Annals of Operations Research
https://doi.org/10.1007/s10479-022-05072-w

ORIGINAL RESEARCH

Country rankings according to well-being evolution: composite indicators from a functional data analysis perspective

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Accepted: 8 November 2022

Abstract
The paper suggests the use of the functional data analysis approach to study the evolution of well being indicators, visualizing their behaviour over time. Thus, an evolutionary well-being indicator is proposed by complement the original data with information concerning the first derivative. The second task is to provide an overall ranking of the countries over time using two functional tools: the area under the curve and functional depth, which return two distinct rankings. A simulation study is conducted to evaluate the effectiveness of the area in distinguishing groups of countries with different levels of well-being. The proposed method is employed on a real dataset concerning the human development index of European countries.

Keywords Area under the curve · Functional depth · Functional derivatives · Overall Ranking · Well-being indicators

1 Introduction

It is widely acknowledged that the progress of a country should be evaluated not only from an economic but also from a social and environmental point of view. Consequently, well-being has become an increasingly involved concept in any world development consideration and several measures have been proposed for its quantification (Gasper, 2004). Well-being indicators are widely used to describe complex phenomena, evaluate the performance of countries and support decision making. Although there is a growing agreement that well-being indicators should be included in a country's development evaluation, they are generally considered from a static point of view that disregards their temporal dynamics. Measures of

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Published online: 15 November 2022

well-being taken on a single occasion provide circumscribed information (Diener, 2006) that does not emphasise whether a country is experiencing an improvement, decline or stability in its level of well-being. Meanwhile, the study of well-being becomes meaningful from a comparative perspective across times and countries. Therefore, the analysis of well-being evolution over time is crucial to evaluate the improvement of countries and the impact of national policies. Although these considerations are widely recognised, suitable measures for assessing changes in well-being over time and across countries are lacking (Boarini et al., 2006). In the literature, the comparison across countries and periods is carried out through synthetic measures, such as average annual well-being growth, absolute difference between well-being values for two time points, and percentage changes in well-being values in different countries and periods. However, these measures could lead to paradoxical conclusions (Tsui, 1996). For example, when a country starts from a very low level of well-being, even a slight increase can lead to high percentage changes. Thus, a country with a low level of well-being that experiences a small increase in the index may find itself in a better position than a country that experiences a greater absolute improvement starting from an already high level of well-being. Low developed countries tend to experience the highest increase in well-being, although this growth is insufficient to transform them into more developed well-being countries (Chakraborty, 2011). It is important to stress here that as well-being becomes high, it becomes more difficult to raise it further (Sen, 1981; Chakraborty, 2011) because most components of well-being, such as life expectancy, cannot grow unlimitedly.

In this paper, the functional data analysis (FDA) approach is proposed to study the evolutionary aspect of well-being indicators. FDA refers to the analysis of curves or functions in a continuous domain and assumes the existence of unknown smooth functions, which generate and underlie the data (see (Ramsay & Silverman, 2005) and (Ferraty & Vieu, 2006) for a more detailed introduction to the FDA). Within this framework, well-being indicators are considered as functions rather than scalar vectors and are treated as single entities (Ramsay & Silverman, 2005). Although the FDA approach is widely used in many fields (Ullah & Finch, 2013; Ramsay & Silverman, 2002), it is a novel perspective in well-being processing. Nevertheless, it is quite natural to consider a well-being indicator as a function because both its annual values and the latent phenomenon that it measures lie in a temporal domain (Fortuna et al., 2022). Moreover, the fact that the analysis of well-being can be considered functional in nature, also emerges from the evidence that some information can be deduced from the behaviour of the functions rather than from the raw data. The development of a functional approach within the context of well-being indicators provides several advantages. First, it allows us to visualize the indicator's behaviour over time by highlighting its evolution. Second, the functional approach can tackle cases where data are not sampled at equally spaced time points, while multivariate techniques assume that the dimension of all data vectors are the same. Third, it is possible to introduce new analytical tools that may sometimes complement the original data with useful information (Fortuna et al., 2018; Di Battista et al., 2016). With reference to the latter, three functional tools are considered in this paper: the derivatives, the area under the curves and the functional depth. The derivatives are particularly relevant to well-being indicators because they are potential quantifications of the function behaviour from an evolutionary perspective. Indeed, the derivatives allow us to highlight a growth, a decrease or constant trend of the indicator, depending on whether they are positive, negative or equal to zero. We claim that both the well-being level and its evolution are essential to analyze the well-being of a country. Starting from these considerations, we propose an evolutionary well-being indicator that integrates the information provided by the well-being indicator with the information concerning its temporal evolution, as reflected by the first derivative of the well-being curve. In this way, each well-being indicator is comple-

mented to discount (reward) for a decreasing (increasing) evolution. To provide a country's ranking according to both the level and the temporal dynamic of their well-being, the area under the curve and the functional depth are considered. This allows us to order the functions by reflecting their behaviour for the whole time span, hence providing an overall ranking. Specifically, the area under the curve reflects both the level and evolutionary dynamics of the trajectories, whereas, the functional depth reflects the centrality of the observations.
In conclusion, this paper introduces a new methodological approach for the study of well-being indicators with a twofold purpose: provide an evolutionary indicator and an overall ranking across times and countries.
The rest of this paper is organized as follows. Section 2 illustrates the analysis of well-being indicators in a functional framework and introduces the evolutionary index. Section 3 proposes a dynamic comparison across countries with an overall ranking using both the area under the curve and functional depth. In Sect. 4, a simulation study is conducted to investigate the reliability of the proposed method. Section 5 shows the main results that are obtained by applying the proposed approach to the time series of the Human Development Index (HDI) for European countries from 1990 to 2019. Finally, Sect. 6 presents the conclusions of this study and makes some recommendations for further developments.

2 Functional well-being indicators

Starting from the temporal sequences of a well-being indicator, observed for the i-th unit, $i = 1, 2, ..., n$, we construct continuous and differentiable curves, $WI_i(t)$ with $t \in \mathcal{T}$, where $\mathcal{T}$ is a real interval on which data are collected. In this way, it is possible to focus on the characteristics of the functions rather than on the simple raw data. Although the values of $WI_i(t)$ exist for all $t \in \mathcal{T}$, in real applications, sample curves are observed with error; that is, only in specific discrete points, $t_l \in \mathcal{T}$, $l = 1, 2, .., L$. Thus, the observation of the i-th function consists of L noisy pairs (WI_{il}, t_{il}), so that:

$$WI_{il} = WI_i(t_{il}) + \epsilon_{il}, \quad l = 1, ..., L;\; i = 1, ..., n, \tag{1}$$

where WI_{il} is the observed value for the i-th unit at the sampling point t_{il}, $WI_i(t_{il})$ is the value of a smooth function at t_{il} and ϵ_{il} is the measurement error for the i-th unit at the sampling point t_{il}. We assume that the smooth functions belong to the Hilbert space of square integrable functions, $L^2(\mathcal{T})$, with the usual inner product $< f, g >= \int_{\mathcal{T}} f(t)g(t)\,dt, \forall f, g \in L^2(\mathcal{T})$ and the L^2-norm $||f|| =< f, f >^{1/2} < \infty$.
A usual solution to reconstruct the functional form starting from the discrete and noisy observations is to assume that sample paths belong to a finite-dimension space spanned by a suitable basis, $\{\phi_1(t), \phi_2(t), \cdots, \phi_B(t)\}$, so that the reconstructed smooth function for the i-th unit can be expressed as a linear combination of certain basis functions, as follows:

$$WI_i(t) = \sum_{b=1}^{B} c_{ib}\phi_b(t), \tag{2}$$

where $WI_i(t)$, c_{ib} and $\phi_b(t)$ represent the reconstructed smooth function, the basis coefficients and the basis functions for the i-th unit, respectively, and B is the total number of basis functions. Various basis systems can be adopted, depending on the characteristics of the curves (Aguilera & Aguilera-Morillo, 2013). For the functional representation of well-being indicators, cubic B-splines basis have been considered because of their flexibility and mathematical properties (De Boor, 2001). Once a suitable basis system has been chosen, the functional form of the curves is determined by the basis coefficients, which can be obtained by least squares approximation; that is, by minimizing, for each i, the following sum of squared error:

$$SSE_i = \sum_{l=1}^{L}\Big(y_{il} - \sum_{b=1}^{B} c_{ib}\phi_b(t)\Big)^2. \tag{3}$$

The advantage of the FDA approach is that it uses functional tools, which may reveal crucial additional information better than the original data. Given that we aim to evaluate the well-being evolution, our attention is focused on the first derivatives of well-being curves, which can yield valuable insights into the time dynamics of functional data. The first derivative may reveal a growth, a deceleration or a constant trend of the indicator, depending on whether it is positive, negative, or equal to zero. Specifically, first derivatives are computed on the spline approximation, as follows:

$$WI_i'(t) = \sum_{b=1}^{B} \widehat{c}_{ib}\phi_b'(t) \tag{4}$$

where $WI_i'(t)$ and $\boldsymbol{\phi}'(t)$ denote the first derivative of the i-th function and basis functions, respectively. The analysis of $WI_i'(t)$ allows us to catch the trend differences among units, as follows: two functions with similar trajectories may have very different derivatives, showing differences in their trends.

Although the information provided by the first derivative is very useful to reveal the evolution of well-being, it neglects many of the features of the curves themselves. In assessing well-being, it is essential to consider a measure that is able to account for both the level and the evolutionary dynamic of the indicator. To this end, we suggest to integrate the information provided by the well-being indicator with its temporal evolution, as reflected by the first derivative. Specifically, we propose an evolutionary well-being indicator, say EWI, that is defined for each $t \in \mathcal{T}$ as follows:

$$EWI_{il} = WI_{il}(1 + \alpha_{il}), \quad i = 1, 2, ..., n; \quad l = 1, 2, ..., L, \tag{5}$$

where WI_{il} is the value of the well-being indicator for the i-th country at time l and α_{il} is a weight determined by the value of the first derivative of the i-th well-being curve at the sampling point l. EWI_{il} is a scalar measure that is observed over time but which can be itself regarded as a function. In this case, EWI_{il} represents a snapshot of the i-th smooth function at time l, which can be reconstructed via basis expansion techniques. Because the well-being of a country changes continuously but slowly, the evolutionary well-being indicator can be computed by setting α_{il} equal to the average of the first derivatives calculated over a certain number p of previous years, that is:

$$EWI_{il}^p = WI_{il}(1 + \alpha_{il}^p) = WI_{il}\left(1 + \frac{\sum_{j=1}^{p} WI_i'(t_{l-j})}{p}\right), \quad l > 1;\ j = 1, 2, ..., p. \tag{6}$$

From a theoretical point of view, EWI_{il} in Eq. (5) and EWI_{il}^p in Eq. (6) could take a negative value when a country experiences a sharp decline in its well-being. To avoid this drawback, the weight α_{il} or α_{il}^p could be set as a fraction of the first derivatives since we believe that the overall reward or penalty for an increasing or decreasing evolution should not outweigh the original indicator WI_{il}. However, in practice, social phenomena undergo slow changes over time, so the value of derivatives is rather small and there is no need to reweigh the derivatives.

3 Functional tools for a dynamic comparison across countries

Despite the difficulty of establishing a unique ranking among countries and periods, the analysis of well-being in the functional framework can yield a solution to this issue thanks to the additional information provided by the functional tools. Specifically, two functional instruments are considered: the area under the curve and functional depth. These are scalar measures that are able to reflect the behaviour of the trajectories in the entire temporal domain, thus providing a single ordering of the functions for the whole time span. However, the resulting rankings are substantially different. Indeed, the area provides an ordering that takes into account both the level and the evolutionary dynamics of the trajectories, whereas the functional depth ranks the observations from the most central to the most extreme. Starting from a sample of n well-being functions, $WI_1(t)$, $WI_2(t)$, ..., $WI_n(t)$, countries can be sorted in descending order according to the area under the curve, say A_i, (Di Battista et al., 2017):

$$A_i = \int_{\mathcal{T}} WI_i(t)\,dt, \quad i = 1, 2, ..., n. \tag{7}$$

Clearly, the greater the area under the curve, the greater the average well-being. The empirical distribution of the area can be used to define groups of countries with different levels of well-being, by establishing cutoff points, such as quartiles. If we are interested in defining a local ordering, then we can resort to a truncated version of A_i by defining the integral in (7) for distinct intervals of the domain. Thus, the area under the curve allows us to capture long and short term trends.

In addition to the overall ranking returned by the area, the FDA approach allows to consider functional depth rank, which reflects the centrality of functional observations with respect to the sample and hence, ranks the observations from the most central to the most extreme. In other words, countries in the top positions of the depth ranking show median levels of well-being; whereas countries in the last positions show atypical behavior compared to the rest of the observed sample. Different depth notions for functional data have been proposed in the literature (Zuo & Serfling, 2000; Cuevas et al., 2007). The order statistics induced by a depth start from the most central sample curve and then move outward in all directions. Thus, observations with a large depth are found near the centre of the sample, whereas low depth observations are outliers. To obtain a depth-based rank, the functional integrated depth (Fraiman & Muniz, 2001) is considered. This computes the integration of an univariate depth along the time axis, as follows:

$$I_i = \int_{\mathcal{T}} D_i(t)\,dt; \quad \forall t \in \mathcal{T}, \quad i = 1, 2, ..., n, \tag{8}$$

where $D_i(t) = 1- \mid 0.5 - F_{n,t} WI_i(t) \mid$ is the univariate depth of $WI_i(t)$ at t, and $F_{n,t}$ is the empirical distribution of the functions. Thus, at each single time point, the values $WI_1(t)$, ..., $WI_n(t)$, are ranked according to their univariate depth. The functional depth-based rank is then obtained by sorting I_i in descending order; that is, by ranking the curves from the most central to the most outlying one.

4 Simulation study

To evaluate the effectiveness of the area in distinguishing groups of countries with different levels of well-being, a simulation study has been performed in the R environment (R Core Team, 2020). A population of n countries has been divided into three groups of well-being

Table 1 Cutoff points and sample sizes for the three well-being groups for each scenario

Scenario	Cutoff points			Group sample size		
	Hight	Medium	Low	Hight	Medium	Low
S_1	≥ 0.700	0.550-0.699	< 0.550	$n/3$	$n/3$	$n/3$
S_2	≥ 0.700	0.550-0.699	< 0.550	20% of n	47% of n	33% of n
S_3	≥ 0.667	0.334-0.666	< 0.334	$n/3$	$n/3$	$n/3$

and three different scenarios have been examined. In the literature, it is common to consider four well-being levels: very high, high, medium and low. However, for illustrative purposes, the very high and high group have not been distinguished in the simulation study. For each scenario, the values of a well-being index have been simulated by drawing from an Uniform distribution with different parameters, which are specified by the cutoff points of each well-being group and considering the $L = 20$ annual values of the index. For the first two scenarios, S_1 and S_2, the cutoff points correspond to the percentiles of the index distribution: less than 0.550 for the low well-being group, from 0.550 to 0.699 for the medium group, and from 0.700 to 1.000 for the high well-being group. In the third scenario, S_3, the cutoff points have been defined by splitting the index distribution into three parts of the same width, so that the low well-being group ranges from 0.001 to 0.333, the medium one from 0.334 to 0.666 and the high one from 0.667 to 1. In S_1 and S_3, the three well-being groups have the same sample sizes, whereas in S_2 the high group represents 20% of the population, the medium group 47% and the low group represents 33% of the population. Table 1 reports the cutoff points and sample sizes of each well-being group in the three scenarios.

In each scenario, eight countries have been forced to a specific trend, leaving the population structure in the three groups unchanged.

Specifically, one country in the high development group, say H_1, has been forced to a decreasing trend, which starts at the upper limit of the group, reaches the lower limit of the group in the last four years and then remains constant in this final time range. A second country with high well-being, say H_2, has been forced to have a constant decreasing trend from the upper to the lower bound of the group. In the opposite group, one country with low well-being, L_1 presents an increasing trend, which starts from the lower bound of the group, ends at the upper one and then remains constant for the last 4 years of the temporal domain. Another country of the low group, L_2, has been forced to have a constant increasing trend from 0.001 to the upper bound of the group. Finally, in the medium group, two countries, M_{H1} and M_{H2}, have been forced as in the case of the high group, whereas other two observations, M_{L1} and M_{L2}, have been forced as in the case of the low group, clearly taking into account the cutoff points of the medium group. For each scenario, three sampling sizes of the population have been considered, $n = 30, 90, 180$, replicating the procedure $R = 5000$ times. Figure 1 displays the simulated data in the three well-being groups for S_1 with $n = 30$ by considering one simulation. The eight observations with a prefixed trend have been highlighted in bold black.

For each scenario, simulated temporal sequences of well-being index are converted into a sample of n functions according to Equation (2), considering $B = 5$ cubic B-splines basis functions chosen by cross validation (Ramsay & Silverman, 2005). Then, first derivatives have been computed as in Equation (4). For each scenario and sample size, Table 2 presents the minimum and maximum values of the first derivatives computed across the R replications, that is:

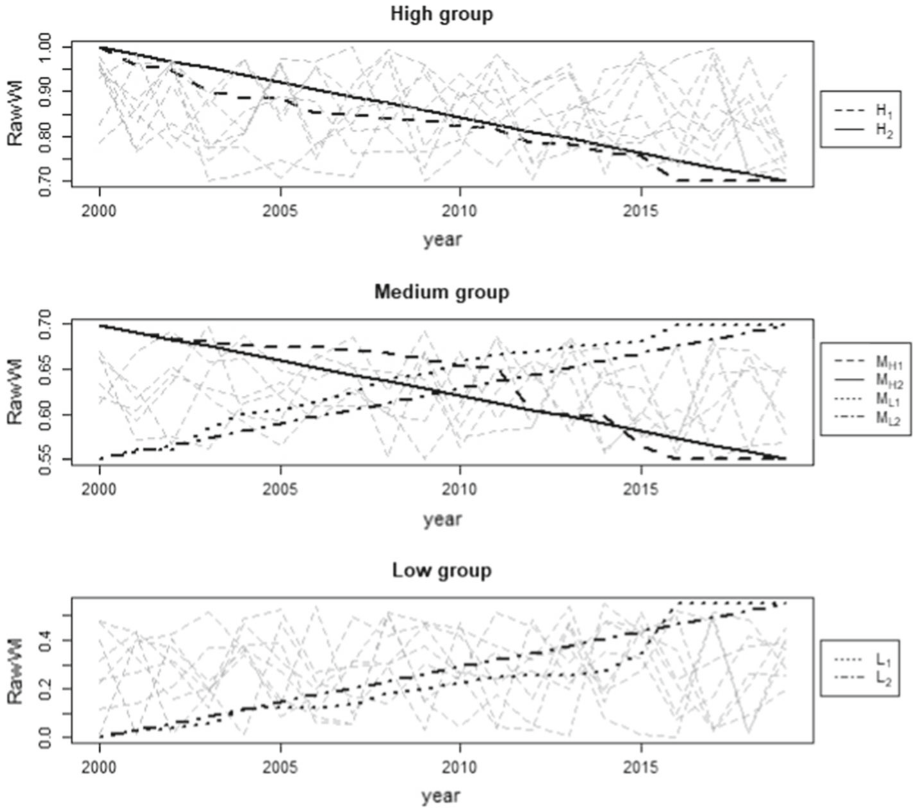

Fig. 1 Simulated well-being indicators of each well-being group for S_1 with $n = 30$ in the r-th simulation, with $r = 1$

$$\overline{\min(WI'(t))} = \frac{1}{R}\sum_{r=1}^{R}\min\Big(WI'_r(t)\Big), \tag{9}$$

$$\overline{\max(WI'(t))} = \frac{1}{R}\sum_{r=1}^{R}\max\Big(WI'_r(t)\Big), \tag{10}$$

where $\min\Big(WI'_r(t)\Big)$ and $\max\Big(WI'_r(t)\Big)$ are the minimum and maximum values of the first derivatives in the entire dataset for the r-th replication, $r = 1, 2, ..., R$. To quantify the between-simulation variability, the following squared error measures have been provided:

$$SSE_{min} = \frac{1}{R-1}\sum_{r=1}^{R}\Big(\min\Big(WI'_r(t)\Big) - \overline{\min(WI'(t))}\Big)^2, \tag{11}$$

$$SSE_{max} = \frac{1}{R-1}\sum_{r=1}^{R}\Big(\max\Big(WI'_r(t)\Big) - \overline{\max(WI'(t))}\Big)^2. \tag{12}$$

SSE_{min} and SSE_{max} are reported in Table 2 for each scenario and sample size.

Table 2 Minimum and maximum values of the first derivatives with their estimation error across the R replications for each scenario and sample size

Scenario	n	$\overline{\min(WI'(t))}$	SSE_{min}	$\overline{\max(WI'(t))}$	SSE_{max}
S_1	30	−0.171	0.002	0.171	0.002
	90	−0.214	0.001	0.213	0.001
	180	−0.235	0.001	0.235	0.001
S_2	30	−0.169	0.002	0.170	0.002
	90	−0.213	0.001	0.214	0.001
	180	−0.236	0.001	0.236	0.001
S_3	30	-0.124	0.001	0.125	0.001
	90	−0.149	0.001	0.148	0.001
	180	−0.160	0.001	0.160	0.001

Although the first derivatives present rather small values, it is noticeable that they reach lower values and increase their range of variation as n increases. The minimum and maximum values of the first derivatives in S_3 are lower in absolute terms than in S_1 and S_2, due to the different specification of the cutoff points of the groups. Starting from the values of the first derivatives, the evolutionary well-being indicators, EWI, and EWI^p, with $p = 5$, have been computed as in Equations (5) and (6). Subsequently, for each replication, $r = 1, 2, ..., R$, the area under the functional counterparts of WI, EWI and EWI^p have been obtained following Equation (7). A final estimate of the area has been computed as follows:

$$\hat{A}_i(f) = \frac{1}{R}\sum_{r=1}^{R} A_i(f_r), \quad i = 1, 2, ..., n, \tag{13}$$

where f is the functional counterpart of a specified well-being indicator ($f = WI(t), EWI(t), EWI^p(t)$) and $A_i(f_r) = \int_{\mathcal{T}} f_{ir}(t)\,dt$. The quantiles of the area distribution for the different functional indicators have then been computed to identify three groups of well-being, which are able to reflect the dynamic of the indicator in the entire domain. The cutoff points of the groups according to the area under the functional indicators have been defined as follows: less than the first quartile for the low group, from the first quartile to the third quartile for the medium group and greater than the third quartile for the high well-being group. In this context, the aim is to check whether and which area manages to capture the fixed trends of the eight countries and thus to declassify observations showing a strongly decreasing trend into the next lower group, and vice versa.

For each country with a prefixed trend, Table 3 shows its new group assignment according to the area under the different functional indicators in each scenario. The sample size of the population does not affect the results. In fact, the group assignment remains unchanged for $n = 30, 90, 180$. The two scenarios with well-being groups of equal sample size, S_1 and S_3, present the same group assignment for the eight observations of prefixed trend. S_1 and S_3 differ for the range of variation of the cutoff points, and hence do not affect the results. For both S_1 and S_3, regardless of the functional indicator, there is no group change for the four countries of the medium well-being group; that is, M_{H1}, M_{H2}, M_{L1} and M_{L2}. The area computed on $WI(t)$ is affected by the trend of H_1 and L_1, which are both assigned to the medium well-being level, reflecting their decreasing and increasing trends, respectively. Meanwhile, the area calculated on $EWI(t)$ and $EWI^p(t)$ also highlights the behaviour of

Table 3 Group assignment of the 8 countries with a prefixed trend according to the area under the different functional indicators in each scenario

Country	S_1 and S_3			S_2		
	$\hat{A}(WI(t))$	$\hat{A}(EWI(t))$	$\hat{A}(EWI^p(t))$	$\hat{A}(WI(t))$	$\hat{A}(EWI(t))$	$\hat{A}(EWI^p(t))$
H_1	**M**	**M**	**M**	H	H	H
H_2	H	**M**	**M**	H	H	H
M_{H1}	M	M	M	M	M	M
M_{H2}	M	M	M	M	M	M
M_{L1}	M	M	M	**H**	**H**	**H**
M_{L2}	M	M	M	M	**H**	**H**
L_1	**M**	**M**	**M**	**M**	**M**	**M**
L_2	L	**M**	**M**	L	**M**	**M**

H_2 and L_2; that is, the two countries with constant decreasing and increasing trend over time, and assigns them to the medium well-being group. The scenario with unequal group sample sizes, S_2, shows different results. In the group with the lowest sample size—that is, the high group (20% of n)—H_1 and H_2 do not undergo assignment changes with any functional indicators. Meanwhile, some countries change well-being group by increasing the number of observations in the middle group. Specifically, the area calculated on $WI(t)$ manages to account for the increasing trends of M_{L1} and L_1, assigning one to the high group and the other to the medium group. The area computed on $EWI(t)$ and $EWI^p(t)$ is also able to capture the trend of M_{L2} and L_2. Meanwhile, M_{H1} and M_{H2} remains in the medium group. In conclusion, the group assignment results are not influenced by the sample size of the population, n, or by the width of the cutoff points, but they are influenced by the the sample sizes of the well-being groups. For each scenario, the group assignments obtained with $EWI(t)$ and $EWI^p(t)$ are the same, whereas there are differences between $WI(t)$ and the two evolutionary functional indicators. Specifically, $EWI(t)$ and $EWI^p(t)$ are able to reflect particular behaviour of the well-being trajectories, yielding changes of group assignment for a greater number of countries with a prefixed trend.

The depth-based rank has not been considered because the focus of the simulation is not to highlight behavior which deviate from the median levels of well-being, whereas to verify the capability of $EWI(t)$ to declassify observations showing a strongly decreasing trend into the next lower group, and vice versa.

5 Application to *HDI* of the European countries

The Human Development Index (HDI), which was devised by the United Nations Development Program (UNDP, 2020), is one of the most widely-used and well-known composite indicators. The HDI is based on Sen's theoretical work (Sen, 1999) on the important capabilities required for an individual to achieve optimal functioning. As we can read on the UNDP's website http://hdr.undp.org/en/humandev, the human development approach is aims to expand the richness of human life, improve the lives that people lead rather than assuming that economic growth will lead, which will automatically lead to a greater well-being for all. The HDI is a summary measure of average achievement in three key dimensions of human development: longevity and health (i.e., life expectancy at birth), access to knowledge (i.e.,

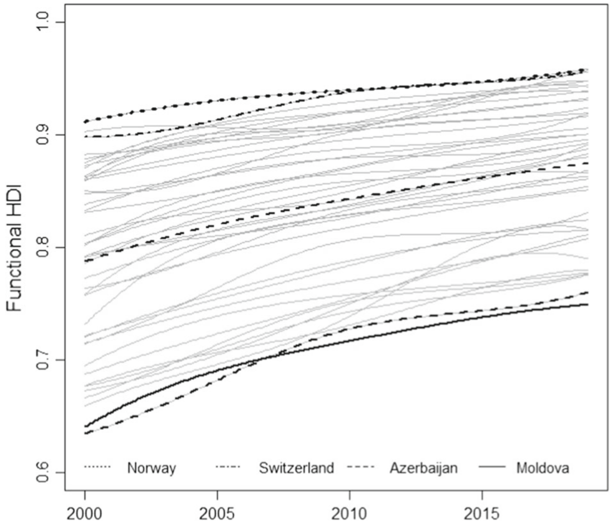

Fig. 2 Functional HDI of the European countries from 2000 to 2019

mean years of schooling for adults and expected years of schooling for children at school entering age) and a decent standard of living (i.e., Gross National Income per capita, adjusted for purchasing power). Since 2010, the HDI has been the geometric mean of normalized indices for each of the three dimensions, whereas it was previously calculated as a weighted average of the three dimensions.

In this section, the annual values of the HDI of 44 European countries from 2000 to 2019 have been considered (data are available at http://www.hdr.undp.org/en/indicators/137506#). The list of countries considered is available in Table 4. The HDI time series of each country can be considered as continuous functions observed at $L = 20$ discrete points. Thus, raw temporal sequences of HDI have been converted into a sample of functions adopting a basis expansion, as in (2), with $WI_i(t) = HDI_i(t)$, and considering cubic B-splines basis, whose coefficients have been obtained by least square approximation. The order of the spline system ($B = 5$) has been chosen to ensure sufficient flexibility to represent the behavior of the functions and, at the same time, to obtain a continuous estimate of the first derivatives.

Figure 2 shows the reconstructed functional HDI for each country. The central dashed-black line represents the functional mean. This presents a slight increasing trend, which starts from 0.79 and goes up to 0.87. The highest and lowest trajectories have been highlighted in Fig. 2. The highest curves are those of Norway (dotted-black line) and Switzerland (dot-dashed-black line). The trajectories of these two countries intersect each other, so it is not possible to identify the country with the highest human development over the whole domain. The lowest curves are those of Azerbaijan (dashed-black line) and of the Republic of Moldova (solid line). Also in this case, the two curves intersect each other: Azerbaijan presents the lowest human development until 2007, while from 2008 onwards the Republic of Moldova shows the lowest level of human development.

Figure 3 shows the first derivatives of the functional HDI, computed as in (4). The range of the first derivatives is rather limited because their values are between -0.007 and 0.024. Some peculiar behaviors of the derivatives are highlighted with black lines in Fig. 3. Turkey (solid-black line) shows a trend of the first derivative, which decreases until 2003, increases

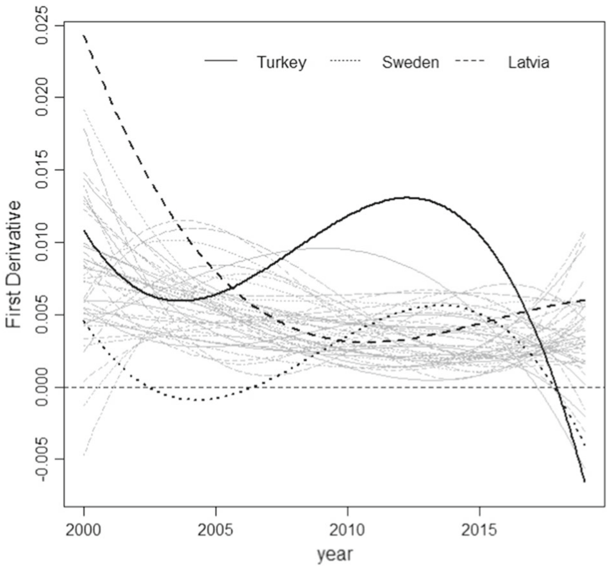

Fig. 3 First derivatives of the functional HDI for the European countries

until 2010 and then decreases after 2015. Moreover, in the time interval 2011-2015, Turkey had the highest value of the first derivative. The first derivative of Latvia (dashed line) exhibits a strong decreasing trend until 2008, and then remains constant for the last years. Sweden (dotted line) presents the lowest value of the first derivatives for $t \in [2002, 2018]$. Using the information provided by the first derivatives, the evolutionary indicators, $EHDI$ and $EHDI^p$, with $p = 5$, have been computed as in (5) and (6), for all t_l, $l = 2000, ..., 2019$. A comparison between the rankings obtained with the standard HDI and those resulting from the two evolutionary indicators has been considered by analyzing the number of times in which a country experiences a mismatch between the two rankings, focusing on the entire time span. Comparing the rankings obtained with HDI and $EHDI$, the Netherlands records the highest number of mismatches, changing its positions for 13 years, while loosing no more than two positions and gaining no more than three positions. Meanwhile, Azerbaijan, Croatia, Luxembourg and the Republic of Moldova change position according to the two indicators in only one year. Specifically, all of these above countries lose one position according to the $EHDI$, except for the Republic of Moldova, which in 2013 goes from position 44 to position 43 according to $EHDI$. On average, countries change position in their ranking 12.1 times, with a standard deviation equal to 3.7. The comparison between the rankings of HDI and $EHDI^p$, with $p = 5$, is clearly made over 15 years. The Republic of Moldova and Ukraine do not undergo changes, whereas the country with the higher number of mismatches is Iceland, which changes its position 11 times, loosing no more than three positions and gaining no more than one position. On average, countries change position in their ranking 11.7 times, with a standard deviation of 2.9.

To provide an overall ranking, the area in (7) has been computed for each country, considering the different functional indicators: $HDI(t)$, $EHDI(t)$ and $EHDI^p(t)$. The three obtained rankings (see Table 4) agree in the first three and in the last three positions. In particular, the countries in the first three positions are, respectively: Norway, Switzerland and Germany. Meanwhile, in the last three positions are: the Republic of Moldova, preceded

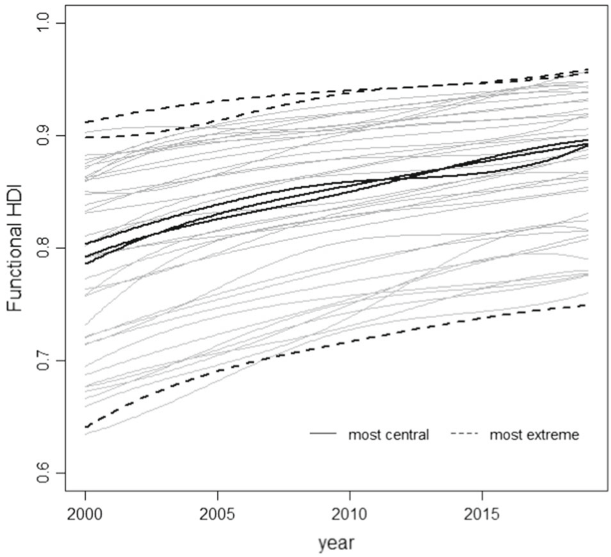

Fig. 4 Most central and most external HDI functions according to the depth-based rank

by Azerbaijan, and Bosnia and Herzegovina. It is important to note that, thanks to the area under the curves, we can establish that, considering the entire domain, Norway has a higher level of human development than Switzerland, while the Republic of Moldova has a lower level of human development than Azerbaijan.

As shown in Table 4, the overall rankings obtained with $HDI(t)$ and $EHDI(t)$ are quite similar; in fact, only seven mismatches occur. According to $EHDI(t)$, Albania, Hungary, Slovakia and Sweden lose one position; Denmark and Turkey gain one position; and Latvia rise two positions. The overall rankings obtained with $HDI(t)$ and $EHDI^p(t)$ are quite different. In this case, the number of mismatches rises to 21. As shown in Table 4, according to $EHDI^p(t)$, 10 countries gain one position; Turkey and Latvia gain two and three positions, respectively; Bulgaria, Italy, Luxembourg and Poland lose one position; Cyprus, Hungary, Portugal and Ukraine lose two positions; and Sweden is penalized by three positions. Comparing the overall rankings obtained with $EHDI(t)$ and $EHDI^p(t)$, it can be seen that the second functional indicator tends to reward countries: 12 countries rise in the ranking, while nine countries are penalised. A further overall ranking can be obtained using the functional depth in (8). We remark that the functional depth ranks the observations from the most central to the most extreme. As shown in Table 5, the depth-based rankings obtained with $HDI(t)$, $EHDI(t)$ differ for 15 countries, eight of which lose positions, which indicates a move away from the central positions according to the $EHDI(t)$. The depth-based rankings obtained with $HDI(t)$, $EHDI^p(t)$ differ by 26 countries, 12 of which gain positions in the ranking, indicating their greater centrality according to the $EHDI^p(t)$.

However, regardless of the functional indicator, Cyprus, Estonia and Malta are the countries in the first three position; that is, the countries near the centre of the sample as can be seen in Fig. 4. Meanwhile, Norway, Switzerland and the Republic of Moldova are in the lowest positions. These three countries represent functional outliers because their behavior is atypical with respect to the rest of the sample. The depth-based rank does not specify the direction of departure from the central observations. Thus, in the last three positions there

Table 4 Overall rankings and group assignment of the European countries according to the area computed on $HDI(t)$, $EHDI(t)$ and $EHDI^p(t)$

Country	Ranking			Group	
	$A(HDI(t))$	$A(EHDI(t))$	$A(EHDI^p(t))$	$A(HDI(t))$; $A(EHDI(t))$	$A(EHDI^p(t))$
Albania	38	39	38	L	L
Andorra	24	24	24	M	M
Armenia	40	40	40	L	L
Austria	14	14	13	H	H
Azerbaijan	43	43	43	L	L
Belgium	10	10	10	VH	VH
Bosnia and H.	42	42	42	L	L
Bulgaria	33	33	34	M	L
Croatia	31	31	31	M	M
Cyprus	21	21	23	H	M
Czechia	19	19	19	H	H
Denmark	5	4	4	VH	VH
Estonia	22	22	21	H	H
Finland	7	7	6	VH	VH
France	16	16	16	H	H
Georgia	36	36	36	L	L
Germany	3	3	3	VH	VH
Greece	20	20	20	H	H
Hungary	28	29	30	M	M
Iceland	9	9	9	VH	VH
Ireland	8	8	8	VH	VH
Italy	17	17	18	H	H
Latvia	30	28	27	M	M
Liechtenstein	12	12	12	H	H
Lithuania	26	26	25	M	M
Luxembourg	13	13	14	H	H
Malta	23	23	22	M	H
Moldova	44	44	44	L	L
Netherlands	6	6	5	VH	VH
North Macedonia	41	41	41	L	L
Norway	1	1	1	VH	VH
Poland	25	25	26	M	M
Portugal	27	27	29	M	M
Romania	32	32	32	M	M
Russian F.	34	34	33	L	M
Serbia	35	35	35	L	L
Slovakia	29	30	28	M	M
Slovenia	15	15	15	H	H

Table 4 continued

Country	Ranking			Group	
	$A(HDI(t))$	$A(EHDI(t))$	$A(EHDI^p(t))$	$A(HDI(t))$; $A(EHDI(t))$	$A(EHDI^p(t))$
Spain	18	18	17	H	H
Sweden	4	5	7	VH	VH
Switzerland	2	2	2	VH	VH
Turkey	39	38	37	L	L
Ukraine	37	37	39	L	L
UK	11	11	11	VH	VH

Table 5 Depth-based rankings of the European countries according to $HDI(t)$, $EHDI(t)$ and $EHDI^p(t)$

Country	Depth-based Ranking			Country	Depth-based Ranking		
	$HDI(t)$	$EHDI(t)$	$EHDI^p(t)$		$HDI(t)$	$EHDI(t)$	$EHDI^p(t)$
Albania	33	30	30	Latvia	13	11	9
Andorra	6	4	4	Liechtenstein	23	23	23
Armenia	34	34	33	Lithuania	7	7	5
Austria	19	20	21	Luxembourg	18	18	18
Azerbaijan	41	41	41	Malta	3	3	3
Belgium	29	27	26	Moldova	42	42	42
Bosnia and H.	36	36	38	Netherlands	35	35	35
Bulgaria	22	22	22	North Macedonia	38	38	39
Croatia	16	16	16	Norway	44	44	44
Cyprus	1	2	1	Poland	5	5	6
Czechia	8	8	8	Portugal	9	9	12
Denmark	37	37	37	Romania	20	19	19
Estonia	2	1	2	Russian F.	21	21	20
Finland	30	31	34	Serbia	24	24	25
France	15	15	15	Slovakia	10	13	11
Georgia	26	26	27	Slovenia	17	17	17
Germany	40	40	40	Spain	11	12	13
Greece	4	6	7	Sweden	39	39	36
Hungary	14	14	14	Switzerland	43	43	43
Iceland	28	28	31	Turkey	32	32	28
Ireland	31	33	32	Ukraine	27	29	29
Italy	12	10	10	UK	25	25	24

is no distinction on the level of the functions, which may therefore be considerably lower or higher. The last positions of the depth-based rank coincide with the extreme positions highlighted by the three area-based ranks, which place Norway and Switzerland in the first two places and the Republic of Moldova in the last place.

Finally, European countries have been classified into four human development groups according to the quartiles of the empirical distribution of the area under the different functional indicators. Specifically, the cutoff points of the groups have been defined as follows: less than

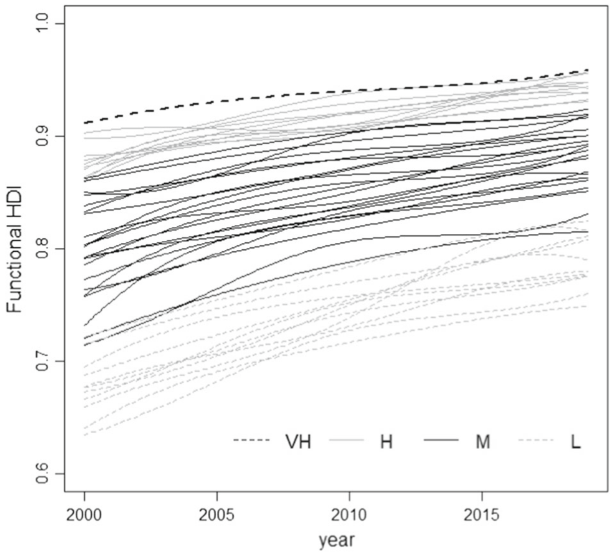

Fig. 5 Classification of $HDI_i(t)$ in 4 development groups according to the quantiles of $A(HDI(t))$

the first quartile for the low group, from the first quartile to the second quartile for the medium group, greater than the second quartile to the third quartile for the high group and greater than the third quartile for the very high group. The country assignment in the four groups is shown in Table 4. The area computed on $HDI(t)$ and $EHDI(t)$ leads to the same group assignment.

Figure 5 displays the configuration distinguishing the functional HDI in the four groups. The very high group (dashed-black lines) consists of 11 Northern Europe countries; the high group (solid-grey lines) is mainly represented by Southern European countries; both the medium (solid-black lines) and the low (dashed-grey lines) groups are predominantly characterized by Eastern European countries. The area calculated on $EHDI^p(t)$ leads to the same results, except for four countries: Cyprus is placed in the medium group rather than in the high group; Bulgaria is declassified in the low group; Malta goes from the medium to the high group; and the Russian Federation moves from the low to the medium group.

From the analysis of Fig. 5, it can be noticed that some functions intersect with those belonging to another group. For example, Bulgaria and Romania, which belong to the medium group, intersect with the functions of the low group. In particular, the curve of Romania is below that of Bulgaria and it is assigned to the low group if one considers the classification according to the $EHDI^p(t)$. Cyprus and Estonia, which belong to the high group, intersect with the functions of the medium group. Also in this case, the $EHDI^p(t)$ assigns Cyprus to the medium group.

In conclusion, The FDA approach allows us to establish a ranking of European countries based on both the level and the evolution dynamics of human development. The overall ranking resulting from the area under the functional indicators highlights that Norway and the Republic of Moldova are the countries with the highest and lowest level of human development, respectively. This result is confirmed by the functional depth, which identifies Norway and Moldova as the functions more extreme, with a high and low level of human development, respectively. The overall rankings resulting from the area calculated on the three functional indicators not only agree in the first and last positions but they also present some differences, which become more evident in the case of $EHDI^p(t)$. The $EHDI^p(t)$ tends to reward countries. In fact, when comparing it with $HDI(t)$ and with $EHDI^p(t)$, 12 countries gain ranking positions. Therefore, although the range of the first derivatives is limited, considering their trend over the previous 5 years, we can see a positive trend in some countries. If we compare the annual rankings resulting from the scalar versions of the two evolutionary indices and those obtained with the raw HDI, then some differences become apparent. This is particularly evident for the Netherlands and Ireland, which show a greater number of mismatches in the pairwise comparisons between HDI and $EHDI$, and between HDI and $EHDI^p$. However, countries with the lowest levels of human development tend to have a stable position in the annual orders, regardless of the index considered, as is the case of the Republic of Moldova. The identification of four groups of human development using the quartiles of the area under the three functional indicators characterizes the Northern European countries as very high, the Southern countries as high, and the Eastern countries as medium and low. The results obtained with $HDI(t)$ and $EHDI(t)$ are the same, while $EHDI^p(t)$ differs for four countries: Malta and the Russian Federation are assigned to a group with a higher human development level than the classification obtained with the other two functional indicators, while Cyprus and Bulgaria are declassified to a group with a lower level.

6 Concluding remarks and suggestions for further developments

This paper aimed to study the evolution of well-being indicators by means of some functional data analysis tools. The scientific literature on well-being indicators is usually concerned with the definition of the composite indicator (i.e., the manifest variables it is based on, the aggregation function and the weights) and with comparisons among geographical areas or social groups from a static point of view. Instead, this paper has focused is on both the time evolution of the indicator and cross-country comparison in a specific temporal domain. This double purpose is achieved through functional data analysis instruments: first, the functional form of the well-being indicator is derived by means of B-splines; and second, country rankings are provided by means of the area under the curve and the functional depth. Specifically, the area reflects both the level and the evolutionary dynamics of the trajectories, whereas the depth measures the ‘outlyingness’ of a curve within a set of functions. Moreover, growth or decrease matter at least as much as the level for well being indicators, and therefore an evolutionary well being indicator is defined using the derivatives of the functions.

The use of functional instruments represents a valid tool to identify an overall ranking of countries based on their level of human development. Indeed, although the FDA approach allows us to visualize the trend of the indicator over time, it is not always possible to identify a univocal ordering among the functions without the aid of additional functional tools. This happens when functions intersect each other. However, thanks to the area under the functional indicators, it is possible to obtain an unambiguous ranking over the entire reference domain. The simulation study shows that the two proposed evolutionary functional indicators are very competitive with respect to the non-evolutionary functional indicator. Indeed, the area under the two functional evolutionary indicators is able to catch particular behaviour of the well-being curves, penalizing or rewarding countries with a decreasing or increasing trend, respectively. We therefore conclude that some of the variation from curve to curve can be explained at the level of certain derivatives. The fact that derivatives are of interest is further reason to think of the records as functions rather than as vectors of observations in discrete time.

The application of the functional tools to the HDI of European countries provided an overall ranking that allows to identify the countries with the higher and lower level of human development in the entire temporal domain. The rankings obtained with the evolutionary and non-evolutionary functional HDI curves present some differences but agree in the extreme positions. Moreover, the ranking resulting from the area under the $EHDI^p(t)$ tends to reward countries, which emphasizes that growth in well-being is a slow process.

Further research will go in two directions. The first is to provide deeper insight into the evolutionary indicator to fine-tune the discount/reward due to its trend. In other words, the task is to define an optimal weight for the evolutionary component so that it does not overpower the indicator. The second research direction is to analyse the rank dynamics. Thus, for each unit, the smooth rank trajectory can be estimated by starting from the ranking at each fixed time. In this way, the relative performance of countries can be compared throughout the time period by evaluating the rank stability on the basis of the number of intersections between paths.

Funding Open access funding provided by Universit[Pleaseinsertintopreamble] degli Studi Roma Tre within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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