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Recognizing DAGs with page-number 2 is NP-complete [☆,☆☆]

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ABSTRACT

The page-number of a directed acyclic graph (a DAG, for short) is the minimum k for which the DAG has a topological order and a k-coloring of its edges such that no two edges of the same color cross, i.e., have alternating endpoints along the topological order. In 1999, Heath and Pemmaraju conjectured that the recognition of DAGs with page-number 2 is NP-complete and proved that recognizing DAGs with page-number 6 is NP-complete (Heath and Pemmaraju (1999) [15]). Binucci et al. recently strengthened this result by proving that recognizing DAGs with page-number k is NP-complete, for every $k \geq 3$ (Binucci et al. (2019) [6]). In this paper, we finally resolve Heath and Pemmaraju's conjecture in the affirmative. In particular, our NP-completeness result holds even for st-planar graphs and planar posets.

1. Introduction

The problem of embedding graphs in books [26] has a long history of research with early results dating back to the 1970's. Such embeddings are specified by a linear order of the vertices along a line, called *spine*, and by a partition of the edges into sets, called *pages*, such that the edges in each page are drawn crossing-free in a half-plane delimited by the spine. The *page-number* of a graph is the minimum number of pages over all its book embeddings, while the page-number of a graph family is the maximum page-number over its members.

An important branch of literature focuses on the page-number of planar graphs. An upper bound of 4 was known since 1986 [29], while a matching lower bound was only recently proposed [3,30]. Better bounds are known for several families of planar graphs [13,14]. A special attention has been devoted to the planar graphs with page-number 2 [2,7,10,12,18,20,24,27]. These have been characterized as the subgraphs of the Hamiltonian planar graphs [16] and hence are called *subhamiltonian*. Recognizing subhamiltonian graphs turns out to be NP-complete [28].

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If the input graph is directed and acyclic (a DAG, for short), then the linear vertex order of a book embedding is required to be a *topological order* of it [25]. Heath and Pemmaraju [15] showed that there exist *planar DAGs* (i.e., DAGs whose underlying graph is planar) whose page-number is linear in the input size. Certain subfamilies of planar DAGs, however, have bounded page-number [1,5,9,17]. Further, it was recently shown that *upward planar graphs* (i.e., DAGs that admit an upward planar drawing, where *upward* means that each edge is represented by a curve whose y-coordinates monotonically increase from the source to the sink of the edge) have sublinear page-number [19], improving upon previous bounds [11]. From an algorithmic point of view, testing whether a DAG has page-number k is NP-complete for every fixed value of $k \geq 3$ [6], linear-time solvable for $k = 1$ [15], and fixed-parameter tractable with respect to the vertex cover number for every k [5] and with respect to the treewidth for *st-graphs* (i.e., DAGs with a single source and a single sink) when $k = 2$ [6]. In contrast to the undirected setting, however, for $k = 2$ the complexity question has remained open since 1999, when Heath and Pemmaraju posed the following conjecture.

Conjecture 1 *(Heath and Pemmaraju [15]). Deciding whether a DAG has page-number 2 is* NP*-complete.*

Our contribution. In this work, we settle Conjecture 1 in the affirmative. More precisely, in Section 3, we show that testing whether an *st-planar graph* (i.e., an *st*-graph that admits a planar drawing in which the source and the sink are incident to the outer face) admits a 2-page book embedding is an NP-complete problem. In Section 4, we further show that the problem is NP-complete also for *planar posets* (i.e., upward planar graphs with no transitive edges). Section 2 contains some definitions and preliminaries, while Section 5 presents some conclusions and open problems.

2. Preliminaries

A *combinatorial embedding* of a graph is an equivalence class of planar drawings of the graph, where two drawings are equivalent if they define the same clockwise order of the incident edges at each vertex. A *plane embedding* of a connected graph is an equivalence class of planar drawings of the graph, where two drawings are equivalent if they define the same combinatorial embedding and the same clockwise order of the vertices along the outer face. The *flip* of a plane embedding produces a plane embedding in which the clockwise order of the incident edges at each vertex and the clockwise order of the vertices along the outer face is the reverse of the original one. An *upward planar embedding* is an equivalence class of upward planar drawings of a DAG, where two drawings are equivalent if they define the same plane embedding and the same left-to-right order of the outgoing (and incoming) edges at each vertex. A *plane DAG* is a DAG together with an upward planar embedding. It is known that every *st*-planar graph is upward planar [8,21]. An *st-plane* graph is an *st*-planar graph together with an upward planar embedding in which s and t are incident to the outer face.

As in the undirected case, a DAG G has page-number 2 if it is *subhamiltonian*, i.e., it is a spanning subgraph of an *st*-planar graph $\overline{G}$ that has a directed Hamiltonian *st*-path P [23]. In the previous definition, if G has a prescribed plane embedding, we additionally require that the plane embedding of $\overline{G}$ restricted to G coincides with the one of G. We say that P is a *subhamiltonian path* for G, and we refer to the edges of P that are not in G as *augmenting edges*. Further, $\overline{G}$ is called an *HP-completion* of G.

A *generalized diamond* is an *st*-plane graph consisting of three directed paths from a vertex v_s to a vertex v_t, one of which is the edge $v_s v_t$ and appears between the other two paths in the upward planar embedding; see Fig. 1a.

Unless otherwise specified, by *face* of a plane DAG we always mean an *internal face*. A face of a plane DAG whose boundary consists of two directed paths is an *st-face*. An *st*-face is *transitive* if one of these paths is an edge; otherwise, it is *non-transitive* (see Fig. 1b). A *rhombus* is a non-transitive *st*-face whose boundary paths have length 2; see Fig. 1c. The following property follows from Theorem 1 in [23].

Property 2. *A Hamiltonian st-plane graph contains only transitive faces and no generalized diamond.*

From the above property we can deduce the following.

Property 3. *Let G be a plane DAG and P be a subhamiltonian path for G. If G contains a rhombus (v_s, v_l, v_r, v_t) with source v_s and sink v_t, then P contains either the edge $v_l v_r$ or the edge $v_r v_l$, i.e., v_l and v_r are consecutive in P.*

The next property follows directly from Theorem 1 in [22] and Property 2.

Property 4. *Let G be a plane DAG and P be a subhamiltonian path for G. If G contains a non-transitive face f with boundaries (v_s, w, v_t) and $(v_s, v_1, \ldots, v_r, v_t)$, then the augmenting edges of P inside f are either (i) the edge $w v_1$, or (ii) the edge $v_r w$, or (iii) edges $v_i w$ and $w v_{i+1}$ for some $1 \leq i < r$.*

Proof. Consider any HP-completion $\overline{G}$ of G with subhamiltonian path P. By Property 2, we have that $\overline{G}$ does not contain any non-transitive face and any generalized diamond. Since f is non-transitive and $\overline{G}$ cannot contain the edge $v_s v_t$ inside f, as this would create a generalized diamond with the boundary paths of f, it follows that P has at least one augmenting

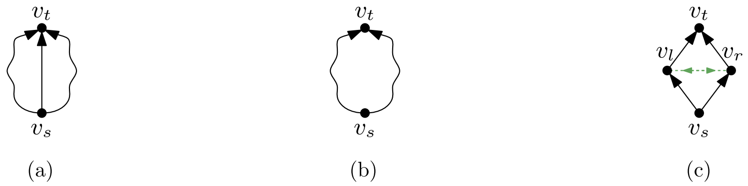
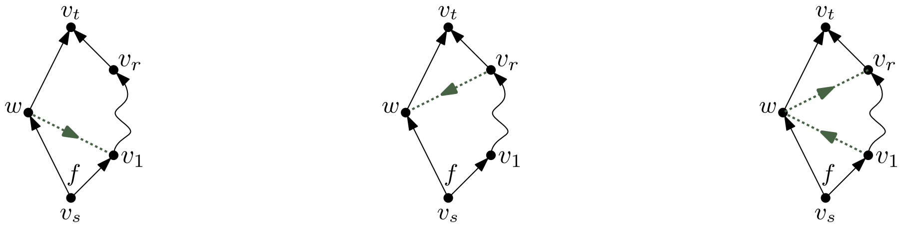


Fig. 1. (a) A generalized diamond, (b) a non-transitive face, and (c) a rhombus; curly curves represent paths and straight-line segments represent edges.



(a) Case (i) (b) Case (ii) (c) Case (iii)

Fig. 2. Illustrations for Property 4.

edge inside f that connects w and a vertex of its right boundary. Since w can be incident to at most two edges of P, there can be at most two augmenting edges of P inside f.

If there is only one such edge, again by Property 2, this edge must split f into two transitive faces. This is achieved only by the edges wv_1 and v_rw, which implies cases (i) and (ii) of the statement; see Figs. 2a and 2b. On the other hand, if there are two such edges, say v_iw and wv_j with $1 \leq i < j \leq r$, then $j = i + 1$ holds (refer to Fig. 2c), as otherwise $\overline{G}$ would contain a non-transitive face with left boundary (v_i, w, v_j) and right boundary $(v_i, v_{i+1}, \ldots, v_j)$, contradicting Property 2. Hence case (iii) holds. □

3. NP-completeness proof for planar *st*-graphs

Let ϕ be a Boolean 3-SAT formula with n variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$. A clause of ϕ is *positive* (*negative*) if it has only positive (negative) literals. The *incidence graph* G_ϕ of ϕ is the graph that has *variable vertices* $x_1, \ldots, x_n$, *clause vertices* $c_1, \ldots, c_m$, and has an edge (c_j, x_i) for each clause c_j containing x_i or $\overline{x}_i$. Note that we use the same notation for variables (clauses) in ϕ and variable vertices (clause vertices) in G_ϕ. If ϕ has clauses with less than three literals, we introduce parallel edges in G_ϕ so that all clause vertices have degree 3 in G_ϕ; see, e.g., the dotted edge (c_2, x_4) in Fig. 3. The formula ϕ is an instance of the NP-complete Planar Monotone 3-SAT problem [4] if each clause of ϕ is positive or negative, and G_ϕ has a plane embedding $\mathcal{E}_\phi$ to which the edges of a cycle $\mathcal{C}_\phi := x_1, \ldots, x_n$ can be added that separates positive and negative clause vertices. The problem asks whether ϕ is satisfiable.

Next, we present gadgets that we are going to use in a reduction from Planar Monotone 3-SAT to the problem of deciding whether a given *st*-planar graph admits a 2-page book embedding.

Double ladder A double ladder of even length ℓ is defined as follows; see Fig. 4a. Its vertex set consists of two sources, s_1 and s_2, two sinks, t_1 and t_2, and vertices in $\cup_{i=0}^{\ell}\{u_i, v_i, w_i\}$. Its edge set consists of edges s_1u_0, s_1v_0, s_2v_0, s_2w_0, $u_\ell t_1$, $v_\ell t_1$, $v_\ell t_2$, $w_\ell t_2$, and $\cup_{i=0}^{\ell-1}\{u_iu_{i+1}, v_iu_{i+1}, v_iv_{i+1}, w_iv_{i+1}, w_iw_{i+1}\}$.

Property 5. *The double ladder has a unique upward planar embedding (up to a flip), shown in Fig. 4a.*

Proof. The embedding shown in Fig. 4a clearly is an upward planar embedding of the double ladder. The underlying graph of the double ladder has four combinatorial embeddings, which are obtained from the embedding in Fig. 4a, by possibly moving the path $u_1u_0s_1v_0$ inside the cycle $u_1u_2v_1v_0$ and the path $w_{\ell-1}w_\ell t_2v_\ell$ inside the cycle $w_{\ell-1}w_{\ell-2}v_{\ell-1}v_\ell$. However, these movements respectively force s_1v_0 and $v_\ell t_2$ to point downward, hence the resulting combinatorial embeddings do not correspond to upward planar embeddings. Finally, since the outer face of the embedding in Fig. 4a is the only face containing at least one source and one sink of the double ladder, the claim follows. □

Property 6. *Let G be a plane DAG with a subhamiltonian path P. If G contains a double ladder of length ℓ, then P contains the pattern $[\ldots u_iv_iw_i \ldots w_{i+1}v_{i+1}u_{i+1} \ldots]$ or $[\ldots w_iv_iu_i \ldots u_{i+1}v_{i+1}w_{i+1} \ldots]$ for $i = 0, \ldots, \ell - 1$.*

Proof. By Properties 3 and 5, we have that P contains either the subpath $u_iv_iw_i$ or the subpath $w_iv_iu_i$, for $i = 0, \ldots, \ell$. The edge u_iu_{i+1} then implies that the vertices u_i, v_i, w_i precede the vertices $u_{i+1}, v_{i+1}, w_{i+1}$ in P. So, it remains to rule

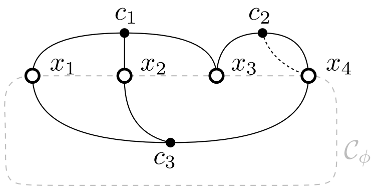
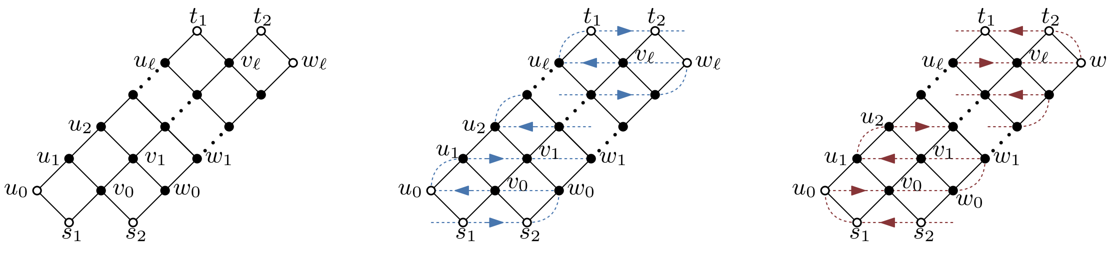


Fig. 3. The incidence graph G_ϕ of an instance ϕ of Planar Monotone 3-SAT with $\phi = c_1 \wedge c_2 \wedge c_3$, $c_1 = (x_1 \vee x_2 \vee x_3)$, $c_2 = (x_3 \vee x_4)$, and $c_3 = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_4)$.



(a) A double ladder of length ℓ (b) `true` assignment (c) `false` assignment

Fig. 4. (a) A double ladder of length ℓ, and (b)-(c) the two subhamiltonian paths of it; edges with no arrow are directed upward, also in subsequent figures.

out patterns $[\dots u_i v_i w_i \dots u_{i+1} v_{i+1} w_{i+1} \dots]$ and $[\dots w_i v_i u_i \dots w_{i+1} v_{i+1} u_{i+1} \dots]$. If P contains one of them, then any book embedding of G in which the order of the vertices along the spine is the one in P requires at least 3 pages, given that the edges $u_i u_{i+1}$, $v_i v_{i+1}$ and $w_i w_{i+1}$ pairwise cross. This contradicts the fact that P is a subhamiltonian path for G. □

Corollary 7. *There exist two subhamiltonian paths for the double ladder, shown in Figs. 4b and 4c.*

Variable gadget Let $x \in \{x_1, \dots, x_n\}$. The variable gadget L_x for x is the double ladder of length $4d_x$, where d_x is the degree of x in G_ϕ. To distinguish between vertices of different variable gadgets, we denote the vertices of L_x with the superscript x, as in Fig. 5. Vertices s^x_1, s^x_2, u^x_0 are the *bottom connectors* and $w^x_{4d_x}$, t^x_1, t^x_2 are the *top connectors* of L_x. The two subhamiltonian paths of Corollary 7 correspond to the truth assignments of x; Fig. 4b corresponds to `true`, while Fig. 4c to `false`. Also, we refer to the edges of L_x that are part of the subhamiltonian path of Fig. 4b (of Fig. 4c) as *true edges* (*false edges*, respectively). In particular, $u^x_{2j} u^x_{2j+1}$ and $w^x_{2j+1} w^x_{2j+2}$ are true edges of L_x, while $u^x_{2j+1} u^x_{2j+2}$ and $w^x_{2j} w^x_{2j+1}$ are false edges of L_x, for $j = 0, \dots, 2d_x - 1$.

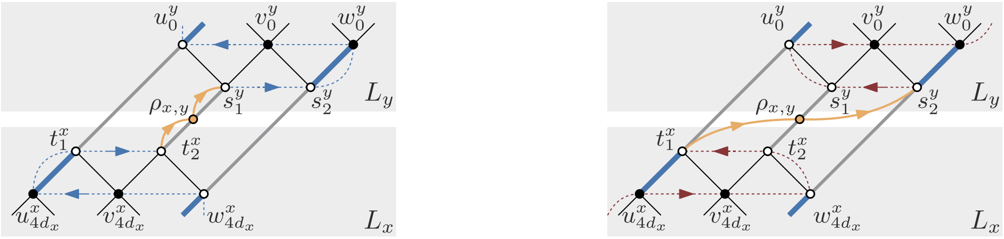
Connector gadget A connector gadget *joins* two variable gadgets L_x and L_y by means of three paths from the top connectors of L_x to the bottom connectors of L_y; see Fig. 5. These paths are: the edge $t^x_1 u^y_0$, the length-2 path $t^x_2 \rho_{x,y} s^y_1$, where $\rho_{x,y}$ is a newly introduced vertex, and the edge $w^x_{4d_x} s^y_2$.

Property 8. *Given subhamiltonian paths P_x for L_x and P_y for L_y, there is a subhamiltonian path P containing P_x and P_y for the graph obtained by joining L_x and L_y by means of a connector gadget.*

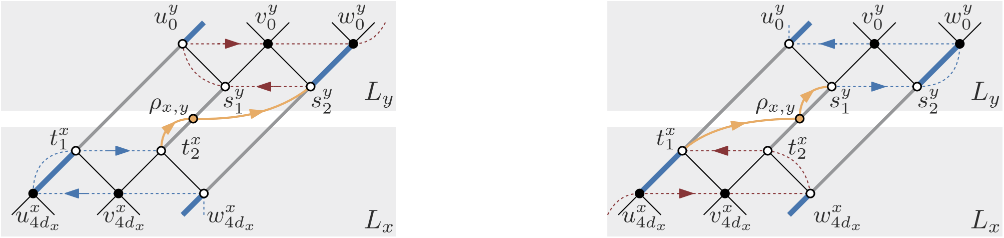
Proof. Each of P_x and P_y is one of the two subhamiltonian paths of Corollary 7; see Fig. 4. In particular, the last vertex of P_x is t^x_1 or t^x_2, and the first vertex of P_y is s^y_1 or s^y_2. We obtain P by adding a directed edge from the last vertex of P_x to $\rho_{x,y}$ and a direct edge from $\rho_{x,y}$ to the first vertex of P_y, as illustrated in Fig. 5. □

Clause gadget Let c be a positive (negative) clause of ϕ. Assume that the variables x, y and z of c appear in this order along $\mathcal{C}_\phi$, when traversing $\mathcal{C}_\phi$ from x_1 towards x_n. In $\mathcal{E}_\phi$, the edges between x and the positive (negative) clause vertices of G_ϕ appear consecutively around x. Assume that the edge (c, x) is the $(i+1)$-th such edge in a clockwise (counter-clockwise) traversal of the edges around x starting at the edge of $\mathcal{C}_\phi$ incoming x; note that $i \in \{0, \dots, d_x - 1\}$. Similarly, define indices j and k for y and z, respectively. For convenience, we refer to i, j and k as the *clause-indices* of x, y and z, respectively. Let L_x, L_y, and L_z be the three variable gadgets for x, y, and z, respectively.

The clause gadget C_c for c consists of an *anchor vertex* a_c, and four edges. If c is positive, these edges are $u^x_{4i} a_c$, $a_c u^z_{4k+1}$, $u^x_{4i+1} u^y_{4j}$ and $u^y_{4j+1} u^z_{4k}$ (curved edges in Fig. 6a); otherwise, they are $w^x_{4i} a_c$, $a_c w^z_{4k+1}$, $w^x_{4i+1} w^y_{4j}$ and $w^y_{4j+1} w^z_{4k}$ (curved edges in Fig. 6b). Note that C_c creates a non-transitive face f_c, called *anchor face*, whose boundary is delimited by the two newly-introduced edges incident to a_c and by a directed path whose edges alternate between three true edges (if c is positive) or three false edges (if c is negative) and the two newly-introduced edges not incident to a_c. We refer to the three true (or false) edges on the boundary of f_c stemming from L_x, L_y, and L_z as the *base-edges* of C_c. The length of the double

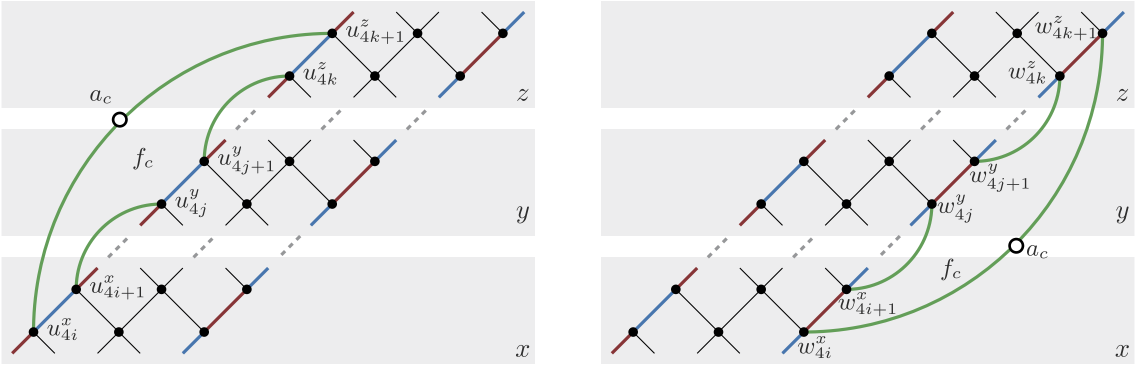


(a) $L_x \leftarrow$ `true`; $L_y \leftarrow$ `true` (b) $L_x \leftarrow$ `false`; $L_y \leftarrow$ `false`



(c) $L_x \leftarrow$ `true`; $L_y \leftarrow$ `false` (d) $L_x \leftarrow$ `false`; $L_y \leftarrow$ `true`

Fig. 5. A subhamiltonian path for the graph composed of two variable gadgets joined by a connector gadget, if the subhamiltonian paths for the variable gadgets represent (a)-(b) the same truth assignment, and (c)-(d) the opposite truth assignment.



(a) $(x \vee y \vee z)$ (b) $(\overline{x} \vee \overline{y} \vee \overline{z})$

Fig. 6. Clause gadgets for (a) a positive clause and (b) a negative clause.

ladders ensures that, if $x = y$ (which implies that $j = i+1$), then vertices u_{4i+1}^x and u_{4j}^y (w_{4i+1}^x and w_{4j+1}^y) are not adjacent in L_x and the edge $u_{4i+1}^x u_{4j}^y$ ($w_{4i+1}^x w_{4j+1}^y$) is well defined; this is the reason that we do not use vertices with indices $2, 3$ mod 4.

We are now ready to prove the main theorem of this section.

Theorem 9. *Recognizing whether a DAG has page-number* 2 *is* NP*-complete, even if the input is an st-planar graph.*

Proof. The problem clearly belongs to NP, as a non-deterministic Turing machine can guess an order of the vertices of an input graph and a partition of its edges into two pages, and check in polynomial time whether the order is a topological order and if so, whether any two edges in the same page cross.

Given an instance ϕ of Planar Monotone 3-SAT, we construct in polynomial time an st-planar graph H that has page-number 2 if and only if ϕ is satisfiable; see Fig. 7. We consider the variable gadgets $L_{x_1}, \dots, L_{x_n}$, where $x_1, \dots, x_n$ is the order of the variables of ϕ along the cycle $\mathcal{C}_\phi$; for $i = 1, \dots, n-1$, we connect L_{x_i} with $L_{x_{i+1}}$ using a connector gadget. For each positive (negative) clause c of ϕ, we add a clause gadget C_c. By the choice of the clause-indices, we can deduce that the resulting graph is a plane DAG containing two sources $s_1^{x_1}$ and $s_2^{x_1}$ and two sinks $t_1^{x_n}$ and $t_2^{x_n}$. We add a source s connected with outgoing edges to $s_1^{x_1}$ and $s_2^{x_1}$, and a sink t connected with incoming edges from $t_1^{x_n}$ and $t_2^{x_n}$. The constructed graph H is st-planar. Since the underlying graph of H is a subdivision of a triconnected planar graph and since only one

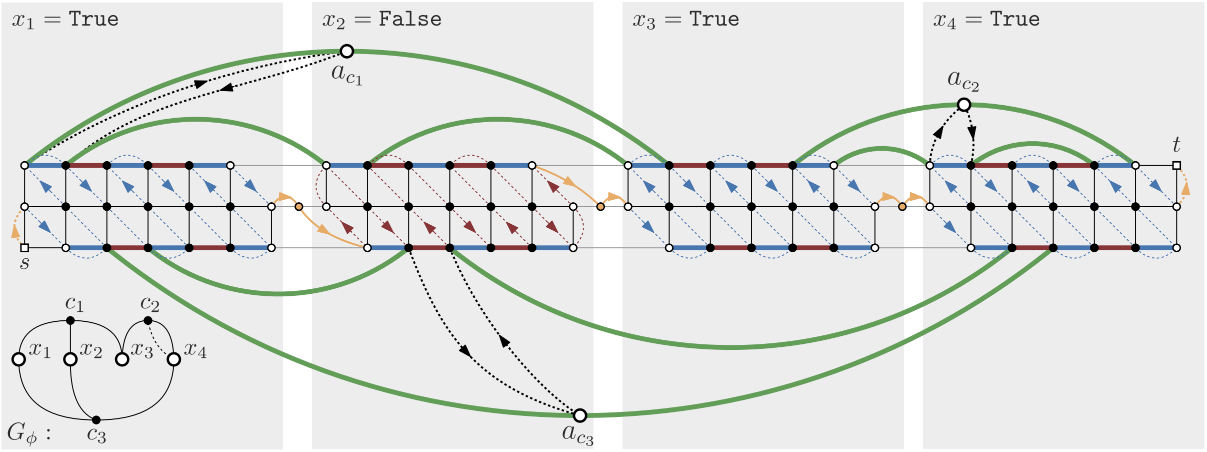


Fig. 7. The graph H obtained from the instance of Planar Monotone 3-SAT illustrated in Fig. 3. For space reasons, the double ladders of the variable gadgets have smaller length and the drawing is rotated by 45°.

face of H contains s and t, it follows that H has a unique upward planar embedding. We next prove that ϕ is satisfiable if and only if H is subhamiltonian (and therefore has page-number 2).

For the forward direction, assuming that ϕ admits a satisfying truth assignment, we show how to construct a subhamiltonian path P for H. For $i = 1, \dots, n$, we have that P contains the subhamiltonian path P_i for L_{x_i} shown in Fig. 4b if x_i is `true`, and the one shown in Fig. 4c otherwise. By Property 8, there is a subhamiltonian path P for the subgraph of H induced by the vertices of all variable and connector gadgets, containing $P_1, \dots, P_n$ as subpaths. The path P starts from a source of L_{x_1} and ends at a sink of L_{x_n}; hence we can extend P to include s and t as its first and last vertices, respectively. We now extend P to a subhamiltonian path for H by including the anchor vertex of each clause gadget. Consider a positive clause $c = (x \vee y \vee z)$ with anchor vertex a_c; the case of a negative clause is analogous. As ϕ is satisfied, at least one of x, y and z is `true`; assume that x is `true`, as the other two cases are analogous. By construction, the anchor face f_c of C_c is non-transitive, with the anchor vertex a_c on its left boundary, and the three base-edges of C_c along its right boundary. Recall that each of these base-edges belong to L_x, L_y, and L_z, respectively. Let $i \in \{0, \dots, d_x - 1\}$ be such that $u^x_{4i}u^x_{4i+1}$ is the (true) base-edge of L_x on the right boundary of f_c. Since x is `true`, the vertices u^x_{4i} and u^x_{4i+1} are consecutive in P. We extend P by visiting vertex a_c after u^x_{4i} and before u^x_{4i+1}. This corresponds to adding two augmenting edges $u^x_{4i}a_c$ and $a_c u^x_{4i+1}$ of P in the interior of f_c; see the black dashed edges of Fig. 7. At the end of this process, P is extended to a subhamiltonian path for H.

For the other direction, assume that H is subhamiltonian and let P be subhamiltonian path for it. For each variable gadget L_{x_i}, P induces a subhamiltonian path P_i for L_{x_i}. By Corollary 7, P_i is one of the two subhamiltonian paths of Fig. 4. We assign to x_i the value `true` if P_i is the path of Fig. 4b and `false` if P_i is the path of Fig. 4c. We claim that this truth assignment satisfies ϕ. Assume, for a contradiction, that there exists a clause c that is not satisfied. Also assume that c is a positive clause $(x \vee y \vee z)$ (the case of c being negative is analogous). Without loss of generality, we can further assume that x, y and z appear in this order in C_ϕ, and that the base-edges of the clause gadget C_c along the right boundary of the anchor face f_c are the true edges $u^x_{4i}u^x_{4i+1}$, $u^y_{4j}u^y_{4j+1}$, and $u^z_{4k}u^z_{4k+1}$ of L_x, L_y, and L_z, respectively. Since clause c that is not satisfied, x, y and z are `false`, which implies that the corresponding subhamiltonian paths P_x, P_y and P_z of L_x, L_y and L_z are the ones of Fig. 4c. Hence, P contains the augmenting edges $u^x_{4i}v^x_{4i}$ and $v^x_{4i+1}u^x_{4i+1}$ of P_x, $u^y_{4j}v^y_{4j}$ and $v^y_{4j+1}u^y_{4j+1}$ of P_y and $u^z_{4k}v^z_{4k}$ and $v^z_{4k+1}u^z_{4k+1}$ of P_z. By Property 4 for the non-transitive face f_c, P contains either (i) the augmenting edge $a_c u^x_{4i+1}$, or (ii) the augmenting edge $u^z_{4k}a_c$, or (iii) for a pair of consecutive vertices, say u and u', along the right boundary of f_c, the augmenting edges ua_c and $a_c u'$. Cases (i) and (ii) contradict the existence of augmenting edges $v^x_{4i+1}u^x_{4i+1}$ and $u^z_{4k}v^z_{4k}$ of P, respectively. Further, in case (iii) the augmenting edges of P that belong to P_x, P_y, and P_z imply that $u \notin \{u^x_{4i}, u^y_{4j}, u^z_{4k}\}$ and $u' \notin \{u^x_{4i+1}, u^y_{4j+1}, u^z_{4k+1}\}$. Hence $u = u^x_{4i+1}$ and $u' = u^y_{4j}$ holds, or $u = u^y_{4j+1}$ and $u' = u^z_{4k}$ holds. In both cases, the HP-completion of H contains a generalized diamond with $v_s = u$ and $v_t = u'$ (the two directed paths on the sides of the edge $v_s v_t$ are $v_s a_c v_t$ and the path composed of the edges of the ladder and connector gadgets from v_s to v_t), violating Property 2. Hence at least one of variables x, y and z must be `true`, contradicting our assumption that c is not satisfied. □

4. NP-completeness proof for planar posets

In this section, we show that the problem of determining whether a DAG has page-number 2 is NP-complete also if the input graph is a planar poset. To show this, we increase the length of the double ladders used for the variable gadgets and

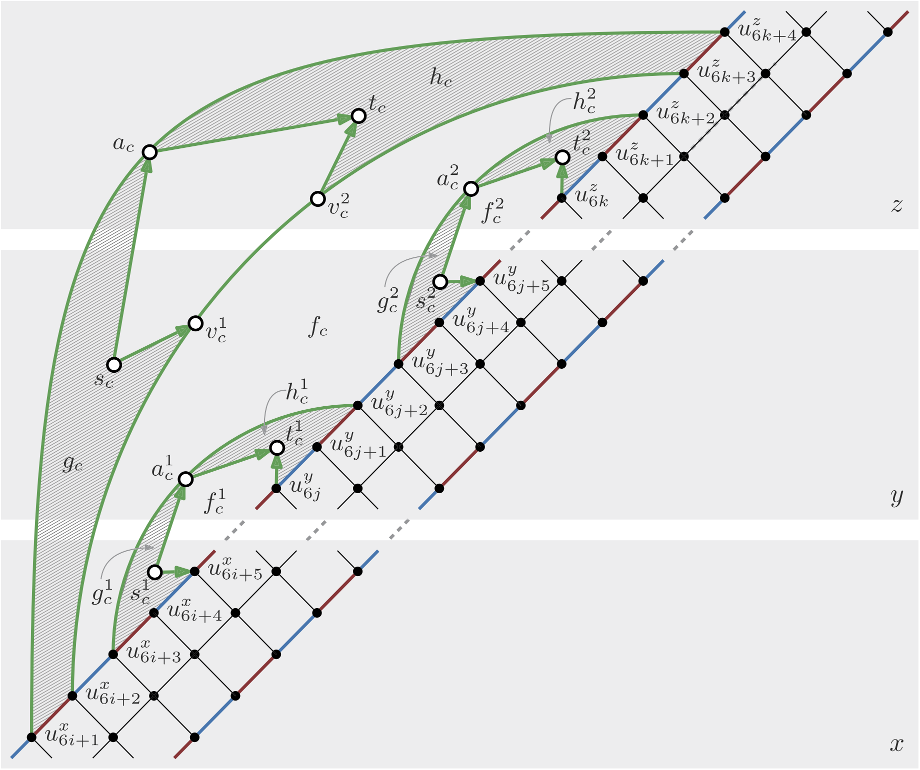


Fig. 8. Clause gadget with sources s_c, s^1_c and s^2_c, sinks t_c, t^1_c and t^2_c, and without transitive edges.

modify the clause gadget, while keeping the same connector gadget. In particular for a variable $x \in \{x_1, \dots, x_n\}$, the variable gadget L_x is the double ladder of length $6d_x$, where d_x is the degree of x in the incidence graph G_ϕ.

For a clause c of ϕ whose variables x, y and z appear in this order along $\mathcal{C}_\phi$ starting from x_1 towards x_n, define the clause-indices i, j and k of c in the exact same way as in Section 3. Then, the clause gadget C_c corresponding to c consists of 11 vertices a_c, a^1_c, a^2_c, s_c, s^1_c, s^2_c, t_c, t^1_c, t^2_c, v^1_c and v^2_c (white circles in Fig. 8) and 21 edges defined as follows. If c is positive, these edges are $u^x_{6i+1}a_c$, $a_c u^z_{6k+4}$, $s_c a_c$, $s_c v^1_c$, $a_c t_c$, $v^2_c t_c$, $v^1_c v^2_c$, $v^2_c u^z_{6k+3}$, $u^x_{6i+3}a^1_c$, $a^1_c u^y_{6j+2}$, $s^1_c a^1_c$, $s^1_c u^x_{6i+5}$, $a^1_c t^1_c$, $u^y_{6j} t^1_c$, $u^y_{6j+3}a^2_c$, $a^2_c u^y_{6k+2}$, $s^2_c a^2_c$, $s^2_c u^y_{6i+5}$, $a^2_c t^2_c$, $u^z_{6k} t^2_c$; otherwise, they are $w^x_{6i+1}a_c$, $a_c w^z_{6k+4}$, $s_c a_c$, $s_c v^1_c$, $a_c t_c$, $v^2_c t_c$, $v^1_c v^2_c$, $v^2_c w^z_{6k+3}$, $w^x_{6i+3}a^1_c$, $a^1_c w^y_{6j+2}$, $s^1_c a^1_c$, $s^1_c w^x_{6i+5}$, $a^1_c t^1_c$, $w^y_{6j} t^1_c$, $w^y_{6j+3}a^2_c$, $a^2_c w^y_{6k+2}$, $s^2_c a^2_c$, $s^2_c w^y_{6i+5}$, $a^2_c t^2_c$, $w^z_{6k} t^2_c$. By construction, s_c, s^1_c and s^2_c are sources, while t_c, t^1_c and t^2_c are sinks. The remaining vertices of the clause gadget ensure the absence of transitive edges, as required in the construction.

Theorem 10. *Recognizing whether a DAG has page-number* 2 *is* NP*-complete, even if the input is a planar poset.*

Proof. Given an instance ϕ of Planar Monotone 3-SAT, we construct a plane DAG H similarly as in the proof of Theorem 9. The graph H is upward planar, it has multiple sources and sinks, and it does not contain any transitive edges, i.e., it is a planar poset. In particular, the absence of transitive edges and the presence of multiple sources and sinks derive from the design of the clause gadget, as we have already mentioned. In the following, we focus on proving that ϕ is satisfiable if and only if H is subhamiltonian.

First, we show how to construct a subhamiltonian path for H, given ϕ admits a truth assignment. As the variable and connector gadgets are similar to those used for the proof of Theorem 9, it suffices to show how to include the vertices of each clause gadget in the directed path P that starts at s and ends at t passing through all the vertices of the variable gadgets. Consider a positive clause $c = (x \vee y \vee z)$ and let i, j, and k be the clause-indices of C_c; the case of a negative clause is symmetric. We first show how to include in P the vertices a^1_c and s^1_c (the vertices a^2_c and s^2_c can be included analogously).

- If x is `true`, then the vertices u^x_{6i+4} and u^x_{6i+5} are consecutive in P. This allows us to extend P so as to include s^1_c and a^1_c consecutively between u^x_{6i+4} and u^x_{6i+5}, by adding to P the augmenting edges $u^x_{6i+4}s^1_c$ and $a^1_c u^x_{6i+5}$; see Fig. 9a.
- If x is `false`, then vertices u^x_{6i+3} and u^x_{6i+4} are consecutive in P; the same holds for vertices u^x_{6i+5} and u^x_{6i+6}. This allows us to include s^1_c between u^x_{6i+3} and u^x_{6i+4}, and a^1_c between u^x_{6i+5} and u^x_{6i+6}, by adding to P the augmenting edges $u^x_{6i+3}s^1_c$, $s^1_c u^x_{6i+4}$, $u^x_{6i+5}a^1_c$, and $s^1_c u^x_{6i+4}$; see Fig. 9b.

(a) $x =$ `true` (b) $x =$ `false`

Fig. 9. Extending subhamiltonian path P to include a_c^1 and s_c^1 in the case in which x is (a) `true`, and (b) `false`.

Next, we show how to include in P the vertex t_c^1 (the vertex t_c^2 can be included analogously).

- If y is `true`, then the vertices u^y_{6j} and u^y_{6j+1} are consecutive in P. This allows us to extend P so as to include t_c^1 between u^y_{6j} and u^y_{6j+1}, by adding to P the augmenting edge $t_c^1 u^y_{6j+1}$.
- If y is `false`, then the vertices u^y_{6j+1} and u^y_{6j+2} are consecutive in P. This allows us to extend P to include t_c^1 between u^y_{6j+1} and u^y_{6j+2}, by adding to P the augmenting edges $u^y_{6j+1} t_c^1$ and $t_c^1 u^y_{6j+2}$.

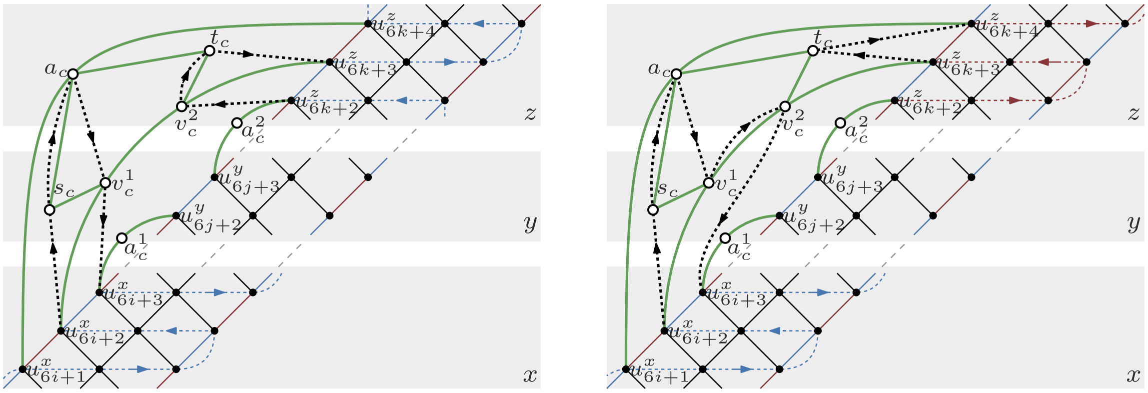
We next focus on the remaining vertices of clause gadget C_c, namely a_c, s_c, t_c, v_c^1 and v_c^2. We distinguish three cases depending on the truth assignments for x and z; see Figs. 10a, 10b and 10d.

- Suppose first that both x and z are `true`; see Fig. 10a. In this case, the vertices u^x_{6i+2} and u^x_{6i+3} of L_x are consecutive along P; the same holds for the vertices u^z_{6k+2} and u^z_{6k+3} of L_z. This allows us to extend P to include s_c, a_c, and v_c^1 consecutively between u^x_{6i+2} and u^x_{6i+3}, by adding to P the augmenting edges $u^x_{6i+2} s_c$, $a_c v_c^1$, and $v_c^1 u^x_{6i+3}$, and to include v_c^2 and t_c consecutively between u^z_{6k+2} and u^z_{6k+3}, by adding to P the augmenting edges $u^z_{6k+2} v_c^2$ and $t_c u^z_{6k+3}$. Note that the described extension of P is independent of the truth assignment for y.
- Suppose next that exactly one of x and z is `true`. We describe the case in which x is `true` and z is `false` (see Fig. 10b); the case in which x is `false` and z is `true` is analogous (see Fig. 10c). In this case, the vertices u^x_{6i+2} and u^x_{6i+3} of L_x are consecutive along P; the same holds for the vertices u^z_{6k+3} and u^z_{6k+4} of L_z. This allows us to extend P to include s_c, a_c, v_c^1, and v_c^2 consecutively between u^x_{6i+2} and u^x_{6i+3}, by adding to P the augmenting edges $u^x_{6i+2} s_c$, $a_c v_c^1$, and $v_c^2 u^x_{6i+3}$, and to include t_c between u^z_{6k+3} and u^z_{6k+4}, by adding to P the augmenting edges $u^z_{6k+3} t_c$ and $t_c u^z_{6k+4}$. Again, the extension is independent of the truth assignment for y.
- Suppose finally that both x and z are `false`; see Fig. 10d. Since c is satisfied, y is `true`. In this case, the vertices u^x_{6i+1} and u^x_{6i+2} of L_x are consecutive along P; the same holds for the vertices u^y_{6j+2} and u^y_{6j+3} of L_y and for the vertices u^z_{6k+3} and u^z_{6k+4} of L_z. This allows us to extend P to include s_c between u^x_{6i+1} and u^x_{6i+2}, by adding to P the augmenting edges $u^x_{6i+1} s_c$ and $s_c u^x_{6i+2}$, to include v_c^1, a_c, and v_c^2 consecutively between u^y_{6j+2} u^y_{6j+3}, by adding to P the augmenting edges $u^y_{6j+2} v_c^1$, $v_c^1 a_c$, $a_c v_c^2$, and $v_c^2 u^y_{6j+3}$, and to finally include t_c between u^z_{6k+3} and u^z_{6k+4}, by adding to P the augmenting edges $u^z_{6k+3} t_c$ and $t_c u^z_{6k+4}$.

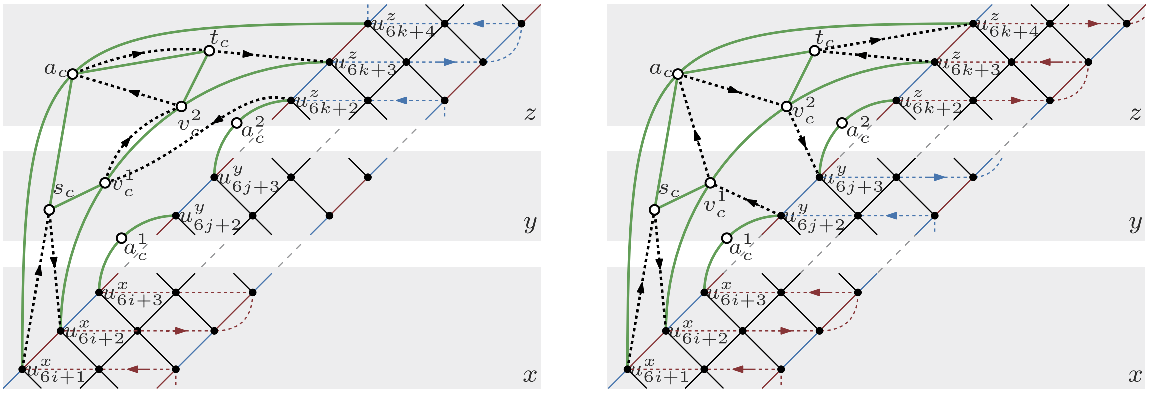
We now prove the other direction, that is, that if H is subhamiltonian, then ϕ is satisfiable. We start by introducing three useful properties of H.

Property 11. *The DAG H has a unique upward planar embedding (up to a flip).*

Proof. Since G_ϕ is planar, we have that H admits an upward planar embedding $\mathcal{E}_H$ with s and t on its outer face. Since the underlying graph of H is a subdivision of a triconnected planar graph, it has a unique combinatorial embedding. Hence, any upward planar embedding of H might differ from $\mathcal{E}_H$ only by the choice of the outer face. The only internal faces of $\mathcal{E}_H$ that are incident both to a source and to a sink of H are the faces of a clause gadget C_c incident to s_c and t_c, or to s_c^1 and t_c^1, or to s_c^2 and t_c^2. Suppose, for a contradiction, that an upward planar embedding $\mathcal{E}'_H$ of H exists in which the outer face is the face of C_c incident to s_c and t_c; the argument for the other two cases is analogous. Then the outer face of $\mathcal{E}'_H$ is delimited by the directed paths $s_c a_c t_c$ and $s_c v_c^1 v_c^2 t_c$. Note that s_c is only incident to the outer face of $\mathcal{E}'_H$ and to an internal face whose incident vertices are s_c, a_c, u^x_{6i+1}, u^x_{6i+2}, and v_c^1 (the face labeled g_c in Fig. 8). Consider the directed graph K obtained from H by removing the vertex s_c and let $\mathcal{E}'_K$ be the upward planar embedding of K obtained from $\mathcal{E}'_H$ by removing s_c. Then the vertices incident to the outer face of $\mathcal{E}'_K$ are a_c, u^x_{6i+1}, u^x_{6i+2}, v_c^1, v_c^2, and t_c. However, none of these vertices is a source of K, which contradicts the fact that $\mathcal{E}'_K$ is an upward planar embedding. □



(a) $x =$ `true`, $z =$ `true`　(b) $x =$ `true`, $z =$ `false`



(c) $x =$ `false`, $z =$ `true`　(d) $x =$ `false`, $z =$ `false`

Fig. 10. Different cases that occur while extending subhamiltonian path P to include the remaining vertices of a clause gadget (i.e., those different than a^1_c and s^1_c).

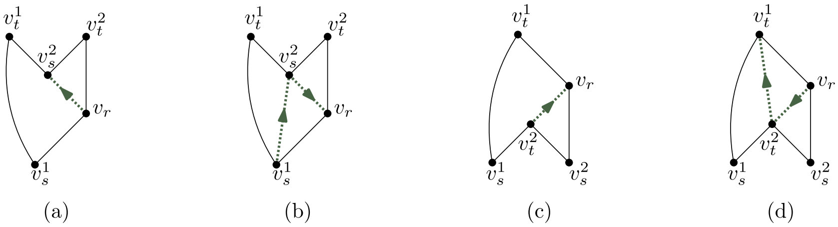


Fig. 11. Possible augmenting edges of (a-b) a $\mathbb{V}$-face, and (c-d) a $\mathbb{A}$-face.

By Property 11, in the unique upward planar embedding $\mathcal{E}_H$ of H there exist several faces, formed by clause gadgets, consisting of five vertices, out of which two are sources for the face, two are sinks for the face, while the fifth one is neither a source nor a sink for the face (refer, e.g., to the shaded in gray in Fig. 8). Consider such a face f and denote by v^1_s, v^2_s its two sources, and by v^1_t, v^2_t its two sinks; see Fig. 11 for an illustration. We call f a *$\mathbb{V}$-face* if the edge $v^1_s v^2_s$ or the edge $v^2_s v^1_s$ can be added inside f while preserving the upward planarity of H (see Figs. 11a and 11b); otherwise we call f a *$\mathbb{A}$-face* (see Figs. 11c and 11d). The faces that are denoted by g_c, g^1_c and g^2_c in Fig. 8 are $\mathbb{V}$-faces, while the ones denoted by h_c, h^1_c and h^2_c are $\mathbb{A}$-faces.

Property 12. *Consider any $\mathbb{V}$-face f of $\mathcal{E}_H$ with sources v^1_s, v^2_s and sinks v^1_t, v^2_t in which the edge $v^1_s v^2_s$ can be added inside f while preserving the upward planarity of H. Any subhamiltonian path P for H contains either (i) only the augmenting edge $v_r v^2_s$, or (ii) the two augmenting edges $v^1_s v^2_s$ and $v^2_s v_r$, where v_r is the fifth vertex of f.*

Proof. Since f is an internal face of $\mathcal{E}_H$, vertex v^2_s can be reached only from v_r or v^1_s. The first case (see Fig. 11a) yields the augmenting edge $v_r v^2_s$ of part (i) of the statement. In this case, no other augmenting edge can be added; indeed, the edge $v^1_s v^2_s$ would let v^2_s have two incoming edges in P, the edge $v_r v^1_t$ would let v_r have two outgoing edges in P, and the edge

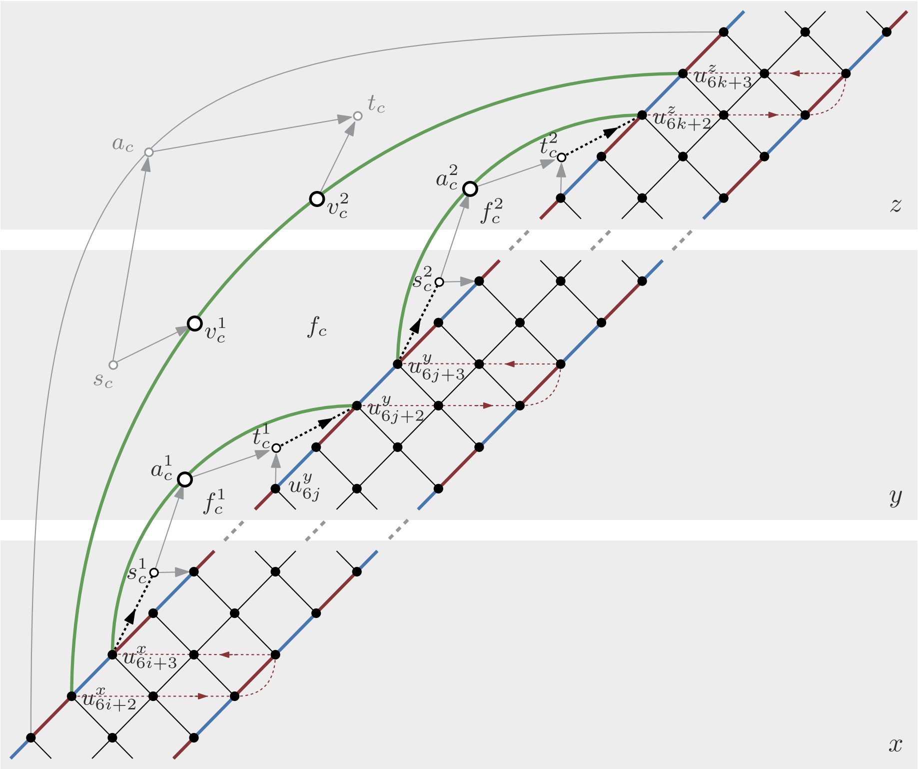


Fig. 12. An unsatisfied positive clause $c = (x \vee y \vee z)$. Vertices u^x_{6i+3}, u^y_{6j+2}, u^y_{6j+3} and u^z_{6k+2} are incident to two augmenting edges of P that are not in the interior of the anchor face f_c.

$v_t^1 v_r$ would create a directed cycle $v_t^1 v_r v_s^2 v_t^1$. In the second case (see Fig. 11b), vertices v_s^1, v_s^2, v_r, v_t^2 create a rhombus and, by Property 3, the augmenting edge $v_s^2 v_r$ or $v_r v_s^2$ must also be present, however the latter would let v_s^2 have two incoming edges in P. □

Symmetrically, one can prove the following.

Property 13. *Consider any ⩓-face f with sources v_s^1, v_s^2 and sinks v_t^1, v_t^2 of $\mathcal{E}_H$ in which the edge $v_t^2 v_t^1$ can be added inside f while preserving the upward planarity of H. Any subhamiltonian path P for H contains either (i) only the augmenting edge $v_t^2 v_r$, or (ii) the two augmenting edges $v_r v_t^2$ and $v_t^2 v_t^1$, where v_r is the fifth vertex of f.*

To complete the proof of the theorem, assume that there exists a subhamiltonian path P for H, from s to t. We compute a truth assignment as described in the proof of Theorem 9. It suffices to prove that all the clauses of ϕ are satisfied. Let $c = (x \vee y \vee z)$ be a positive clause, where x, y and z are all `false`; the case of an unsatisfied negative clause is analogous. By Property 12, in the ⩔-face g_c there exists either the augmenting edge $u^x_{6i+2} s_c$ or the augmenting edges $u^x_{6i+1} s_c$ and $s_c u^x_{6i+2}$. As x is `false`, P already contains the augmenting edge $u^x_{6i+2} v^x_{6i+2}$, which implies that $u^x_{6i+1} s_c$ and $s_c u^x_{6i+2}$ are augmenting edges of P. Leveraging Properties 12 and 13, we can argue similarly for all other ⩔-faces and ⩓-faces of C_c, as shown in Fig. 12. In particular, $u^x_{6i+3} s_c^1$, $u^y_{6j+3} s_c^2$, $t_c^1 u^y_{6j+2}$ and $t_c^2 u^z_{6k+2}$ are augmenting edges of P. Now consider the face f_c, whose left boundary contains the vertices u^x_{6i+2}, v_c^1, v_c^2 and u^z_{6k+3}, and whose right boundary contains the vertices u^x_{6i+2}, u^x_{6i+3}, a_c^1, u^y_{6j+2}, u^y_{6j+3}, a_c^2, u^z_{6k+2} and u^z_{6k+3}. Since f_c is non-transitive, there exists at least one augmenting edge of P, say e, inside f_c, connecting a vertex of its left boundary to a vertex of its right boundary. We already proved that edge $u^x_{6i+3} s_c^1$ is an augmenting edge of P in f_c^1, and since x is `false`, the same holds for $v^x_{6i+3} u^x_{6i+3}$. Hence, we have identified the two edges incident to u^x_{6i+3} that belong to P, and none of them can be the edge e. The same holds for vertices u^y_{6j+2}, u^y_{6j+3} and u^z_{6k+2}. As a consequence e connects a vertex of the left boundary of f_c to a_c^1 or a_c^2. Assume that a_c^1 is an endpoint of e, as the case in which a_c^2 is an endpoint of e is analogous. As a_c^1 belongs to the non-transitive face f_c^1, by Property 4, there is exactly one augmenting edge of P in f_c^1, and this edge is either $a_c^1 u^x_{6i+5}$ or $u^y_{6j} a_c^1$. However, this is not possible since x and y are `false` and P contains augmenting edges $v^x_{6i+5} u^x_{6i+5}$ and $u^y_{6j} v^y_{6j}$; a contradiction. We conclude that c is satisfied, thus completing the proof of the theorem. □

5. Conclusions

In this paper, settling a long-standing conjecture of Heath and Pemmaraju [15] and improving upon previous results by Heath and Pemmaraju [15] and by Binucci et al. [6], we have proved that deciding whether a DAG has page-number 2 is NP-complete. Indeed, we have proved that the problem is NP-hard even for *st*-planar graphs and for planar posets.

Whether our two hardness results can be combined into a single, and stronger, hardness result remains open. That is: What is the complexity of deciding whether an *st*-planar graph without transitive edges has page-number 2?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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