.9), if we show that its Fréchet differential $d\mathscr{F}$ satisfies

$$\sup_{G\in\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})} |d\mathscr{F}(G)|_{op} \leq K < 1,$$

by Lagrange's inequality, one has

$$|\mathscr{F}(G) - \mathscr{F}(G')|_{r-\rho,s}^{\infty,w} \leq \sup_{G\in\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})} |d\mathscr{F}(G)|_{op}|G-G'|_{r-\rho,s}^{\infty,w} \leq K|G-G'|_{r-\rho,s}^{\infty,w}.$$

Namely, $\mathscr{F}$ would turn out to be a contraction with a Lipschitz constant equal to $K < 1$. We make use of (1.20) for obtaining a uniform bound for the operator norm of the Fréchet differential of $\mathscr{F}$. First, note that

$$\min_{G\in\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}_{r-\rho,s}^{\infty,w})} \rho_G = \min\{\rho/3, r/2\} = \rho/3.$$

[3] Actually, Gateaux differentiability would have been sufficient for our purposes.

Then, (1.20) gives

$$\sup_{G\in\overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}^{\infty,w}_{r-\rho,s})} |d\mathscr{F}(G)|_{\mathrm{op}} \le \frac{2}{e}\frac{r}{r-\rho/3}\frac{3|F|^{\infty,w}_{r,s}}{\rho} = \frac{2}{e}\frac{r}{r-\rho/3} < \frac{8}{3e} < 1.$$

Therefore, there exists $\hat{G} \in \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}^{\infty,w}_{r-\rho,s})$ such that, for all $G \in \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}^{\infty,w}_{r-\rho,s})$, the sequence $(G_n)_{n\in\mathbb{N}}$ defined in (1.24) satisfies

$$\lim_{n\to\infty} |G_n - \hat{G}|^{\infty,w}_{r-\rho,s} = 0.$$

Of course, $\hat{G}$ satisfies $\hat{G} = \Phi^*_{\hat{G}}F$. ■

Let us address the claim (a) of Theorem 1.3.2. Namely, we prove the following proposition, which is of interest in its own right.

Corollary 1.3.11 (Level sets as graphs)**.** *Given $r, s > 0$, let $P \in \mathscr{U}_{r/4}(\mathcal{A}^{\infty,w}_{r,s})$ and consider*

$$\Gamma_P : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \to \ell^\infty_w, \qquad \Gamma_P(y, \pi_\infty(\varphi)) := y + P(y,\varphi)\ .$$

Then, there exists a unique $A \in \mathcal{A}^{\infty,w}_{r-\rho,s}$, where $\rho := 3|P|^{\infty,w}_{r,s}$ such that, $\forall \gamma \in \mathscr{U}_{r-\rho}(\ell^\infty_w)$,

$$\Gamma_P(\gamma - A(\gamma,\varphi), \pi_\infty(\varphi)) = \gamma\ , \qquad \forall \varphi \in \ell^\infty. \tag{1.25}$$

Proof. Parametrizing y as $y = \gamma - A(\gamma,\varphi)$, with $A \in \mathscr{U}_{2\rho/3}(\mathcal{A}^{\infty,w}_{r-\rho,s})$, one has the following:

$$\Gamma_P(\gamma - A(\gamma,\varphi), \pi_\infty(\varphi)) = \gamma - A(\gamma,\varphi) + P(\gamma - A(\gamma,\varphi),\varphi), \quad \forall \gamma \in \mathscr{U}_{r-\rho}(\ell^\infty_w)\ ,$$

which, when substituted into equation (1.25), gives

$$A(\gamma,\varphi) = P(\gamma - A(\gamma,\varphi),\varphi).$$

The claim follows by Lemma 1.3.10, by setting $F = -P$ and $\hat{G} = -A$. ■

Let us show that a function $\Gamma_P : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \to \ell^\infty_w$ as the one in the above theorem induces a change of coordinates. More precisely, we have the following

Theorem 1.3.12. *Given $r_0, s_0 > 0$, if $|P|^{\infty,w}_{r_0,s_0} < r_0/6$, the map*

$$\Phi_P : \mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (y,\varphi) \mapsto (y + P(y,\varphi), \varphi),$$

is a change of coordinates, with $\rho = 3|P|^{\infty,w}_{r_0,s_0}$, $\sigma = 0$, and inverse given by

$$\Phi_{-A} : \mathscr{U}_{r_0-\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (y,\varphi) \mapsto (y - A(y,\varphi), \varphi),$$

where $A \in \overline{\mathscr{U}}_{2\rho/3}(\mathcal{A}^{\infty,w}_{r_0-\rho,s_0})$ is defined by $(\Phi_{-A})^*P = A$. Moreover, $\Phi_{-A}(\mathscr{U}_{r_0-\rho}(\ell^\infty_w) \times \ell^\infty)$ is an open neighborhood of $\{0\} \times \ell^\infty$.

Proof. Since for all $r \le r_0$ and $s \le s_0$, $|P|^{\infty,w}_{r,s} \le |P|^{\infty,w}_{r_0,s_0} \le \rho/3$, one has $|y + P|^{\infty,w}_{r,s} \le r + \rho/3 \le r + \rho$. Then

$$\Phi_P : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho} \times \ell^\infty\ , \text{ for all } 0 < r \le r_0\ .$$

Moreover, by Proposition 1.2.19 with $N = P$ and $\Omega = 0$, for all $r \in (\rho, r_0 + \rho]$, $s \in (0, s_0]$, and every $f \in \mathcal{A}_{r,s}$

$$|\Phi^*_P f|_{r-\rho,s} \le 2|f|_{r,s}.$$

Note that, by Lemma 1.3.10, A is well defined and satisfies $|A|^{\infty,w}_{r_0-\rho,s_0} \leq 2\rho/3$. Let us prove that Φ_{-A} satisfies all the properties stated in (b) of Definition 1.2.26. Since for all $r \leq r_0 - \rho$ and $s \leq s_0$, $|y - A|^{\infty,w}_{r,s} \leq r + 2\rho/3 \leq r + \rho$, one has

$$\Phi_{-A} : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho} \times \ell^\infty .$$

Proposition 1.2.19, with $N = -A$ and $\Omega = 0$ gives that, for all $r \in (\rho, r_0]$, $s \in (0, s_0]$, and every $f \in \mathcal{A}_{r,s}$

$$|(\Phi_{-A})^* f|_{r-\rho,s} \leq 2|f|_{r,s}.$$

The fact that Φ_{-A} is the inverse of Φ_P follows from Lemma 1.3.10. The further condition $|P|^{\infty,w}_{r_0,s_0} < r_0/6$ ensures that the left inverse is also well defined. To conclude, note that, given a change of coordinates $\Phi \in \mathcal{C}_{r_0,s_0}(\mathcal{A})$ with inverse $\Psi \in \mathcal{C}_{r_0-\rho,s_0-\sigma}(\mathcal{A})$, one has that

$$\{0\} \times \ell^\infty \subset \mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty = \Psi \circ \Phi(\mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty) \subset \Psi(\mathscr{U}_{r_0-\rho} \times \ell^\infty) . \tag{1.26}$$

The last inclusion follows because $\Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho}(\ell^\infty_w) \times \ell^\infty$ for all $r \leq r_0$. ■

The preceding results provide sufficient information to establish the main theorem.

Proof of Theorem 1.3.2. Let us prove that, given $r_0, s_0 > 0$, for all $P \in \mathscr{U}_{r_0/6}(\mathcal{A}^{\infty,w}_{r_0,s_0})$, claims (a) and (b) of Theorem 1.3.2 hold for all $r \leq r_0 - 3|P|^{\infty,w}_{r_0,s_0}$. The first point was proved in Corollary 1.3.11. To prove that $\mathscr{N}_0 := \Gamma_P^{-1}(\mathscr{U}_r(\ell^\infty_w))$ is a neighborhood of $\{0\} \times \mathbb{T}^\infty$, it is sufficient to note that

$$\Gamma_P^{-1}(\mathscr{U}_r(\ell^\infty_w)) = q \circ \Phi_{-A}(\mathscr{U}_r(\ell^\infty_w) \times \ell^\infty) .$$

Indeed, by Theorem 1.3.12, $\Phi_{-A}(\mathscr{U}_r(\ell^\infty_w) \times \ell^\infty)$ is an open neighborhood of $\{0\} \times \ell^\infty$, and the image of a neighborhood of $\{0\} \times \ell^\infty$ under the action of q is a neighborhood of $\{0\} \times \mathbb{T}^\infty$. Finally, the diffeomorphism χ that makes the diagram in (b) of Theorem 1.3.2 commute is the one defined by $\chi \circ q = q \circ \Phi_P$, namely $\chi = \tilde{\Phi}_P$ (see Remark 1.3.7). ■

Theorem 1.3.2 provides the essential geometric framework for extending the *Liouville-Arnold* theorem to an infinite-dimensional setting by interpreting Γ_P as a momentum map. Nevertheless, a fundamental distinction from Arnold's original derivation (see, e.g., [Arn89]) lies in the characterization of the level sets of the momentum map: while the classical (finite-dimensional) approach recovers the tori as a quotient of the group action defined by the joint flow (an algebraic-geometric construction) with respect to the stabilizer, our method is fundamentally analytic, describing the tori as graphs of majorant functions. It should be noted, however, that this result is perturbative in nature, as it requires the momentum map to be a small perturbation of the projection onto the first factor, thus restricting the class of systems to which it applies. A primary focus of the next chapter is the construction of an infinite-dimensional version of *action-angle* coordinates adapted to a foliation by tori, such as the one investigated in this chapter.

Chapter 2

Local Liouville-Arnold Theorem in Infinite Dimension

2.1 Puroposes, General Strategy and Formal Framework

As established in the previous chapter, more precisely, in Theorem 1.3.2, the level sets of an analytic map from an open neighbourhood of $\{0\} \times \mathbb{T}^\infty$ to ℓ^∞_w , that is sufficiently close to the projection onto the first factor, are analytically diffeomorphic to $\mathbb{T}^\infty$ and form an analytic foliation. Let us recall the precise statement. Let Γ_P denote the map $\Gamma_P : \mathscr{U}_{r_0}(\ell^\infty_w) \times \mathbb{T}^\infty \to \ell^\infty_w$ defined by

$$\Gamma_P(y, \pi_\infty(\varphi)) = y + P(y, \varphi) \ , \qquad P \in \mathcal{A}^\infty_{r_0, s_0}$$

for all $y \in \mathscr{U}_{r_0}(\ell^\infty_w)$, $\varphi \in \ell^\infty$. Accordingly, Γ_0 denotes the projection onto the first factor. Then, Theorem 1.3.2 asserts that

Given $r_0 > 0$, if $P \in \mathscr{U}_{r_0/6}(\mathcal{A}^{\infty,w}_{r_0,s_0})$, for some $s_0 > 0$, then for all $r \le r_0 - 3|P|^{\infty,w}_{r_0,s_0}$ one has that:
- for all $\gamma \in \mathscr{U}_r(\ell^\infty_w)$, the level set $\Gamma_P^{-1}(\gamma)$ is the graph of a majorant analytic function from $\mathbb{T}^\infty$ to ℓ^∞_w and hence it is analytically diffeomorphic to $\mathbb{T}^\infty$;
- $\mathscr{N}_0 := \Gamma_P^{-1}(\mathscr{U}_r(\ell^\infty_w))$ is a neighbourhood of $\{0\} \times \mathbb{T}^\infty$, and there is a bi-analytic map χ such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{N}_0 \arrow[r, "\chi"] \arrow[d, "\Gamma_P"'] & \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \arrow[dl, "\Gamma_0"] \\
\mathscr{U}_r(\ell^\infty_w) &
\end{tikzcd}$$

More precisely, the bi-analytic map χ is defined by the relation $\chi(y, \pi_\infty(\varphi)) = \Phi_P(y, \varphi)$ for all $(y, \varphi) \in \mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty$, where Φ_P is given by

$$\Phi_P(y, \varphi) = (\Gamma_P(y, \pi_\infty(\varphi)), \varphi).$$

Moreover, it was shown in Theorem 1.3.12 that Φ_P is a well-defined change of coordinates.

In the following discussion, for the sake of simplicity, we shall slightly abuse notation by using the same symbol for functions defined on the torus $\mathbb{T}^\infty$ and their associated multi-periodic counterparts defined on ℓ^∞.

The primary goal of this chapter is to show that, under a further smallness condition on the perturbation relative to a more restrictive norm (see Definition 2.3.8), and assuming that the components of the map Γ_P Poisson-commute, namely

$$\{\Gamma_P^{(m)}, \Gamma_P^{(n)}\} = 0 \qquad \text{for all } \ m, n \in \mathbb{N},$$

there exists a symplectic straightening of the foliation provided by the aforementioned theorem. More precisely, we construct a coordinate transformation Δ such that the composition

$$\Phi := \Delta \circ \Phi_P \tag{2.1}$$

defines a symplectomorphism (cf. Definition 2.3.18). This map induces a new coordinate system (x, θ) where x depends on the original variables (y, φ) exclusively through Γ_P. Such a coordinate frame, which we refer to as *action-angle* coordinates, is adapted to the foliation: the coordinates $x = (x_k)_{k\in\mathbb{N}}$ act as labels for the leaves, while $\theta = (\theta_k)_{k\in\mathbb{N}}$ serves as the coordinate system along each leaf. To this end, we lift the classical generating function framework, as presented in the introduction (see Subsection 2 or [Arn89] for more detailed discussion) to an infinite-dimensional setting by adapting the standard Hamiltonian machinery to our context. It is clear that in infinite dimension one has to carefully check that all the algebraic manipulations make sense at a quantitative level. This is were the functional setting we are going to introduce, and in particular the notion of majorant analyticity and regularity (see Definitions 2.3.4, 2.3.8 and 2.3.11), come into the foreground. To provide some intuition, let us sketch the underlying heuristic, which follows the ideas developed for the finite-dimensional setting.

2.1.1 Candidate action coordinates

Using the same notation as in Theorem 1.3.2, the statement of which is recalled above, for all $\gamma \in \mathscr{U}_r(\ell_w^\infty)$, we endow the infinite torus $\Gamma_P^{-1}(\gamma)$ with the pullback measure $\lambda := p_\varphi^* \mu$, where p_φ is the projection onto the second factor $p_\varphi : \ell_w^\infty \times \mathbb{T}^\infty \to \mathbb{T}^\infty$ and μ is the Lebesgue product measure of the infinite torus, namely

$$\int_{\mathbb{T}^\infty} d\mu = \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} d\varphi_1 \dots d\varphi_N \ .$$

In this way, for all $\gamma \in \mathscr{U}_r(\ell_w^\infty)$ and for all $f \in \mathcal{A}_{r+\rho, s_0}$, with $\rho := 3|P|_{r_0,s_0}^{\infty,w}$, one has that

$$\int_{\Gamma_P^{-1}(\gamma)} f \, d\lambda := \int_{\mathbb{T}^\infty} f \circ (\Phi_P|_{\Gamma_P^{-1}(\gamma)})^{-1} \, d\mu = \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} f(\gamma - A(\gamma,\varphi), \varphi) \, d\varphi_1 \dots d\varphi_N \ ,$$

where the map $A \in \mathcal{A}_{r,s_0}$ is obtained in Proposition 1.3.11 and it is uniquely defined by

$$\Gamma_P(\gamma - A(\gamma,\varphi), \varphi) = \gamma \quad \text{ for all } (\gamma, \varphi) \in \mathscr{U}_r(\ell_w^\infty) \times \ell^\infty \ . \tag{2.2}$$

Drawing an analogy with statistical mechanics, we refer to the quantity above, which is well defined by Lemma 1.2.9, as the average of f over the infinite torus $\Gamma_P^{-1}(\gamma)$.

In the following discussions, we shall assume that components of Γ_P pairwise Poisson commute:

$$\{\Gamma_P^m, \Gamma_P^{(n)}\} := \sum_{k\in\mathbb{N}} \left(\frac{\partial \Gamma^{(m)}}{\partial y_k} \frac{\partial \Gamma^{(n)}}{\partial \varphi_k} - \frac{\partial \Gamma^{(m)}}{\partial \varphi_k} \frac{\partial \Gamma^{(n)}}{\partial y_k} \right) = 0 \qquad \text{for all } m, n \in \mathbb{N}. \tag{2.3}$$

Consider the map $\gamma \mapsto x(\gamma) = (x_k(\gamma))_{k\in\mathbb{N}}$ defined by

$$\mathscr{U}_r(\ell_w^\infty) \ni \gamma \mapsto x(\gamma) := \left(\int_{\Gamma_P^{-1}(\gamma)} y_k \, d\lambda \right)_{k\in\mathbb{N}} \in \ell_w^\infty \tag{2.4}$$

and note that, since for all $k \in \mathbb{N}$ we have

$$y_k \circ (\Phi_P|_{\Gamma_P^{-1}(\gamma)})^{-1}(\gamma, \varphi) = \gamma_k - A^{(k)}(\gamma, \varphi), \tag{2.5}$$

the definition of the integral over $\Gamma_P^{-1}(\gamma)$ with respect to the pullback measure λ yields the explicit formula

$$x_k(\gamma) = \gamma_k - A_0^{(k)}(\gamma), \quad \forall \gamma \in \mathscr{U}_r(\ell_w^\infty),\ k \in \mathbb{N}. \tag{2.6}$$

Notice that definition of the map $x(\gamma)$ in (2.4) provides the infinite-dimensional analogue of the integral of the Liouville one-form $\sum_n y_n d\varphi_n$ along the k^{th}-cycle of the torus defined by the level set $\Gamma_P^{-1}(\gamma)$. It is important to remark that the consistency of this identification relies on the commuting property (2.3). Indeed, as a consequence of the relations (2.12) (which we shall derive in the next subsections), the integral of the Liouville form along the full cycle parametrized by φ_k results in a constant value. Since the torus is endowed with a probability measure, this constant must coincide with the average over the torus, justifying the expression for the candidate action coordinates x_k as the mean values of the momenta y_k.

We shall prove that, under a suitable smallness condition on the perturbation P with respect to an appropriate stronger norm, as specified in Definition 2.3.8, the map

$$\mathscr{U}_r(\ell_w^\infty) \times \ell^\infty \ni (\gamma, \varphi) \mapsto (x(\gamma), \varphi) \in \ell_w^\infty \times \ell^\infty$$

indeed defines a change of coordinates. We claim that the coordinates $x = (x_k)_{k\in\mathbb{N}}$ defined above will play the role of *action* coordinates for the given foliation. However, the definition of action variables is inherently incomplete without their canonical counterparts; indeed, their characterization as actions is only fully realized when they are paired with the corresponding angle variables $\theta = (\theta_k)_{k\in\mathbb{N}}$ that satisfy the standard Poisson bracket relations. To complete this framework, we now briefly illustrate the construction of the associated angles. We achieve this by adapting the classical generating function method to the present infinite-dimensional setting.

2.1.2 Candidate angle coordinates

We seek an analytic generating function $S = S(x, \varphi)$ that satisfies

$$y_k(x, \varphi) = \frac{\partial S}{\partial \varphi_k}(x, \varphi) \quad \text{for all } k \in \mathbb{N} \tag{2.7}$$

and such that the relations

$$\theta_k(x, \varphi) = \frac{\partial S}{\partial x_k}(x, \varphi) \quad \text{for all } k \in \mathbb{N} \tag{2.8}$$

are invertible in ℓ^∞ with respect to φ. Assuming that, for all $k \in \mathbb{N}$, the functions $\theta_k(x, \varphi) - \varphi_k$ and $y_k(x, \varphi)$ are 2π-periodic in each φ_j-variable, $j \in \mathbb{N}$, equations (2.7) and (2.8) define a symplectomorphism, and the list $(\theta_k)_{k\in\mathbb{N}}$ constitutes a system of angle coordinates (with respect to the actions $(x_k)_{k\in\mathbb{N}}$ defined above) on each leaf of the foliation. Let us show the validity of this assertion. For the moment, we disregard any issues concerning the regularity of the maps and the convergence of the series involved, which will be addressed rigorously later in Section 2.5.2. At a formal level, as in finite dimension, the preservation of the Poisson brackets and the invariance of the symplectic form are equivalent. Therefore, we verify the symplectomorphism property by showing the invariance of the canonical symplectic form, which streamlines the algebraic manipulations.

$$\begin{aligned}\sum_{k\in\mathbb{N}} dy_k\wedge d\varphi_k &= \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}\frac{\partial^2 S}{\partial x_\ell\partial\varphi_k}dx_\ell\wedge d\varphi_k = \sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}\frac{\partial\theta_\ell}{\partial\varphi_k}dx_\ell\wedge d\varphi_k\\ &= \sum_{\ell\in\mathbb{N}} dx_\ell\wedge\sum_{k\in\mathbb{N}}\frac{\partial\theta_\ell}{\partial\varphi_k}d\varphi_k = \sum_{\ell\in\mathbb{N}} dx_\ell\wedge\left(d\theta_\ell-\sum_{k\in\mathbb{N}}\frac{\partial\theta_\ell}{\partial x_k}dx_k\right)\\ &= \sum_{\ell\in\mathbb{N}} dx_\ell\wedge d\theta_\ell-\sum_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{N}}\frac{\partial^2 S}{\partial x_k\partial x_\ell}dx_\ell\wedge dx_k\\ &= \sum_{\ell\in\mathbb{N}} dx_\ell\wedge d\theta_\ell\ .\end{aligned}$$

The last equality follows by the antisymmetry property of the wedge product. Hence, the map $(y,\varphi)\mapsto(x,\theta)$ is a symplectomorphism. The role of $x=(x_k)_{k\in\mathbb{N}}$ as a label for the leaves is guaranteed by the fact that x depends on the original variables only through Γ_P . Finally, since, for all $k\in\mathbb{N}$, the functions $\theta_k(x,\varphi)-\varphi_k$ are 2π-periodic in each φ_j variable, $j\in\mathbb{N}$, $(\theta_k)_{k\in\mathbb{N}}$ are indeed angles that parametrize each torus.

We assume that the generating function S is an analytic perturbation of the generator of the identity,

$$S(x,\varphi)=\sum_{k\in\mathbb{N}} x_k\varphi_k-W(x,\varphi), \tag{2.9}$$

The perturbation W is assumed to be 2π-periodic in each φ_k-variable, $k\in\mathbb{N}$. Then, Equation (2.7) reads

$$y_k(x,\varphi)=x_k-\frac{\partial W}{\partial\varphi_k}(x,\varphi)\quad\text{for all } k\in\mathbb{N}\ . \tag{2.10}$$

Given $m\in\mathbb{N}$, let Λ_m be the subset of $\mathbb{Z}^{(\mathbb{N})}$ defined by

$$\Lambda_m:=\{\nu\in\mathbb{Z}^{(\mathbb{N})}\ :\ \nu_m\neq 0\,,\nu_j=0\ \text{ for all } j<m\}\ .$$

We claim that the function

$$W(x,\varphi)=\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m}\frac{A_\nu^{(m)}(\gamma(x))}{i\nu_m}e^{i\nu\cdot\varphi} \tag{2.11}$$

satisfies equation (2.10). To justify this assertion at a formal level, deferring analytic technicalities to Section 2.5.2, we first show that if the components of Γ_P Poisson commute, the function $A\in\mathcal{A}_{r,s}$, whose definition is recalled in (2.2), satisfies the relations

$$\nu_\ell A_\nu^{(m)}=\nu_m A_\nu^{(\ell)}\quad\text{for all } \ell,m\in\mathbb{N}\ . \tag{2.12}$$

Note that, by the definition of the function A , see (2.2), it follows that

$$\left.\frac{\partial}{\partial\varphi_k}\right|_{\gamma=\Gamma_P(y,\varphi)}\Gamma_P(\gamma-A(\gamma,\varphi),\varphi)=0\ ,$$

which implies

$$\sum_{\ell\in\mathbb{N}}\frac{\partial\Gamma_P^{(m)}}{\partial y_\ell}(y,\varphi)\frac{\partial A^{(\ell)}}{\partial\varphi_k}(\Gamma_P(y,\varphi),\varphi)=\frac{\partial\Gamma_P^{(m)}}{\partial\varphi_k}(y,\varphi)\ ,\quad\text{for all } m\in\mathbb{N}\ . \tag{2.13}$$

To ease the notation, in the following computations it is understood that the derivatives of A are pre-composed with the map $(y,\varphi)\mapsto(\Gamma_P(y,\varphi),\varphi)$, even if not explicitly indicated. Using the assumption that the components of Γ_P Poisson commute, we have:

$$\begin{aligned}
0=\{\Gamma_P^{(m)},\Gamma_P^{(n)}\}&=\sum_{k\in\mathbb{N}}\left(\frac{\partial\Gamma_P^{(m)}}{\partial y_k}\frac{\partial\Gamma_P^{(n)}}{\partial\varphi_k}-\frac{\partial\Gamma_P^{(m)}}{\partial\varphi_k}\frac{\partial\Gamma_P^{(n)}}{\partial y_k}\right)\\
&\overset{(2.13)}{=}\sum_{k\in\mathbb{N}}\frac{\partial\Gamma_P^{(m)}}{\partial y_k}\sum_{\ell\in\mathbb{N}}\frac{\partial\Gamma_P^{(n)}}{\partial y_\ell}\frac{\partial A^{(\ell)}}{\partial\varphi_k}-\sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}\frac{\partial\Gamma_P^{(m)}}{\partial y_\ell}\frac{\partial A^{(\ell)}}{\partial\varphi_k}\frac{\partial\Gamma_P^{(n)}}{\partial y_k}\\
&=\sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}\frac{\partial\Gamma_P^{(m)}}{\partial y_k}\frac{\partial\Gamma_P^{(n)}}{\partial y_\ell}\left(\frac{\partial A^{(\ell)}}{\partial\varphi_k}-\frac{\partial A^{(k)}}{\partial\varphi_\ell}\right).
\end{aligned}$$

Notice that the operator $\partial_y\Gamma_P$ is invertible, being a small bounded perturbation of the identity. Then the above equalities yield

$$\frac{\partial A^{(\ell)}}{\partial\varphi_k}=\frac{\partial A^{(k)}}{\partial\varphi_\ell}\,,$$

which proves formula (2.12). Consequently, the following equalities hold for all $k\in\mathbb{N}$:

$$\begin{aligned}
\frac{\partial W}{\partial\varphi_k}(x,\varphi)&=\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m}i\nu_k\frac{A_\nu^{(m)}(\gamma(x))}{i\nu_m}e^{i\nu\cdot\varphi}\overset{(2.12)}{=}\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m}A_\nu^{(k)}(\gamma(x))e^{i\nu\cdot\varphi}\\
&=\sum_{\nu\neq0}A_\nu^{(k)}(\gamma(x))e^{i\nu\cdot\varphi}=A^{(k)}(\gamma(x),\varphi)-A_0^{(k)}(\gamma(x))\\
&=\gamma_k(x)-A_0^{(k)}(\gamma(x))-(\gamma_k(x)-A^{(k)}(\gamma(x),\varphi))\\
&=x_k-y_k(x,\varphi)\,.
\end{aligned}$$

The last equality follows from Equations (2.5) and (2.6). Hence, the claim is formally proved. According to (2.8), and recalling (2.9) and (2.11), we can assert that the candidate angle variables are formally defined by

$$\theta_k(x,\varphi)=\varphi_k-\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m}\frac{1}{\mathrm{i}\nu_m}\frac{\partial A_\nu^{(m)}(\gamma(x))}{\partial x_k}e^{\mathrm{i}\nu\cdot\varphi}\,,\quad k\in\mathbb{N}\,. \tag{2.14}$$

2.1.3 Complete integrability in infinite-dimension

The symplectic straightening provides the geometric basis for a formulating an infinite-dimensional version of the Liouville-Arnold theorem. In Section 2.4, we extend the definition of Hamilton's equations to our infinite-dimensional framework. This is a delicate point; indeed, in infinite dimensions, even the analyticity of the Hamiltonian is not sufficient to guarantee local well-posedness.

In section 2.6, we prove the main theorem about the complete integrability of Hamiltonian systems in infinite dimension. By using the symplectic straightening theorem 2.5.12, we show that, once the momentum map is in its canonical form, any Hamiltonian H that commutes with Γ can be easily integrated. In these new coordinates, the dynamics of the original system is conjugate to a Kronecker flow. This result, though presented here informally, is, to our knowledge, the first example of a local Liouville-Arnold theorem in an infinite-dimensional setting.

We shall devote the next sections to the introduction of the functional setting and the quantitative treatment of the analytical issues, aimed at completing the argument rigorously.

2.2 Vector Fields and Lie Derivative

We begin with a brief introduction to vector fields on Banach spaces and state the fundamental result concerning the existence and uniqueness of solution curves. Afterwards, we introduce a particular class of analytic vector fields that are 2π-periodic with respect to each of the φ_k-variables, $k \in \mathbb{N}$, and study the properties of the Lie derivative of an analytic function $f \in \mathcal{A}$ in the direction of a vector field in this class, establishing the foundations required to proceed to the more advanced topics discussed in the next section.

Following [PT87], let $\mathscr{U}$ be an open subset of a real Banach space X. A vector field on $\mathscr{U}$ is a map

$$V : \mathscr{U} \to X.$$

A solution curve of the vector field V with initial value $\underline{x}$ in $\mathscr{U}$ is a differentiable map

$$\phi_V(\underline{x}) : (-\varepsilon, \varepsilon) \ni t \to \phi_V^t(\underline{x}) \in \mathscr{U}$$

such that $\phi_V^0(\underline{x}) = \underline{x}$ and

$$\frac{d}{dt}\phi_V^t(\underline{x}) = V(\phi_V^t(\underline{x})), \quad t \in (-\varepsilon, \varepsilon).$$

The local existence and uniqueness of solution curves is established by the following

Theorem 2.2.1 (Local Existence and Uniqueness Theorem, [PT87] Appendix B)**.** *Suppose V is a locally Lipschitz vector field on $\mathscr{U}$. Then, for every $\underline{x} \in \mathscr{U}$, there exists $\varepsilon > 0$ and a unique solution curve*

$$\phi_V(\underline{x}) : (-\varepsilon, \varepsilon) \to \mathscr{U}.$$

If V is (real) analytic, then ϕ_V is a real analytic function of t and $\underline{x}$.

Let $q : \ell^\infty_w \times \ell^\infty \to \ell^\infty_w \times \mathbb{T}^\infty$ be the map $q(y, \varphi) = (y, \pi_\infty(\varphi))$ introduced in Definition 1.3.5. Motivated by the considerations in Remark 1.1.5, from now on, we shall identify analytic vector fields defined on an open neighbourhood $\mathscr{N}_0 \subset \ell^\infty_w \times \mathbb{T}^\infty$ of $\{0\} \times \mathbb{T}^\infty$ with analytic vector fields on the lifted neighbourhood $q^{-1}(\mathscr{N}_0) \subset \ell^\infty_w \times \ell^\infty$ that are 2π-periodic in each variable φ_k , $k \in \mathbb{N}$.

Definition 2.2.2 (Space of analytic vector fields)**.** *Consider the real Banach space $\ell^\infty_w \times \ell^\infty$ and two real numbers $r, s > 0$. We denote by $\mathfrak{X}_{r,s}$ the space of vector fields*

$$V : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad V(y,\varphi) = (V_y(y,\varphi), V_\varphi(y,\varphi)),$$
$$V_y(y,\varphi) = (V_y^{(k)}(y,\varphi))_{k\in\mathbb{N}}, \quad V_\varphi(y,\varphi) = (V_\varphi^{(k)}(y,\varphi))_{k\in\mathbb{N}},$$

such that,

$$\forall k \in \mathbb{N},\ V_y^{(k)}, V_\varphi^{(k)} \in \mathcal{A}_{r,s}, \quad \textit{with} \quad \max\left\{|V_y|^{\infty,w}_{r,s}, |V_\varphi|^\infty_{r,s}\right\} < \infty. \tag{2.15}$$

Accordingly, we denote by $\mathfrak{X}$ the space of vector fields

$$\mathfrak{X} := \bigcup_{r,s>0} \mathfrak{X}_{r,s}\ .$$

Note that, as a consequence of Theorem 1.2.24, any vector field in $\mathfrak{X}$ is real analytic as a map from $\mathscr{U}_r(\ell^\infty_w) \times \ell^\infty$ to $\ell^\infty_w \times \ell^\infty$ and hence admits a solution curve.

Definition 2.2.3 (Lie derivative)**.** *Given a vector field $V \in \mathfrak{X}_{r,s}$, we define the Lie derivative of any $f \in \mathcal{A}_{r,s}$ in direction of V as*

$$\mathscr{L}_V f := \frac{d}{dt}\bigg|_{t=0} (\phi_V^t)^* f.$$

Proposition 2.2.4. *Given $r, s > 0$, the Lie derivative of $f \in \mathcal{A}_{r,s}$ in the direction of $V \in \mathfrak{X}_{r,s}$ is the function defined on $\mathscr{U}_r(\ell_w^\infty) \times \ell^\infty$ given by*

$$\mathscr{L}_V f = \sum_{k\in\mathbb{N}} \left(\frac{\partial f}{\partial y_k} V_y^{(k)} + \frac{\partial f}{\partial \varphi_k} V_\varphi^{(k)} \right) . \tag{2.16}$$

Moreover, for all $V \in \mathfrak{X}_{r,s}$, the map $f \to \mathscr{L}_V f$ is a bounded linear operator

$$\mathscr{L}_V : \mathcal{A}_{r,s} \to \mathcal{A}_{r-\rho,s-\sigma}$$

for all $\rho \in (0, r)$ and every $\sigma \in (0, s)$. More precisely, the following estimate holds

$$|\mathscr{L}_V f|_{r-\rho,s-\sigma} \le \left(\frac{r}{e\rho(r-\rho)} |V_y|_{r,s}^{\infty,w} + \frac{1}{e\sigma} |V_\varphi|_{r,s}^\infty \right) |f|_{r,s} . \tag{2.17}$$

Proof. By the analyticity of ϕ_V with respect to t, one can easily verify that (2.16) holds pointwise for any monomial $f(y, \varphi) = y^\alpha e^{i\nu\cdot\varphi}$, with $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and $\nu \in \mathbb{Z}^{(\mathbb{N})}$.

Given $(y, \varphi) \in \mathscr{U}_r(\ell_w^\infty) \times \ell^\infty$, we denote $(y(t), \varphi(t)) = \phi_V^t(y, \varphi)$. Suppose that the series

$$\sum_{\nu,\alpha} f_{\nu\alpha} (y(t))^\alpha e^{\mathrm{i}\nu\cdot\varphi(t)} \tag{2.18}$$

$$\sum_{\nu,\alpha} f_{\nu\alpha} \sum_{k\in\mathbb{N}} \alpha_k V_y^{(k)}(y(t), \varphi(t))(y(t))^{\alpha-e_k} e^{\mathrm{i}\nu\cdot\varphi(t)} + \mathrm{i}\nu_k V_\varphi^{(k)}(y(t), \varphi(t))(y(t))^\alpha e^{\mathrm{i}\nu\cdot\varphi(t)} , \tag{2.19}$$

converge totally in a neighborhood of $t = 0$. Then, the derivative of $f \circ \phi_V^t$ with respect to t is equal to the series of the derivatives of each monomial. Furthermore, in this case, the order of the summations $\sum_{\nu,\alpha}$ and $\sum_k$ can be swapped, yielding formula (2.16).

Let us prove that the series in (2.18) and (2.19) converge totally in a neighbourhood of $t = 0$. Let $\varepsilon > 0$ be such that $y(t) \in \mathscr{U}_r(\ell_w^\infty)$ for all $|t| \le \varepsilon$. Then one has

$$\sup_{|t|\le\varepsilon} \left| \sum_{\nu,\alpha} f_{\nu\alpha} (y(t))^\alpha e^{\mathrm{i}\nu\cdot\varphi(t)} \right| \le \sup_{y\in\mathscr{U}_r(\ell_w^\infty)} \sum_{\nu,\alpha} |f_{\nu\alpha} y^\alpha| \le |f|_{r,s}$$

and, for all $r' < r$ such that $\sup_{|t|\le\varepsilon} |y(t)|_{\ell_w^\infty} \in \mathscr{U}_{r'}(\ell_w^\infty)$,

$$\begin{aligned} \sup_{|t|\le\varepsilon} \Bigg| \sum_{\nu,\alpha} f_{\nu\alpha} \sum_{k\in\mathbb{N}} \Big(&\alpha_k V_y^{(k)}(y(t), \varphi(t)) y(t)^{\alpha-e_k} e^{\mathrm{i}\nu\cdot\varphi(t)} \\ &+ \mathrm{i}\nu_k V_\varphi^{(k)}(y(t), \varphi(t)) y(t)^\alpha e^{\mathrm{i}\nu\cdot\varphi(t)} \Big) \Bigg| \\ \le \sup_{y\in\mathscr{U}_{r'}(\ell_w^\infty)} \sup_{\varphi\in\mathbb{T}^\infty} \sum_{\nu,\alpha} |f_{\nu\alpha}| &\sum_{k\in\mathbb{N}} \Big(\alpha_k |V_y^{(k)}(y,\varphi) y^{\alpha-e_k}| + |\nu_k| |V_\varphi^{(k)}(y,\varphi) y^\alpha| \Big) \end{aligned}$$

which is bounded by

$$\begin{aligned} &\sum_{\nu,\alpha} |f_{\nu\alpha}| \sum_{k\in\mathbb{N}} \alpha_k \sup_{(y,\varphi)} |V_y^{(k)}(y,\varphi)| v_{w,r'}^{\alpha-e_k} e^{s'|\nu|} + \sum_{\nu,\alpha} |f_{\nu\alpha}| \sum_{k\in\mathbb{N}} |\nu_k| \sup_{(y,\varphi)} |V_\varphi^{(k)}(y,\varphi)| v_{w,r'}^\alpha e^{s'|\nu|} \\ &\le |V_y|_{r',s'}^{\infty,w} \sum_{\nu,\alpha} |f_{\nu\alpha}| \sum_{k\in\mathbb{N}} \alpha_k w_k^{-1} v_{w,r'}^{\alpha-e_k} e^{s'|\nu|} + |V_\varphi|_{r',s'}^\infty \sum_{\nu,\alpha} |f_{\nu\alpha}| |\nu| v_{w,r'}^\alpha e^{s'|\nu|} , \end{aligned}$$

for all $s' < s$. Therefore, by Lemma 1.2.13 with $n = 1$, we get that the latter quantity is smaller or equal than

$$\frac{r}{e(r-r')r'}|V_y|_{r,s}^{\infty,w}|f|_{r,s} + \frac{1}{e(s-s')}|V_\varphi|_{r,s}^{\infty}|f|_{r,s} .$$

Then, (2.16) holds pointwise for all $f \in \mathcal{A}_{r,s}$.

Finally,

$$\begin{aligned}
|\mathscr{L}_V f|_{r-\rho,s-\sigma} &\leq \sum_{k\in\mathbb{N}} \left(\left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma} |V_y^{(k)}|_{r-\rho,s-\sigma} + \left|\frac{\partial f}{\partial \varphi_k}\right|_{r-\rho,s-\sigma} |V_\varphi^{(k)}|_{r-\rho,s-\sigma} \right) \\
&\leq |V_y|_{r,s}^{\infty,w} \sum_{k\in\mathbb{N}} w_k^{-1} \left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma} + |V_\varphi|_{r,s}^{\infty} \sum_{k\in\mathbb{N}} \left|\frac{\partial f}{\partial \varphi_k}\right|_{r-\rho,s-\sigma} \\
&\leq \left(\frac{r}{e\rho(r-\rho)} |V_y|_{r,s}^{\infty,w} + \frac{1}{e\sigma}|V_\varphi|_{r,s}^{\infty} \right) |f|_{r,s} ,
\end{aligned}$$

which proves the bound in (2.17). ■

In the next section, we turn to the introduction of a symplectic and a Poisson structure, with the aim of defining a Hamiltonian dynamics on our phase space.

2.3 Symplectic and Poisson structure

In this section we set the stage for studying infinite-dimensional Hamiltonian dynamics. We begin by endowing our infinite-dimensional phase space with a symplectic form, defined by considering a direct infinite-dimensional generalization of the usual symplectic form on the cotangent bundle of the torus. To this end, we shall need to require that $\mathsf{c}_w = \sum_{k\in\mathbb{N}} w_k^{-1} < \infty$. In fact, as already observed in the introduction of Chapter 1, this condition allows us to interpret $\ell_w^\infty \times \mathbb{T}^\infty$ as a subset of the weak*-cotangent bundle of the infinite torus $\mathbb{T}^\infty$. Then we give the definition of Hamiltonian vector field and we introduce a subclass $\mathcal{R} \subset \mathcal{A}$ of functions, which we call *regular functions*, that generate Hamiltonian vector fields. Following the standard construction, the Poisson brackets are defined via the underlying symplectic structure. We prove that the space of regular functions possesses the structure of Poisson algebra. Furthermore, we remark that one is actually allowed to define the Poisson bracket between any analytic function $H \in \mathcal{A}$ and any regular function f , by interpreting the Poisson bracket $\{f, H\}$ as the Lie derivative of H in the direction of the Hamiltonian vector field associated with f . If one takes this as the definition of the Poisson bracket, the condition $\mathsf{c}_w < \infty$ is no longer necessary. In this sense, the finiteness of c_w serves only to provide a symplectic interpretation, but it is not a requirement for our main analysis. Indeed, we shall see that the Poisson structure, which is sufficient for establishing our main results, remains well-defined even when $\mathsf{c}_w = \infty$. Finally, we consider bi-analytic diffeomorphisms of the phase space that preserve the Poisson bracket and we present a result that mirrors the classical finite-dimensional result. Namely, in Proposition 2.3.20 we prove that a regular change of coordinates preserving the fundamental Poisson bracket is indeed a symplectomorphism. This result is particularly useful in practical applications, providing an operative way to verify if a given transformation is a symplectomorphism.

2.3.1 Symplectic form

Let us assume that the weight w is such that $\mathsf{c}_w := \sum_{k\in\mathbb{N}} w_k^{-1} < \infty$. Here we introduce a symplectic structure on the infinite-dimensional phase space under consideration and give the definition of *Hamiltonian vector field*.

Definition 2.3.1 (Symplectic form)**.** *We endow the phase space* $\ell^\infty_w \times \mathbb{T}^\infty$ *with the symplectic form* ω *, defined on pairs of vector fields* $V, W \in \mathfrak{X}$ *as*

$$\omega(V,W) := \sum_{k\in\mathbb{N}} \left(V_y^{(k)} W_\varphi^{(k)} - V_\varphi^{(k)} W_y^{(k)} \right).$$

Let us briefly discuss the well-posedness of the definition above. First, note that ω is the infinite-dimensional analogue of the standard symplectic form

$$\sum_{k=1}^n dy_k \wedge d\varphi_k.$$

To ensure the infinite sum is well-defined, we observe that for all $r, s > 0$, the map

$$\omega : \mathfrak{X}_{r,s} \times \mathfrak{X}_{r,s} \to \mathcal{A}_{r,s}$$

is bounded. In fact, by Proposition 1.2.6, the bound (2.15), and the assumption that $\mathsf{c}_w = \sum_{k\in\mathbb{N}} w_k^{-1}$ is finite, we have

$$\begin{aligned} |\omega(V,W)|_{r,s} &\le \sum_{k\in\mathbb{N}} |V_y^{(k)}|_{r,s} |W_\varphi^{(k)}|_{r,s} + \sum_{k\in\mathbb{N}} |V_\varphi^{(k)}|_{r,s} |W_y^{(k)}|_{r,s} \\ &\le \mathsf{c}_w \left(|V_y|_{r,s}^{\infty,w} |W_\varphi|_{r,s}^{\infty} + |V_\varphi|_{r,s}^{\infty} |W_y|_{r,s}^{\infty,w} \right) < \infty \,. \end{aligned}$$

Remark 2.3.2. *The closed* 2*-form introduced in Definition 2.3.1 is actually a weak-symplectic form. Indeed, the map* $\omega^\# : \ell^\infty_w \times \ell^\infty \to (\ell^\infty_w)^* \times (\ell^\infty)^*$ *defined by* $\omega^\#(V) = \omega(\cdot, V)$ *is injective but fails to be surjective.*

2.3.2 Space of regular functions

We now define a class of functions that generate Hamiltonian vector fields and we show that functions in this class form a Banach algebra with respect to the product of functions.

Definition 2.3.3 (Hamiltonian vector field)**.** *A vector field* $V \in \mathfrak{X}$ *is said to be hamiltonian if there exists* $f \in \mathcal{A}$ *such that*

$$\omega(\cdot, V) = df[\,\cdot\,].$$

Definition 2.3.4 (Regular functions)**.** *We say that a function* f *is regular if there exist* $r, s > 0$ *such that* $f \in \mathcal{A}_{r,s}$ *and*

$$\|f\|_{r,s} := \max\left\{ |f|_{r,s},\ \sup_{k\in\mathbb{N}} w_k \left|\frac{\partial f}{\partial \varphi_k}\right|_{r,s},\ r \sup_{k\in\mathbb{N}} \left|\frac{\partial f}{\partial y_k}\right|_{r,s} \right\} < \infty.$$

We denote such space by $\mathcal{R}$ *. Moreover, for any* $r, s > 0$ *,* $\mathcal{R}_{r,s}$ *denotes the subspace of* $\mathcal{R}$ *of functions with* $\|f\|_{r,s} < \infty$ *.*

Remark 2.3.5. *To any regular function* $f \in \mathcal{R}$ *there is an associated Hamiltonian vector field* $V^f \in \mathfrak{X}$ *, namely*

$$\omega(\cdot, V^f) = df[\,\cdot\,] \,.$$

By a direct computation, one can easily verify that

$$(V_y^f)^{(k)} = -\frac{\partial f}{\partial \varphi_k}, \quad (V_\varphi^f)^{(k)} = \frac{\partial f}{\partial y_k} \,. \tag{2.20}$$

In contrast with the finite-dimensional case, the Hamiltonian vector field associated with a general analytic function is not necessarily a well-defined map from the phase space into its tangent bundle. Indeed, the differential of a real analytic function defined on a Banach space X is a map taking values in the dual space X^*. This problem is absent in finite dimensions, where the dual of a vector space is isomorphic to the space itself, and the non-degeneracy of the symplectic form always ensures that the Hamiltonian vector field associated with any analytic Hamiltonian is a well-defined section of the tangent bundle.

Remark 2.3.6. *Note that the condition $\mathtt{c}_w < \infty$ is irrelevant for the definition of regular function, while it is necessary to define the associated Hamiltonian vector field. To overcome this issue, can also take* (2.20) *as a definition of Hamiltonian vector field, forgetting about the symplectic structure.*

The following proposition, which also holds for every choice of the weight w, shows that the space of regular functions possesses a Banach algebra structure.

Proposition 2.3.7 (Algebra of regular functions)**.** *For any $r, s > 0$, $\mathcal{R}_{r,s}$ is a Banach algebra with respect to the product of functions.*

Proof. Given $f, g \in \mathcal{R}_{r,s}$, by the triangular inequality and the algebra property enjoyed by the norm $|\cdot|_{r,s}$ (see Proposition 1.2.6), we have

$$r \sup_{k\in\mathbb{N}} \left|\frac{\partial fg}{\partial y_k}\right|_{r,s} \le r \sup_{k\in\mathbb{N}} \left|\frac{\partial f}{\partial y_k}\right|_{r,s} |g|_{r,s} + |f|_{r,s}\, r \sup_{k\in\mathbb{N}} \left|\frac{\partial g}{\partial y_k}\right|_{r,s} \le 2\|f\|_{r,s}\|g\|_{r,s}.$$

The same estimate holds for the derivative with respect to φ. Thus, by Proposition 1.2.6, one obtains

$$\|fg\|_{r,s} \le 2\|f\|_{r,s}\|g\|_{r,s} \,,$$

which gives the claim. ∎

2.3.3 Regular changes of coordinates

Following the approach of the previous chapter, we aim to define appropriate norms for controlling changes of coordinates that preserve the regularity of functions in the sense of Definition 2.3.4. Given that the $\|\cdot\|_{r,s}$-norm controls the Hamiltonian vector fields associated with regular functions, and that such vector fields transform via the adjoint of the differential of the change of coordinates, we seek to define norms that ensure the boundedness of this operator. We remark that the finiteness of $\mathtt{c}_w$ is note required throughout this subsection.

Definition 2.3.8. *Given $r, s > 0$, we denote by $\mathcal{R}^{\infty,w}_{r,s}$ the space of functions*

$$N : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w, \quad N(y,\varphi) = (F^{(m)}(y,\varphi))_{m\in\mathbb{N}},$$

such that, for any $m \in \mathbb{N}$, $N^{(m)} \in \mathcal{A}_{r,s}$, with

$$\|N\|^{\infty,w}_{r,s} := \max\left\{ |N|^{\infty,w}_{r,s},\ r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\frac{\partial N^{(m)}}{\partial y_k}\right|_{r,s},\ \sup_{k\in\mathbb{N}} w_k \sum_{m=1}^{\infty} \left|\frac{\partial N^{(m)}}{\partial \varphi_k}\right|_{r,s} \right\} < \infty \,.$$

Accordingly, $\mathcal{R}^{\infty}_{r,s}$ denotes the space of functions

$$\Omega : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty, \quad \Omega(y,\varphi) = (\Omega^{(m)}(y,\varphi))_{m\in\mathbb{N}},$$

such that, for any $m \in \mathbb{N}$, $\Omega^{(m)} \in \mathcal{A}_{r,s}$, with

$$\|\Omega\|^{\infty}_{r,s} := \max\left\{ |\Omega|^{\infty}_{r,s},\ r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} w_m^{-1} \left|\frac{\partial \Omega^{(m)}}{\partial y_k}\right|_{r,s},\ \sup_{k\in\mathbb{N}} w_k \sum_{m=1}^{\infty} w_m^{-1} \left|\frac{\partial \Omega^{(m)}}{\partial \varphi_k}\right|_{r,s} \right\} < \infty \,.$$

Given $r, s > 0$, we denote by $\mathcal{C}_{r,s}(\mathcal{R})$ the affine space passing through the identity and directed by $\mathcal{R}^{\infty,w}_{r,s} \times \mathcal{R}^{\infty}_{r,s}$, namely

$$\mathcal{C}_{r,s}(\mathcal{R}) := \{\Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty \ : \ \Phi = \mathrm{Id} + (N, \Omega) \ , \ N \in \mathcal{R}^{\infty,w}_{r,s}, \ \Omega \in \mathcal{R}^{\infty}_{r,s}\} \ .$$

For any $\rho, \sigma > 0$, let $\mathcal{C}^{\rho,\sigma}_{r,s}(\mathcal{R})$ be the subset of those maps $\Phi = \mathrm{Id} + (N, \Omega) \in \mathcal{C}_{r,s}(\mathcal{R})$ such that

$$|N|^{\infty,w}_{r,s} \le \rho \qquad \text{and} \qquad |\Omega|^{\infty}_{r,s} \le \sigma \ .$$

In this case, we say that Φ is close to identity.
Accordingly, we denote by $\mathcal{C}(\mathcal{R})$ the space of functions

$$\mathcal{C}(\mathcal{R}) := \bigcup_{r,s>0} \mathcal{C}_{r,s}(\mathcal{R}) \ .$$

Remark 2.3.9. *A map $\Phi \in \mathcal{C}(\mathcal{R})$ has the property that the adjoint of its differential induces a bounded map from $\mathfrak{X}_{r,s}$ to itself. This will be more clear after the proof of the following proposition.*

Proposition 2.3.10 (Estimate on the pullback of a regular function)**.** *Given $r_0, s_0 > 0$, let $\rho \in [0, r_0)$ and $\sigma \in [0, s_0)$. Consider a function $\Psi = \mathrm{Id} + (N, \Omega) \in \mathcal{C}^{\rho,\sigma/2}_{r_0-\rho,s_0-\sigma}(\mathcal{R})$. Then, for all $r \in (\rho, r_0]$, $s \in (\sigma, s_0]$, and every $f \in \mathcal{R}_{r,s}$, the pullback of f by Ψ belongs to $\mathcal{R}_{r-\rho,s-\sigma}$ with estimate*

$$\|\Psi^* f\|_{r-\rho,s-\sigma} \le 2(1 + \rho/r + \sigma/2)\|f\|_{r,s}.$$

Proof. The hypotheses of this proposition ensure that all the assumptions needed for applying Proposition 1.2.19 are fulfilled. Then, $|\Psi^* f|_{r-\rho,s-\sigma} \le 2|f|_{r,s}$. Let us perform estimates on the derivatives of $\Psi^* f$. Let x, θ be the coordinates defined by $x = y + N(y, \varphi)$, $\theta = \varphi + \Omega(y, \varphi)$. By the triangular inequality applied to the formula given by the chain rule, the algebra property proved in Proposition 1.2.6 and the estimate for the pullback of a majorant analytic function stated in Proposition 1.2.19, one has

$$\begin{aligned}
r \sup_{k\in\mathbb{N}} \left|\frac{\partial\, \Psi^* f}{\partial y_k}\right|_{r-\rho,s-\sigma} &\le r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\Psi^* \frac{\partial f}{\partial x_m}\right|_{r-\rho,s-\sigma} \left|\delta_{mk} + \frac{\partial N^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma} \\
&\quad + r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\Psi^* \frac{\partial f}{\partial \theta_m}\right|_{r-\rho,s-\sigma} \left|\frac{\partial \Omega^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma} \\
&\le 2r \sup_{k\in\mathbb{N}} \left|\frac{\partial f}{\partial x_k}\right|_{r,s} \\
&\quad + 2r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\frac{\partial f}{\partial x_m}\right|_{r,s} \left|\frac{\partial N^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma} + \left|\frac{\partial f}{\partial \theta_m}\right|_{r,s} \left|\frac{\partial \Omega^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma} .
\end{aligned}$$

By the regularity of f, and applying the assumptions, the above quantity is bounded by

$$\begin{aligned}
2\|f\|_{r,s} \left(1 + \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\frac{\partial N^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma} + r \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} w_m^{-1} \left|\frac{\partial \Omega^{(m)}}{\partial y_k}\right|_{r-\rho,s-\sigma}\right) \\
\le 2(1 + \rho/r + \sigma/2)\|f\|_{r,s} \ .
\end{aligned}$$

At this stage it only remains to give a suitable bound for the term of the norm which involves the derivatives with respect to the periodic variables. By the same argument of before we get

$$\begin{aligned}
\sup_{k\in\mathbb{N}} w_k \left|\frac{\partial\, \Psi^* f}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} &\le \sup_{k\in\mathbb{N}} w_k \sum_{m=1}^{\infty} \left|\Psi^* \frac{\partial f}{\partial x_m}\right|_{r-\rho,s-\sigma} \left|\frac{\partial N^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} \\
&\quad + \sup_{k\in\mathbb{N}} w_k \sum_{m=1}^{\infty} \left|\Psi^* \frac{\partial f}{\partial\theta_m}\right|_{r-\rho,s-\sigma} \left|\delta_{mk} + \frac{\partial\Omega^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} \\
&\le 2 \sup_{k\in\mathbb{N}} w_k \left|\frac{\partial f}{\partial\varphi_k}\right|_{r,s} \\
&\quad + 2 \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\frac{\partial f}{\partial x_m}\right|_{r,s} \left|\frac{\partial N^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} + \left|\frac{\partial f}{\partial\theta_m}\right|_{r,s} \left|\frac{\partial\Omega^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} ,
\end{aligned}$$

which is bounded by

$$\begin{aligned}
2\|f\|_{r,s} \Bigg(1 + \frac{1}{r} \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} \left|\frac{\partial N^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma} &+ \sup_{k\in\mathbb{N}} \sum_{m=1}^{\infty} w_m^{-1} \left|\frac{\partial\Omega^{(m)}}{\partial\varphi_k}\right|_{r-\rho,s-\sigma}\Bigg) \\
&\le 2(1+\rho/r+\sigma/2)\|f\|_{r,s} \,.
\end{aligned}$$

The claim is proved. ■

Let us introduce a class of change of coordinates which preserves the regularity of functions belonging to the class $\mathcal{R}$.

Definition 2.3.11 (Regular changes of coordinates)**.** *A function* $\Phi \in \mathcal{C}_{r_0,s_0}(\mathcal{R})$ *is a regular change of coordinates if there exist* $\rho \in [0, r_0/2)$ *and* $\sigma \in [0, s_0/2)$ *such that*

(a) for all $r \le r_0$ *,*

$$\Phi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho}(\ell^\infty_w) \times \ell^\infty$$

with the property that, for all $r \in (\rho, r_0+\rho]$ *,* $s \in (\sigma, s_0+\sigma]$ *, and every* $f \in \mathcal{R}_{r,s}$ *,* $\Phi^* f \in \mathcal{R}_{r-\rho,s-\sigma}$ *with*

$$\|\Phi^*\|^{\rho,\sigma}_{r,s} := \sup_{f\neq 0} \frac{\|\Phi^* f\|_{r-\rho,s-\sigma}}{\|f\|_{r,s}} < \infty \,;$$

(b) there exists $\Psi \in \mathcal{C}_{r_0-\rho,s_0-\sigma}(\mathcal{R})$ *such that, for all* $r \le r_0 - \rho$ *,*

$$\Psi : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathscr{U}_{r+\rho}(\ell^\infty_w) \times \ell^\infty$$

and, for all $r \in (\rho, r_0]$ *,* $s \in (\sigma, s_0]$ *, and every* $f \in \mathcal{R}_{r,s}$ *,* $\Psi^* f \in \mathcal{R}_{r-\rho,s-\sigma}$ *with*

$$\|\Psi^*\|^{\rho,\sigma}_{r,s} := \sup_{f\neq 0} \frac{\|\Psi^* f\|_{r-\rho,s-\sigma}}{\|f\|_{r,s}} < \infty,$$

which is the inverse of Φ *in the following sense:*

$$\Phi\circ\Psi = \mathrm{Id}_{\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty}, \quad \forall r \le r_0 - \rho \,, \qquad \Psi\circ\Phi = \mathrm{Id}_{\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty}, \quad \forall r \le r_0 - 2\rho \,.$$

Remark 2.3.12. *Note that, given* $r_0, s_0 > 0$ *,* $\rho \in [0, r_0)$ *and* $\sigma \in [0, s_0)$*, any invertible map* $\Phi \in \mathcal{C}^{\rho,\sigma/2}_{r_0,s_0}(\mathcal{R})$ *with inverse* $\Psi \in \mathcal{C}^{\rho,\sigma/2}_{r_0-\rho,s_0-\sigma}(\mathcal{R})$ *is a (close to identity) regular change of coordinates, as follows by Proposition 2.3.10.*

2.3.4 Poisson braket

In this subsection, we endow the space of regular functions $\mathcal{R}$ with a Poisson algebra structure. Additionally, we establish the well definition of the Poisson bracket between analytic and regular functions, and we show that it can be expressed in terms of the Poisson bracket between the fundamental power-trigonometric monomials.

Definition 2.3.13 (Poisson bracket)**.** *For any $f, g \in \mathcal{R}$, we define their Poisson bracket*

$$\{f, g\} := \omega(V^f, V^g) = \sum_{k\in\mathbb{N}} \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial \varphi_k} - \frac{\partial f}{\partial \varphi_k}\frac{\partial g}{\partial y_k}. \tag{2.21}$$

Note that the Lie derivative of a function $g \in \mathcal{R}$ in the direction of the hamiltonian vector field V^f associated with $f \in \mathcal{R}$ can be expressed in terms of the Poisson bracket. Indeed, by Equations (2.16), (2.20) and (2.21), one has that

$$\mathscr{L}_{V^f} g := \frac{d}{dt}_{|0}(\phi^t_{V^f})^* g = \sum_{k\in\mathbb{N}} \frac{\partial f}{\partial y_k}(V^f_y)^{(k)} + \frac{\partial f}{\partial \varphi_k}(V^f_\varphi)^{(k)} = \sum_{k\in\mathbb{N}} \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial \varphi_k} - \frac{\partial f}{\partial \varphi_k}\frac{\partial g}{\partial y_k} = \{f, g\}.$$

While Definition 2.3.13 requires the symplectic form to be well-defined (i.e., $\mathtt{c}_w < \infty$), one may alternatively adopt the right-hand side of (2.21) as a definition of the Poisson bracket. By Proposition 2.2.4 and Remark 2.3.6, it follows that this second definition is well-posed even in the case $\mathtt{c}_w = \infty$. Since the Poisson structure is sufficient for establishing our main results, we shall not assume the finiteness of $\mathtt{c}_w$ in the sequel; our results will thus remain valid for arbitrary choices of the weight w. Let us show the Poisson algebra structure of the space of regular functions.

Proposition 2.3.14. *The space of regular functions $\mathcal{R}$ equipped with the product of functions and the bracket* (2.21) *is a Poisson algebra. Namely, for all $r, s > 0$ and every $f, g \in \mathcal{R}_{r,s}$, we have that $\{f, g\} \in \mathcal{R}_{r-\rho,s-\sigma}$ and the following estimate holds*

$$\|\{f, g\}\|_{r-\rho,s-\sigma} \le \frac{2}{e}\left(\frac{r}{r-\rho}\frac{1}{\rho} + \frac{1}{r\sigma}\right)\|f\|_{r,s}\|g\|_{r,s} \, .$$

for all $\rho \in (0, r)$ and $\sigma \in (0, s)$.

Proof. Let $r, s > 0$ and $f, g \in \mathcal{R}_{r,s}$. The following bounds hold for any $\rho \in (0, r)$, $\sigma \in (0, s)$, by triangular inequality and Proposition 1.2.6.

$$|\{f, g\}|_{r-\rho,s-\sigma} \le \sum_{k\in\mathbb{N}} \left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial g}{\partial \varphi_k}\right|_{r-\rho,s-\sigma} + \left|\frac{\partial f}{\partial \varphi_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial g}{\partial y_k}\right|_{r-\rho,s-\sigma}, \tag{2.22}$$

$$\begin{aligned}\sup_{\ell\in\mathbb{N}}\left|\frac{\partial\{f, g\}}{\partial y_\ell}\right|_{r-\rho,s} &\le \sup_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{N}}\left|\frac{\partial^2 f}{\partial y_\ell \partial y_k}\right|_{r-\rho,s}\left|\frac{\partial g}{\partial \varphi_k}\right|_{r-\rho,s} \\ &\quad + \sup_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{N}}\left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s}\left|\frac{\partial^2 g}{\partial y_\ell \partial \varphi_k}\right|_{r-\rho,s} + f \leftrightarrow g\, ,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\sup_{\ell\in\mathbb{N}} w_\ell\left|\frac{\partial\{f, g\}}{\partial \varphi_\ell}\right|_{r,s-\sigma} &\le \sup_{\ell\in\mathbb{N}} w_\ell\sum_{k\in\mathbb{N}}\left|\frac{\partial^2 f}{\partial \varphi_\ell \partial y_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial g}{\partial \varphi_k}\right|_{r-\rho,s-\sigma} \\ &\quad + \sup_{\ell\in\mathbb{N}} w_\ell\sum_{k\in\mathbb{N}}\left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial^2 g}{\partial \varphi_\ell \partial \varphi_k}\right|_{r-\rho,s-\sigma} + f \leftrightarrow g\, .\end{aligned} \tag{2.24}$$

By using the regularity of f and g, and Cauchy estimate on the derivatives with respect to the φ_k-variables, $k \in \mathbb{N}$, one has

$$(2.22) \leq \frac{2}{e}\frac{1}{r\sigma}\|f\|_{r,s}\|g\|_{r,s}.$$

By invoking Lemma 1.2.13, we obtain the following estimates:

$$\begin{aligned}(r-\rho)\times(2.23) =&(r-\rho)\sup_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{N}} w_k^{-1}\left|\frac{\partial^2 f}{\partial y_\ell \partial y_k}\right|_{r-\rho,s-\sigma} w_k\left|\frac{\partial g}{\partial \varphi_k}\right|_{r-\rho,s-\sigma}\\ &+(r-\rho)\sup_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{N}}\left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial^2 g}{\partial y_\ell \partial \varphi_k}\right|_{r-\rho,s-\sigma} + f\leftrightarrow g\\ &\leq \frac{2}{e}\left(\frac{1}{\rho}+\frac{1}{r\sigma}\right)\|g\|_{r,s}\|f\|_{r,s}\end{aligned}$$

and

$$\begin{aligned}(2.24) =&\sup_{\ell\in\mathbb{N}} w_\ell\sum_{k\in\mathbb{N}} w_k^{-1}\left|\frac{\partial^2 f}{\partial \varphi_\ell \partial y_k}\right|_{r-\rho,s-\sigma} w_k\left|\frac{\partial g}{\partial \varphi_k}\right|_{r-\rho,s-\sigma}+\\ &\sup_{\ell\in\mathbb{N}} w_\ell\sum_{k\in\mathbb{N}}\left|\frac{\partial f}{\partial y_k}\right|_{r-\rho,s-\sigma}\left|\frac{\partial^2 g}{\partial \varphi_\ell \partial \varphi_k}\right|_{r-\rho,s-\sigma} + f\leftrightarrow g\\ &\leq \frac{2}{e}\left(\frac{r}{r-\rho}\frac{1}{\rho}+\frac{1}{r\sigma}\right)\|g\|_{r,s}\|f\|_{r,s}\ .\end{aligned}$$

Hence the assertion follows. ■

Remark 2.3.15. *Motivated by Definition 2.3.13, given differentiable, real valued functions f and g defined on $\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty$, for some $r>0$, we denote by $\{f,g\}$ any expression of the form*

$$\sum_{k\in\mathbb{N}}\frac{\partial f}{\partial y_k}\frac{\partial g}{\partial \varphi_k}-\frac{\partial f}{\partial \varphi_k}\frac{\partial g}{\partial y_k}$$

whenever it is well defined. It follows that:

(a) by Proposition 2.2.4 and equation (2.20)*, the Lie derivative of an analytic function $H\in\mathcal{A}$ (not necessarily regular), in the direction of the hamiltonian vector field V^f associated with the regular function $f\in\mathcal{R}$, can be expressed as*

$$\mathscr{L}_{V^f}H=\{f,H\}\ ;$$

(b) the partial derivatives of any analytic function $H\in\mathcal{A}$ can be written as

$$\frac{\partial H}{\partial y_k}=\{H,\varphi_k\},\quad \frac{\partial H}{\partial \varphi_k}=-\{H,y_k\}\ .$$

Remark 2.3.16. *According to the above remark, given $r,s>0$, for any $H\in\mathcal{A}_{r,s}$ and $f\in\mathcal{R}_{r,s}$, by the estimate in Proposition 2.2.4 and the definition of the $\|\cdot\|_{r,s}$-norm, the following estimate holds*

$$|\{H,f\}|_{r-\rho,s-\sigma}\leq\frac{1}{e}\left(\frac{r}{r-\rho}\frac{1}{\rho}+\frac{1}{r\sigma}\right)\|f\|_{r,s}|H|_{r,s}$$

for all $\rho\in(0,r)$ and $\sigma\in(0,s)$.

Now we prove that, the knowledge of the action of the Poisson bracket on any pair of monomials of the form $y^\alpha e^{\mathrm{i}\nu\cdot\varphi}$, with $(\nu,\alpha)\in\mathbb{Z}^{(\mathbb{N})}\times\mathbb{N}_0^{(\mathbb{N})}$, is sufficient to calculate the Poisson bracket $\{H,f\}$, for all $H\in\mathcal{A}$, $f\in\mathcal{R}$. More precisely, we prove the following

Lemma 2.3.17. *Given $r,s>0$, the Poisson bracket of $H\in\mathcal{A}_{r,s}$ with $f\in\mathcal{R}_{r,s}$ is the function defined on $\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty$ by*

$$\{H,f\}=\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}\{y^\alpha e^{\mathrm{i}\nu\cdot\varphi},y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}\}.$$

Proof. By an argument analogous to the one used in the proof of Proposition 2.2.4, it can be shown that, for all $y\in\mathscr{U}_r(\ell^\infty_w)$ and every $\varepsilon>0$ such that $\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w)\subset\mathscr{U}_r(\ell^\infty_w)$, the series

$$\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}y^{\alpha+\alpha'}e^{\mathrm{i}(\nu+\nu')\cdot\varphi}$$

$$\sum_{k\in\mathbb{N}}\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}\left(\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial y_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial\varphi_k}-\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial\varphi_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial y_k}\right)$$

converge totally on $\mathscr{U}_{\varepsilon,y}(\ell^\infty_w)\times\ell^\infty$. Then one has

$$\begin{aligned}
\{H,f\}&=\sum_{k\in\mathbb{N}}\frac{\partial H}{\partial y_k}\frac{\partial f}{\partial\varphi_k}-\frac{\partial H}{\partial\varphi_k}\frac{\partial f}{\partial y_k}\\
&=\sum_{k\in\mathbb{N}}\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}\left(\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial y_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial\varphi_k}-\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial\varphi_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial y_k}\right)\\
&=\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}\sum_{k\in\mathbb{N}}\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial y_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial\varphi_k}-\frac{\partial y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial\varphi_k}\frac{\partial y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}}{\partial y_k}\\
&=\sum_{\nu,\alpha}\sum_{\nu',\alpha'}H_{\nu\alpha}f_{\nu'\alpha'}\{y^\alpha e^{\mathrm{i}\nu\cdot\varphi},y^{\alpha'}e^{\mathrm{i}\nu'\cdot\varphi}\}\ ,
\end{aligned}$$

as desired. ■

2.3.5 Symplectomorphisms

We now provide the definition of *symplectomorphism.* In the finite-dimensional setting, it is well-known that a transformation is a symplectomorphism if and only if it preserves the Poisson bracket. This equivalence, however, does not extend to our infinite-dimensional framework because, as already noted in Remark 2.3.5, the Hamiltonian vector field of a general analytic function does not necessarily take values in the tangent space. Thus, for our purposes, it is more convenient to define the condition for a map to be a symplectomorphism in terms of the preservation of the Poisson bracket.

Definition 2.3.18 (Symplectomorphisms)**.** *A regular change of coordinates $\Phi\in\mathcal{C}_{r_0,s_0}(\mathcal{R})$ is a symplectomorphism if, for all $r\in(\rho,r_0+\rho]$ and $s\in(\sigma,s_0+\sigma]$, with σ and ρ as in Definition 2.3.11, the following equalities hold in $\mathscr{U}_{r-\rho}(\ell^\infty_w)\times\ell^\infty$:*

$$\begin{aligned}
&(i)\quad \Phi^*\{H,f\}=\{\Phi^*H,\Phi^*f\}\ , &&\textit{for all } H\in\mathcal{A}_{r,s}\ ,\ f\in\mathcal{R}_{r,s}\ ,\\
&(ii)\quad \Phi^*\{H,\varphi_m\}=\{\Phi^*H,\Phi^*\varphi_m\}\ , &&\textit{for all } H\in\mathcal{A}_{r,s}\ ,\ m\in\mathbb{N}\ .
\end{aligned}$$

Remark 2.3.19. *Note that, if $\Phi\in\mathcal{C}_{r_0,s_0}(\mathcal{R})$ is a symplectomorphism, then its inverse $\Psi\in\mathcal{C}_{r_0-\rho,s_0-\sigma}(\mathcal{R})$ satisfies (i) and (ii) of the above definition for all $r\in(\rho,r_0]$ and all $s\in(\sigma,s_0]$.*

The following proposition establishes a sufficient condition for a regular map to be a symplectomorphism. Specifically, we show that, in analogy with the finite-dimensional case, the symplectic nature of a transformation can be determined by checking whether it preserves the fundamental Poisson brackets of the coordinate components.

Proposition 2.3.20. *Let* $\Phi = \mathrm{Id} + (N, \Omega) : \mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty$ *be a regular change of coordinates with* $|N|^{\infty,w}_{r_0,s_0} < \rho$ *. Let us denote by* $\hat{N}$ *and* $\hat{\Omega}$ *the functions* $\hat{N}(y,\varphi) = y + N(y,\varphi)$ *and* $\hat{\Omega}(y,\varphi) = \varphi + \Omega(y,\varphi)$ *, and assume that*

$$\{\hat{N}^{(\ell)}, \hat{\Omega}^{(m)}\} = \delta_{\ell m}, \qquad \{\hat{N}^{(\ell)}, \hat{N}^{(m)}\} = 0, \qquad \{\hat{\Omega}^{(\ell)}, \hat{\Omega}^{(m)}\} = 0 \,,$$

for all $\ell, m \in \mathbb{N}$ *. Then,* Φ *is a symplectomorphism.*

The proof of this proposition is deferred to the end of this section, following the proof of the next lemma, which provides a key technical tool for the argument.

Lemma 2.3.21. *Given* $r_0, s_0 > 0$ *, let* $\rho \in [0, r_0)$ *. Consider* $\Phi = \mathrm{Id} + (N, \Omega) \in \mathcal{C}_{r_0,s_0}(\mathcal{R})$ *such that* $|N|^{\infty,w}_{r_0,s_0} < \rho$ *. Then, for all* $r \in (\rho, r_0 + \rho]$ *,* $s > 0$ *,* $H \in \mathcal{A}_{r,s}$ *and every* $f \in \mathcal{R}_{r,s}$ *, one has that*

$$\{\Phi^* H, \Phi^* f\} = \sum_{\nu,\alpha} \sum_{\nu',\alpha'} H_{\nu\alpha} f_{\nu\alpha} \{\Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}, \Phi^* y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi}\} \,.$$

Proof. Since $\Phi \in \mathcal{C}_{r_0,s_0}(\mathcal{R})$, and in particular $\Phi \in \mathcal{C}_{r_0,s_0}(\mathcal{A})$ with $|N|^{\infty,w}_{r_0,s_0} < \rho$, it follows by Proposition 1.2.20 that the derivatives of the series

$$\sum_{\nu,\alpha} H_{\nu\alpha} \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi} \qquad \text{and} \qquad \sum_{\nu,\alpha} f_{\nu\alpha} \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}$$

are equal to series of the derivatives. Then, to prove the claim it is sufficient to show that the series

$$\sum_{k\in\mathbb{N}} \sum_{\nu,\alpha} \sum_{\nu',\alpha'} H_{\nu\alpha} f_{\nu'\alpha'} \Big(\frac{\partial\, \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial y_k} \frac{\partial\, \Phi^* y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi}}{\partial \varphi_k} - \frac{\partial\, \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial \varphi_k} \frac{\partial\, \Phi^* y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi}}{\partial y_k} \Big)$$

converges absolutely. Let us show that, given $r \in (\rho, r_0 + \rho]$, for all $y \in \mathscr{U}_{r-\rho}(\ell^\infty_w)$ and every $\varepsilon > 0$ such that $\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w) \subset \mathscr{U}_{r-\rho}(\ell^\infty_w)$, the above series converges totally on $\mathscr{U}_{\varepsilon,y}(\ell^\infty_w) \times \ell^\infty$. We denote by $\hat{N}$ and $\hat{\Omega}$ the functions $\hat{N}(y,\varphi) = y + N(y,\varphi)$ and $\hat{\Omega}(y,\varphi) = \varphi + \Omega(y,\varphi)$. By the Leibniz rule, for all $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and every $\nu \in \mathbb{Z}^{(\mathbb{N})}$, one has that

$$\begin{aligned}
\frac{\partial\, \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial y_k} &= \frac{\partial\, \hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}}}{\partial y_k} = \sum_{m\in\mathbb{N}} \alpha_m \frac{\partial \hat{N}^{(m)}}{\partial y_k} \hat{N}^{\alpha - e_m} e^{\mathrm{i}\nu\cdot\hat{\Omega}} + \sum_{m\in\mathbb{N}} \mathrm{i}\nu_m \frac{\partial \Omega^{(m)}}{\partial y_k} \hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}} \\
\frac{\partial\, \Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}}{\partial \varphi_k} &= \frac{\partial\, \hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}}}{\partial \varphi_k} = \sum_{m\in\mathbb{N}} \alpha_m \frac{\partial N^{(m)}}{\partial \varphi_k} \hat{N}^{\alpha - e_m} e^{\mathrm{i}\nu\cdot\hat{\Omega}} + \sum_{m\in\mathbb{N}} \mathrm{i}\nu_m \frac{\partial \hat{\Omega}^{(m)}}{\partial \varphi_k} \hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}} \,.
\end{aligned}$$

Therefore, the terms of the series decompose into several components. Since the estimation procedure is essentially the same for all of them, we shall only detail the argument for the term

$$\sum_{k\in\mathbb{N}} \sum_{\nu,\alpha} \sum_{\nu',\alpha'} \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \left| H_{\nu\alpha} f_{\nu'\alpha'} \alpha_m \frac{\partial \hat{N}^{(m)}}{\partial y_k} \hat{N}^{\alpha - e_m} e^{\mathrm{i}\nu\cdot\hat{\Omega}} \alpha'_n \frac{\partial N^{(n)}}{\partial \varphi_k} \hat{N}^{\alpha' - e_n} e^{\mathrm{i}\nu'\cdot\hat{\Omega}} \right| ; \tag{2.25}$$

the bounds for the other terms can be obtained similarly. Clearly, the above expression is equal to

$$\sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} \left| \frac{\partial N^{(n)}}{\partial \varphi_k} \right| \Big(\sum_{\nu,\alpha} |H_{\nu\alpha}| \sum_{m\in\mathbb{N}} \alpha_m \left| \frac{\partial \hat{N}^{(m)}}{\partial y_k} \right| |\hat{N}^{\alpha - e_m}| \Big) \Big(\sum_{\nu',\alpha'} |f_{\nu'\alpha'}| \alpha'_n |\hat{N}^{\alpha' - e_n}| \Big) \,,$$

which is bounded by

$$\sup_{k\in\mathbb{N}} w_k \sum_{n\in\mathbb{N}} \left|\frac{\partial N^{(n)}}{\partial \varphi_k}\right| \Bigg(\sum_{\nu,\alpha} |H_{\nu\alpha}| \sum_{m\in\mathbb{N}} \alpha_m \sum_{k\in\mathbb{N}} w_k^{-1} \left|\frac{\partial \hat{N}^{(m)}}{\partial y_k}\right| |\hat{N}^{\alpha-e_m}| \Bigg) \\ \times \Bigg(\sup_{n\in\mathbb{N}} \sum_{\nu',\alpha'} |f_{\nu'\alpha'}| \alpha'_n |\hat{N}^{\alpha'-e_n}| \Bigg) .$$

Now, given $y \in \mathscr{U}_{r-\rho}(\ell^\infty_w)$, let $\varepsilon > 0$ be such that $\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w) \subset \mathscr{U}_{r-\rho}(\ell^\infty_w)$. Then

$$\begin{aligned} &\sup_{k\in\mathbb{N}} w_k \sum_{n\in\mathbb{N}} \sup_{\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w)\times\ell^\infty} \left|\frac{\partial N^{(n)}}{\partial \varphi_k}\right| \Bigg(\sum_{\nu,\alpha} |H_{\nu\alpha}| \sum_{m\in\mathbb{N}} \alpha_m \sum_{k\in\mathbb{N}} w_k^{-1} \sup_{\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w)\times\ell^\infty} \left|\frac{\partial \hat{N}^{(m)}}{\partial y_k}\right| |\hat{N}^{\alpha-e_m}| \Bigg) \\ &\qquad \times \Bigg(\sup_{n\in\mathbb{N}} \sum_{\nu',\alpha'} |f_{\nu'\alpha'}| \alpha'_n \sup_{\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w)\times\ell^\infty} |\hat{N}^{\alpha'-e_n}| \Bigg) \\ &\leq \|N\|^{\infty,w}_{r_0,s_0} \Bigg(\sum_{\nu,\alpha} |H_{\nu\alpha}| \sum_{m\in\mathbb{N}} \alpha_m \sum_{k\in\mathbb{N}} w_k^{-1} \left|\frac{\partial \hat{N}^{(m)}}{\partial y_k}\right|_{r-\rho-\rho'} \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell-\delta_{m\ell}}_{r-\rho} \Bigg) \\ &\qquad \times \Bigg(\sup_{n\in\mathbb{N}} \sum_{\nu',\alpha'} |f_{\nu'\alpha'}| \alpha'_n \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha'_\ell-\delta_{n\ell}}_{r-\rho} \Bigg) \end{aligned}$$

for all $\rho' > 0$ such that $\overline{\mathscr{U}}_{\varepsilon,y}(\ell^\infty_w) \subset \mathscr{U}_{r-\rho-\rho'}(\ell^\infty_w)$. By performing a Cauchy estimate on the derivative of $\hat{N}^{(m)}$ we get that there exists a constant c such that the above quantity is bounded by

$$\begin{aligned} &c\,\|N\|^{\infty,w}_{r_0,s_0} \Bigg(\sum_{\nu,\alpha} |H_{\nu\alpha}| |\alpha| \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha_\ell}_{r-\rho} \Bigg) \Bigg(\sup_{n\in\mathbb{N}} \sum_{\nu',\alpha'} |f_{\nu'\alpha'}| \alpha'_n \prod_{\ell\in\mathbb{N}} |\hat{N}^{(\ell)}|^{\alpha'_\ell-\delta_{n\ell}}_{r-\rho} \Bigg) \\ &\leq c\,\|N\|^{\infty,w}_{r_0,s_0} \Bigg(\sum_{\nu\alpha} |H_{\nu\alpha}| |\alpha| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) v_{w,1} \Big)^{\alpha} \Bigg) \\ &\qquad \times \Bigg(\sum_{\nu\alpha} |f_{\nu\alpha}| \alpha'_n \prod_{\ell\in\mathbb{N}} \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) w_k^{-1} \Big)^{\alpha'_\ell-\delta_{n\ell}} \Bigg) . \end{aligned} \tag{2.26}$$

Since $|N|^{\infty,w}_{r_0,s_0} < \rho$, we have that $(r-\rho+|N|^{\infty,w}_{r_0,s_0})/r < 1$, and hence

$$c' := \sup_{|\alpha|} |\alpha| \left(\frac{r-\rho+|N|^{\infty,w}_{r_0,s_0}}{r} \right)^{|\alpha|} < \infty .$$

Therefore

$$\sum_{\nu\alpha} |H_{\nu\alpha}| |\alpha| \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) v_{w,1} \Big)^{\alpha} \leq c' \sum_{\nu\alpha} |H_{\nu\alpha}| v^{\alpha}_{w,r} \leq c' |H|_{r,s} < \infty ,$$

and

$$\sup_{n\in\mathbb{N}} \sum_{\nu\alpha} |f_{\nu\alpha}| \alpha'_n \prod_{\ell\in\mathbb{N}} \Big((r-\rho+|N|^{\infty,w}_{r_0,s_0}) w_k^{-1} \Big)^{\alpha'_\ell-\delta_{n\ell}} \leq \sup_{n\in\mathbb{N}} \sum_{\nu\alpha} |f_{\nu\alpha}| \alpha'_n v^{\alpha-e_n}_{w,r} \leq \|f\|_{r,s} .$$

Then we finally get

$$(2.26) \leq cc'\,\|N\|^{\infty,w}_{r_0,s_0} |H|_{r,s} \|f\|_{r,s} ,$$

which proves the total convergence of (2.25) in the open set $\mathscr{U}_{\varepsilon,y}(\ell^\infty_w) \times \ell^\infty$. The claim follows by the fact that this argument holds for all $y \in \mathscr{U}_{r-\rho}(\ell^\infty_w)$. ■

We conclude this section with the proof of Proposition 2.3.20.

Proof of Proposition 2.3.20. We prove this statement in several steps. First, we show that

$$\{\Phi^* y_\ell, \Phi^* e^{\mathrm{i}\varphi_m}\} = \Phi^*\{y_\ell, e^{\mathrm{i}\varphi_m}\} \qquad \text{and} \qquad \{\Phi^* e^{\mathrm{i}\varphi_\ell}, \Phi^* e^{\mathrm{i}\varphi_m}\}$$

for all $m, \ell \in \mathbb{N}$. Then, by the Leibniz rule we shall get that

$$\{\Phi^*(y^\alpha e^{\mathrm{i}\nu\cdot\varphi}), \Phi^*(y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi})\} = \Phi^*\{y^\alpha e^{\mathrm{i}\nu\cdot\varphi}, y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi}\}$$

holds for all $(\nu, \alpha) \in \mathbb{Z}^{(\mathbb{N})} \times \mathbb{N}_0^{(\mathbb{N})}$. Then, the thesis will follow by Lemmas 2.3.17 and 2.3.21.

Let $m \in \mathbb{N}$. Since the power series defining the exponential function converges in any ball, and given that $\Omega^{(m)}$, together with its derivatives, is bounded on $\mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty$, the series

$$\sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!}(\hat\Omega^{(m)})^n\,, \qquad \sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!}\frac{\partial(\hat\Omega^{(m)})^n}{\partial y_k} \qquad \text{and} \qquad \sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!}\frac{\partial(\hat\Omega^{(m)})^n}{\partial \varphi_k}\,,$$

converge uniformly on $\mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty$ for all $k \in \mathbb{N}$. Moreover, for all $\ell, m \in \mathbb{N}$, both the series

$$\sum_{k\in\mathbb{N}}\sum_{n=0}^\infty \frac{\partial \hat N^{(\ell)}}{\partial y_k}\frac{1}{n!}(\hat\Omega^{(m)})^n\frac{\partial\hat\Omega^{(m)}}{\partial\varphi_k} \qquad \text{and} \qquad \sum_{k\in\mathbb{N}}\sum_{n=0}^\infty \frac{\partial \hat N^{(\ell)}}{\partial \varphi_k}\frac{1}{n!}(\hat\Omega^{(m)})^n\frac{\partial\hat\Omega^{(m)}}{\partial y_k}$$

converge absolutely for all $(y, \varphi) \in \mathscr{U}_{r_0}(\ell^\infty_w) \times \ell^\infty$. Indeed, consider for e.g. the first one. We have that

$$\begin{aligned}
&\sum_{k\in\mathbb{N}}\sum_{n=0}^\infty \left|\frac{\partial \hat N^{(\ell)}}{\partial y_k}(y,\varphi)\right|\frac{1}{n!}|\hat\Omega^{(m)}(y,\varphi)|^n\left|\frac{\partial\hat\Omega^{(m)}}{\partial\varphi_k}(y,\varphi)\right| \\
&= e^{|\hat\Omega^{(m)}(y,\varphi)|}\sum_{k\in\mathbb{N}}\left|\delta_{\ell k} + \frac{\partial N^{(\ell)}}{\partial y_k}(y,\varphi)\right|\left|\delta_{mk} + \frac{\partial\Omega^{(m)}}{\partial\varphi_k}(y,\varphi)\right| \\
&\le e^{|\hat\Omega^{(m)}(y,\varphi)|}\left(\delta_{\ell m} + \left|\frac{\partial N^{(\ell)}}{\partial y_m}(y,\varphi)\right| + \left|\frac{\partial\Omega^{(m)}}{\partial\varphi_\ell}(y,\varphi)\right| + \sum_{k\in\mathbb{N}}\left|\frac{\partial N^{(\ell)}}{\partial y_k}(y,\varphi)\right|\left|\frac{\partial\Omega^{(m)}}{\partial\varphi_k}(y,\varphi)\right|\right)\,.
\end{aligned}$$

The latter quantity is finite because $N^{(\ell)}$ and $\Omega^{(m)}$ are regular for all $\ell, m \in \mathbb{N}$. Similar estimates give that the other series is also absolutely convergent. Consequently, the order of the summations over n and k may be interchanged, yielding

$$\{\hat N^{(\ell)}, e^{i\hat\Omega^{(m)}}\} = \sum_{n=0}^\infty \frac{i^n}{n!}\{\hat N^{(\ell)}, (\hat\Omega^{(m)})^n\}\,,$$

for all $\ell, m \in \mathbb{N}$. Then we have

$$\begin{aligned}
\{\Phi^* y_\ell, \Phi^* e^{\mathrm{i}\varphi_m}\} &= \{\hat N^{(\ell)}, e^{\mathrm{i}\hat\Omega^{(m)}}\} = \sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!}\{\hat N^{(\ell)}, (\hat\Omega^{(m)})^n\} \\
&= \sum_{n=0}^\infty \frac{\mathrm{i}^n}{n!} n (\hat\Omega^{(m)})^{n-1}\{\hat N^{(\ell)}, \hat\Omega^{(m)}\} = \mathrm{i}\delta_{\ell m}\sum_{n=0} \frac{i^n}{n!}(\hat\Omega^{(m)})^n \\
&= \mathrm{i}\delta_{\ell m} e^{\mathrm{i}\hat\Omega^{(m)}} = \mathrm{i}\delta_{\ell m}\Phi^* e^{\mathrm{i}\varphi_m} = \Phi^*\{y_\ell, e^{\mathrm{i}\varphi_m}\}\,.
\end{aligned} \tag{2.27}$$

Following an analogous argument, one can prove that

$$\{\Phi^* e^{\mathrm{i}\varphi_\ell}, \Phi^* e^{\mathrm{i}\varphi_m}\} = \{e^{\mathrm{i}\hat\Omega^{(\ell)}}, e^{\mathrm{i}\hat\Omega^{(m)}}\} = \Phi^*\{e^{\mathrm{i}\varphi_\ell}, e^{\mathrm{i}\varphi_m}\} = 0\,. \tag{2.28}$$

By a direct calculation, and by using the Leibniz rule, it is easy to prove that the results in (2.27) and (2.28) imply that

$$\{\Phi^*(y^\alpha e^{\mathrm{i}\nu\cdot\varphi}), \Phi^*(y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi})\} = \Phi^*\{y^\alpha e^{\mathrm{i}\nu\cdot\varphi}, y^{\alpha'} e^{\mathrm{i}\nu'\cdot\varphi}\}$$

holds for all $(\nu, \alpha) \in \mathbb{Z}^{(\mathbb{N})} \times \mathbb{N}_0^{(\mathbb{N})}$. The above equation together with Lemmas 2.3.17 and 2.3.21 gives formula (i) of Definition 2.3.18. Moreover,

$$\begin{aligned}\{\Phi^* H, \Phi^*\varphi_m\} &= \{\Phi^* \sum_{\nu,\alpha} H_{\nu\alpha} y^\alpha e^{\mathrm{i}\nu\cdot\varphi}, \Phi^*\varphi_m\} = \sum_{\nu,\alpha} H_{\nu\alpha}\{\Phi^* y^\alpha e^{\mathrm{i}\nu\cdot\varphi}, \Phi^*\varphi_m\} \\ &= \sum_{\nu,\alpha} H_{\nu\alpha}\{\hat{N}^\alpha e^{\mathrm{i}\nu\cdot\hat{\Omega}}, \hat{\Omega}^{(m)}\} = \sum_{\nu,\alpha} H_{\nu\alpha}\{\hat{N}^\alpha, \hat{\Omega}^{(m)}\} e^{\mathrm{i}\nu\cdot\hat{\Omega}} .\end{aligned}$$

By the Leibniz rule, and by the fact that $\{\hat{N}^{(\ell)}, \hat{\Omega}^{(m)}\} = \delta_{\ell m}$, it follow that $\{\hat{N}^\alpha, \hat{\Omega}^{(m)}\} = \alpha_m \hat{N}^{\alpha - e_m}$. Therefore,

$$\sum_{\nu,\alpha} H_{\nu\alpha}\{\hat{N}^\alpha, \hat{\Omega}^{(m)}\} e^{\mathrm{i}\nu\cdot\hat{\Omega}} = \sum_{\nu\alpha} H_{\nu\alpha}\alpha_m \hat{N}^{\alpha-e_m} e^{\mathrm{i}\nu\cdot\hat{\Omega}} = \Phi^*\frac{\partial H}{\partial y_k} = \Phi^*\{H, \varphi_m\} ,$$

which yields formula (ii) of Definition 2.3.18. This concludes the proof. ■

We now have the necessary tools to introduce a Hamiltonian dynamics on our phase space. In the next section, we shall employ the Poisson structure developed herein to introduce an infinite-dimensional version of the Hamilton's equations.

2.4 Hamilton's Equations

In this section, we extend the classical Hamiltonian formalism to our infinite-dimensional framework, defining the equations of motion and investigating their behavior under symplectic transformations.

Remark 2.4.1. *In light of the discussion in Section 1.1 and to ease the notation, throughout the remainder of this chapter we shall identify the elements of the torus* $\mathbb{T}^\infty$ *with their representatives in the universal cover* ℓ^∞*. With a slight abuse of notation, we shall use the same symbol* φ *for both, as well as for functions defined on the torus and their associated multi-periodic counterparts defined on* ℓ^∞*, provided that this does not lead to confusion.*

2.4.1 Hamilton's equations and Hamiltonian flow

Given $r, s > 0$ and a function $H \in \mathcal{A}_{r,s}$, we say that H induces a *Hamiltonian flow* ϕ_H on an open neighbourhood $\mathcal{N}_0 \subseteq \mathscr{U}_r(\ell^\infty_{\mathtt{w}}) \times \mathbb{T}^\infty$ of $\{0\} \times \mathbb{T}^\infty$ if there exists a group action ϕ_H of $\mathbb{R}$ on $\mathcal{N}_0$

$$\mathbb{R} \times \mathcal{N}_0 \ni (t, y, \varphi) \mapsto \phi_H^t(y, \varphi) \in \mathcal{N}_0$$

such that, denoting $\phi_H^t(y, \varphi) = (y_1(t), \ldots, y_k(t), \ldots, \varphi_1(t), \ldots, \varphi_k(t), \ldots)$, the maps $t \mapsto y_k(t)$ and $t \mapsto \varphi_k(t)$ are differentiable functions for all $k \in \mathbb{N}$ and satisfy the *Hamilton's equations*

$$\begin{cases} \dot{y}_k = \{H, y_k\} \\ \dot{\varphi}_k = \{H, \varphi_k\} \end{cases} , \quad k \in \mathbb{N} .$$

In the following, we shall refer to H as the *Hamiltonian function*.

Note that, in general, a Hamiltonian flow is not even continuous in time as a function of the time $t \in \mathbb{R}$ taking values in the Banach manifold. This is essentially due to the fact that, as already pointed out in Remark 2.3.5, the gradient of an analytic function defined on an infinite-dimensional Banach space X does not take values in X, in general. Let us see this with a simple example. Given an unbounded sequence of real numbers $\omega = (\omega_j)_{j\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} \omega_j w_j^{-1}$ is finite, consider the Hamiltonian function

$$H = \sum_{j\in\mathbb{N}} \omega_j y_j \ , \qquad y \in \mathscr{U}_r(\ell^\infty_w) \ . \tag{2.29}$$

Clearly, H belongs to $\mathcal{A}_{r,s}$ for all $r, s > 0$. The Hamilton's equations associated with the Hamiltonian H are given by

$$\begin{cases} \dot{y}_k = 0 \\ \dot{\varphi}_k = \omega_k \end{cases} , \quad k \in \mathbb{N} ,$$

which can be easily integrated yielding the solutions $y_k(t) = \text{const}$, $\varphi_k(t) = \omega_k t + \text{const}$. Therefore, for all $r > 0$, H induces the Hamiltonian flow on $\mathscr{U}_r(\ell^\infty_{\mathtt{w}}) \times \mathbb{T}^\infty$ defined by

$$\phi^t_H(y, \varphi) = (y, \varphi_1 + \omega_1 t, \ldots, \varphi_k + \omega_k t, \ldots) \ .$$

Since $(\omega_j)_{j\in\mathbb{N}}$ is unbounded, this flow cannot be continuous in time. Indeed, for any $(y, \varphi) \in \ell^\infty_w \times \ell^\infty$, any two distinct points on the orbit of (y, φ) under the action of ϕ_H are infinitely separated. Specifically, for any $t_1 \neq t_2$, one has

$$\|\varphi + \omega t_1 - \varphi - \omega t_2\|_{\ell^\infty} = \sup_{j\in\mathbb{N}} |\omega_j(t_1 - t_2)| = \infty.$$

For the sake of clarity, we emphasize that, in accordance with standard arguments from the theory of differentiable functions between Banach spaces, the gradient of the Hamiltonian defined in (2.29) belongs to the topological dual of $\ell^\infty_w \times \ell^\infty$. This follows directly from the fact that the sum $\sum_{j\in\mathbb{N}} \omega_j w_j^{-1}$ is finite.

2.4.2 Invariance of the Hamilton's equations under symplectomorphisms

We now show that the Hamilton's equations are invariant under symplectic changes of coordinates. Namely, we prove that for any $H \in \mathcal{A}$ inducing a Hamiltonian flow and any symplectomorphism Φ, the Hamilton's equations in the coordinate frame $(x, \theta) := \Phi(y, \varphi)$ take the form

$$\begin{cases} \dot{x}_k = \{H \circ \Phi^{-1}, x_k\} \\ \dot{\theta}_k = \{H \circ \Phi^{-1}, \theta_k\} \end{cases} , \quad k \in \mathbb{N}.$$

To this end, we first establish the following lemma, which is central to our argument.

Lemma 2.4.2. *Let $r, s > 0$ and let $H \in \mathcal{A}_{r,s}$ be a Hamiltonian function that induces a Hamiltonian flow. Then, for all $f \in \mathcal{R}_{r,s}$,*

$$\frac{d}{dt}(f \circ \phi^t_H) = \{H, f\} \circ \phi^t_H \ .$$

Proof. The above equality follows easily for any power-trigonometric polynomial. The claim then follows for all $f \in \mathcal{R}_{r,s}$ by an argument strictly analogous to the proof of Proposition 2.2.4. ∎

With the help of the result established in the above lemma, let us prove the invariance of the Hamilton's equations under symplectic changes of coordinates. More precisely, we prove the following proposition, which is sufficient and crucial for our purposes.

Proposition 2.4.3. *Given* $r, s > 0$ *, let* $H \in \mathcal{A}_{r,s}$ *and consider a symplectomorphism* $\Phi \in \mathcal{C}_{r,s}(\mathcal{R})$ *. Assume that* $H \circ \Phi^{-1}$ *induces a Hamiltonian flow* $\phi_{H\circ\Phi^{-1}}$ *on* $\mathscr{U}_{r-\rho}(\ell_w^\infty) \times \mathbb{T}^\infty$ *, with* $\rho > 0$ *as in Definition 2.3.11. Then,* H *induces a Hamiltonian flow on the neighbourhood* $\mathscr{N}_0 := \Phi^{-1}(\mathscr{U}_{r-\rho}(\ell_w^\infty) \times \mathbb{T}^\infty)$ *of* $\{0\} \times \mathbb{T}^\infty$ *. In particular, we have that*

$$\phi_H^t = \Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t \circ \Phi$$

holds for all $t \in \mathbb{R}$ *.*

Proof. First, note that $H \circ \Phi^{-1}$ is a well defined analytic function belonging to $\mathcal{A}_{r-\rho, s-\sigma}$, with $\sigma > 0$ as in Definition 2.3.11. Moreover, since Φ is in particular a change of coordinates, as already noted in the proof of Theorem 1.3.12 (see (2.33)), the open set $\Phi^{-1}(\mathscr{U}_{r-\rho}(\ell_w^\infty) \times \mathbb{T}^\infty)$ is indeed a neighbourhood of $\{0\} \times \mathbb{T}^\infty$. Now, by assumption, $H \circ \Phi^{-1}$ induces a Hamiltonian flow. Hence, the above Lemma implies

$$\frac{d}{dt}(y_k \circ \Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t) = \{H \circ \Phi^{-1}, y_k \circ \Phi^{-1}\} \circ \phi_{H\circ\Phi^{-1}}^t \,.$$

Since Φ is a symplectomorphism, by Remark 2.3.19, the right-hand side of the above equation is equal to

$$\{H, y_k\} \circ \Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t$$

By a similar, slightly refined argument one can prove that

$$\frac{d}{dt}(\varphi_k \circ \Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t) = \{H, \varphi_k\} \circ \Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t \,.$$

We remark that the argument requires refinement because, in general, $\varphi_k \circ \Phi^{-1}$ does not belong to $\mathcal{R}$. This precludes the direct application of Lemma 2.4.2. Nevertheless, this is not a real obstacle: by the definition of change of coordinates and denoting the new angular variables by θ_k (with $k \in \mathbb{N}$), it follows that $\varphi_k \circ \Phi^{-1}$ is simply the projection onto the θ_k coordinate plus a function in $\mathcal{R}_{r,s}$. Therefore, we conclude that $\Phi^{-1} \circ \phi_{H\circ\Phi^{-1}}^t \circ \Phi$ is the Hamiltonian flow induced by H . ■

Let us make some comments on the above proposition. The fact that $H \circ \Phi^{-1}$ induces a Hamiltonian flow on $\mathscr{U}_{r-\rho} \times \mathbb{T}^\infty$ means that there exists a group action $\phi_{H\circ\Phi^{-1}}$

$$\mathbb{R} \times \mathscr{U}_{r-\rho} \times \mathbb{T}^\infty \ni (t, x, \theta) \mapsto \phi_{H\circ\Phi^{-1}}^t(x, \theta) \in \mathscr{U}_{r-\rho} \times \mathbb{T}^\infty$$

such that, denoting $\phi_{H\circ\Phi^{-1}}^t(x, \theta) = (x_1(t), \ldots, x_k(t), \ldots, \theta_1(t), \ldots, \theta_k(t), \ldots)$, the maps $t \mapsto x_k(t)$ and $t \mapsto \theta_k(t)$ are differentiable functions for all $k \in \mathbb{N}$ and satisfy the Hamilton's equations

$$\begin{cases} \dot{x}_k = \{H \circ \Phi^{-1}, x_k\} \\ \dot{\theta}_k = \{H \circ \Phi^{-1}, \theta_k\} \end{cases} , \quad k \in \mathbb{N} \,. \tag{2.30}$$

Then, by Proposition 2.4.3, the Hamilton's equations associated with the Hamiltonian H , namely,

$$\begin{cases} \dot{y}_k = \{H, y_k\} \\ \dot{\varphi}_k = \{H, \varphi_k\} \end{cases} , \quad k \in \mathbb{N} \,,$$

admit a solution $(y_1(t), \ldots, y_k(t), \ldots, \varphi_1(t), \ldots, \varphi_k(t), \ldots)$ for any initial datum (y, φ) in the open neighbourhood $\mathscr{N}_0 := \Phi^{-1}(\mathscr{U}_{r-\rho}(\ell_w^\infty) \times \mathbb{T}^\infty)$ of $\{0\} \times \mathbb{T}^\infty$. Specifically, the solution is given by

$$\phi_H^t(y, \varphi) = \Phi^{-1}(\phi_{H\circ\Phi^{-1}}^t(\Phi(y, \varphi))) \,, \qquad \text{for all } (y, \varphi) \in \mathscr{N}_0 \,,$$

and ϕ_H is a group action of $\mathbb{R}$ on $\mathscr{N}_0$.

2.4.3 Action-angle coordinates

The above argument suggests that a good strategy to solve Hamilton's equations for a given Hamiltonian $H \in \mathcal{A}$ consists in trying to find a symplectomorphism Φ such that the Hamilton's equations for the new Hamiltonian $H \circ \Phi^{-1}$ are easily solvable. For instance, if $H \circ \Phi^{-1}$ does not depend on the variables θ_k , $k \in \mathbb{N}$, the infinite system of ODE in (2.30) admit the unique solution

$$\phi^t_{H\circ\Phi^{-1}}(x,\theta) = (x, \theta_1 + \omega_1(x)t, \ldots, \theta_k + \omega_k(x)t, \ldots) \ , \quad \text{where } \omega_k(x) := \frac{\partial H \circ \Phi^{-1}}{\partial x_k}(x) \ , \ k \in \mathbb{N} \, .$$

Namely, $H \circ \Phi^{-1}$ induces a Hamiltonian flow $\phi_{H\circ\Phi^{-1}}$ such that, for all x , the leaf $\{x\} \times \mathbb{T}^\infty$ is invariant and the restriction of the Hamiltonian flow on such invariant manifold, namely

$$\{x\} \times \mathbb{T}^\infty \in \theta \mapsto \phi^t_{H\circ\Phi^{-1}}(x,\theta) \in \{x\} \times \mathbb{T}^\infty \ ,$$

is simply a linear flow on the infinite torus. Therefore, $\mathscr{N}_0$ is foliated by infinite tori of the form $\Phi^{-1}(\{x\} \times \mathbb{T}^\infty)$ and, by virtue of Proposition 2.4.3, each leaf is invariant for the Hamiltonian flow ϕ_H , which is conjugated by Φ to a linear flow on the infinite torus. The set of coordinates (x, θ) defined by such symplectomorphism are called *action-angle coordinates*.

A final observation is in order: as previously observed, a Hamiltonian flow is not even continuous in time, in general. Thus, at this stage, we cannot talk about topological conjugacy, even if the conjugating map has very strong regularity. We shall see in Chapter 4 that such kind of dynamics is always continuous if one endows the infinite torus with the product topology.

In the next sections, we provide conditions for a Hamiltonian $H \in \mathcal{A}$ to admit a symplectomorphism Φ such that the new Hamiltonian $H \circ \Phi^{-1}$ is independent of the variables θ_k for $k \in \mathbb{N}$. The strategy follows the generating function method as developed in the context of classical Hamiltonian mechanics (see, e.g., [Arn89]). Specifically, the construction is divided into two parts: in Section 2.5, we establish a general geometric result that provides an infinite-dimensional version of *symplectic straightening* for a complete set of Poisson-commuting integrals. Then, in Section 2.6, we show that if a Hamiltonian H Poisson-commutes with these integrals, the resulting symplectic transformation yields a set of action-angle coordinates, effectively linearizing the dynamics.

2.5 Symplectic Straightening of Foliations by Infinite Tori

Here we give the precise statement of the aforementioned theorem about the symplectic straightening and we address all the technical issues needed to prove that the set of coordinates formally defined in the introduction to the chapter are indeed a set of action-angle coordinates for the foliation induced by the map Γ_P .

2.5.1 A technical lemma

Here we prove a lemma that is fundamental to bridge the results of the previous chapter to the present. We recall that, given $G \in \mathcal{A}^{\infty;w}_{r,s}$, we denote by Φ_G the map

$$\Phi_G(y,\varphi) = (y + G(y,\varphi), \varphi) \ , \qquad \forall \, y \in \mathscr{U}_r(\ell^\infty_w) \ , \ \varphi \in \ell^\infty \, .$$

Then we have the following estimate for the fixed point of the equation $G = \Phi_G^* F := F \circ \Phi_G$, where F is a sequence of regular functions.

Lemma 2.5.1. *Given $r, s > 0$, let $0 \le \rho < r$. Consider $F \in \mathscr{U}_{r/2}(\mathcal{R}^{\infty,w}_{r,s})$ and let $G \in \overline{\mathscr{U}}_{\rho}(\mathcal{A}^{\infty}_{r-\rho,s})$. Assume that $G = \Phi_G^* F := F \circ \Phi_G$. Then, $G \in \mathcal{R}^{\infty,w}_{r-\rho,s}$ and satisfies the estimate*

$$\|G\|^{\infty,w}_{r-\rho,s} \le \frac{2}{1-(2\|F\|^{\infty,w}_{r,s}/r)}\|F\|^{\infty,w}_{r,s} \ ,$$

Proof. Let us begin by noting that the bound

$$|G|^{\infty,w}_{r-\rho,s} = |\Phi_G^* F|^{\infty,w}_{r-\rho,s} \le 2|F|^{\infty,w}_{r,s}$$

follows by Proposition 1.2.19. We now give suitable estimates for the derivatives. We adopt the notation $(x,\theta) = \Phi_G(y,\varphi)$. Then, for all $m, k \in \mathbb{N}$, by the chain rule one has that

$$\begin{aligned}
\frac{\partial G^{(m)}}{\partial y_k} &= \frac{\partial\, \Phi_G^* F^{(m)}}{\partial y_k} = \sum_{\ell\in\mathbb{N}} \Phi_G^* \frac{\partial F^{(m)}}{\partial x_\ell}\left(\delta_{\ell k} + \frac{\partial G^{(\ell)}}{\partial y_k}\right) \\
\frac{\partial G^{(m)}}{\partial \varphi_k} &= \frac{\partial\, \Phi_G^* F^{(m)}}{\partial \varphi_k} = \sum_{\ell\in\mathbb{N}} \Phi_G^* \frac{\partial F^{(m)}}{\partial x_\ell}\ \frac{\partial G^{(\ell)}}{\partial \varphi_k} + \Phi_G^* \frac{\partial F^{(m)}}{\partial \varphi_k}
\end{aligned}$$

The above equations can also be written as

$$\begin{aligned}
\sum_{\ell\in\mathbb{N}}\left(\delta_{m\ell} - \Phi_G^*\frac{\partial F^{(m)}}{\partial x_\ell}\right)\frac{\partial G^{(\ell)}}{\partial y_k} &= \Phi_G^*\frac{\partial F^{(m)}}{\partial x_k} \\
\sum_{\ell\in\mathbb{N}}\left(\delta_{m\ell} - \Phi_G^*\frac{\partial F^{(m)}}{\partial x_\ell}\right)\frac{\partial G^{(\ell)}}{\partial \varphi_k} &= \Phi_G^*\frac{\partial F^{(m)}}{\partial \varphi_k} \ ,
\end{aligned}$$

and they can be inverted by Neumann series, namely

$$\begin{aligned}
\frac{\partial G}{\partial y} &= \sum_{n=0}^{\infty}\left(\Phi_G^*\frac{\partial F}{\partial x}\right)^n \Phi_G^*\frac{\partial F}{\partial x} \\
\frac{\partial G}{\partial \varphi} &= \sum_{n=0}^{\infty}\left(\Phi_G^*\frac{\partial F}{\partial x}\right)^n \Phi_G^*\frac{\partial F}{\partial \varphi} \ .
\end{aligned}$$

Indeed, by the triangular inequality and by the algebra property established in Proposition 1.2.6, one has

$$\begin{aligned}
\sup_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}}\left|\frac{\partial G^{(m)}}{\partial y_k}\right|_{r-\rho,s} &\le \sup_{k\in\mathbb{N}}\sum_{n=0}^{\infty}\sum_{\substack{m\in\mathbb{N}\\ \ell_1,\dots,\ell_n\in\mathbb{N}}}\left|\Phi_G^*\frac{\partial F^{(m)}}{\partial x_{\ell_1}}\dots\Phi_G^*\frac{\partial F^{(\ell_{n-1})}}{\partial x_{\ell_n}}\ \Phi_G^*\frac{\partial F^{\ell_n}}{\partial x_k}\right|_{r-\rho,s} \\
&\le \sum_{n=0}^{\infty}\sup_{\ell_{n+1}\in\mathbb{N}}\sum_{\ell_0,\ell_1,\dots,\ell_n\in\mathbb{N}}\prod_{j=0}^{n}\left|\Phi_G^*\frac{\partial F^{(\ell_j)}}{\partial x_{\ell_{j+1}}}\right|_{r-\rho,s} .
\end{aligned}$$

By Proposition 1.2.19, above quantity is bounded by

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2^{n+1}\sup_{\ell_{n+1}\in\mathbb{N}}\left(\sum_{\ell_0,\dots,\ell_n\in\mathbb{N}}\prod_{j=0}^{n}\left|\frac{\partial F^{(\ell_j)}}{\partial x_{\ell_{j+1}}}\right|_{r,s}\right) \\
&\qquad\le \sum_{n=0}^{\infty} 2^{n+1}\prod_{j=0}^{n}\sup_{\ell_{j+1}\in\mathbb{N}}\sum_{\ell_j\in\mathbb{N}}\left|\frac{\partial F^{(\ell_j)}}{\partial x_{\ell_{j+1}}}\right|_{r,s} \\
&\qquad\le \sum_{n=0}^{\infty}\left(\frac{2\|F\|^{\infty,w}_{r,s}}{r}\right)^{n+1} = \frac{2\|F\|^{\infty,w}_{r,s}}{r-2\|F\|^{\infty,w}_{r,s}} \ .
\end{aligned}$$

Therefore,

$$r \sup_{k\in\mathbb{N}} \sum_{m\in\mathbb{N}} \left| \frac{\partial G^{(m)}}{\partial y_k} \right|_{r-\rho,s} \leq \frac{2}{1-(2\|F\|^{\infty,w}_{r,s}/r)} \|F\|^{\infty,w}_{r,s} \ .$$

By analogous computations one can prove the same bound for the derivatives with respect to the periodic variables, yielding the claim. ■

This result connects the fixed-point argument developed in the first chapter (see First Fixed Point Lemma 1.3.10) with the specific analytical framework of the current one. While the previous chapter established the existence of the fixed point in a broader functional space, this lemma ensures that the fixed point inherits the more stringent regularity properties of F . In particular, we have shown that if F is bounded with respect to the stronger norm $\|\cdot\|^{\infty,w}_{r,s}$, the same holds for the fixed point. This consistency is what allows us to carry out the subsequent steps of the analysis in the present setting.

2.5.2 Symplectic straightening

Let us start by proving that the map

$$\mathscr{U}_r(\ell^\infty_w) \ni \gamma \mapsto x(\gamma) := \left(\int_{\Gamma_P^{-1}(\gamma)} y_k \, d\lambda \right)_{k\in\mathbb{N}} \in \ell^\infty_w$$

induces a regular change of coordinates. Recalling that, for all $k \in \mathbb{N}$, we have the explicit formula

$$\int_{\Gamma_P^{-1}(\gamma)} y_k \, d\lambda = \gamma_k - A_0^{(k)}(\gamma) \ ,$$

where A_0 is the average of the function A defined by $A = \Phi^*_{-A}P$ (see Lemma 1.3.10), it is clear that the following theorem gives the desired claim.

Theorem 2.5.2. *Given $r_0, s_0 > 0$, let $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ satisfy $\|P\|^{\infty,w}_{r_0,s_0} < r_0/24$. Then the map*

$$\Phi_{-A_0} : \mathscr{U}_{r_0-\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (\gamma,\varphi) \mapsto (\gamma - A_0(\gamma), \varphi),$$

is a regular change of coordinates for all ρ such that $7\|P\|^{\infty,w}_{r_0,s_0} \leq \rho < r_0/2$ and $0 \leq \sigma < s_0/2$. Its inverse is given by

$$\Phi_B : \mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (x,\varphi) \mapsto (x + B(x), \varphi),$$

where $B \in \mathcal{R}^{\infty,w}_{r_0-2\rho,s_0}$ is defined by $B = \Phi^*_B A_0$. Furthermore, the following bounds hold:

$$\|A_0\|^{\infty,w}_{r_0-\rho,s_0} \leq \|A\|^{\infty,w}_{r_0-\rho,s_0} \leq \frac{24}{11}\|P\|^{\infty,w}_{r_0,s_0}, \qquad \|B\|^{\infty,w}_{r_0-2\rho,s_0} \leq \frac{48}{7}\|P\|^{\infty,w}_{r_0,s_0}. \tag{2.31}$$

Proof. We first verify that Φ_{-A_0} is a well-defined change of coordinates. Recalling that $A = \Phi^*_{-A}P$, Proposition 1.2.19 implies that $|A|^{\infty,w}_{r_0-\rho,s_0}$ is bounded by $2|P|^{\infty,w}_{r_0,s_0}$, which leads to the chain of inequalities

$$|A_0|^{\infty,w}_{r_0-\rho,s_0} \leq |A|^{\infty,w}_{r_0-\rho,s_0} \leq 2|P|^{\infty,w}_{r_0,s_0}.$$

Under the smallness assumption $\|P\|^{\infty,w}_{r_0,s_0} < r_0/24$, it follows directly from the above inequalities that

$$|A_0|^{\infty,w}_{r_0-\rho,s_0} < \frac{r_0-\rho}{6}.$$

Thus, the hypotheses of Theorem 1.3.12 are satisfied. Consequently, the map

$$\Phi_{-A_0} : \mathscr{U}_{r_0-\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty$$

is a change of coordinates with $\rho' = 3|A_0|^{\infty,w}_{r_0-\rho,s_0}$ and $\sigma = 0$[1]. Its inverse is

$$\Phi_B : \mathscr{U}_{r_0-\rho-\rho'}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty,$$

where $B \in \mathcal{A}^{\infty,w}_{r_0-\rho-\rho',s_0}$ is defined by $B = \Phi_B^* A_0$ and satisfies the bound $|B|^{\infty,w}_{r_0-\rho-\rho',s_0} \leq 2|A_0|^{\infty,w}_{r_0-\rho,s_0}$.

Finally, since $\rho' \leq \rho$, to complete the proof, it suffices to show that Φ_{-A_0} is also a regular change of coordinates with $7\|P\|^{\infty,w}_{r_0,s_0} \leq \rho < r_0/2$. To this end, we first note that the smallness condition on P and Lemma 2.5.1 directly imply the bounds in (2.31). In particular, the norms $\|A_0\|^{\infty,w}_{r_0-\rho,s_0}$ and $\|B\|^{\infty,w}_{r_0-2\rho,s_0}$ are smaller than ρ . Then Remark 2.3.12 ensures that Φ_{-A_0} is indeed a regular change of coordinates with inverse Φ_B. This completes the proof. ■

It is worth highlighting that the estimates stated in above theorem imply that the proposed generating function S defined by

$$S(x,\varphi) = \sum_{k\in\mathbb{N}} x_k\varphi_k - W(x,\varphi), \qquad W(x,\varphi) := \sum_{m\in\mathbb{N}} \sum_{\nu\in\Lambda_m} \frac{A^{(m)}_\nu(\gamma(x))}{i\nu_m} e^{i\nu\cdot\varphi}, \tag{2.32}$$

is indeed a real majorant analytic function $S : \mathscr{U}_r(\ell^\infty_w) \times \ell^\infty \to \mathbb{R}$, $r \leq r_0 - 2\rho$: In fact, by the nature of the elements involved, the first sum is totally convergent, and $|W|_{r,s}$ can be bounded by $\|A\|^{\infty,w}_{r,s}$, up to a multiplicative constant, for any $r \leq r_0 - 2\rho$.

We now prove an important intermediate result which plays a crucial role in the construction of the aimed symplectic straightening of the foliation induced by Γ_P.

Theorem 2.5.3. *Given $r_0, s_0 > 0$, let $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ satisfy $\|P\|^{\infty,w}_{r_0,s_0} < r_0/27$. Then the map*

$$\Phi_{-A_0} \circ \Phi_P : \mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty$$

is a regular change of coordinates for all ρ such that $9\|P\|^{\infty,w}_{r_0,s_0} \leq \rho < r_0/3$ and $0 \leq \sigma < s_0/2$. Its inverse is given by

$$\Phi_{-A} \circ \Phi_B : \mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty.$$

Moreover, the set $(\Phi_{-A} \circ \Phi_B)(\mathscr{U}_{r_0-3\rho}(\ell^\infty_w) \times \ell^\infty)$ is an open neighbourhood of $\{0\} \times \ell^\infty$.
Finally, assuming that

$$\{\Gamma^{(m)}_P, \Gamma^{(n)}_P\} = 0 \qquad \textit{for all } m,n \in \mathbb{N}\,,$$

and denoting $(x,\varphi) = (\Phi_{-A_0} \circ \Phi_P)\,(y,\varphi)$, the following hold

$$\{x_n, x_m\} = 0 \qquad \textit{for all } n,m \in \mathbb{N}.$$

Proof. First, note that the maps defined above are indeed the inverse of each other. In fact, by Theorem 2.5.2, we have:

$$\begin{aligned}\Phi_{-A_0} \circ \Phi_P \circ \Phi_{-A} \circ \Phi_B &\overset{\text{Thm1.3.12}}{=} \Phi_{-A_0} \circ \mathrm{Id}_{\mathscr{U}_{r_0-\rho}(\ell^\infty_w)\times\ell^\infty} \circ \Phi_B = \Phi_{-A_0} \circ \Phi_B \\ &\overset{\text{Thm2.5.2}}{=} \mathrm{Id}_{\mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty}\end{aligned}$$

[1]Note that there is no loss of regularity with respect to the variables φ_k , $k \in \mathbb{N}$, because the transformation does not change the angles.

$$\Phi_{-A}\circ\Phi_B\circ\Phi_{-A_0}\circ\Phi_P \overset{\text{Thm2.5.2}}{=} \Phi_{-A}\circ \mathrm{Id}_{\mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty}\circ\Phi_P = \Phi_{-A}\circ\Phi_P$$
$$\overset{\text{Thm1.3.12}}{=} \mathrm{Id}_{\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty} \ .$$

We now verify that the composition

$$\Phi_{-A_0}\circ\Phi_P : \mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty \to \ell^\infty_w\times\ell^\infty$$

is indeed a regular change of coordinates. We write the explicit action of the maps involved:

$$(\Phi_{-A_0}\circ\Phi_P)(y,\varphi) = (y+P(y,\varphi)-A_0(y+P(y,\varphi)),\varphi)\ ,$$
$$(\Phi_{-A}\circ\Phi_B)(x,\varphi) = (x+B(x)-A(x+B(x),\varphi),\varphi)\ ,$$

for all $(y,\varphi)\in\mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty$. Therefore, by defining the maps N and M as

$$N(y,\varphi) = P(y,\varphi)-A_0(y+P(y,\varphi))\ ,\qquad M(x,\varphi)=B(x)-A(x+B(x),\varphi)\ ,$$

we have that $\Phi_{-A_0}\circ\Phi_P = \mathrm{Id}+(N,0)$ and $\Phi_{-A}\circ\Phi_B = \mathrm{Id}+(M,0)$. By using the bounds in (2.31) and estimating the composition of functions with respect to the $\|\cdot\|^{\infty,w}_{r,s}$-norm, it follows that both $\|N\|^{\infty,w}_{r_0-2\rho,s_0}$ and $\|M\|^{\infty,w}_{r_0-2\rho,s_0}$ are smaller than ρ. Consequently, the first claim follows from Remark 2.3.12.

To prove the second claim, note that, given a change of coordinates $\Phi\in\mathcal{C}^{\rho,0}_{r_0-2\rho,s_0}(\mathcal{A})$ with inverse $\Psi\in\mathcal{C}^{\rho,0}_{r_0-2\rho,s_0-\sigma}(\mathcal{A})$, one has that

$$\{0\}\times\ell^\infty\subset\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty = \Psi\circ\Phi(\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty)\ \subset\Psi(\mathscr{U}_{r_0-2\rho}\times\ell^\infty)\ . \tag{2.33}$$

The last inclusion follows because $\Phi:\mathscr{U}_r(\ell^\infty_w)\times\ell^\infty\to\mathscr{U}_{r+\rho}(\ell^\infty_w)\times\ell^\infty$ for all $r\le r_0-2\rho$.

We conclude the proof by showing that the coordinate functions $(x_k)_{k\in\mathbb{N}}$ induced by the map $\Phi_{-A_0}\circ\Phi_P$ Poisson commute. To simplify the notation, in the following computations we assume that the derivatives of x_k with respect to γ_ℓ (for $k,\ell\in\mathbb{N}$) are evaluated at $(\Gamma_P(y,\varphi),\varphi)$, even if not explicitly indicated. Then we have:

$$\begin{aligned}\{x_n,x_m\} &= \sum_{k\in\mathbb{N}}\frac{\partial x_n}{\partial y_k}\frac{\partial x_m}{\partial\varphi_k}-\frac{\partial x_n}{\partial\varphi_k}\frac{\partial x_m}{\partial y_k}\\ &=\sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}}\sum_{h\in\mathbb{N}}\frac{\partial x_n}{\partial\gamma_\ell}\frac{\partial x_m}{\partial\gamma_h}\left(\frac{\partial\Gamma_P^{(\ell)}}{\partial y_k}\frac{\partial\Gamma_P^{(h)}}{\partial\varphi_k}-\frac{\partial\Gamma_P^{(\ell)}}{\partial\varphi_k}\frac{\partial\Gamma_P^{(h)}}{\partial y_k}\right)\ .\end{aligned}$$

Since, by assumption, the components of Γ_P Poisson commute, the latter expression is formally equal to 0. By performing Cauchy estimates and exploiting the strong regularity of the involved maps (expressed in terms of the $\|\cdot\|^{\infty,w}_{r,s}$-norms), one can prove, by following the same argument as in the proof of Lemma 2.3.21, that for every $(y,\varphi)\in\mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty$, there exists a neighbourhood in which the series is totally convergent. This concludes the proof. ■

We now turn to the issue of proving that the angles defined formally by (see Equation (2.14) and the previous discussion)

$$\theta_k(x,\varphi)=\varphi_k-\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m}\frac{1}{\mathrm{i}\nu_m}\frac{\partial A^{(m)}_\nu(\gamma(x))}{\partial x_k}e^{\mathrm{i}\nu\cdot\varphi}\ ,\quad k\in\mathbb{N} \tag{2.34}$$

define indeed a regular change of coordinates. Let us start by justifying the algebraic manipulations that we performed for proving the relations in (2.12), which we recall below

$$\nu_\ell A^{(m)}_\nu = \nu_m A^{(\ell)}_\nu \quad \text{for all } \ell,m\in\mathbb{N}\ . \tag{2.35}$$

Based on the construction of the candidate angle coordinates presented in Subsection 2.1.2, to prove (2.12), we must show that involved formal series are in fact totally convergent.

Lemma 2.5.4. *Given* $r, s > 0$ *, and* $0 < \rho < r$ *, let* $P \in \mathcal{R}^{\infty,w}_{r,s}$ *be such that* $|P|^{\infty,w}_{r,s} < \rho$ *and consider* $A \in \mathcal{A}^{\infty,w}_{r-\rho,s}$ *. Then, for all* $(y,\varphi) \in \mathscr{U}_{r-2\rho}(\ell^\infty_w) \times \ell^\infty$ *, there exists a neighbourhood such that the formal series*

$$\sum_{\ell\in\mathbb{N}} \frac{\partial \Gamma_P^{(m)}}{\partial y_\ell}(y,\varphi) \frac{\partial A^{(\ell)}}{\partial \varphi_k}(\Gamma_P(y,\varphi),\varphi) , \qquad m \in \mathbb{N} ,$$

$$\sum_{k\in\mathbb{N}}\sum_{\ell\in\mathbb{N}} \frac{\partial \Gamma_P^{(m)}}{\partial y_k}(y,\varphi) \frac{\partial \Gamma_P^{(n)}}{\partial y_\ell}(y,\varphi) \left(\frac{\partial A^{(\ell)}}{\partial \varphi_k}(\Gamma_P(y,\varphi),\varphi) - \frac{\partial A^{(k)}}{\partial \varphi_\ell}(\Gamma_P(y,\varphi),\varphi) \right) , \quad m, n \in \mathbb{N} ,$$

are totally convergent.

Proof. Regarding the first series, the statement actually holds for all $P \in \mathscr{U}_\rho(\mathcal{A}^{\infty,w}_{r,s})$; that is, the regularity of P is not required. In fact, total convergence in a neighbourhood of any point in the given domain can be shown as follows: following the same argument as in the proof of Lemma 2.3.21, by performing a Cauchy estimate on the derivative of A and subsequently providing a bound for the composition (see Proposition 1.2.19), the analyticity of Γ_P allows one to complete the argument with a further Cauchy estimate. For the second series, the reasoning is similar, with the difference that one must exploit the regularity of P to achieve the desired estimate. ■

Adopting the same notations as in Theorems 2.5.2 and 2.5.3, we now show that the function Q : $\mathscr{U}_{r_0-3\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty$, defined for all $(x,\varphi) \in \mathscr{U}_{r_0-2\rho}(\ell^\infty_w) \times \ell^\infty$ by

$$Q^{(k)}(x,\varphi) = \sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m} \frac{1}{i\nu_m} \left.\frac{\partial A^{(m)}_\nu \circ \Phi_B}{\partial x_k}\right|_x e^{i\nu\cdot\varphi}, \quad k \in \mathbb{N}, \tag{2.36}$$

belongs to $\mathcal{R}^\infty_{r_0-3\rho,s_0}$. More precisely, we have the following result:

Proposition 2.5.5. *Given* $r_0, s_0 > 0$ *, let* $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ *satisfy* $\|P\|^{\infty,w}_{r_0,s_0} < r_0/28$ *. Assume that the components of* Γ_P *Poisson commute. Then the function* Q *defined in* (2.36) *belongs to* $\mathcal{R}^\infty_{r_0-3\rho,s_0}$ *, for all* ρ *such that* $7\|P\|^{\infty,w}_{r_0,s_0} \le \rho < r_0/4$ *, and satisfies the bound*

$$\|Q\|^\infty_{r_0-3\rho,s_0} \le c\|P\|^{\infty,w}_{r_0,s_0} ,$$

for some constant $c > 1$.

Proof. First, note that all assumptions of Theorem 2.5.2 are satisfied; thus, the function B is well-defined and belongs to $\overline{\mathscr{U}}_\rho(\mathcal{R}^{\infty,w}_{r_0-2\rho,s_0})$. Consequently, the composition $A \circ \Phi_B \in \mathcal{R}^{\infty,w}_{r_0-2\rho,s_0}$ is also well-defined. The additional condition $\rho < r_0/4$ ensures sufficient analyticity width to perform Cauchy estimates that are uniform with respect to ρ. Without this condition, the constant c would depend on ρ such that $\lim_{\rho\to r_0/3} c(\rho) = \infty$. By the aforementioned lemma, the fact that the components of Γ_P Poisson commute implies that the formulas in (2.35) hold. Consequently, for any $k \in \mathbb{N}$, the derivative of Q with respect to φ_k can be expressed as

$$\sum_{m\in\mathbb{N}}\sum_{\nu\in\Lambda_m} \frac{\mathrm{i}\nu_k}{i\nu_m} \left.\frac{\partial A^{(m)}_\nu \circ \Phi_B}{\partial x_k}\right|_x e^{i\nu\cdot\varphi} = \sum_{\nu\neq 0} \left.\frac{\partial A^{(k)}_\nu \circ \Phi_B}{\partial x_k}\right|_x e^{i\nu\cdot\varphi} .$$

Therefore, the claim then follows by applying the properties of the $\|\cdot\|^{\infty,w}_{r,s}$-norm under composition, following the same idea as in the proof of Lemma 2.5.1. ■

We now discuss the invertibility of the map $(x,\varphi) \mapsto (x,\theta)$, where the coordinates $\theta = (\theta_k)_{k\in\mathbb{N}}$, according to (2.34) and using the notation (2.36), are defined by

$$\theta_k(x,\varphi) = \varphi_k - Q(x,\varphi) \ , \qquad \text{for all } \ k \in \mathbb{N} \ .$$

Then, parametrizing φ as $\varphi = \theta + R(x,\theta)$, we want to solve

$$R(x,\theta) = Q(x,\theta + R(x,\theta)) \ . \tag{2.37}$$

Let us introduce a notation:

Notation 2.5.6. *Given* $G \in \mathcal{A}^\infty_{r,s}$ *, we denote by* Ψ_G *the map*

$$\Psi_G(x,\varphi) = (x, \varphi + G(x,\varphi)) \ , \qquad \forall \, x \in \mathscr{U}_r(\ell^\infty_w) \ , \ \varphi \in \ell^\infty \ .$$

Therefore, Equation (2.37) can be written as $R = \Psi^*_R Q$, where

$$\Psi^*_R Q(x,\theta) := \left(\Psi^*_R Q^{(m)}(x,\theta) \right)_{m\in\mathbb{N}} = (Q^{(m)}(x,\theta + R(x,\theta))_{m\in\mathbb{N}} \ .$$

As in the previous chapter, the strategy consists in formalizing our question as a fixed-point problem for the nonlinear operator $G \mapsto \Psi^*_G Q$ acting on a closed ball of the Banach space $\mathcal{A}^\infty_{r,s}$. In order to do this, we show that it is Fréchet differentiable and provide a suitable estimate for the differential. These technical results are contained in Proposition 2.5.7 and Lemma 2.5.8. Once the existence and uniqueness of the fixed point in the regularity class $\mathcal{A}^\infty$ have been established, we shall prove, by applying Lemma 2.5.9 (which is analogous to Lemma 2.5.1), that the fixed point inherits the stronger regularity properties of Q. Namely, we show that if Q is bounded with respect to the stronger norm $\|\cdot\|^\infty_{r,s}$, the same property holds for the fixed point.

Our first step is to establish the differentiability of this operator. We shall omit the details of the proof because it follows by the same argument as in Proposition 1.3.9.

Proposition 2.5.7. *Given* $r,s > 0$ *, let* $F \in \mathcal{A}^\infty_{r,s}$*. Then, for any* $\sigma \in (0,s)$*, the map*

$$\mathscr{F} : \mathscr{U}_{\sigma/2}(\mathcal{A}^\infty_{r,s-\sigma}) \ni G \mapsto \Psi^*_G F \in \mathcal{A}^\infty_{r,s-\sigma}$$

is differentiable in the sense of Fréchet. Moreover, the following estimate holds

$$|d\mathscr{F}(G)|_{op} \leq \frac{2}{e(\sigma - 2|G|^\infty_{r,s-\sigma})} |F|^\infty_{r,s} \ . \tag{2.38}$$

The estimate (2.38) suggests that the operator $\mathscr{F}$ is a contraction provided that F is sufficiently small. This ensure the existence and uniqueness of the fixed point, as proved in the following lemma.

Lemma 2.5.8 (Second Fixed Point Lemma)**.** *Given* $r,s > 0$*, let* $F \in \mathscr{U}_{s/8}(\mathcal{A}^\infty_{r,s})$ *and define* $\sigma := 8|F|^{\infty,w}_{r,s}$*. Consider the map*

$$\mathscr{F} : \overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^\infty_{r,s-\sigma}) \ni G \mapsto \Psi^*_G F \in \mathcal{A}^\infty_{r,s-\sigma}$$

with Ψ_G *as in 2.5.6. Then, there exists a unique* $\hat{G} \in \overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^\infty_{r,s-\sigma})$ *such that*

$$\hat{G} = \Psi^*_{\hat{G}} F.$$

Proof. We follow the same strategy pursued for proving Lemma 1.3.10. Namely, we first we show that

$$\sup_{G\in\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})} |\Psi_G^* F|^{\infty}_{r,s-\sigma} \le \sigma/4 \; ;$$

then, we prove that $\mathscr{F}$ acts as a contraction on $\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})$ and the thesis will follow by the fact that a closed ball in a Banach space is indeed a complete metric space.

The first point follows as an immediate consequence of Proposition 1.2.19 by setting, with the same notations of Proposition 1.2.19, $\Psi = \Psi_G$, i.e., $N = 0$, $\Omega \equiv G$. Indeed,

$$|G|^{\infty}_{r,s-\sigma} \le \sigma/4 \le \sigma/2 \; .$$

Then, for all $G \in \overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})$, the sequence $(G_n)_{n\in\mathbb{N}}$ defined by

$$G_{n+1} := \mathscr{F}(G_n), \qquad G_1 := G \tag{2.39}$$

is contained in $\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})$.

Since $\mathscr{F}$ is Fréchet differentiable, if we show that its Fréchet differential $d\mathscr{F}$ satisfies

$$\sup_{G\in\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})} |d\mathscr{F}(G)|_{op} \le K < 1,$$

by Lagrange's inequality, one has

$$|\mathscr{F}(G) - \mathscr{F}(G')|^{\infty}_{r,s-\sigma} \le \sup_{G\in\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})} |d\mathscr{F}(G)|_{op} |G - G'|^{\infty}_{r,s-\sigma} \le K|G - G'|^{\infty}_{r,s-\sigma}.$$

Namely, $\mathscr{F}$ would turn out to be a contraction with a Lipschitz constant equal to $K < 1$. Note that, by (2.38) we have that

$$\sup_{G\in\overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})} |d\mathscr{F}(G)|_{\rm op} \le \frac{4}{e\sigma}|F|^{\infty}_{r,s} = \frac{1}{2e} < 1 \; .$$

Therefore, there exists $\hat{G} \in \overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})$ such that, for all $G \in \overline{\mathscr{U}}_{\sigma/4}(\mathcal{A}^{\infty}_{r,s-\sigma})$, the sequence $(G_n)_{n\in\mathbb{N}}$ defined in (2.39) satisfies

$$\lim_{n\to\infty} |G_n - \hat{G}|^{\infty}_{r,s-\sigma} = 0.$$

Of course, $\hat{G}$ satisfies $\hat{G} = \Psi^*_{\hat{G}} F$. ■

We now focus on the regularity of the fixed point $\hat{G}$. The next lemma shows that if F is bounded and sufficiently small with respect to the $\|\cdot\|^{\infty,w}_{r,s}$-norm, then $\hat{G}$ is also bounded with respect to the same norm. Since this result is analogous to Lemma 2.5.1 and follows the same logic, we omit the proof.

Lemma 2.5.9. *Given $r, s > 0$, let $0 \le \sigma < s$. Consider $F \in \mathscr{U}_{1/2}(\mathcal{R}^{\infty}_{r,s})$ and let $G \in \overline{\mathscr{U}}_{\sigma/2}(\mathcal{A}^{\infty}_{r,s-\sigma})$. Assume that $G = \Psi_G^* F$. Then, $G \in \mathcal{R}^{\infty}_{r,s-\sigma}$ and satisfies the estimate*

$$\|G\|^{\infty}_{r,s-\sigma} \le \frac{2}{1 - 2\|F\|^{\infty}_{r,s}} \|F\|^{\infty}_{r,s} \; .$$

With these technical results at hand, we now have all the necessary ingredients to prove the theorem concerning the invertibility of the map $(x, \varphi) \mapsto (x, \theta)$.

Theorem 2.5.10. *Given $r_0, s_0 > 0$, let $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ satisfy*

$$\|P\|^{\infty,w}_{r_0,s_0} < \min\left\{\frac{r_0}{28}\,,\,\frac{s_0}{16c}\,,\,\frac{1}{4c}\right\}$$

and define $r_1 := r_0 - 27\|P\|^{\infty,w}_{r_0,s_0}$. Assume that the components of Γ_P Poisson-commute. Then the map

$$\Psi_{-Q} : \mathscr{U}_{r_1}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (x,\varphi) \mapsto (x, \varphi - Q(x,\varphi)),$$

is a regular change of coordinates with $\rho = 0$ and $8c\|P\|^{\infty,w}_{r_0,s_0} \le \sigma < s_0/2$. Its inverse is given by

$$\Psi_R : \mathscr{U}_{r_1}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty, \quad (x,\theta) \mapsto (x, \theta + R(x,\theta)),$$

where $R \in \mathcal{R}^\infty_{r_1,s_0-\sigma}$ is defined by $R = \Psi_R^* Q$. Furthermore, the following bounds hold:

$$\|Q\|^\infty_{r_1,s_0} \le c\|P\|^{\infty,w}_{r_0,s_0}, \qquad \|R\|^\infty_{r_1,s_0-\sigma} \le 4c\|P\|^{\infty,w}_{r_0,s_0}. \tag{2.40}$$

Proof. Note that all the assumptions of Theorem 2.5.2 and Proposition 2.5.5 are satisfied. Thereferore, using the notation established previously, the maps A and B are well-defined together with $Q \in \mathcal{R}^\infty_{r_1,s_0}$, ensuring that Ψ_{-Q} is also well-defined.
Moreover, the condition $\|P\|^{\infty,w}_{r_0,s_0} < s_0/(16c)$ together with the bound $\|Q\|_{r_1,s_0} \le c\|P\|^{\infty,w}_{r_0,s_0}$ implies that $\|Q\|_{r_1,s_0} < s_0/16$. Therefore, we can apply the Second Fixed Point Lemma (see 2.5.8, setting $F = Q$ and $\hat{G} = R$) to conclude that Ψ_{-Q} is a change of coordinates with inverse Ψ_R, where $R \in \mathcal{A}^\infty_{r_1,s_0-\sigma}$ satisfies the bound $|R|^\infty_{r_1,s_0-\sigma} \le 2|Q|^\infty_{r_1,s_0}$. Note that while $|Q|_{r_1,s_0} < s_0/8$ would suffice to apply Lemma 2.5.8, the stronger requirement $\|P\|^{\infty,w}_{r_0,s_0} < s_0/(16c)$ ensures that $2\sigma < s_0$, so that the domain of the inverse is well-defined.

The first bound in (2.40) was established in Proposition 2.5.5. The second bound follows from Lemma 2.5.9, which is applicable since

$$|R|^\infty_{r_1,s_0-\sigma} \le 2|Q|^\infty_{r_1,s_0} \le 2\|Q\|^\infty_{r_1,s_0} \le 2c\|P\|^{\infty,w}_{r_0,s_0} = \frac{2c\sigma}{8c} = \frac{\sigma}{4} \le \frac{\sigma}{2}\,.$$

Indeed, the assumption $\|P\|^{\infty,w}_{r_0,s_0} \le 1/(4c)$ combined with the first bound of (2.40) implies $\|Q\|^\infty_{r_1,s_0} \le 1/4$. Thus, Lemma 2.5.9 yields

$$\|R\|^\infty_{r_1,s_0-\sigma} \le \frac{2}{1 - 2\|Q\|^\infty_{r_1,s_0}}\|Q\|^\infty_{r_1,s_0} \le \frac{2}{1-1/2}\|Q\|^\infty_{r_1,s_0} = 4\|Q\|^\infty_{r_1,s_0} \le 4c\|P\|^{\infty,w}_{r_0,s_0}\,.$$

Since the right-hand side of these inequalities is equal to $\sigma/2$, both $\|R\|^\infty_{r_1,s_0-\sigma}$ and $\|Q\|^\infty_{r_1,s_0}$ are bounded by $\sigma/2$. This implies that $\Psi_{-Q} \in \mathcal{C}^{0,\sigma/2}_{r_1,s_0}(\mathcal{R})$ and $\Psi_R \in \mathcal{C}^{0,\sigma/2}_{r_1,s_0-\sigma}(\mathcal{R})$, and the claim follows from Remark 2.3.12. ■

To make a connection with the discussion at the beginning of this chapter, we emphasize that the composition $\Psi_{-Q} \circ \Phi_{-A_0}$ provides the explicit construction of the map Δ (see (2.1) and the related discussion). This transformation is precisely the regular change of coordinates required to straighten the foliation, ensuring that the resulting map $\Phi = \Delta \circ \Phi_P$ defines a symplectomorphism. By gathering the arguments developed so far, we are now in a position to prove the following theorem; from this result, the *Symplectic Straightening Theorem*, which was informally stated in the introduction, will follow as a direct corollary.

Theorem 2.5.11. *Given $r_0, s_0 > 0$, let $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ and assume that the components of Γ_P pairwise Poisson-commute. Then, there exists a constant $C > 1$ such that, provided*

$$\|P\|^{\infty,w}_{r_0,s_0} < \frac{1}{4C}\min\{r_0, s_0, 1\},$$

the map

$$\Psi_{-Q} \circ \Phi_{-A_0} \circ \Phi_P : \mathscr{U}_{r_0-4\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty \tag{2.41}$$

is a symplectomorphism with inverse given by

$$\Phi_{-A} \circ \Phi_B \circ \Psi_R : \mathscr{U}_{r_0-3\rho}(\ell^\infty_w) \times \ell^\infty \to \ell^\infty_w \times \ell^\infty.$$

for every ρ and σ satisfying $C\|P\|^{\infty,w}_{r_0,s_0} \le \rho < r_0/4$ and $2C\|P\|^{\infty,w}_{r_0,s_0} \le \sigma < s_0/2$.

Proof. Note that, by assuming $C \ge \max\{9, 4c\}$, all the hypotheses of Theorems 2.5.2, 2.5.3, and 2.5.10 are simultaneously satisfied. Consequently, both maps introduced in the statement are well-defined. To show they are the inverse of each other, we note that

$$\begin{aligned}
\Psi_{-Q} \circ \Phi_{-A_0} \circ \Phi_P \circ \Phi_{-A} \circ \Phi_B \circ \Psi_R &\overset{Thm1.3.12}{=} \Psi_{-Q} \circ \Phi_{-A_0} \circ \mathrm{Id}_{\mathscr{U}_{r_0-2\rho}(\ell^\infty_w)\times\ell^\infty} \circ \Phi_B \circ \Psi_R \\
&= \Psi_{-Q} \circ \Phi_{-A_0} \circ \Phi_B \circ \Psi_R \\
&\overset{Thm2.5.2}{=} \Psi_{-Q} \circ \mathrm{Id}_{\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty} \circ \Psi_R \\
&= \Psi_{-Q} \circ \Psi_R \overset{Thm2.5.10}{=} \mathrm{Id}_{\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty} ,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{-A} \circ \Phi_B \circ \Psi_R \circ \Psi_{-Q} \circ \Phi_{-A_0} \circ \Phi_P &\overset{Thm2.5.10}{=} \Phi_{-A} \circ \Phi_B \circ \mathrm{Id}_{\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty} \circ \Phi_{-A_0} \circ \Phi_P \\
&= \Phi_{-A} \circ \Phi_B \circ \Phi_{-A_0} \circ \Phi_P \\
&\overset{Thm2.5.2}{=} \Phi_{-A} \circ \mathrm{Id}_{\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\ell^\infty} \circ \Phi_P \\
&= \Phi_{-A} \circ \Phi_P \overset{Thm1.3.12}{=} \mathrm{Id}_{\mathscr{U}_{r_0-4\rho}(\ell^\infty_w)\times\ell^\infty} .
\end{aligned}$$

Next, recall the maps N and M introduced in the proof of Theorem 2.5.3:

$$N(y,\varphi) = P(y,\varphi) - A_0(y + P(y,\varphi)) , \qquad M(x,\varphi) = B(x) - A(x + B(x), \varphi) .$$

In terms of these maps, we can write

$$\Phi_{-Q} \circ \Phi_{-A_0} \circ \Phi_P = \mathrm{Id} + (N, Q \circ \Phi_{-A_0} \circ \Phi_P), \qquad \text{and} \qquad \Phi_{-A} \circ \Phi_B \circ \Psi_R = \mathrm{Id} + (M \circ \Psi_R, R) .$$

To establish that $\Psi_{-Q} \circ \Phi_{-A_0} \circ \Phi_P$ is a regular change of coordinates, it remains to verify the following smallness conditions:

$$\max\{\|N\|^{\infty,w}_{r_0-4\rho}, \|M \circ \Psi_R\|^{\infty,w}_{r_0-4\rho}\} \le \rho$$

$$\max\{\|Q \circ \Phi_{-A_0} \circ \Phi_P\|^{\infty}_{r_0-3\rho}, \|R\|^{\infty}_{r_0-3\rho}\} \le \frac{\sigma}{2} .$$

If these hold, the thesis follows from Remark 2.3.12. Clearly, by using bounds established previously, and performing estimates for the norms of the compositions, one can prove that there exist constants $C_1, C_2 > 1$ such that

$$\max\{\|N\|^{\infty,w}_{r_0-4\rho}, \|M \circ \Psi_R\|^{\infty,w}_{r_0-4\rho}\} \le C_1\|P\|^{\infty,w}_{r_0,s_0}$$

$$\max\{\|Q\circ\Phi_{-A_0}\circ\Phi_P\|^{\infty}_{r_0-3\rho}, \|R\|^{\infty}_{r_0-3\rho}\} \le C_2\|P\|^{\infty,w}_{r_0,s_0}\,.$$

Thus, by defining

$$C := \max\{9, 4c, C_1, C_2\}, \tag{2.42}$$

we deduce that $\Psi_{-Q}\circ\Phi_{-A_0}\circ\Phi_P$ is indeed a regular change of coordinates.

To conclude, we observe that the map preserves the Poisson brackets by construction. Indeed, this change of variables is induced by the generating function S in (2.32), which, as shown at the beginning of this chapter, formally defines a symplectomorphism. The estimates provided above ensure that these algebraic manipulations are well-defined on the respective domains, thereby completing the proof. ■

Consistent with the explanation in Remark 2.4.1, we utilize a simplified notation exploiting the correspondence between $\mathbb{T}^\infty$ and its universal cover ℓ^∞ established in the section dedicated in the first chapter.

Theorem 2.5.12 (Symplectic Straightening). *Let $r_0, s_0 > 0$, $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$ and let*

$$\Gamma_P : \mathscr{U}_{r_0}(\ell^\infty_w)\times\mathbb{T}^\infty \to \ell^\infty_w \qquad (y,\varphi)\mapsto \Gamma_P(y,\varphi) = y + P(y,\varphi)$$

be such that $\{\Gamma_P^{(m)}, \Gamma_P^{(n)}\} = 0$ for all $m,n \in \mathbb{N}$. Then, there is a constant $C > 1$ such that, the smallness condition

$$\|P\|^{\infty,w}_{r_0,s_0} < \frac{1}{4C}\min\{r_0, s_0, 1\}$$

implies that, for all ρ,σ such that

$$C\|P\|^{\infty,w}_{r_0,s_0} \le \rho < r_0/4 \qquad \text{and} \qquad 2C\|P\|^{\infty,w}_{r_0,s_0} \le \sigma < s_0/2,$$

there exists a symplectomorphism

$$\Phi : \mathscr{U}_{r_0-4\rho}(\ell^\infty_w)\times\mathbb{T}^\infty \to \ell^\infty_w\times\mathbb{T}^\infty, \quad (y,\varphi)\mapsto\Phi(y,\varphi) = (x,\theta),$$

that straightens the foliation induced by the map Γ_P. Namely for all $\gamma\in\mathscr{U}_{r_0-\rho}(\ell^\infty_w)$

$$\Phi(\Gamma_P^{-1}(\gamma)) = \{x\}\times\mathbb{T}^\infty, \quad x = \gamma - A_0(\gamma),$$

where A_0 is defined in Theorem 2.5.2.
In particular Φ is a regular change of coordinates of class $\mathcal{C}^{\rho,\sigma/2}_{r_0-4\rho,s_0}(\mathcal{R})$, with $\Phi^{-1}\in\mathcal{C}^{\rho,\sigma/2}_{r_0-3\rho,s_0-\sigma}(\mathcal{R})$.

Proof. The result follows directly from the construction developed in the previous paragraphs. By Theorem 2.5.11, the composition $\Delta := \Psi_{-Q}\circ\Phi_{-A_0}$ provides the regular change of coordinates required to define the symplectomorphism $\Phi = \Delta\circ\Phi_P$. The straightening property is a consequence of the fact that Φ_P maps the level sets of Γ_P into the standard horizontal leaves $\{\gamma = \text{const}\}\times\mathbb{T}^\infty$, while the transformation Δ preserves this foliation while restoring the symplectic structure. ■

In this sense, the transformation $(x,\theta) = \Phi(y,\varphi)$ defines a set of action-angle coordinates adapted to the foliation, where the geometry of the perturbed level sets is reduced to the standard product structure of the action-angle space. This geometric rectification has profound implications in Hamiltonian dynamics. In the following section, we shall see how the existence of such a straightening transformation allows us to integrate the equations of motion for a class of infinite-dimensional systems, leading to a natural generalization of the Liouville-Arnold Theorem.

2.6 An Infinite-Dimensional Liouville-Arnold Theorem

In this section, we provide the dynamical counterpart to the geometric results established in Section 2.5. In the classical finite-dimensional setting, the Liouville-Arnold Theorem ensures that a Hamiltonian system admitting a complete set of Poisson-commuting integrals is integrable by quadratures and can be described in terms of action-angle variables. Here, we prove that under suitable smallness conditions on the perturbation, an analogous conclusion holds for systems defined on $\ell^\infty_w \times \mathbb{T}^\infty$. Specifically, we show that if a Hamiltonian H commutes with a set of integrals that define a Poisson-commuting foliation, the straightening symplectomorphism Φ constructed in Theorem 2.5.12 acts as the linearizing transformation for the Hamilton's equations.

We shall slightly abuse notation in the sense explained in Remark 2.4.1.

Theorem 2.6.1 (Complete Integrability). *Let $r, s > 0$ and consider a majorant real analytic Hamiltonian*

$$H : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \to \mathbb{R}$$

with $H \in \mathcal{A}_{r,s}$. Assume that there exists $P \in \mathcal{R}^{\infty,w}_{r_0,s_0}$, with $r_0 \in (0, r]$ and $s_0 \in (0, s]$, satisfying the following properties:

(a.1) the following bound holds

$$\|P\|^{\infty,w}_{r_0,s_0} < \frac{1}{4C} \min\{r_0, s_0, 1\}, \tag{2.43}$$

where the constant C is defined in (2.42)*;*

(a.2) the map Γ_P defined by

$$\Gamma_P : \mathscr{U}_{r_0}(\ell^\infty_w) \times \mathbb{T}^\infty \to \ell^\infty_w \qquad \Gamma_P(y, \varphi) = y + P(y, \varphi)$$

is such that, for all $m, n \in \mathbb{N}$, $\{\Gamma_P^{(m)}, \Gamma_P^{(n)}\} = 0$ and $\{\Gamma_P^{(m)}, H\} = 0$ for all $m \in \mathbb{N}$.

Then,

(i) H induces a Hamiltonian flow ϕ_H on a neighbourhood $\mathscr{N}_0$ of $\{0\} \times \mathbb{T}^\infty$;

(ii) there is a neighbourhood $\mathscr{U}_0$ of $\gamma = 0$ in ℓ^∞_w such that, for all $\gamma \in \mathscr{U}_0$, the level set $\Gamma_P^{-1}(\gamma)$ is an analytic Banach manifold, diffeomorphic to the infinite torus $\mathbb{T}^\infty$, which is invariant for the Hamiltonian flow ϕ_H;

(iii) there exists a set of action-angle *coordinates (x, θ) in which Hamilton's equations read*

$$\begin{cases} \dot{x}_k = 0 \\ \dot{\theta}_k = \dfrac{H \circ \Phi^{-1}}{\partial x_k} \end{cases} , \quad k \in \mathbb{N}, \tag{2.44}$$

where Φ is the symplectomorphism defined in (2.41) *and $(x, \theta) = \Phi(y, \varphi)$.*

In particular the Hamilton's equations can be integrated by quadratures and the Hamiltonian flow reduces to a linear flow on the infinite torus $\mathbb{T}^\infty$.

Proof. To prove (i), we first address point (iii). In the new coordinates $(x, \theta) = \Phi(y, \varphi)$ provided by Theorem 2.5.12, the Hamiltonian takes the form $K := H \circ \Phi^{-1}$. The fact that $\{H, \Gamma_P^{(m)}\} = 0$ implies, via the symplecticity of Φ, that $\{K, x_m\} = 0$ for all $m \in \mathbb{N}$. Consequently, K is independent of the angular variables θ. As discussed in the dedicated section, the Hamilton's equations for K take the form (2.44) and such a Hamiltonian K induces a flow ϕ_K on $\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\mathbb{T}^\infty$, with $\rho = C\|P\|^{\infty,w}_{r_0,s_0}$, which is easily solvable and linear in the angles. Therefore, by applying Proposition 2.4.3, the symplectomorphism Φ conjugates the flow of K to the flow of H on the neighbourhood of $\{0\}\times\mathbb{T}^\infty$ given by $\mathscr{N}_0 := \Phi^{-1}(\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\mathbb{T}^\infty)$.

Point (ii) follows from the regularity of Φ and the straightening property: since Φ is a diffeomorphism and, for all $\gamma \in \mathscr{U}_0 := \Delta^{-1}(\mathscr{U}_{r_0-3\rho}(\ell^\infty_w)\times\mathbb{T}^\infty)$, maps $\Gamma_P^{-1}(\gamma)$ to the torus $\{x = \text{const}\}\times\mathbb{T}^\infty$, the level sets are indeed analytic Banach manifolds diffeomorphic to $\mathbb{T}^\infty$. The invariance of these manifolds under ϕ_H is a direct consequence of the fact that $\dot{x}_k = 0$ for all $k \in \mathbb{N}$. ■

Despite the formal analogy with the classical Liouville-Arnold Theorem, some crucial distinctions must be emphasized. In the finite-dimensional case, integrability is often a global or semi-global geometric property derived from the algebraic structure of the symmetry groups involved. In this infinite-dimensional setting, however, our result relies heavily on the specific functional form of the momentum map Γ_P. The requirement that P belongs to the class of majorant analytic functions $\mathcal{R}^{\infty,w}_{r_0,s_0}$ and satisfies the smallness condition (a.1) indicates that our approach is inherently perturbative, reflecting the technical challenges posed by the infinite-dimensional setting. Nonetheless, this version of the Liouville-Arnold Theorem provides a rigorous framework for studying several physically relevant models. As we shall see in the next chapter, while the ℓ^2 topology presents obstructions to the existence of foliations by infinite tori, our setting allows us to recover a form of complete integrability for systems defined on infinite lattices and for certain classes of Hamiltonian PDEs on the circle.

Chapter 3

Hamiltonian Systems on Infinite Lattices and PDEs on the Circle

A broad class of physical models characterized by an infinite number of degrees of freedom admits a Hamiltonian formulation. During the 1960s, it was discovered that certain nonlinear evolution equations possess an infinite number of conserved quantities ([Gar71]); consequently, their integrability has been rigorously investigated through diverse mathematical frameworks. Depending on the topological properties of the underlying manifolds, these approaches include algebraic geometry, inverse scattering transforms, and Riemann-Hilbert problems ([AS81]). Such integrable structures arise both in continuous models—notably in the context of shallow water waves described by the *Korteweg–de Vries* (KdV) equation—and in discrete lattices, such as the *Toda lattice*, which models the dynamics of an infinite chain of interacting particles [Tod89]. The case of Hamiltonian PDEs on the circle is particularly interesting because, in an appropriate set of coordinates, these infinite dimensional Hamiltonian systems look like infinitely many coupled harmonic oscillators, namely systems on an infinite lattice. In this chapter, we exploit this correspondence by embedding these systems into a weighted complex Banach space of sequences framework and provide conditions that allow to construct a set of action-angle coordinates.

3.1 Systems on Infinite Lattices

Building upon the results of the previous chapter, we provide here a class of Hamiltonian systems on infinite lattices that admit action-angle coordinates on an open subset of the phase space. Let us introduce the setting.

3.1.1 The phase space

Given a sequence of positive real numbers $(\eta_j)_{j\in\mathbb{N}}$, we consider as a phase space the complex weighted Banach space $\ell^\infty_\eta(\mathbb{C})$ defined by

$$\ell^\infty_\eta(\mathbb{C}) := \left\{ u \in \mathbb{C}^{\mathbb{N}} : \|u\|_{\ell^\infty_\eta(\mathbb{C})} := \sup_{j\in\mathbb{N}} \eta_j |u_j| < \infty \right\}.$$

Note that $\ell^\infty_\eta(\mathbb{C})$ can be embedded into $\ell^2(\mathbb{C})$ if the weight η satisfies the condition

$$\mathsf{c}_\eta := \sum_{j\in\mathbb{N}} \eta_j^{-2} < \infty.$$

From this embedding, one can inherit the standard symplectic form of $\ell^2(\mathbb{C})$, namely

$$\omega := \mathrm{i} \sum_{j \in \mathbb{N}} du_j \wedge d\bar{u}_j,$$

by pulling it back onto $\ell^\infty_\eta(\mathbb{C})$.

Remark 3.1.1. *The embedding $\iota : \ell^\infty_\eta(\mathbb{C}) \hookrightarrow \ell^2(\mathbb{C})$ is compact, and hence the pull back of the standard symplectic form of $\ell^2(\mathbb{C})$ to $\ell^\infty_\eta(\mathbb{C})$ is represented by a bounded operator $(\iota^*\omega)^\# : \ell^\infty_\eta(\mathbb{C}) \to (\ell^\infty_\eta(\mathbb{C}))^*$, $\omega^\#(V) = \omega(\cdot, V)$, which is compact. Therefore $(\iota^*\omega)^\#$ is injective but not surjective, making $\iota^*\omega$ a weak symplectic form on $\ell^\infty_\eta(\mathbb{C})$.*

Let us begin to draw a connection with the phase space $\ell^\infty_w \times \mathbb{T}^\infty$ introduced in the previous chapter. Let $\xi = (\xi_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that

$$0 < \inf_{k\in\mathbb{N}} \eta_k^2 \xi_k \le \sup_{k\in\mathbb{N}} \eta_k^2 \xi_k < \infty. \tag{3.1}$$

It then follows that the set

$$\mathcal{T}_\xi := \left\{ u \in \ell^\infty_\eta(\mathbb{C}) : |u_k|^2 = \xi_k \text{ for all } k \in \mathbb{N} \right\} \tag{3.2}$$

is analytically diffeomorphic to $\mathbb{T}^\infty$ via the map $\mathbb{T}^\infty \ni \varphi \mapsto (\sqrt{\xi_k} e^{\mathrm{i}\varphi_k})_{k\in\mathbb{N}}$. Indeed, by Theorem 1.2.24, this map is analytic as it is defined by a uniformly bounded family of analytic functions. The condition on the infimum in (3.1) ensures both the invertibility of the map and the analyticity of its inverse, $(u_k \mapsto -\mathrm{i}\ln(u_k/\sqrt{\xi_k}))_{k\in\mathbb{N}}$, which again follows from Theorem 1.2.24. Consequently, $\mathcal{T}_\xi$ is an analytic Banach submanifold.[1]

Furthermore, the condition $0 < \inf_{k\in\mathbb{N}} \eta_k |u_k|$ defines an open subset of $\ell^\infty_\eta(\mathbb{C})$, which allows for the possibility of a foliation whose leaves are analytically diffeomorphic to $\mathbb{T}^\infty$. Specifically, any ξ satisfying (3.1) possesses an open neighborhood foliated by infinite tori. To see this, let w be the weight defined by $w_k = \eta_k^2$ for all $k \in \mathbb{N}$ and note that, for any $r < \inf_{k\in\mathbb{N}} \eta_k^2 \xi_k$, the map

$$\Phi_\xi : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \ni (y, \varphi) \mapsto (\sqrt{\xi_k + y_k} e^{\mathrm{i}\varphi_k})_{k\in\mathbb{N}} \in \ell^\infty_\eta(\mathbb{C}) \tag{3.3}$$

is bi-analytic onto its image, providing the desired local coordinates for the foliation. This property is remarkable for the following reason: while $\ell^2(\mathbb{C})$ appears to be the most natural candidate for an infinite-dimensional Hamiltonian phase space, as it inherently provides a symplectic form via its Hermitian product, its topology does not allow for a foliation by infinite tori. Indeed, in $\ell^2(\mathbb{C})$, the set of finite-dimensional tori of the form (3.2) (with $\xi_k = 0$ for all but finitely many k) is dense. Equivalently, as pointed out in [HK14], the locus

$$\eth := \{u \in \ell^2(\mathbb{C}) : u_k = 0 \text{ for some } k \in \mathbb{N}\}$$

is dense in the strong topology. This density constitutes a topological obstruction that prevents Liouville-Arnold integrability, directing research in recent years towards the closely related concept of Birkhoff integrability for infinite-dimensional Hamiltonian systems.

Finally, provided that η is such that c_η is finite, the symplectic form defined in the previous chapter (see Definition 2.3.1) can be obtained by pulling back the symplectic form $\mathrm{i}\sum_k du_k \wedge d\overline{u}_k$ through Φ_ξ, where ξ satisfies (3.1), leading to the conclusion that Φ_ξ is a symplectomorphism.

[1] We refer the reader to [PT87] for a comprehensive treatment of analytic submanifolds in Banach spaces.

3.1.2 Space of Hamiltonians

For $R > 0$, let $\mathcal{H}_R$ be the space of Hamiltonians

$$H : \mathscr{U}_R(\ell^\infty_\eta(\mathbb{C})) \to \mathbb{R}$$

such that there exists a pointwise absolutely convergent power series expansion[2]

$$H(u) = \sum_{a,b\in\mathbb{N}_0^{(\mathbb{N})}} H_{ab} u^a \overline{u}^b$$

satisfying the following properties:

(i) Reality condition:

$$H_{a,b} = \overline{H_{b,a}}$$

(ii) Majorant analyticity: Given H as above, we define its majorant as

$$H_M(u) = \sum_{a,b} |H_{ab}| u^a \overline{u}^b \ .$$

Then we require that

$$\lfloor H \rceil_R := \sup_{u\in\mathscr{U}_R(\ell^\infty_\eta(\mathbb{C}))} |H_M(u)| < \infty \ .$$

Note that, given $R > 0$, if we let $v_{\eta,R} \in \ell^\infty_\eta(\mathbb{C})$ be the vector defined by $(v_{\eta,R})_j = R\eta_j^{-1}$, then

$$\lfloor H \rceil_R = \sum_{a,b} |H_{ab}| v_{\eta,R}^{a+b}.$$

It is worth stressing that the finiteness of the $\lfloor\cdot\rceil_R$-norm guarantees that the function is majorant-analytic in the ball $\mathscr{U}_R(\ell^\infty_\eta(\mathbb{C}))$, meaning that its majorant is itself an analytic function, as it is the uniform limit of polynomials (see, e.g., [Muj86]).

We now show that, given $\xi = (\xi_k)_{k\in\mathbb{N}}$, $\xi_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, satisfying (3.1), the pullback of any majorant analytic Hamiltonian $H \in \mathcal{H}_R$ by Φ_ξ admits a Taylor-Fourier series which belongs to the class $\mathcal{A}$. More precisely, we fix a sequence of positive real numbers $\eta = (\eta_k)_{k\in\mathbb{N}}$ and we prove the following

Proposition 3.1.2. *Let $\xi_-, \xi_+ \in \mathbb{R}$ satisfy $0 < \xi_- \leq \xi_+$. Then, for every sequence of real numbers $\xi = (\xi_k)_{k\in\mathbb{N}}$ such that*

$$0 < \xi_- \leq \inf_{k\in\mathbb{N}} \eta_k^2 \xi_k \leq \sup_{k\in\mathbb{N}} \eta_k^2 \xi_k \leq \xi_+ \ ,$$

and for all $R, r, s > 0$ satisfying

$$\xi_+^{1/2} e^s \sum_{m=0}^{\infty} |a_m| \left(\frac{r}{\xi_-}\right)^m \leq R, \tag{3.4}$$

where $(a_m)_{m=0}^\infty$ are the Taylor coefficients of the function $x \mapsto \sqrt{1+x}$, the pullback of any majorant analytic Hamiltonian $H \in \mathcal{H}_R$ by the map

$$\Phi_\xi : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \ni (y,\varphi) \mapsto (\sqrt{\xi_k + y_k} e^{\mathrm{i}\varphi_k})_{k\in\mathbb{N}} \in \ell^\infty_\eta(\mathbb{C}), \qquad w_k = \eta_k^2 \text{ for all } k \in \mathbb{N},$$

belongs to $\mathcal{A}_{r,s}$ and satisfies the bound

$$|\Phi_\xi^* H|_{r,s} \leq \lfloor H \rceil_R \ .$$

[2] For the sake of brevity, the index set in summations will be omitted in the following.

Proof. At a purely formal level, one has

$$
\begin{aligned}
(H \circ \Phi_\xi)(y,\varphi) &= \sum_{a,b} H_{ab} \prod_{k\in\mathbb{N}} (\xi_k + y_k)^{(a_k+b_k)/2}\, e^{\mathrm{i}(a-b)\cdot\varphi} \\
&= \sum_{\nu\in\mathbb{Z}} \sum_{a-b=\nu} H_{ab} \prod_{k\in\mathbb{N}} (\xi_k + y_k)^{(a_k+b_k)/2}\, e^{\mathrm{i}\nu\cdot\varphi} \\
&= \sum_{\nu\in\mathbb{Z}} \sum_{a-b=\nu} H_{ab} \prod_{k\in\mathbb{N}} \left(\xi_k^{1/2} \sqrt{1+\frac{y_k}{\xi_k}}\right)^{(a_k+b_k)} e^{\mathrm{i}\nu\cdot\varphi} \\
&= \sum_{\nu\in\mathbb{Z}} \sum_{a-b=\nu} H_{ab} \prod_{k\in\mathbb{N}} \left(\xi_k^{1/2} \sum_{m=0}^{\infty} a_m \left(\frac{y_k}{\xi_k}\right)^m\right)^{(a_k+b_k)} e^{\mathrm{i}\nu\cdot\varphi}.
\end{aligned}
$$

The latter series is formally equal to

$$
\sum_{\nu,\alpha} K_{\nu\alpha} y^\alpha e^{\mathrm{i}\nu\cdot\varphi}
$$

for suitable coefficients $K_{\nu\alpha}$, with $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and $\nu \in \mathbb{Z}^{(\mathbb{N})}$. Clearly, by the triangular inequality, and by using the assumptions on R, r and s, it follows that

$$
\begin{aligned}
\sum_{\nu,\alpha} |K_{\nu\alpha}| v_{w,r}^{\alpha} e^{s|\nu|} &\le \sum_{\nu\in\mathbb{Z}} \sum_{a-b=\nu} |H_{ab}| \prod_{k\in\mathbb{N}} \left(\xi_k^{1/2} \sum_{m=0}^{\infty} |a_m| \left(\frac{r w_k^{-1}}{\xi_k}\right)^m\right)^{(a_k+b_k)} e^{s|\nu|} \\
&\le \sum_{a,b} |H_{ab}| \prod_{k\in\mathbb{N}} \left(\xi_+^{1/2} \eta_k^{-1} e^s \sum_{m=0}^{\infty} |a_m| \left(\frac{r}{\xi_-}\right)^m\right)^{(a_k+b_k)} \\
&\le \sum_{a,b} |H_{ab}| \prod_{k\in\mathbb{N}} v_{\eta,R}^{a+b} = \lfloor H \rceil_R.
\end{aligned}
$$

Therefore, $H \circ \Phi_\xi$ admits a power-trigonometric series representation, namely $(H \circ \Phi_\xi)_{\nu\alpha} = K_{\nu\alpha}$, and belongs to $\mathcal{A}_{r,s}$ with $|\Phi_\xi^* H|_{r,s} \le \lfloor H \rceil_R$. ■

3.1.3 Hamiltonian vector fields

In this paragraph we assume that the constant c_η is finite. A vector field $V : \mathscr{U}_R(\ell_\eta^\infty(\mathbb{C})) \to \ell_\eta^\infty(\mathbb{C})$ is Hamiltonian if there exists $H \in \mathcal{H}_R$ such that

$$
\omega(\cdot, V) = dH[\cdot]. \tag{3.5}
$$

In this case, we say that the vector field is associated with the Hamiltonian H and denote it by V_H. It should be noted that, in general, not all Hamiltonians in $\mathcal{H}_R$ induce a Hamiltonian vector field. Indeed, the majorant-analyticity property only ensures that the differential belongs to the topological dual of the phase space. In the following, in line with the previous chapter, we introduce a class of analytic Hamiltonians that do induce Hamiltonian vector fields.

3.1.4 Space of regular Hamiltonians

We say that a Hamiltonian $f \in \mathcal{H}_R$ is regular if the norm

$$
\llbracket f \rrbracket_R := \max\left\{ \lfloor f \rceil_R,\ R \sup_{k\in\mathbb{N}} \eta_k \left\lfloor \frac{\partial f}{\partial \overline{u}_k} \right\rceil_R \right\}
$$

is finite. Note that, if $\mathsf{c}_\eta < \infty$, the finiteness of the $\llbracket \cdot \rrbracket_R$-norm ensures that the function defines a Hamiltonian vector field in the sense of (3.5), as the gradient of any such regular Hamiltonian is necessarily an element of $\ell^2(\mathbb{C})$. Explicitly, the Hamiltonian vector field associated with a regular Hamiltonian f is given by

$$V_f^{(k)} = \mathrm{i}\frac{\partial f}{\partial \overline{u}_k}, \quad k \in \mathbb{N}. \tag{3.6}$$

It is worth noting that, even when $\mathsf{c}_\eta = \infty$, the regularity of f ensures that the associated Hamiltonian vector field is analytic as a map $V_f : \mathscr{U}_R(\ell^\infty_\eta(\mathbb{C})) \to \ell^\infty_\eta(\mathbb{C})$. This is in contrast to the case where f is merely analytic; in the latter situation, the Hamiltonian vector field, defined as the sequence of derivatives as in (3.6), is only guaranteed to be bounded when viewed as a map taking values into the dual space $(\ell^\infty_\eta(\mathbb{C}))^*$.

With the aim of exploiting the analytical background developed in the previous chapter, we now prove that given any regular Hamiltonian f, the composition $f \circ \Phi_\xi$ defines indeed a regular function in the sense of the previous chapter. Let us first recall that given $r, s > 0$ the $|\cdot|_{r,s}$-norm of a regular function $f \in \mathcal{R}_{r,s}$ is defined as

$$\|f\|_{r,s} := \max\left\{ |f|_{r,s},\ \sup_{k\in\mathbb{N}} w_k \left|\frac{\partial f}{\partial \varphi_k}\right|_{r,s},\ r\sup_{k\in\mathbb{N}}\left|\frac{\partial f}{\partial y_k}\right|_{r,s} \right\} < \infty.$$

Proposition 3.1.3. *With the same notations of Proposition 3.1.2, and under the same assumptions, we have that the pullback of any regular Hamiltonian* $f : \mathscr{U}_R(\ell^\infty_\eta) \to \mathbb{R}$ *by the map* Φ_ξ *belongs to* $\mathcal{R}_{r,s}$ *and satisfies the bound*

$$\|\Phi_\xi^* f\|_{r,s} \le 2\,\llbracket f \rrbracket_R\,.$$

Proof. It follows by Proposition 3.1.2 that $|\Phi_\xi^* f|_{r,s} \le \lfloor f \rfloor_R \le \llbracket f \rrbracket_R$. Furthermore, by using the relations

$$\frac{\partial}{\partial y_k} = \frac{1}{2}\left(\frac{1}{\overline{u}_k}\frac{\partial}{\partial u_k} + \frac{1}{u_k}\frac{\partial}{\partial \overline{u}_k}\right), \qquad \frac{\partial}{\partial \varphi_k} = \mathrm{i}\left(u_k\frac{\partial}{\partial u_k} + \overline{u}_k\frac{\partial}{\partial \overline{u}_k}\right), \qquad \text{for all } k \in \mathbb{N},$$

and the reality condition $f_{ab} = \overline{f_{ab}}$, one can follows the same argument as in the proof of Proposition 3.1.2 to prove the following bounds

$$\begin{aligned} r\sup_{k\in\mathbb{N}}\left|\frac{\partial \Phi_\xi^* f}{\partial y_k}\right|_{r,s} &\le \frac{r}{R^2} R \sup_{k\in\mathbb{N}} \eta_k \left\lfloor \frac{\partial f}{\partial \overline{u}_k}\right\rfloor_R \le \frac{r}{R^2}\,\llbracket f \rrbracket_R \\ \sup_{k\in\mathbb{N}} w_k \left|\frac{\partial \Phi_\xi^* f}{\partial \varphi_k}\right|_{r,s} &\le 2R\sup_{k\in\mathbb{N}} \eta_k \left\lfloor \frac{\partial f}{\partial \overline{u}_k}\right\rfloor_R \le 2\,\llbracket f \rrbracket_R\,. \end{aligned}$$

Since the assumption (3.4) implies that $r/R^2 < 1$, the claim follows. ∎

In particular, the Hamiltonian vector field associated with a regular Hamiltonian $H \in \mathcal{H}_R$ is pushed forward to $\ell^\infty_w \times \mathbb{T}^\infty$ by Φ_ξ^{-1}, and the resulting vector field is analytic and bounded. More precisely, it belongs to the space of vector field $\mathfrak{X}_{r,s}$ (see Definition 2.2.2).

3.1.5 Poisson bracket

As usual, the symplectic form induces a Poisson bracket on the space of *regular* functions by setting

$$\{f, g\} := \omega(V_f, V_g)\,.$$

By a direct calculation, one discovers that

$$\{f,g\} = \mathrm{i}\sum_{k\in\mathbb{N}} \frac{\partial f}{\partial \overline{u}_k}\frac{\partial g}{\partial u_k} - \frac{\partial f}{\partial u_k}\frac{\partial g}{\partial \overline{u}_k}\,. \tag{3.7}$$

According to the identification between the symplectic structures $\mathrm{i}\sum_k du_k \wedge d\overline{u}_k$ and $\sum_k dy_k \wedge d\varphi_k$ just established, we have that

$$\Phi_\xi^*\left(\mathrm{i}\sum_{k\in\mathbb{N}} \frac{\partial f}{\partial \overline{u}_k}\frac{\partial g}{\partial u_k} - \frac{\partial f}{\partial u_k}\frac{\partial g}{\partial \overline{u}_k}\right) = \sum_{k\in\mathbb{N}} \frac{\partial \Phi_\xi^* f}{\partial y_k}\frac{\partial \Phi_\xi^* g}{\partial \varphi_k} - \frac{\partial \Phi_\xi^* f}{\partial \varphi_k}\frac{\partial \Phi_\xi^* g}{\partial y_k}\,.$$

Therefore, in line with the observations in Subsection 2.3.4, one can take (3.7) as a definition on the Poisson bracket, overcoming then the necessity of the requirement $c_\eta < \infty$. From now on, we shall dispense with the embedding into $\ell^2(\mathbb{C})$, as our results do not depend on such an assumption. Another important property that we inherit from the analysis developed in Section 2.5 is the possibility of extending the operation induced by the Poisson bracket between pairs of functions in which one of the two is regular and the other one belongs to $\mathcal{H}_R$ for some $R > 0$. This can be easily seen by interpreting this operation as the Lie derivative of an analytic function in the direction of the Hamiltonian vector field associated with a regular function.

3.1.6 Hamilton's equations

Given $R > 0$ and a function $H \in \mathcal{H}_R$, we say that H induces a *Hamiltonian flow* ϕ_H on an open set $\mathscr{U} \subseteq \ell^\infty_\eta(\mathbb{C})$ if there exists a group action ϕ_H of $\mathbb{R}$ on $\mathscr{U}$

$$\mathbb{R} \times \mathscr{U} \ni (t,u) \mapsto \phi_H^t(u) \in \mathscr{U}$$

such that, denoting $\phi_H^t(u) = (u_1(t), \ldots, u_k(t), \ldots)$, the maps $t \mapsto u_k(t)$ are differentiable functions for all $k \in \mathbb{N}$ and satisfy the *Hamilton's equations*

$$\dot{u}_k = \{H, u_k\}, \qquad k \in \mathbb{N}.$$

According to the notation previously introduced, we denote by $V_H := (\{H, u_k\})_{k\in\mathbb{N}}$ the (typically unbounded) Hamiltonian vector field associated with H. Clearly,

$$V_H^{(k)} = \mathrm{i}\frac{\partial H}{\partial \overline{u}_k}.$$

The same discussion and remarks of Section 2.4 on the regularity and on the existence of the Hamiltonian flow hold literally in this setting as well.

3.1.7 The main theorem: Liouville-Arnold integrability

In this section we prove a theorem about the complete integrability (à la Liouville-Arnold) of Hamiltonian systems on infinite lattices. The idea is the following: when describing the phase space $\ell^\infty_\eta(\mathbb{C})$, we observed that, given any torus of the form $\mathcal{T}_\xi$ (see (3.2)), there is a tubular neighbourhood which is analytically and identified with the product space $\mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty$ via the symplectomorphism Φ_ξ defined in (3.3). Hence, in light of the results in Propositions 3.1.2 and 3.1.3, Φ_ξ^{-1} pushes the Hamiltonian vector field $\mathrm{i}\partial_{\overline{u}}H$ defined on $\ell^\infty_\eta(\mathbb{C})$ forward to $\ell^\infty_w \times \mathbb{T}^\infty$. Of course, the Hamilton's equations are preserved.

Theorem 3.1.4 (Complete Integrability for Systems on Infinite Lattices)**.** *Let $R > 0$ and consider a majorant real analytic Hamiltonian*

$$H : \mathscr{U}_R(\ell^\infty_\eta(\mathbb{C})) \to \mathbb{R}$$

with $H \in \mathcal{H}_R$. Suppose that there are functions $(f^{(m)})_{m\in\mathbb{N}}$, where $f^{(m)} \in \mathcal{H}_R$ for all $m \in \mathbb{N}$, satisfying the following properties:

(a.1) the Taylor series of every $f^{(m)}$ starts at order larger or equal than three.

(a.2) denoting $f = (f^{(m)})_{m\in\mathbb{N}}$, the following quantity is bounded

$$\|f\|_R := \max\left\{ \sup_{m\in\mathbb{N}} \eta_m \lfloor f^{(m)} \rceil_R,\ R \sup_{k\in\mathbb{N}} \eta_k \sum_{m\in\mathbb{N}} \left\lfloor \frac{\partial f^{(m)}}{\partial \bar{u}_k} \right\rceil_R \right\};$$

(a.3) let $w_k = \eta_k^2$ for all $k \in \mathbb{N}$, then the map $\Gamma : \mathscr{U}_R(\ell^\infty_\eta(\mathbb{C})) \to \ell^\infty_w$ defined by

$$\Gamma^{(m)}(u) = |u_m|^2 + f^{(m)}(u)$$

is such that, for all $m, n \in \mathbb{N}$, $\{\Gamma^{(m)}, \Gamma^{(n)}\} = 0$ and $\{\Gamma^{(m)}, H\} = 0$ for all $m \in \mathbb{N}$.

Then, there exists $\xi_-, \xi_+ \in \mathbb{R}$, with $0 < \xi_- \le \xi_+ < R^2$, such that for every sequence of real numbers $\xi = (\xi_k)_{k\in\mathbb{N}}$ satisfying

$$\xi_- \le \inf_{k\in\mathbb{N}} \eta_k^2 \xi_k \le \sup_{k\in\mathbb{N}} \eta_k^2 \xi_k \le \xi_+$$

(i) H induces a Hamiltonian flow ϕ_H on a neighbourhood $\mathscr{M}_\xi$ of $\mathcal{T}_\xi$;

(ii) there is a neighbourhood $\mathscr{U}_\xi$ of $\gamma = \xi$ in ℓ^∞_w such that, for all $\gamma \in \mathscr{U}_\xi$, the level set $\Gamma^{-1}(\gamma)$ is an analytic Banach manifold, diffeomorphic to the infinite torus $\mathbb{T}^\infty$, which is invariant for the Hamiltonian flow ϕ_H;

(iii) there exists a set of action-angle *coordinates (x, θ) in which Hamilton's equations read*

$$\begin{cases} \dot{x}_k = 0 \\ \dot{\theta}_k = \dfrac{H \circ \Phi_\xi \circ \Phi^{-1}}{\partial x_k} \end{cases}, \quad k \in \mathbb{N}, \tag{3.8}$$

where Φ and Φ_ξ are the symplectomorphisms defined in (2.41) *and 3.3, respectively, and $(x, \theta) = \Phi \circ \Phi_\xi^{-1}(u)$.*

In particular the Hamilton's equations can be integrated by quadratures and the Hamiltonian flow reduces to a linear flow on the infinite torus $\mathbb{T}^\infty$.

Proof. Note that the requirement that the perturbations $f^{(m)}$, for all $m \in \mathbb{N}$, are of order three or higher implies that the quantity $R^{-2}\|f\|_R$ can be made arbitrarily small by simply reducing the radius R. Then, there exists $R_0 > 0$ such that

$$R_0 \le \min\{R, 2^{5/2}\sqrt{\ln(2)}\} \tag{3.9}$$

$$R_0^{-2}\|f\|_{R_0} \le \frac{1}{2^9 C}\ , \tag{3.10}$$

where the constant C is defined in (2.42). Let $\xi_- := 32C\|f\|_{R_0}$ and $\xi_+ := R_0^2/16$. Clearly, (3.10) implies that $\xi_- \le \xi_+$. Finally, define $r_0 = \xi_-/2$ and let $s_0 := \ln(2)$. It is immediate to verify that

R_0, r_0, s_0 satisfies (3.4). Therefore, by setting $P = \Phi_\xi^* f$ and $\Gamma_P = \Phi_\xi^* \Gamma$, it follows directly from Proposition 3.1.3 that

$$\|P\|_{r_0,s_0}^{\infty,w} \le 2\|f\|_{R_0}.$$

This bound allows the smallness assumption of Theorem 2.5.12 to be met. Indeed, one has the following chains on inequalities

$$\|P\|_{r_0,s_0}^{\infty,w} \le 2\|f\|_{R_0} = \frac{\xi_-}{16C} = \frac{r_0}{8C} < \frac{r_0}{4C}$$
$$\|P\|_{r_0,s_0}^{\infty,w} \le 2\|f\|_{R_0} \overset{(3.10)}{\le} \frac{R_0^2}{2^8 C} \overset{(3.9)}{\le} \frac{\ln(2)}{8C} < \frac{\ln(2)}{4C} < \frac{1}{4C},$$

which imply

$$\|P\|_{r_0,s_0}^{\infty,w} < \frac{1}{4C}\min\{r_0, s_0, 1\}.$$

Thus, since Φ_ξ is a symplectomorphism, the claim follows by Theorem 2.6.1 and Proposition 2.4.3 by setting, with the same notations of Theorem 2.6.1, $\mathscr{M}_\xi := \Phi_\xi(\mathscr{N}_0)$ and $\mathscr{U}_\xi = \xi + \mathscr{U}_0$. ■

It is worth stressing that, besides providing a connection (established in Proposition 3.1.3) with the theory developed in the previous chapter, the boundedness of the $\|\cdot\|_R$ norm ensures that the momentum map is a majorant analytic map taking values in the space of actions, and that the adjoint of the differential of Γ induces a bounded operator from the space of analytic vector fields to itself. This property, together with result established in the above theorem, should be compared with [BS20], where he authors show how a family of commuting analytic vector fields can be reduced to a normal form near the origin via an analytic coordinate transformation. Their approach provides an abstract framework to construct Birkhoff coordinates for infinite-dimensional systems in a small neighbourhood around the origin, with direct applications to the KdV, the NLS equation, and the Toda lattice. Similarly to their approach, our result is local in nature; however, it applies to an open region of the phase space that excludes the origin, whose existence is fundamentally linked to the ℓ^∞-based topology. On the other hand, our construction provides a more detailed geometric description in terms of foliations into infinite-dimensional invariant submanifolds.

Another remarkable point is that, by interpreting the functions $u \mapsto |u_m|^2$ for all $m \in \mathbb{N}$ as Hamiltonians of harmonic oscillators, the first integrals $f^{(m)}$ can be seen as small (with respect to the $\|\cdot\|_R$ norm) perturbations of these systems. Finally, we point out that one has the explicit expression for the actions

$$x_m = \int_{\Gamma^{-1}(\gamma)} (|u_m|^2 - \xi_m)\, d\lambda', \qquad m \in \mathbb{N},$$

where λ' is the pushforward by Φ_ξ of the measure λ defined in Subsection 2.1.1. This directly follows by the definition of the actions (see (2.4)) and by the fact that $|u_m|^2 = \xi_m + y_m$ for all $m \in \mathbb{N}$. Since, by (3.8), these actions are preserved by the Hamiltonian flow, we find that, although the energies of the harmonic oscillators $u \mapsto |u_m|^2$ are not preserved, the averages of such energies on the invariant tori are constant in time.

3.1.8 Connection with Hamiltonian PDEs on the circle

Many Hamiltonian PDEs on the circle can be viewed as Hamiltonian systems on infinite lattices, where this correspondence is realized via the Fourier transform. Consider, for instance, the Schrödinger equation for the wave function $u \in H_0^1(\mathbb{T}, \mathbb{C})$[3] of a free particle in a box of length 2π with periodic boundary

[3] As usual, for any $N > 0$, $H_0^N(\mathbb{T}, \mathbb{C})$ denotes the Hilbert space of 2π-periodic functions with zero mean whose N-th weak derivative is in $L^2(\mathbb{T}, \mathbb{C})$.

conditions, which, in natural units where $\hbar = 1$[4], reads as

$$\mathrm{i}u_t = u_{xx}, \qquad t \in \mathbb{R},\ x \in \mathbb{T}.$$

By expanding the wave function u in a Fourier series as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{\mathrm{i}kx}$, the Schrödinger equation for the Fourier coefficients becomes

$$\dot{u}_k = \mathrm{i}k^2 u_k, \qquad \text{for all } k \in \mathbb{Z}.$$

It is then clear that this dynamics can be viewed as a Hamiltonian system on the infinite lattice $\mathbb{Z}$ with Hamiltonian H and symplectic form ω given by

$$H(u) = \sum_{k\in\mathbb{Z}} k^2|u_k|^2, \qquad \omega = \mathrm{i}\sum_{k\in\mathbb{Z}} du_k \wedge d\bar{u}_k. \tag{3.11}$$

One can easily see that the solutions of the above equation are given by

$$u_k(t) = u_k(0)e^{\mathrm{i}k^2 t}, \qquad \text{for all } k \in \mathbb{Z},$$

where $u(0,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{\mathrm{i}kx}$ represents the state of the particle at time $t = 0$.

By embedding h^1 into ℓ^∞_η, which is possible if

$$\sum_{k\in\mathbb{Z}} k^2\eta_k^{-2} \tag{3.12}$$

is finite[5], it is easy to see that this system is indeed completely integrable in the sense of Theorem 3.1.4. Indeed, condition (3.12) ensures that the Hamiltonian in (3.11) is analytic in any ball centered at the origin of $\ell^\infty_\eta(\mathbb{C})$. Furthermore, with the same notation as in Theorem 3.1.4, the momentum map Γ is given by

$$\Gamma^{(m)}(u) = |u_m|^2, \qquad \text{for all } m \in \mathbb{Z},$$

and, for every sequence of real numbers $\xi = (\xi_k)_{k\in\mathbb{N}}$ satisfying

$$0 < \inf_{k\in\mathbb{N}} \eta_k^2\xi_k \leq \sup_{k\in\mathbb{N}} \eta_k^2\xi_k < \infty,$$

the parametrization $u_k = \sqrt{\xi_k + x_k}e^{\mathrm{i}\theta_k}$, $k \in \mathbb{Z}$, provides a set of action-angle variables in a neighborhood of the torus

$$\mathcal{T}_\xi := \left\{ u \in \ell^\infty_\eta(\mathbb{C}) : |u_k|^2 = \xi_k \text{ for all } k \in \mathbb{N} \right\}. \tag{3.13}$$

Finally, such a neighborhood is foliated by infinite-dimensional invariant tori, analytically diffeomorphic to $\mathbb{T}^\infty$, which support the linear dynamics

$$\phi^t_H(x,\theta) = (x, (\theta_k + k^2 t)_{k\in\mathbb{Z}}),$$

also known as Kronecker flow (see Chapter 4 for an in-depth investigation of this topic).

According to the discussion in Subsection 2.4.1, such a flow is clearly not continuous in time as a function $\mathbb{R} \ni t \mapsto \ell^\infty_\eta(\mathbb{C})$ because the sequence of frequencies $(k^2)_{k\in\mathbb{Z}}$ is unbounded. It is worth stressing that this kind of dynamics is always continuous with respect to the weak* topology, which is the topology on $\ell^\infty_\eta(\mathbb{C})$ defined by the neighborhood system consisting of all sets of the form

$$\mathscr{U}_{A,\epsilon}(u) := \left\{ v \in \ell^\infty_\eta(\mathbb{C}) \ :\ \left| \sum_{k\in\mathbb{Z}} (u_k - v_k)\phi_k \right| < \epsilon\ ,\ \forall \phi \in A \right\}, \quad u \in \ell^\infty_\eta(\mathbb{C}),$$

[4] According to the standard literature, $\hbar$ denotes the reduced Planck's constant.

[5] Given $N > 0$, we denote by h^N the complex Hilbert space of Fourier coefficients associated with functions in $H^N_0(\mathbb{T}, \mathbb{C})$.

where A is an arbitrary finite subset of the Banach space

$$\ell^1_{\eta^{-1}}(\mathbb{C}) := \left\{ \phi = (\phi_k)_{k\in\mathbb{Z}} \ : \ \forall k \in \mathbb{Z}, \phi_k \in \mathbb{C}, \text{ with } \sum_{k\in\mathbb{Z}} \eta_k^{-1}|\phi_k| < \infty \right\}.$$

Indeed, $\ell^\infty_\eta(\mathbb{C})$ is the dual space of $\ell^1_{\eta^{-1}}(\mathbb{C})$. This topology coincides with the topology of pointwise convergence; hence, a function with values in $\ell^\infty_\eta(\mathbb{C})$ is continuous with respect to the weak* topology if and only if all its component functions are continuous. We shall prove in Appendix D that the invariant tori of the form (3.13), equipped with the subspace topology induced by the weak*, are homeomorphic to the infinite torus endowed with the product topology. It will turn out that the restriction of the action-angle map on each invariant torus provides a topological conjugacy between the Hamiltonian dynamical system and the associated Kronecker flow.

3.2 The KdV Equation

In this section we shall apply the results established thus far for showing that the KdV equation on the circle is a completely integrable infinite-dimensional Hamiltonian system in the sense of Liouville-Arnold (see Theorem 3.2.7). By the nature of our results, complete integrability is to be intended in a region of small data bounded away from the origin, structured as a product of thin complex annuli of the form $0 < r < |u_m| < R$ for each Fourier component.

3.2.1 Introduction and state of the art

We consider the Korteweg-de Vries (KdV) equation with periodic boundary conditions for a real-valued function u with zero mean:

$$u_t + u_{xxx} - 6uu_x = 0, \quad x \in \mathbb{R}/\mathbb{Z}, \quad \int_0^1 u(x,t)dx = 0. \tag{3.14}$$

The condition $\int_0^1 u(x,0)dx = 0$ is preserved by the KdV flow. Indeed, since $6uu_x - u_{xxx} = (3u^2 - u_{xx})_x$, the total mass is a conserved quantity. By multiplying both sides of the equation by u and integrating, one can also find the conserved Hamiltonian:

$$H(u) = \int_0^1 \left(\frac{u_x^2}{2} + u^3 \right) dx.$$

In [Gar71], Gardner showed that $L^2_0(\mathbb{R}/\mathbb{Z},\mathbb{R})$[6] carries a Poisson structure defined by the bracket

$$\{F, G\} := \int_0^1 \nabla_{L^2} F \cdot \partial_x \nabla_{L^2} G \, dx,$$

where ∇_{L^2} denotes the L^2-gradient, making KdV an infinite-dimensional Hamiltonian system with Hamiltonian vector field $V_H = \partial_x \nabla_{L^2} H$.

Defining the Fourier coefficients as $u_j := \int_0^1 u(x)e^{-\mathrm{i}jx}dx$, $j \in 2\pi\mathbb{Z}$, the Hamiltonian can be expressed as

$$H(u) = \frac{1}{2} \sum_{j\in 2\pi\mathbb{Z}\setminus\{0\}} j^2|u_j|^2 + \sum_{\substack{j,k,\ell\in 2\pi\mathbb{Z}\setminus\{0\} \\ j+k+\ell=0}} u_j u_k u_\ell.$$

[6] We denote by $L^2_0(\mathbb{R}/\mathbb{Z},\mathbb{R})$ the subspace of $L^2(\mathbb{R}/\mathbb{Z},\mathbb{R})$ of functions with zero mean.

The first term represents an infinite system of independent harmonic oscillators, while the second term introduces a coupling. In these coordinates, the Hamilton's equations read

$$\dot{u}_j = \mathrm{i}j^3 u_j + 3\mathrm{i}j \sum_{k\in 2\pi\mathbb{Z}\setminus\{0\}} u_{j-k}u_k, \quad j \in 2\pi\mathbb{Z}\setminus\{0\}.$$

A natural question is whether this system is integrable in the Liouville-Arnold sense. Specifically, one seeks a bi-analytic diffeomorphism that decouples the system into a Birkhoff normal form:

$$H(z) = \sum_{j\in 2\pi\mathbb{N}} \omega_j |z_j|^2 + Z(I), \qquad \omega = \mathrm{i} \sum_{j\in 2\pi\mathbb{N}} dz_j \wedge dz_{-j},$$

where Z depends only on the actions $I_j = |z_j|^2$. If such coordinates exist, the set

$$\mathcal{T}_I := \left\{ z : |z_k|^2 = I_k \text{ for all } k \in 2\pi\mathbb{N} \right\}$$

are invariant tori for the Hamiltonian flow induced by the KdV equation.

In [Lax68], Lax showed that u satisfies KdV if and only if the pair of operators

$$L_u(t) := -\frac{\partial^2}{\partial x^2} + u(x,t), \quad B_u(t) := -4\frac{\partial^3}{\partial x^3} + 3u(x,t)\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}(u(x,t)\cdot\,)$$

satisfies the Lax equation $\frac{d}{dt}L_u(t) = [B_u(t), L_u(t)]$. Since L_u is self-adjoint and B_u is anti-self-adjoint, the spectrum of L_u, denoted by $(\lambda_n)_{n\in\mathbb{N}}$, remains constant in time. This isospectral property provides an infinite family of conserved quantities, implying that the KdV flow evolves on the isospectral sets

$$\mathrm{Iso}(u_0) := \{u : \mathrm{spec}(L_u) = \mathrm{spec}(L_{u_0})\}.$$

To characterize these sets, one typically appeals to Floquet theory, which describes the spectrum as a union of disjoint intervals (bands) separated by spectral gaps. Specifically, the gaps are defined as the intervals of the real line (E_{2n-1}, E_{2n}) that are not contained in the spectrum of the Schrödinger operator. The endpoints of the spectral gaps correspond to the pure point spectrum of L_u when its action is restricted on the space of 2-periodic functions, and exhibit the asymptotic behavior

$$E_{2n-1}(u), E_{2n}(u) = \pi^2 n^2 + \int_0^1 u(x,0)dx + \ell_n^2,$$

where $E_0 < E_1 \le E_2 < E_3 \le E_4 \ldots$, and ℓ_n^2 denotes the n-th entry of an ℓ^2 sequence. A key result in [GT84] establishes that the spectrum of the periodic Schrödinger operator is entirely determined by the mean value of the potential u and by the gap lengths, which are defined by

$$g_n := E_{2n} - E_{2n-1}, \qquad n \in \mathbb{N},$$

allowing representing the level sets of the conserved quantities as

$$\mathrm{Iso}(u_0) = \{u : g_n(u) = g_n(u_0)\, \forall n \in \mathbb{N}\}.$$

A closer inspection of the isospectral sets shows that for any finite N, they are compact analytic submanifolds homeomorphic to N-dimensional tori, where N corresponds to the number of open gaps. While this topological identification also holds for $N = \infty$ (see [MT76]), such level sets cannot be analytic submanifolds of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ since an infinite-dimensional compact subset of a Banach space can never be a submanifold (see e.g. Appendix C of [PT87]).

If we vary the widths of the open gaps $(g_1, \dots, g_N) \in U \subset \mathbb{R}^N$, we obtain a set of functions isomorphic to $U \times \mathbb{T}^N$. This subset is invariant for the dynamics, and the restriction of the KdV system is a finite ($2N$-)dimensional integrable system. In [MT76], it was shown that the squared gap lengths g_j^2 are real analytic functions of u that Poisson commute with each other, representing the meeting point between the Lax pair structure and the Hamiltonian framework.

In [FM76], Flaschka and McLaughlin provided explicit action-angle variables $(I_k, \theta_k)_{k=1}^N$ in terms of spectral gaps. One might think of repeating their construction for all N and then taking the limit $N \mapsto \infty$ to cover the whole phase space. However, the limit is singular in the dense subset of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ given by

$$\Game := \{u : g_n(u) = 0 \text{ for some } n \in \mathbb{N}\}.$$

The solution, as developed in [KP03],[7] is the introduction of the *Birkhoff coordinates*

$$x_k = \sqrt{2I_k}\cos(\theta_k), \quad y_k = \sqrt{2I_k}\sin(\theta_k),$$

which are well-defined and analytic on the entire phase space. In these variables, the symplectic form becomes $\omega = \sum_k dx_k \wedge dy_k$ and the Hamiltonian H depends only on the actions. This fundamental breakthrough is established by the following theorem, which provides a global Birkhoff normal form for the KdV equation. As usual, H_0^N denotes the Sobolev-Hilbert spaces of zero-mean functions with the N^{th} weak derivative in L_0^2 and h^N its image under the Fourier transform.

Theorem ([KP03], Theorem 12.6)**.** *There exists a diffeomorphism $\Phi : L_0^2 \to h^{1/2}$ with the following properties:*

(i) Φ is one-to-one, onto, bi-analytic, and preserves the Poisson bracket.

(ii) For each $N \geq 0$, the restriction of Φ to H_0^N, denoted by the same symbol, is a map

$$\Phi : H_0^N \to h^{N+1/2},$$

which is one-to-one, onto, and bi-analytic as well.

(iii) The coordinates (x, y) in $h^{3/2}$ are global Birkhoff coordinates for the KdV equation. That is, the transformed KdV Hamiltonian $H \circ \Phi^{-1}$ depends only on $x_n^2 + y_n^2$, $n \geq 1$, with (x, y) being canonical coordinates in $h^{3/2}$.

We remark that the actions are asymptotically equivalent to the rescaled squared gap lengths, i.e.

$$I_k = x_k^2 + y_k^2 \sim \frac{1}{k} g_k^2 \qquad \text{for large } k,$$

which are themselves real analytic integrals in involution. The regularity of the gap lengths coincides with the regularity of the Fourier coefficients, as proved in [MO75], thereby providing a complete link between the spectral and Hamiltonian perspectives.

In this section, we place the KdV equation within our ℓ^∞-weighted space setting. We show that, when viewed as a system on an infinite lattice via the Fourier transform, it defines a completely integrable infinite-dimensional Hamiltonian system in the sense of Theorem 2.6.1. To this end, it is necessary to investigate the structure and regularity of the spectral gaps of the associated periodic Schrödinger operator. We achieve this by interpreting the spectral problem as a first-order dynamical system, describing a perturbed harmonic oscillator where the potential u acts as an external periodic force. We then apply

[7] See [BKM96] for the original paper. For a different proof of the same result, see [KM01].

a reduction scheme to transform this system into one with constant coefficients. Such a reduction significantly simplifies the analysis of the spectral gaps, which, in this setting, correspond precisely to the instability intervals of the associated dynamical system.

For the sake of notational simplicity and to avoid the proliferation of factors of π, we prefer to work with 2π-periodic functions rather than 1-periodic ones. This choice allows us to identify the Fourier lattice directly with $\mathbb{Z}$ instead of $2\pi\mathbb{Z}$. To this end, we perform a rescaling of the parameters by setting $t \to (2\pi)^3 t$, $u \to (2\pi)^{-2}u$, and $x \to 2\pi x$. One can easily verify that this transformation leaves the form of the KdV equation, including all its coefficients, invariant. Therefore, in what follows, we shall adopt this scaling without changing the notation for the variables.

3.2.2 Spectral problem and Floquet theory

Consider the Schrödinger operator $L_u = -\partial_{xx} + u$, where u is 2π-periodic, with zero mean and analytic. The eigenvalue equation

$$-\psi_{xx} + u\psi = E\psi$$

can be reformulated as a first-order dynamical system. Setting $t = x$, $q = \psi$ and $p = \psi_t$, we obtain:

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -E & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}\right]\begin{pmatrix} q \\ p \end{pmatrix}, \tag{3.15}$$

namely the Hamilton's equation for a harmonic oscillator forced by a periodic perturbation. From Floquet theory, it is well known that the spectrum is composed of bands and gaps, that correspond to bounded and unbounded solution for the associated dynamical system (3.15). For the unperturbed case $u = 0$, the periodic and anti-periodic eigenvalues are doubly degenerate and equal to $E_n = n^2/4$, with associated eigenfunctions $e^{\pm int/2}$, for $n \in \mathbb{N}$. Clearly, for even $n \in \mathbb{N}$, one has 2π-periodic eigenfunctions, while for odd n, the eigenfunctions are 2π-anti-periodic. There is no spectrum for $E < 0$, while $n = 1$ yields the first anti-periodic eigenvalue $E_1 = 1/4$. For small u, the effect of the perturbation is that of open gaps between the unperturbed doubly degenerate eigenvalues. Moreover, E_n remains close to its unperturbed value. Then the study of the spectrum amounts to identifying the regions of $\lambda > 0$, where $E = \lambda^2$, that yield bounded eigenfunctions. Clearly, the study of the spectral gaps coincides with the identification of the complementary region.

3.2.3 Complex coordinates and Lie algebra structure

Let us introduce the complex coordinates

$$z = \frac{1}{\sqrt{2}}\left(\frac{p}{\sqrt{\lambda}} + \mathrm{i}\sqrt{\lambda}q\right) \qquad \text{and} \qquad \bar{z} = \frac{1}{\sqrt{2}}\left(\frac{p}{\sqrt{\lambda}} - \mathrm{i}\sqrt{\lambda}q\right).$$

Under this change of variables, the eigenvalue equation $L_u\psi = \lambda^2\psi$ recasts as:

$$\frac{d}{dt}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \left[\lambda\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} + \frac{u}{2\lambda}\begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}\right]\begin{pmatrix} z \\ \bar{z} \end{pmatrix}. \tag{3.16}$$

This system represents a two-parameter family of Hamiltonian dynamical systems, characterized by the parameters u and λ. The corresponding Hamiltonian function is given by

$$H(z, \bar{z}\,; u, \lambda) = \left(\frac{u}{2\lambda} - \lambda\right)|z|^2 - \frac{u}{2\lambda}\mathrm{Re}(z^2),$$

and the Hamiltonian vector field is defined via the standard symplectic form $\omega = \mathrm{i}dz \wedge d\bar{z}$. Clearly, such vector field is time dependent and depends on t only through the external force u. In this representation the state vector evolves in $\mathbb{C}^2$, and the matrices driving the evolution must preserve the symplectic form ω. Namely the flow is represented by an element of the group $SU(1,1)$,[8] which is defined as:

$$SU(1,1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,:\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Consequently, the matrices inducing the dynamics, being the infinitesimal generators of such a flow, must belong to the corresponding Lie algebra $\mathfrak{su}(1,1)$ given by:

$$\mathfrak{su}(1,1) := \left\{ \begin{pmatrix} \mathrm{i}a & b \\ \bar{b} & -\mathrm{i}a \end{pmatrix} \,:\, a \in \mathbb{R},\ b \in \mathbb{C} \right\}.$$

To make this more explicit, we can project the dynamics onto a specific basis (over $\mathbb{R}$) $\{J_1, J_2, J_3\}$ for $\mathfrak{su}(1,1)$:

$$J_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

In this basis, the Hamilton's equations (3.16) take the form

$$\frac{d}{dt} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = D(u, \lambda) \begin{pmatrix} z \\ \bar{z} \end{pmatrix},$$

where the matrix $D(u, \lambda) \in \mathfrak{su}(1,1)$ can be expressed as

$$D(u, \lambda) = \left(\lambda - \frac{u}{2\lambda} \right) J_1 + \frac{u}{2\lambda} J_3,$$

highlighting the underlying algebraic structure.

3.2.4 Purposes and basic ideas of the strategy

We seek a time-dependent change of coordinates in $SU(1,1)$ that reduces the Hamilton's equations (3.16) to a system with constant coefficients. Assuming such a transformation exists, the new equations of motion will take the form:

$$\frac{d}{dt} \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = \mathcal{D}(u, \lambda) \begin{pmatrix} w \\ \bar{w} \end{pmatrix},$$

where $\mathcal{D}(u, \lambda)$ has constant coeffiecients. Since the transformation is performed via an element of the group $SU(1,1)$, the constant matrix $\mathcal{D}$ must belong to the Lie algebra $\mathfrak{su}(1,1)$. In particular, $\mathcal{D}$ is traceless, and its eigenvalues are given by:

$$\mu_{\pm}(u, \lambda) = \pm\sqrt{-\det(\mathcal{D}(u, \lambda))}.$$

Therefore, if $\det(\mathcal{D}(u, \lambda)) > 0$, the dynamics is equivalent to that of a harmonic oscillator, with the frequency determined by the eigenvalues. Conversely, if $\det(\mathcal{D}(u, \lambda)) < 0$, the eigenvalues are real, leading to unbounded dynamics. The intervals of $E = \lambda^2$ for which this occurs correspond to the spectral gaps.

1. Outline of the strategy for the reduction to constant coeffiecients. In order to make explicit the perturbative nature of the external force, we define the matrix

$$P(u, \lambda) = \frac{u}{2\lambda} \begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} = \frac{u}{2\lambda}(J_3 - J_1). \tag{3.17}$$

[8]The notation is motivated by the fact that, from a complex geometry perspective, ω represents a $(1,1)$-differential form.

With this choice, the Hamilton's equations (3.16) recast as:

$$\frac{d}{dt}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \Big[\lambda J_1 + P(u,\lambda)\Big]\begin{pmatrix} z \\ \bar{z} \end{pmatrix}. \tag{3.18}$$

The goal is to find a change of coordinates in $SU(1,1)$, such that the transformed system has constant coefficients. Evidently, this change of coordinates will inherit the 2π-periodicity of the perturbation.

Given a differentiable, 2π-periodic map $\mathbb{R} \ni t \mapsto B(t) \in SU(1,1)$, let $\xi, \bar{\xi}$ be the set of coordinates defined by $(\xi,\bar{\xi})^T = B(z,\bar{z})^T$. In this set of coordinates, the equations of motion read

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = D_1(\lambda)\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \qquad D_1(\lambda) = B\,(\lambda J_1 + P(\lambda))B^{-1} - B\frac{d}{dt}B^{-1}. \tag{3.19}$$

At this stage, we proceed by considering only the formal algebraic structure of the transformations. A rigorous justification, including the definition of the appropriate functional setting and the proof of convergence for the series involved, will be provided in the next subsections.

Let us assume that $B = \exp(-A)$, for some $A \in \mathfrak{su}(1,1)$. By defining the adjoint action as $\mathrm{ad}_{-A}M = [-A, M]$, and recalling that

$$e^{-A}Me^{A} = \sum_{k=0}^{\infty}\frac{(\mathrm{ad}_{-A})^k}{k!}M \qquad \text{and} \qquad e^{-A}\frac{d}{dt}e^{A} = \sum_{k=1}^{\infty}\frac{(\mathrm{ad}_{-A})^{k-1}}{k!}\dot{A},$$

equation (3.19) reads

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \left(\sum_{k=0}^{\infty}\frac{(\mathrm{ad}_{-A})^k}{k!}(\lambda J_1 + P) - \sum_{k=1}^{\infty}\frac{(\mathrm{ad}_{-A})^{k-1}}{k!}\dot{A}\right)\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}. \tag{3.20}$$

Assuming that $A = O(|P|)$, the transformation intended to eliminate the time dependence yields the following homological equation at order $O(|P|)$:

$$\dot{A} + \lambda[A, J_1] = P - \mathscr{Z}_0, \tag{3.21}$$

where the matrix $\mathscr{Z}_0$ is introduced to account for the (possibly) non-trivial kernel of the operator

$$\mathcal{L}_\lambda := \frac{d}{dt} + \lambda[\,\cdot\,, J_1]. \tag{3.22}$$

A primary issue emerges at this stage: if $\lambda = n/2$ for some $n \in \mathbb{N}$, the kernel of $\mathcal{L}_\lambda$, acting on the space of 2π-periodic functions with values in $\mathfrak{su}(1,1)$, consists of time-dependent functions. Specifically,

$$\mathrm{Ker}(\mathcal{L}_{n/2}) = \left\{\begin{pmatrix} \mathrm{i}a & be^{\mathrm{i}nt} \\ \bar{b}e^{-\mathrm{i}nt} & -\mathrm{i}a \end{pmatrix} \;:\; a \in \mathbb{R},\ b \in \mathbb{C}\right\}.$$

This property prevents the conjugation of the system to a constant coefficients matrix using only transformations of the form e^{-A} with $A \in \mathfrak{su}(1,1)$.

To address this, rather than seeking a global reduction for all values of λ, we focus on a neighborhood of each specific resonance to perform the reduction to constant coefficients. Specifically, for a fixed $n \in \mathbb{N}$, we restrict our study to the region:

$$|2\lambda - n| < \gamma, \quad \frac{1}{4} \le \gamma \le \frac{1}{2}. \tag{3.23}$$

This condition ensures that the operator $\mathcal{L}_\lambda$ encounters no singularities other than the resonance at $\lambda = n/2$, providing the necessary information to characterize the n-th spectral gap. The lower bound for γ in (3.23) ensures that the region under consideration is wide enough to encompass the entire gap. Indeed, according to Floquet theory, the spectrum is expected to persist in regions away from the resonances by a distance of order $O(|P|)$. Thus, for u sufficiently small, no relevant spectral features are excluded and the complement of our study region lies entirely within the spectral bands. This local analysis serves as the basis for the subsequent study of the global map that associates the potential u with the entire sequence of its spectral gaps.

Our strategy proceeds by performing a contractive iteration in the space of coordinate transformations to eliminate the time-dependent terms lying in the range of the operator $\mathcal{L}_{n/2}$. Let us outline the procedure: the matrix, intended to have constant coefficients in (3.20), can be recast as follows:

$$\lambda J_1 - \sum_{k=1}^{\infty} \frac{\mathrm{ad}_{-A}^{k-1}}{k!}(\dot{A} + \lambda[A, J_1]) + \sum_{k=0}^{\infty} \frac{\mathrm{ad}_{-A}^{k}}{k!} P. \tag{3.24}$$

Recalling the definition of $\mathcal{L}_\lambda$ in (3.22), and noting that $J_1 \in \mathrm{Ker}(\mathcal{L}_\lambda)$ for all $\lambda > 0$, and in particular for $\lambda = n/2$, the $\mathfrak{su}(1,1)$-valued function A defining the desired change of coordinates must solve the equation

$$\Pi_{rg}\Big(-\sum_{k=1}^{\infty} \frac{\mathrm{ad}_{-A}^{k-1}}{k!} \mathcal{L}_\lambda A + e^{\mathrm{ad}_{-A}} P \Big) = 0,$$

where Π_{rg} denotes the projector onto the range of $\mathcal{L}_{n/2}$. Denoting by $\mathcal{B}(A)$ the inverse of the operator $X \mapsto \sum_{k=1}^{\infty} (\mathrm{ad}_{-A}^{k-1})/k! X$ (see Lemma A.5.1 for details), and noting that the operator $\mathcal{L}_\lambda$ admits an inverse $\mathcal{L}_\lambda^{-1}$ defined on its range, we can derive an implicit fixed-point equation for the generator. Specifically, if A is such that

$$\mathcal{L}_\lambda A = \mathcal{B}(A)\Pi_{rg}\left(e^{\mathrm{ad}_{-A}} P\right),$$

by applying the linear operator $\mathcal{L}_\lambda^{-1}$ to both sides, one can rephrase the question on the reducibility into a fixed point problem for the map ϕ_P defined by

$$\phi_P(A) = \mathcal{L}_\lambda^{-1}\Pi_{rg}\left(\mathcal{B}(A)\Pi_{rg}\left(e^{\mathrm{ad}_{-A}} P\right)\right).$$

After having introduced the appropriate functional setting in the following subsections, we prove in Theorem 3.2.4 that if the perturbation P is sufficiently small and regular, the map ϕ_P is a contraction. Consequently, the unique fixed point A_P ensures that the transformed vector field lies entirely within the kernel of the operator $\mathcal{L}_{n/2}$. Since the range of $\mathcal{L}_{n/2}$ has codimension two in the space of 2π-periodic analytic functions with values in $\mathfrak{su}(1,1)$, this iterative procedure effectively simplifies the dynamics. Specifically, this process leads to a vector field of the form:

$$\lambda J_1 + \tilde{\mathcal{Z}}(u, \lambda, t),$$

where the residual perturbation $\tilde{\mathcal{Z}}$ belongs to $\mathrm{Ker}(\mathcal{L}_{n/2})$. As observed above, the elements of the kernel are still time-dependent; however, this dependence is purely oscillatory with frequency n. We can then reach a frame where the kernel is stationary by applying the additional $SU(1,1)$ transformation:

$$S_n(t) = \begin{pmatrix} e^{-int/2} & 0 \\ 0 & e^{int/2} \end{pmatrix}. \tag{3.25}$$

Under this final change of variables, the terms in the kernel are mapped to constant matrices, effectively completing the reduction to constant coefficients as desired. Namely, in Theorem 3.2.6 we prove that the transformation defined by the composition $\eta = S_n e^{-A_P} z$ reduces the Hamiltonian system (3.33) to a constant coefficient system of the form:

$$\frac{d}{dt}\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix} = [(\lambda - n/2)J_1 + \mathcal{Z}_0(u,\lambda) + \mathcal{P}_\infty(u,\lambda)]\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}, \tag{3.26}$$

where

$$\mathcal{Z}_0 = \begin{pmatrix} 0 & P_n^{+-} \\ P_n^{+-} & 0\end{pmatrix} = \frac{1}{2\lambda}\begin{pmatrix} 0 & \mathrm{i}u_n \\ -\mathrm{i}\bar{u}_n & 0\end{pmatrix}$$

is the leading order correction, while $\mathcal{P}_\infty$ is of order $O(|P|^2)$.

2. Spectral gaps and integrability for the KdV equation. The reduction of the system to the constant coefficient matrix (3.26) allows for an explicit characterization of the instability zones in the energy spectrum. Indeed, the endpoints of the n-th spectral gap correspond to the values of λ for which the determinant of the reduced matrix vanishes. By analyzing the structure of (3.26), we can derive an implicit equation for the gap boundaries (see (3.47)), which we will be able to solve, further deducing the regularity of the solution, thanks to the chosen functional setting and the fixed-point structure of the reduced constant-coefficient matrix.

However, in order to establish the complete integrability of the KdV equation (3.14) exploiting our analysis of the spectral gaps and the theory developed in the previous chapters, it is necessary to express these spectral quantities in terms of variables satisfying the standard canonical relations. A detail previously omitted in the introduction to the KdV section is that the symplectic form induced on the space of Fourier coefficients by the Gardner Poisson brackets,

$$\{F, G\} := \frac{1}{2\pi}\int_0^{2\pi} \nabla_{L^2} F \cdot \partial_x \nabla_{L^2} G \, dx,$$

is not the standard Darboux one, but is given by $\mathrm{i}\sum_{k\in\mathbb{N}} k^{-1} du_k \wedge du_{-k}$. To recover the canonical structure, we first perform the change of variables

$$z_\ell = \langle \ell \rangle^{-1/2} u_\ell.$$

Clearly, since the potential u is real-valued and has zero mean, we have $z_{-\ell} = \bar{z}_\ell$ for all $\ell \in \mathbb{Z}$, and $z_0 = 0$. In these coordinates, the symplectic form assumes the standard Darboux form $\omega = \mathrm{i}\sum_{\ell\in\mathbb{N}} dz_\ell \wedge dz_{-k}$, and the KdV equation can be rewritten as an infinite system of coupled oscillators:

$$\dot{z}_k = \mathrm{i}k^3 z_k + 3\mathrm{i}k \sum_{\substack{\ell+m=k\\ \ell,m\neq 0}} \sqrt{|\ell m|}\, z_\ell z_m, \quad k \in \mathbb{Z}\setminus\{0\},$$

where the corresponding Hamiltonian is given by:

$$H_{KdV}(z) = \sum_{k\in\mathbb{N}} k^3 |z_k|^2 + \sum_{\substack{k+\ell+m=0\\ k,\ell,m\neq 0}} \sqrt{|k\ell m|}\, z_k z_\ell z_m.$$

Our goal is to prove the Liouville-Arnold integrability of the KdV equation (Theorem 3.2.7) by showing that the map constructed from the complete list of the rescaled squared gaps,[9] expressed in symplectic polar coordinates, satisfies the requirements of the Complete Integrability Theorem 2.6.1.

A significant challenge in this direction is the potential lack of analyticity of the gaps at the origin ($z = 0$), due to the implicit definition involving the modulus in (3.42). However, for our purposes, we

[9] The squared gap lengths must be appropriately rescaled to match the dimension of $|z|^2$.

can bypass this singularity by restricting the phase space to a neighborhood of an infinite-dimensional torus $\mathcal{T}_\xi$, effectively working in an open set where the moduli $|z_k|$ are quantitatively far from 0 for all $k \in \mathbb{Z}$. In this setting, the required regularity and smallness conditions are established by showing that the derivatives of the gap map are "controlled" by a family of majorant analytic functions $(f^{(n)}(z))_{n\in\mathbb{N}}$. Namely, we prove the applicability of the integrability theorem by showing the finiteness of the following quantity:

$$\sup_{j\in\mathbb{N}} \langle j\rangle^p e^{a|j|} \sum_{n=1}^{\infty} \left\| \frac{\partial f^{(n)}}{\partial z_j} \right\|_{a,p,R} < \infty,$$

where $p > 3/2$, $a \geq 0$ and $\|\cdot\|_{a,p,R}$ denotes the majorant norm introduced in Subsection 3.2.6, formula (3.32). This condition should be compared with the "summably normally analytic" property used in [BS20]. The verification of these bounds, detailed in Subsection 3.2.8, leads to the final result of this chapter: the integrability of the KdV equation in the sense of a foliation of an open set of the phase space by infinite-dimensional invariant tori, and the existence of a set of action-angle coordinates adapted to such a foliation. This result is stated in Theorem 3.2.7.

3.2.5 Homological equation and spectral gaps at the leading order

While the full reduction is established through the convergence of the contractive scheme, it is highly instructive to examine the result at the first order in $|P|$. Solving the homological equation (3.21) explicitly at this order provides immediate insight into the leading order behaviour of the spectral gaps and the role of the Fourier coefficients of the potential.

1. Homological equation. We first introduce a notation: for any matrix $W \in \mathrm{Mat}(2,\mathbb{C})$, we write

$$W = \begin{pmatrix} W^{++} & W^{+-} \\ W^{-+} & W^{--} \end{pmatrix},$$

or $W = (W^{\sigma\sigma'})_{\sigma\sigma'=\pm1}$. Accordingly, every matrix W in the Lie algebra $\mathfrak{su}(1,1)$ satisfies $W^{--} = -W^{++} \in \mathrm{i}\mathbb{R}$ and $W^{-+} = \overline{W^{+-}} \in \mathbb{C}$. To streamline the presentation, in the following calculations we shall drop the explicit dependence of the maps on u and λ.

Following the strategy outlined above, we now examine how the system is transformed by solving the homological equation (3.21), which we recall being

$$\mathcal{L}_\lambda A = P - \mathcal{Z}_0.$$

Since u is analytic, the matrix-valued function $t \mapsto P(t)$ can be expanded in a Fourier series:

$$P(t) = \sum_{\ell\in\mathbb{Z}} e^{\mathrm{i}\ell t} \begin{pmatrix} P_\ell^{++} & P_\ell^{+-} \\ P_\ell^{-+} & P_\ell^{--} \end{pmatrix},$$

where, from the definition (3.17), we have for all $\ell \in \mathbb{Z}$:

$$P_\ell^{++} = P_\ell^{-+} = \frac{-\mathrm{i}u_\ell}{2\lambda} \qquad \text{and} \qquad P_\ell^{+-} = P_\ell^{--} = \frac{\mathrm{i}u_\ell}{2\lambda}.$$

By substituting this expansion into $\mathcal{L}_\lambda A = P - \mathcal{Z}_0$, we obtain a system of algebraic equations for the Fourier coefficients of the generator A:

$$\mathrm{i}(\ell - (\sigma - \sigma')\lambda) A_\ell^{\sigma\sigma'} = P_\ell^{\sigma\sigma'} - \tilde{\mathcal{Z}}_{0,\ell}^{\sigma\sigma'}.$$

Since u has zero mean, the diagonal part of the equation is solved by setting:

$$A_\ell^{++} = \frac{P_\ell^{++}}{\mathrm{i}\ell}, \quad A_\ell^{--} = \frac{P_\ell^{--}}{\mathrm{i}\ell}, \quad \tilde{\mathcal{Z}}_{0,\ell}^{++} = \tilde{\mathcal{Z}}_{0,\ell}^{--} = 0.$$

For the off-diagonal components, we set for all $\ell \neq \pm n$:

$$A_\ell^{+-} = \frac{P_\ell^{+-}}{\mathrm{i}(\ell - 2\lambda)}, \quad A_\ell^{-+} = \frac{P_\ell^{-+}}{\mathrm{i}(\ell + 2\lambda)}, \quad \tilde{\mathcal{Z}}_{0,\ell}^{+-} = \tilde{\mathcal{Z}}_{0,\ell}^{-+} = 0.$$

The resonant modes $\ell = \pm n$ are then handled by isolating the terms in the kernel $\tilde{\mathcal{Z}}_0(t)$:

$$A_n^{+-} = 0, \quad A_n^{-+} = \frac{P_n^{-+}}{\mathrm{i}(n + 2\lambda)}, \quad \tilde{\mathcal{Z}}_{0,n}^{+-} = P_n^{+-}, \quad \tilde{\mathcal{Z}}_{0,n}^{--} = 0,$$
$$A_{-n}^{+-} = -\frac{P_{-n}^{+-}}{\mathrm{i}(n + 2\lambda)}, \quad A_{-n}^{-+} = 0, \quad \tilde{\mathcal{Z}}_{0,-n}^{++} = 0, \quad \tilde{\mathcal{Z}}_{0,-n}^{-+} = P_{-n}^{-+}.$$

Consequently, the matrix $\tilde{\mathcal{Z}}_0(t)$ representing the projection of $P(t)$ onto the kernel of $\mathcal{L}_\lambda$ takes the form:

$$\tilde{\mathcal{Z}}_0(t) = \begin{pmatrix} 0 & P_n^{+-}e^{\mathrm{i}nt} \\ P_{-n}^{-+}e^{-\mathrm{i}nt} & 0 \end{pmatrix}.$$

Under this transformation, the vector field is mapped to $\lambda J_1 + \tilde{\mathcal{Z}}_0 + \mathcal{R}$, where the remainder $\mathcal{R}$ collects all higher-order terms:

$$\mathcal{R} = [-A, P] + \sum_{k \geq 2} \left(\frac{(\mathrm{ad}_{-A})^k}{k!}(\lambda J_1 + P) - \frac{(\mathrm{ad}_{-A})^{k-1}}{k!}\dot{A} \right).$$

Finally, by moving to the rotating frame as defined in (3.25), the dynamics is described by:

$$\frac{d}{dt}\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = \left((\lambda - n/2)J_1 + \mathcal{Z}_0 + S_n \mathcal{R} S_n^{-1} \right) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}, \tag{3.27}$$

where $\mathcal{Z}_0$ is the constant, off-diagonal matrix:

$$\mathcal{Z}_0 = \begin{pmatrix} 0 & P_n^{+-} \\ P_{-n}^{-+} & 0 \end{pmatrix} = \frac{1}{2\lambda}\begin{pmatrix} 0 & \mathrm{i}u_n \\ -\mathrm{i}\bar{u}_n & 0 \end{pmatrix}.$$

The term $S_n \mathcal{R} S_n^{-1}$ represents the transformed remainder, which is (formally) of order $O(|P|^2)$.

2. Spectral gap at order $O(P)$. We denote by D_1 the matrix defining the leading order of the vector filed (3.27), namely

$$D_1(\lambda) = (\lambda - n/2)J_1 + \mathcal{Z}_0(\lambda).$$

Our objective is to determine the squared gap between the boundaries of the interval of the squared parameter λ^2 for which $\det(D_1(\lambda)) \leq 0$. This requirement is satisfied whenever $-|u_n| \leq 2\lambda^2 - n\lambda \leq |u_n|$. Geometrically, this requires the parabola $f(\lambda) = 2\lambda^2 - n\lambda$ to be bounded by the horizontal lines $\pm|u_n|$. Recalling that we are focusing on the region $|2\lambda - n| < 1/2$, the endpoints of this instability interval, λ_+ and λ_-, correspond to

$$\lambda_+ = \frac{n + \sqrt{n^2 + 8|u_n|}}{4}, \quad \lambda_- = \frac{n + \sqrt{n^2 - 8|u_n|}}{4}.$$

Defining $E_+ = \lambda_+^2$ and $E_- = \lambda_-^2$, the squared gap is given by $g_n^2 = (E_+ - E_-)^2$. By using the identity $E_\pm = \frac{n\lambda_\pm \pm |u_n|}{2}$, we obtain the exact expression for the difference:

$$g_n = E_+ - E_- = |u_n| + \frac{n^2}{8}\left(\sqrt{1+\frac{8|u_n|}{n^2}} - \sqrt{1-\frac{8|u_n|}{n^2}}\right)$$

The Taylor series for $\sqrt{1+x}$ and $\sqrt{1-x}$ centered at $x = 0$ are:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$
$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \dots$$

When calculating the difference $\sqrt{1+x} - \sqrt{1-x}$, all terms involving even powers of x (including the constant 1) cancel out exactly. Conversely, the terms with odd powers of x have opposite signs and thus reinforce each other:

$$\sqrt{1+x} - \sqrt{1-x} = x + \frac{1}{8}x^3 + \mathcal{O}(x^5)$$

Substituting $x = \frac{8|u_n|}{n^2}$ back into the expression for g_n, we obtain:

$$g_n = |u_n| + \frac{n^2}{8}\left[\frac{8|u_n|}{n^2} + \frac{1}{8}\left(\frac{8|u_n|}{n^2}\right)^3 + O\left(\left(\frac{|u_n|}{n^2}\right)^5\right)\right] = 2|u_n| + \frac{8|u_n|^3}{n^4} + O\left(\left(\frac{|u_n|}{n^2}\right)^5\right)$$

This simple argument shows that, at the leading order, the n-th spectral gap is directly proportional to the n-th Fourier coefficient of the potential, serving as the formal bridge that links the regularity of the potential u to the decay properties of the spectral gaps. This expansion provides a heuristic insight into the classical result by Marchenko and Ostrovskii [MO75]. In their seminal work, they established a characterization of the spectrum, proving that the sequence of spectral gaps belongs to the Sobolev space h^s if and only if the potential u belongs to H^s for $s \geq 0$. Thus, the identity $g_n \approx 2|u_n|$ is consistent with the $H^s \leftrightarrow h^s$ correspondence.

The fact that g_n is a series of odd powers of the modulus $|u_n|$ implies that the n^{th} spectral gap is analytic in a punctured neighborhood of the origin, possessing a branch-point singularity at $u_n = 0$. Conversely, the squared gap g_n^2 can be expressed as a power series involving only even powers of $|u_n|$:

$$g_n^2 = 4|u_n|^2 + \sum_{k=2}^{\infty} \frac{c_{2k}}{n^{4(k-1)}}|u_n|^{2k} \tag{3.28}$$

and hence it is a majorant analytic function in the whole ball of radius $R = n^2/8$ centered at the origin, where the radius of convergence R for this series is determined by the branch point where the discriminant $\Delta_- = n^2 - 8|u_n|$ vanishes.

However, a fundamental distinction arises when considering the analyticity of the gap map $u \mapsto (g_n(u))_{n\in\mathbb{N}}$. In the standard ℓ^2 framework (or h^0 for the coefficients), the set

$$\{u \in \ell^2 : u_n = 0 \text{ for some } n \in \mathbb{N}\}$$

is dense. Consequently, the presence of a singularity at $u_n = 0$ implies that there is no open neighborhood in ℓ^2 where the gap map is analytic, a fact that reflects the singular nature of the action-angle variables $(I_n, \theta_n)_{n\in\mathbb{N}}$ constructed by Flaschka and McLaughlin [FM76] in the limit $N \to \infty$. Conversely, in our setting based on ℓ^∞, the situation is qualitatively different. The condition $\inf_n |u_n| > 0$ defines an open set in ℓ^∞ which is disjoint from the singular locus. Within this open set, the gap map is not only well-defined but also majorant analytic.

3.2.6 Functional setting

Let $p > 1$ and $s \geq 0$. For any $\ell \in \mathbb{Z}$, we denote the standard weight as $\langle \ell \rangle := \max\{1, |\ell|\}$.

1. The spaces $\ell^\infty_{s,p}(\mathscr{B})$. Let $(\mathscr{B}, |\cdot|_{\mathscr{B}}, \cdot)$ be a Banach algebra with constant $C_{\text{alg}}(\mathscr{B}) \in \mathbb{R}^+$, that is, for all $x, y \in \mathscr{B}$, one has

$$|x \cdot y|_{\mathscr{B}} \leq C_{\text{alg}}(\mathscr{B}) |x|_{\mathscr{B}} |y|_{\mathscr{B}}.$$

We define $\ell^\infty_{s,p}(\mathscr{B})$ as the Banach space of sequences $X = (X_\ell)_{\ell \in \mathbb{Z}}$, $X_\ell \in \mathscr{B}$ for all $\ell \in \mathbb{Z}$, equipped with the norm:

$$\|X\|^{\mathscr{B}}_{s,p} := \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} |X_\ell|_{\mathscr{B}} < \infty.$$

This space serves as the space of Fourier coefficients for the classes of periodic functions considered herein.

The convolution of two sequences $X, Y \in \ell^\infty_{s,p}(\mathscr{B})$ is the sequence $Z = X \star Y$ defined componentwise as

$$Z_\ell = \sum_{k \in \mathbb{Z}} X_{\ell-k} Y_k.$$

The definition is well posed because

$$\begin{aligned}
\sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} |Z_\ell| &\leq C_{\text{alg}}(\mathscr{B}) \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \sum_{k \in \mathbb{Z}} |X_{\ell-k}|_{\mathscr{B}} |Y_k|_{\mathscr{B}} \\
&\leq C_{\text{alg}}(\mathscr{B}) \|X\|^{\mathscr{B}}_{s,p} \|Y\|^{\mathscr{B}}_{s,p} \sup_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \frac{\langle \ell \rangle^p}{\langle \ell - k \rangle^p \langle k \rangle^p} e^{s(|\ell| - |\ell - k| - |k|)} \\
&\leq C_{\text{alg}}(\mathscr{B}) \|X\|^{\mathscr{B}}_{s,p} \|Y\|^{\mathscr{B}}_{s,p} \sup_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \frac{\langle \ell \rangle^p}{\langle \ell - k \rangle^p \langle k \rangle^p} \\
&\leq C_\star(p) C_{\text{alg}}(\mathscr{B}) \|X\|^{\mathscr{B}}_{s,p} \|Y\|^{\mathscr{B}}_{s,p},
\end{aligned} \tag{3.29}$$

where

$$C_\star(p) := 2^p (1 + 2\zeta(p)) \tag{3.30}$$

denotes the convolution algebra constant of $\ell^\infty_{s,p}$ and $\zeta(p) = \sum_{n \geq 1} n^{-p}$ is the *Riemann function*.

Concretely, in the following discussion we shall consider the cases $\mathscr{B} = \mathbb{C}$, where $C_{\text{alg}}(\mathbb{C}) = 1$, and $\mathscr{B} = \text{Mat}(2, \mathbb{C})$, endowed with the norm

$$[\![W]\!] := \max_{\sigma, \sigma' = \pm 1} |W^{\sigma\sigma'}|,$$

for which $C_{\text{alg}}(\text{Mat}(2, \mathbb{C})) = 2$.

For the case $\mathscr{B} = \mathbb{C}$, we adopt the shortened notation $\ell^\infty_{s,p}(\mathbb{C}) = \ell^\infty_{s,p}$.

2. The algebra $\mathscr{S}_{s,p}$. We define $\mathscr{S}_{s,p}$ as the Banach space of 2π-periodic functions $S : \mathbb{R} \to \mathfrak{su}(1,1)$, $S(t) = \sum_{\ell \in \mathbb{Z}} S_\ell e^{\mathrm{i}\ell t}$, with $S_\ell = (S^{\sigma\sigma'}_\ell)_{\sigma, \sigma' = \pm 1} \in \mathfrak{sl}(2, \mathbb{C})$, satisfying

$$|S|_{s,p} := \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \max_{\sigma, \sigma' = \pm 1} |S^{\sigma\sigma'}_\ell|.$$

It is clear that the list of matrices $(S_\ell)_{\ell \in \mathbb{Z}}$ must necessarily satisfy the constrains

$$S^{--}_\ell = -S^{++}_\ell, \quad \text{and} \quad \overline{S^{-+}_{-\ell}} = S^{+-}_\ell, \qquad \text{for all } \ell \in \mathbb{Z},$$

in order to define a $\mathfrak{su}(1,1)$ valued function via the Fourier series.

3. The algebra $\hat{\mathscr{S}}_{s,p}$. We denote by $\hat{\mathscr{S}}_{s,p}$ the subspace of $\ell^\infty_{s,p}(\text{Mat}(2, \mathbb{C}))$ consisting of matrix-valued sequences $X = (X_\ell)_{\ell \in \mathbb{Z}}$ such that, for every $\ell \in \mathbb{Z}$, the matrix X_ℓ satisfies:

1. $X_\ell^{--} = -X_\ell^{++}$ (Trace-free condition);
2. $X_\ell^{++} = -\overline{X_{-\ell}^{++}}$ (Diagonal antisymmetry);
3. $X_\ell^{-+} = \overline{X_{-\ell}^{+-}}$ (Off-diagonal conjugation).

These relations ensure that $\hat{\mathscr{S}}_{s,p}$ is the space of Fourier coefficients for maps taking values in the Lie algebra $\mathfrak{su}(1,1)$. The following lemma implies that the Fourier transform defines an isomorphism of Banach-Lie algebras between $\hat{\mathscr{S}}_{s,p}$ and $\mathscr{S}_{s,p}$.

Lemma 3.2.1. *The space $\hat{\mathscr{S}}_{s,p}$, equipped with the norm*

$$[\![X]\!]_{s,p} := \sup_{\ell\in\mathbb{Z}} \langle \ell\rangle^p e^{s|\ell|} [\![X_\ell]\!] = \|X\|_{s,p}^{Mat(2,\mathbb{C})},$$

is a Banach-Lie algebra with respect to the Lie bracket $[X,Y]_\star = X\star Y - Y\star X$. In particular, the bracket is continuous and satisfies the estimate:

$$[\![[X,Y]_\star]\!]_{s,p} \le 4C_\star(p)[\![X]\!]_{s,p}[\![Y]\!]_{s,p}, \tag{3.31}$$

where $C_\star(p)$ the constant defined in (3.30).

Proof. The space $\hat{\mathscr{S}}_{s,p}$ is a closed subspace of the Banach space $\ell^\infty_{s,p}(\mathrm{Mat}(2,\mathbb{C}))$. To show it is a Banach-Lie algebra, we verify the algebraic closure and the continuity of the bracket.
Algebraic closure: For any $X, Y \in \hat{\mathscr{S}}_{s,p}$, let $Z = [X,Y]_\star$. The condition $\mathrm{Tr}(Z_\ell) = 0$ follows from the linearity of the trace and the identity $\mathrm{Tr}([A,B]) = 0$. Regarding the diagonal antisymmetry, a direct computation on the $(++)$ component of the convolution yields:

$$\begin{aligned}
-\overline{(X\star Y)_{-\ell}^{++}} &= -\sum_{k\in\mathbb{Z}} \left(\overline{X_{-\ell-k}^{++}Y_k^{++}} + \overline{X_{-\ell-k}^{+-}Y_k^{-+}}\right) \\
&= -\sum_{k\in\mathbb{Z}} \left((-X_{\ell+k}^{++})(-Y_{-k}^{++}) + X_{\ell+k}^{-+}Y_{-k}^{+-}\right) = -(Y\star X)_\ell^{++}.
\end{aligned}$$

It follows that $-\overline{Z_{-\ell}^{++}} = -(Y\star X)_\ell^{++} + (X\star Y)_\ell^{++} = Z_\ell^{++}$. Similarly, evaluating the conjugation of the $(+-)$ component:

$$\begin{aligned}
\overline{(X\star Y)_{-\ell}^{+-}} &= \sum_{k\in\mathbb{Z}} \left(\overline{X_{-\ell-k}^{++}Y_k^{+-}} + \overline{X_{-\ell-k}^{+-}Y_k^{--}}\right) \\
&= \sum_{j\in\mathbb{Z}} \left(X_{\ell-j}^{++}Y_j^{-+} + X_{\ell-j}^{-+}Y_j^{--}\right) = (Y\star X)_\ell^{-+}.
\end{aligned}$$

Thus $\overline{Z_{-\ell}^{+-}} = (Y\star X)_\ell^{-+} - (X\star Y)_\ell^{-+} = -Z_\ell^{-+}$, confirming that $Z_\ell^{-+} = \overline{Z_{-\ell}^{+-}}$.
Continuity of the bracket: The sub-multiplicativity of the convolution product established in (3.29) implies:

$$\begin{aligned}
[\![[X,Y]_\star]\!]_{s,p} &\le [\![X\star Y]\!]_{s,p} + [\![Y\star X]\!]_{s,p} \\
&\le 2C_\star(p)C_{\mathrm{alg}}(\mathrm{Mat}(2,\mathbb{C}))[\![X]\!]_{s,p}[\![Y]\!]_{s,p}.
\end{aligned}$$

Substituting $C_{\mathrm{alg}}(\mathrm{Mat}(2,\mathbb{C})) = 2$, we obtain the bound (3.31). The boundedness of the bracket ensures that $\hat{\mathscr{S}}_{s,p}$ is a Banach-Lie algebra. ∎

4. The functional space $\mathcal{H}_{s,p,R}$. Given $R > 0$, we denote by $\mathcal{H}_{s,p,R}$ the Banach space of functions

$$f : \mathscr{U}_R(\ell^\infty_{s,p}) \to \mathbb{C}$$

of the form $f(u) = \sum_{\alpha\in\mathbb{N}_0^{\mathbb{Z}}} f_\alpha u^\alpha$, with $f_\alpha \in \mathbb{C}$, satisfying

$$\|f\|_{s,p,R} = \sum_{\alpha\in\mathbb{N}^{\mathbb{Z}}} |f_\alpha| v^\alpha_{s,p,R} < \infty, \tag{3.32}$$

where $v_{s,p,R} := (\langle\ell\rangle^{-p} e^{-s|\ell|} R)_{\ell\in\mathbb{Z}}$. Note that, if one introduces the *majorant* $\underline{f}$ of f as the function defined by

$$\underline{f}(u) = \sum_{\alpha\in\mathbb{N}_0^{\mathbb{Z}}} |f_\alpha| u^\alpha,$$

the condition (3.32) can be rewritten as

$$\|f\|_{s,p,R} = \sup_{\|u\|_{s,p}\le R} |\underline{f}(u)| < \infty.$$

Crucially, the majorant of a product of functions precedes the product of the respective majorants., i.e., $\underline{fg} \preceq \underline{f}\,\underline{g}$. This notation means that the inequality holds coefficient-wise for the corresponding formal power series. Consequently, for any u with non-negative components (such as the weight vector $v_{s,p,R}$), one has $\underline{fg}(u) \le \underline{f}(u)\underline{g}(u)$. This property ensures that $\mathcal{H}_{s,p,R}$ is a Banach algebra, namely

$$\|fg\|_{s,p,R} \le \|f\|_{s,p,R}\|g\|_{s,p,R}.$$

It is worth stressing that any $f \in \mathcal{H}_{s,p,R}$ is majorant analytic, as its majorant $\underline{f}$ is the uniform limit of continuous polynomials.

5. The Banach-Lie Algebra $\underline{\mathcal{A}}^\pi_R(\ell^\infty_{s,p}, \hat{\mathscr{S}}_{s,p})$. Given $R > 0$, we define the space $\underline{\mathcal{A}}^\pi_R(\ell^\infty_{s,p}, \hat{\mathscr{S}}_{s,p})$ as the space of analytic maps $X : \mathscr{U}_R(\ell^\infty_{s,p}) \to \hat{\mathscr{S}}_{s,p}$. An element X in this space is identified with a sequence of matrix-valued power series $X(u) = (X_\ell(u))_{\ell\in\mathbb{Z}}$, where each component $X_\ell(u)$ satisfies a *momentum conservation condition*. Specifically, the Taylor expansion of the entries $X^{\sigma\sigma'}_\ell \in \mathcal{H}_{s,p,R}$ involves only multi-indices $\alpha \in \mathbb{N}^{\mathbb{Z}}$ whose *momentum* $\pi(\alpha)$ matches the Fourier index ℓ:

$$X^{\sigma\sigma'}_\ell(u) = \sum_{\alpha\in\mathscr{M}_\ell} X^{\sigma\sigma'}_{\ell,\alpha} u^\alpha, \quad \text{with} \quad \mathscr{M}_\ell := \Big\{\alpha \in \mathbb{N}_0^{\mathbb{Z}} : \pi(\alpha) := \sum_{m\in\mathbb{Z}} m\alpha_m = \ell\Big\}.$$

The momentum conservation condition ensures that each Fourier component X_ℓ of the map transforms, with respect to a time shift t, with the same frequency as the phase factor $e^{\mathrm{i}\ell t}$. This implies that the map depends on t only through u, as shown by the following chain of equalities:

$$e^{\mathrm{i}\ell t} X_\ell(u) = \sum_{\alpha\in\mathscr{M}_\ell} X_{\ell,\alpha} u^\alpha e^{\mathrm{i}\pi(\alpha)t} = \sum_{\alpha\in\mathscr{M}_\ell} X_{\ell,\alpha} \prod_m (u_m e^{\mathrm{i}mt})^{\alpha_m} = X_\ell(u(t)).$$

To equip this space with a Banach structure, we introduce the *majorant* $\underline{X}$ as the sequence of maps $\underline{X} = (\underline{X_\ell})_{\ell\in\mathbb{Z}}$ obtained by replacing each entry with its majorant in $\mathcal{H}_{s,p,R}$, namely $\underline{X}^{\sigma\sigma'}_\ell(u) := \sum_{\alpha\in\mathscr{M}_\ell} |X^{\sigma\sigma'}_{\ell,\alpha}| u^\alpha$. The majorant $\underline{X}$ allows us to define the norm $|X|_{s,p,R}$ by considering the supremum of the $\hat{\mathscr{S}}_{s,p}$-norm over the ball of radius R:

$$|X|_{s,p,R} := \sup_{\|u\|_{s,p}\le R} [\![\underline{X}(u)]\!]_{s,p} = \sup_{\ell\in\mathbb{Z}} \langle\ell\rangle^p e^{s|\ell|} \max_{\sigma,\sigma'=\pm 1} \|X^{\sigma\sigma'}_\ell\|_{s,p,R},$$

which, due to the non-negativity of the coefficients, coincides with evaluating the majorant on the weight vector $v_{s,p,R}$.

It should be observed that any $X \in \underline{\mathcal{A}}^{\pi}_{R}$ is majorant analytic. In fact, Theorem 1.2.24 ensures that $\underline{X}$ is analytic as it is defined by the uniformly bounded family of (majorant) analytic functions $X^{\sigma\sigma'}_{\ell} \in \mathcal{H}_{s,p,R}$.

Lemma 3.2.2. *The space $(\underline{\mathcal{A}}^{\pi}_{R}, |\cdot|_{s,p,R})$ is a Banach-Lie algebra with respect to the pointwise Lie bracket $[X,Y]_\star(u) = X(u) \star Y(u) - Y(u) \star X(u)$, satisfying:*

$$|[X,Y]_\star|_{s,p,R} \leq K|X|_{s,p,R}|Y|_{s,p,R},$$

with $K = 4C_\star(p)$ (see (3.30)*).*

Proof. The momentum condition is preserved by the convolution product: if $\pi(\alpha) = \ell - k$ and $\pi(\beta) = k$, then $\pi(\alpha+\beta) = \ell$, and hence $[X,Y]_\star(u) \in \hat{\mathscr{S}}_{s,p}$ by Lemma 3.2.1. To prove the continuity, we recall that for any $f, g \in \mathcal{H}_{s,p,R}$ one has $\underline{fg} \preceq \underline{f}\underline{g}$, which implies that $\underline{X \star Y} \preceq \underline{X} \star \underline{Y}$ in the sense of power series. Consequently, for any u with non-negative components (and thus for the supremum over the ball), it follows that:

$$\begin{aligned}
|[X,Y]_\star|_{s,p,R} &= \sup_{\|u\|_{s,p}\leq R} [\![\underline{[X,Y]_\star(u)}]\!]_{s,p} \\
&\leq \sup_{\|u\|_{s,p}\leq R} \left([\![(\underline{X}\star\underline{Y})(u)]\!]_{s,p} + [\![(\underline{Y}\star\underline{X})(u)]\!]_{s,p}\right) \\
&\leq 4C_\star(p) \sup_{\|u\|_{s,p}\leq R} [\![\underline{X}(u)]\!]_{s,p} \sup_{\|u\|_{s,p}\leq R} [\![\underline{Y}(u)]\!]_{s,p},
\end{aligned}$$

where we used the algebra property of the $\hat{\mathscr{S}}_{s,p}$ norm established in Lemma 3.2.1 and the fact that the majorant of a product precedes the product of the majorants. ∎

3.2.7 Reduction to constant coefficients

In this subsection, we establish the reducibility of the linear Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \left[\lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{u}{2\lambda}\begin{pmatrix} -i & i \\ -i & i \end{pmatrix}\right]\begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \tag{3.33}$$

where $u = u(t) \in \mathbb{R}$ is a zero-mean, analytic, 2π-periodic function whose sequence of Fourier coefficients is small in the $\ell^{\infty}_{s,p}$ norm. We provide quantitative estimates for the reducing transformation; these results will be crucial in the next subsection to investigate the regularity of the spectral gaps with respect to the potential u. It is worth stressing that, in this dynamical setting, such gaps correspond precisely to the instability intervals of the periodic forcing. For any fixed $n \in \mathbb{N}$, the results hold for all spectral parameters $\lambda \in \mathbb{R}^+$ satisfying $|2\lambda - n| < \gamma \leq 1/2$. Accordingly, throughout the following discussion, we fix a natural number n and we assume that λ remains restricted to this region.

Following the strategy outlined in Subsection 3.2.4, we aim to find a 2π-periodic change of coordinates that reduces the Hamiltonian system (3.33) to constant coefficients. To preserve the algebraic structure, this transformation must take values in the Lie group $SU(1,1)$.

Given the identification between the space of 2π-periodic functions $\mathscr{S}_{s,p}$ and its Fourier coefficients space $\hat{\mathscr{S}}_{s,p}$ established in Subsection 3.2.6 (see Lemma 3.2.1), we seek the generator of this transformation within the functional space $\underline{\mathcal{A}}^{\pi}_{R}(\ell^{\infty}_{s,p}, \hat{\mathscr{S}}_{s,p})$. This choice allows us to track the regularity of the transformation with respect to the potential u and eventually derive estimates for the spectral gaps. In this regard, it is worth noting that the perturbation

$$P(u,\lambda) = \frac{u}{2\lambda}\begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} = \frac{u}{2\lambda}(J_3 - J_1).$$

indeed belongs to $\underline{\mathcal{A}}^{\pi}_R(\ell^\infty_{s,p}, \hat{\mathscr{S}}_{s,p})$, with norm $|P|_{s,p,R} = \frac{R}{2\lambda}$.

Given a map $A \in \underline{\mathcal{A}}^{\pi}_R$, we consider a change of variables $(\xi,\bar{\xi})^T = e^{-A(u,\lambda)}(z,\bar{z})^T$. Here, the exponential e^{-A} and the transformation law are to be understood by identifying the sequence $A(u,\lambda) \in \hat{\mathscr{S}}_{s,p}$ with its corresponding matrix-valued periodic function in $\mathscr{S}_{s,p}$ via the Fourier isomorphism. With this identification, the time derivative $\dot{A}$ acts as a diagonal operator on the Fourier coefficients, namely $(\dot{A})_\ell = i\ell A_\ell$. Under this transformation, the matrix $D(u,\lambda)$ associated with the linear vector field inducing the Hamiltonian dynamics defined in (3.18) is transformed into

$$\mathcal{D}(u,\lambda) = e^{-A(u,\lambda)}D(u,\lambda)e^{A(u,\lambda)} - e^{-A(u,\lambda)}\frac{d}{dt}e^{A(u,\lambda)}. \tag{3.34}$$

By defining the adjoint action as $\mathrm{ad}_A(M) := [A,M]_\star$, where the Lie bracket $[\cdot,\cdot]_\star$ is defined in Lemma 3.2.2, and recalling the standard expansion for the adjoint action and the derivative of the exponential map:

$$e^{-A}Me^{A} = \sum_{k=0}^{\infty}\frac{(\mathrm{ad}_{-A})^k}{k!}M, \qquad e^{-A}\frac{d}{dt}e^{A} = \sum_{k=1}^{\infty}\frac{(\mathrm{ad}_{-A})^{k-1}}{k!}\dot{A},$$

the equation (3.34) can be expressed as:

$$\mathcal{D}(\lambda) = \sum_{k=0}^{\infty}\frac{(\mathrm{ad}_{-A})^k}{k!}(\lambda J_1 + P(\lambda)) - \sum_{k=1}^{\infty}\frac{(\mathrm{ad}_{-A})^{k-1}}{k!}\dot{A}. \tag{3.35}$$

Thanks to the Banach-Lie algebra structure of $\underline{\mathcal{A}}^{\pi}_R$, the series in (3.35) is well-defined and convergent, provided that the norm $|A|_{s,p,R}$ is sufficiently small (see Appendix A for details).

We remark that we restrict our search to generators A belonging to the space $\underline{\mathcal{A}}^{\pi}_R$, thereby enforcing the momentum conservation condition. This assumption is not strictly necessary for the reduction to constant coefficients. However, since the initial perturbation trivially satisfies this property and the Lie bracket preserves it, the property remains valid at each step of the contractive iteration we shall implement to construct the solution. While not required for the reducibility itself, this selection rule ensures that the transformed system inherits the same symmetry, which will be crucial in the subsequent sections when studying the derivatives of the spectral gaps.

As already noted in Subsection 3.2.4, the reduction to constant coefficients for the component lying in the range of $\mathcal{L}_{n/2}$ can be recast as a fixed-point problem for the operator ϕ_P, defined as

$$\phi_P(A) = \mathcal{L}_\lambda^{-1}\Pi_{rg}\left(\mathcal{B}(A)\Pi_{rg}\left(e^{\mathrm{ad}_{-A}}P\right)\right),$$

where Π_{rg} denotes the projector onto the range of the operator $\mathcal{L}_{n/2}$. Here, $\mathcal{B}(A)$ represents the inverse of the map $X \mapsto \sum_{k=1}^{\infty}\frac{1}{k!}(\mathrm{ad}_{-A})^{k-1}X$ (the details of which are provided in Lemma A.5.1), while $\mathcal{L}_\lambda^{-1}$ is the inverse of $\mathcal{L}_\lambda$, which is well defined on $\mathrm{range}(\mathcal{L}_{n/2})$.

Before stating and proving the fixed-point theorem, it is of crucial importance to make some comments on the operator $\mathcal{L}_\lambda^{-1}$: since $\mathcal{L}_\lambda$ acts diagonally on the components of the generator A, namely

$$(\mathcal{L}_\lambda A)_\ell^{\sigma\sigma'} = \mathrm{i}(\ell - \lambda(\sigma-\sigma'))A^{\sigma\sigma'}, \quad \text{for all } \sigma,\sigma' = \pm 1,\ \ell\in\mathbb{Z},$$

$\mathcal{L}_\lambda^{-1}$ naturally preserves both the momentum conservation condition and the range of the operator $\mathcal{L}_{n/2}$. Furthermore, we have the following lemma:

Lemma 3.2.3. *The inverse $\mathcal{L}_\lambda^{-1}$, viewed as an operator from $\Pi_{rg}(\underline{\mathcal{A}}_R^\pi)$ into itself, is bounded. Specifically, for any $X \in \Pi_{rg}(\underline{\mathcal{A}}_R^\pi)$, we have the estimate:*

$$|\mathcal{L}_\lambda^{-1} X|_{s,p,R} \le 2|X|_{s,p,R}.$$

Proof. Recall that for $X \in \Pi_{rg}(\underline{\mathcal{A}}_R^\pi)$, the action of $\mathcal{L}_\lambda^{-1}$ on the Fourier components $X_\ell^{\sigma\sigma'}$ is given by the division by the factor $\mathrm{i}(\ell - (\sigma - \sigma')\lambda)$. Consequently, the majorant of the transformed component satisfies:

$$\underline{\left(\frac{1}{\mathrm{i}(\ell - (\sigma - \sigma')\lambda)} X_\ell^{\sigma\sigma'}\right)} = \frac{1}{|\ell - (\sigma - \sigma')\lambda|} \underline{X_\ell^{\sigma\sigma'}}.$$

Noting that $|\ell - (\sigma - \sigma')\lambda| \ge 1/2$ for all terms in the range of $\mathcal{L}_{n/2}$ within the region $|2\lambda - n| < \gamma \le 1/2$, we obtain:

$$\begin{aligned}
|\mathcal{L}_\lambda^{-1} X|_{s,p,R} &= \sup_{\|u\|_{s,p} \le R} [\![\underline{\mathcal{L}_\lambda^{-1} X}(u)]\!]_{s,p} \\
&= \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \max_{\sigma,\sigma'=\pm 1} \sup_{\|u\|_{s,p} \le R} \left| \underline{\left(\frac{1}{\mathrm{i}(\ell - (\sigma - \sigma')\lambda)} X_\ell^{\sigma\sigma'}\right)}(u) \right| \\
&= \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \max_{\sigma,\sigma'=\pm 1} \frac{1}{|\ell - (\sigma - \sigma')\lambda|} \sup_{\|u\|_{s,p} \le R} |\underline{X_\ell^{\sigma\sigma'}}(u)| \\
&= \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \max_{\sigma,\sigma'=\pm 1} \frac{1}{|\ell - (\sigma - \sigma')\lambda|} \|X_\ell^{\sigma\sigma'}\|_{s,p,R} \\
&\le 2 \sup_{\ell \in \mathbb{Z}} \langle \ell \rangle^p e^{s|\ell|} \max_{\sigma,\sigma'=\pm 1} \|X_\ell^{\sigma\sigma'}\|_{s,p,R} = 2|X|_{s,p,R},
\end{aligned}$$

yielding the claim. ■

Given $n \ge 1$, in the following theorem, we require the smallness condition on P to be satisfied for all $\lambda \in \mathbb{R}^+$ such that $|2\lambda - n| \le 1/2$. This ensures that both the existence of the fixed point and the derived estimates are valid uniformly throughout this spectral region.

Theorem 3.2.4 (Reduction to constant coefficients in the range of $\mathcal{L}_{n/2}$)**.** *Let $|P|_{s,p,R} \le (48K)^{-1}$, where the constant K is defined in Lemma 3.2.2. Then the map ϕ_P has a unique fixed point A_P in the ball $\overline{\mathscr{U}}_{8|P|_{s,p,R}}(\Pi_{rg}(\underline{\mathcal{A}}_R^\pi))$. The fixed point is such that $A_{P=0} = 0$ and satisfies the following asymptotic expansion for small P:*

$$A_P = \mathcal{L}_\lambda^{-1} \Pi_{rg} P + O(|P|_{s,p,R}^2).$$

Furthermore, A_P is analytic in λ and satisfies the estimate

$$|\partial_\lambda A_P|_{s,p,R} \le 96|P|_{s,p,R}.$$

Finally, $\mathcal{L}_\lambda A_P$, which is also analytic in λ, is estimated as $|\mathcal{L}_\lambda A_P|_{s,p,R} \le 4|P|_{s,p,R}$ and satisfies the following asymptotic expansion for small P:

$$\mathcal{L}_\lambda A_P = \Pi_{rg} P + O(|P|_{s,p,R}^2).$$

Proof. The proof follows from the application of the general fixed point framework developed in Appendix A to the Banach-Lie algebra $(\underline{\mathcal{A}}_R^\pi, |\cdot|_{s,p,R})$. According to Lemma 3.2.2, this space satisfies the sub-multiplicative property with constant K. We identify the operators in the appendix with our specific case by setting $\Pi = \Pi_{rg}$, $T = \mathcal{L}_\lambda^{-1}$, and $X = A$.

Existence and uniqueness. By hypothesis, $|P|_{s,p,R} \le (48K)^{-1} < (24K)^{-1}$. This satisfies the smallness condition required by the Fixed Point Theorem A.6.1. Consequently, ϕ_P is a contraction on the ball

$\overline{\mathscr{U}}_{8|P|_{s,p,R}}$, ensuring the existence of a unique fixed point A_P such that $A_{P=0} = 0$. The first-order expansion follows directly from the same theorem:

$$A_P = \mathcal{L}_\lambda^{-1}\Pi_{rg}P + O(|P|^2_{s,p,R}).$$

Analyticity in λ and estimate for the derivative. Since $\mathcal{L}_\lambda^{-1}$ and P are analytic in λ, Corollary A.6.2 implies that A_P is also analytic in λ. To derive the bound on $\partial_\lambda A_P$, we use the estimate provided in the same corollary:

$$|\partial_\lambda A_P|_{s,p,R} \leq \frac{4\|\partial_\lambda \mathcal{L}_\lambda^{-1}\||P|_{s,p,R} + 8|\partial_\lambda P|_{s,p,R}}{1 - 24K|P|_{s,p,R}}.$$

Given the condition $|P|_{s,p,R} \leq (48K)^{-1}$, the denominator is bounded as $(1-24K|P|)^{-1} \leq (1-1/2)^{-1} = 2$. Using the specific bounds for the derivatives of the operators in this functional setting, and the fact that $\partial_\lambda P = -\lambda^{-1}P$, where $\lambda \geq 1/4$, we obtain:

$$|\partial_\lambda A_P|_{s,p,R} \leq 96|P|_{s,p,R}.$$

Estimates for $\mathcal{L}_\lambda A_P$. Setting $U = \mathcal{L}_\lambda$ as the inverse of T on range$(\mathcal{L}_{n/2})$, we apply Lemma A.7.1. Since A_P is in the range of $\mathcal{L}_\lambda^{-1}$, the expression $\mathcal{L}_\lambda A_P$ is well-defined and satisfies:

$$|\mathcal{L}_\lambda A_P|_{s,p,R} \leq 4|P|_{s,p,R}.$$

Finally, the asymptotic expansion for $\mathcal{L}_\lambda A_P$ is given by:

$$\mathcal{L}_\lambda A_P = \Pi_{rg}P + O(|P|^2_{s,p,R}),$$

which concludes the proof. ■

As a consequence of Theorem 3.2.4, the matrix associated with the linear vector field (3.33) is now reduced to the form

$$\lambda J_1 + \tilde{\mathcal{Z}}(u,\lambda,t), \qquad \text{with} \quad \tilde{\mathcal{Z}} \in \mathrm{Ker}(\mathcal{L}_{n/2}).$$

By construction, and considering the expression of the conjugated matrix given in (3.24), $\tilde{\mathcal{Z}}$ is defined by

$$\tilde{\mathcal{Z}} = \Pi_{ker}\bigg(-\sum_{k=1}^{\infty}\frac{\mathrm{ad}^{k-1}_{-A}}{k!}\mathcal{L}_\lambda A + e^{\mathrm{ad}_{-A}}P\bigg), \tag{3.36}$$

where Π_{ker} denotes the projector onto the kernel of $\mathcal{L}_{n/2}$, and A denotes the fixed point provided by Theorem 3.2.4. Using the fact that $\mathcal{L}_\lambda A \in \mathrm{range}(\mathcal{L}_{n/2})$, and isolating the term of order $O(|P|_{s,p,R})$, we get

$$\tilde{\mathcal{Z}} = \Pi_{ker}P + \tilde{\mathcal{P}}_\infty := \Pi_{ker}P + \Pi_{ker}\bigg((e^{\mathrm{ad}_{-A}} - \mathrm{Id})P - \sum_{k=2}^{\infty}\frac{\mathrm{ad}^{k-1}_{-A}}{k!}\mathcal{L}_\lambda A\bigg). \tag{3.37}$$

The following corollary, proved in Appendix A (see Lemma A.7.2), shows that $\tilde{\mathcal{P}}_\infty$ is indeed of order $O(|P|^2_{s,p,R})$ and provides an asymptotic expansion near $P=0$.

Corollary 3.2.5. *The following bound holds:*

$$|\tilde{\mathcal{P}}_\infty|_{s,p,R} \leq 64K|P|^2_{s,p,R}.$$

Furthermore, $\tilde{\mathcal{P}}_\infty$ satisfies the following asymptotic expansion for small P:

$$\tilde{\mathcal{P}}_\infty = \Pi_{ker}\bigg([\Pi_{ker}P, \mathcal{L}_\lambda^{-1}\Pi_{rg}P]_\star + \frac{1}{2}[\Pi_{rg}P, \mathcal{L}_\lambda^{-1}\Pi_{rg}P]_\star\bigg) + O(|P|^3_{s,p,R}).$$

To understand the final step of the reduction, it is useful to recall the explicit structure of the kernel of $\mathcal{L}_{n/2}$, namely

$$\operatorname{Ker}(\mathcal{L}_{n/2}) = \left\{ \begin{pmatrix} X_0^{++} & X_n^{+-}e^{\mathrm{i}nt} \\ X_{-n}^{-+}e^{-\mathrm{i}nt} & X_0^{--} \end{pmatrix} \; : \; X_0^{--} = -X_0^{++} \in \mathrm{i}\mathbb{R}, \; X_{-n}^{-+} = \overline{X_n^{+-}} \in \mathbb{C} \right\}. \tag{3.38}$$

The elements in this kernel are still time-dependent, but we can then reach a frame where the system is stationary by applying an additional $SU(1,1)$ transformation, specifically a rotation $S_n(t)$ that counteracts these oscillations:

$$S_n(t) = \begin{pmatrix} e^{-\mathrm{i}nt/2} & 0 \\ 0 & e^{\mathrm{i}nt/2} \end{pmatrix}.$$

Under this final change of variables, the resonant terms are mapped to constant matrices, effectively completing the reduction to constant coefficients. This leads to the conclusive theorem for this subsection. As before, given $n \geq 1$, we assume the smallness condition on P to hold for all $\lambda \in \mathbb{R}^+$ within the closed interval $|2\lambda - n| \leq 1/2$. Then the following result holds for all λ in this region.

Theorem 3.2.6 (Reducibility to constant coefficients). *Given $p > 1$ and $s \geq 0$, let $|P|_{s,p,R} \leq (48K)^{-1}$, where the constant K is defined in Lemma 3.2.2, and let A_P be the fixed point provided by Theorem 3.2.4. Then, the transformation defined by the composition $\eta = S_n e^{-A_P} z$ reduces the Hamiltonian system* (3.33) *to a constant coefficient system of the form:*

$$\frac{d}{dt}\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = [(\lambda - n/2)J_1 + \mathcal{Z}_0(u,\lambda) + \mathcal{P}_\infty(u,\lambda)] \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix},$$

where,

$$\mathcal{Z}_0 = \begin{pmatrix} 0 & P_n^{+-} \\ \overline{P_n^{+-}} & 0 \end{pmatrix} = \frac{1}{2\lambda}\begin{pmatrix} 0 & \mathrm{i}u_n \\ -\mathrm{i}\bar{u}_n & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P}_\infty = \begin{pmatrix} \mathrm{i}a & b \\ \bar{b} & -\mathrm{i}a \end{pmatrix}. \tag{3.39}$$

The coefficients $a(u,\lambda) \in \mathbb{R}$ and $b(u,\lambda) \in \mathbb{C}$ are majorant analytic in u and λ, more precisely they belong to $\mathcal{H}_{s,p,R}$, and satisfy the following estimates:

$$\|a\|_{s,p,R}, \|b\|_{s,p,R}, \|\partial_\lambda a\|_{s,p,R}, \|\partial_\lambda b\|_{s,p,R} \leq C|P|^2_{s,p,R},$$

for a suitable constant $C > 0$, uniformly in the region $|2\lambda - n| \leq 1/2$.

Proof. Let $\tilde{\mathcal{D}} = \lambda J_1 + \tilde{\mathcal{Z}}$ be the resonant form provided by Theorem 3.2.4, with $\tilde{\mathcal{Z}} = \Pi_{ker}P + \tilde{\mathcal{P}}_\infty$ defined in (3.36) and (3.37). Following the change of variables $\eta = S_n e^{-A} z$, the reduced matrix is obtained as

$$\mathcal{D} = S_n \tilde{\mathcal{D}} S_n^{-1} + \dot{S}_n S_n^{-1}.$$

The diagonal part λJ_1 commutes with S_n, while the derivative of the rotation contributes the shift:

$$\dot{S}_n S_n^{-1} = \begin{pmatrix} -\mathrm{i}n/2 & 0 \\ 0 & \mathrm{i}n/2 \end{pmatrix} = -\frac{n}{2}J_1,$$

yielding the term $(\lambda - n/2)J_1$. For the resonant part, the adjoint action of S_n on any $M \in \operatorname{Ker}(\mathcal{L}_{n/2})$ cancels the time-dependence $e^{\pm \mathrm{i}nt}$. Specifically,

$$\mathcal{Z}_0 := S_n(\Pi_{ker}P)S_n^{-1} \quad \text{and} \quad \mathcal{P}_\infty := S_n \tilde{\mathcal{P}}_\infty S_n^{-1} \tag{3.40}$$

are constant matrices. The explicit form of $\mathcal{Z}_0$ follows from the Fourier expansion of the perturbation P in the kernel of $\mathcal{L}_{n/2}$. The majorant analyticity and the bounds for the coefficients of $\mathcal{P}_\infty$ are a direct consequence of the estimates for $\tilde{\mathcal{P}}_\infty$ established in Corollary 3.2.5 and the analyticity of the fixed point A_P with respect to λ. ■

3.2.8 Analysis of the spectral gaps and Liouville-Arnold integrability for the KdV on $\mathbb{T}$

In this subsection, we derive regularity properties for the spectral gaps of the Schrödinger operator with potential $u(t) = \sum_{\ell\in\mathbb{Z}} e^{i\ell t} u_\ell$, where u is identified with the sequence of its Fourier coefficients $(u_\ell)_{\ell\in\mathbb{Z}} \in \ell^\infty_{s,p}$, with $p > 1, s \geq 0$, which are such that $u_{-\ell} = \bar{u}_\ell$ and $u_0 = 0$. These regularity results are essential for applying the theorems established in the previous chapter. In particular, they allow us to prove the complete integrability of the KdV equation, characterized by a foliation into infinite-dimensional invariant tori and the existence of analytic action-angle coordinates that are well-defined in an open set of the phase space in the sense of Theorem 2.6.1.

As already pointed out in Subsection 3.2.4, to establish the complete integrability of the KdV equation by applying the theory developed in the previous chapters, it is necessary to express these spectral quantities in terms of variables that satisfy the standard canonical relations. To this end, we first perform the change of variables induced by the operator $|\partial|^{1/2} : \ell^\infty_{s,p} \to \ell^\infty_{s,p-1/2}$, acting as a Fourier multiplier:

$$u := |\partial|^{1/2} z, \quad \text{with} \quad u_\ell = \langle \ell \rangle^{1/2} z_\ell.$$

Theorem 3.2.7 (Liouville-Arnold Integrability of KdV on $\mathbb{T}$)**.** *Let $p > 2$ and $s \geq 0$. Consider the KdV equation on $\mathbb{T}$ expressed in the canonical variables $z \in \ell^\infty_{s,p}$:*

$$\dot{z}_k = i\frac{\partial H_{KdV}}{\partial \bar{z}_k}, \quad H_{KdV}(z) = \sum_{k\in\mathbb{N}} k^3 |z_k|^2 + \sum_{\substack{k+\ell+m=0 \\ k,\ell,m\neq 0}} \sqrt{|k\ell m|}\, z_k z_\ell z_m.$$

There exist $0 < \xi_- \leq \xi_+$ such that, for any sequence $\xi = (\xi_k)_{k\in\mathbb{N}}$ of positive real numbers satisfying

$$\xi_- \leq \inf_{k\in\mathbb{N}} e^{2s|k|} \langle k \rangle^{2p} \xi_k \leq \sup_{k\in\mathbb{N}} e^{2s|k|} \langle k \rangle^{2p} \xi_k \leq \xi_+,$$

the following hold in a neighborhood of the torus $\mathcal{T}_\xi = \{z \in \ell^\infty_{s,p} : |z_k|^2 = \xi_k\}$.

(i) ***Foliation by Infinite Invariant Tori:*** *The phase space is locally foliated by analytic invariant tori analytically diffeomorphic to $\mathbb{T}^\infty$. Each torus is a level set of the spectral gaps and supports an almost-periodic Hamiltonian flow.*

(ii) ***Action-Angle Coordinates:*** *There exists an analytic symplectomorphism $\Psi : z \mapsto (x, \theta)$ that introduces action-angle coordinates in a neighborhood of $\mathcal{T}_\xi$.*

(iii) ***Integration by Quadratures:*** *In these coordinates, the Hamiltonian $H_{KdV} \circ \Psi^{-1}$ depends only on the actions x, and the Hamilton's equations reduce to the linear flow:*

$$\begin{cases} \dot{x}_k = 0 \\ \dot{\theta}_k = \omega_k(x) \end{cases}, \quad k \in \mathbb{N},$$

where the frequencies are given by $\omega_k(x) = \partial_{x_k}(H_{KdV} \circ \Psi^{-1})$.

The heuristic expansion $g_n = 2|u_n| + O(|u_n|^3)$ discussed in Subsection 3.2.5 provides a formal bridge between the decay of the potential's Fourier coefficients and the size of the spectral gaps, recovering the $H^s \leftrightarrow h^s$ correspondence established in [MO75]. However, a rigorous proof of complete integrability in our ℓ^∞-based framework requires moving beyond the leading-order approximation of the spectral gaps.

According to Theorem 3.2.6, for any $n \in \mathbb{N}$, the Hamiltonian system (3.33) can be reduced to constant coefficients for all λ in the region $|2\lambda - n| \leq 1/2$. This reduction yields a linear system whose dynamics is governed by the matrix:

$$\mathcal{D}(u, \lambda) = (\lambda - n/2)J_1 + \mathcal{Z}_0(u, \lambda) + \mathcal{P}_\infty(u, \lambda).$$

Explicitly, using the same notation as in formula (3.39),

$$\mathcal{D}(u, \lambda) = \begin{pmatrix} \mathrm{i}(\lambda - n/2 + a(u, \lambda)) & \mathrm{i}\dfrac{u_n}{2\lambda} + b(u, \lambda) \\ -\mathrm{i}\dfrac{\bar{u}_n}{2\lambda} + \bar{b}(u, \lambda) & -\mathrm{i}(\lambda - n/2 + a(u, \lambda)) \end{pmatrix}. \tag{3.41}$$

Since the constant matrix $\mathcal{D}$ is an element of the Lie algebra $\mathfrak{su}(1, 1)$, it is traceless with eigenvalues satisfying:

$$\mu_\pm(u, \lambda) = \pm\sqrt{-\det(\mathcal{D}(u, \lambda))}.$$

The spectral nature of the system is thus encoded in the sign of this determinant. When $\det(\mathcal{D}) > 0$, the eigenvalues are purely imaginary, and the system's evolution is conjugate to a harmonic oscillator. On the other hand, a negative determinant, $\det(\mathcal{D}) < 0$, results in real eigenvalues and an unbounded dynamics. The regions in the energy spectrum $E = \lambda^2$ characterized by this instability correspond exactly to the spectral gaps.

By imposing the instability condition $\det(\mathcal{D}(u, \lambda)) \leq 0$, we can determine the endpoints of the n-th spectral gap. Given the structure of the reduced matrix in (3.41), the determinant is given by

$$\det(\mathcal{D}(u, \lambda)) = \left(\lambda - \frac{n}{2} + a(u, \lambda)\right)^2 - \left|\frac{\mathrm{i}u_n}{2\lambda} + b(u, \lambda)\right|^2.$$

The boundaries of the gap, λ_+ and λ_-, are the roots of the equation $\det(\mathcal{D}(u, \lambda)) = 0$. In the region $|2\lambda - n| \leq 1/2$, these roots satisfy the equation

$$\lambda_\pm - \frac{n}{2} + a(u, \lambda_\pm) = \pm\left|\frac{\mathrm{i}u_n}{2\lambda_\pm} + b(u, \lambda_\pm)\right|. \tag{3.42}$$

The presence of the term $a(u, \lambda)$ induces a shift in the center of the gap, while the modulus on the right-hand side, which includes the higher-order corrections $b(u, \lambda)$, determines the width of the instability interval.

Therefore, by looking at the structure of the equation (3.42) defining the end points of the spectral gaps, it is clear that the first and most fundamental step is to investigate the regularity of the coefficients α and β defined by:

$$\begin{pmatrix} \mathrm{i}\alpha & \beta \\ \bar{\beta} & -\mathrm{i}\alpha \end{pmatrix} := \mathcal{P}_\infty \circ |\partial|^{1/2}.$$

From now on, the proof is articulated in the following main steps:

1. Analysis of the regularity of the coefficients α e β.

2. Majorant analyticity of the spectral gaps.

3. Derivatives of the spectral gaps and proof of the main theorem.

Step 1. Let us start by proving the following proposition, whose technical ingredients are provided in Appendix B.

Proposition 3.2.8 (Regularity of the coefficients α and β)**.** *Given $p > 3/2$ and $s \geq 0$, for sufficiently small $R > 0$, the maps $\alpha, \beta : \mathscr{U}_R(\ell^\infty_{s,p}) \to \mathbb{C}$ (where α is real-valued) are majorant analytic and depends analytically on λ. Specifically, $\alpha, \beta \in \mathcal{H}_{s,p,R}$ and satisfy the bounds:*

$$\|\alpha\|_{s,p,R} \leq \frac{n}{\lambda^2} R^2, \quad \|\beta\|_{s,p,R} \leq \frac{n^{3/2}}{\lambda^2} R^2 \frac{e^{-s|n|}}{\langle n\rangle^p} \tag{3.43}$$

and

$$\|\partial_{z_j}\alpha\|_{s,p,R} \leq \frac{n}{\lambda^2} R \frac{e^{-s|j|}}{\langle j\rangle^p}, \quad \|\partial_{z_j}\beta\|_{s,p,R} \leq \frac{n^{3/2}}{\lambda^2} R \frac{e^{-s|j-n|}}{\langle j-n\rangle^p}. \tag{3.44}$$

Proof. The majorant analyticity of α and β follows from Theorem 3.2.6 and from the fact that if $f \in \mathcal{H}_{s,p-1/2,R}$, then $f \circ |\partial|^{1/2} \in \mathcal{H}_{s,p}$. Analyticity in λ is also inherited from Theorem 3.2.6.

Using the structure of $\ker(\mathcal{L}_{n/2})$ given in (3.38), and the fact that $\mathcal{P}_\infty = S_n \tilde{\mathcal{P}}_\infty S_n^{-1}$ (see (3.40)), one immediately deduces that

$$\mathrm{i}\alpha = \tilde{\mathcal{P}}^{++}_{\infty,0} \circ |\partial|^{1/2} \quad \text{and} \quad \beta = \tilde{\mathcal{P}}^{+-}_{\infty,n} \circ |\partial|^{1/2}. \tag{3.45}$$

Moreover, by Lemma B.4.1, one can deduce the following bound:

$$\sup_{\ell\in\mathbb{Z}} \langle \ell\rangle^{p-1/2} e^{s|\ell|} \max_{\sigma\sigma'=\pm 1} \|\tilde{\mathcal{P}}^{\sigma\sigma'}_{\infty,\ell} \circ |\partial|^{1/2}\|_{s,p,R} \leq \frac{n}{\lambda^2} R^2, \tag{3.46}$$

for $R > 0$ sufficiently small. Therefore, combining (3.46) and (3.45), we get the bounds (3.43).

Lemma B.4.1, states also that

$$\sup_{\ell\in\mathbb{Z}} \langle \ell\rangle^{p-1/2} e^{s|\ell|} \max_{\sigma\sigma'=\pm 1} \|\partial_{z_j}(\tilde{\mathcal{P}}^{\sigma\sigma'}_{\infty,\ell} \circ |\partial|^{1/2})\|_{s,p,R} \leq \frac{n}{\lambda^2} R,$$

from which one can easily derive the bounds in (3.44). ■

Step 2. We now have all the necessary ingredients to study the regularity of the spectral gaps.

Remark 3.2.9. *Throughout this subsection, K denotes a generic positive constant, independent of n, z, and R, whose value may change at each occurrence. Furthermore, since we need to recall the majorant norm $|\cdot|_{r,s}$ for the power-trigonometric series (introduced in Chapter 1), we denote the majorant analytic norm in the z variables by $\|\cdot\|_{a,p,R}$ and the corresponding weights by $\langle j\rangle^p e^{a|j|}$, where $a \geq 0$. This choice of notation is intended to avoid any confusion between the decay of the z variables and the regularity parameter s.*

Let us rewrite formula (3.42) as

$$\phi(z, \lambda_\pm) = \pm \left| \frac{i\sqrt{n} z_n}{2} + \lambda_\pm \beta(z, \lambda_\pm) \right|, \tag{3.47}$$

where ϕ is the function defined by

$$\phi(z, \lambda) := \lambda(\lambda - n/2 + \alpha(z, \lambda)). \tag{3.48}$$

Since, by Proposition 3.2.8, α is analytic as a function of λ within the region $|2\lambda - n| \leq 1/2$, it follows that ϕ inherits the same regularity properties. In particular, ϕ is differentiable with respect to λ and its derivative is

$$\partial_\lambda \phi(z, \lambda) = 2\lambda - n/2 + \alpha(z, \lambda) + \lambda \partial_\lambda \alpha(z, \lambda).$$

This derivative can be bounded from below by a constant proportional to n, provided the domain is suitably restricted. For instance, assuming $|2\lambda - n| \leq 1/8$, one ensures that $2\lambda - n/2 \geq n/4$ for all $n \geq 1$. Furthermore, from Proposition 3.2.8, we have $\|\alpha\|_{a,p,R} \leq n\lambda^{-2}R^2$. Since in the considered region $\lambda \sim n/2$, this term is $O(R^2 n^{-1})$. The derivative $\partial_\lambda \alpha$ can be estimated via a standard Cauchy estimate, yielding a bound proportional to R^2. Consequently, for R sufficiently small, the perturbation terms $\alpha + \lambda\partial_\lambda\alpha$ are small compared to $n/4$. Thus, we conclude that

$$\inf_{\substack{\|z\|_{a,p}\leq R \\ |2\lambda - n|\leq 1/8}} |\partial_\lambda \phi(z,\lambda)| \geq Kn,$$

ensuring the invertibility of the map.

This lower bound on the derivative allows us to control the separation between the roots $\lambda_+(z)$ and $\lambda_-(z)$. By the Mean Value Theorem applied to ϕ with respect to λ, there exists a value $\bar{\lambda}$ in the segment $[\lambda_-(z), \lambda_+(z)]$ such that:

$$\phi(z,\lambda_+(z)) - \phi(z,\lambda_-(z)) = \partial_\lambda \phi(z,\bar{\lambda})(\lambda_+(z) - \lambda_-(z)).$$

Recalling the definition of $\phi(z,\lambda_\pm)$ and the lower bound $|\partial_\lambda\phi(z,\bar{\lambda})| \geq Kn$, we have:

$$\begin{aligned}
|\lambda_+(z) - \lambda_-(z)| &= \frac{|\phi(z,\lambda_+(z)) - \phi(z,\lambda_-(z))|}{|\partial_\lambda \phi(z,\bar{\lambda})|} \\
&\leq \frac{1}{Kn}\left(\left|\frac{i\sqrt{n}z_n}{2} + \lambda_+\beta(z,\lambda_+)\right| + \left|\frac{i\sqrt{n}z_n}{2} + \lambda_-\beta(z,\lambda_-)\right|\right) \\
&\leq \frac{1}{Kn}\left(\sqrt{n}|z_n| + |\lambda_+\beta(z,\lambda_+)| + |\lambda_-\beta(z,\lambda_-)|\right).
\end{aligned}$$

From Proposition 3.2.8, we have $\|\beta\|_{a,p,R} \leq Kn^{3/2}\lambda^{-2}R^2 e^{-a|n|}\langle n\rangle^{-p}$. Since $\lambda \sim n/2$, the terms $\lambda_\pm\beta(z,\lambda_\pm)$ are of order $\sqrt{n}R^2 e^{-a|n|}\langle n\rangle^{-p}$. Similarly, the contribution from the potential is given by $\sqrt{n}|z_n| \leq \sqrt{n}Re^{-a|n|}\langle n\rangle^{-p}$.

By combining these contributions, the numerator is bounded by $K\sqrt{n}(R + R^2)e^{-a|n|}\langle n\rangle^{-p}$. We then conclude that the following estimate holds uniformly for all $z \in \mathscr{U}_R(\ell^\infty_{a,p})$:

$$|\lambda_+(z) - \lambda_-(z)| \leq KR\frac{e^{-a|n|}}{\sqrt{n}\langle n\rangle^p}.$$

Finally, the estimate for the sum $\lambda_+(z)+\lambda_-(z)$ follows directly from the fact that both roots belong to the region $|2\lambda - n| \leq 1/8$. This condition implies $|\lambda_\pm| \leq n/2 + 1/16$, and summing these contributions yields:

$$|\lambda_+(z) + \lambda_-(z)| \leq |\lambda_+| + |\lambda_-| \leq n + 1/8 \leq Kn,$$

which again holds uniformly for all $z \in \mathscr{U}_R(\ell^\infty_{a,p})$ and $n \geq 1$.

By combining the previous results, we can now estimate the term $(\lambda_+(z)^2 - \lambda_-(z)^2)$. Using the identity $a^2 - b^2 = (a-b)(a+b)$, we can write:

$$|\lambda_+(z)^2 - \lambda_-(z)^2| = |\lambda_+(z) - \lambda_-(z)| \cdot |\lambda_+(z) + \lambda_-(z)|.$$

By substituting the uniform estimates obtained above, we get:

$$|\lambda_+(z)^2 - \lambda_-(z)^2| \leq KR\sqrt{n}\frac{e^{-a|n|}}{\langle n\rangle^p}.$$

These quantities will be used to construct the action variables for the Korteweg-de Vries equation. In order for the actions to have the correct physical dimension (consistent with the order of $|z|^2$) and to map properly into the considered sequence spaces, it is necessary to introduce a rescaling factor n^{-1}. Indeed, by squaring the previous estimate and dividing by n, we obtain the bound for rescaled squared gap lengths:

$$\gamma_n(z) := \frac{(\lambda_+(z)^2 - \lambda_-(z)^2)^2}{n} \leq KR^2 \frac{e^{-2a|n|}}{\langle n\rangle^{2p}}, \tag{3.49}$$

which holds uniformly for all $z \in \mathscr{U}_R(\ell^\infty_{a,p})$.

However, a significant challenge arises from the implicit definition of the gaps via the modulus $|\cdot|$ in (3.42), which introduces a singularity at the origin ($z = 0$). To establish the regularity of the gaps without explicitly addressing the singularity, we exploit the possibility of constructing local symplectic polar coordinates (y, φ) around any infinite torus which is sufficiently far form the origin, as explained in Subsection 3.1.1. Namely, let $w = (w_k)_{k\in\mathbb{N}}$ be the weight defined by $w_k = e^{2a|k|}\langle k\rangle^{2p}$. For a fixed sequence $\xi = (\xi_k)_{k\in\mathbb{N}}$ of strictly positive real numbers, we assume that there exist constants $0 < \xi_- \leq \xi_+$ such that:

$$0 < \xi_- \leq \inf_{k\in\mathbb{N}} w_k\xi_k \leq \sup_{k\in\mathbb{N}} w_k\xi_k \leq \xi_+ < \frac{R^2}{4}. \tag{3.50}$$

Under this assumption, for small $r > 0$, we define the local chart Φ_ξ as in (3.3)

$$\Phi_\xi : \mathscr{U}_r(\ell^\infty_w) \times \mathbb{T}^\infty \ni (y, \varphi) \mapsto z = (\sqrt{\xi_k + y_k}e^{\mathrm{i}\varphi_k})_{k\in\mathbb{N}} \in \ell^\infty_{a,p}$$

in a neighbourhood of the torus

$$\mathcal{T}_\xi := \left\{z \in \ell^\infty_{a,p} : |z_k|^2 = \xi_k \text{ for all } k \in \mathbb{N}\right\}.$$

The regularity of the spectral gaps in these coordinates is then established through the norm defining the Banach space $\mathcal{A}_{r,s}$ of power-trigonometric series introduced in Definition 3.1.2. Recall that for $f \in \mathcal{A}_{r,s}$, the majorant norm is given by:

$$|f|_{r,s} := \sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}} \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |f_{\nu\alpha}| v^\alpha_{w,r} e^{s|\nu|}, \quad (v_{w,r})_j = rw_j^{-1}.$$

Crucially, the right-hand sides of our estimates for γ_n are composed of functions (such as z_n and $\beta(z, \lambda)$) that are majorant analytic in the z variables even at the origin. This allows us to bound their $|\cdot|_{r,s}$ norm by further estimating them with the $\|\cdot\|_{a,p,R}$ norm.

The transition between these two norms is governed by Proposition 3.1.2. Specifically, for any Hamiltonian $H(z)$ that is majorant analytic in $\mathscr{U}_R(\ell^\infty_{a,p})$, its pullback $\Phi^*_\xi H$ belongs to $\mathcal{A}_{r,s}$ and satisfies:

$$|\Phi^*_\xi H|_{r,s} \leq \|H\|_{a,p,R},$$

provided the parameters r, s, ξ, R satisfy the consistency condition (3.4). This framework leads to the following regularity result:

Theorem 3.2.10. *Given $p > 3/2$ and $a \geq 0$, let $\xi \in \ell^\infty_{2a,2p}$ be a sequence satisfying the condition* (3.50)*, and let $R > 0$ be the analyticity radius of the coefficients α and β (see Proposition 3.2.8). Then, for $r > 0$ and $s \geq 0$ sufficiently small (depending on R, as explained in Proposition 3.1.2), the rescaled squared gaps in the coordinate system (y, φ), namely*

$$\Phi^*_\xi\gamma_n := \frac{((\Phi^*_\xi\lambda_+)^2 - (\Phi^*_\xi\lambda_-)^2)^2}{n},$$

belong to the space $\mathcal{A}_{r,s}$ (see Definition 1.2.1) and satisfy the following bound:

$$\sup_{n\in\mathbb{N}} e^{2a|n|}\langle n\rangle^{2p}|\Phi_\xi^*\gamma_n|_{r,s} \le KR^2,$$

for a suitable positive constant K.

The Theorem above ensures that the rescaled squared gaps are well-behaved as elements of $\mathcal{A}_{r,s}$ and the map $\Gamma := (\Phi_\xi^*\gamma_n)_{n\in\mathbb{N}}$ take values in $\ell^\infty_{2a,2p}$.

Step 3. In order to satisfy the smallness and regularity requirements of Theorem 2.6.1, we must also ensure a suitable bound for the derivatives of Γ.

Specifically, the applicability of the construction of action-angle coordinates requires the finiteness of the following quantity:

$$\mathcal{K}(\gamma) := \max\left\{ r\sup_{j\in\mathbb{N}}\sum_{n=1}^\infty \left|\frac{\partial\Phi_\xi^*\gamma_n}{\partial y_j}\right|_{r,s},\ \sup_{j\in\mathbb{N}} e^{2a|j|}\langle j\rangle^{2p}\sum_{n=1}^\infty\left|\frac{\partial\Phi_\xi^*\gamma_n}{\partial\varphi_j}\right|_{r,s}\right\}. \tag{3.51}$$

To estimate (3.51), we relate the derivatives in the local chart (y,φ) to those in the canonical variables $(z_j,\bar z_j)$ via the relations:[10]

$$\frac{\partial}{\partial y_j} = \frac{1}{2}\left(\frac{1}{\bar z_j}\frac{\partial}{\partial z_j} + \frac{1}{z_j}\frac{\partial}{\partial\bar z_j}\right) \quad\text{and}\quad \frac{\partial}{\partial\varphi_j} = \mathrm{i}\left(z_j\frac{\partial}{\partial z_j} - \bar z_j\frac{\partial}{\partial\bar z_j}\right),$$

and we perform our calculations in the variables z. As already observed, for the rescaled squared gaps γ_n the majorant analytic norm $\|\cdot\|_{a,p,R}$ in the z variables is not well defined due to the implicit presence of the modulus in (3.42), which prevents the Taylor expansion around the origin. Nevertheless, as we shall see from the explicit calculations, the derivatives of γ_n can be bounded by functions involving the coefficients α,β and their derivatives $\partial_{z_j}\alpha, \partial_{z_j}\beta$, for which the majorant norm $\|\cdot\|_{a,p,R}$ is well defined.

Specifically, we shall prove that the terms in (3.51) are dominated by the majorant norms of these bounding functions. Namely, by virtue of Proposition 3.1.3, if we denote such functions by $f^{(n)}$, the quantity $\mathcal{K}(\gamma)$ satisfies:

$$\mathcal{K}(\gamma) \le 2R\sup_{j\in\mathbb{N}}\langle j\rangle^p e^{a|j|}\sum_{n=1}^\infty\left\|\frac{\partial f^{(n)}}{\partial z_j}\right\|_{a,p,R}. \tag{3.52}$$

By substituting the bounds from Proposition 3.2.8 into the expressions for $f^{(n)}$, we shall see that this sum is finite, thus ensuring the applicability of Theorem 2.6.1.

Let us proceed with the explicit calculation of the gradient of the rescaled squared gaps with respect to the canonical variables z_j. From the definition of $\gamma_n(z)$ (see (3.49)), we have:

$$\partial_{z_j}\gamma_n = \frac{(\lambda_+^2-\lambda_-^2)}{n}\left(2\lambda_+\partial_{z_j}\lambda_+ - 2\lambda_-\partial_{z_j}\lambda_-\right). \tag{3.53}$$

The regularity of $\partial_{z_j}\gamma_n$ depends on the behavior of the gradients $\partial_{z_j}\lambda_\pm$. To compute these, we differentiate the relation (3.42) squared:

$$\lambda_\pm^2\left(\lambda_\pm - \frac{n}{2} + \alpha(z,\lambda_\pm)\right)^2 = \left|\frac{i\sqrt{n}z_n}{2} + \lambda_\pm\beta(z,\lambda_\pm)\right|^2. \tag{3.54}$$

[10] We recall that, being the potential u a real-valued function, we have that $z_{-\ell} = \bar z_\ell$.

By differentiating the left-hand side of (3.54) with respect to z_j, and recalling the definition of $\phi(z,\lambda)$ from (3.48), we obtain:

$$2\phi(z,\lambda_\pm)\left[\Big(2\lambda_\pm-\frac{n}{2}+\alpha(z,\lambda_\pm)+\lambda_\pm\partial_\lambda\alpha(z,\lambda_\pm)\Big)\,\partial_{z_j}\lambda_\pm+\lambda_\pm\partial_{z_j}\alpha(z,\lambda_\pm)\right].$$

Similarly, differentiating the right-hand side of (3.54) yields:

$$\begin{aligned}&\Big(-\mathrm{i}\sqrt{n}\bar{z}_n+2\lambda_\pm\bar{\beta}(z,\lambda_\pm)\Big)\left[\frac{\mathrm{i}\sqrt{n}}{2}\delta_{jn}+\Big(\beta(z,\lambda_\pm)+\lambda_\pm\partial_\lambda\beta(z,\lambda_\pm)\Big)\partial_{z_j}\lambda_\pm+\lambda_\pm\partial_{z_j}\beta(z,\lambda_\pm)\right]\\&+\Big(\mathrm{i}\sqrt{n}z_n+2\lambda_\pm\beta(z,\lambda_\pm)\Big)\left[\Big(\bar{\beta}(z,\lambda_\pm)+\lambda_\pm\partial_\lambda\bar{\beta}(z,\lambda_\pm)\Big)\partial_{z_j}\lambda_\pm+\lambda_\pm\partial_{z_j}\bar{\beta}(z,\lambda_\pm)\right].\end{aligned}\tag{3.55}$$

Let us introduce the phase factor

$$e^{\mathrm{i}\theta_\pm}:=\frac{\mathrm{i}\sqrt{n}z_n+2\lambda_\pm\beta(z,\lambda_\pm)}{|\mathrm{i}\sqrt{n}z_n+2\lambda_\pm\beta(z,\lambda_\pm)|},$$

Note that $e^{\mathrm{i}\theta_\pm}$ is well-defined as long as the denominator is non-zero; this condition is guaranteed within the neighborhood of the torus $\mathcal{T}_\xi$ defined by the chart Φ_ξ. Dividing both sides of (3.55) by the factor $\phi(z,\lambda_\pm)=\pm|i\sqrt{n}z_n/2+\lambda_\pm\beta(z,\lambda_\pm)|$, we can isolate the derivative $\partial_{z_j}\lambda_\pm$. The resulting equation for the gradient is:

$$\begin{aligned}&\left[\Big(2\lambda_\pm-\frac{n}{2}+\alpha+\lambda_\pm\partial_\lambda\alpha\Big)-\operatorname{Re}\Big(e^{-\mathrm{i}\theta_\pm}(\beta+\lambda_\pm\partial_\lambda\beta)\Big)\right]\partial_{z_j}\lambda_\pm\\&=-\lambda_\pm\partial_{z_j}\alpha+\frac{e^{-\mathrm{i}\theta_\pm}}{2}\sqrt{n}\delta_{jn}+\operatorname{Re}\Big(e^{\mathrm{i}\theta_\pm}\lambda_\pm\partial_{z_j}\beta\Big).\end{aligned}\tag{3.56}$$

Following the same argument used for the invertibility of ϕ, the coefficient in the square brackets on the left-hand side of (3.56) is bounded from below by Kn. Consequently, the derivative $\partial_{z_j}\lambda_\pm$ can be expressed as:

$$\partial_{z_j}\lambda_\pm=\frac{-\lambda_\pm\partial_{z_j}\alpha+\frac{e^{-\mathrm{i}\theta_\pm}}{2}\sqrt{n}\delta_{jn}+\operatorname{Re}\left(e^{\mathrm{i}\theta_\pm}\lambda_\pm\partial_{z_j}\beta\right)}{\left(2\lambda_\pm-\frac{n}{2}+\alpha+\lambda_\pm\partial_\lambda\alpha\right)-\operatorname{Re}\left(e^{-\mathrm{i}\theta_\pm}(\beta+\lambda_\pm\partial_\lambda\beta)\right)}.\tag{3.57}$$

By applying the estimates for α and β from Proposition 3.2.8 to (3.57), we obtain:

$$|\partial_{z_j}\lambda_\pm|\le\frac{K}{n}\left[\sqrt{n}\delta_{jn}+R\left(\frac{e^{-a|j|}}{\langle j\rangle^p}+\sqrt{n}\frac{e^{-a|j-n|}}{\langle j-n\rangle^p}\right)\right].$$

Substituting this bound and the estimate for the gap length $|\lambda_+^2-\lambda_-^2|\le KR\sqrt{n}e^{-a|n|}\langle n\rangle^{-p}$ into (3.53), we get:

$$|\partial_{z_j}\gamma_n(z)|\le KR\frac{e^{-a|n|}}{\langle n\rangle^p}\delta_{jn}+KR^2\left(\frac{1}{\sqrt{n}}\frac{e^{-a(|j|+|n|)}}{\langle n\rangle^p\langle j\rangle^p}+\frac{e^{-a(|j-n|+|n|)}}{\langle n\rangle^p\langle j-n\rangle^p}\right).\tag{3.58}$$

As previously discussed, the right-hand side of (3.57) consists of functions that are majorant analytic in the z variables, thus providing the required bounding functions $f^{(n)}(z)$ for the estimate (3.52). To verify the convergence of the quantity that gives the bound for $\mathcal{K}(\gamma)$, we observe that:

$$\sup_{j\in\mathbb{N}}\langle j\rangle^p e^{a|j|}\sum_{n\in\mathbb{N}}\frac{e^{-a(|n|+|j|)}}{\langle n\rangle^p}\delta_{jn}\le 1,\tag{3.59}$$

and the other terms satisfy:

$$\sup_{j\in\mathbb{N}}\langle j\rangle^p e^{a|j|}\sum_{n\in\mathbb{N}}\left(\frac{1}{\sqrt{n}}\frac{e^{-a(|j|+|n|)}}{\langle n\rangle^p\langle j\rangle^p}+\frac{e^{-a(|j-n|+|n|)}}{\langle n\rangle^p\langle j-n\rangle^p}\right)\le\sum_{n\in\mathbb{N}}\frac{1}{\langle n\rangle^{p+1/2}}+\sum_{n\in\mathbb{N}}\frac{\langle j\rangle^p}{\langle j-n\rangle^p\langle n\rangle^p}<\infty.\tag{3.60}$$

These bounds ensure that $\mathcal{K}(\gamma) < \infty$, proving the desired regularity requirement for the map $\Gamma = (\Phi_\xi^*\gamma_n)_{n\in\mathbb{N}}$ in the local coordinates (y, φ).

From the estimate (3.58), we observe that the leading contribution (of order R) is purely diagonal and is given by the term $KRe^{-a|n|}\langle n\rangle^{-p}\delta_{jn}$. This term corresponds to the derivative of $4|z_n|^2$, which is consistent with the linear approximation derived in Subsection 3.2.4. Indeed, as shown in (3.28) when studying the spectral gaps g_n of the approximated system:

$$g_n^2 = 4|u_n|^2 + O(|u|^4),$$

where $u_k = \langle k\rangle^{1/2} z_k$ and, in our rescaled setting, $\gamma_n \sim g_n^2/n$. This identification allows us to decompose the map Γ as $\Gamma(y,\varphi) = 4(\xi+y)+P(y,\varphi)$, where $4(\xi+y)$ is the pullback of the leading term $4|z|^2$. The perturbation $P(y,\varphi)$ collects all higher-order terms and, by (3.52), (3.58), (3.59) and (3.60), satisfies:

$$\max\left\{ r\sup_{j\in\mathbb{N}}\sum_{n=1}^{\infty}\left|\frac{\partial P^{(n)}}{\partial y_j}\right|_{r,s},\ \sup_{j\in\mathbb{N}} e^{2a|j|}\langle j\rangle^{2p}\sum_{n=1}^{\infty}\left|\frac{\partial P^{(n)}}{\partial \varphi_j}\right|_{r,s}\right\} \le KR^3. \tag{3.61}$$

This bound with R^3 is crucial for ensuring the smallness condition needed for the applicability of Theorem 2.6.1.

We can now summarize these results in the following Theorem.

Theorem 3.2.11. *Let ξ be a sequence satisfying* (3.50)*. For $r, s > 0$ sufficiently small, the rescaled squared gaps $(\gamma_n)_{n\in\mathbb{N}}$ satisfy*

$$\sup_{j\in\mathbb{N}}\sum_{n=1}^{\infty}\left|\frac{\partial \Phi_\xi^*\gamma_n}{\partial y_j}\right|_{r,s} < \infty \quad \textit{and} \quad \sup_{j\in\mathbb{N}} e^{2a|j|}\langle j\rangle^{2p}\sum_{n=1}^{\infty}\left|\frac{\partial \Phi_\xi^*\gamma_n}{\partial \varphi_j}\right|_{r,s} < \infty.$$

Moreover, the map $\Gamma = (\Phi_\xi^*\gamma_n)_{n\in\mathbb{N}}$ can be written as

$$\Gamma(y,\varphi) = 4(\xi + y) + P(y,\varphi),$$

where the perturbation P is majorant analytic and satisfies the bound

$$\|P\|_{r,s}^{\infty,w} \le KR^3 \tag{3.62}$$

for a suitable positive constant K, where $w_j = \langle j\rangle^{2p}e^{2a|j|}$, for all $j \in \mathbb{N}$, and the norm $\|\cdot\|_{r,s}^{\infty,w}$ is defined in Definition 2.3.8.

Theorems 3.2.10 and 3.2.11 provide the final verification of the assumptions of Theorem 2.6.1. Specifically, the decomposition of the map Γ into a leading diagonal part plus a perturbation P of order R^3 ensures that the smallness condition (a.1) is satisfied for a sufficiently small but finite radius R. Furthermore, it is well-known that the squared gap lengths are integrals of the motion in involution, i.e., $\{\gamma_m, \gamma_n\} = 0$ and $\{\gamma_n, H_{KdV}\} = 0$ for all $m, n \in \mathbb{N}$. Since the pullback via Φ_ξ preserves the Poisson brackets (see the discussion in 3.1.5), the map Γ provides a complete set of integrals in involution for the Hamiltonian $H_{KdV} \circ \Phi_\xi$, fulfilling condition (a.2) of Theorem 2.6.1.

Proof of Theorem 3.2.7. The proof consists in verifying that the Hamiltonian H_{KdV} and the map $\Gamma = (\Phi_\xi^*\gamma_n)_{n\in\mathbb{N}}$ satisfy the hypotheses of Theorem 2.6.1.

First, we observe that the Hamiltonian H_{KdV} is majorant analytic on $\ell_{a,p}^\infty$ if $p > 2$. Its pullback via the local chart, $\Phi_\xi^* H_{KdV}$, thus belongs to the class $\mathcal{A}$.

By Theorems 3.2.10 and 3.2.11 the map can be decomposed as $\Gamma(y,\varphi) = 4(\xi + y) + P(y,\varphi)$, where P satisfies the cubic bound $\|P\|_{r,s}^{\infty,w} \le KR^3$ (see (3.61)), with $R > 0$ being the analyticity radius of the coefficients α and β determined in Proposition 3.2.8.

Recalling the conditions (2.43), together with (3.62), and (3.4), one can find $0 < \xi_- \le \xi_+$ such that, there exist $R' \le R$, $r' \le r$ and $s' \le s$ satisfying

$$\xi_+^{1/2} e^{s'} \sum_{m=0}^{\infty} |a_m| \left(\frac{r'}{\xi_-}\right)^m \le R' \quad \text{and} \quad K(R')^3 \le \frac{1}{4C} \min\{r', s', 1\},$$

ensuring the condition $(a.1)$ of Theorem 2.6.1. Finally, the commutativity condition (a.2) follows from the standard spectral theory: the squared gap lengths γ_n are integrals of motion in involution. Since the map Φ_ξ is a symplectomorphism, it preserves the Poisson brackets, ensuring that the components of Γ commute among themselves and with the Hamiltonian. ■

Chapter 4

Kronecker Flow on the Infinite Torus

4.1 Introduction

Kronecker flows play a central role in Hamiltonian dynamics as they describe the almost-periodic motion on compact, regular common level sets of complete systems of Poisson-commuting first integrals. The Kronecker flow on the regular sets is defined by a particular Hamiltonian which admits these first integrals. In the finite-dimensional case, KAM theory establishes the persistence of most of such almost-periodic solutions under small perturbations. In general, this is not the case in infinite-dimensional Hamiltonian systems. Still, the search for almost-periodic solutions is an active and challenging topic in the study of global solutions for evolution PDEs. Therefore, understanding Kronecker flows on infinite tori is crucial for gaining insights into the dynamical and topological properties of almost-periodic solutions of Hamiltonian PDEs and systems on infinite lattices.

When the dimension of the torus is finite, Kronecker flows satisfy the following properties.

A The (non-resonance) condition that the frequencies are rationally independent is equivalent to topological transitivity, to minimality, and to unique ergodicity (see [Laz93] for a modern exposition).

B Any Kronecker flow is topologically conjugate by an automorphism of the torus to another Kronecker flow having the property that the non-zero frequencies are rationally independent (see, for e.g., [FM10]). Consequently, by Property A, the closure of any orbit is still a torus.

Preliminaries studies of the infinite-dimensional analogue are due to Jessen [Jes34], who essentially proves a special case of Birkhoff's Ergodic Theorem. The ergodic properties of Kronecker flows on infinite tori with the product topology are also studied in [Koz21] and [SV24]. In the same paper [SV24], Sakbaev and Volovich also show that there is a new type of trajectories that is absent in the finite-dimensional case, namely, orbits whose closure is not a torus. Therefore, the properties of Kronecker flow in the infinite-dimensional setting are much more involved than in the finite-dimensional case.

4.1.1 Purposes and Informal Short Summary of the Main Results

We consider Kronecker flows on (countably) infinite tori endowed with the product topology. This particular choice of the topology, which coincides with the one considered in [Jes34], [Koz21] and [SV24], is motivated in Section 4.1.3 and it is related to the topology of the functional spaces typically considered in the study of Hamiltonian PDEs. We refer to the next section for precise definitions.

1. The first question addressed in this work can be informally rephrased as follows:

Does there exist a class of Kronecker flows on infinite tori which share the same properties [1] *as in the finite-dimensional case?*

We prove that this question has positive answer (see Theorem A and B). Namely, we identify this special class as the class of Kronecker flows whose frequency module — the $\mathbb{Z}$-module of all integer (finite) linear combinations of the frequencies — is a free abelian group. In this light, Properties A and B of the finite-dimensional case follow form the structure theorem for finitely generated abelian groups without torsion, which ensures that the frequency module is always free in the case of Kronecker flows with only finitely many non-zero frequencies (see Remark 4.2.11).

2. It is not difficult to provide examples of Kronecker flows on infinite tori whose associated frequency module is not free. This remark motivates the next question tackled in this chapter.

In the case of Kronecker flows with non-free frequency module, what is the topological structure of the closure of the orbits?

In Section 4.4.2 and 4.4.3 we construct a class of Kronecker flows with the property that the closure of any orbit is a product of circles and solenoids[2]. This can be viewed as an extension of the result of Sakbaev and Volovich [SV24], who showed that for linear flows on the infinite torus there exist orbits whose closure is not a torus (see Remark 4.4.26). Indeed, we exhibit a specific class of such flows and we provide a more refined understanding of the topological structure of such orbits (see Theorem C).

3. At this stage, the following natural question arises.

Are we able to exhibit examples of Hamiltonian PDEs admitting almost-periodic solutions whose closure is homeomorphic to a product of circles and solenoids?

The Benjamin-Ono equation with periodic boundary condition for a real valued function

$$u_t = \mathrm{H}(u_{xx}) - 2uu_x, \quad x \in \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}, \tag{4.1}$$

where $\mathrm{H}(\cdot)$ denotes the Hilbert transform

$$\mathrm{H}(u)(x,t) = -\frac{\mathrm{i}}{2\pi} \sum_{j\in\mathbb{Z}} \mathrm{sgn}(j) \int_0^{2\pi} u(x',t) e^{\mathrm{i}j(x-x')} dx', \tag{4.2}$$

is an infinite-dimensional, completely integrable (see [GK21]), Hamiltonian system that describes one-dimensional, periodic internal waves in deep water approximation. In Theorem BO of Section 4.4.4, we show that it admits a family of (non-typical) almost-periodic solutions that are dense on invariant subsets of $L^2(\mathbb{R}/2\pi\mathbb{Z})$ homeomorphic to the product of a circle and a solenoid. It may be interesting to understand if such family of almost-periodic solutions can be characterized by some physical feature. One can think of some new type of turbulence.

4. The last question we answer consists in a classification problem.

[1] In the sense of Properties A and B.

[2] Solenoids are compact, connected subgroups of the infinite torus. We give the definition in Section 4.4.2 and we also study their local topological properties (see Proposition 4.4.8). For an intrinsic definition, see for e.g. [HR79], in which they go under the name of ***a****-adic Solenoids*.

Given two Kronecker flows, are we able to determine whether the closures of their orbits are homeomorphic?

In Section 4.4.5 we prove that the above question is equivalent to the highly non-trivial classification problem for countably infinite abelian groups without torsion. To date, a complete classification is known only for the rank 1 case (see [Bae37], [BZ51] and [Fuc15]), which corresponds to the class of Kronecker flows whose orbits are dense in a product of circles and solenoids, as we show in Section 4.4.3. The key result that bridges such correspondence is Theorem D, which we believe is interesting by itself.

4.1.2 Setting

Let us give the fundamental definitions to rigorously introduce the subject of concern.

Notation 4.1.1. *In what follows, we denote by $\mathbb{N} := \{1, 2, \dots\}$ the set of positive integers.*

The symbol $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ denotes the circle group, identified with the additive group of real numbers modulo 2π and endowed with the quotient topology.

Given an additive abelian group G, a subgroup $H \subseteq G$, and an element $g \in G$, we denote by $g + H$ the coset (or equivalence class) of g modulo H, defined as:

$$g + H := \{g + h \, : \, h \in H\}.$$

This corresponds to the equivalence relation where $g \sim g'$ if and only if $g - g' \in H$.

1. *Torus.* A torus is a countable direct product of circle groups. More precisely, given a countable set of indices J, the torus $\mathbb{T}^J$ is the topological abelian group defined as

$$\mathbb{T}^J := \prod_{j \in J} \mathbb{T}_j, \quad \mathbb{T}_j = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad \text{for all } j \in J.$$

endowed with the product topology, that is, the topology of pointwise convergence. Note that any torus is metrisable and, by Tychonoff's theorem, it is compact. A basis for the product topology is given by the open cylinders, where a cylinder is a subset $\prod_{j \in J} X_j$ of $\mathbb{T}^J$ such that, for each $j \in J$, $X_j \subseteq \mathbb{T}$, with $X_j = \mathbb{T}$ for all but finitely many $j \in J$, and an open cylinder is a cylinder in which all the X_j are open. See Appendix C for more details about the product topology.

Notation 4.1.2. *Throughout this work, given $\theta = (\theta_j)_{j \in J} \in \mathbb{T}^J$, we denote by $\Theta := (\Theta_j)_{j \in J}$ any list of real numbers such that $\theta_j = \Theta_j + 2\pi\mathbb{Z}$ for all $j \in J$.*

With this notation, we define the group operation $*$ on $\mathbb{T}^J$ componentwise: for any $\theta, \theta' \in \mathbb{T}^J$, their product $\theta * \theta'$ is the element of $\mathbb{T}^J$ whose j-th component is given by

$$(\theta * \theta')_j := \Theta_j + \Theta'_j + 2\pi\mathbb{Z}, \quad \text{for all } j \in J.$$

It is straightforward to verify that this definition is well-defined, as it is independent of the choice of the representatives Θ_j and Θ'_j.

Remark 4.1.3. *In the following, we shall improperly use the term "dimension" to refer to the rank (in the group-theoretic sense) of the torus. This is motivated by the fact that, if the cardinality of J is finite, the rank of $\mathbb{T}^J$ coincides with its dimension when viewed as a topological manifold. Note that, if the cardinality of J is not finite, $\mathbb{T}^J$ cannot be a topological manifold, i.e., locally homeomorphic to a topological vector space (TVS). Indeed, any open cylinder of $\mathbb{T}^J$ contains non contractible loops while TVS are simply connected.*

2. *Kronecker Flow.* Given $\omega = (\omega_j)_{j\in J} \in \mathbb{R}^J$, we define the Kronecker flow associated with the "frequency vector" ω as the topological dynamical system $(\mathbb{T}^J, \Phi_\omega)$ induced by the continuous $\mathbb{R}$-action

$$\Phi_\omega : \mathbb{R} \times \mathbb{T}^J \to \mathbb{T}^J, \quad (t,\theta) \mapsto \Phi_\omega^t(\theta) := (\Theta_j + \omega_j t + 2\pi\mathbb{Z})_{j\in J}.$$

The real numbers ω_j are called "frequencies".

Remark 4.1.4. *The closure of any orbit induced by a Kronecker flow is homeomorphic to the closure of the orbit through the origin. This follows from the invariance by translations.*

4.1.3 Motivations

In the sixties, it was realized that many Hamiltonian systems with infinitely many degrees of freedom possess an infinite number of constants of motion. Some of these systems exhibit characteristics very similar to those of finite-dimensional completely integrable Hamiltonian systems. Namely, the solutions are supported on invariant tori of possibly infinite dimension; in this case, if an appropriate topology is induced on them, they turn out to be homeomorphic to $\mathbb{T}^J$ for some countable set of indices $J \subseteq \mathbb{Z}$, and the dynamics on them is conjugated to a Kronecker flow. Examples of systems having these features are provided by Hamiltonian PDEs on the circle having an elliptic fixed point, such as the Korteweg-de Vries equation (KdV), the Nonlinear Schrödinger equation (NLS) and the Benjamin-Ono equation (BO) (4.1) with periodic boundary conditions.

These equations have been extensively studied in the Sobolev-Hilbert space $H^s_0(\mathbb{T})$,

$$H^s_0(\mathbb{T}) := \left\{ u \in H^s(\mathbb{T}) \ : \ \int_0^{2\pi} u(x)\,dx = 0 \right\},$$

for suitable $s \geq 0$.[3] For each of the above equations, there exists $s' \geq 0$ and a global bi-analytic diffeomorphism

$$\Psi : H^s_0(\mathbb{T}) \ni u \mapsto \Psi(u) \in h^{s'} := \left\{ z = (z_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ : \ \|z\|^2_{h^{s'}} := \sum_{j\in\mathbb{N}} j^{2s'} |z_j|^2 < \infty \right\},$$

a "Birkhoff map",[4] such that , for any initial data $\underline{u} \in H^s_0(\mathbb{T})$, the solution is supported on the invariant set $\Psi^{-1}(\mathcal{T}_I)$. Here $I = (I_j)_{j\in\mathbb{N}}$ is the list defined by $I_j := |\Psi_j(\underline{u})|^2$, where $\Psi(\underline{u}) = (\Psi_j(\underline{u}))_{j\in\mathbb{N}}$, and

$$\mathcal{T}_I := \{ z \in h^{s'} \ : \ |z_j|^2 = I_j, \ \forall j \in \mathbb{N} \}. \tag{4.3}$$

Remark 4.1.5. *The sets defined in* (4.3)*, as showed in Appendix D, when considered with the subspace topology induced by the* $\|\cdot\|_{h^{s'}}$*-norm, are homeomorphic to the possibly infinite torus* $\mathbb{T}^{supp(I)}$*, where* $supp(I)$ *denotes the support of the sequence* I*. Namely, the map*

$$\Xi : \mathbb{T}^{supp(I)} \ni \theta \mapsto \Xi(\theta) := (\sqrt{I_j} e^{\mathrm{i}\Theta_j})_{j\in\mathbb{N}} \in \mathcal{T}_I \subset h^{s'},$$

where one chooses $\Theta_j := 0$ *for all* $j \notin supp(I)$*, is a topological embedding. Whenever the cardinality of* $supp(I)$ *is infinite, the torus* $\mathbb{T}^{supp(I)}$ *has infinite dimension.*

[3] The flows of the equations KdV, NLS and BO preserve the average of the solutions.

[4] For the case of the NLS, being the unknown a complex valued function, $\mathbb{N}$ has to be replaced with $\mathbb{Z} \setminus \{0\}$.

With respect to the coordinate system induced by the Birkhoff map, the solution for the given initial data $\underline{u}$ is continuous in time as a map with values in $h^{s'}$ and is given by

$$\Psi_j(u(t)) = \Psi_j(\underline{u})e^{\mathrm{i}\omega_j(I)t}, \quad \omega_j(I) := \frac{\partial H \circ \Psi^{-1}}{\partial |z_j|^2}(\sqrt{I_1}, ..., \sqrt{I_n}, ...), \quad \forall j \in \mathbb{N},$$

where $H : H_0^s(\mathbb{T}) \to \mathbb{R}$ is the Hamiltonian function of the system under consideration. For references, see [KP03],[GK21],[GK14] and [KLTZ09]. Therefore, since the Birkhoff map Ψ is a bi-analytic diffeomorphism, and hence, a homeomorphism, the Hamiltonian flow on the invariant torus $\Psi^{-1}(\mathcal{T}_I)$ is topologically conjugate by $\mathscr{H} := \Psi^{-1} \circ \Xi$ to the Kronecker flow

$$\Phi_\omega : \mathbb{R} \times \mathbb{T}^{\mathrm{supp}(I)} \to \mathbb{T}^{\mathrm{supp}(I)}, \quad (t, \theta) \mapsto \Phi_\omega^t(\theta) := (\Theta_j + \omega_j(I)t + 2\pi\mathbb{Z})_{j \in \mathrm{supp}(I)},$$

with $\omega = (\omega_j(I))_{j \in \mathbb{N}}$.

In [KM18], the global weak* $\mathscr{C}^0$-wellposedness of the periodic KdV equation in the space of pseudomeasures on $\mathbb{T}$ is proved. For any given initial data, the solution is supported on an invariant subset that, when endowed with the subspace topology induced by the weak* topology, is homeomorphic to a torus (possibly of infinite dimension) with the product topology. Furthermore, the KdV flow on these invariant sets is topologically conjugate to a Kronecker flow.

To underline the ubiquity of Kronecker flows in the context of infinite Hamiltonian systems, we point out that there are also results about the construction of invariant tori for non-integrable Hamiltonian PDEs. For instance, in [Bou05], Bourgain constructs invariant tori of "full dimension" in Gevrey regularity for the $1D$ periodic NLS equation with convolution potential $V = (V_j)_{j \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ whose coefficients are considered as external parameters. In [BMP21], the authors extend the result of Bourgain to an entire ball of initial data in the phase space of Fourier coefficients

$$\mathtt{w}_{p,s,a}^\infty := \left\{ u = (u_j)_{j \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z} \ : \ \|u\|_{\mathtt{w}_{p,s,a}^\infty} := \sup_{j \in \mathbb{Z}} |u_j| \langle j \rangle^p e^{a|j| + s\langle j \rangle^\alpha} \right\}, \quad \langle j \rangle := \max\{1, |j|\},$$

with $p > 1$, $s > 0$, $0 < \alpha < 1$ and suitable $a \geq 0$. Namely, for most choices of $\omega = (\omega_j)_{j \in \mathbb{N}} \in \mathbb{R}^\mathbb{Z}$ satisfying $\sup_{j \in \mathbb{Z}} |\omega_j - j^2| \leq 1/2$, and for all list $I = (I_j)_{j \in \mathbb{Z}}$ of non-negative real numbers such that $(\sqrt{I_j})_{j \in \mathbb{Z}}$ belongs to a suitable neighbourhood of the origin in $\mathtt{w}_{p,s,a}^\infty$, there exists $V \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ such that the set $\mathcal{T}_I$ defined by

$$\mathcal{T}_I := \{u = (u_j)_{j \in \mathbb{Z}} \in \mathtt{w}_{p,s,a}^\infty \ : \ |u_j|^2 = I_j\}$$

is invariant for the NLS flow with convolution potential V. Furthermore, when $\mathcal{T}_I$ is endowed with the subspace topology induced by the weak* topology τ_*,[5] the dynamics on such invariant set is topologically conjugate to a Kronecker flow on $\mathbb{T}^{\mathrm{supp}(I)}$ (with $\omega_j \sim j^2$ for large $|j|$) through the homeomorphism associated with the embedding[6]

$$\mathbb{T}^{\mathrm{supp}(I)} \ni \theta \mapsto (\sqrt{I_j} e^{\mathrm{i}\Theta_j})_{j \in \mathbb{Z}} \in \mathcal{T}_I \subset (\mathtt{w}_{p,s,a}^\infty, \tau_*), \quad \Theta_j := 0, \text{ for all } j \notin \mathrm{supp}(I).$$

The analysis carried out in the present chapter yields a connection between algebraic properties of the characteristic frequencies of the infinite Hamiltonian system under consideration and the topological

[5] Being the topological dual of the Banach space

$$\ell_{p,s,a}^1 := \{v = (v_j)_{j \in \mathbb{Z}} \ : \ \|v\|_{\ell_{p,s,a}^1} := \textstyle\sum_{j \in \mathbb{Z}} |v_j| \langle j \rangle^{-p} e^{-a|j| - s\langle j \rangle^\alpha} < \infty\},$$

$\mathtt{w}_{p,s,a}^\infty$ can be endowed with the weak* topology, i.e., the coarsest topology that makes $\mathtt{w}_{p,s,a}^\infty \ni u \mapsto u(v) \in \mathbb{C}$ continuous for all $v \in \ell_{p,s,a}^1$.

[6] See Appendix D.

structure of the minimal invariant sets. In Section 4.4.4 we apply our results to the Benjamin-Ono equation (4.1).

Before moving on, it is worth noting that Kronecker flows on infinite tori are deeply connected with the theory of almost-periodic functions. We refer the interested reader to [Fin74].

4.2 Preliminaries

4.2.1 Basics of Topological Dynamics

A topological dynamical system is a pair (X, Φ), where X is a topological space and Φ is a continuous flow $\Phi : \mathbb{R} \times X \to X$. An invariant probability measure for (X, Φ) is a probability Borel measure μ on X such that, for any t in $\mathbb{R}$ and any Borel subset $U \subseteq X$, $\mu((\Phi^t)^{-1}(U)) = \mu(U)$. In terms of the pushforward measure, this states that $\Phi^t_*(\mu) = \mu$ for all $t \in \mathbb{R}$.

We now recall the definitions of the properties of a topological dynamical system that will be relevant to our study.

A topological dynamical system (X, Φ) is

- topologically transitive if there exists a dense orbit, and minimal if every orbit is dense;
- uniquely ergodic if it admits a unique invariant probability measure.

Let μ be an invariant probability measure for (X, Φ). Then, (X, Φ, μ) is

- ergodic if the only invariant sets have measure 1 or 0.

Remark 4.2.1. *A topological dynamical system $\Phi : \mathbb{R} \times X \to X$ admitting a unique invariant probability measure must be ergodic. Indeed, let μ be an invariant probability measure for (X, Φ). If the system is not ergodic, there exists $U \subset X$ invariant and such that $0 < \mu(U) < \mu(X) = 1$. But then, we can define another invariant probability measure simply normalizing by $\mu(U)$, namely, $\mu_U(V) := (\mu(U))^{-1}\mu(V \cap U)$, $\forall V \subset X$.*

A central notion in topological dynamics is that of topological conjugacy. In particular, all the aforementioned properties of a topological dynamical system are preserved under topological conjugacy.

Definition 4.2.2. *Two topological dynamical systems (X, Φ) and (Y, Ψ) are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that, for any $t \in \mathbb{R}$,*

$$h \circ \Phi^t = \Psi^t \circ h.$$

4.2.2 Resonance Module and Frequency Module of a Kronecker flow

In this subsection, we introduce two abelian groups that one can associate with every Kronecker flow: the resonance module and the frequency module, both of which play a crucial role in the following discussion.

Remark 4.2.3. *Given a countable set of indices J, and a countable family of abelian groups $(G_j)_{j\in J}$, the direct sum $\bigoplus_{j\in J} G_j$ is the abelian subgroup of $\prod_{j\in J} G_j$ consisting of all $(g_j)_{j\in J} \in \prod_{j\in J} G_j$ such that g_j differs from 0 only for finitely many $j \in J$.*

Notation 4.2.4. *Given a set of indices J and an abelian group G, we adopt the convention*

$$G^J := \prod_{j\in J} G := \prod_{j\in J} G_j, \quad G_j = G,\ \forall j \in J,$$
$$G^{(J)} := \bigoplus_{j\in J} G := \bigoplus_{j\in J} G_j, \quad G_j = G,\ \forall j \in J.$$

If $J = \{1, ..., n\}$, with $n \in \mathbb{N}$, we set $G^n := G^J = G^{(J)}$.

Definition 4.2.5 (Resonance module)**.** *Given $\omega \in \mathbb{R}^J$, we define the resonance module associated with ω as*

$$\mathscr{R}_\omega := \left\{ \nu \in \mathbb{Z}^{(J)} \ : \ \sum_{j\in J} \omega_j \nu_j = 0 \right\}. \tag{4.4}$$

We say that, the real numbers $(\omega_j)_{j\in J}$ are rationally independent if and only if $\mathscr{R}_\omega = \{0\}$, in which case, we say that $\omega = (\omega_j)_{j\in J}$ is non-resonant, otherwise we say that ω is resonant. Accordingly, the Kronecker flow $(\mathbb{T}^J, \Phi_\omega)$ associated with ω is said to be (non-)resonant if ω is (non-)resonant.

Definition 4.2.6. *An abelian group is said to be free if it admits a basis. Namely, an abelian group G is free if and only if there exists a set of indices J and an isomorphism of groups between G and $\mathbb{Z}^{(J)}$ (see Corollary E.2.9).*

Remark 4.2.7. *Being a subgroup of the free abelian group $\mathbb{Z}^{(J)}$, the resonance module $\mathscr{R}_\omega$ is itself free as an abelian group (see, for e.g., [Rot02]).*

Definition 4.2.8 (Frequency Module)**.** *The frequency module associated with $\omega \in \mathbb{R}^J$ is the subgroup of $(\mathbb{R}, +)$ generated by the integer (finite) linear combinations of the frequencies, i.e.,*

$$\mathscr{M}_\omega := \left\{ \sum_{j\in J} \omega_j \nu_j \ : \ \nu \in \mathbb{Z}^{(J)} \right\} \subset \mathbb{R}.$$

Remark 4.2.9. *As the map $\mathbb{Z}^{(J)} \ni \nu \mapsto \omega \cdot \nu \in (\mathbb{R}, +)$ is a group homomorphism, by the first isomorphism theorem, the frequency module $\mathscr{M}_\omega$ is isomorphic to the quotient group $\mathbb{Z}^{(J)}/\mathscr{R}_\omega$, where $\mathscr{R}_\omega$ is the resonance module defined in* (4.4)*.*

Remark 4.2.10. *$\mathscr{M}_\omega$ is countable and without torsion, namely, for each element $g \in \mathscr{M}_\omega$, if there exists a non-zero integer n such that $ng = 0$, then, $g = 0$.*

The following remark highlights the most fundamental difference between the finite-dimensional and infinite-dimensional settings and, as we shall see in Theorem B, reveals the reasoning behind why the simple, well known properties A and B that hold for Kronecker flows on finite-dimensional tori do not apply in general to this infinite-dimensional context.

Remark 4.2.11. *If the cardinality of J is finite, then $\mathscr{M}_\omega$ is a free abelian group: this follows from Remark 4.2.10 and the general structure theorem for finitely generated abelian groups without torsion, noting that the finiteness of the cardinality of J ensures that the frequency module is finitely generated.*

If the cardinality of J is not finite, $\mathscr{M}_\omega$ may be not free: it is no longer guaranteed that $\mathscr{M}_\omega$ is finitely generated. As an example, for $J = \mathbb{N}$ and $\omega = (\omega_j)_{j\in\mathbb{N}}$ with $\omega_j = 1/j$, one has $\mathscr{M}_\omega = (\mathbb{Q}, +)$.

4.2.3 The Finite-Dimensional Case

Let $n \in \mathbb{N}$ and $\omega = (\omega_j)_{j=1}^n \in \mathbb{R}^n$. For the Kronecker flow Φ_ω on $\mathbb{T}^n$, topological transitivity, minimality, and unique ergodicity are equivalent to the non-resonance condition $\mathscr{R}_\omega = \{0\}$.

Furthermore, if $1 \leq \operatorname{rank}(\mathscr{R}_\omega) =: d < n$, there exists an automorphism $\mathscr{A}$ of $\mathbb{T}^n$ that conjugates $(\mathbb{T}^n, \Phi_\omega)$ to another Kronecker flow $(\mathbb{T}^n, \Phi_{\tilde{\omega}})$; specifically,

$$\mathscr{A} \circ \Phi_\omega^t = \Phi_{\tilde{\omega}}^t \circ \mathscr{A} \quad \text{for all } t \in \mathbb{R},$$

where $\tilde{\omega} = (0_d, \bar{\omega})$, $\bar{\omega}$ is non-resonant, and 0_d is the null vector in $\mathbb{R}^d$. In particular, if we denote by q^d the projection

$$q^d : \mathbb{T}^n \to \mathbb{T}^d, \quad (\theta_j)_{j=1}^n \mapsto (\theta_j)_{j=1}^d,$$

the continuous surjection $q^d \circ \mathscr{A} : \mathbb{T}^n \to \mathbb{T}^d$ defines a (trivial) fiber bundle with fibers homeomorphic to $\mathbb{T}^{n-d}$. Each fiber is a minimal invariant set for the Kronecker flow $(\mathbb{T}^n, \Phi_\omega)$, and the restriction of the flow to each fiber is topologically conjugate to the non-resonant Kronecker flow $(\mathbb{T}^{n-d}, \Phi_{\bar{\omega}})$.

We now provide a characterization of the group $\operatorname{Aut}(\mathbb{T}^\mathbb{N})$ of automorphisms of the topological group $\mathbb{T}^\mathbb{N}$ (see Proposition 4.2.19) and apply this result to develop an algorithm for reducing resonant Kronecker flows (see Proposition 4.2.23).

4.2.4 Characterization of $\operatorname{Aut}(\mathbb{T}^\mathbb{N})$

Notation 4.2.12. *We shall adopt the following notations throughout this section:*

- *For any $j \in \mathbb{N}$, $q_j : \mathbb{T}^\mathbb{N} \to \mathbb{T}$ denotes the projection onto the j-th factor, namely, $q_j(\theta) = \theta_j$.*
- *For any $m \in \mathbb{N}$, $q^m : \mathbb{T}^\mathbb{N} \to \mathbb{T}^m$ denotes the projection $(\theta_j)_{j\in\mathbb{N}} \mapsto (\theta_j)_{j=1}^m$.*
- *For any $m \in \mathbb{N}$, $i^m : \mathbb{R}^m \to \mathbb{R}^\mathbb{N}$ denotes the inclusion $(\Theta_j)_{j=1}^m \mapsto (\Theta_1, \dots, \Theta_m, 0, \dots)$.*
- *For any $m \in \mathbb{N}$, $\pi_m : \mathbb{R}^m \to \mathbb{T}^m$ denotes the standard quotient projection $\Theta \mapsto \Theta + (2\pi\mathbb{Z})^m$.*
- *$\pi_\mathbb{N} : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}/(2\pi\mathbb{Z})^\mathbb{N}$ denotes the projection $\Theta \mapsto \Theta + (2\pi\mathbb{Z})^\mathbb{N}$.*

Remark 4.2.13. *All the spaces defined above are endowed with the product topology. This choice ensures that all the maps introduced in Notation 4.2.12 are continuous.*

We first show that $\mathbb{T}^\mathbb{N}$ is naturally isomorphic as a topological group to the quotient $\mathbb{R}^\mathbb{N}/(2\pi\mathbb{Z})^\mathbb{N}$, where $\mathbb{R}^\mathbb{N}$ is endowed with the product topology and the quotient carries the induced quotient topology.

Proposition 4.2.14 (Infinite torus as quotient space)**.** *The torus $\mathbb{T}^\mathbb{N}$ is isomorphic as a topological group to the quotient $\mathbb{R}^\mathbb{N}/(2\pi\mathbb{Z})^\mathbb{N}$.*

Proof. The algebraic group isomorphism is immediate. To prove that the map

$$\Phi : \mathbb{R}^\mathbb{N}/(2\pi\mathbb{Z})^\mathbb{N} \to \mathbb{T}^\mathbb{N}, \quad \Theta + (2\pi\mathbb{Z})^\mathbb{N} \mapsto (\Theta_j + 2\pi\mathbb{Z})_{j\in\mathbb{N}}$$

is a homeomorphism, we recall that a set $U \subseteq \mathbb{R}^\mathbb{N}/(2\pi\mathbb{Z})^\mathbb{N}$ is open if and only if its preimage $\pi_\mathbb{N}^{-1}(U)$ is open in $\mathbb{R}^\mathbb{N}$.

Since the product topology on $\mathbb{T}^\mathbb{N}$ is generated by the subbase of open cylinders of the form $q_j^{-1}(A)$ (with $A \subseteq \mathbb{T}$ open), it suffices to check that their preimages under Φ are open. To this end, observe that

$$\pi_\mathbb{N}^{-1}\left(\Phi^{-1}(q_j^{-1}(A))\right) = \{\Theta \in \mathbb{R}^\mathbb{N} \;:\; \Theta_j \in \pi^{-1}(A)\},$$

where $\pi : \mathbb{R} \to \mathbb{T}$ is the standard projection. This set is an open cylinder in $\mathbb{R}^{\mathbb{N}}$; hence, Φ is continuous.

Conversely, the image under $\Phi \circ \pi_{\mathbb{N}}$ of any open cylinder in $\mathbb{R}^{\mathbb{N}}$ is an open cylinder in $\mathbb{T}^{\mathbb{N}}$. Since $\Phi \circ \pi_{\mathbb{N}}$ maps the basis of the product topology of $\mathbb{R}^{\mathbb{N}}$ to the basis of $\mathbb{T}^{\mathbb{N}}$, it follows that $\Phi \circ \pi_{\mathbb{N}}$ is an open map. By the properties of the quotient topology, this implies that Φ is an open map as well, and thus a homeomorphism. ■

Remark 4.2.15 (Not a covering map)**.** *To avoid misunderstandings, we stress that the previous proposition does not imply that $\pi_{\mathbb{N}}$ is a covering map. In fact, $\pi_{\mathbb{N}}$ cannot be a local homeomorphism, since $\mathbb{R}^{\mathbb{N}}$ is simply connected, whereas inside each cylinder of the torus one can find non-contractible loops.*

Remark 4.2.16 (Abuse of notation)**.** *In what follows, we shall identify $\mathbb{T}^{\mathbb{N}}$ with $\mathbb{R}^{\mathbb{N}}/(2\pi\mathbb{Z})^{\mathbb{N}}$ and $\mathbb{T}^m$ with $\mathbb{R}^m/(2\pi\mathbb{Z})^m$ in the sense of Proposition 4.2.14.*

Theorem 4.2.17 (Lifts)**.** *Given a continuous function $F : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$ such that $F(0) = 0$, there exists a unique continuous function $\hat{F} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, which we call "lift", such that $\pi_{\mathbb{N}} \circ \hat{F} = F \circ \pi_{\mathbb{N}}$ and $\hat{F}(0) = 0$.*

Proof. For every $m \in \mathbb{N}$, consider the map

$$f^m := q^m \circ F \circ \pi_{\mathbb{N}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{T}^m.$$

Since $\mathbb{R}^{\mathbb{N}}$ is connected and simply connected, and $\pi_m : \mathbb{R}^m \to \mathbb{T}^m$ is the universal covering of $\mathbb{T}^m$ with $\pi_m(0) = 0$, there exists a unique continuous lift $\hat{f}^m : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^m$ such that $\pi_m \circ \hat{f}^m = f^m$ and $\hat{f}^m(0) = 0$.

We then define $\hat{F}^m : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ via the inclusion i^m:

$$\hat{F}^m := i^m \circ \hat{f}^m.$$

By construction, each $\hat{F}^m$ is continuous and satisfies $\hat{F}^m(0) = 0$. We now show that for any $j \le m$, the j-th component $\hat{F}^m_j$ is independent of m.

Observe that for $j \le m$, the component $f^m_j = (F \circ \pi_{\mathbb{N}})_j$ does not depend on m. Thus, both $\hat{F}^m_j$ and $\hat{F}^{m+1}_j$ are lifts of the same map $f^m_j : \mathbb{R}^{\mathbb{N}} \to \mathbb{T}$. Specifically:

$$\hat{F}^m_j(\Theta) + 2\pi\mathbb{Z} = f^m_j(\Theta) = f^{m+1}_j(\Theta) = \hat{F}^{m+1}_j(\Theta) + 2\pi\mathbb{Z}.$$

Since $\mathbb{R}^{\mathbb{N}}$ is connected and $\hat{F}^m_j(0) = \hat{F}^{m+1}_j(0) = 0$, the uniqueness of lifts implies that:

$$\hat{F}^m_j = \hat{F}^{m+1}_j \quad \text{for all } m \ge j.$$

This consistency allows us to well-define the map $\hat{F} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by setting, for each $j \in \mathbb{N}$:

$$\hat{F}_j := \hat{F}^m_j \quad \text{for any } m \ge j.$$

By definition, $\hat{F}(0) = 0$ and $\pi_{\mathbb{N}} \circ \hat{F} = F \circ \pi_{\mathbb{N}}$ holds component-wise. Finally, $\hat{F}$ is continuous because each of its components $\hat{F}_j$ is continuous, which is the defining property of the product topology.

Regarding uniqueness, any other lift $\tilde{F}$ satisfying the same conditions would necessarily have components $\tilde{F}_j$ which are continuous lifts of $f_j = (F \circ \pi_{\mathbb{N}})_j$ such that $\tilde{F}_j(0) = 0$. Since $\pi : \mathbb{R} \to \mathbb{T}$ is a covering map and $\mathbb{R}^{\mathbb{N}}$ is connected, each component $\tilde{F}_j$ is uniquely determined, whence $\tilde{F} = \hat{F}$. ■

Let $\mathrm{Aut}(\mathbb{T}^{\mathbb{N}})$ be the group of automorphisms of $\mathbb{T}^{\mathbb{N}}$, namely, the set of bijective maps $\mathscr{A} : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$ such that both $\mathscr{A}$ and $\mathscr{A}^{-1}$ are continuous group homomorphisms. To characterize these automorphisms, we first introduce a particular group of infinite invertible matrices.

Definition 4.2.18 (The group $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$)**.** *We denote by* $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ *the set of infinite matrices* $A = (a_{ij})_{i,j\in\mathbb{N}}$ *with* $a_{ij} \in \mathbb{Z}$ *satisfying the following conditions:*

1. *each row of A has finite support, i.e., for every $i \in \mathbb{N}$, the set $\{j \in \mathbb{N} : a_{ij} \neq 0\}$ is finite;*
2. *the matrix A is invertible, meaning there exists an infinite matrix $A^{-1} = (a'_{ij})_{i,j\in\mathbb{N}}$ with integer entries and finite support rows such that $AA^{-1} = A^{-1}A = I$, where I is the infinite identity matrix.*

The condition on the finite support of the rows ensures that the action of the matrix on $\mathbb{R}^{\mathbb{N}}$ is well-defined. Specifically, for any $\Theta = (\Theta_j)_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, the i-th component of the image $A\Theta$ is given by the sum:

$$(A\Theta)_i = \sum_{j\in\mathbb{N}} a_{ij}\Theta_j.$$

Since each row has finite support, this sum is always finite, and thus $A\Theta$ is a well-defined element of $\mathbb{R}^{\mathbb{N}}$. Furthermore, the resulting linear map $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is continuous with respect to the product topology, as each component $(A\Theta)_i$ depends only on a finite number of coordinates of Θ.

Matrix multiplication for $A, B \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ is defined by the standard rule $(AB)_{ij} = \sum_{k\in\mathbb{N}} a_{ik}b_{kj}$. This sum is always well-posed because each row of A contains only finitely many non-zero entries. Moreover, $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ is closed under multiplication: the i-th row of the product AB is a finite linear combination of the rows of B, namely

$$(AB)_{i\cdot} = \sum_{k\in S_i} a_{ik}(B)_{k\cdot},$$

where $S_i := \{k \in \mathbb{N} : a_{ik} \neq 0\}$ is the finite support of the i-th row of A. Since the union of finitely many finite sets is itself finite, the i-th row of AB also has finite support. This structure, combined with the existence of the inverse (which by definition belongs to the same set), ensures that $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ forms a group under matrix multiplication.

Proposition 4.2.19. $\mathrm{Aut}(\mathbb{T}^{\mathbb{N}})$ *is isomorphic to* $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ *via the correspondence*

$$\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z}) \ni A = (a_{ij})_{i,j\in\mathbb{N}} \mapsto \mathscr{A} \in \mathrm{Aut}(\mathbb{T}^{\mathbb{N}}), \tag{4.5}$$

where, for all $\theta = (\Theta_j + 2\pi\mathbb{Z})_{j\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, the map $\mathscr{A}$ is defined by

$$\mathscr{A}(\theta) = \left(\sum_{j\in\mathbb{N}} a_{ij}\Theta_j + 2\pi\mathbb{Z}\right)_{i\in\mathbb{N}}. \tag{4.6}$$

Proof. It is immediate to verify that any $A \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ defines, through (4.5) and (4.6), an automorphism of $\mathbb{T}^{\mathbb{N}}$.

Let us prove the converse. Given $\mathscr{A} \in \mathrm{Aut}(\mathbb{T}^{\mathbb{N}})$, we first recall the identification $\mathbb{T}^{\mathbb{N}} \cong \mathbb{R}^{\mathbb{N}}/(2\pi\mathbb{Z})^{\mathbb{N}}$ established in Proposition 4.2.14. By Theorem 4.2.17, there exists a unique continuous lift $\hat{\mathscr{A}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ of $\mathscr{A}$ such that $\hat{\mathscr{A}}(0) = 0$ and $\pi_{\mathbb{N}} \circ \hat{\mathscr{A}} = \mathscr{A} \circ \pi_{\mathbb{N}}$. Since $\mathbb{R}^{\mathbb{N}}$ is connected and $\mathscr{A}$ is an endomorphism, $\hat{\mathscr{A}}$ is an endomorphism as well; its continuity then ensures that it is an $\mathbb{R}$-linear map.

Let e_j be the j-th canonical vector in $\mathbb{R}^{\mathbb{N}}$. For any $\Theta \in \mathbb{R}^{\mathbb{N}}$, let $(\Theta^n)_{n\in\mathbb{N}}$, where $\Theta^n := \sum_{j=1}^n \Theta_j e_j$, be the sequence of its truncations. By the definition of the product topology, $\Theta^n \to \Theta$ as $n \to \infty$. Since $\hat{\mathscr{A}}$ is continuous, it preserves the limits of convergent sequences; thus, for each component $i \in \mathbb{N}$, we have

$$(\hat{\mathscr{A}}(\Theta))_i = \left(\hat{\mathscr{A}}\left(\lim_{n\to\infty} \Theta^n\right)\right)_i = \lim_{n\to\infty} (\hat{\mathscr{A}}(\Theta^n))_i. \tag{4.7}$$

By the linearity of $\hat{\mathscr{A}}$ and the fact that Θ^n has finite support, it follows that, for any $n \in \mathbb{N}$, $(\hat{\mathscr{A}}(\Theta^n))_i = \sum_{j=1}^{n} a_{ij}\Theta_j$, where $a_{ij} := (\hat{\mathscr{A}}(e_j))_i$. Substituting this into (4.7), we obtain $(\hat{\mathscr{A}}(\Theta))_i = \sum_{j\in\mathbb{N}} a_{ij}\Theta_j$.

The convergence of this series for every $\Theta \in \mathbb{R}^{\mathbb{N}}$ implies that each row of A must have finite support. Indeed, if for some i there were infinitely many non-zero entries $(a_{ij_k})_{k\in\mathbb{N}}$, one could define $\Theta \in \mathbb{R}^{\mathbb{N}}$ by setting $\Theta_{j_k} = k/a_{ij_k}$ (and $\Theta_j = 0$ otherwise), making the i-th component of $\hat{\mathscr{A}}(\Theta)$ diverge, which is a contradiction.

Furthermore, the relation $\pi_{\mathbb{N}} \circ \hat{\mathscr{A}} = \mathscr{A} \circ \pi_{\mathbb{N}}$ requires that $\hat{\mathscr{A}}$ maps the kernel of the projection $\pi_{\mathbb{N}}$, namely $(2\pi\mathbb{Z})^{\mathbb{N}}$, into itself; hence $a_{ij} \in \mathbb{Z}$ for all $i, j \in \mathbb{N}$. Finally, since $\mathscr{A}$ is an automorphism, there exists a continuous inverse $\mathscr{A}^{-1}$ which, by the same arguments, possesses a unique continuous linear lift $(\widehat{\mathscr{A}^{-1}})$ represented by an integer matrix A' with finite support rows. By the uniqueness of lifts, the relations $\mathscr{A}\circ\mathscr{A}^{-1} = \mathrm{Id}_{\mathbb{T}^{\mathbb{N}}}$ and $\mathscr{A}^{-1}\circ\mathscr{A} = \mathrm{Id}_{\mathbb{T}^{\mathbb{N}}}$ imply that $\hat{\mathscr{A}}\circ(\widehat{\mathscr{A}^{-1}}) = \mathrm{Id}_{\mathbb{R}^{\mathbb{N}}}$ and $(\widehat{\mathscr{A}^{-1}})\circ\hat{\mathscr{A}} = \mathrm{Id}_{\mathbb{R}^{\mathbb{N}}}$, respectively. In terms of matrices, this yields $AA' = A'A = I$, confirming that $A \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$. ■

4.2.5 Reduction of Resonant Kronecker Flows

Let $\mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$ be the group of bijective maps $\mathscr{B}$ from $\mathbb{Z}^{(\mathbb{N})}$ to itself such that both $\mathscr{B}$ and $\mathscr{B}^{-1}$ are homomorphisms of $\mathbb{Z}$-modules.

Example 1 (Permutations in $\mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$)**.** *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. We define the operator P_σ by its action on the basis: $P_\sigma(e_j) = e_{\sigma(j)}$, extending it by linearity to any $\nu = \sum \nu_j e_j \in \mathbb{Z}^{(\mathbb{N})}$.*

Since any $\nu \in \mathbb{Z}^{(\mathbb{N})}$ has finite support, its image $P_\sigma(\nu)$ also has finite support, making P_σ a well-defined $\mathbb{Z}$-module homomorphism from $\mathbb{Z}^{(\mathbb{N})}$ into itself. The associated matrix $(b_{ij})_{i,j\in\mathbb{N}}$ is:

$$b_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

Since σ is a bijection, each row i has exactly one non-zero entry at column $\sigma^{-1}(i)$. This ensures that the inverse operator $P_{\sigma^{-1}}$ is also a well-defined homomorphism on $\mathbb{Z}^{(\mathbb{N})}$, thus $P_\sigma \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$.

Proposition 4.2.20. *An operator $\mathscr{B}$ belongs to $\mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$ if and only if its representing matrix $B = (b_{ij})_{i,j\in\mathbb{N}}$ with respect to the canonical basis $\{e_j\}_{j\in\mathbb{N}}$ satisfies $B^* \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$, where B^* denotes the transpose matrix of B.*

Proof. ($\Longrightarrow$) Let $\mathscr{B} \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$. As a $\mathbb{Z}$-module homomorphism, $\mathscr{B}$ is uniquely determined by its action on the canonical basis $\{e_j\}_{j\in\mathbb{N}}$. Since $\mathscr{B}(e_j) \in \mathbb{Z}^{(\mathbb{N})}$ for all $j \in \mathbb{N}$, we can write its image as a finite linear combination of the basis vectors:

$$\mathscr{B}(e_j) = \sum_{i\in\mathbb{N}} b_{ij} e_i,$$

where $b_{ij} \in \mathbb{Z}$ are zero for all but finitely many i. This defines the representing matrix $B = (b_{ij})_{i,j\in\mathbb{N}}$, which has integer entries and columns with finite support. Furthermore, as $\mathscr{B}$ is an automorphism, there exists an inverse homomorphism $\mathscr{B}^{-1}$ on $\mathbb{Z}^{(\mathbb{N})}$ whose matrix $B^{-1} = (b'_{jk})_{j,k\in\mathbb{N}}$ must also have integer entries and columns with finite support. The infinite matrix products BB^{-1} and $B^{-1}B$ are well-defined because each entry is a sum with only finitely many non-zero terms: specifically, $(BB^{-1})_{ik} = \sum_{j\in\mathbb{N}} b_{ij}b'_{jk}$ involves only finitely many terms due to the finite support of the columns of B^{-1}, while $(B^{-1}B)_{ik} = \sum_{j\in\mathbb{N}} b'_{ij}b_{jk}$ is finite due to the finite support of the columns of B. This property ensures that the standard algebraic identity for the transpose of a product $(BB^{-1})^* = (B^{-1})^*B^*$ holds. Thus, transposing $BB^{-1} = B^{-1}B = I$ yields $(B^{-1})^*B^* = B^*(B^{-1})^* = I$, showing that B^* is invertible in the sense of $\mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$ with $(B^*)^{-1} = (B^{-1})^*$. Since the rows of B^* and $(B^*)^{-1}$ are the columns of B and B^{-1} respectively, we have $B^* \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$.

($\Longleftarrow$) Conversely, suppose $B^* \in \mathrm{FGL}_{\mathbb{N}}(\mathbb{Z})$. By definition, both B^* and its inverse $(B^*)^{-1}$ have integer entries and rows with finite support. This ensures that $B = (b_{ij})_{i,j\in\mathbb{N}}$ and the matrix $B' := ((B^*)^{-1})^*$ have integer entries and columns with finite support. The finite support of the columns of B, together with the fact that $b_{ij} \in \mathbb{Z}$ for all $i, j \in \mathbb{N}$, ensures that the linear extension $\mathscr{B}$ of the map $e_j \mapsto \sum_i b_{ij}e_i$ is a well-defined homomorphism from $\mathbb{Z}^{(\mathbb{N})}$ into itself; similarly, the operator $\mathscr{B}'$ associated to B' is a well-defined homomorphism on $\mathbb{Z}^{(\mathbb{N})}$. Again, the finite support of the columns of B and B' justifies the identity $(B'B)^* = B^*(B')^* = B^*(B^*)^{-1} = I$, which implies $B'B = I$. A symmetric argument shows $BB' = I$, proving that $\mathscr{B}'$ is the inverse of $\mathscr{B}$ and thus $\mathscr{B} \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$. ■

Lemma 4.2.21. *Let $\nu \in \mathbb{Z}^{(\mathbb{N})} \setminus \{0\}$. Then, there exists $\mathscr{B} \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$ such that $\tilde{\nu} := \mathscr{B}\nu$ satisfies $\tilde{\nu}_1 \neq 0$ and $\tilde{\nu}_j = 0$ for all $j \geq 2$.*

Proof. If the support of ν consists of a single element, say j^*, then it is sufficient to consider the permutation operator P_σ (as in Example 1.1) that swaps the first and the j^*-th component. By Proposition 4.2.20, $P_\sigma \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$.

Consider now the case where the cardinality of the support of ν is $N > 1$. Let B_1 be the matrix that assigns to the given ν the list $(|\nu_{j_1}|, \dots, |\nu_{j_N}|, 0, \dots)$, where indices are chosen such that $1 \leq |\nu_{j_1}| \leq \dots \leq |\nu_{j_N}|$. This matrix represents an automorphism of $\mathbb{Z}^{(\mathbb{N})}$ as it is a finite composition of permutations P_σ and a diagonal matrix with ± 1 on the diagonal. Define $\nu^1 \in \mathbb{Z}^{(\mathbb{N})}$ by setting $\nu^1_k = |\nu_{j_k}|$ for $k = 1, \dots, N$ and $\nu^1_k = 0$ for $k > N$, and let

$$S^1 := \sum_{j=1}^{N} \nu^1_j.$$

Consider the transformation $\nu^1 \mapsto \nu^2$ defined by $\nu^2_1 = \nu^1_1$, $\nu^2_j = \nu^1_j - \nu^1_1$ for $2 \leq j \leq N$, and $\nu^2_j = 0$ for $j > N$. This transformation is induced by a matrix B_2 representing an automorphism of $\mathbb{Z}^{(\mathbb{N})}$:

$$B_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ -1 & 1 & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & & \\ -1 & \vdots & & \ddots & \\ 0 & \vdots & & & \ddots \\ \vdots & \vdots & & & \end{pmatrix} \qquad B_2^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & & \\ 1 & \vdots & & \ddots & \\ 0 & \vdots & & & \ddots \\ \vdots & \vdots & & & \end{pmatrix}$$

Note that $\nu^2_1 = \nu^1_1 \geq 1$ and, for $j = 2, \dots, N$, we have $0 \leq \nu^2_j < \nu^1_j$. Thus, if we define $S^2 := \sum_{j=1}^{N} \nu^2_j$, we obtain $1 \leq S^2 < S^1$.

At this stage, two cases may occur:

1. If ν^2 has only one non-zero component, then by construction this component is ν^2_1 and the process terminates.

2. If ν^2 still has more than one non-zero component, we denote by $K \geq 0$ the number of components that have become zero (i.e., $\nu^2_j = 0$ for some $j \in \{2, \dots, N\}$). We then apply a permutation P_σ to move the $N - K$ remaining non-zero components to the first $N - K$ positions.

After this rearrangement, we reorder the remaining non-zero components in non-decreasing order and repeat the entire procedure. Let S^α denote the sum of the components of the resulting vector at the end of the α-th step. By iterating the process, we generate a strictly decreasing sequence of positive integers:

$$S^1 > S^2 > S^3 > \cdots > S^\alpha > \cdots \geq 1.$$

This procedure must terminate in a finite number of steps $\bar{\alpha}$ because a strictly decreasing sequence of positive integers cannot be infinite. Specifically, the process stops when the sum $S^{\bar{\alpha}}$ consists of only one non-zero term.

By construction, this remaining term is moved to the first position. Since the overall transformation $\mathscr{B}$ is a composition of a finite number of automorphisms, it follows from Proposition 4.2.20 and the group property of $\mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$ that $\mathscr{B} \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$, which completes the proof. ■

Lemma 4.2.22. *Given $\omega \in \mathbb{R}^\mathbb{N}$, if there exists $\nu \in \mathbb{Z}^{(\mathbb{N})}$ such that $\sum_{j\in\mathbb{N}} \omega_j \nu_j = 0$, then, there is an $\tilde{\omega} \in \mathbb{R}^\mathbb{N}$, with $\tilde{\omega}_1 = 0$, such that the Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ is topologically conjugate to $(\mathbb{T}^\mathbb{N}, \Phi_{\tilde{\omega}})$.*

Proof. Let $\nu \in \mathbb{Z}^{(\mathbb{N})} \setminus \{0\}$ be such that $\sum_{j\in\mathbb{N}} \omega_j \nu_j = 0$. By Lemma 4.2.21, there exists $\mathscr{B} \in \mathrm{Aut}(\mathbb{Z}^{(\mathbb{N})})$ such that $\tilde{\nu} := \mathscr{B}\nu$ satisfies $\tilde{\nu}_1 \neq 0$ and $\tilde{\nu}_j = 0$ for all $j > 1$. Remark 4.2.20 tells us that such a $\mathscr{B}$ is represented by a matrix B such that $B^* \in \mathrm{FGL}_\mathbb{N}(\mathbb{Z})$. By Proposition 4.2.19, there exists an automorphism $\mathscr{A}$ of $\mathbb{T}^\mathbb{N}$ that is represented by B^*. But then, the Kronecker flows $(\mathbb{T}^\mathbb{N}, \Phi_{\tilde{\omega}})$, with $\tilde{\omega} := B^*\tilde{\omega}$, and $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ are topologically conjugate, and $\tilde{\omega}_1 = 0$. ■

Proposition 4.2.23. *Let $\omega \in \mathbb{R}^\mathbb{N}$ be such that $1 \leq d := \mathrm{rank}(\mathscr{R}_\omega) < \infty$. Then, $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ is topologically conjugate to another Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_{\tilde{\omega}})$ with $\tilde{\omega} = (0_d, \bar{\omega})$, where $\bar{\omega}$ is non-resonant and 0_d is the null vector in $\mathbb{R}^d$.*

Proof. By the above lemma, a resonant Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ is topologically conjugate to another Kronecker flow such that the first component of the frequency vector vanishes. If the restriction to the support of such a new frequency vector is still resonant, we get rid of another frequency following the same procedure as before. If the dimension of the resonance module is finite, this process will end after a number of iteration equal to the dimension of $\mathscr{R}_\omega$, because it is sufficient to do it for a basis of $\mathscr{R}_\omega$. The resulting frequency vector $\tilde{\omega}$ will be such that its non-vanishing frequencies are rationally independent by construction. ■

4.3 The Non-Resonant Case

1. *The Haar Measure.* Let $\mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R})$ be the space of trigonometric polynomials on $\mathbb{T}^\mathbb{N}$, i.e.,

$$\mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R}) := \left\{ p : \mathbb{T}^\mathbb{N} \to \mathbb{R} \ : \ p(\theta) = \sum_{\nu\in\mathbb{Z}^{(\mathbb{N})}} a_\nu e^{\mathrm{i}\Theta\cdot\nu}, \ (a_\nu)_{\nu\in\mathbb{Z}^{(\mathbb{N})}} \in \bigoplus_{\nu\in\mathbb{Z}^{(\mathbb{N})}} \mathbb{C}, \ a_\nu = \overline{a_{-\nu}} \right\}.$$

Consider the linear functional

$$\tilde{\Lambda} : \mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R}) \ni p \mapsto \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} p(\Theta)\, d\Theta_1 \ldots d\Theta_N. \tag{4.8}$$

It is easy to verify that the limit exists for any trigonometric polynomial. In fact, given $\nu \in \mathbb{Z}^{(\mathbb{N})}$, there exists $M < \infty$ such that $e^{\mathrm{i}\Theta\cdot\nu} = e^{\mathrm{i}\nu_1\Theta_1} e^{\mathrm{i}\nu_2\Theta_1} \cdots e^{\mathrm{i}\nu_M\Theta_M}$. Thus, for $N \geq M$,

$$\lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} e^{\mathrm{i}\Theta\cdot\nu}\, d\Theta_1 \ldots d\Theta_N = \lim_{N\to\infty} \prod_{j=1}^{M} \int_0^{2\pi} e^{\mathrm{i}\nu_j\Theta_j} \frac{d\Theta_j}{2\pi} \prod_{k=M+1}^{N} \int_0^{2\pi} \frac{d\Theta_k}{2\pi}.$$

The right-hand side of this equality can only take the values 0, if there is $\nu_j \neq 0$, or 1, if $\nu = 0$. In particular, denoting by $\mathbb{1}$ the trigonometric polynomial identically equal to one, we have

$$\tilde{\Lambda}(\mathbb{1}) = 1. \tag{4.9}$$

Furthermore, $\tilde{\Lambda}$ is Lipschitz relative to the uniform norm, i.e.

$$\left| \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} p(\Theta)\, d\Theta_1 \dots d\Theta_N \right| \leq \|p\|_\infty \cdot 1.$$

Since $\mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R})$ is a unital subalgebra of $\mathscr{C}^0(\mathbb{T}^\mathbb{N}, \mathbb{R})$ which separates the points of the compact space $\mathbb{T}^\mathbb{N}$, by the Stone-Weierstrass Theorem, each continuous function on $\mathbb{T}^\mathbb{N}$ is a uniform limit of some sequence of trigonometric polynomials. Then, $\tilde{\Lambda}$ extends uniquely to a Lipschitz (and hence, continuous) linear functional

$$\Lambda : \mathscr{C}^0(\mathbb{T}^\mathbb{N}, \mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) := \lim_{n\to\infty} \tilde{\Lambda}(p_n), \tag{4.10}$$

where $(p_n)_{n\in\mathbb{N}}$ is any sequence in $\mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R})$ converging uniformly to f. Moreover, Λ enjoys also the property well explicated in the proposition just below.

Proposition 4.3.1. *The linear functional Λ is positive, meaning that, if $f \in \mathscr{C}^0(\mathbb{T}^\mathbb{N}, \mathbb{R})$ is such that $\inf_{\theta\in\mathbb{T}^\mathbb{N}} f(\theta) \geq 0$, then $\Lambda(f) \geq 0$.*

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be any sequence in $\mathscr{P}(\mathbb{T}^\mathbb{N}, \mathbb{R})$ uniformly converging to f. Then,

$$\begin{aligned} \Lambda(f) = \lim_{n\to\infty} \tilde{\Lambda}(p_n) &= \lim_{n\to\infty} \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int_{\mathbb{T}^N} p_n(\Theta)\, d\Theta_1 \dots d\Theta_N \\ &\geq \lim_{n\to\infty} \inf_{\theta\in\mathbb{T}^\mathbb{N}} p_n(\theta) = \inf_{\theta\in\mathbb{T}^\mathbb{N}} f(\theta) \geq 0\,. \end{aligned}$$

■

In what follows, we will repeatedly apply the following celebrated theorem.

Theorem 4.3.2 (Riesz Representation Theorem [Rud87]). *Let X be a locally compact space, and let Λ be a positive and continuous linear functional on $\mathscr{C}^0(X, \mathbb{R})$. Then there exists a σ-algebra $\mathfrak{M}$ on X which contains all Borel subsets of X, and there exists a unique positive measure λ on $\mathfrak{M}$ which represents Λ in the sense that*

$$\Lambda f = \int_X f\, d\lambda, \quad \forall f \in \mathscr{C}^0(X, \mathbb{R}),$$

and which has the following additional properties:

(a) $\lambda(V) := \sup\{\Lambda(f) \ :\ supp(f) \subset V\}$ for any open set $V \subset X$;

(b) $\lambda(K) < \infty$ for every compact set $K \subset X$;

(c) λ is inner and outer regular, meaning that for every $E \subset \mathfrak{M}$ we have

$$\lambda(E) = \inf\{\lambda(V) \ :\ E \subset V,\ V \text{ open}\} = \sup\{\lambda(K) \ :\ K \subset E,\ K \text{ compact}\}.$$

Hence, $\mathbb{T}^{\mathbb{N}}$ admits a σ-algebra $\mathfrak{M}$ containing all Borel sets such that there exists a unique positive measure λ on $\mathfrak{M}$ associated to the linear functional Λ (defined in (4.8) and (4.10)) in the sense of the above theorem. Thus, by restricting λ to the Borel algebra $\mathfrak{B}$ and taking (4.9) into account, we conclude that $\mu := \lambda_{|\mathfrak{B}}$ is the unique regular Borel probability measure on $\mathbb{T}^{\mathbb{N}}$ that represents Λ, i.e.,

$$\Lambda(f) = \int_{\mathbb{T}^{\mathbb{N}}} f \, d\mu, \quad \forall f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R}).$$

Theorem 4.3.3. *Let $\theta' \in \mathbb{T}^{\mathbb{N}}$ and let $R_{\theta'} : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$ be the translation defined by $R_{\theta'}(\theta) = \theta * \theta'$, for all $\theta \in \mathbb{T}^{\mathbb{N}}$. The unique Borel measure μ associated to Λ (as defined in* (4.10)*) in the sense of Theorem 4.3.2 is $R_{\theta'}$-invariant.*

Proof. By Theorem 4.3.2, specifically properties (a) and (c), it suffices to show that $\Lambda(f \circ R_{\theta'}) = \Lambda(f)$ for all $f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$. First, observe that this equality clearly holds for any trigonometric polynomial $p \in \mathscr{P}(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$.

Now, let $f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence of trigonometric polynomials in $\mathscr{P}(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$ that converges uniformly to f. It follows that the sequence $(p_n \circ R_{\theta'})_{n\in\mathbb{N}}$ converges uniformly to $f \circ R_{\theta'}$. Consequently, by the continuity of the functional Λ, we have:

$$\Lambda(f \circ R_{\theta'}) = \lim_{n\to\infty} \Lambda(p_n \circ R_{\theta'}) = \lim_{n\to\infty} \Lambda(p_n) = \Lambda(f).$$

■

In particular, μ is Φ^t_ω-invariant for all $\omega \in \mathbb{R}^{\mathbb{N}}$ and any time t. Thus, (4.8) and (4.10) define a probability invariant Borel measure for any Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$.

Remark 4.3.4. *The measure just constructed is nothing but the Haar measure on the infinite torus, and corresponds to the invariant integral introduced by Jessen in [Jes34]. For references on the Haar measure, see for e.g. [Die68].*

2. *Proof of Theorem A.* Let us first prove that, given $f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$, its time average along any orbit of a non-resonant Kronecker flow converges uniformly to the integral of f with respect to the Haar measure over the whole torus. This step is crucial for the proof of Theorem A.

Theorem 4.3.5. *If $\mathscr{R}_\omega = \{0\}$, then $\forall f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$*

$$\lim_{T\to\infty} \left\| \frac{1}{T} \int_0^T f \circ \Phi^t_\omega \, dt - \int_{\mathbb{T}^{\mathbb{N}}} f \, d\mu \right\|_\infty = 0. \tag{4.11}$$

Proof. Fix $f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$. If f is constant, then (4.11) is trivial. Moreover, $\forall \nu \in \mathbb{Z}^{(\mathbb{N})} \setminus \{0\}$,

$$\lim_{T\to\infty} \sup_{\theta\in\mathbb{T}^{\mathbb{N}}} \left| \frac{1}{T} \int_0^T e^{\mathrm{i}\nu\cdot(\Theta+\omega t)} \, dt - \int_{\mathbb{T}^{\mathbb{N}}} e^{\mathrm{i}\nu\cdot\Theta} \, d\mu \right| = \lim_{T\to\infty} \sup_{\theta\in\mathbb{T}^{\mathbb{N}}} \left| \frac{1}{T} e^{\mathrm{i}\nu\cdot\Theta} \frac{e^{\mathrm{i}\nu\cdot\omega T} - 1}{\mathrm{i}\omega\cdot\nu} \right| = 0.$$

This is also true for any finite linear combination, i.e., for any trigonometric polynomial. Now, given $\varepsilon > 0$, let $p \in \mathscr{P}(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$ be such that $\|f - p\|_\infty < \frac{\varepsilon}{3}$ and let $T_{\frac{\varepsilon}{3}}$ be such that

$$\left\| \frac{1}{T} \int_0^T p \circ \Phi^t_\omega \, dt - \int_{\mathbb{T}^{\mathbb{N}}} p \, d\mu \right\|_\infty < \frac{\varepsilon}{3} \quad \forall T > T_{\frac{\varepsilon}{3}}.$$

Then,

$$\begin{aligned}\left\|\frac{1}{T}\int_0^T f\circ\Phi_\omega^t\,dt-\int_{\mathbb{T}^{\mathbb{N}}} f\,d\mu\right\|_\infty &\le \left\|\frac{1}{T}\int_0^T (f-p)\circ\Phi_\omega^t\,dt\right\|_\infty\\ &+\left\|\frac{1}{T}\int_0^T p\circ\Phi_\omega^t\,dt-\int_{\mathbb{T}^{\mathbb{N}}} p\,d\mu\right\|_\infty+\left|\int_{\mathbb{T}^{\mathbb{N}}}(f-p)\,d\mu\right|\\ &\le 2\|f-p\|_\infty+\left\|\frac{1}{T}\int_0^T p\circ\Phi_\omega^t\,dt-\int_{\mathbb{T}^{\mathbb{N}}} p\,d\mu\right\|_\infty<\varepsilon.\end{aligned}$$

Therefore, for all $\varepsilon>0$, we have

$$\left\|\frac{1}{T}\int_0^T f\circ\Phi_\omega^t\,dt-\int_{\mathbb{T}^{\mathbb{N}}} f\,d\mu\right\|_\infty<\varepsilon\quad\forall T>T_{\frac{\varepsilon}{3}}.$$

■

Theorem A. *The following properties of a Kronecker flow on $\mathbb{T}^{\mathbb{N}}$ are equivalent:*

(a) it is non-resonant;

(b) it is uniquely ergodic;

(c) it is minimal;

(d) it is topologically transitive.

Proof. $(a)\Rightarrow(b)$. Let μ' be an invariant regular probability Borel measure for the Kronecker flow $(\mathbb{T}^{\mathbb{N}},\Phi_\omega)$. Then, for any continuous function $f:\mathbb{T}^{\mathbb{N}}\to\mathbb{R}$, one has

$$\int_{\mathbb{T}^{\mathbb{N}}} f\,d\mu'=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_{\mathbb{T}^{\mathbb{N}}} f\,d\mu'\right)dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_{\mathbb{T}^{\mathbb{N}}} f\circ\Phi_\omega^t\,d\mu'\right)dt\,. \tag{4.12}$$

Since $|f\circ\Phi_\omega^t|$ is a continuous function on the compact domain $[0,T]\times\mathbb{T}^{\mathbb{N}}$, its integral over the product Borel measure space $[0,T]\times\mathbb{T}^{\mathbb{N}}$ is finite. Therefore, by Fubini's Theorem, we can exchange the order of integration, namely,

$$(4.12)=\lim_{T\to\infty}\int_{\mathbb{T}^{\mathbb{N}}}\frac{1}{T}\left(\int_0^T f\circ\Phi_\omega^t\,dt\right)d\mu'.$$

For all $\theta\in\mathbb{T}^{\mathbb{N}}$ and $T>0$, we observe that

$$\left|\frac{1}{T}\int_0^T f\circ\Phi_\omega^t(\theta)\,dt\right|\le\sup_{\theta\in\mathbb{T}^{\mathbb{N}}}|f(\theta)|=\|f\|_\infty.$$

Since $f\in\mathscr{C}^0(\mathbb{T}^{\mathbb{N}},\mathbb{R})$ and $\mathbb{T}^{\mathbb{N}}$ is compact, the constant $\|f\|_\infty$ is finite and thus provides an integrable dominating function on the probability space $(\mathbb{T}^{\mathbb{N}},\mu')$.

Furthermore, if ω is non-resonant, Theorem 4.3.5 ensures that the time average converges uniformly (and thus pointwise) to $\int_{\mathbb{T}^{\mathbb{N}}} f\,d\mu$ as $T\to\infty$. Since the map

$$T\mapsto\frac{1}{T}\int_0^T f\circ\Phi_\omega^t\,dt$$

is continuous and its limit as $T\to\infty$ exists, this limit is uniquely determined and coincides with the limit along any sequence $(T_n)_{n\in\mathbb{N}}$ such that $T_n\to\infty$ as $n\to\infty$. This allows us to invoke the Lebesgue's

Dominated Convergence Theorem for any such sequence, effectively justifying the interchange of the limit and the integral over $\mathbb{T}^{\mathbb{N}}$. Therefore, we obtain:

$$(4.12) = \int_{\mathbb{T}^{\mathbb{N}}} \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ \Phi_\omega^t \, dt \right) d\mu' = \int_{\mathbb{T}^{\mathbb{N}}} \left(\int_{\mathbb{T}^{\mathbb{N}}} f \, d\mu \right) d\mu'.$$

Using the fact that μ' is a probability measure, we finally get

$$\int_{\mathbb{T}^{\mathbb{N}}} f \, d\mu' = \int_{\mathbb{T}^{\mathbb{N}}} f \, d\mu \,.$$

Since $f \in \mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$ is arbitrary, by Theorem 4.3.2, we conclude that $\mu' = \mu$.

$(b) \Rightarrow (a)$. Assume that there exists $\nu \in \mathbb{Z}^{(\mathbb{N})} \setminus \{0\}$ such that $\omega \cdot \nu = 0$. By Lemma 4.2.22, $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ is topologically conjugate to another Kronecker flow whose frequency vector $\tilde{\omega}$ has the first component equal to zero. Given $x \in \mathbb{T}$, define the linear functional

$$\tilde{\Lambda}_x : \mathscr{P}(\mathbb{T}^{\mathbb{N}}, \mathbb{R}) \ni p \mapsto \lim_{N\to\infty} \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{T}^{N-1}} p(x, \Theta_2, \Theta_3, \dots) \, d\Theta_2 \dots d\Theta_N .$$

As in the case discussed in the introduction to this section, all these functionals extend to positive linear functionals Λ_x on the entire space $\mathscr{C}^0(\mathbb{T}^{\mathbb{N}}, \mathbb{R})$. To each of them, there is associated a unique regular Borel probability measure μ_x in the sense of Theorem 4.3.2.

Moreover, these measures are all invariant under the Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$. Let $\mathscr{A}$ be the automorphism of $\mathbb{T}^{\mathbb{N}}$ that conjugates $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ to $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$. Then, for any $x \in \mathbb{T}$, the pullback probability Borel measures $\mathscr{A}^*\mu_x$ are invariant under $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$. Clearly, for distinct values of $x \in \mathbb{T}$, the resulting pullback measures $\mathscr{A}^*\mu_x$ are distinct, thereby proving the existence of multiple invariant probability measures for the flow $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$.

$(a) \Rightarrow (c)$. Let $\omega \in \mathbb{R}^{\mathbb{N}}$ be non-resonant. Suppose that there exists a non-empty open set $V \subset \mathbb{T}^{\mathbb{N}}$ that is never visited by a certain orbit. Then the statement of Theorem 4.3.5 would be violated, for instance, by any continuous function f which is strictly positive on a compact subset $K \subset V$ and such that $f|_{(\mathbb{T}^{\mathbb{N}} \setminus V)} \equiv 0$; such a function exists by Urysohn's Lemma.

$(c) \Rightarrow (a)$. We prove the contrapositive. As in the proof of $(b) \Rightarrow (a)$, if a Kronecker flow is resonant, it is topologically conjugate through a certain $\mathscr{A} \in \mathrm{Aut}(\mathbb{T}^{\mathbb{N}})$ to another Kronecker flow whose frequency vector has the first component equal to zero. Let $\theta \in \mathbb{T}^{\mathbb{N}}$. Then, the open set $\mathscr{A}^{-1}(C)$, where C is the open cylinder

$$C := (\mathbb{T} \setminus \{\Theta_1 + 2\pi\mathbb{Z}\}) \times \prod_{j=2}^{\infty} \mathbb{T} \subset \mathbb{T}^{\mathbb{N}},$$

is never visited by the orbit of $\mathscr{A}^{-1}(\theta)$.

$(d) \Rightarrow (c)$. It is an immediate consequence of Remark 4.1.4.

$(c) \Rightarrow (d)$. It follows trivially by definition. ■

4.4 The Resonant Case: an Algebraic Viewpoint

4.4.1 The "Free" Case

The main goal of this subsection is to give a proof of Theorem B, which gives a dynamical characterization of the class of Kronecker flows whose frequency module is a free abelian group.

Notation 4.4.1 (Closure of the orbit through the origin)**.** *Given a frequency vector $\omega \in \mathbb{R}^{\mathbb{N}}$, we denote by $\mathcal{O}_\omega$ the compact subgroup of $\mathbb{T}^{\mathbb{N}}$ obtained by taking the closure of the orbit through the origin induced by the Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$:*

$$\mathcal{O}_\omega := \overline{\{\Phi_\omega^t(0) \ : \ t \in \mathbb{R}\}}.$$

Remark 4.4.2. *To see that $\mathcal{O}_\omega$ is a subgroup, let $E = \{\Phi_\omega^t(0) : t \in \mathbb{R}\}$. Since the map $t \mapsto \Phi_\omega^t(0)$ is a continuous group homomorphism, its image E is a subgroup of $\mathbb{T}^{\mathbb{N}}$. Now, let $\theta, \varphi \in \mathcal{O}_\omega$. By definition of closure, there exist sequences $(\theta_n)_{n\in\mathbb{N}}, (\varphi_n)_{n\in\mathbb{N}} \subset E$ such that $\theta_n \to \theta$ and $\varphi_n \to \varphi$ as $n \to \infty$. Due to the continuity of the group operations in $\mathbb{T}^{\mathbb{N}}$, we have:*

1. *$\theta_n * \varphi_n \to \theta * \varphi$. The fact that E is a subgroup ensures $(\theta_n * \varphi_n) \in E$ for all $n \in \mathbb{N}$, so $(\theta * \varphi) \in \overline{E} = \mathcal{O}_\omega$.*

2. *$-\theta_n \to -\theta$. Since E is a subgroup, $-\theta_n \in E$ for all $n \in \mathbb{N}$, hence $-\theta \in \overline{E} = \mathcal{O}_\omega$.*

Thus, $\mathcal{O}_\omega$ is a subgroup. Its compactness follows from being a closed subset of the compact space $\mathbb{T}^{\mathbb{N}}$.

Lemma 4.4.3. *Let $\omega, \tilde{\omega} \in \mathbb{R}^{\mathbb{N}}$ be two frequency vectors. Assume that:*

(i) the Kronecker flows $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ and $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ are topologically conjugate via a homeomorphism $h : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$ such that $h(0) = 0$;

(ii) the non-vanishing components $(\tilde{\omega}_j)_{j\in J}$ are rationally independent, where $J \subseteq \mathbb{N}$ is the support of $\tilde{\omega}$.

Then, $\mathscr{M}_\omega = \mathscr{M}_{\tilde{\omega}}$.

Proof. By the conjugacy relation $h \circ \Phi_{\tilde{\omega}}^t = \Phi_\omega^t \circ h$, h maps the orbit of the origin of the first flow into the orbit of the origin of the second. Since the infinite torus is compact, the continuity of h implies $h(\mathcal{O}_{\tilde{\omega}}) = \mathcal{O}_\omega$.

We first prove that the restriction $h|_{\mathcal{O}_{\tilde{\omega}}}$ is a group isomorphism. Let $\theta \in \mathcal{O}_{\tilde{\omega}}$; by definition of the closure of the orbit, there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\theta = \lim_{k\to\infty} \Phi_{\tilde{\omega}}^{t_k}(0)$. Now, let $\varphi \in \mathcal{O}_{\tilde{\omega}}$ be any other point. Using the conjugacy and the fact that $\Phi^t(\varphi) = \Phi^t(0) * \varphi$, we compute:

$$\begin{aligned} h(\theta * \varphi) &= h\left(\lim_{k\to\infty} \Phi_{\tilde{\omega}}^{t_k}(0) * \varphi\right) = \lim_{k\to\infty} h(\Phi_{\tilde{\omega}}^{t_k}(\varphi)) \\ &= \lim_{k\to\infty} \Phi_\omega^{t_k}(h(\varphi)) = \lim_{k\to\infty} (\Phi_\omega^{t_k}(0) * h(\varphi)) \\ &= \left(\lim_{k\to\infty} h(\Phi_{\tilde{\omega}}^{t_k}(0))\right) * h(\varphi) \\ &= h(\theta) * h(\varphi). \end{aligned}$$

Thus, $h|_{\mathcal{O}_{\tilde{\omega}}}$ is an injective group homomorphism. Since h is a homeomorphism and $h(\mathcal{O}_{\tilde{\omega}}) = \mathcal{O}_\omega$, the same argument applied to h^{-1} shows that $h|_{\mathcal{O}_{\tilde{\omega}}}$ is an isomorphism of topological groups between $\mathcal{O}_{\tilde{\omega}}$ and $\mathcal{O}_\omega$.

Now, let $\hat{h} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the unique lift of h that fixes the origin (see Theorem 4.2.17). Since $h|_{\mathcal{O}_{\tilde{\omega}}}$ is a continuous homomorphism and $\mathcal{O}_{\tilde{\omega}} = \pi_{\mathbb{N}}(E)$ for $E = \prod_{j\in\mathbb{N}} X_j$ (where $X_j = \mathbb{R}$ if $j \in J$ and $\{0\}$ otherwise), the restriction $\hat{h}|_E$ is linear. From the conjugacy, we obtain $\hat{h}(\tilde{\omega}) = \omega$.

Since h is a continuous map on $\mathbb{T}^{\mathbb{N}}$ that restricts to a homomorphism on $\mathcal{O}_{\tilde{\omega}}$, its lift $\hat{h}$ on E is represented by an integer matrix $A = (a_{ij})_{i\in\mathbb{N}, j\in J}$ where the continuity in the product topology ensures that each row has finite support (this follows by an argument analogous to that of Proposition 4.2.19).

Thus, $\omega_i = \sum_{j\in J} a_{ij}\tilde{\omega}_j$ for all $i \in \mathbb{N}$. Similarly, for h^{-1}, there exists an integer matrix $B = (b_{ij})_{i\in J, j\in\mathbb{N}}$ with rows of finite support such that $\tilde{\omega}_i = \sum_{j\in\mathbb{N}} b_{ij}\omega_j$ for all $i \in J$.

Substituting ω_j into the expression for $\tilde{\omega}_i$ ($i \in J$):

$$\tilde{\omega}_i = \sum_{j\in\mathbb{N}} b_{ij} \sum_{k\in J} a_{jk}\tilde{\omega}_k \implies \sum_{k\in J}\left(\sum_{j\in\mathbb{N}} b_{ij}a_{jk} - \delta_{ik}\right)\tilde{\omega}_k = 0.$$

By the rational independence of $(\tilde{\omega}_k)_{k\in J}$, we have $\sum_{j\in\mathbb{N}} b_{ij}a_{jk} = \delta_{ik}$. This implies that for any $\tilde{\nu} \in \mathbb{Z}^{(J)}$ there exists $\nu \in \mathbb{Z}^{(\mathbb{N})}$ such that $\tilde{\nu}_j = \sum_{i\in\mathbb{N}} \nu_i a_{ij}$ (specifically, $\nu_i = \sum_{j\in J} \tilde{\nu}_j b_{ji}$). Consequently:

$$\mathscr{M}_\omega = \left\{\sum_{i\in\mathbb{N}} \omega_i\nu_i = \sum_{j\in J}\tilde{\omega}_j \sum_{i\in\mathbb{N}} a_{ij}\nu_i \;:\; \nu \in \mathbb{Z}^{(\mathbb{N})}\right\} = \left\{\sum_{j\in J}\tilde{\omega}_j\tilde{\nu}_j \;:\; \tilde{\nu} \in \mathbb{Z}^{(J)}\right\} = \mathscr{M}_{\tilde{\omega}}.$$

This concludes the proof. ■

Note that one can give a shorter and less cumbersome proof of the above lemma by invoking Pontryagin's Duality Theorem.

Remark 4.4.4. *From an algebraic point of view, for a Kronecker flow the property of having non-zero frequencies that are rationally independent implies that the frequency module is free. Indeed, in that case, the map*

$$\bigoplus_{j\in J}\mathbb{Z} \ni \nu \to \sum_{j\in J}\omega_j\nu_j \in \mathscr{M}_\omega,$$

where $J \subseteq \mathbb{N}$ is the support of $\omega \in \mathbb{R}^{\mathbb{N}}$, is an isomorphism of abelian groups.

The observation above inspires the following

Theorem B. *Let $\omega \in \mathbb{R}^{\mathbb{N}}$ and let $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ be the corresponding Kronecker flow. Then, $\mathscr{M}_\omega$ is a free abelian group if and only if $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ is topologically conjugate to another Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ whose non-vanishing frequencies are rationally independent. If this is the case, $\mathscr{M}_\omega$ is isomorphic to $\mathbb{Z}^{(\mathit{supp}(\tilde{\omega}))}$.*

Proof. ($\implies$) Let $(\nu^{(\alpha)})_{\alpha\in\Lambda}$, $\Lambda \subseteq \mathbb{N}$ be a set of elements of $\mathbb{Z}^{(\mathbb{N})}$ such that

$$\left(\sum_{j\in\mathbb{N}}\omega_j\nu_j^{(\alpha)}\right)_{\alpha\in\Lambda}$$

is a basis for $\mathscr{M}_\omega$. Then, $\mathbb{Z}^{(\mathbb{N})}$ is isomorphic to

$$\mathscr{R}_\omega \bigoplus \mathrm{span}_{\mathbb{Z}}(\nu^{(\alpha)})_{\alpha\in\Lambda}. \tag{4.13}$$

Indeed, since $\mathscr{M}_\omega$ is isomorphic to the group of cosets $\mathbb{Z}^{(\mathbb{N})}/\mathscr{R}_\omega$ (Remark 4.2.9), for any $\nu \in \mathbb{Z}^{(\mathbb{N})}$, there exists a unique set of natural numbers $(n_\alpha)_{\alpha\in\Lambda}$ such that $\nu + \mathscr{R}_\omega = \sum_{\alpha\in\Lambda} n_\alpha(\nu^{(\alpha)} + \mathscr{R}_\omega)$, where only finitely many n_α are different from zero. Then, every $\nu \in \mathbb{Z}^{(\mathbb{N})}$ can be decomposed into $\nu = (\nu - \sum_{\alpha\in\Lambda} n_\alpha\nu^{(\alpha)}) + \sum_{\alpha\in\Lambda} n_\alpha\nu^{(\alpha)}$. Of course, $\nu - \sum_{\alpha\in\Lambda} n_\alpha\nu^{(\alpha)} \in \mathscr{R}_\omega$. This proves (4.13). Being a submodule of a free $\mathbb{Z}$-module, $\mathscr{R}_\omega$ is also free (see for e.g. [Rot02]). Let $(\xi^{(\beta)})_{\beta\in\Gamma}$, where Γ is a countable set of indices, be a basis for $\mathscr{R}_\omega$. Consider any matrix A constructed so that each row is either an element of $(\nu^{(\alpha)})_{\alpha\in\Lambda}$ or of $(\xi^{(\beta)})_{\beta\in\Gamma}$, and such that each element of $(\nu^{(\alpha)})_{\alpha\in\Lambda}$ and of $(\xi^{(\beta)})_{\beta\in\Gamma}$

appears exactly once as a row of A. This matrix corresponds to an automorphism $\mathscr{A} \in \mathrm{Aut}(\mathbb{T}^{\mathbb{N}})$ that conjugates $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ to $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde\omega})$, where $\tilde\omega := A\omega$. By construction, the non-vanishing components of $\tilde\omega$ are rationally independent.

($\Longleftarrow$) Let $h : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}^{\mathbb{N}}$ be a homeomorphism such that, for all t in $\mathbb{R}$,

$$h \circ \Phi^t_{\tilde\omega} = \Phi^t_\omega \circ h \ ,$$

where $\tilde\omega \in \mathbb{R}^{\mathbb{N}}$ is such that the non-vanishing components of $\tilde\omega$ are rationally independent. Without loss of generality, we assume $h(0) = 0$. Then, the claim follows from Lemma 4.4.3 and Remark 4.4.4. ■

We already pointed out in the introduction that if $\mathscr{M}_\omega$ is finitely generated, and in particular if ω has finite support, then $\mathscr{M}_\omega$ is free. This follows from the structure theorem for finitely generated abelian groups without torsion.

Finally, if the resonance module $\mathscr{R}_\omega$ has finite rank, Proposition 4.2.23 provides a conjugacy to a flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde\omega})$ whose non-vanishing frequencies are rationally independent. By Theorem B, this ensures that $\mathscr{M}_\omega$ must be a free abelian group, even though it is no longer finitely generated.

4.4.2 Solenoidal Orbits

In the general case, the topological structure of the orbits of a Kronecker flow on the infinite torus can be much more complicated. Inspired by Sakbaev and Volovich [SV24], here we construct a class of Kronecker flows such that the closure of any orbit induced by Kronecker flows in such class is a solenoid, namely a compact, connected subgroup of $\mathbb{T}^{\mathbb{N}}$ which is locally homeomorphic to the product of an interval and a Cantor set (see Proposition 4.4.8). These orbits have the property that they are uniformly in time approximated by periodic ones, with the period diverging as the approximation becomes more accurate. In accordance with the previous section, the frequency modules associated with such Kronecker flows must fail to be free abelian groups. More precisely, we shall see in the next section that they are isomorphic to non-free subgroups of the additive rationals (see Lemma 4.4.24 and Remark 4.4.25).

Notation 4.4.5. *We denote by Σ the set of sequences of natural numbers $\boldsymbol{a} = (a_j)_{j\in\mathbb{N}}$ with $a_1 = 1$, and $a_j > 1$ for all $j > 1$.*

Definition 4.4.6 (Solenoid)**.** *Given $\boldsymbol{a} \in \Sigma$, the solenoid $\sigma_{\boldsymbol{a}}$ is the compact, connected subgroup of $\mathbb{T}^{\mathbb{N}}$ defined as*

$$\sigma_{\boldsymbol{a}} := \left\{ \theta = (\Theta_j + 2\pi\mathbb{Z})_{j\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}} \ : \ \Theta_j = a_{j+1}\Theta_{j+1} \bmod 2\pi, \ \forall j \in \mathbb{N} \right\}.$$

Theorem 4.4.7. *Let $\boldsymbol{a} \in \Sigma$, and consider $\omega(\boldsymbol{a}) = (\omega_j(\boldsymbol{a}))_{j\in\mathbb{N}}$ defined by*

$$\omega_j(\boldsymbol{a}) = \prod_{k=1}^{j} a_k^{-1} \quad \text{for all } j \in \mathbb{N}.$$

Then, the closure of any orbit induced by the Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_{\omega(\boldsymbol{a})})$ is homeomorphic to the solenoid $\sigma_{\boldsymbol{a}}$.

Proof. By Remark 4.1.4, it is sufficient to prove the statement for the orbit through the origin. Note that for any time $t \in \mathbb{R}$, the point $(\omega_j(\boldsymbol{a})t + 2\pi\mathbb{Z})_{j\in\mathbb{N}}$ belongs to $\sigma_{\boldsymbol{a}}$. Since $\sigma_{\boldsymbol{a}}$ is closed, the closure of the orbit through the origin is necessarily contained in $\sigma_{\boldsymbol{a}}$.

To show the reverse inclusion, let $\theta \in \sigma_{\boldsymbol{a}}$. We seek a sequence of real numbers $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}(\omega_j(\boldsymbol{a})t_k + 2\pi\mathbb{Z}) = \theta_j + 2\pi\mathbb{Z}$ for all $j \in \mathbb{N}$. To this end, we first establish a one-to-one correspondence between $\sigma_{\boldsymbol{a}}$ and the set

$$[0,1) \times \prod_{j=2}^{\infty}\{0,1,\ldots,a_j-1\}.$$

Indeed, $\theta \in \sigma_{\boldsymbol{a}}$ if and only if, for each $j \geq 2$, there exists $n_j \in \mathbb{Z}$ such that $a_j\check{\theta}_j - \check{\theta}_{j-1} = n_j$, where $\check{\theta}_j$ is the unique representative of $(2\pi)^{-1}\theta_j$ in $[0,1)$. Since $0 \leq \check{\theta}_j < 1$, it follows that

$$-1 < -\check{\theta}_{j-1} \leq a_j\check{\theta}_j - \check{\theta}_{j-1} = n_j < a_j,$$

which implies $n_j \in \{0,1,\ldots,a_j-1\}$. Thus, each $\theta \in \sigma_{\boldsymbol{a}}$ uniquely determines a pair $(\tau,(n_j)_{j\geq 2})$ with $\tau = \check{\theta}_1 \in [0,1)$ and $n_j = a_j\check{\theta}_j - \check{\theta}_{j-1}$.

Conversely, given $\tau \in [0,1)$ and $(n_j)_{j\geq 2}$ in the specified ranges, we associate the unique $\theta \in \sigma_{\boldsymbol{a}}$ such that $\check{\theta}_1 = \tau$ and

$$\check{\theta}_j = \omega_j(\boldsymbol{a})\left(\tau + \sum_{m=2}^{j}\frac{n_m}{\omega_{m-1}(\boldsymbol{a})}\right) \quad \text{for all } j \geq 2. \tag{4.14}$$

This association is well-defined since $\check{\theta}_j$ belongs to $[0,1)$, as shown by the following estimate:

$$0 \leq \tau + \sum_{m=2}^{j}\frac{n_m}{\omega_{m-1}(\boldsymbol{a})} = \tau + \sum_{m=2}^{j} n_m \prod_{k=1}^{m-1} a_k < 1 + \sum_{m=2}^{j}(a_m-1)\prod_{k=1}^{m-1} a_k = \prod_{k=1}^{j} a_k = \frac{1}{\omega_j(\boldsymbol{a})}.$$

It is straightforward to check that $\theta \in \sigma_{\boldsymbol{a}}$ and that these two mappings are inverses of each other.

Finally, for a fixed $\theta \in \sigma_{\boldsymbol{a}}$, we define the sequence of times $(t_k)_{k\in\mathbb{N}}$ as

$$t_1 := 2\pi\check{\theta}_1, \quad t_k := 2\pi\left(\check{\theta}_1 + \sum_{m=2}^{k}\frac{n_m}{\omega_{m-1}(\boldsymbol{a})}\right) \quad \text{for all } k \geq 2.$$

By construction, for any fixed $j \in \mathbb{N}$, we have $\lim_{k\to\infty}(\omega_j(\boldsymbol{a})t_k + 2\pi\mathbb{Z}) = \theta_j + 2\pi\mathbb{Z}$ in the product topology. This confirms that $\mathcal{O}_{\omega(\boldsymbol{a})} = \sigma_{\boldsymbol{a}}$. ∎

Let us give a simple description of the local topological structure of solenoids.

Proposition 4.4.8. *For all $\boldsymbol{a} \in \Sigma$, any point of $\sigma_{\boldsymbol{a}}$ admits a neighbourhood which is homeomorphic to the product of an interval and a Cantor set.*

Proof. Let $\varphi = (\varphi_j)_{j\in\mathbb{N}} \in \sigma_{\boldsymbol{a}}$. Without loss of generality, we can always assume that $\varphi_1 = \pi + 2\pi\mathbb{Z}$. Consider the open neighbourhood U of φ in $\sigma_{\boldsymbol{a}}$ obtained by taking the intersection

$$U := \left[\left(\mathbb{T}\setminus\{0+2\pi\mathbb{Z}\}\right) \times \prod_{j=2}^{\infty}\mathbb{T}\right] \cap \sigma_{\boldsymbol{a}}.$$

We shall show that U is homeomorphic to the product of the interval $(0,1)$ and the Cantor set $K := \prod_{j=2}^{\infty}\{0,1,\ldots,a_j-1\}$. By the same argument used in the proof of Theorem 4.4.7, U is in a one-to-one correspondence with $(0,1)\times K$.

Let us show that the map

$$f : U \ni \theta \mapsto (\check{\theta}_1,(a_j\check{\theta}_j - \check{\theta}_{j-1})_{j=2}^{\infty}) \in (0,1)\times K$$

is a homeomorphism, where $(0,1)$ carries the standard topology, each $\{0,1,\dots,a_j-1\}$ is considered as discrete, and the infinite product is endowed with the product topology. Consider $\theta \in U$ and let $(\theta^k)_{k\in\mathbb{N}}$, $\theta^k \in U$ for all $k \in \mathbb{N}$, be a sequence converging to θ. Then,

$$\lim_{k\to\infty} (\theta_1^k + 2\pi\mathbb{Z}) = \theta_1 + 2\pi\mathbb{Z},$$
$$\lim_{k\to\infty} (\theta_j^k + 2\pi\mathbb{Z}) = \theta_j + 2\pi\mathbb{Z}, \quad \forall j > 1.$$

Since $\theta \in U$, the first limit reads $\lim_{k\to\infty} \check{\theta}_1^k = \check{\theta}_1$. The second limit is

$$\lim_{k\to\infty} \left(\omega_j \sum_{m=2}^{j} \frac{n_m^k}{\omega_{m-1}} + \mathbb{Z} \right) = \omega_j \sum_{m=2}^{j} \frac{n_m}{\omega_{m-1}} + \mathbb{Z}, \quad \forall j \geq 2, \tag{4.15}$$

where $n_m^k = a_m\check{\theta}_m^k - \check{\theta}_{m-1}^k$ and $n_m = a_m\check{\theta}_m - \check{\theta}_{m-1}$. But, for all $k \in \mathbb{N}$, we have

$$\omega_j \sum_{m=2}^{j} \frac{n_m^k}{\omega_{m-1}} \leq 1 - \omega_j < 1, \quad \text{for all } j \geq 2.$$

Thus, (4.15) reads

$$\lim_{k\to\infty} \sum_{m=2}^{j} \frac{n_m^k}{\omega_{m-1}} = \sum_{m=2}^{j} \frac{n_m}{\omega_{m-1}}, \quad \forall j \geq 2,$$

which means that $\lim_{k\to\infty} n_j^k = n_j$ for all $j \geq 2$ because the n_j^k take values in a discrete set. This proves that f is continuous. The continuity of f^{-1} follows from the fact that it is defined by the map (4.14), which is a composition of continuous functions.

It remains to show that $K = \prod_{j=2}^{\infty}\{0,1,\dots,a_j-1\}$ is a Cantor set, namely, that it is compact, totally disconnected, and each point is an accumulation point. Compactness follows from Tychonoff's theorem. To see that K is totally disconnected, which is a standard result for the product of discrete spaces, let $(n_j)_{j=2}^{\infty} \in K$ and consider $X \subseteq K$ such that X contains $(n_j)_{j=2}^{\infty}$ and at least another distinct point $(n'_j)_{j=2}^{\infty}$. Let $k \in \mathbb{N}, k \geq 2$, be such that $n_k \neq n'_k$, and consider the open cylinder

$$C := \{(m_j)_{j=2}^{\infty} \in K : m_k = n_k\}.$$

Since each factor carries the discrete topology, the complement $\overline{C}$ of C is also an open cylinder. Then, X can be written as a union of disjoint non-empty open subsets $X = (C \cap X) \cup (\overline{C} \cap X)$. Finally, we observe that each point of K is an accumulation point. Consider a list $(n_j)_{j=2}^{\infty} \in K$ and an open subset X containing it. X can be written as a union of open cylinders $\cup_\alpha C_\alpha$ such that $(n_j)_{j=2}^{\infty}$ belongs to some C_α. Since $C_\alpha = \prod_{j=2}^{\infty} X_j$, where $X_j \neq \{0,1,\dots,a_j-1\}$ only for finitely many j, there always exists $(n'_j)_{j=2}^{\infty} \in C_\alpha$ such that $n_k \neq n'_k$ for some $k \geq 2$. This concludes the proof. ■

4.4.3 Orbits as Pontryagin Duals

The first goal of this section is to show that, given a Kronecker flow on the infinite torus, the closure of any orbit is homeomorphic to the Pontryagin dual of the associated frequency module (Proposition 4.4.23). Thereafter, given any countable family $(G_i)_{i\in I}$ of subgroups of $(\mathbb{Q},+)$, we construct a Kronecker flow such that its frequency module is isomorphic to $\bigoplus_{i\in I} G_i$. Then, by using Pontryagin's theory of locally compact abelian groups, together with Theorem 4.4.7 and Proposition 4.4.23, we give the topological structure of the orbits. It turns out that the class of Kronecker flows so obtained is characterized by the fact that the closure of any orbit is a product of circles and solenoids (see Theorem C).

1. We begin by reviewing some fundamental definitions and results from Pontryagin's theory of locally compact abelian groups. For references, see the original paper by Lev Pontryagin [Pon34], and [Mor77] for a modern formulation of the theory.

Notation 4.4.9. *In the following, the symbol $\cong$ denotes an isomorphism of topological groups. Consistently with the definition of $\mathbb{T}$ given in 4.1.1, the group operation on the one dimensional torus is always expressed in additive notation and denoted by $+$.*

Definition 4.4.10 (Character)**.** *Let G be a locally compact abelian group. A character on G is a continuous homomorphism of groups $\chi : G \to \mathbb{T}$.*

Definition 4.4.11 (Pontryagin Dual)**.** *The Pontryagin dual G^* of a locally compact abelian group G is the group of all characters on G endowed with the compact-open topology. The group operation $*$ is defined by*

$$(\chi * \psi)(g) := \chi(g) + \psi(g), \quad \forall \chi, \psi \in G^*, \quad g \in G.$$

Theorem 4.4.12. *The Pontryagin dual of a locally compact abelian group is indeed a locally compact abelian group.*

Theorem 4.4.13. *The Pontryagin dual of a discrete abelian group is compact.*

Example 2. *The Pontryagin dual of $\mathbb{Z}$ is the torus $\mathbb{T}$, with the dual coupling given by*

$$\theta : \mathbb{Z} \ni n \mapsto n\Theta + 2\pi\mathbb{Z} \in \mathbb{T}.$$

Theorem 4.4.14. *Let J be a countable set of indices and consider $(G_j)_{j\in J}$ a family of discrete abelian groups. Then, the direct sum $\bigoplus_{j\in J} G_j$ is a discrete abelian group when considered with the product topology, and its Pontryagin dual is the (compact) abelian group $\prod_{j\in J}(G_j)^*$ endowed with the product topology.*

Example 3. *Let J be a countable set of indices. By Example 2 and Theorem 4.4.14, the Pontryagin dual of $\bigoplus_{j\in J} \mathbb{Z}$ is the torus $\prod_{j\in J} \mathbb{T}$, with dual coupling is given by*

$$\theta : \bigoplus_{j\in J} \mathbb{Z} \ni \nu \mapsto \sum_{j\in J} \nu_j \Theta_j + 2\pi\mathbb{Z} \in \mathbb{T}.$$

Definition 4.4.15 (Evaluation Character/Map)**.** *Given an element g of a locally compact abelian group G, the associated evaluation character $\mathrm{ev}_G(g)$ is the character on G^* defined by*

$$\mathrm{ev}_G(g)(\chi) := \chi(g), \quad \forall \chi \in G^*.$$

The map $\mathrm{ev}_G : G \to G^{**}$ is called evaluation map.

Theorem 4.4.16 (Pontryagin Duality)**.** *For any locally compact abelian group, the evaluation map is an isomorphism of topological groups.*

Definition 4.4.17 (Annihilator)**.** *Let G be a locally compact abelian group and H a closed subgroup. The annihilator of H is the (locally compact) subgroup of G^* defined by*

$$Ann(H) := \{\chi \in G^* \ : \ \chi(g) = 0, \ \forall g \in H\}.$$

Theorem 4.4.18. *Let G be a locally compact abelian group and H a closed subgroup. Then,*

$$Ann(Ann(H)) \cong H.$$

Proposition 4.4.19. *Let G be a locally compact abelian group and H a closed subgroup. Then, G/H is a locally compact abelian group.*

Theorem 4.4.20. *Let G be a locally compact abelian group and H a closed subgroup. Then, $(G/H)^* \cong Ann(H)$.*

2. With the above classical results in the theory of locally compact abelian groups, we can now proceed with exploring their implications in our context.

Lemma 4.4.21. *Let $\omega := (\omega_j)_{j\in\mathbb{N}}$, be a list of real numbers and $\mathcal{O}_\omega$ the closure of the orbit through the origin induced by the Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$. Then, the annihilator of $\mathcal{O}_\omega$ is the resonance module*

$$\mathscr{R}_\omega := \left\{\nu \in \mathbb{Z}^{(\mathbb{N})} \ : \ \sum_{j\in\mathbb{N}} \nu_j\omega_j = 0\right\}.$$

Proof. Let $\nu \in Ann(\mathscr{O}_\omega)$. Then, $t\sum_{j\in\mathbb{N}}\omega_j\nu_j \in 2\pi\mathbb{Z}$ for all $t \in \mathbb{R}$. This implies that $\nu \in \mathscr{R}_\omega$. Now, assume that $\nu \in \mathscr{R}_\omega$. For any $\theta \in \mathcal{O}_\omega$ there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that, for all $j \in \mathbb{N}$, $\Theta_j + 2\pi\mathbb{Z} = \lim_{k\to\infty} t_k\omega_j + 2\pi\mathbb{Z}$. Then,

$$\sum_{j\in\mathbb{N}} \Theta_j\nu_j + 2\pi\mathbb{Z} = \sum_{j\in\mathbb{N}} \lim_{k\to\infty} t_k\omega_j\nu_j + 2\pi\mathbb{Z} = 0 + 2\pi\mathbb{Z} = 0,$$

which ends the proof. ■

Proposition 4.4.22. *Given a Kronecker flow $(\mathbb{T}^\mathbb{N}, \Phi_\omega)$ on the infinite torus, the closure of the orbit through the origin is isomorphic as a topological group to $\mathscr{M}_\omega^*$, where the frequency module $\mathscr{M}_\omega$ is endowed with the discrete topology.*

Proof. First observe that, by the first isomorphism theorem (see Remark 4.2.9), $\mathscr{M}_\omega$ is isomorphic to $\mathbb{Z}^{(\mathbb{N})}/\mathscr{R}_\omega$. Then,

$$\mathscr{M}_\omega^* \cong \left(\mathbb{Z}^{(\mathbb{N})}/\mathscr{R}_\omega\right)^* \cong Ann(\mathscr{R}_\omega) \cong Ann(Ann(\mathcal{O}_\omega)) \cong \mathcal{O}_\omega.$$

The last isomorphisms follow from Theorem 4.4.20, Lemma 4.4.21, and Theorem 4.4.18, respectively. ■

Proposition 4.4.23. *Given a Kronecker flow Φ_ω on the infinite torus, the closure of any orbit is homeomorphic to $\mathscr{M}_\omega^*$.*

Proof. The claim follows by Remark 4.1.4 and Proposition 4.4.22. ■

3. As a consequence of a characterization of all the subgroups of the additive rationals (see [BZ51]), we have the following lemma, which is proved in Appendix E.3.

Lemma 4.4.24. *Any subgroup of $(\mathbb{Q}, +)$ is either free or isomorphic to*

$$Q(\boldsymbol{a}) := \left\{\frac{m}{\prod_{j=1}^N a_j} \ : \ m \in \mathbb{Z} \text{ and } N \in \mathbb{N}\right\},$$

for some sequence $\boldsymbol{a} \in \Sigma$.

Remark 4.4.25. *Given $\boldsymbol{a} \in \Sigma$, consider the non-free subgroup $Q(\boldsymbol{a})$ of the additive rational. Then, there exists a Kronecker flow whose frequency module is isomorphic to the given subgroup. Indeed, if $\omega = \omega(\boldsymbol{a})$, where*

$$\omega(\boldsymbol{a}) = (\omega_j(\boldsymbol{a}))_{j\in\mathbb{N}}, \quad \text{with} \quad \omega_j(\boldsymbol{a}) = \prod_{k=1}^{j} a_k^{-1} \quad \text{for all } j \in \mathbb{N},$$

then $\mathscr{M}_\omega = Q(\boldsymbol{a})$.

A priori, there may be other possible choices of the frequency vector ω that give the same frequency module. Nevertheless, Proposition 4.4.23 ensures that the topological structure of the closure of the orbits of any Kronecker flow is uniquely determined by the frequency module. Finally, Theorem 4.4.7 implies that the Pontryagin dual of $Q(\boldsymbol{a})$ is the solenoid $\sigma_{\boldsymbol{a}}$.

Remark 4.4.26. *The key idea of the construction of Sakbaev and Volovich [SV24], consists in considering orbits that are (uniformly in time) approximated by periodic ones, whose period diverges as the approximation becomes more accurate. In particular, they provide two examples, namely, $\omega_j = 1/j!$, $j \in \mathbb{N}$, and $\omega'_j = \boldsymbol{p}_{2j}/\boldsymbol{p}_{2j-1}$, $j \in \mathbb{N}$, where $(\boldsymbol{p}_j)_{j\in\mathbb{N}}$ is the list of all prime numbers in ascending order. Note that the closures of the orbits induced by Φ_ω and $\Phi_{\omega'}$ are homeomorphic to the solenoids $\sigma_{\boldsymbol{a}}$, with $\boldsymbol{a} = (1, 2, 3, 4, ...)$, and $\sigma_{\boldsymbol{a}'}$, with $\boldsymbol{a}' = (1, \boldsymbol{p}_1, \boldsymbol{p}_3, \boldsymbol{p}_5, ...)$, respectively. Indeed, with the same notations of Lemma 4.4.24, it is easy to verify that*

$$\mathscr{M}_\omega = Q(\boldsymbol{a}), \quad \mathscr{M}_{\omega'} = Q(\boldsymbol{a}').$$

The conclusion follows by Proposition 4.4.23 and Remark 4.4.25.

To conclude this section, we show that, given a countable family $(G_i)_{i\in I}$ of subgroups of $(\mathbb{Q}, +)$, we are able to construct a Kronecker flow such that its frequency module is isomorphic to $\bigoplus_{i\in I} G_i$ and we give the general structure of the orbits. First, we need the following

Proposition 4.4.27. *Let G be a subgroup of $(\mathbb{R}, +)$. Assume that there exist a set of indices I and a family of rationally independent numbers $(\alpha_i)_{i\in I}$ such that for any $g \in G$ there is a finite subset $J \subseteq I$ and $(r_i)_{i\in J}$, $r_i \in \mathbb{Q}$ for all $i \in J$, satisfying $g = \sum_{i\in J} r_i\alpha_i$. Then, G is isomorphic to $\bigoplus_{i\in I} R_i$, where, for any $i \in I$, R_i is the following subgroup of $(\mathbb{Q}, +)$:*

$$R_i := \alpha_i^{-1}\{g \in G \ : \ g = \alpha_i r, \text{ for some } r \in \mathbb{Q}\}.$$

Proof. Since $(\alpha_i)_{i\in I}$ are rationally independent, then for any finite subset $J \subseteq I$, the equation $\sum_{i\in J} \alpha_i r_i = 0$ has a unique solution in $\bigoplus_{i\in J} \mathbb{Q}$, that is, $r_i = 0$ for all $i \in J$. ■

Theorem C. *Let I be a countable set of indices and $(G_i)_{i\in I}$ a family of subgroups of $(\mathbb{Q}, +)$. Then, there exists a Kronecker flow whose frequency module is isomorphic to $\bigoplus_{i\in I} G_i$. Moreover, the closure of any orbit is homeomorphic to a product $\prod_{i\in I} X_i$, where each factor X_i is a circle if G_i is free, and a solenoid otherwise.*

Proof. By taking an arbitrary enumeration of I, we can think at I as a subset of $\mathbb{N}$. By Lemma 4.4.24, $\bigoplus_{i\in I} G \cong \bigoplus_{k\in K} \mathbb{Z} \oplus \bigoplus_{n\in J} Q(\boldsymbol{a}^n)$, for some $J, K \subseteq I$, $J \cap K = \emptyset$, and some family $(\boldsymbol{a}^n)_{n\in J}$ of sequences of natural numbers such that, for any $n \in J$, $\boldsymbol{a}^n \in \Sigma$. Let $(\mathbf{p}_n)_{n\in\mathbb{N}}$ be the list of all prime numbers in ascending order. Then, define $\omega \in \mathbb{R}^{\mathbb{N}}$ as

$$\omega_j := \frac{\sqrt{\mathbf{p}_n}}{\prod_{k=1}^{N} a_k^n} \quad \text{if } j = \mathbf{p}_n^N, \text{ for some } n \in J, \text{ and } N \in \mathbb{N},$$

and we complete the construction of ω by associating powers of π, i.e., π^k, for all $k \in K$, and zeros to all the remaining components. By Proposition 4.4.27, an easy calculation gives

$$\mathscr{M}_\omega \cong \bigoplus_{k\in K} \mathbb{Z} \oplus \bigoplus_{n\in J} \left\{ \frac{m}{\prod_{k=1}^{N} a_k^n} \ : \ m \in \mathbb{Z} \text{ and } N \in \mathbb{N} \right\}.$$

Furthermore, by Proposition 4.4.23, Proposition 4.4.27 and Remark 4.4.25, the closure of any orbit induced by $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ is homeomorphic to the direct product of circles and solenoids

$$\prod_{k\in K} \mathbb{T} \times \prod_{n\in J} \sigma_{\boldsymbol{a}^n}.$$

■

4.4.4 Solenoidal Solutions of the Benjamin-Ono Equation

As an application of the results obtained in the previous section, we show that the Benjamin-Ono equation for a 2π-periodic in space, real valued function

$$u_t = \mathrm{H}(u_{xx}) - 2uu_x, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{4.16}$$

where $\mathrm{H}(\,\cdot\,)$ denotes the Hilbert transform

$$\mathrm{H}(u)(x,t) = -\frac{\mathrm{i}}{2\pi} \sum_{j\in\mathbb{Z}} \operatorname{sgn}(j) \int_0^{2\pi} u(x',t) e^{\mathrm{i}j(x-x')} dx', \tag{4.17}$$

admits a family of (non-typical) almost-periodic solutions that are dense on invariant subsets of $L^2(\mathbb{T})$ which are homeomorphic to the product of a circle and a solenoid.

Notation 4.4.28. *In the following, $L_0^2(\mathbb{T})$ denotes the space of functions $u \in L^2(\mathbb{T})$ with $\int_0^{2\pi} u(x)dx = 0$.*

The following result is proved in [GK21] by Kappeler and Gerard.

Theorem 4.4.29 ([GK21])**.** *There exists a global bi-analytic diffeomorphism*

$$\Psi : L_0^2(\mathbb{T}) \ni u \mapsto z \in h^{1/2} := \left\{ z \in \mathbb{C}^{\mathbb{N}} \ : \ \|z\|_{1/2}^2 := \sum_{j\in\mathbb{N}} j|z_j|^2 < \infty \right\},$$

called "Birkhoff map", such that, for any initial data $\underline{u} \in L_0^2(\mathbb{T})$, the solution of the Benjamin-Ono equation (4.16) *— that is supported on the invariant torus $\Psi^{-1}(\mathcal{T}_\gamma)$*

$$\mathcal{T}_\gamma := \{ z \in h^{1/2} \ : \ |z_j|^2 = \gamma_j := |\Psi_j(\underline{u})|^2,\ \forall j \in \mathbb{N} \} \tag{4.18}$$

whose dimension is equal to the number of non-zero "Birkhoff coodinates" associated with the initial data — is given by

$$\Psi_j(u(t)) = \Psi_j(\underline{u}) e^{\mathrm{i}\omega_j(\gamma)t}, \quad \forall j \in \mathbb{N},$$

with frequencies

$$\omega_j(\gamma) := j^2 - 2\sum_{k\in\mathbb{N}} \min(j,k)\gamma_k, \quad \forall j \in \mathbb{N}. \tag{4.19}$$

Remark 4.4.30. *It is easy to verify that the tori defined in* (4.18)*, when considered with the subspace topology induced by the* $\|\cdot\|_{1/2}$*-norm, are homeomorphic to the, possibly infinite, torus* $\prod_{j\in supp(\gamma)}\mathbb{T}$*. Namely, the map*

$$\Xi:\prod_{j\in supp(\gamma)}\mathbb{T}\ni\theta\mapsto\Xi(\gamma):=(\sqrt{\gamma_j}e^{\mathrm{i}\Theta_j})_{j\in\mathbb{N}}\in\mathcal{T}_\gamma\subset h^{1/2},$$

where $\Theta_j:=0$ *for all* $j\notin supp(\gamma)$*, is a topological embedding. Therefore, since the Birkhoff map* Ψ *is a homeomorphism, the dynamics induced by the Benjamin-Ono equation on any invariant set* $\Psi^{-1}(\mathcal{T}_\gamma)$ *is topologically conjugate by*

$$\mathscr{H}:=\Psi^{-1}\circ\Xi$$

to the Kronecker flow

$$\Phi_{\omega(\gamma)}:\mathbb{R}\times\prod_{j\in supp(\gamma)}\mathbb{T}\to\prod_{j\in supp(\gamma)}\mathbb{T}\,,\quad(t,\theta)\mapsto\Phi^t_{\omega(\gamma)}(\theta):=(\Theta_j+\omega_j(\gamma)t+2\pi\mathbb{Z})_{j\in supp(\gamma)}.$$

with $\omega(\gamma)$ *given by* (4.19).

The goal of this section is to give a proof of the following theorem.

Theorem BO. *Consider the Benjamin-Ono equation for a real-valued function* $u(t,x)$ *that is* 2π*-periodic in space:*

$$u_t=\mathrm{H}(u_{xx})-2uu_x,\quad x\in\mathbb{T},\ t\in\mathbb{R},$$

where $\mathrm{H}(\cdot)$ *denotes the Hilbert transform defined in* (4.17)*. Let*

$$\Psi:L^2_0(\mathbb{T})\ni u\mapsto z\in h^{1/2}$$

be its Birkhoff map [GK21]. Given $z\in h^{1/2}$*, if there exists* $\beta\in\mathbb{R}\setminus\mathbb{Q}$ *such that* $|z_j|^2\in\beta\mathbb{Q}\setminus\{0\}$ *for all* $j\in\mathbb{N}$*, then the closure of the orbit with initial data* $u=\Psi^{-1}(z)$ *under the Benjamin-Ono flow is homeomorphic to the product of a circle and a solenoid.*

In the above theorem, the closure is taken with respect to the subspace topology induced by the L^2-norm. Then, by Remark 4.4.30, it is sufficient to study the orbits of a Kronecker flow with frequencies defined as in (4.19) on the infinite torus $\mathbb{T}^{\mathbb{N}}$ endowed with the product topology.

Proposition 4.4.31. *Let* α *be an irrational number and* $\omega=(\omega_j)_{j\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$ *a list defined by*

$$\omega_j=j^2+\alpha\sum_{k\in\mathbb{N}}\min(j,k)s_k$$

where $(s_j)_{j\in\mathbb{N}}$ *is a list of rational numbers such that* $\sum_{j\in\mathbb{N}}js_j$ *is finite. Then, the frequency module* $\mathscr{M}_\omega$ *is isomorphic to* $\mathbb{Z}\oplus R$*, where* R *is the subgroup of* $(\mathbb{Q},+)$ *defined by*

$$R:=\left\{\sum_{j\in\mathbb{N}}\nu_j\sum_{k\in\mathbb{N}}\min(j,k)s_k\ :\ \nu\in\mathbb{Z}^{(\mathbb{N})}\right\}.\tag{4.20}$$

Proof. The claim follows from Proposition 4.4.27 and from the fact that

$$\left\{\sum_{j\in\mathbb{N}}\nu_j j^2:\nu\in\mathbb{Z}^{(\mathbb{N})}\right\}=\mathbb{Z},$$

which holds since the set of squares $\{j^2 : j \in \mathbb{N}\}$ generates the entire additive group $(\mathbb{Z}, +)$. This is immediately verified by considering the map $n \mapsto ne_1$, where e_1 is the first vector of the canonical basis of $\mathbb{Z}^{(\mathbb{N})}$. Since the weighted sum associated with e_1 is $1^2 = 1$, and 1 is the generator of $\mathbb{Z}$, the image of the mapping $\nu \mapsto \sum \nu_j j^2$ must coincide with $\mathbb{Z}$. Formally, the existence of e_1 such that $\sum (e_1)_j j^2 = 1$ ensures that the generator of the target group is reached, making any further contributions from $j > 1$ redundant for the purpose of surjectivity. ■

Proposition 4.4.32. *Given a Kronecker flow Φ_ω on the infinite torus, with*

$$\omega_j := j^2 + \alpha \sum_{k \in \mathbb{N}} \min(j,k) s_k, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

where $(s_j)_{j\in\mathbb{N}}$ is a list of rational numbers such that the sum $\sum_{j\in\mathbb{N}} j s_j$ is finite, the closure of any orbit is homeomorphic to $\mathbb{T} \times R^*$, where R is defined in (4.20).

Proof. By Proposition 4.4.23 the closure of any orbit is homeomorphic to the Pontryagin dual of $\mathbb{Z} \oplus R$, which, by Theorem 4.4.14, Proposition 4.4.31, and Example 2, turns out to be equal to $\mathbb{T} \times R^*$. ■

Now we prove that, if the support of $(s_j)_{j\in\mathbb{N}}$ is not finite, and each s_j is non-negative, the subgroup $R \subset (\mathbb{Q}, +)$, defined in (4.20), is not free. The strategy is based on the following lemma, which is proved in Appendix E.3.

Lemma 4.4.33. *Let G be a subgroup of $(\mathbb{Q}, +)$. If there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset G$, $g_n \neq 0$ for all $n \in \mathbb{N}$, converging to 0 with respect to the subspace topology induced by the standard one on $\mathbb{R}$, then, G is not free.*

Proposition 4.4.34. *With the same notations as in Proposition 4.4.32, if moreover the support of $(s_j)_{j\in\mathbb{N}}$ is not finite, and $s_j \geq 0$ $\forall j \in \mathbb{N}$, the subgroup $R \subset (\mathbb{Q}, +)$ defined in* (4.20) *is not free.*

Proof. Consider the sequence $(\nu^n)_{n\in\mathbb{N}} \subset \mathbb{Z}^{(\mathbb{N})}$ defined by $\nu^n = e_{n+1} - e_n$. Then,

$$\begin{aligned}
g_n : &= \sum_{j=1}^{\infty} \nu_j^n \sum_{k=1}^{\infty} \min(j,k) s_k = \sum_{k=1}^{\infty} \min(n+1,k) s_k - \sum_{k=1}^{\infty} \min(n,k) s_k \\
&= \sum_{k=1}^{n+1} k s_k + (n+1) \sum_{k=n+2}^{\infty} s_k - \sum_{k=1}^{n} k s_k - n \sum_{k=n+1}^{\infty} s_k \\
&= (n+1) s_{n+1} + n \left(\sum_{k=n+2}^{\infty} s_k - \sum_{k=n+1}^{\infty} s_k \right) + \sum_{k=n+2}^{\infty} s_k \\
&= (n+1) s_{n+1} - n s_{n+1} + \sum_{k=n+2}^{\infty} s_k \\
&= \sum_{k>n} s_k.
\end{aligned}$$

By assumption, the support of $(s_j)_{j\in\mathbb{N}}$ is not finite, and each s_j is non-negative. Then, $g_n \neq 0$ for all $n \in \mathbb{N}$, and the sequence $(g_n)_{n\in\mathbb{N}} \subset R$ converges to 0. The claim follows by Lemma 4.4.33. ■

Proposition 4.4.35. *Let Φ_ω be a Kronecker flow on the infinite torus where ω satisfies the same assumptions as in Proposition 4.4.32. If we also assume that the support of $(s_j)_{j\in\mathbb{N}}$ is not finite, and each s_j is non-negative, then the closure of any orbit is homeomorphic to the product of a circle and a solenoid.*

Proof. The claim follows directly from Propositions 4.4.32 and 4.4.34, Lemma 4.4.24 and Remark 4.4.25. ■

We are now in a position to prove Theorem BO.

Proof of Theorem BO. Let $z \in h^{1/2}$ be such that $|z_j|^2 \in \beta\mathbb{Q}\setminus\{0\}$, for all $j \in \mathbb{N}$, for a certain $\beta \in \mathbb{R}\setminus\mathbb{Q}$. Let $(\gamma_j)_{j\in\mathbb{N}}$, $\gamma_j := |z_j|^2$ for all $j \in \mathbb{N}$, and define $\mathcal{T}_\gamma$ as

$$\mathcal{T}_\gamma := \{w \in h^{1/2} \ : \ |w_j|^2 = \gamma_j, \ \forall j \in \mathbb{N}\}.$$

As already observed in Remark 4.4.30, the map Ξ defined by

$$\Xi : \prod_{j\in\mathbb{N}} \mathbb{T} \ni \theta \mapsto \Xi(\gamma) := (\sqrt{\gamma_j} e^{\mathrm{i}\Theta_j})_{j\in\mathbb{N}} \in \mathcal{T}_\gamma \subset h^{1/2}$$

is a topological embedding. Therefore, the dynamics induced by the Benjamin-Ono equation on the invariant torus $\Psi^{-1}(\mathcal{T}_\gamma)$ is topologically conjugate by

$$\mathscr{H} := \Psi^{-1} \circ \Xi$$

to a Kronecker flow $\Phi_{\omega(\gamma)}$ with frequencies $\omega(\gamma) = (\omega_j(\gamma))_{j\in\mathbb{N}}$, where

$$\omega_j(\gamma) := j^2 - 2\sum_{k\in\mathbb{N}} \min(j,k)\gamma_k \quad \text{for all} j \in \mathbb{N}.$$

Therefore, the claim follows from Proposition 4.4.35. ■

4.4.5 A Classification Problem

Here we address a classification problem for Kronecker flows on the infinite torus. The questions is: given two Kronecker flows, are we able to determine whether the closures of their orbits are homeomorphic? To address this problem, we make use of Pontryagin's Duality Theorem 4.4.16, along with a result by Scheinberg [Sch74] concerning the equivalence of the notions of homeomorphism and isomorphism for connected, locally compact abelian groups.

Theorem D. *Given $\omega, \tilde{\omega} \in \mathbb{R}^{\mathbb{N}}$, let $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ and $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$ be two Kronecker flows on the infinite torus. The closure of any orbit of Φ_ω is homeomorphic to the closure of any orbit of $\Phi_{\tilde{\omega}}$ if and only if the associated frequency modules $\mathscr{M}_\omega$ and $\mathscr{M}_{\tilde{\omega}}$ are isomorphic as abelian groups.*

Proof. By Remark 4.1.4, it is sufficient to prove the statement in the case of orbits through the origin. In [Sch74] it is proved that two connected, locally compact abelian groups are homeomorphic if and only if they are isomorphic as topological groups. Then, the statement of Theorem D is equivalent to the following: given two Kronecker flows on the infinite torus, the closures of the orbits through the origin are isomorphic as topological groups if and only if their frequency modules are isomorphic. Let us prove it.

Let $\tilde{\omega}, \omega \in \mathbb{R}^{\mathbb{N}}$ and consider the Kronecker flows $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$, $(\mathbb{T}^{\mathbb{N}}, \Phi_{\tilde{\omega}})$. If $\mathscr{M}_\omega \cong \mathscr{M}_{\tilde{\omega}}$, then, by Proposition 4.4.22,

$$\mathcal{O}_\omega \cong (\mathscr{M}_\omega)^* \cong (\mathscr{M}_{\tilde{\omega}})^* \cong \mathcal{O}_{\tilde{\omega}}.$$

Now, assume that $\mathcal{O}_\omega \cong \mathcal{O}_{\tilde{\omega}}$. Then, by Theorem 4.4.16 and Proposition 4.4.22,

$$\mathscr{M}_\omega \cong (\mathscr{M}_\omega)^{**} \cong (\mathcal{O}_\omega)^* \cong (\mathcal{O}_{\tilde{\omega}})^* \cong (\mathscr{M}_{\tilde{\omega}})^{**} \cong \mathscr{M}_{\tilde{\omega}}.$$

■

We now describe how to associate to each countable abelian group G without torsion a Kronecker flow $(\mathbb{T}^{\mathbb{N}}, \Phi_\omega)$ in such a way that G is isomorphic to $\mathscr{M}_\omega$. Any countable abelian group G without torsion embeds canonically into a countable vector space $\mathscr{V}$ over $\mathbb{Q}$. This procedure, briefly explained in Appendix E, is known in commutative algebra as "localization of an abelian group". Assuming Zorn's lemma, there exists a Hamel (algebraic) basis for $\mathscr{V}$. Thus, there exists a countable set of indices J such that $\mathscr{V}$ is isomorphic to $\mathbb{Q}^{(J)}$. In particular, any element $x \in \mathscr{V}$ is uniquely represented by a family of rational numbers $(x_j)_{j\in J}$ such that $x_j \neq 0$ only for finitely many $j \in J$. Now, observe that the map

$$\mathbf{h} : \mathscr{V} \ni (x_j)_{j\in J} \mapsto \sum_{j\in J} \alpha_j x_j \in (\mathbb{R}, +),$$

where $(\alpha_j)_{j\in J}$ is a family of rationally independent real numbers, is an injective group homomorphism, and so is the composition $\mathbf{H} := \mathbf{h} \circ \imath$, where $\imath : G \to \mathscr{V}$ is the injective homomorphism $\imath : G \to \mathscr{V}$ that embeds G into $\mathscr{V}$. Then, by the first isomorphism theorem, G is isomorphic to the image of $\mathbf{H}$ in $(\mathbb{R}, +)$. Finally, we define the list of real numbers $\omega \in \mathbb{R}^{\mathbb{N}}$ by taking an arbitrary enumeration of the image of $\mathbf{H}$. By construction, $\mathscr{M}_\omega \cong G$.

Then, this argument, together with Theorem D, implies that completely classifying Kronecker flows on $\mathbb{T}^{\mathbb{N}}$ up to homeomorphism of the closure of their orbits is a highly complex issue, as it is equivalent to the classification of countable abelian groups without torsion. To date, a complete classification of such groups is known only for the rank 1 case, i.e., subgroups of $(\mathbb{Q}, +)$ (for references, see [Bae37], [BZ51] and [Fuc15])[7], which corresponds to the class of Kronecker flows (constructed in Section 4.4.3) whose orbits are dense in a product of circles and solenoids.

[7] For the general theory of infinite abelian groups we refer to Kaplansky's book [Kap54].

Appendix A

A Fixed Point Theorem in Banach-Lie Algebras

In this appendix, we investigate the existence and uniqueness of a solution for a nonlinear equation of the form $X = \phi(X)$. We operate within a Banach Lie algebra $(\mathcal{V}, [\cdot, \cdot]| \cdot |)$ characterized by a sub-multiplicative norm such that $|[X,Y]| \leq K|X||Y|$ for all $X, Y \in \mathcal{V}$. Within this framework, we consider two analytic operator-valued maps $\mathcal{E}$ and $\mathcal{B}$, and we construct formally the map $\phi_P : \mathscr{U} \subset \mathcal{V} \to \mathcal{V}$ in a suitable neighbourhood $\mathscr{U}$ of the origin as:

$$\phi_P(X) = T\Pi\left(\mathcal{B}(X)\Pi(\mathcal{E}(X)P)\right)$$

where $P \in \mathcal{V}$ is a fixed element, and $\Pi \in \mathcal{L}(\mathcal{V})$ is a bounded linear operator satisfying $\Pi^2 = \Pi$, while $T : \mathrm{range}(\Pi) \to \mathcal{V}$ is a bounded linear operator defined on the range of Π. Our objective is to prove that ϕ_P admits a unique fixed point $X^* \in \mathcal{V}$ by showing that it is a contraction. To achieve this, we first establish the analyticity of the constituent operators; second, we derive their Fréchet derivatives; and third, we provide quantitative norm estimates to satisfy the hypotheses of the Banach Fixed Point Theorem.

A.1 Functional framework

Let $(\mathcal{V}, [\cdot, \cdot], | \cdot |)$ be a Banach Lie algebra with sub-multiplicative norm: $|[X,Y]| \leq K|X||Y|$ for all $X, Y \in \mathcal{V}$. We denote by $\mathcal{L}(\mathcal{V})$ the space of bounded linear operators on $\mathcal{V}$ with the operator norm $\|\cdot\|$. For $X \in \mathcal{V}$, the adjoint operator $\mathrm{ad}_X \in \mathcal{L}(\mathcal{V})$ is defined by $\mathrm{ad}_X(Y) = [X,Y]$. Its norm satisfies:

$$\|\mathrm{ad}_X\| \leq K|X|. \tag{A.1}$$

A.2 Definition and analyticity of map $\mathcal{E}$ and $\mathcal{B}$

For $X \in \mathcal{V}$, we formally define two operators $\mathcal{E}(X)$ and $\mathcal{B}(X)$ via the Taylor series of two complex functions $f(z) = e^z$ and $g(z) = \frac{z}{e^z - 1}$.

It is well known that:

1. $f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$ has radius of convergence $R_f = \infty$;
2. let $g(z) = \sum_{k=0}^{\infty} \beta_k z^k$ be the Taylor expansion of the function g around $z = 0$. Then the radius of convergence is $R_g = 2\pi$.

The operators are then defined as $\mathcal{E}(X) = f(\mathrm{ad}_{-X})$ and $\mathcal{B}(X) = g(\mathrm{ad}_{-X})$.

Proposition A.2.1 (Analyticity of $\mathcal{E}$ and $\mathcal{B}$)**.** *The maps $\mathcal{E} : \mathcal{V} \to \mathcal{L}(\mathcal{V})$ and $\mathcal{B} : \mathscr{U}_{2\pi/K}(\mathcal{V}) \to \mathcal{L}(\mathcal{V})$ are analytic*

Proof. Let $h(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series with radius of convergence R. Define the sequence of partial sums $S_n(X) = \sum_{k=0}^{n} c_k(\mathrm{ad}_{-X})^k$.

1. **Analyticity of polynomials:** Since $X \mapsto \mathrm{ad}_{-X}$ is a bounded linear map, each term $c_k(\mathrm{ad}_{-X})^k$ is a continuous polynomial in X. In any Banach space, continuous polynomials are analytic functions. Thus, $S_n(X)$ is analytic for every $n \in \mathbb{N}$.

2. **Uniform convergence:** Let $r < R$ and consider $X \in \mathscr{U}_{r/K}(\mathcal{V})$. From the bound $\|\mathrm{ad}_{-X}\| \leq K|X|$ established in (A.1), we have $K|X| \leq r$, which implies:

$$\sum_{k=0}^{\infty} \|c_k(\mathrm{ad}_{-X})^k\| \leq \sum_{k=0}^{\infty} |c_k|\|\mathrm{ad}_{-X}\|^k \leq \sum_{k=0}^{\infty} |c_k| r^k.$$

   Since $r < R$, the numerical series $\sum |c_k| r^k$ converges. By the Weierstrass M-test, the series of operators converges absolutely and uniformly in the operator norm $\|\cdot\|$ on $\mathscr{U}_{r/K}(\mathcal{V})$. This proves that the sequence of partial sums $S_n(X)$ converges to a limit operator $S(X)$ for every X in the open ball $\mathscr{U}_{R/K}(\mathcal{V})$.

3. **Limit property:** According to a fundamental result in complex analysis (cf. Mujica, *Complex Analysis in Banach Spaces*), if a sequence of analytic functions $\{S_n\}$ on an open set Ω converges uniformly to a function S, then the limit S is analytic on Ω.

Since the analyticity radii of $f(z)$ and $g(z)$ are $R_f = \infty$ and $R_g = 2\pi$ respectively, the maps $\mathcal{E}$ (defined on the whole $\mathcal{V}$) and $\mathcal{B}$ (defined on $\mathscr{U}_{2\pi/K}(\mathcal{V})$) are analytic. ∎

A.3 Definition and analyticity of the map ϕ_P

Let $P \in \mathcal{V}$. Consider a bounded linear operator $\Pi \in \mathcal{L}(\mathcal{V})$ such that $\Pi^2 = \Pi$ (a projection), and a linear operator $T : \mathrm{range}(\Pi) \to \mathcal{V}$ which satisfies the norm bound $\|T\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})} \leq 2$. We also assume $\|\Pi\| \leq 1$.

Definition A.3.1. *We define the map $\phi_P : \mathscr{U}_{2\pi/K}(\mathcal{V}) \to \mathcal{V}$ as follows:*

$$\phi_P(X) = T\Pi\left(\mathcal{B}(X)\Pi(\mathcal{E}(X)P)\right),$$

where the non-linear analytic maps $\mathcal{E}$ and $\mathcal{B}$ are defined in Subsection A.2.

The analyticity of ϕ_P follows directly from its construction as a composition of analytic operator-valued maps and constant, bounded linear operators applied to an element $P \in \mathcal{V}$.

A.4 Differentiation of $\mathcal{E}$ and $\mathcal{B}$

To differentiate the map ϕ_P, we must first establish the precise form of the Fréchet derivative of the operator-valued power series.

Derivative of the adjoint operator. The map $\alpha : \mathcal{V} \to \mathcal{L}(\mathcal{V})$ defined by $\alpha(X) = \mathrm{ad}_{-X}$ is a bounded linear operator. By the definition of the Fréchet derivative for linear maps, we have:

$$\mathrm{d}\alpha(X)[H] = \mathrm{ad}_{-H}, \quad \forall H \in \mathcal{V}.$$

The Leibniz rule for operator powers. Consider the k-th power of the adjoint operator as a product of k linear maps. Applying the Leibniz rule for linear maps in a Banach algebra of operators, the derivative in the direction H is the sum of k terms, where the derivative ad_{-H} occupies each possible position in the product:

$$\mathrm{d}\alpha^k(X)[H] = \sum_{j=0}^{k-1} (\mathrm{ad}_{-X})^j \circ (\mathrm{ad}_{-H}) \circ (\mathrm{ad}_{-X})^{k-1-j}.$$

This generalized Leibniz formula accounts for the non-commutativity of the operators within $\mathcal{L}(\mathcal{V})$.

Term-wise differentiation of the series. Assuming that we can differentiate term-by-term, the derivative of $\mathcal{E}$ and $\mathcal{B}$ are

$$\mathrm{d}\mathcal{E}(X)[H] = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=0}^{k-1} (\mathrm{ad}_{-X})^j \circ (\mathrm{ad}_{-H}) \circ (\mathrm{ad}_{-X})^{k-1-j}, \tag{A.2}$$

$$\mathrm{d}\mathcal{B}(X)[H] = \sum_{k=1}^{\infty} \beta_k \sum_{j=0}^{k-1} (\mathrm{ad}_{-X})^j \circ (\mathrm{ad}_{-H}) \circ (\mathrm{ad}_{-X})^{k-1-j}, \tag{A.3}$$

where β_k are the Taylor coefficients of $g(z) = z/(e^z - 1)$. In the next subsection, we provide quantitative bounds to establish that these operators are indeed the Fréchet derivatives of $\mathcal{E}$ and $\mathcal{B}$.

A.5 Estimates

In this subsection, we derive explicit bounds for the maps $\mathcal{E}$ and $\mathcal{B}$ and their differentials. These estimates are fundamental to establishing the contraction property of the map ϕ_P.

Estimates for $\mathcal{E}$ and $\mathcal{B}$. Let $\rho = \|\mathrm{ad}_{-X}\| \leq K|X|$. For $\rho < 1/2$, the series defining $\mathcal{E}(X)$ and $\mathcal{B}(X)$ are totally convergent in $\mathcal{L}(\mathcal{V})$. This implies that the following norm estimates hold uniformly for all X in the ball $\mathscr{U}_{1/(2K)}(\mathcal{V})$:

- By the sub-multiplicative property and the triangle inequality, the norm of the power series is bounded by the power series of the norms:

$$\|\mathcal{E}(X)\| = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_{-X})^k \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|\mathrm{ad}_{-X}\|^k = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} = e^{\rho}.$$

  For $\rho < 1/2$, we have $e^{\rho} < e^{1/2}$, thus $\|\mathcal{E}(X)\| < 2$.

- The Taylor coefficients β_k satisfy $|\beta_k| \leq 1/(2\pi)^k \leq 1$. To obtain a robust upper bound, we compare the series with the dominant geometric series:

$$\|\mathcal{B}(X)\| = \left\| \sum_{k=0}^{\infty} \beta_k (\mathrm{ad}_{-X})^k \right\| \leq \sum_{k=0}^{\infty} |\beta_k| \rho^k \leq \sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho}.$$

  For $\rho < 1/2$, the term $1/(1-\rho) < 2$.

Estimates for the differential of the k-th power. To estimate the norm of the operator $\mathrm{d}\alpha^k(X)$, we rely on the Banach algebra structure of the space of bounded operators $\mathcal{L}(\mathcal{V})$. A key property of such an algebra is that the norm is sub-multiplicative: $\|S \circ T\| \leq \|S\|\|T\|$.

Applying this property to each individual term in the Leibniz expansion of the k-th power, and using the bound $\|\mathrm{ad}_{-H}\| \leq K|H|$ from (A.1), we have:

$$\|(\mathrm{ad}_{-X})^j \circ (\mathrm{ad}_{-H}) \circ (\mathrm{ad}_{-X})^{k-1-j}\| \leq \|\mathrm{ad}_{-X}\|^j \|\mathrm{ad}_{-H}\| \|\mathrm{ad}_{-X}\|^{k-1-j} \leq K|H| \|\mathrm{ad}_{-X}\|^{k-1}.$$

Then, by applying the triangle inequality, we obtain the fundamental bound:

$$\|\mathrm{d}\alpha^k(X)[H]\| \leq \sum_{j=0}^{k-1} \|(\mathrm{ad}_{-X})^j \circ (\mathrm{ad}_{-H}) \circ (\mathrm{ad}_{-X})^{k-1-j}\| \leq kK|H| \|\mathrm{ad}_{-X}\|^{k-1}. \tag{A.4}$$

Bounds for the differentials $\mathrm{d}\mathcal{E}$ and $\mathrm{d}\mathcal{B}$. Recalling that the derivative of ad_{-X} is ad_{-H}, and substituting (A.4) into (A.2) and (A.3), we obtain:

- Substituting the bound $kK|H|\rho^{k-1}$ for each term $k \geq 1$:

$$\|\mathrm{d}\mathcal{E}(X)[H]\| \leq \sum_{k=1}^{\infty} \frac{1}{k!}(kK|H|\rho^{k-1}) = K|H| \sum_{k=1}^{\infty} \frac{k}{k!}\rho^{k-1}.$$

  Since $k/k! = 1/(k-1)!$, we shift the index $m = k-1$ to recover the exponential series:

$$\|\mathrm{d}\mathcal{E}(X)[H]\| \leq K|H| \sum_{m=0}^{\infty} \frac{\rho^m}{m!} = K|H|e^{\rho}.$$

  For $\rho < 1/2$, we have $K|H|e^{\rho} < K|H|e^{1/2} < 2K|H|$.

- Using the same term-wise estimation and the property $|\beta_k| \leq 1$:

$$\|\mathrm{d}\mathcal{B}(X)[H]\| \leq \sum_{k=1}^{\infty} |\beta_k|(kK|H|\rho^{k-1}) \leq K|H| \sum_{k=1}^{\infty} k\rho^{k-1}.$$

  The sum $\sum_{k=1}^{\infty} k\rho^{k-1}$ is the derivative of the geometric series $\frac{d}{d\rho}(1-\rho)^{-1} = (1-\rho)^{-2}$. Therefore:

$$\|\mathrm{d}\mathcal{B}(X)[H]\| \leq K|H| \frac{1}{(1-\rho)^2}.$$

  For $\rho < 1/2$, the factor $1/(1-\rho)^2 < 4$, yielding the bound $4K|H|$.

Invertibility and Estimates for $\mathcal{B}(X)^{-1}$. Let us show that $\mathcal{B}(X)$ is invertible and provide quantitative bounds for its norm and its Fréchet differential.

Lemma A.5.1. *For any fixed $X \in \mathscr{U}_{2\pi/K}(\mathcal{V})$, the operator $\mathcal{B}(X) \in \mathcal{L}(\mathcal{V})$ is invertible. Its inverse is given by the power series:*

$$\mathcal{B}(X)^{-1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}_{-X})^k, \tag{A.5}$$

which defines an analytic map $X \mapsto \mathcal{B}(X)^{-1}$ on the whole space $\mathcal{V}$. Moreover, let $\rho = \|\mathrm{ad}_{-X}\| \leq K|X|$; for $\rho < 1/2$, the following estimates hold:

1. $\|\mathcal{B}(X)^{-1}\| \le \frac{1}{1-\rho} < 2$;

2. $\|\mathrm{d}(\mathcal{B}(X)^{-1})[H]\| \le \frac{K|H|}{(1-\rho)^2} < 4K|H|$.

Proof. The existence of the inverse follows from the functional identity $g(z)q(z) = 1$, where $g(z) = z/(e^z - 1)$ and $q(z) = (e^z - 1)/z$. The Taylor expansion of $q(z)$ is $q(z) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} z^k$, which represents an entire function with infinite radius of convergence.

Let us derive the quantitative estimates:

$$\|\mathcal{B}(X)^{-1}\| \le \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \rho^k \le \sum_{k=0}^{\infty} \rho^k = (1-\rho)^{-1}.$$

The estimate for the Fréchet derivative is obtained by term-wise differentiation of the series (A.5). Using the bound $\|\mathrm{d}\alpha^k(X)[H]\| \le kK|H|\rho^{k-1}$ previously established, we obtain:

$$\|\mathrm{d}(\mathcal{B}(X)^{-1})[H]\| \le K|H| \sum_{k=1}^{\infty} \frac{k}{(k+1)!} \rho^{k-1} \le K|H| \sum_{k=1}^{\infty} k\rho^{k-1} = K|H|(1-\rho)^{-2}.$$

Under the assumption $\rho < 1/2$, we immediately recover the bounds $\|\mathcal{B}(X)^{-1}\| < 2$ and $\|\mathrm{d}(\mathcal{B}(X)^{-1})[H]\| < 4K|H|$. ■

A.6 The Fixed Point Theorem

Differentiation of the map ϕ_P. The Fréchet derivative $\mathrm{d}\phi_P$ is computed by applying the Leibniz rule to the expression of ϕ_P. Since T, Π, and P are constant, we only differentiate $\mathcal{B}(X)$ and $\mathcal{E}(X)$ according to their position in the product:

$$\mathrm{d}\phi_P(X)[H] = T\Pi \left[\Big(\mathrm{d}\mathcal{B}(X)[H] \Big) \Pi(\mathcal{E}(X)P) + \mathcal{B}(X)\Pi\Big(\mathrm{d}\mathcal{E}(X)[H]P \Big) \right]. \tag{A.6}$$

Convergence via the Banach Fixed Point Theorem. To prove that ϕ_P is a contraction, we employ the Mean Value Inequality for Banach spaces: given any two points X_1, X_2 in the closed ball $\overline{\mathscr{U}}_{8|P|}(\mathcal{V})$, the segment connecting them is entirely contained within the ball due to convexity. The inequality states:

$$|\phi_P(X_1) - \phi_P(X_2)| \le \left(\sup_{X \in \overline{\mathscr{U}}_{8|P|}(\mathcal{V})} \|\mathrm{d}\phi_P(X)\| \right) |X_1 - X_2|. \tag{A.7}$$

By finding a suitable uniform bound for $\|\mathrm{d}\phi_P(X)\|$ that holds for every X in the ball, we establish that ϕ_P is a contraction.

Theorem A.6.1 (Fixed Point Theorem). *Let $|P| < (24K)^{-1}$. Then the map ϕ_P has a unique fixed point $X(P)$ in the ball $\overline{\mathscr{U}}_{8|P|}(\mathcal{V})$. Moreover, the fixed point is such that $X(0) = 0$ and satisfies the following asymptotic expansion for small P:*

$$X(P) = T\Pi P + O(|P|^2).$$

Proof. For any $X \in \overline{\mathscr{U}}_{8|P|}(\mathcal{V})$, we have $\|X\| \le 8|P|$, which implies $\rho = \|\mathrm{ad}_{-X}\| \le 8K|P|$. Under the smallness condition on P, we have $\rho < 1/3 < 1/2$. Therefore, we can apply the uniform norm bounds for $\mathcal{B}$ and $\mathcal{E}$ derived in the previous subsection:

$$|\phi_P(X)| \le \|T\|_{\mathcal{L}(\mathrm{range}(\Pi), \mathcal{V})} \|\Pi\| \|\mathcal{B}(X)\| \|\Pi\| \|\mathcal{E}(X)\| |P| \le 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2|P| = 8|P|.$$

This confirms that ϕ_P maps the ball $\overline{\mathscr{U}}_{8|P|}(\mathcal{V})$ into itself.

We now derive a uniform bound for the norm of the operator $\mathrm{d}\phi_P(X)$ for an arbitrary $X \in \overline{\mathscr{U}}_{8|P|}(\mathcal{V})$. Applying the triangle inequality and the sub-multiplicative property to the Leibniz expansion (A.6), and recalling that $\|T\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})} \leq 2$ and $\|\Pi\| \leq 1$:

$$|\mathrm{d}\phi_P(X)[H]| \leq \|T\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})}\|\Pi\| \left(\|\mathrm{d}\mathcal{B}(X)[H]\| \cdot \|\Pi\| \cdot |\mathcal{E}(X)P| + \|\mathcal{B}(X)\| \cdot \|\Pi\| \cdot |\mathrm{d}\mathcal{E}(X)[H]P|\right).$$

Substituting the quantitative bounds for the differentials and operators found in the previous subsection, and noting that these bounds are valid throughout the ball since $\rho < 1/3 < 1/2$:

$$|\mathrm{d}\phi_P(X)[H]| \leq 2 \cdot 1 \cdot ([4K|H|] \cdot 1 \cdot (2|P|) + 2 \cdot 1 \cdot ([2K|H|] \cdot |P|)) = 24K|P||H|.$$

Since this bound holds for all $X \in \overline{\mathscr{U}}_{8|P|}(\mathcal{V})$, the Lipschitz constant in (A.7) is bounded by $\gamma = 24K|P|$. The condition $\gamma < 1$ is satisfied by the hypothesis on P. Therefore, by the Banach Fixed Point Theorem, ϕ_P is a contraction on a complete metric space, ensuring the existence of a unique fixed point $X(P)$.

To establish the asymptotic expansion, we first observe that for $P = 0$, the map $\phi_0(X)$ vanishes identically for any X, since P enters the definition as a linear factor. This immediately implies that $X(0) = \phi_0(X(0)) = 0$.

For $P \neq 0$, we expand ϕ_P around the origin $X = 0$ using Taylor's theorem:

$$X(P) = \phi_P(0) + \mathcal{R}_1(X(P)), \quad \text{where} \quad \mathcal{R}_1(X(P)) = \int_0^1 \mathrm{d}\phi_P(tX(P))[X(P)]\mathrm{d}t.$$

Evaluating ϕ_P at $X = 0$, and noting that $\mathrm{ad}_0 = 0$ implies $\mathcal{B}(0) = \mathcal{E}(0) = \mathrm{Id}$, we obtain the first-order term:

$$\phi_P(0) = T\Pi(\mathrm{Id}\ \Pi(\mathrm{Id}\, P)) = T\Pi^2 P = T\Pi P.$$

The remainder $\mathcal{R}_1$ accounts for higher-order terms. Using the previously derived bound $\|\mathrm{d}\phi_P\| \leq 24K|P|$ and the fact that $|X(P)| \leq 8|P|$, we estimate the norm of the integral:

$$|\mathcal{R}_1(X(P))| \leq \sup_{t\in[0,1]} \|\mathrm{d}\phi_P(tX(P))\||X(P)| \leq (24K|P|)(8|P|) = 192K|P|^2.$$

Thus, $\mathcal{R}_1 = O(|P|^2)$, confirming that $X(P) = T\Pi P + O(|P|^2)$. ■

Corollary A.6.2. *Suppose that $T = T(\lambda)$ and $P = P(\lambda)$ depend analytically on a real parameter λ for λ in a closed interval $\Lambda \subset \mathbb{R}$ and satisfy the bounds $\|T(\lambda)\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})} \leq 2$, $|P(\lambda)| < (24K)^{-1}$ for all $\lambda \in \Lambda$. Then the unique fixed point $X(P, \lambda)$ is analytic in λ. Furthermore, its derivative satisfies the bound:*

$$\left|\frac{dX}{d\lambda}\right| \leq \frac{4\|T'\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})}|P| + 8|P'|}{1 - 24K|P|}$$

where $T' = \frac{d}{d\lambda}T$ and $P' = \frac{d}{d\lambda}P$.

Proof. The proof starts by ensuring that the fixed point is analytic in λ. Since $T(\lambda)$ and $P(\lambda)$ are analytic, the operator $\phi_{P,\lambda}$ depends analytically on λ. Moreover, the condition $|P(\lambda)| < (24K)^{-1}$ on the closed interval Λ ensures that $\phi_{P,\lambda}$ is always a contraction with a constant $\gamma = 24K|P| < 1$, uniformly in λ. Therefore, the Implicit Function Theorem guarantees that the fixed point $X(P, \lambda)$ is analytic in λ.

Since T and P are analytic on the closed interval Λ, their derivatives T' and P' are continuous and their norms remain finite for all $\lambda \in \Lambda$. To find the bound for the derivative of the fixed point, we differentiate both sides of the identity $X(\lambda) = \phi_{P,\lambda}(X(\lambda))$ with respect to λ using the chain rule:

$$\frac{dX}{d\lambda} = \frac{\partial \phi_{P,\lambda}}{\partial \lambda} + \mathrm{d}\phi_{P,\lambda}(X)\left[\frac{dX}{d\lambda}\right].$$

Here, $\partial_\lambda \phi_{P,\lambda}$ represents the derivative with respect to the external parameter λ, while $\mathrm{d}\phi_{P,\lambda}$ is the Fréchet differential with respect to X. Moving the terms involving $dX/d\lambda$ to the left, we get:

$$(\mathrm{Id} - \mathrm{d}\phi_{P,\lambda}(X))\,\frac{dX}{d\lambda} = \frac{\partial \phi_{P,\lambda}}{\partial \lambda}.$$

Since the norm of the operator $\mathrm{d}\phi_{P,\lambda}$ is bounded by $24K|P| < 1$, the linear operator $\mathrm{Id} - \mathrm{d}\phi_{P,\lambda}(X)$ is invertible. Using the usual geometric series estimate for the Neumann series that defines the inverse, we obtain the bound

$$\left|\frac{dX}{d\lambda}\right| \leq \frac{1}{1 - 24K|P|}\left|\frac{\partial \phi_{P,\lambda}}{\partial \lambda}\right|.$$

The partial derivative $\partial_\lambda \phi_{P,\lambda}$ is calculated by differentiating $\phi_P(X) = T\Pi(\mathcal{B}(X)\Pi(\mathcal{E}(X)P))$ while keeping X fixed:

$$\frac{\partial \phi_{P,\lambda}}{\partial \lambda} = T'\Pi\left(\mathcal{B}(X)\Pi(\mathcal{E}(X)P)\right) + T\Pi\left(\mathcal{B}(X)\Pi(\mathcal{E}(X)P')\right).$$

Using the bounds $\|\Pi\| \leq 1$, $\|T\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})} \leq 2$, $\|\mathcal{B}\| < 2$, and $\|\mathcal{E}\| < 2$, we find:

$$\left|\frac{\partial \phi_{P,\lambda}}{\partial \lambda}\right| \leq \|T'\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})} \cdot (1 \cdot 2 \cdot 1 \cdot 2|P|) + 2 \cdot (1 \cdot 2 \cdot 1 \cdot |P'|) = 4\|T'\|_{\mathcal{L}(\mathrm{range}(\Pi),\mathcal{V})}|P| + 8|P'|.$$

Plugging this bound into the inequality for $|dX/d\lambda|$ gives the final estimate. ■

A.7 Additional Estimates for Applications

Throughout this subsection, we assume that T maps the range of Π into itself and that there exists a left inverse U such that $U \circ T = \mathrm{Id}$ on $\mathrm{range}(\Pi)$. We do not require U to be a bounded operator on $\mathcal{V}$.

If X is the unique fixed point established in Theorem A.6.1, then $X \in \mathrm{range}(T)$, which ensures that the expression UX is always well-defined. We can therefore characterize the fixed point through the identity:

$$UX = \Pi\left(\mathcal{B}(X)\Pi\mathcal{E}(X)P\right).$$

Lemma A.7.1 (Estimates for UX)**.** *Let $|P| < (24K)^{-1}$ and let X be the unique fixed point of ϕ_P in $\overline{\mathscr{U}}_{8|P|}$. Then, $|UX| \leq 4|P|$ and satisfies the following asymptotic expansion for small P:*

$$UX = \Pi P + O(|P|^2).$$

Proof. To prove the bound, we use the property $\|\Pi\| \leq 1$ and the bounds $\|\mathcal{B}(X)\| \leq 2$, $\|\mathcal{E}(X)\| \leq 2$, which are valid since $K|X| < 1/2$ for any X in the ball $\overline{\mathscr{U}}_{8|P|}$. We have:

$$|UX| \leq \|\Pi\|^2\|\mathcal{B}(X)\|\|\mathcal{E}(X)\||P| \leq 1 \cdot 2 \cdot 2|P| = 4|P|.$$

For the asymptotic expansion, we observe that if $P = 0$, then $X(0) = 0$ and UX vanishes. For $P \neq 0$, we expand UX around $X = 0$ using the fundamental theorem of calculus:

$$UX = (UX)\big|_{X=0} + \mathcal{R}_1(X,P), \quad \text{where} \quad \mathcal{R}_1(X,P) = \int_0^1 \frac{d}{dt}\left[\Pi\left(\mathcal{B}(tX)\Pi\mathcal{E}(tX)P\right)\right]\mathrm{d}t.$$

Since $\mathcal{B}(0) = \mathcal{E}(0) = \mathrm{Id}$, the first-order term is $(UX)\big|_{X=0} = \Pi^2 P = \Pi P$. By applying the Leibniz rule, the integrand becomes:

$$\frac{d}{dt}\left[\Pi\mathcal{B}(tX)\Pi\mathcal{E}(tX)P\right] = \Pi\left(\mathrm{d}\mathcal{B}(tX)[X]\right)\Pi\mathcal{E}(tX)P + \Pi\mathcal{B}(tX)\Pi\left(\mathrm{d}\mathcal{E}(tX)[X]\right)P.$$

Recalling the estimates from the previous section, for $X \in \overline{\mathscr{U}}_{8|P|}$ and $t \in [0,1]$, we have the bounds $\|\mathrm{d}\mathcal{B}(tX)[X]\| \leq 4K|X|$ and $\|\mathrm{d}\mathcal{E}(tX)[X]\| \leq 2K|X|$, as well as $\|\mathcal{B}(tX)\| \leq 2$ and $\|\mathcal{E}(tX)\| \leq 2$. It follows that:

$$\begin{aligned}\left|\frac{d}{dt}\left[\Pi\mathcal{B}(tX)\Pi\mathcal{E}(tX)P\right]\right| &\leq \|\Pi\|^2\|\mathrm{d}\mathcal{B}(tX)[X]\|\cdot\|\mathcal{E}(tX)\||P| + \|\Pi\|^2\|\mathcal{B}(tX)\|\cdot\|\mathrm{d}\mathcal{E}(tX)[X]\||P| \\ &\leq 1\cdot(4K|X|)\cdot 2|P| + 1\cdot 2\cdot(2K|X||P|) \\ &= 8K|X||P| + 4K|X||P| = 12K|X||P|.\end{aligned}$$

Integrating over $t \in [0,1]$ and using the bound $|X| \leq 8|P|$ from the fixed point theorem, we obtain:

$$|\mathcal{R}_1(X,P)| \leq \int_0^1 12K|X||P|\,\mathrm{d}t \leq 12K(8|P|)|P| = 96K|P|^2.$$

This confirms the expansion $UX = \Pi P + O(|P|^2)$. ■

We conclude with the following technical lemma:

Lemma A.7.2 (Estimates for Δ)**.** *Let $|P| < (24K)^{-1}$ and X be the unique fixed point. Under the assumption $\|\operatorname{Id}-\Pi\| \leq 1$, the quantity*

$$\Delta := (\operatorname{Id}-\Pi)\big((\mathcal{E}(X)-\operatorname{Id})P - (\mathcal{B}(X)^{-1}-\operatorname{Id})UX\big)$$

satisfies $|\Delta| \leq 64K|P|^2$. Furthermore, Δ satisfies the following asymptotic expansion for small P:

$$\Delta = (\operatorname{Id}-\Pi)\left([(\operatorname{Id}-\Pi)P, T\Pi P] + \frac{1}{2}[\Pi P, T\Pi P]\right) + O(|P|^3).$$

Proof. First, we substitute the fixed point identity $UX = \Pi\left(\mathcal{B}(X)\Pi\mathcal{E}(X)P\right)$ into the definition of Δ:

$$\Delta = (\operatorname{Id}-\Pi)(\mathcal{E}(X)-\operatorname{Id})P - (\operatorname{Id}-\Pi)(\mathcal{B}(X)^{-1}-\operatorname{Id})\Pi\mathcal{B}(X)\Pi\mathcal{E}(X)P.$$

We now use the triangle inequality to split the norm into two pieces:

$$|\Delta| \leq |\underbrace{(\operatorname{Id}-\Pi)(\mathcal{E}(X)-\operatorname{Id})P}_{\Delta_1}| + |\underbrace{(\operatorname{Id}-\Pi)(\mathcal{B}(X)^{-1}-\operatorname{Id})\Pi\mathcal{B}(X)\Pi\mathcal{E}(X)P}_{\Delta_2}|.$$

Let $\rho = \|\mathrm{ad}_{-X}\| \leq 8K|P|$. Since $|P| < (24K)^{-1}$, we have $\rho < 1/3$ and, as established in Lemma A.5.1, the bound $\|(\mathcal{B}(X))^{-1}\| \leq \frac{1}{1-\rho} < 2$ holds.

Term Δ_1. The operator $\mathcal{E}(X)-\operatorname{Id}$ is the series $\sum_{k=1}^{\infty}\frac{1}{k!}(\mathrm{ad}_{-X})^k$. Taking the norm and using the bound $\|\operatorname{Id}-\Pi\| \leq 1$:

$$|\Delta_1| \leq 1\cdot\left(\sum_{k=1}^{\infty}\rho^k\right)|P| = \frac{\rho}{1-\rho}|P|.$$

Substituting $\rho \leq 8K|P|$ and $\frac{1}{1-\rho} < 2$, we get:

$$|\Delta_1| \leq 2\cdot(8K|P|)\cdot|P| = 16K|P|^2.$$

Term Δ_2. To estimate this piece, we use the property from Lemma A.5.1 where $\|(\mathcal{B}(X))^{-1}-\operatorname{Id}\| \leq \frac{\rho}{1-\rho}$. Combining this with $\|\mathcal{B}(X)\| \leq 2$, and $|\mathcal{E}(X)P| \leq 2|P|$:

$$|\Delta_2| \leq \frac{\rho}{1-\rho}\cdot 1\cdot 2\cdot 1\cdot 2|P| = 4\frac{\rho}{1-\rho}|P| \leq 48K|P|^2.$$

Summing the estimates yields $|\Delta| \le 64K|P|^2$.

Asymptotic Expansion. To derive the leading order for small P, we perform a Taylor expansion of Δ around $X = 0$. Since $\Delta(0) = 0$, we have $\Delta(X) = \mathrm{d}\Delta(0)[X] + \mathcal{R}_2(X)$. We differentiate the two components Δ_1 and Δ_2 separately:

$$\begin{aligned}\Delta_1(X) &= (\mathrm{Id} - \Pi)(\mathcal{E}(X) - \mathrm{Id})P,\\ \Delta_2(X) &= (\mathrm{Id} - \Pi)(\mathcal{B}(X)^{-1} - \mathrm{Id})\Pi\mathcal{B}(X)\Pi\mathcal{E}(X)P.\end{aligned}$$

From Section A.2, we have $\mathrm{d}\mathcal{E}(0)[H] = \mathrm{ad}_{-H}$. Then,

Differential of Δ_1*:* Differentiating Δ_1 at $X = 0$ in the direction H:

$$\mathrm{d}\Delta_1(0)[H] = (\mathrm{Id} - \Pi)\left(\mathrm{d}\mathcal{E}(0)[H]\right)P = (\mathrm{Id} - \Pi)\mathrm{ad}_{-H}P = (\mathrm{Id} - \Pi)[P, H].$$

Differential of Δ_2*:* We apply the Leibniz rule to the product of operators in Δ_2 and evaluate at $X = 0$. Recalling that $\mathcal{B}(0) = \mathcal{E}(0) = \mathrm{Id}$, we compute the differential. Since $(\mathcal{B}(0)^{-1} - \mathrm{Id}) = 0$, only the term where the derivative acts on the first factor remains non-zero:

$$\begin{aligned}\mathrm{d}\Delta_2(0)[H] &= (\mathrm{Id} - \Pi)\left[\left(\mathrm{d}(\mathcal{B}(0)^{-1})[H]\right)\Pi\mathcal{B}(0)\Pi\mathcal{E}(0)P\right]\\ &= (\mathrm{Id} - \Pi)\left(\mathrm{d}(\mathcal{B}(0)^{-1})[H]\right)\Pi P.\end{aligned}$$

Now, from the analytic expansion in Lemma A.5.1, we substitute $\mathrm{d}(\mathcal{B}(0)^{-1})[H] = \frac{1}{2}\mathrm{ad}_{-H}$ to obtain

$$\mathrm{d}\Delta_2(0)[H] = (\mathrm{Id} - \Pi)\left(\tfrac{1}{2}\mathrm{ad}_{-H}\Pi P\right) = \tfrac{1}{2}(\mathrm{Id} - \Pi)[\Pi P, H].$$

Differential of Δ*:* Combining the differentials of Δ_1 and Δ_2, we have $\mathrm{d}\Delta(0)[H] = \mathrm{d}\Delta_1(0)[H] - \mathrm{d}\Delta_2(0)[H]$. Substituting $X = T\Pi P + O(|P|^2)$ into the linear expressions:

$$\mathrm{d}\Delta(0)[X] = (\mathrm{Id} - \Pi)\left(\frac{1}{2}[\Pi P, T\Pi P] + [(\mathrm{Id} - \Pi)P, T\Pi P]\right) + O(|P|^3).$$

Estimate for the reminder: The remainder $\mathcal{R}_2(X)$ is defined by the integral formula:

$$\mathcal{R}_2(X) = \int_0^1 (1-t)\mathrm{d}^2\Delta(tX)[X, X]\mathrm{d}t.$$

The second Fréchet derivative $\mathrm{d}^2\Delta(X)[X, X]$ is a sum of terms involving double commutators with P, of the form $[X, [X, P]]$, multiplied by operators $\mathcal{E}, \mathcal{B}, \mathcal{B}(X)^{-1}$ and their derivatives. Given the algebra property and the fact that these operators are analytic and bounded for $\rho < 1/3$, the norm of the second derivative satisfies:

$$\|\mathrm{d}^2\Delta(X)[X, X]\| \le CK^2|X|^2|P|,$$

where C is a constant. Substituting this bound into the integral:

$$\begin{aligned}|\mathcal{R}_2(X)| &\le \int_0^1 (1-t)\|\mathrm{d}^2\Delta(tX)[X, X]\|\mathrm{d}t\\ &\le \left(\int_0^1 (1-t)\mathrm{d}t\right)\cdot \sup_{t\in[0,1]}\left(CK^2|X|^2|P|\right)\\ &= \frac{1}{2}\cdot CK^2|X|^2|P|.\end{aligned}$$

Using the bound for the fixed point $|X| \leq 8|P|$ we get

$$|\mathcal{R}_2(X)| \leq \frac{1}{2} \cdot CK^2(8|P|)^2|P| = 32CK^2|P|^3.$$

This shows that $|\mathcal{R}_2(X)| \leq M|P|^3$ for a constant M, which confirms $\mathcal{R}_2 = O(|P|^3)$. ∎

Remark A.7.3. *If the operators T and P depend analytically on a real parameter $\lambda \in \Lambda$ (where Λ is a closed interval of the real line), the fixed point $X(\lambda)$ inherits the analyticity, as proved in Corollary A.6.2. Consequently, UX and Δ are analytic in λ since they are obtained by composing analytic maps (the operators $\mathcal{B}$ and $\mathcal{E}$) with the analytic functions $X(\lambda)$ and $P(\lambda)$. Moreover, the constants appearing in the estimates depend only on the uniform bounds of the operators for $\lambda \in \Lambda$, ensuring that all the asymptotic expansions hold uniformly in the parameter λ.*

Appendix B

Technical Lemmas for the Analysis of the Spectral Gaps

To ensure a self-contained treatment, we detail here the functional framework and provide the proofs for several auxiliary results. These technical lemmas are fundamental to the study of the spectral gaps developed in Subsection 3.2.8.

To ease the notation, we suppress the explicit dependence on the parameter s in the Banach spaces we consider and their respective norms, since it remains constant and plays no role throughout this appendix; nevertheless, we emphasize that all subsequent results hold for any $s \geq 0$.

Definition B.0.1. *Given $R > 0$, $p > 1$, and $s \geq 0$, for any $j \in \mathbb{Z}$ and $\tau = \pm 1/2$, we denote by $\mathcal{M}^{\tau,j}_{p,R}$ the space of maps $X : \mathscr{U}_R(\ell^\infty_p) \to \prod_{\ell\in\mathbb{Z}} \mathrm{Mat}(2,\mathbb{C})$ such that each entry $X^{\sigma\sigma'}_\ell \in \mathcal{H}_{p,R}$ satisfies the momentum conservation condition $\pi(\alpha) = \ell - j$. We equip this space with a Banach structure by introducing the majorant $\underline{X} = (\underline{X}_\ell)_{\ell\in\mathbb{Z}}$, where $\underline{X}^{\sigma\sigma'}_\ell(z) := \sum_{\alpha\in\mathscr{M}_{\ell-j}} |X^{\sigma\sigma'}_{\ell\alpha}| z^\alpha$.*

To define the norm, we first generalize the norm on the space of sequences of 2×2 complex matrices $W = (W_\ell)_{\ell\in\mathbb{Z}}$ to the (τ, j)-weighted case:

$$\|W\|^{\tau,j}_p := \sup_{\ell\in\mathbb{Z}} \langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^\tau \max_{\sigma,\sigma'=\pm1} |W^{\sigma\sigma'}_\ell|.$$

The norm on $\mathcal{M}^{\tau,j}_{p,R}$ is then given by:

$$|X|^{\tau,j}_{p,R} := \sup_{\|z\|_p\leq R} \|\underline{X}(z)\|^{\tau,j}_p = \sup_{\ell\in\mathbb{Z}} \langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^\tau \max_{\sigma,\sigma'=\pm1} \|X^{\sigma\sigma'}_\ell\|_{p,R}, \tag{B.1}$$

which, due to the non-negativity of the coefficients, coincides with evaluating the majorant on the weight vector $v_{p,R}$. The space $\mathcal{M}^{\tau,j}_{p,R}$ consists of all such maps for which this norm is finite.

Remark B.0.2. *Any $X \in \mathcal{M}^{\tau,0}_{p,R}$ is a majorant analytic function $X : \ell^\infty_p \to \ell^\infty_{p+\tau,R}(\mathrm{Mat}(2,\mathbb{C}))$. In fact, Theorem 1.2.24 ensures that $\underline{X}$ is analytic as it is defined by the uniformly bounded family of analytic functions $\underline{X}^{\sigma\sigma'}_\ell \in \mathcal{H}_{p,R}$.*

B.1 Estimates for $P \circ |\partial|^{1/2}$ and $\partial_{z_j}(P \circ |\partial|^{1/2})$

Given a positive real number λ, let P be the map from ℓ^∞_p to the space of sequences of 2×2 complex matrices, defined for all $u \in \ell^\infty_p$ by its components:

$$P_\ell(u) = \frac{u_\ell}{2\lambda} \begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad \text{for all } \ell \in \mathbb{Z}.$$

In the following, we denote by $|\partial|^{1/2} : \ell^\infty_p \to \ell^\infty_{p-1/2}$ the operator acting as a Fourier multiplier:

$$u := |\partial|^{1/2} z, \quad \text{with} \quad u_\ell = \langle \ell \rangle^{1/2} z_\ell.$$

Lemma B.1.1. *For any $R > 0$, $P \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R}$ and $\partial_{z_j}(P \circ |\partial|^{1/2}) \in \mathcal{M}^{-1/2,j}_{p,R}$. Specifically, the following estimates hold:*

$$|P \circ |\partial|^{1/2}|^{-1/2,0}_{p,R} = \frac{R}{2\lambda}, \quad |\partial_{z_j}(P \circ |\partial|^{1/2})|^{-1/2,j}_{p,R} \le \frac{1}{2\lambda}.$$

Proof. The first equality follows directly from the definition of P, noting that the weight of the norm compensates for the action of $|\partial|^{1/2}$. By the same reasoning, the second bound is immediate, as the derivative produces a $\delta_{j\ell}$ which is trivially bounded by $\langle \ell - j \rangle^{-p} e^{-s|\ell - j|}$. ■

B.2 Estimates for $A_P \circ |\partial|^{1/2}$ and $\mathcal{L}_\lambda A_P \circ |\partial|^{1/2}$

Fix $n \in \mathbb{N}$ and let $\lambda \in \mathbb{R}^+$ be such that $|2\lambda - n| \le 1/2$. Let $\mathcal{L}_\lambda$ be the operator defined in Subsection 3.2.4 (see formula (3.22)). By identifying periodic functions taking values in the Lie algebra $\mathrm{Mat}(2, \mathbb{C})$ with the sequences of their Fourier coefficients, the operator $\mathcal{L}_\lambda$ induces an action on the space of matrix sequences $W = (W_\ell)$ defined by

$$(\mathcal{L}_\lambda W)^{\sigma\sigma'}_\ell = i(\ell - \lambda(\sigma - \sigma'))W^{\sigma\sigma'}_\ell.$$

We have already observed that, if $\lambda = n/2$, this operator has a non-trivial kernel. Namely, a sequence W belongs to $\mathrm{range}(\mathcal{L}_{n/2})$ if and only if its components $W^{\sigma\sigma'}_\ell$ are zero whenever $\ell - \frac{n}{2}(\sigma - \sigma') = 0$.

From the definition of the action of $\mathcal{L}_\lambda$ on the Fourier coefficients, it follows that the formal inverse $\mathcal{L}_\lambda^{-1}$ acts diagonally as:

$$(\mathcal{L}_\lambda^{-1} W)^{\sigma\sigma'}_\ell = \frac{1}{i(\ell - \lambda(\sigma - \sigma'))} W^{\sigma\sigma'}_\ell,$$

provided that the denominator does not vanish. For any λ such that $|2\lambda - n| \le 1/2$, we define $\mathcal{L}_\lambda^{-1}$ on $\mathrm{range}(\mathcal{L}_{n/2})$. Clearly, both $\mathcal{L}_\lambda$ and its inverse preserve $\mathrm{range}(\mathcal{L}_{n/2})$.

For λ in the specified region, the divisors for indices in the range of the operator satisfy the following estimate:

$$\frac{\langle \ell \rangle}{|\ell - \lambda(\sigma - \sigma')|} \le 4n. \tag{B.2}$$

This bound shows that the inverse operator compensates for the loss of regularity, effectively shifting the weight from $\langle \ell \rangle^{-1/2}$ to $\langle \ell \rangle^{+1/2}$ at the cost of a factor n, as established in the following lemma.

Lemma B.2.1. *Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ satisfy $|2\lambda - n| \le 1/2$. Then, $\mathcal{L}_\lambda^{-1}$ is a bounded linear operator*

$$\mathcal{L}_\lambda : \mathcal{M}^{-1/2,j}_{p,R} \cap \mathrm{range}(\mathcal{L}_{n/2}) \to \mathcal{M}^{+1/2,j}_{p,R} \cap \mathrm{range}(\mathcal{L}_{n/2})$$

In particular, there exists a constant $C > 0$ such that

$$|\mathcal{L}_\lambda^{-1} X|^{+1/2,j}_{p,R} \le 4n |X|^{-1/2,j}_{p,R}$$

for every $X \in \mathcal{M}^{-1/2,j}_{p,R} \cap \mathrm{range}(\mathcal{L}_{n/2})$.

Proof. By the definition of the norm on $\mathcal{M}^{\tau,j}_{p,R}$ given in (B.1), we have

$$|\mathcal{L}_\lambda^{-1} X|^{+1/2,j}_{p,R} = \sup_{\ell\in\mathbb{Z}} \left(\langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^{1/2} \max_{\sigma,\sigma'} \|(\mathcal{L}_\lambda^{-1} X)^{\sigma\sigma'}_\ell\|_{p,R} \right).$$

Recall that $(\mathcal{L}_\lambda^{-1} X)^{\sigma\sigma'}_\ell = \frac{1}{i(\ell-\lambda(\sigma-\sigma'))} X^{\sigma\sigma'}_\ell$. Substituting this into the expression, we obtain

$$|\mathcal{L}_\lambda^{-1} X|^{+1/2,j}_{p,R} = \sup_{\ell\in\mathbb{Z}} \left(\langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^{1/2} \max_{\sigma,\sigma'} \frac{1}{|\ell-\lambda(\sigma-\sigma')|} \|X^{\sigma\sigma'}_\ell\|_{p,R} \right). \tag{B.3}$$

We can rewrite the terms inside the supremum by multiplying and dividing by $\langle \ell\rangle^{-1/2}$ to recover the norm of X:

$$\text{(B.3)} = \sup_{\ell\in\mathbb{Z}} \left(\frac{\langle \ell\rangle}{|\ell-\lambda(\sigma-\sigma')|} \cdot \langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^{-1/2} \max_{\sigma,\sigma'} \|X^{\sigma\sigma'}_\ell\|_{p,R} \right).$$

Using the estimate (B.2), we get

$$|\mathcal{L}_\lambda^{-1} X|^{+1/2,j}_{p,R} \le 4n \sup_{\ell\in\mathbb{Z}} \left(\langle \ell - j\rangle^p e^{s|\ell-j|} \langle \ell\rangle^{-1/2} \max_{\sigma,\sigma'} \|X^{\sigma\sigma'}_\ell\|_{p,R} \right).$$

The supremum on the right-hand side is precisely the norm $|X|^{-1/2,j}_{p,R}$. Therefore, we get

$$|\mathcal{L}_\lambda^{-1} X|^{+1/2,j}_{p,R} \le 4n |X|^{-1/2,j}_{p,R},$$

which ends the proof. ■

Let A_P be the fixed point provided by Theorem 3.2.4, which we recall to be defined by

$$A_P = \phi_P(A_P) = \mathcal{L}_\lambda^{-1} \Pi_{rg} \left(\mathcal{B}(A_P) \Pi_{rg} \left(e^{\mathrm{ad}_{-A_P}} P \right) \right), \tag{B.4}$$

where the operators $\mathcal{E}(X)$ and $\mathcal{B}(X)$ are formally defined via the Taylor series of the complex functions $f(z) = e^z = \sum_{k=0}^\infty \frac{1}{k!} z^k$ and $g(z) = \frac{z}{e^z-1} = \sum_{k=0}^\infty \beta_k z^k$ (see Section A.2). Specifically, $\mathcal{E}(X) = f(\mathrm{ad}_{-X})$ and $\mathcal{B}(X) = g(\mathrm{ad}_{-X})$.

In the following lemma, we first derive a bound for $\mathcal{L}_\lambda A_P \circ |\partial|^{1/2}$. We then apply the smoothing property of $\mathcal{L}_\lambda^{-1}$ established in Lemma B.2.1 to prove that $A_P \circ |\partial|^{1/2}$ maps into a sequence space with stronger decay, with a resulting bound constant that grows linearly with n.

Lemma B.2.2. *Let $n \in \mathbb{N}$ and $\lambda > 0$ such that $|2\lambda - n| \le 1/2$. Then, for all $p > 3/2$, we have that $\mathcal{L}_\lambda A_P \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R}$ and $A_P \circ |\partial|^{1/2} \in \mathcal{M}^{+1/2,0}_{p,R}$, with the following estimates:*

$$|\mathcal{L}_\lambda A_P \circ |\partial|^{1/2}|^{-1/2,0}_{p,R} \le \frac{2R}{\lambda}, \quad |A_P \circ |\partial|^{1/2}|^{+1/2,0}_{p,R} \le 8R\frac{n}{\lambda}.$$

Proof. By Theorem 3.2.4, $|\mathcal{L}_\lambda A_P|_{p',R} \le 4|P|_{p',R}$ for all $p' > 1$. Then, the first bound follows by the fact that, if $X \in \underline{\mathcal{A}}^\pi_R$ satisfies $|X|_{p-1/2,R} < \infty$, then $X \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R}$, with the identity

$$|X \circ |\partial|^{1/2}|^{-1/2,0}_{p,R} = |X|_{p-1/2,R}$$

holding by definition, combined with Lemma B.1.1. The second estimate is obtained by writing $A_P = (\mathcal{L}_\lambda^{-1} \circ \mathcal{L}_\lambda) A_P$ and applying the first bound together with the one provided by Lemma B.2.1. ■

B.3 Estimates for $\partial_{z_j}(A_P \circ |\partial|^{1/2})$ and $\partial_{z_j}(\mathcal{L}_\lambda A_P \circ |\partial|^{1/2})$

In this section, we provide estimates for the maps $\partial_{z_j}(A_P \circ |\partial|^{1/2})$ and $\partial_{z_j}(\mathcal{L}_\lambda A_P \circ |\partial|^{1/2})$, where A_P is the fixed point of the operator from Theorem 3.2.4, whose definition is recalled in (B.4). Since it is implicitly defined through commutators, the main task is to provide estimates for the latter by understanding how the convolution product works in our weighted spaces. The following lemmas are the key tool for all the calculations in this section.

Lemma B.3.1 (Convolution and Algebra Property)**.** *Let $X \in \mathcal{M}_{p,R}^{-1/2,j}$ and $Y \in \mathcal{M}_{p,R}^{+1/2,j'}$. The convolution product $Z = X \star Y$, defined by the sequence of matrices $Z_\ell(z) = \sum_{k\in\mathbb{Z}} X_{\ell-k}(z)Y_k(z)$, belongs to $\mathcal{M}_{p,R}^{-1/2,j+j'}$. More precisely, it satisfies the following properties:*

- ***Momentum Conservation:*** *For each $\ell \in \mathbb{Z}$, the entries of $Z_\ell(z)$ belong to $\mathcal{H}_{p,R}$ and their Taylor expansion involves only multi-indices α such that $\pi(\alpha) = \ell - (j + j')$.*
- ***Estimate:*** *The following estimates hold:*

$$|X \star Y|_{p,R}^{-1/2,j+j'}, |Y \star X|_{p,R}^{-1/2,j'+j} \le 2C_\star(p)|X|_{p,R}^{-1/2,j}\,|Y|_{p,R}^{+1/2,j'},$$

where $C_\star(p) = 2^p(1 + 2\zeta(p))$ denotes the convolution algebra constant of $\ell^\infty_{s,p}$ and $\zeta(p) = \sum_{n\ge1} n^{-p}$ is the Riemann zeta function.

Proof. Momentum Conservation. Let $X \in \mathcal{M}_{p,R}^{-1/2,j}$ and $Y \in \mathcal{M}_{p,R}^{+1/2,j'}$. By definition, the entries of the convolution $Z = X \star Y$ are given by

$$Z_\ell^{\sigma\sigma'}(z) = \sum_{k\in\mathbb{Z}}\sum_{\rho\in\{\pm1\}} X_{\ell-k}^{\sigma\rho}(z)Y_k^{\rho\sigma'}(z).$$

Substituting the Taylor expansions of the entries, we obtain

$$Z_\ell^{\sigma\sigma'}(z) = \sum_{k\in\mathbb{Z}}\sum_{\rho\in\{\pm1\}} \left(\sum_{\alpha\in\mathscr{M}_{\ell-k-j}} X_{\ell-k,\alpha}^{\sigma\rho} z^\alpha\right)\left(\sum_{\beta\in\mathscr{M}_{k-j'}} Y_{k,\beta}^{\rho\sigma'} z^\beta\right).$$

The product of two monomials $z^\alpha z^\beta = z^{\alpha+\beta}$ satisfies the condition $\pi(\alpha+\beta) = \pi(\alpha) + \pi(\beta) = (\ell - k - j) + (k - j') = \ell - (j + j')$. Thus, each term in the expansion of Z_ℓ belongs to $\mathscr{M}_{\ell-(j+j')}$. Since $\mathcal{H}_{p,R}$ is an algebra, the series converges in $\mathcal{H}_{p,R}$, and the momentum condition is satisfied.

Estimate. To estimate the norm $|X\star Y|_{p,R}^{-1/2,j+j'}$, we consider the majorant $\underline{Z}_\ell$. By the triangle inequality and the definition of the majorant norm, we have

$$\begin{aligned}\max_{\sigma,\sigma'} |\underline{Z}_\ell^{\sigma\sigma'}(z)| &\le 2\sum_{k\in\mathbb{Z}} \max_{\sigma,\rho} |\underline{X}_{\ell-k}^{\sigma\rho}(z)| \max_{\rho,\sigma'} |\underline{Y}_k^{\rho\sigma'}(z)| \\ &\le 2\sum_{k\in\mathbb{Z}} \frac{\|\underline{X}(z)\|_p^{-1/2,j}}{\langle \ell-k-j\rangle^p e^{s|\ell-k-j|}\langle \ell-k\rangle^{-1/2}} \cdot \frac{\|\underline{Y}(z)\|_p^{+1/2,j'}}{\langle k-j'\rangle^p e^{s|k-j'|}\langle k\rangle^{1/2}}.\end{aligned}$$

Multiplying both sides by the weight $\langle \ell-(j+j')\rangle^p e^{s|\ell-(j+j')|}\langle \ell\rangle^{-1/2}$ and taking the supremum over ℓ, we obtain:

$$\|\underline{Z}(z)\|_p^{-1/2,j+j'} \le 2\cdot 2^p \left(\sup_{\ell\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{1}{\langle \ell-k-j\rangle^p}\right) \|\underline{X}(z)\|_p^{-1/2,j}\|\underline{Y}(z)\|_p^{+1/2,j'}.$$

The sum in the parentheses is independent of ℓ and converges for any $p > 1$. In particular, we get:

$$\|\underline{Z}(z)\|_p^{-1/2,j+j'} \le 2C_\star(p)\|\underline{X}(z)\|_p^{-1/2,j}\|\underline{Y}(z)\|_p^{+1/2,j'},$$

Finally, taking the supremum over $\|z\|_p \le R$ yields the desired estimate. ■

Lemma B.3.2. *Let $R > 0$ and $X \in \mathcal{M}_{p,R}^{+1/2,0}$. The adjoint operator $\mathrm{ad}_X : Y \mapsto X \star Y - Y \star X$ is a bounded linear operator from $\mathcal{M}_{p,R}^{-1/2,j}$ to itself for every $j \in \mathbb{Z}$. Its operator norm, defined as*

$$\|\mathrm{ad}_X\|_R := \sup_{j\in\mathbb{Z}} \sup_{|Y|_{p,R}^{-1/2,j}\le 1} |X \star Y - Y \star X|_{p,R}^{-1/2,j},$$

satisfies the estimate

$$\|\mathrm{ad}_X\|_R \le 4C_\star(p)|X|_{p,R}^{+1/2,0}.$$

Proof. Let $Y \in \mathcal{M}_{p,R}^{-1/2,j}$. By Lemma B.3.1, the star product satisfies $|X\star Y|_{p,R}^{-1/2,j} \le C|X|_{p,R}^{+1/2,0}|Y|_{p,R}^{-1/2,j}$. Consequently, the commutator can be estimated as:

$$|X \star Y - Y \star X|_{p,R}^{-1/2,j} \le |X \star Y|_{p,R}^{-1/2,j} + |Y \star X|_{p,R}^{-1/2,j} \le 4C_\star(p)|X|_{p,R}^{+1/2,0}|Y|_{p,R}^{-1/2,j}.$$

By dividing by the norm of Y and taking the supremum over $Y \in \mathcal{M}_{p,R}^{-1/2,j} \setminus \{0\}$, we obtain:

$$\sup_{|Y|_{p,R}^{-1/2,j}\le 1} |X \star Y - Y \star X|_{p,R}^{-1/2,j} \le 4C_\star(p)|X|_{p,R}^{+1/2,0}.$$

Since the right-hand side does not depend on the momentum index j, the inequality remains valid when taking the supremum over all $j \in \mathbb{Z}$. This concludes the proof that $\|\mathrm{ad}_X\|_R \le 4C_\star(p)|X|_{p,R}^{+1/2,0}$. ■

Lemma B.3.3 (Estimates for ϕ_Y and its derivative)**.** *Let $n \in \mathbb{N}$ and $\lambda > 0$ such that $|2\lambda - n| \le 1/2$. Given $Y \in \bigcup_{j\in\mathbb{Z}} \mathcal{M}_{p,R}^{-1/2,j}$ and $X \in \mathcal{M}_{p,R}^{+1/2,0}$, consider the map:*

$$\phi_Y(X) = \mathcal{L}_\lambda^{-1}\Pi_{rg}\left(\mathcal{B}(X)\Pi_{rg}\left(e^{\mathrm{ad}_{-X}}Y\right)\right).$$

Then, the following properties hold:

1. *For any fixed $X \in \mathcal{M}_{p,R}^{+1/2,0}$ satisfying $4C_\star(p)|X|_{p,R}^{+1/2,0} \le 1/2$, the map $Y \mapsto \phi_Y(X)$ defines a bounded linear operator from $\mathcal{M}_{p,R}^{-1/2,j}$ to $\mathcal{M}_{p,R}^{+1/2,j}$ for all $j \in \mathbb{Z}$. In particular:*

$$|\phi_Y(X)|_{p,R}^{+1/2,j} \le 16n|Y|_{p,R}^{-1/2,j}.$$

2. *For a fixed $Y \in \mathcal{M}_{p,R}^{-1/2,0}$, the Fréchet derivative of ϕ_Y at $X \in \mathcal{M}_{p,R}^{+1/2,0}$, with $4C_\star(p)|X|_{p,R}^{+1/2,0} \le 1/2$, denoted by $\mathrm{d}\phi_Y(X)$, satisfies:*

$$\|\mathrm{d}\phi_Y(X)\|_R := \sup_{j\in\mathbb{Z}} \sup_{|H|_{p,R}^{+1/2,j}\le 1} |\mathrm{d}\phi_Y(X)[H]|_{p,R}^{+1/2,j} \le 256nC_\star(p)|Y|_{p,R}^{-1/2,0}.$$

Proof. 1. Estimate for $\phi_Y(X)$. Let $X \in \mathcal{M}^{+1/2,0}_{p,R}$ be fixed and consider $Y \in \mathcal{M}^{-1/2,j}_{p,R}$. From Lemma B.3.2, the adjoint operator ad_{-X} is a bounded linear operator with norm:

$$\|\mathrm{ad}_{-X}\|_R \leq 4C_\star(p)|X|^{+1/2,0}_{p,R} \leq \frac{1}{2}.$$

Consequently, the operators $\mathcal{E}(X) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}_{-X})^k$ and $\mathcal{B}(X) = \sum_{k=0}^{\infty} \beta_k(\mathrm{ad}_{-X})^k$ are well-defined and bounded. In fact, under the assumption $4C_\star(p)|X|^{+1/2,0}_{p,R} \leq 1/2$, we provide a coarse estimate for their norms by dominating the Taylor coefficients with 1:

$$\|\mathcal{E}(X)\|_R,\ \|\mathcal{B}(X)\|_R \leq \sum_{k=0}^{\infty} \|\mathrm{ad}_{-X}\|_R^k \leq 2. \tag{B.5}$$

Let $Z = \Pi_{rg}\mathcal{B}(X)\Pi_{rg}(\mathcal{E}(X)Y)$. Since both the adjoint action and the projectors preserve the momentum conservation condition, $Z \in \mathcal{M}^{-1/2,j}_{p,R} \cap \mathrm{range}(\mathcal{L}_{n/2})$. Using $\|\Pi_{rg}\| \leq 1$, we obtain:

$$|Z|^{-1/2,j}_{p,R} \leq \|\mathcal{B}(X)\|_R\|\mathcal{E}(X)\|_R|Y|^{-1/2,j}_{p,R} \leq 4|Y|^{-1/2,j}_{p,R}.$$

Finally, by applying the smoothing estimate for $\mathcal{L}_\lambda^{-1}$ from Lemma B.2.1, we have

$$|\phi_Y(X)|^{+1/2,j}_{p,R} \leq 4n|Z|^{-1/2,j}_{p,R} \leq 16n|Y|^{-1/2,j}_{p,R},$$

yielding the first claim.

2. Estimate for the derivative $\mathrm{d}\phi_Y(X)$. Fix $Y \in \mathcal{M}^{-1/2,0}_{p,R}$. The Fréchet derivative $\mathrm{d}\phi_Y(X)$ in a direction $H \in \mathcal{M}^{+1/2,j}_{p,R}$ is computed via the Leibniz rule applied to $\mathcal{T}(X) = \mathcal{B}(X)\Pi_{rg}\mathcal{E}(X)$:

$$\mathrm{d}\mathcal{T}(X)[H] = (\mathrm{d}\mathcal{B}(X)[H])\Pi_{rg}\mathcal{E}(X) + \mathcal{B}(X)\Pi_{rg}(\mathrm{d}\mathcal{E}(X)[H]).$$

To justify the estimates for the differentials, consider a generic power series $\mathcal{S}(X) = \sum_{k=0}^{\infty} a_k(\mathrm{ad}_{-X})^k$. Its Fréchet derivative in the direction H is obtained by differentiating each term of the expansion:

$$\mathrm{d}\mathcal{S}(X)[H] = \sum_{k=1}^{\infty} a_k \sum_{m=0}^{k-1} (\mathrm{ad}_{-X})^m \circ \mathrm{ad}_{-H} \circ (\mathrm{ad}_{-X})^{k-1-m}.$$

By taking the operator norm $\|\cdot\|_R$, and using the sub-multiplicativity of the norm, we have:

$$\|\mathrm{d}\mathcal{S}(X)[H]\|_R \leq \|\mathrm{ad}_{-H}\|_R \sum_{k=1}^{\infty} |a_k| k \|\mathrm{ad}_{-X}\|_R^{k-1}.$$

Dominating the coefficients $|a_k|$ with 1 and recalling the identity for the derivative of a geometric series $\sum_{k=1}^{\infty} k\rho^{k-1} = (1-\rho)^{-2}$, we obtain the bound:

$$\|\mathrm{d}\mathcal{S}(X)[H]\|_R \leq \frac{\|\mathrm{ad}_{-H}\|_R}{(1 - \|\mathrm{ad}_{-X}\|_R)^2}.$$

In view of Lemma B.3.2, we have $\|\mathrm{ad}_{-X}\|_R \leq 4C_\star(p)|X|^{+1/2,0}_{p,R}$, thus the differentials of $\mathcal{B}(X)$ and $\mathcal{E}(X)$ satisfy:

$$\|\mathrm{d}\mathcal{B}(X)[H]\|_R \leq \frac{\|\mathrm{ad}_{-H}\|_R}{(1 - 4C_\star(p)|X|^{+1/2,0}_{p,R})^2}, \quad \|\mathrm{d}\mathcal{E}(X)[H]\|_R \leq \frac{\|\mathrm{ad}_{-H}\|_R}{(1 - 4C_\star(p)|X|^{+1/2,0}_{p,R})^2}.$$

Combining these with (B.5) and using $4C_\star(p)|X|_{p,R}^{+1/2,0} \leq 1/2$, we find:

$$\begin{aligned}\|\mathrm{d}\mathcal{T}(X)[H]\|_R &\leq \frac{\|\mathrm{ad}_{-H}\|_R}{(1-4C_\star(p)|X|_{p,R}^{+1/2,0})^2}\frac{1}{(1-4C_\star(p)|X|_{p,R}^{+1/2,0})} \\ &+ \frac{1}{(1-4C_\star(p)|X|_{p,R}^{+1/2,0})}\frac{\|\mathrm{ad}_{-H}\|_R}{(1-4C_\star(p)|X|_{p,R}^{+1/2,0})^2} \\ &= 16\|\mathrm{ad}_{-H}\|_R.\end{aligned}$$

Applying the bound $\|\mathrm{ad}_{-H}\|_R \leq 4C_\star(p)|H|_{p,R}^{+1/2,j}$ and Lemma B.2.1, we obtain

$$|\mathrm{d}\phi_Y(X)[H]|_{p,R}^{+1/2,j} \leq 256nC_\star(p)|Y|_{p,R}^{-1/2,0}|H|_{p,R}^{+1/2,j},$$

which proves the estimate for $\|\mathrm{d}\phi_Y(X)\|_R$ and ends the proof. ∎

We are now in a position to prove the desired bounds for $\partial_{z_j}(A_P \circ |\partial|^{1/2})$ and $\partial_{z_j}(\mathcal{L}_\lambda A_P \circ |\partial|^{1/2})$.

Lemma B.3.4. *Given $p > 3/2$ and $s \geq 0$, let $n \in \mathbb{N}$ and $\lambda > 0$ be such that $|2\lambda - n| \leq 1/2$. Provided that $R \leq (1024C_\star(p))^{-1}$, for every $j \in \mathbb{Z}$, the fixed point A_P satisfies $\partial_{z_j}(A_P \circ |\partial|^{1/2}) \in \mathcal{M}_{p,R}^{+1/2,j}$ and $\partial_{z_j}(\mathcal{L}_\lambda A_P \circ |\partial|^{1/2}) \in \mathcal{M}_{p,R}^{-1/2,j}$. Furthermore, the following estimates hold:*

$$|\partial_{z_j}(A_P \circ |\partial|^{1/2})|_{p,R}^{+1/2,j} \leq 16\frac{n}{\lambda}, \quad |\partial_{z_j}(\mathcal{L}_\lambda A_P \circ |\partial|^{1/2})|_{p,R}^{-1/2,j} \leq \frac{4}{\lambda}.$$

Proof. Let $X_P = A_P \circ |\partial|^{1/2}$ and $Y_P = P \circ |\partial|^{1/2}$. From Theorem 3.2.4, the fixed point satisfies $X_P = \phi_{Y_P}(X_P)$. By differentiating this identity with respect to z_j, we obtain:

$$(\mathrm{Id} - \mathrm{d}\phi_{Y_P}(X_P))\,[\partial_{z_j}X_P] = \phi_{\partial_{z_j}Y_P}(X_P). \tag{B.6}$$

Let us first verify the regularity of the terms involved. By Lemma B.1.1, we have $|Y_P|_{p,R}^{-1/2,0} = R/(2\lambda)$ and $|\partial_{z_j}Y_P|_{p,R}^{-1/2,j} \leq 1/(2\lambda)$. Furthermore, Lemma B.2.2 ensures that $X_P \in \mathcal{M}_{p,R}^{+1/2,0}$ with the bound $|X_P|_{p,R}^{+1/2,0} \leq 8Rn/\lambda \leq 32R$ (using $n/\lambda \leq 4$).

The requirements of Lemma B.3.3 are widely satisfied if $R \leq (1024C_\star)^{-1}$. This ensures first that $4C_\star(p)|X_P|_{p,R}^{+1/2,0} \leq 128C_\star(p)R \leq 1/8 \leq 1/2$, so that the map ϕ and its Fréchet derivative are well-defined. Moreover, by item (2) of Lemma B.3.3, we have:

$$\|\mathrm{d}\phi_{Y_P}(X_P)\|_R \leq 128nC|Y_P|_{p,R}^{-1/2,0} = 256C_\star(p)\frac{nR}{2\lambda} \leq 512C_\star(p)R \leq \frac{1}{2}.$$

This proves that the operator $(\mathrm{Id} - \mathrm{d}\phi_{Y_P}(X_P))$ is invertible via a Neumann series with norm bounded by 2.

We can now derive the first estimate. Inverting (B.6) and applying point (1) of Lemma B.3.3, we get:

$$|\partial_{z_j}X_P|_{p,R}^{+1/2,j} \leq 2|\phi_{\partial_{z_j}Y_P}(X_P)|_{p,R}^{+1/2,j} \leq 2(16n)|\partial_{z_j}Y_P|_{p,R}^{-1/2,j}.$$

Using the bound $|\partial_{z_j}Y_P|_{p,R}^{-1/2,j} \leq 1/(2\lambda)$ from Lemma B.1.1, we obtain:

$$|\partial_{z_j}X_P|_{p,R}^{+1/2,j} \leq 16\frac{n}{\lambda}.$$

For the second estimate, we define $Z_P = \mathcal{L}_\lambda X_P$. Recalling the definition of the map $\phi_{Y_P}(X_P)$, we have $Z_P = \Pi_{rg}\left(\mathcal{B}(X_P)\Pi_{rg}(e^{\mathrm{ad}_{-X_P}}Y_P)\right)$. To compute $\partial_{z_j} Z_P$, we apply the Leibniz rule to the composition of operators:

$$\partial_{z_j} Z_P = \Pi_{rg}\left((\partial_{z_j}\mathcal{B}(X_P))\Pi_{rg}(e^{\mathrm{ad}_{-X_P}}Y_P) + \mathcal{B}(X_P)\Pi_{rg}\partial_{z_j}(e^{\mathrm{ad}_{-X_P}}Y_P)\right).$$

By further expanding the second term, we obtain three contributions:

$$\begin{aligned} I &= (\mathrm{d}\mathcal{B}(X_P)[\partial_{z_j} X_P])e^{\mathrm{ad}_{-X_P}}Y_P, \\ II &= \mathcal{B}(X_P)(\mathrm{d}e^{\mathrm{ad}_{-X_P}}[\partial_{z_j} X_P])Y_P, \\ III &= \mathcal{B}(X_P)e^{\mathrm{ad}_{-X_P}}(\partial_{z_j} Y_P). \end{aligned}$$

We now estimate each term using the bounds $|\partial_{z_j} X_P|_{p,R}^{+1/2,j} \le 16\frac{n}{\lambda} \le 64$ and $R \le (1024C_\star(p))^{-1}$.

For the terms I and II, the Fréchet derivatives of the analytic operators $\mathcal{B}$ and e^{ad} are bounded by $16C_\star(p)$. Combined with the fact that $|Y_P|_{p,R}^{-1/2,0} = R/(2\lambda)$, and the norms of the operators themselves are bounded by 2, we have:

$$|I|_{p,R}^{-1/2,j} + |II|_{p,R}^{-1/2,j} \le 2\left((16C_\star(p)\cdot 64)\cdot 2\cdot\frac{R}{2\lambda}\right) = \frac{2048C_\star(p)R}{\lambda} \le \frac{2}{\lambda}$$

Using $\|\mathcal{B}(X_P)\|_R \le 2$, $\|e^{\mathrm{ad}_{-\mathrm{X_P}}}\|_R \le 2$ and the bound $|\partial_{z_j} Y_P|_{p,R}^{-1/2,j} \le 1/(2\lambda)$ from Lemma B.1.1, we get:

$$|III|_{p,R}^{-1/2,j} \le 2\cdot 2\cdot\frac{1}{2\lambda} = \frac{2}{\lambda}.$$

Summing the contributions from I, II, and III, we obtain the claim. ■

B.4 Estimate for the Correction to the Kernel Term

In this final section, we establish the fundamental estimates for the term $\tilde{\mathcal{P}}_\infty$, defined as:

$$\tilde{\mathcal{P}}_\infty = \Pi_{ker}\left((e^{\mathrm{ad}_{-A_P}} - \mathrm{Id})P - \sum_{k=2}^{\infty}\frac{\mathrm{ad}_{-A_P}^{k-1}}{k!}\mathcal{L}_\lambda A_P\right).$$

This $\mathfrak{su}(1,1)$-valued map represents the $O(|P|^2)$ correction to the linear vector field resulting from the reducibility procedure performed on the range of the operator $\mathcal{L}_{n/2}$. Controlling this term is essential for the analysis of the spectral gaps when viewed as functions of the variable z, defined by the scaling $|\partial|^{1/2}z = u$.

The main goal is to provide suitable bounds for the norm of $\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2}$ and its derivative with respect to z_j. The result relies on a key observation regarding the action of ∂_{z_j} on the functional spaces defined in this Appendix. Specifically, while the perturbation $P \circ |\partial|^{1/2}$ belongs to $\mathcal{M}_{p,R}^{-1/2,0}$, its derivative with respect to z_j belongs to $\mathcal{M}_{p,R}^{-1/2,j}$. Similarly, the fixed point $A_P \circ |\partial|^{1/2}$ belongs to $\mathcal{M}_{p,R}^{+1/2,0}$, while its derivative $\partial_{z_j}(A_P \circ |\partial|^{1/2})$ belongs to $\mathcal{M}_{p,R}^{+1/2,j}$.

These properties, proved in Lemmas B.1.1 and B.2.2, ensure that each term in the series defining $\tilde{\mathcal{P}}_\infty$ is well-defined and bounded. More precisely, we have the following lemma.

Lemma B.4.1. *Let $n \in \mathbb{N}$ and $\lambda > 0$ such that $|2\lambda - n| \leq 1/2$. For all $p > 3/2$ and every $s \geq 0$, if $R \leq (1024 C_\star(p))^{-1}$, then*

$$\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R} \quad \textit{and} \quad \partial_{z_j}(\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2}) \in \mathcal{M}^{-1/2,j}_{p,R}.$$

Moreover, the following bounds hold:

$$|\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2}|^{-1/2,0}_{p,R} \leq \frac{nR^2}{\lambda^2}, \quad |\partial_{z_j}(\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2})|^{-1/2,j}_{p,R} \leq \frac{nR}{\lambda^2}.$$

Proof. Let $X_P := A_P \circ |\partial|^{1/2} \in \mathcal{M}^{+1/2,0}_{p,R}$, $Y_P := P \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R}$, and $Z_P := \mathcal{L}_\lambda A_P \circ |\partial|^{1/2} \in \mathcal{M}^{-1/2,0}_{p,R}$. We consider the operator norm $\rho := \|\mathrm{ad}_{-X_P}\|_R$ acting on the spaces $\mathcal{M}^{-1/2,j}_{p,R}$. From Lemma B.3.2 and Lemma B.2.2, we have:

$$\|\mathrm{ad}_{-X_P}\|_R \leq 4C_\star(p)|X_P|^{+1/2,0}_{p,R} \leq 32 C_\star(p) R\frac{n}{\lambda} \leq \frac{1}{8},$$

where we used the hypothesis $R \leq (1024 C_\star(p))^{-1}$.

Since $\rho \leq 1/8 < 1/2$, the sum of the geometric series $\sum_{k=1}^\infty \rho^k = \rho/(1-\rho)$ is bounded by 2ρ. For the first estimate, we sum the contributions of the two series defining $\tilde{\mathcal{P}}_\infty$:

- $\left|\sum_{k=1}^\infty \frac{\mathrm{ad}^k_{-X_P}}{k!} Y_P\right|^{-1/2,0}_{p,R} \leq 2\rho |Y_P|^{-1/2,0}_{p,R} \leq 2\left(32C_\star(p)R\frac{n}{\lambda}\right)\frac{R}{2\lambda} = 32C_\star(p)R\frac{nR^2}{\lambda^2}.$
- $\left|\sum_{k=2}^\infty \frac{\mathrm{ad}^{k-1}_{-X_P}}{k!} Z_P\right|^{-1/2,0}_{p,R} \leq 2\rho |Z_P|^{-1/2,0}_{p,R} \leq 2\left(32C_\star(p)R\frac{n}{\lambda}\right)\frac{2R}{\lambda} = 128C_\star(p)R\frac{nR^2}{\lambda^2}.$

The total norm in $\mathcal{M}^{-1/2,0}_{p,R}$ is bounded by $160C_\star(p)R\frac{nR^2}{\lambda^2}$, which is strictly less than nR^2/λ^2.

For the second estimate, we differentiate the operator $\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2}$ with respect to z_j. Let $H_j := \partial_{z_j} X_P \in \mathcal{M}^{+1/2,j}_{p,R}$ be the direction of differentiation. We first recall the operator bounds derived from the previous lemmas. Let $\rho := \|\mathrm{ad}_{-X_P}\|_R$; from Lemma B.3.2 and Lemma B.2.2, we have $\rho \leq 4C_\star(p)|X_P|^{+1/2,0}_{p,R} \leq 32C_\star(p)R\frac{n}{\lambda} \leq 1/8$. Furthermore, for any operator $\mathcal{S}(X) = \sum a_k(\mathrm{ad}_{-X})^k$ with $|a_k| \leq 1$, its Fréchet derivative in the direction H_j satisfies

$$\|\mathrm{d}\mathcal{S}(X)[H_j]\|_R \leq \frac{\|\mathrm{ad}_{-H_j}\|_R}{(1-\rho)^2} \leq 16C_\star(p)|H_j|^{+1/2,j}_{p,R}, \tag{B.7}$$

where we used the estimate $(1-1/8)^{-2} \leq 4$ and the commutator bound $\|\mathrm{ad}_{-H_j}\|_R \leq 4C_\star(p)|H_j|^{+1/2,j}_{p,R}$ from Lemma B.3.2. Applying the Leibniz rule to the definition of $\tilde{\mathcal{P}}_\infty$, we obtain:

$$\begin{aligned}\partial_{z_j}(\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2}) = \Pi_{ker}\Big(&\underbrace{\mathrm{d}(e^{\mathrm{ad}_{-X_P}} - \mathrm{id})[H_j]Y_P}_{I} + \underbrace{(e^{\mathrm{ad}_{-X_P}} - \mathrm{id})\partial_{z_j} Y_P}_{II} \\ &- \underbrace{\mathrm{d}\left(\sum_{k=2}^\infty \frac{\mathrm{ad}^{k-1}_{-X_P}}{k!}\right)[H_j]Z_P}_{III} - \underbrace{\left(\sum_{k=2}^\infty \frac{\mathrm{ad}^{k-1}_{-X_P}}{k!}\right)\partial_{z_j} Z_P}_{IV}\Big).\end{aligned}$$

Using (B.7) and the operator norm bound $\frac{\rho}{1-\rho} \leq 2\rho$, we estimate each term in $\mathcal{M}^{-1/2,j}_{p,R}$ using the estimates from Lemma B.1.1, Lemma B.2.2 and Lemma B.3.4:

i) $|I|_{p,R}^{-1/2,j} \leq 16C_\star(p)\left(\frac{nR}{\lambda}\right)|Y_P|_{p,R}^{-1/2,0} \leq 16C_\star(p)\left(\frac{nR}{\lambda}\right)\left(\frac{R}{2\lambda}\right) = 8C_\star(p)R\frac{nR}{\lambda^2}$.

ii) $|II|_{p,R}^{-1/2,j} \leq 2\rho|\partial_{z_j}Y_P|_{p,R}^{-1/2,j} \leq \left(64C_\star(p)R\frac{n}{\lambda}\right)\frac{1}{2\lambda} = 32C_\star(p)R\frac{n}{\lambda^2}$.

iii) $|III|_{p,R}^{-1/2,j} \leq 16C_\star(p)\left(\frac{nR}{\lambda}\right)|Z_P|_{p,R}^{-1/2,0} \leq 16C_\star(p)\left(\frac{nR}{\lambda}\right)\left(\frac{2R}{\lambda}\right) = 32C_\star(p)R\frac{nR}{\lambda^2}$.

iv) $|IV|_{p,R}^{-1/2,j} \leq 2\rho|\partial_{z_j}Z_P|_{p,R}^{-1/2,j} \leq \left(64C_\star(p)R\frac{n}{\lambda}\right)\frac{4}{\lambda} = 256C_\star(p)R\frac{n}{\lambda^2}$.

Summing these four contributions and using $R \leq (1024C_\star(p))^{-1}$ we get

$$|\partial_{z_j}(\tilde{\mathcal{P}}_\infty \circ |\partial|^{1/2})|_{p,R}^{-1/2,j} \leq (288 + 40R)C_\star(p)R\frac{n}{\lambda^2} \leq 328C_\star(p)R\frac{n}{\lambda^2} \leq \frac{n}{\lambda^2},$$

concluding the proof. ∎

Appendix C

Product Topology

Definition C.0.1 (Product Topology)**.** *Let $(X_k, \tau_k)_{k\in\mathbb{N}}$ be a family of topological spaces and $X := \prod_{k\in\mathbb{N}} X_k$. The product topology τ_{prod} on X is the coarsest topology that makes all the projections $q_k : X \to X_k$ continuous. It is then clear that a basis for the product topology consists of all sets obtained as finite intersections of sets of the form $q_k^{-1}(A_k)$, where A_k is open in X_k. An open cylinder is a subset of X of the form $\prod_{k\in\mathbb{N}} A_k$, where, for all $k \in \mathbb{N}$, $A_k \in \tau_k$, and $A_k \neq X_k$ only for a finite number of $k \in \mathbb{N}$. Therefore, the product topology on X is generated by the open cylinders, i.e., $U \in \tau_{prod}$ if and only if U is an arbitrary union of open cylinders.*

Remark C.0.2. *Let $(X_k, \tau_k)_{k\in\mathbb{N}}$ be a family of topological spaces and $X := \prod_{k\in\mathbb{N}} X_k$. If, for any $k \in \mathbb{N}$, (X_k, τ_k) is Hausdorff, then, (X, τ_{prod}) is Hausdorff.*

Proposition C.0.3 (Convergence)**.** *A sequence $(x^n)_{n\in\mathbb{N}}$ in X converges if and only if, for every $k \in \mathbb{N}$, $(x_k^n)_{n\in\mathbb{N}}$ converges in X_k.*

Proof. Since all the projections $q_k : X \to X_k$ are continuous, if $(x^n)_{n\in\mathbb{N}}$ converges to x in X, then $(x_k^n)_{n\in\mathbb{N}}$ converges to x_k in X_k for each $k \in \mathbb{N}$. Conversely, let $x \in X$ and consider a sequence $(x^n)_{n\in\mathbb{N}}$ such that, $\forall k \in \mathbb{N}$, the sequence $(x_k^n)_{n\in\mathbb{N}}$ converges to x_k in X_k. Since any open set is union of open cylinders, it is sufficient to show that, for any cyclinder containing x, $\exists N \in \mathbb{N}$ such that $x^n \in C\ \forall n > N$. So, let $C = \prod_{k\in\mathbb{N}} A_k$ be an open cylinder such that, for all $k \in \mathbb{N}$, $x_k \in A_k$. Of course, there exists $K \in \mathbb{N}$ such that $A_k = X_k$ for all $k > K$. Now, by hypothesis, for any $k \in \mathbb{N}$ there is $N(A_k)$ such that $x_k^n \in A_k$ for all $n > N(A_k)$. But then, it is sufficient to take $N := \sup_{k\leq K} N(A_k)$. ■

Remark C.0.4. *Let (X, d) be a metric space. The topology generated by the metric d coincides with the topology generated by $d' := d/(1+d)$. In particular, if the goal is to study only the topological structure of a metric space with metric d, one can always consider the metric d', which takes values in $[0, 1)$.*

Let $(X_k, d_k)_{k\in\mathbb{N}}$ be a family of metric spaces such that, for every $k \in \mathbb{N}$, $d_k : X_k \times X_k \to [0, 1)$. For every $k \in \mathbb{N}$, let τ_k be the topology on X_k generated by the metric d_k. Let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of real numbers such that, $\forall k \in \mathbb{N}$, $\rho_k > 0$, and $\sum_{k\in\mathbb{N}} \rho_k < \infty$. We define the metric on $X := \prod_{k\in\mathbb{N}} X_k$

$$d_\rho : X \times X \to [0, \infty) \ , \quad d_\rho(x, y) := \sum_{k\in\mathbb{N}} \rho_k d_k(x_k, y_k)$$

for every $x = (x_k)_{k\in\mathbb{N}}, y = (y_k)_{k\in\mathbb{N}} \in X$.

Theorem C.0.5. *The topology on X generated by d_ρ coincides with τ_{prod}.*

Proof. Let $C = \prod_{k\in\mathbb{N}} A_k \in \tau_{prod}$ be an open cylinder such that each A_k is different from the empty set, and take an arbitrary $x = (x_k)_{k\in\mathbb{N}} \in C$. We show that there exists $\epsilon > 0$ such that the d_ρ-ball of radius ϵ centered at x, i.e., $B_\epsilon(x, d_\rho)$, is entirely contained in C. Now, for every $k \in \mathbb{N}$, there exists $\epsilon_k > 0$ such that $B_{\epsilon_k}(x_k, d_k) \subset A_k$, because A_k belongs to τ_k, which is the topology generated by the metric d_k. Let $\epsilon := \inf\{\ \rho_k\epsilon_k\ :\ k \in \mathbb{N}$ such that $A_k \neq X_k\}$. Note that $\epsilon > 0$ because $A_k \neq X_k$ only for a finite number of $k \in \mathbb{N}$. Now, consider a generic $y \in B_\epsilon(x, d_\rho)$. For every k such that $A_k \neq X_k$, we have that $\rho_k d_k(x_k, y_k) \le d_\rho(x, y) < \epsilon \le \rho_k\epsilon_k$ (by construction). That is, $d_k(x_k, y_k) < \epsilon_k$. In particular $y_k \in A_k$. If instead k is such that $A_k = X_k$, there is nothing to check. Therefore, $B_\epsilon(x, d_\rho) \subset C$.

Now, we show that, given $x \in X$ and $\epsilon > 0$, there exists an open cylinder $C = \prod_{k\in\mathbb{N}} A_k \in \tau_{prod}$ containing x such that $C \subset B_\epsilon(x, d_\rho)$.

Let $\Lambda \in \mathbb{N}$ be such that

$$\sum_{k>\Lambda}^{\infty} \rho_k < \epsilon/2.$$

Let $\|\rho\|_\infty := \sup_{k\in\mathbb{N}} \rho_k$ and define a family $(A_k)_{k\in\mathbb{N}}$ such that, for every $k \le \Lambda$, $A_k := B_{\epsilon/(2\|\rho\|_\infty\Lambda)}(x_k, d_k)$, and $A_k = X_k$ for all $k > \Lambda$. Clearly, $x \in C := \prod_{k\in\mathbb{N}} A_k$.

To show that the open cylinder C is entirely contained in $B_\epsilon(x, d_\rho)$, consider an arbitrary $y \in C$. By construction, for every $k \le \Lambda$, we have $d_k(x_k, y_k) < \epsilon/(2\|\rho\|_\infty\Lambda)$. It follows that

$$\begin{aligned}
d_\rho(x, y) &= \sum_{k\in\mathbb{N}} \rho_k d_k(x_k, y_k) = \sum_{k=1}^{\Lambda} \rho_k d_k(x_k, y_k) + \sum_{k>\Lambda}^{\infty} \rho_k d_k(x_k, y_k) \\
&\le \|\rho\|_\infty \sum_{k=1}^{\Lambda} d_k(x_k, y_k) + \sum_{k>\Lambda}^{\infty} \rho_k < \|\rho\|_\infty \Lambda \frac{\epsilon}{2\|\rho\|_\infty\Lambda} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

thereby establishing the claim. ∎

Appendix D

Realizations of $\mathbb{T}^{\mathbb{N}}$ into Sequence Spaces

We explore here some specific settings where the infinite torus $\mathbb{T}^{\mathbb{N}}$ endowed with the product topology arises. To this end, let us first define the sequence spaces that will provide the framework for our constructions.

For $s \in \mathbb{R}$, we denote by h^s the Sobolev-Hilbert space of sequences $u = (u_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} k^{2s}|u_k|^2 < \infty$, endowed with its natural norm topology.

Furthermore, given a sequence of positive real numbers $\eta = (\eta_k)_{k\in\mathbb{N}}$, we define the weighted space

$$\ell^\infty_\eta := \left\{ u = (u_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ : \ \sup_{k\in\mathbb{N}} \eta_k |u_k| < \infty \right\},$$

which can be endowed either with the norm topology or with the weak-* topology.

To define the latter, we introduce the pre-dual space $\ell^1_{1/\eta}$ as the Banach space of sequences $\phi = (\phi_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that

$$\|\phi\|_{\ell^1_{1/\eta}} := \sum_{k\in\mathbb{N}} \eta_k^{-1}|\phi_k| < \infty.$$

The weak-* topology on ℓ^∞_η is then the coarsest topology such that the linear functionals

$$\ell^\infty_\eta \ni u \mapsto \langle u, \phi\rangle_{\ell^\infty_\eta, \ell^1_{1/\eta}} := \sum_{k\in\mathbb{N}} u_k\phi_k \in \mathbb{C}$$

are continuous for every $\phi \in \ell^1_{1/\eta}$.

In what follows, we construct suitable topological embeddings of $\mathbb{T}^{\mathbb{N}}$ into these spaces, considering three distinct cases:

(i) the h^s case with norm topology;

(ii) the ℓ^∞_η case with weak-* topology;

(iii) the ℓ^∞_η case with norm topology.

More specifically, let $\xi = (\xi_k)_{k\in\mathbb{N}}$ be a sequence of non-negative real numbers with $\xi_k > 0$ for infinitely many $k \in \mathbb{N}$. We consider the set

$$\mathcal{T}_\xi := \left\{ u = (u_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \ : \ |u_k|^2 = \xi_k, \ \forall k \in \mathbb{N} \right\},$$

under the assumption that ξ satisfies suitable decay conditions such that $\mathcal{T}_\xi \subset h^s$ or $\mathcal{T}_\xi \subset \ell^\infty_\eta$, depending on the case.

In this appendix, we show that:

- In Cases (i) and (ii), $\mathcal{T}_\xi$ is always a realization of $\mathbb{T}^{\mathbb{N}}$;
- In Case (iii), $\mathcal{T}_\xi$ is a realization of $\mathbb{T}^{\mathbb{N}}$ if and only if $\lim_{k\to\infty} \eta_k^2\xi_k = 0$.

It is worth mentioning that if the sequence of radii is eventually bounded away from zero on the support of ξ, namely if $\inf_{k\in S} \eta_k^2\xi_k > 0$ where $S = \operatorname{supp}(\xi)$, then $\mathcal{T}_\xi$ would instead inherit the structure of a non-compact Banach manifold modeled on ℓ^∞.

To simplify the technical discussion, we observe that it is always possible to restrict our analysis to the following normalized setting:

$$s = 0, \quad \eta_k = 1, \quad \text{and} \quad \xi_k > 0 \quad \forall k \in \mathbb{N}. \tag{D.1}$$

Indeed, the isometric isomorphisms

$$\begin{aligned} h^s \ni u &\mapsto v := (k^s u_k)_{k\in\mathbb{N}} \in \ell^2, \\ \ell^\infty_\eta \ni u &\mapsto v := (\eta_k u_k)_{k\in\mathbb{N}} \in \ell^\infty, \end{aligned} \tag{D.2}$$

are homeomorphisms with respect to the strong topologies and, in the case of (D.2), also with respect to the weak-* topology.

Remark D.0.1. *The map* (D.2) *is a weak-* homeomorphism as it is the adjoint of the isometric isomorphism between the pre-dual spaces $f : \ell^1 \to \ell^1_{1/\eta}$ defined by $(f(\phi))_k = \eta_k\phi_k$ for all $\phi \in \ell^1$ and $k \in \mathbb{N}$. Indeed, for any $u \in \ell^\infty_\eta$ and $\phi \in \ell^1$, the duality pairing satisfies*

$$\langle (\eta_k u_k), \phi\rangle_{\ell^\infty,\ell^1} = \sum_{k\in\mathbb{N}} \eta_k u_k \phi_k = \sum_{k\in\mathbb{N}} u_k(\eta_k\phi_k) = \langle u, f(\phi)\rangle_{\ell^\infty_\eta,\ell^1_{1/\eta}}.$$

Since f is a topological isomorphism between the pre-duals, its adjoint is a homeomorphism between the dual spaces endowed with their weak- topologies.*

Furthermore, let $S := \operatorname{supp}(\xi)$ be the set of indices where $\xi_k > 0$. For any $k \notin S$, the condition $|u_k|^2 = \xi_k = 0$ implies $u_k = 0$. Consequently, the projection onto the coordinates in S,

$$\Pi_S : \mathcal{T}_\xi \to \mathbb{C}^S,$$

is a bijection onto its image and restricts to a homeomorphism in all the topological settings considered. Since S is infinite, we can identify S with $\mathbb{N}$ through a monotonic reindexing, which preserves both the decay properties of the sequences and the respective topologies. By virtue of these identifications, in the following sections we shall assume the conditions in (D.1) to hold without loss of generality.

D.1 Embedding into ℓ^2

In this case, we assume $\xi \in \ell^1$, which ensures $\mathcal{T}_\xi \subset \ell^2$. We consider the map $\mathcal{J} : \mathbb{T}^{\mathbb{N}} \to \ell^2$ defined by

$$\mathcal{J}(\theta) := \left(\sqrt{\xi_k}e^{\mathrm{i}\theta_k}\right)_{k\in\mathbb{N}},$$

which is a bijection between the infinite torus $\mathbb{T}^{\mathbb{N}}$ and $\mathcal{T}_\xi$. To show that $\mathcal{J}$ is a homeomorphism, we compare the product topology on $\mathbb{T}^{\mathbb{N}}$ with the topology induced by the ℓ^2-norm on $\mathcal{T}_\xi$.

Recall that the product topology on $\mathbb{T}^{\mathbb{N}}$ is metrizable; a suitable metric is given by

$$d_{\mathbb{T}^{\mathbb{N}}}(\theta,\varphi) := \sum_{k\in\mathbb{N}} \xi_k \left| e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}\right|,$$

where the weights ξ_k ensure the convergence of the series. On the other hand, the ℓ^2-distance between $u = \mathcal{J}(\theta)$ and $v = \mathcal{J}(\varphi)$ in $\mathcal{T}_\xi$ is

$$d_{\ell^2}(u, v) = \|u - v\|_{\ell^2} = \sqrt{\sum_{k\in\mathbb{N}} \xi_k \,|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}|^2}.$$

We first prove the continuity of $\mathcal{J}$. For any $\epsilon > 0$, we seek $\delta > 0$ such that $d_{\mathbb{T}^\mathbb{N}}(\theta, \varphi) < \delta$ implies $d_{\ell^2}(\mathcal{J}(\theta), \mathcal{J}(\varphi)) < \epsilon$. Since $|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| \le 2$, we have

$$d_{\ell^2}(\mathcal{J}(\theta), \mathcal{J}(\varphi))^2 = \sum_{k\in\mathbb{N}} \xi_k \left|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}\right|^2 \le 2 \sum_{k\in\mathbb{N}} \xi_k \left|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}\right| = 2 d_{\mathbb{T}^\mathbb{N}}(\theta, \varphi).$$

Thus, it suffices to choose $\delta = \epsilon^2/2$.

Conversely, we show that $\mathcal{J}^{-1}$ is continuous. For a fixed $\epsilon > 0$, let $N \in \mathbb{N}$ be large enough such that $2\sum_{k>N} \xi_k < \epsilon/2$. Then, for any $\theta, \varphi \in \mathbb{T}^\mathbb{N}$,

$$\begin{aligned}
d_{\mathbb{T}^\mathbb{N}}(\theta, \varphi) &< \sum_{k=1}^{N} \xi_k \left|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}\right| + \frac{\epsilon}{2} \\
&\le \sqrt{\sum_{k=1}^{N} \xi_k} \sqrt{\sum_{k=1}^{N} \xi_k \,|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}|^2} + \frac{\epsilon}{2} \\
&\le \sqrt{\|\xi\|_{\ell^1}}\, d_{\ell^2}(\mathcal{J}(\theta), \mathcal{J}(\varphi)) + \frac{\epsilon}{2},
\end{aligned}$$

where we applied the Cauchy-Schwarz inequality to the first N terms. By choosing $\delta = \epsilon/(2\sqrt{\|\xi\|_{\ell^1}})$, we obtain that $d_{\ell^2}(u, v) < \delta$ implies $d_{\mathbb{T}^\mathbb{N}}(\theta, \varphi) < \epsilon$.

D.2 Embedding into ℓ^∞ with the Weak-* Topology

We consider the realization of the infinite torus $\mathbb{T}^\mathbb{N}$ as the subset $\mathcal{T}_\xi$ of the Banach space ℓ^∞ endowed with the weak-* topology. To ensure that $\mathcal{T}_\xi \subset \ell^\infty$, we assume that the sequence of radii satisfies $\sup_{k\in\mathbb{N}} \xi_k < \infty$.

First, let us characterize the topology of $\mathbb{T}^\mathbb{N}$, which is defined as the product of circles $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. Since each factor $\mathbb{T}$ is homeomorphic to the unit circle $U(1) \subset \mathbb{C}$ via the map $\theta \mapsto e^{\mathrm{i}\theta}$, the product topology on $\mathbb{T}^\mathbb{N}$ is equivalent to the product topology on $U(1)^\mathbb{N}$. This topology is induced by any metric of the form

$$d_{\mathbb{T}^\mathbb{N}}(\theta, \varphi) = \sum_{k\in\mathbb{N}} \rho_k |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}|, \tag{D.3}$$

where $(\rho_k)_{k\in\mathbb{N}}$ is a summable sequence of positive weights.

Next, we describe the topology on the target space. The weak-* topology on ℓ^∞ is the coarsest topology making the linear functionals $u \mapsto \sum_{k\in\mathbb{N}} u_k \phi_k$ continuous for all $\phi \in \ell^1$. A neighborhood of a point $u \in \ell^\infty$ in this topology is given by a finite intersection of sets of the form

$$V_{\phi,\epsilon}(u) := \left\{ v \in \ell^\infty \ :\ \left|\sum_{k\in\mathbb{N}} (u_k - v_k)\phi_k\right| < \epsilon \right\}, \quad \phi \in \ell^1,\ \epsilon > 0.$$

While the weak-* topology is not metrizable on the whole ℓ^∞, its restriction to any bounded set is metrizable. Since $\mathcal{T}_\xi$ is bounded, we can consider a ball $B_R := \{u \in \ell^\infty : \|u\|_{\ell^\infty} \le R\}$ with

$R > \sqrt{\|\xi\|_{\ell^\infty}}$ and define on it the metric

$$d_{w^*}(u, v) := \sum_{k \in \mathbb{N}} 2^{-k} |u_k - v_k|. \tag{D.4}$$

To show that d_{w^*} induces the weak-* topology on B_R, we verify that the two neighborhood systems generate the same topology:

1. Any neighborhood $V_{\phi,\epsilon}(u)$ contains a metric ball $B_{d_{w^*}}(u, \delta)$. Indeed, for a fixed $\phi \in \ell^1$, there exists $N \in \mathbb{N}$ such that $\sum_{k>N} |\phi_k| < \epsilon/(4R)$. For any $v \in B_R$ such that $d_{w^*}(u, v) < \delta$, the definition of the metric implies $2^{-k}|u_k - v_k| < \delta$ for all $k \in \mathbb{N}$. In particular, for all $k \leq N$, we have the uniform bound $|u_k - v_k| < 2^N \delta$. By choosing $\delta < \epsilon/(2 \cdot 2^N \|\phi\|_{\ell^1})$, we ensure that

$$\left| \sum_{k \in \mathbb{N}} (u_k - v_k)\phi_k \right| \leq \sum_{k=1}^{N} |u_k - v_k||\phi_k| + 2R \sum_{k>N} |\phi_k| < 2^N \delta \sum_{k=1}^{N} |\phi_k| + \frac{\epsilon}{2} < \epsilon.$$

2. Any metric ball $B_{d_{w^*}}(u, \epsilon)$ contains a weak-* neighborhood. To see this, we first observe that for any $u, v \in B_R$, the distance between coordinates is uniformly bounded by $|u_k - v_k| \leq 2R$. We then choose an index $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} 2^{-k}(2R) < \frac{\epsilon}{2}.$$

Now, let $A = \{e_1, \ldots, e_N\} \subset \ell^1$ be the finite set of functionals from the canonical basis, and consider the weak-* neighborhood

$$U_{A,\epsilon/2}(u) := \left\{ v \in B_R \ : \ |u_k - v_k| < \frac{\epsilon}{2}, \ \forall k = 1, \ldots, N \right\}.$$

For any $v \in U_{A,\epsilon/2}(u)$, the metric distance $d_{w^*}(u, v)$ satisfies:

$$\begin{aligned} d_{w^*}(u, v) &= \sum_{k=1}^{N} 2^{-k}|u_k - v_k| + \sum_{k=N+1}^{\infty} 2^{-k}|u_k - v_k| \\ &< \frac{\epsilon}{2} \left(\sum_{k=1}^{N} 2^{-k} \right) + \sum_{k=N+1}^{\infty} 2^{-k}(2R) < \epsilon. \end{aligned}$$

This proves the inclusion $U_{A,\epsilon/2}(u) \subset B_{d_{w^*}}(u, \epsilon)$.

Finally, we consider the map $\mathcal{J} : \mathbb{T}^{\mathbb{N}} \to \mathcal{T}_\xi$ defined by the coordinate-wise assignment

$$\mathcal{J}(\theta) := \left(\sqrt{\xi_k} e^{\mathrm{i}\theta_k} \right)_{k \in \mathbb{N}}.$$

By construction, $\mathcal{J}$ is a bijection. To show it is a homeomorphism, we evaluate the metric d_{w^*} restricted to the set $\mathcal{T}_\xi$. For any two points $u = \mathcal{J}(\theta)$ and $v = \mathcal{J}(\varphi)$ in $\mathcal{T}_\xi$, the distance defined in (D.4) reads

$$d_{w^*}(u, v) = \sum_{k \in \mathbb{N}} 2^{-k} \left| \sqrt{\xi_k} e^{\mathrm{i}\theta_k} - \sqrt{\xi_k} e^{\mathrm{i}\varphi_k} \right| = \sum_{k \in \mathbb{N}} \left(2^{-k} \sqrt{\xi_k} \right) \left| e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k} \right|.$$

By identifying the weights in the product metric (D.3) as $\rho_k := 2^{-k}\sqrt{\xi_k}$, which are strictly positive and summable since $\sup_k \xi_k < \infty$, the map $\mathcal{J}$ becomes an isometry between the metric spaces $(\mathbb{T}^{\mathbb{N}}, d_{\mathbb{T}^{\mathbb{N}}})$ and $(\mathcal{T}_\xi, d_{w^*})$.

Since we have already established that d_{w^*} induces the weak-* topology on bounded sets of ℓ^∞, it follows that $\mathcal{J}$ is a homeomorphism between the infinite torus $\mathbb{T}^{\mathbb{N}}$ and the set $\mathcal{T}_\xi$ endowed with the weak-* topology.

D.3 Embedding into ℓ^∞ with the Strong Topology

In this section, we consider ℓ^∞ endowed with the strong (norm) topology induced by $\|u\|_{\ell^\infty} = \sup_{k\in\mathbb{N}} |u_k|$. We prove that if $\lim_{k\to\infty} \xi_k = 0$, then the map $\mathcal{J} : \mathbb{T}^{\mathbb{N}} \to \mathcal{T}_\xi$ is a homeomorphism.

To show this, we verify that the product topology on $\mathbb{T}^{\mathbb{N}}$ and the norm topology on $\mathcal{T}_\xi$ are equivalent through $\mathcal{J}$.

1. **Continuity of $\mathcal{J}$:** Let $\epsilon > 0$ and $\theta \in \mathbb{T}^{\mathbb{N}}$. Since $\sqrt{\xi_k} \to 0$, there exists an index $N \in \mathbb{N}$ such that $\sqrt{\xi_k} < \epsilon/2$ for all $k > N$. We define a cylinder U in the product topology as:

$$U := \left\{ \varphi \in \mathbb{T}^{\mathbb{N}} \ : \ |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| < \frac{\epsilon}{\max_{j\le N} \sqrt{\xi_j}}, \quad \forall k = 1, \dots, N \right\}.$$

   For any $\varphi \in U$, we evaluate the distance $\|\mathcal{J}(\theta) - \mathcal{J}(\varphi)\|_{\ell^\infty}$ by splitting the supremum:

   - For $k \le N$, the definition of U implies:

$$\sqrt{\xi_k}|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| < \sqrt{\xi_k}\frac{\epsilon}{\max_{j\le N} \sqrt{\xi_j}} \le \epsilon.$$

   - For $k > N$, the condition on the radii ensures:

$$\sqrt{\xi_k}|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| \le 2\sqrt{\xi_k} < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

   Since the supremum over all $k \in \mathbb{N}$ is less than ϵ, it follows that $\|\mathcal{J}(\theta) - \mathcal{J}(\varphi)\|_{\ell^\infty} < \epsilon$, proving continuity.

2. **Continuity of $\mathcal{J}^{-1}$:** Let $u = \mathcal{J}(\theta) \in \mathcal{T}_\xi$ and let C be a basic cylinder centered at θ in $\mathbb{T}^{\mathbb{N}}$, defined by a finite set of indices $K \subset \mathbb{N}$ and open intervals of width $\delta_k > 0$:

$$C = \{\varphi \in \mathbb{T}^{\mathbb{N}} \ : \ |e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| < \delta_k, \ \forall k \in K\}.$$

   To prove that $\mathcal{J}^{-1}$ is continuous, we find $\epsilon > 0$ such that $B_{\ell^\infty}(u, \epsilon) \cap \mathcal{T}_\xi \subset \mathcal{J}(C)$. Since K is a finite set and $\sqrt{\xi_k} > 0$ for all k, we can define:

$$\epsilon := \min_{k\in K} \left(\sqrt{\xi_k}\delta_k\right) > 0.$$

   Now, let $v = \mathcal{J}(\varphi) \in \mathcal{T}_\xi$ be any point such that $\|u - v\|_{\ell^\infty} < \epsilon$. By the definition of the supremum norm, for each $k \in K$ we have:

$$\sqrt{\xi_k}|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| \le \sup_{j\in\mathbb{N}} \sqrt{\xi_j}|e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\varphi_j}| = \|u - v\|_{\ell^\infty} < \epsilon.$$

   Using the definition of ϵ, we observe that $\epsilon \le \sqrt{\xi_k}\delta_k$ for every $k \in K$. It follows that:

$$\sqrt{\xi_k}|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| < \sqrt{\xi_k}\delta_k.$$

   Dividing by $\sqrt{\xi_k}$, we obtain $|e^{\mathrm{i}\theta_k} - e^{\mathrm{i}\varphi_k}| < \delta_k$ for all $k \in K$, which implies $\varphi \in C$. This confirms that $\mathcal{J}^{-1}$ is continuous.

The condition $\lim_{k\to\infty} \xi_k = 0$ is essential. If it were not satisfied, say $\inf_{k\in\mathbb{N}} \xi_k = c > 0$, a norm-ball of radius $\epsilon < c$ would impose a constraint on an infinite number of angles simultaneously. Such a set cannot contain the image of any cylinder (which only constrains finitely many coordinates), and $\mathcal{T}_\xi$ would behave as a non-compact Banach manifold rather than a compact torus.

Appendix E

Algebraic Background

E.1 Localization of Abelian Groups

Definition E.1.1. *Let $S := \mathbb{Z} \setminus \{0\}$ and G an abelian group. The localization of G by S is the set of equivalence classes*

$$S^{-1}G := \left\{ \frac{g}{s},\ g \in G,\ s \in S\ :\ \frac{g}{s} = \frac{g'}{s'} \Leftrightarrow \exists n \in S \text{ such that } n(sg' - s'g) = 0 \text{ in } G \right\}.$$

Proposition E.1.2. *$S^{-1}G$ is an abelian group with the operation*

$$\frac{g}{s} + \frac{h}{r} = \frac{rg + sh}{sr} \tag{E.1}$$

Proof. First, observe that, given $g \in G$ and $s \in S$, for any $m \in S$ we have

$$\frac{g}{s} = \frac{mg}{ms}. \tag{E.2}$$

Now, let $g, h, g', h' \in G$ and $s, r, s', r' \in S$ be such that

$$\frac{g}{s} = \frac{g'}{s'} \quad \text{and} \quad \frac{h}{r} = \frac{h'}{r'}.$$

Then, by definition, there exist $n, m \in S$ such that $n(sg' - s'g) = m(rh' - r'h) = 0$. Thus,

$$\begin{aligned} \frac{g'}{s'} + \frac{h'}{r'} = \frac{r'g' + s'h'}{r's'} = \frac{nmrsr'g' + nmrss'h'}{nmrsr's'} = \frac{mr'rnsg' + ns'smrh'}{nmr's'rs} \\ = \frac{mr'rns'g + ns'smr'h}{nmr's'rs} = \frac{rg + sh}{rs} = \frac{g}{s} + \frac{h}{r}. \end{aligned}$$

So, the group operation (E.1) is well defined and, of course, it is associative. The identity element is given by

$$\frac{0}{1}$$

and, given $g \in G$, $s \in S$,

$$-\frac{g}{s} = \frac{-g}{s} = \frac{g}{-s}.$$

Finally, the commutativity property of the operation just defined follows trivially. ∎

Definition E.1.3. *An abelian group is said to be without torsion if there are no elements $g \in G$, $g \neq 0$, that admit a non-zero integer n such that $ng = 0$.*

Proposition E.1.4. *If G is without torsion, the group homomorphism*

$$\imath : G \to S^{-1}G, \quad g \mapsto \frac{g}{1}$$

is injective.

Proof. Assume $g/1 = 0/1$. Then, there exists $n \in S$ such that $n(1 \cdot g - 1 \cdot 0) = ng = 0$. The statement follows by the fact that G has not torsion elements. ■

Proposition E.1.5. *If G is without torsion, $S^{-1}G$ is a $\mathbb{Q}$-vector space.*

Proof. Being $S^{-1}G$ an abelian group, and hence a $\mathbb{Z}$-module, it is sufficient to define an appropriate notion of multiplication by rational scalars. For any rational number $\mathbf{r} \in \mathbb{Q}$ and any $g \in G$, $s \in S$, we define

$$\mathbf{r}\frac{g}{s} = \frac{pg}{qs},$$

where $p \in \mathbb{Z}$ and $q \in S$ are such that $\mathbf{r} = p/q$. By using (E.2), it is very easy to show that this operation is well defined. ■

Theorem E.1.6. *Any countable abelian group without torsion can be canonically embedded in a countable vector space over $\mathbb{Q}$.*

Proof. The proof follows by Propositions E.1.2, E.1.4 and E.1.5, and by the fact that, if G is countable, then $S^{-1}G$ is also countable. ■

E.2 Basics of Abelian Groups

Definition E.2.1 (Independent Set)**.** *Let G be an abelian group. Given a set of indices J and a subset $\{g_j\}_{j\in J}$ of G, we say that $\{g_j\}_{j\in J}$ is independent if and only if, for any $\nu \in \mathbb{Z}^{(J)}$, $\sum_{j\in J} \nu_j g_j = 0$ implies $\nu = 0$; otherwise, we say that $\{g_j\}_{j\in J}$ is dependent.*

Definition E.2.2 (Maximal Independent Set)**.** *Let $\{g_j\}_{j\in J}$ be an independent subset of the abelian group G. If for any other $g \in G$, $\{g_j\}_{j\in J} \cup \{g\}$ is dependent, then, we say that $\{g_j\}_{j\in J}$ is a maximal independent subset of G. If we allow the group to have an infinite number of independent elements, then we have to assume the axiom of choice for maximal independent sets to be well defined.*

Proposition E.2.3. *Let G be an abelian group without torsion. Then, all its maximal independent subsets have the same cardinality.*

Proof. We show that, for any maximal independent subset $\{g_j\}_{j\in J} \subset G$, $\{g_j/1\}_{j\in J}$ is a basis for $S^{-1}G$. Then, the fact that, for a vector space, any basis has the same cardinality, yields to the conclusion. So, let $\{g_j\}_{j\in J} \subset G$ a maximal independent subset. First, observe that, for any $g/s \in S^{-1}G$, $g \neq 0$, there is a finite family of rational numbers $(r_i)_{i\in I}$, $I \subseteq J$, such that

$$\frac{g}{s} = \sum_{i\in I} r_i \frac{g_i}{1}.$$

Indeed, let $g \neq 0$. Then, there exist $m \in \mathbb{N}$ and $\nu \in \mathbb{Z}^{(J)}$ such that $mg = \sum_{j\in J} \nu_j g_j$. Therefore,

$$\frac{g}{s} = \frac{mg}{ms} = \frac{\sum_{j\in J} \nu_j g_j}{ms} = \sum_{j\in J} \frac{\nu_j}{ms} g_j.$$

Furthermore, let $(p_j/q_j)_{j\in I}$, $I \subseteq J$, be a finite family of rational numbers. Then,

$$\sum_{j\in I} \frac{p_j}{q_j}\frac{g_j}{1} = 0 \;\Rightarrow\; \sum_{j\in I} p_j \left(\prod_{i\neq j} q_i\right) g_j = 0 \;\Rightarrow\; p_j = 0,\ \forall j \in I.$$

This implies that $\{g_j/1\}_{j\in J}$ is a basis for the $\mathbb{Q}$-vector space $S^{-1}G$. ■

Definition E.2.4 (Rank)**.** *The rank of G is the cardinality of any maximal independent subset.*

Proposition E.2.5. *The rank of* $(\mathbb{Q}, +)$*, and that of any subgroup of it, is equal to* 1.

Proof. For any $a/b,\ c/d \in \mathbb{Q}$, there exist $m, n \in \mathbb{Z}$, not both vanishing, such that $m(a/b)+n(c/d) = 0$. In particular, if both $a \neq 0$ and $c \neq 0$, $m = cb, n = -ad$. ■

Proposition E.2.6. *The rank of* $\mathbb{Z}^{(J)}$ *equals the cardinality of the set of indices* J.

Proof. It is sufficient to observe that the subset $\{e_j\}_{j\in J} \subset \mathbb{Z}^{(J)}$, is a maximal independent set. ■

Theorem E.2.7. *Let* $\Phi : G \to G'$ *be an isomorphism of abelian groups. Then, the rank of G coincides with the rank of* G'.

Proof. It is sufficient to prove the followings: (i) if $g, h \in G$ are independent, then $\Phi(g)$ and $\Phi(h)$ are independent; (ii) let $\{g_j\}_{j\in J}$ be a maximal independent subset of G, then, for any $g' \in G'$ there exist $m \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^{(J)}$ such that $mg' - \sum_{j\in J} \nu_j \Phi(g_j) = 0$.

Concerning (i), let $n, m \in \mathbb{Z}$ be such that $n\Phi(g) + m\Phi(h) = 0$. Then,

$$0 = \Phi^{-1}(0) = \Phi^{-1}(n\Phi(g) + m\Phi(h)) = \Phi^{-1}(\Phi(ng + mh)) = ng + mh.$$

This implies that $n = m = 0$ because, by hypothesis, g and h are independent.

Now, we prove (ii). Let $g' \in G'$ and $g := \Phi^{-1}(g')$. Since $\{g_j\}_{j\in J}$ is a maximal independent subset of G, there exist $m \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^{(J)}$ such that $mg - \sum_{j\in J} \nu_j g_j = 0$. Then,

$$0 = \Phi(0) = \Phi\left(mg - \sum_{j\in J} \nu_j g_j\right) = mg' + \sum_{j\in J} \nu_j \Phi(g_j).$$

The proof is concluded. ■

Definition E.2.8 (Free Abelian Group)**.** *An abelian group G is said to be free if there exists a maximal independent subset* $\{g_j\}_{j\in J}$ *of G such that, for any* $g \in G$*, there exists a unique* $\nu \in \mathbb{Z}^{(J)}$ *such that* $g = \sum_{j\in J} \nu_j g_j$*. We call such subset "basis".*

Corollary E.2.9. *Let G be a free abelian group and* $\{g_j\}_{j\in J}$ *a basis. Then* $G \cong \mathbb{Z}^{(J)}$.

Proof. By definition, $\mathbb{Z}^{(J)} \ni \nu \mapsto \sum_{j\in J} \nu_j g_j$ is injective and surjective. ■

E.3 Subgroups of $(\mathbb{Q}, +)$

Proposition E.3.1. *Let G be a free subgroup of $(\mathbb{Q}, +)$. Then, G is isomorphic to $\mathbb{Z}$.*

Proof. By Proposition E.2.5, G has rank 1, and the only rank 1 direct sum of copies of $\mathbb{Z}$ is the direct sum involving only one term (see Proposition E.2.6). Then, the claim follows by Theorem E.2.7. ∎

Corollary E.3.2. *A subgroup G of $(\mathbb{Q}, +)$ is free if and only if there exists $r \in \mathbb{Q}$ such that for any $g \in G$ we have $g = nr$ for some $n \in \mathbb{Z}$.*

Proof. Let $\Phi : \mathbb{Z} \to G$ be an isomorphism. Define $r = \Phi(1)$. Then,

$$G = \Phi(\mathbb{Z}) = \{\Phi(n) \ : \ n \in \mathbb{Z}\} = \{n\Phi(1) \ : \ n \in \mathbb{Z}\} = \{nr \ : \ n \in \mathbb{Z}\}.$$

The claim follows by Proposition E.3.1. ∎

Proof of Lemma 4.4.33. Assume by contradiction that G is free. By Corollary E.3.2, $G = r\mathbb{Z}$ for some $r \in \mathbb{Q}$. This prevents the existence of such a sequence. ∎

Theorem E.3.3 ([Bae37][BZ51][Fuc15]). *Let $\Lambda \in \prod_{j\in\mathbb{N}} \mathbb{N}_0 \cup \{\infty\}$, $(\boldsymbol{p}_j)_{j\in\mathbb{N}}$ the list of all prime numbers in ascending order, and i a natural number that is not divisible by $\boldsymbol{p}_j$ if $\Lambda_j > 0$. Then, non-trivial subgroups of the additive rationals are exactly all those groups of the form*

$$S(i, \Lambda) := \left\{ \frac{ni}{\prod_{j\in\mathbb{N}} \boldsymbol{p}_j^{\lambda_j}} \ : \ \lambda \in \bigoplus_{j\in\mathbb{N}} \mathbb{N}_0 \cap \{0, ..., \Lambda_j\}, \ n \in \mathbb{Z} \right\},$$

Moreover, $S(i, \Lambda) \cong S(i', \Lambda')$ if and only if $\Lambda_j = \Lambda'_j$ for almost all j, and, whenever they are different, both are finite.

Proof of Lemma 4.4.24. With the same notations of Theorem E.3.3, if $\Lambda \in \prod_{j\in\mathbb{N}} \mathbb{N}_0 \cup \{\infty\}$ satisfies $\prod_{j\in\mathbb{N}} \mathbf{p}^{\Lambda_j} < \infty$, then $S(i, \Lambda) = i(\prod_{j\in\mathbb{N}} \mathbf{p}^{\Lambda_j})^{-1}\mathbb{Z}$.

So, let us focus on the case $\prod_{j\in\mathbb{N}} \mathbf{p}^{\Lambda_j} = \infty$. Let $(b_k)_{k=2}^{\infty}$ be the sequence

$$b_k = \begin{cases} \mathbf{p}_\ell & \text{if } k = \mathbf{p}_\ell^{\lambda_\ell}, \quad \text{for some } \ell \in \mathbb{N}, \ \lambda_\ell \in \mathbb{N}, \ \lambda_\ell \leq \Lambda_\ell, \\ 0 & \text{otherwise} \end{cases}$$

and construct $\boldsymbol{a} = (a_j)_{j\in\mathbb{N}}$ by setting $a_1 := 1$, $a_j := b_{k_j}$ for all $j \geq 2$, where $(k_j)_{j\in\mathbb{N}}$ is the support of the sequence $(b_k)_{k\in\mathbb{N}}$, with $k_j < k_{j'}$ if $j < j'$. Now, given $\lambda \in \bigoplus_{j\in\mathbb{N}} \mathbb{N}_0 \cap \{0, ..., \Lambda_j\}$, let N be the natural number such that $k_N = \mathbf{p}_{j^*}^{\lambda_{j^*}}$, with j^* such that $\mathbf{p}_{j^*}^{\lambda_{j^*}} \geq \mathbf{p}_j^{\lambda_j}$ for all $j \in \mathbb{N}$. Then, for any $n \in \mathbb{Z}$,

$$\frac{n}{\prod_{j=1}^{\infty} \mathbf{p}_j^{\lambda_j}} = \frac{n \prod_{j=1}^{\infty} \mathbf{p}_j^{\overline{\lambda}_j - \lambda_j}}{\prod_{j=1}^{N} a_j}, \tag{E.3}$$

where, for all $j \in \mathbb{N}$, $\overline{\lambda}_j := \max\{\lambda \in \{0, ..., \Lambda_j\} \mid \mathbf{p}_j^{\lambda} \leq \mathbf{p}_{j^*}^{\lambda_{j^*}}\}$. On the other hand, given $m \in \mathbb{Z}$ and $N \in \mathbb{N}$, there exists a unique $\lambda \in \bigoplus_{j\in\mathbb{N}} \mathbb{N}_0 \cap \{0, 1, ..., \Lambda_j\}$ such that

$$\frac{m}{\prod_{j=1}^{N} a_j} = \frac{m}{\prod_{j=1}^{\infty} \mathbf{p}_j^{\lambda_j}}. \tag{E.4}$$

Clearly, (E.3) and (E.4) are inverses of each other. Moreover, for all $i \in \mathbb{N}$, the map

$$Q(\boldsymbol{a}) \ni \frac{m}{\prod_{j=1}^{N} a_j} \mapsto \frac{mi}{\prod_{j=1}^{\infty} \mathbf{p}_j^{\lambda_j}} \in S(i,\Lambda),$$

where λ is defined in (E.4), is a homomorphism of groups. Therefore, for all $i \in \mathbb{N}$, $S(i,\Lambda)$ is isomorphic to $Q(\boldsymbol{a})$, with $\boldsymbol{a} \in \Sigma$ defined as above.

Conversely, given $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}} \in \Sigma$, for all $j \in \mathbb{N}$, let $(\Lambda_j^{(k)})_{k\in\mathbb{N}}$ be the list of natural numbers defined by

$$a_k = \prod_{j=1}^{\infty} \mathbf{p}_j^{\Lambda_j^{(k)}}.$$

We define $\Lambda = (\Lambda_j)_{j\in\mathbb{N}} \in \bigoplus_{j\in\mathbb{N}} \mathbb{N}_0 \cup \{\infty\}$ as

$$\Lambda_j := \sum_{k=1}^{\infty} \Lambda_j^{(k)}. \tag{E.5}$$

Note that

$$\prod_{k=1}^{N} a_k = \prod_{k=1}^{N}\prod_{j=1}^{\infty} \mathbf{p}_j^{\Lambda_j^{(k)}} = \prod_{j=1}^{\infty} \mathbf{p}_j^{\sum_{k=1}^{N} \Lambda_j^{(k)}},$$

which implies that, for all $i \in \mathbb{N}$, $Q(\boldsymbol{a})$ is isomorphic to $S(i,\Lambda)$, with Λ as in (E.5), via the map

$$Q(\mathbf{a}) \ni \frac{m}{\prod_{k=1}^{N} a_k} \mapsto \frac{n}{\prod_{j=1}^{\infty} \mathbf{p}_j^{\lambda_j}} \in S(i,\Lambda), \quad n = m, \quad \lambda_j = \sum_{k=1}^{N} \Lambda_j^{(k)}.$$

This ends the proof. ■

Bibliography

[Arn63] V. I. Arnol'd. Small denominators and problems of stability of motion in classical and celestial mechanics. *Uspehi Mat. Nauk*, 18(6(114)):91–192, 1963.

[Arn89] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[AS81] M. J. Ablowitz and H. Segur. *Solitons and the Inverse Scattering Transform*. SIAM, Philadelphia, 1981.

[Bae37] R. Baer. Abelian groups without elements of finite order. *Duke Mathematical Journal*, 3(1):68 – 122, 1937.

[BBP13] M. Berti, L. Biasco, and M. Procesi. KAM theory for the Hamiltonian derivative wave equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(2):301–373, 2013.

[BK79] T. L. Bock and M. D. Kruskal. A two-parameter Miura transformation of the Benjamin-Ono equation. *Phys. Lett. A*, 74(3-4):173–176, 1979.

[BKM96] D. Bättig, T. Kappeler, and B. Mityagin. On the Korteweg-de Vries equation: convergent Birkhoff normal form. *J. Funct. Anal.*, 140(2):335–358, 1996.

[BMP21] L. Biasco, J. E. Massetti, and M. Procesi. Almost periodic invariant tori for the NLS on the circle. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 38(3):711–758, 2021.

[Bou05] J. Bourgain. On invariant tori of full dimension for 1D periodic NLS. *J. Funct. Anal.*, 229(1):62–94, 2005.

[BS20] D. Bambusi and L. Stolovitch. Convergence to normal forms of integrable PDEs. *Comm. Math. Phys.*, 376(2):1441–1470, 2020.

[BZ51] R. A. Beaumont and H. S. Zuckerman. A characterization of the subgroups of the additive rationals. *Pacific J. Math.*, 1:169–177, 1951.

[CGP24] L. Corsi, G. Gentile, and M. Procesi. Maximal tori in infinite-dimensional Hamiltonian systems: a renormalisation group approach. *Regul. Chaotic Dyn.*, 29(4):677–715, 2024.

[CH93] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[DHH02] A. Degasperis, D. D. Holm, and A. N. W. Hone. A new integrable equation with peakon solutions. *Theoretical and Mathematical Physics*, 133(2):1463–1474, 2002.

[Die68] J. Dieudonné. *Éléments d'Analyse*. Gauthier Villars, 1st edition, 1968.

[Fin74] A. M. Fink. *Almost periodic differential equations*, volume Vol. 377 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1974.

[Fla74] H. Flaschka. The Toda lattice. I. Existence of integrals. *Phys. Rev. B (3)*, 9:1924–1925, 1974.

[FM76] H. Flaschka and D. W. McLaughlin. Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions. *Progress of Theoretical Physics*, 55(2):438–456, 1976.

[FM10] A. Fasano and S. Marmi. *Analytical mechanics*. Oxford Graduate Texts. Oxford University Press, Oxford, italian edition, 2010. An introduction, Reprint of the 2006 translation.

[Fuc15] L. Fuchs. *Abelian groups*. Springer Monographs in Mathematics. Springer, Cham, 2015.

[Gar71] C. S. Gardner. Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system. *J. Mathematical Phys.*, 12:1548–1551, 1971.

[GG10] P. Gérard and S. Grellier. The cubic Szegő equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(5):761–810, 2010.

[GK14] B. Grébert and T. Kappeler. *The defocusing NLS equation and its normal form*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2014.

[GK21] P. Gérard and T. Kappeler. On the integrability of the Benjamin-Ono equation on the torus. *Comm. Pure Appl. Math.*, 74(8):1685–1747, 2021.

[GT84] J. Garnett and E. Trubowitz. Gaps and bands of one-dimensional periodic Schrödinger operators. *Comment. Math. Helv.*, 59(2):258–312, 1984.

[HK14] G. Huang and S. Kuksin. The KdV equation under periodic boundary conditions and its perturbations. *Nonlinearity*, 27(9):R61–R88, 2014.

[HR79] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, second edition, 1979. Structure of topological groups, integration theory, group representations.

[Jes34] B. Jessen. The theory of integration in a space of an infinite number of dimensions. *Acta Mathematica*, 63:249–323, 1934.

[Kap54] I. Kaplansky. *Infinite abelian groups*. University of Michigan Press, Ann Arbor, MI, 1954.

[Kap91] T. Kappeler. Fibration of the phase space for the Korteweg-de Vries equation. *Ann. Inst. Fourier (Grenoble)*, 41(3):539–575, 1991.

[KLTZ09] T. Kappeler, P. Lohrmann, P. Topalov, and N. T. Zung. Birkhoff coordinates for the focusing NLS equation. *Comm. Math. Phys.*, 285(3):1087–1107, 2009.

[KM01] T. Kappeler and M. Makarov. On Birkhoff coordinates for KdV. *Ann. Henri Poincaré*, 2(5):807–856, 2001.

[KM18] T. Kappeler and J. Molnar. On the wellposedness of the KdV equation on the space of pseudomeasures. *Selecta Math. (N.S.)*, 24(2):1479–1526, 2018.

[Kol54] A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. *Dokl. Akad. Nauk SSSR (N.S.)*, 98:527–530, 1954.

[Koz21] V. V. Kozlov. On the ergodic theory of equations of mathematical physics. *Russ. J. Math. Phys.*, 28(1):73–83, 2021.

[KP03] T. Kappeler and J. Pöschel. *KdV & KAM*, volume 45 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2003.

[KP10] S. Kuksin and G. Perelman. Vey theorem in infinite dimensions and its application to KdV. *Discrete Contin. Dyn. Syst.*, 27(1):1–24, 2010.

[Lax68] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[Laz93] V. F. Lazutkin. *KAM Theory and Semiclassical Approximations to Eigenfunctions*. Springer, 1st edition, 1993.

[Lio55] J. Liouville. Note sur l'intégration des équations différentielles de la dynamique. *Journal de Mathématiques Pures et Appliquées*, 20:137–138, 1855.

[MO75] V. A. Marčenko and I. V. Ostrovskiĭ. A characterization of the spectrum of the Hill operator. *Mat. Sb. (N.S.)*, 97(139)(4(8)):540–606, 633–634, 1975.

[Mor77] S. A. Morris. *Pontryagin duality and the structure of locally compact abelian groups*, volume No. 29 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York-Melbourne, 1977.

[Mos61] J. Moser. A new technique for the construction of solutions of nonlinear differential equations. *Proc. Nat. Acad. Sci. U.S.A.*, 47:1824–1831, 1961.

[Mos66] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 20:265–315, 1966.

[MP21] R. Montalto and M. Procesi. Linear Schrödinger equation with an almost periodic potential. *SIAM Journal on Mathematical Analysis*, 53(1):386–434, 2021.

[MT76] H. P. McKean and E. Trubowitz. Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points. *Comm. Pure Appl. Math.*, 29(2):143–226, 1976.

[Muj86] J. Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática, 107. [Mathematical Notes].

[Nak79] A. Nakamura. Bäcklund transform and conservation laws of the Benjamin-Ono equation. *J. Phys. Soc. Japan*, 47(4):1335–1340, 1979.

[Pon34] L. Pontrjagin. The theory of topological commutative groups. *Ann. of Math. (2)*, 35(2):361–388, 1934.

[PT87] J. Pöschel and E. Trubowitz. *Inverse spectral theory*, volume 130 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1987.

[Rot02] J. J. Rotman. *Advanced modern algebra*. Prentice Hall, Inc., Upper Saddle River, NJ, 2002.

[Rud87] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[Sch74] S. Scheinberg. Homeomorphism and isomorphism of abelian groups. *Canadian J. Math.*, 26:1515–1519, 1974.

[SV24] V. Sakbaev and I. Volovich. Analogues of Jacobi and Weyl theorems for infinite-dimensional tori. *J. Stoch. Anal.*, 5(1):Art. 4, 16, 2024.

[Tod89] M. Toda. *Theory of nonlinear lattices*, volume 20 of *Springer Series in Solid-State Sciences*. Springer-Verlag, Berlin, second edition, 1989.

[ZF71] V. E. Zaharov and L. D. Faddeev. The Korteweg-de Vries equation is a fully integrable Hamiltonian system. *Funkcional. Anal. i Priložen.*, 5(4):18–27, 1971.

[ZS] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.