Nagata type statements

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Abstract

Nagata solved Hilbert's 14-th problem in 1958 in the negative. The solution naturally lead him to a tantalizing conjecture that remains widely open after more than half a century of intense efforts. Using Nagata's theorem as starting point, and the conjecture, with its multiple variations, as motivation, we explore the important questions of finite generation for invariant rings, for support semigroups of multigraded algebras, and for Mori cones of divisors on blown up surfaces, and the rationality of Waldschimdt constants. Finally we suggest a connection between the Mori cone of the Zariski–Riemann space and the continuity of the Waldschmidt constant as a function on the space of valuations.

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Introduction

Hilbert's 14-th problem on finite generation of algebras that are invariant under the action of some groups was formulated in the middle of La Belle Époque as an algebraic question. A few decades later, just when pop art was sprouting and rock–and–roll music was turning to the surf–rock music, Zariski translated it into a geometric counterpart asking when the total coordinate ring of a projective variety is finitely generated, and Nagata gave a negative answer to it producing certain blowups of projective spaces, which couldn't but spur a renewed interest on the subject.

The first section of these notes is devoted to Nagata's results from a modern point of view, taking into account contributions by Mukai, Ciliberto-Miranda and Ciliberto-Harbourne-Miranda-Roé. We show how one can construct a group G, associated to n points of the complex projective plane with multiplicities (a fat point scheme). An action of G on the ring of polynomials in $2n$ indeterminates is then given, such that the algebra of invariants of G is the Rees algebra of the ideal I of the scheme of points: it is the direct sum of all the symbolic powers of I, thus, it is naturally a bigraded algebra. Its support, that is, the subset of indices such that the corresponding addend is not trivial, is a semigroup. In the case that this semigroup turns out not to be finitely generated, then the same holds for the algebra. One can study the real convex cone spanned by the semigroup: as Nagata observed, if it is not closed, then the semigroup, and hence the algebra, cannot be finitely generated. For suitable choices of the points and their multiplicities, this is exactly the case.

The second section is devoted to the Mori cone of curves on the blowup of the n points, following the work of Waldschmidt, Demailly, Harbourne, de Fernex and Ciliberto-Harbourne-Miranda-Roé. We describe how the real cone of the first section can be understood as a slice of the Mori cone, and then Nagata's conjecture can be interpreted as a statement on the boundary of this cone. Numerical invariants such as Waldschmidt constants or Seshadri constants, which control slopes of certain extremal rays in the Mori cone, then come into play, leading to the question of existence of irrational Waldschmidt and Seshadri constants and to the quest for extremal rays in the Mori cone.

Analogous statements can be made considering valuations as generalizations of points. This point of view was initiated by Dumnicki-Harbourne-Küronya-Roé-Szemberg, and the last two sections are devoted to this subject. It leads to conjectures that make sense for real values $t \geq 1$ of the number of points rather than integral ones, and to the study of cones of effective *b–divisors* on the Zariski–Riemann space of the projective plane.

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1 Nagata's theorem and conjecture

1.1 Nagata's Theorem

Hilbert's 14-th problem

Let k be a field, let $z_1, \ldots, z_\mu$ be indeterminates over k and let $\mathbb{K}$ be an *intermediate field* between k and $k(z_1, \ldots, z_\mu)$, i.e.

$$k \subseteq \mathbb{K} \subseteq k(z_1, \ldots, z_\mu).$$

Hilbert's 14-th problem asks: *is* $\mathbb{K} \cap k[z_1, \ldots, z_\mu]$ *a finitely generated* k*-algebra?*

Hilbert had in mind the following situation coming from invariant theory. Let G be a subgroup of the *affine group*, i.e. the group of automorphisms of $\mathbb{A}^\mu_k$. Then G acts as a set of automorphisms of the k-algebra $k[z_1, \ldots, z_\mu]$, hence on $k(z_1, \ldots, z_\mu)$, and we let $\mathbb{K} = k(z_1, \ldots, z_\mu)^G$ be the field of G-invariant elements. Then the question is: *is*

$$k[z_1, \ldots, z_\mu]^G = \mathbb{K} \cap k[z_1, \ldots, z_\mu]$$

a finitely generated k*-algebra?*

In the case $\mu = 1$, Hilbert's problem has trivially an affirmative answer. The answer is also affirmative for $\mu = 2$, as proved by Zariski in [55]. In [45], [44], Nagata provided counterexamples to the latter formulation of Hilbert's problem. Nagata's *minimal counterexample* has $\mu = 32$ and $\operatorname{tr.deg}(\mathbb{K}/k) = 4$. Several other counterexamples have been given by various authors, too long a story to be reported on here. The most recent one is due to Totaro [53], who shows that Nagata's construction and some of its variations work even over a finite field k.

We now give a streamlined review of Nagata's counterexample drawing on the more general constructions of Mukai [42] and Ciliberto–Harbourne–Miranda–Roé [13]. For the sake of simplicity, we fix the base field to be the complex numbers, $k = \mathbb{C}$.

Nagata's group action

Let $P = (p_{ij})_{1 \le i \le 3; 1 \le j \le n}$ be a $3 \times n$ matrix of complex numbers. Its columns determine n points $p_1, \ldots, p_n$ in $\mathbb{P}^2$; we will assume that they are n distinct points, not all on a hyperplane (in particular, $\operatorname{rank}(P) = 3$). The $(n-3)$–dimensional linear subspace $K = \ker(P)$ of $\mathbb{C}^n$ formed by all vectors $\mathbf{b} = (b_1, \ldots, b_n)$ such that $P \cdot \mathbf{b} = \mathbf{0}$, is said to be *associated* to $p_1, \ldots, p_n$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be vectors of indeterminates, and consider the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ (so that $\mu = 2n$). Initially [45], Nagata considered the *unipotent* action of K on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ given by

$$\begin{aligned} \mathbf{b}(x_i) &= x_i \\ \mathbf{b}(y_i) &= y_i + b_i x_i, \text{ for } 1 \le i \le n. \end{aligned} \tag{1}$$

For adequate choices of n and P the $\mathbb{C}$-algebra $\mathbb{C}[\mathbf{x}, \mathbf{y}]^G$ is not finitely generated, as we shall see.

Later, in [44], with the goal of obtaining examples with smaller transcendence degree, Nagata considered the action of a larger group, which we now introduce in the generalized form of [13]. Fix a vector $\mathbf{v} = (v_1, \ldots, v_n)$ of positive integers ("multiplicities") and consider the following subgroup of the multiplicative group $(\mathbb{C}^*)^n$:

$$H_{\mathbf{v}} = \{(c_1, \ldots, c_n) \in (\mathbb{C}^*)^n \mid c_1^{v_1} \cdots c_n^{v_n} = 1\}.$$

Given $\mathbf{c} \in H_{\mathbf{v}}$ and $\mathbf{b} \in K$, set

$$\begin{aligned} \sigma_{\mathbf{c},\mathbf{b}}(x_i) &= c_i\, x_i, \\ \sigma_{\mathbf{c},\mathbf{b}}(y_i) &= \frac{c_i}{c_1 \cdots c_n}(y_i + b_i x_i), \text{ for } 1 \leq i \leq n. \end{aligned} \tag{2}$$

This defines a semidirect product $G = H_{\mathbf{v}} \ltimes K$, a $(2n-4)$–dimensional subgroup of $(\mathbb{C}^*)^n \ltimes \mathbb{C}^n$, acting linearly on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$; here $\sigma_{\mathbf{c},\mathbf{b}}$ is the image in $\mathrm{GL}_{2n}(\mathbb{C})$ of an element in G. We shall identify the groups $H_{\mathbf{v}}$ and K with their isomorphic images $H_{\mathbf{v}} \times \{0\}$ and $\{1\} \times K$ in $G = H_{\mathbf{v}} \ltimes K$.

Again, for adequate choices of n, $\mathbf{v}$, and P, the $\mathbb{C}$-algebra $\mathbb{C}[\mathbf{x}, \mathbf{y}]^G$ is not finitely generated.

Exercise 1.1. The semidirect product G is determined by an action $\phi : H_{\mathbf{v}} \to \operatorname{Aut} K$ of $H_{\mathbf{v}}$ on K. Make this action and the resulting product $G = H_{\mathbf{v}} \ltimes_{\phi} K$ explicit. Nagata in [44] considered the case $v_1 = \cdots = v_n = 1$. Show that this leads to the trivial action, and hence to the direct product $G = H_{\mathbf{v}} \times K$.

In order to prove non finite generation, Nagata's key insight is to identify $k[\mathbf{x}, \mathbf{y}]^G$ with a graded algebra built from plane geometry; the kind of algebra that will be the main object of study in these notes. The proof then proceeds in two steps. First, sufficient conditions are found for the algebra to be non-finitely generated, expressible in terms of the existence of curves in the projective plane with given degree and multiplicities at the points p_j. The second step consists in actually showing that such sufficient conditions are satisfied for adequate choices of n and $\mathbf{v}$, if P is general enough.

The construction can be carried over using a matrix P with $r \geq 3$ rows, leading to other counterexamples to Hilbert's 14-th problem related to the geometry of projective $(r-1)$–space. This generalization is due to Mukai, who used it in [42] to show counterexamples where the group acting on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ is $K \cong \mathbb{C}^k$ for any $k \geq 3$. It is not known whether there exist counterexamples for the group $\mathbb{C}^2$, while there are none for $\mathbb{C}$ by Weitzenböck's result [54].

The invariant ring of the unipotent action as a Rees algebra

To describe the connection with geometry, let us fix some additional notation. Choose coordinates $\mathbf{w} = (w_1, w_2, w_3)$ on $\mathbb{P}^2$, so that $\mathbb{C}[\mathbf{w}] = \mathbb{C}[\mathbb{P}^2]$ is the homogeneous coordinate ring of $\mathbb{P}^2_{\mathbb{C}}$, and call $I(p_j) \subset \mathbb{C}[\mathbb{P}^2]$ the homogeneous ideal of the point $p_j = [p_{1j}, p_{2j}, p_{3j}]$ for $j = 1, \ldots, n$, where $p_j \neq p_h$ for $j \neq h$. For an arbitrary vector of multiplicities $\mathbf{m} = (m_1, \ldots, m_n)$, by abuse of language, and consistently with the notation in [31], in the rest of the paper we denote by $Z_{\mathbf{m}} = \sum_{j=1}^n m_j p_j$ the 0-dimensional subscheme of $\mathbb{P}^2_{\mathbb{C}}$ (a *fat points scheme*) determined by the homogeneous ideal $I(Z_{\mathbf{m}}) = \bigcap_{j=1}^n I(p_j)^{m_j}$.

For any homogeneous ideal I in a given graded ring, denote as customary I_t its homogeneous component in degree t.

Since the monomials x_j are invariant under the unipotent action (1) of the associated space K, this action can be extended to the ring $\mathbb{C}[\mathbf{x}, \mathbf{y}][x_1^{-1}, \ldots, x_n^{-1}]$. Here K acts by translation, so the invariant ring $\mathbb{C}[\mathbf{x}, \mathbf{y}][x_1^{-1}, \ldots, x_n^{-1}]^K$ can be immediately computed: it is generated by

$$\varpi_1 = \sum_{j=1}^n p_{1j} y_j / x_j; \quad \varpi_2 = \sum_{j=1}^n p_{2j} y_j / x_j; \quad \varpi_3 = \sum_{j=1}^n p_{3j} y_j / x_j$$

over the ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The elements

$$\begin{aligned}\varpi_1 \cdot x_1 \cdots x_n &= p_{1,1} y_1 x_2 \cdots x_n + p_{1,2} x_1 y_2 x_3 \cdots x_n + \cdots + p_{1,n} x_1 \cdots x_{n-1} y_n, \\ \varpi_2 \cdot x_1 \cdots x_n &= p_{2,1} y_1 x_2 \cdots x_n + p_{2,2} x_1 y_2 x_3 \cdots x_n + \cdots + p_{2,n} x_1 \cdots x_{n-1} y_n, \\ \varpi_3 \cdot x_1 \cdots x_n &= p_{3,1} y_1 x_2 \cdots x_n + p_{3,2} x_1 y_2 x_3 \cdots x_n + \cdots + p_{3,n} x_1 \cdots x_{n-1} y_n\end{aligned} \tag{3}$$

are independent linear combinations of the obviously algebraically independent elements

$$y_1 x_2 \cdots x_n, \quad x_1 y_2 \cdots x_n, \quad \dots, \quad x_1 \cdots x_{n-1} y_n,$$

so they are algebraically independent; we identify them with the coordinates w_1, w_2, w_3, i.e.,

$$w_1 = \varpi_1 \cdot x_1 \cdots x_n, \quad w_2 = \varpi_2 \cdot x_1 \cdots x_n, \quad w_3 = \varpi_3 \cdot x_1 \cdots x_n.$$

Since these elements belong to $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ and are invariant under K, they realize $\mathbb{C}[\mathbb{P}^2] = \mathbb{C}[w_1, w_2, w_3]$ as a subring of $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$. Then

$$(\mathbb{C}[\mathbf{x}, \mathbf{y}][x_1^{-1}, \dots, x_n^{-1}])^K = \mathbb{C}[\mathbb{P}^2][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

and thus

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^K = (\mathbb{C}[\mathbf{x}, \mathbf{y}][x_1^{-1}, \dots, x_n^{-1}])^K \cap \mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[\mathbb{P}^2][x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cap \mathbb{C}[\mathbf{x}, \mathbf{y}]. \tag{4}$$

Remark 1.1. The identification of the three forms in (3) with w_1, w_2, w_3, and the identification of the columns of P with points in $\mathbb{P}^2$ are mutually consistent with respect to changes of variables. Indeed, given an invertible matrix $A \in \mathrm{GL}_3$, consider $P' = AP$, which has the same associated space K, hence the same invariant ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$. The new invariant elements $\mathbf{w}' = (w_1', w_2', w_3')$ generate the same invariant subring $\mathbb{C}[\mathbb{P}^2]$, as they satisfy $\mathbf{w}' = A\mathbf{w}$.

For $j = 1, \dots, n$, let V_j be the linear space of homogeneous elements of $\mathbb{C}[\mathbb{P}^2]$ of degree 1 that are divisible by x_j (in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$). Equivalently, V_j is the linear subspace of $\mathbb{C}[\mathbb{P}^2]_1$ formed by elements whose coefficient in the monomial $(x_1 \cdots x_n) y_j / x_j$ vanishes.

Lemma 1.2. *A degree d homogeneous polynomial F in $\mathbb{C}[\mathbb{P}^2]$ vanishes on p_j with multiplicity m_j if and only if it belongs to*

$$[(V_j)^{m_j}]_d = [(x_j)^{m_j} \cap \mathbb{C}[\mathbb{P}^2]]_d.$$

Moreover, in this case $F/x_j^{m_j}$ is in $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$.

Proof. For simplicity of notation we assume that $j = 1$ and set $m_1 = m$. Start with $m = 1$ and consider a homogeneous polynomial of degree d

$$F(w_1, w_2, w_3) = \sum_{a+b+c=d} \alpha_{abc} w_1^a w_2^b w_3^c.$$

Expanding all powers of the w_is' using (3), we see that the only terms that are not divisible by x_1 add up to

$$\sum_{a+b+c=d} \alpha_{abc}\, p_{1,1}^a p_{2,1}^b p_{3,1}^c (y_1 x_2 \cdots x_n)^d,$$

and obviously this vanishes if and only if $F(p_1) = \sum \alpha_{abc}\, p_{1,1}^a p_{2,1}^b p_{3,1}^c = 0$. The fact that $F/x_1 \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^K$ is now immediate by (4).

The argument for $m > 1$ is analogous, but in order to make the computation more transparent we will do a further reduction, and assume that the first point is the coordinate point $p_1 = (1, 0, 0)$ in $\mathbb{P}^2$. By remark 1.1 this is not restrictive.

Now w_2 and w_3 are multiples of x_1 by (3) (since $p_{2,1} = p_{3,1} = 0$), and they span V_1. Therefore $[(V_1)^m]_d \subset [(x_1)^m \cap \mathbb{C}[\mathbb{P}^2]]_d$.

Conversely, assume that F has multiplicity $e < m$ at p_1. In terms of the expansion $F(w_1, w_2, w_3) = \sum \alpha_{abc} w_1^a w_2^b w_3^c$, this means there are nonvanishing terms $\alpha_{d-e,b,c} w_1^a w_2^b w_3^c$ with $b + c = e$. As each of these $w_2^b w_3^c$ is a multiple of x_1^e, and $w_1 = y_1 x_2 \cdots x_n$ modulo x_1, the following equality holds modulo x_1^{e+1}:

$$F(w_1, w_2, w_3) = \sum_{a+b+c=d} \alpha_{abc} w_1^a w_2^b w_3^c = \sum_{b+c=e} \alpha_{d-e,b,c} (y_1 x_2 \cdots x_n)^{d-e} w_2^b w_3^c \neq 0,$$

i.e., $F(w_1, w_2, w_3)$ is not equal to zero modulo x_1^{e+1}, so it is not divisible by x_1^m, and we have proved the inclusion $[(V_1)^m]_d \supset [(x_1)^m \cap \mathbb{C}[\mathbb{P}^2]]_d$. $\square$

Lemma 1.3. *Let as before $Z_{\mathbf{m}} = m_1 p_1 + \cdots + m_n p_n \subset \mathbb{P}^2$; then*

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^K \cong \bigoplus_{\mathbf{m} \in \mathbb{Z}^n} I(Z_{\mathbf{m}}).$$

Proof. By (4), an element $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is invariant by K if and only if there exist nonnegative integers $m_1, \ldots, m_n$ such that $f x_1^{m_1} \cdots x_n^{m_n} \in \mathbb{C}[\mathbb{P}^2]$; this gives

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^K = \mathbb{C}[\mathbb{P}^2][x_1, \ldots, x_n] + \sum_{\mathbf{m} > \underline{0}} ((x_1)^{m_1} \cap \ldots \cap (x_n)^{m_n} \cap \mathbb{C}[\mathbb{P}^2]) x_1^{-m_1} \cdots x_n^{-m_n}.$$

The previous lemma then says that the last expression equals

$$\mathbb{C}[\mathbb{P}^2][x_1, \ldots, x_n] + \sum_{\mathbf{m} > \underline{0}} I(Z_{\mathbf{m}}) x_1^{-m_1} \cdots x_n^{-m_n},$$

that is clearly isomorphic to $\oplus_{\mathbf{m} \in \mathbb{Z}^n} I(Z_{\mathbf{m}})$, as claimed. $\square$

The multigraded algebra of Lemma 1.3 is called the *Rees algebra of the multigraded filtration* $\{I(Z_{\mathbf{m}})\}_{\mathbf{m} \in \mathbb{Z}^n}$. It also inherits the natural grading of $\mathbb{C}[\mathbb{P}^2]$, so that it is in fact a $\mathbb{Z}^{n+1}$–graded algebra:

$$\bigoplus_{\mathbf{m} \in \mathbb{Z}^n} I(Z_{\mathbf{m}}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}^n, d \geq 0} [I(Z_{\mathbf{m}})]_d \,.$$

For details on Rees algebras for general filtrations, for modules, and their connection with blowups, see [28], [22].

The invariant ring of the Nagata action as a Rees algebra

Let us now go back and consider a fixed vector of multiplicities $\mathbf{v}$, and the groups $H_{\mathbf{v}} = \{(c_1, \ldots, c_n) \mid c_1^{v_1} \cdots c_n^{v_n} = 1\}$ and $G = H_{\mathbf{v}} \ltimes K$ acting by (2). The algebra of invariants of G can be described as

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^G = \left(\mathbb{C}[\mathbf{x}, \mathbf{y}]^K\right)^G = \left(\mathbb{C}[\mathbf{x}, \mathbf{y}]^K\right)^{H_{\mathbf{v}}}. \tag{5}$$

The three elements w_1, w_2, w_3 are clearly invariant not only under the action of K, but under the whole group G. Therefore $H_{\mathbf{v}}$ acts on $\mathbb{C}[\mathbb{P}^2][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and in fact the action can be described as follows. For every $\mathbf{c} \in H_{\mathbf{v}}$,

$$\mathbf{c}(w_j) = w_j, \text{ for } 1 \leq j \leq 3,$$
$$\mathbf{c}(x_i) = c_i x_i, \text{ for } 1 \leq i \leq n.$$

Therefore, by the definition of $H_{\mathbf{v}}$,

$$\mathbb{C}[\mathbb{P}^2][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{H_{\mathbf{v}}} = \mathbb{C}[\mathbb{P}^2][t^{\pm 1}], \quad \text{where } t = x_1^{v_1} \cdots x_n^{v_n}. \tag{6}$$

For every nonnegative integer m, let

$$I(mZ_{\mathbf{v}}) = I(Z_{\mathbf{v}})^{(m)} = I(Z_{m\mathbf{v}}) = \bigcap_{j=1}^{n} I(p_j)^{mv_j}$$

be the so-called m-th *symbolic power* of $I(Z_{\mathbf{v}})$. Putting together (4), (5), (6), and Lemma 1.3, the following description of the invariant ring holds.

Proposition 1.4. *Let $Z_{\mathbf{v}} = v_1p_1 + \cdots + v_np_n \subset \mathbb{P}^2$; then*

$$\mathbb{C}[\mathbf{x}, \mathbf{y}]^G \cong \bigoplus_{m\in\mathbb{Z}} I(mZ_{\mathbf{v}}) = \bigoplus_{m\in\mathbb{Z}, d\geq 1} [I(mZ_{\mathbf{v}})]_d. \tag{7}$$

Again, we have identified the invariant ring as a Rees algebra.

1.2 Semigroups, cones and finite generation

The next step is to find sufficient conditions under which the multigraded algebras $\oplus_{\mathbf{m},d}[I(Z_{\mathbf{m}})]_d$ of Lemma 1.3 and $\oplus_{m,d}[I(mZ_{\mathbf{v}})]_d$ of Proposition 1.4 are not finitely generated.

Given a k-algebra $A = \oplus_{\lambda\in\Lambda} A_\lambda$ graded by a free abelian group Λ, the subset $\{\lambda \in \Lambda | A_\lambda \neq 0\}$ of Λ is a semigroup called the *support of* A and denoted by *Supp*(A). Clearly, if A is finitely generated as a ring over k then *Supp*(A) is finitely generated as a semigroup. In our case $\mathcal{S}_K = \mathit{Supp}(\oplus_{\mathbf{m},d}[I(Z_{\mathbf{m}})]_d)$ (respectively $\mathcal{S}_G = \mathit{Supp}(\oplus_{m,d}[I(mZ_{\mathbf{v}})]_d)$) is a semigroup in $\mathbb{Z}^{n+1}$ (respectively in $\mathbb{Z}^2$), and it will be enough to give conditions in order that $\mathcal{S}_K$ or $\mathcal{S}_G$ is not finitely generated. In fact, we shall give sufficient conditions for the *convex cone* spanned by the semigroup *SuppA* in the real vector space $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^N$ to be non finitely generated, which is a stronger condition.

A *convex cone* in a real vector space V is a subset $C \subset V$ closed under nonnegative linear combinations:

$$\forall u, v \in C, \forall a, b \in \mathbb{R}, a, b \geq 0 \Longrightarrow au + bv \in C.$$

Given an arbitrary subset $S \subset V$, the *cone spanned* by S (or *conic hull*) is the set of all nonnegative linear combinations of vectors in S:

$$\mathrm{co}(S) = \left\{ \sum_{i=1}^{k} a_i v_i \,\middle|\, a_i \geq 0, v_i \in S \right\}.$$

The conic hull co(v) of a nonzero vector is called the *ray* spanned by v. Given a cone C, a ray $R \subset C$ is said to be *extremal* if for every $u, v \in C$, $u + v \in R$ implies $u, v \in R$. A cone is *polyhedral* if it can be spanned by a finite set. A polyhedral cone is always closed.

Given two cones C_1, C_2, the cone spanned by their union is denoted by $C_1 + C_2 = \mathrm{co}(C_1 \cup C_2)$, as it coincides with their Minkowski sum as subsets of V.

Consider now the real convex cone spanned by *Supp*(A)

$$\mathrm{co}(\mathit{Supp}(A)) = \left\{ \sum_{i=1}^{k} a_i\lambda_i \,|\, a_i \in \mathbb{R}_{\geq 0}, \lambda_i \in \mathit{Supp}(A) \right\} \subset \Lambda \otimes \mathbb{R} \cong \mathbb{R}^N.$$

Whenever the semigroup $Supp(A)$ is finitely generated, $\mathrm{co}(Supp(A))$ is a closed polyhedral cone, whose extremal rays are spanned by a subset of generators of $Supp(A)$. Nagata's method to prove that $\mathcal{S}_G$ is not finitely generated is to show that $\mathrm{co}(\mathcal{S}_G)$ is not closed. Observe that $\mathrm{co}(\mathcal{S}_G)$ can be understood as the intersection of $\mathrm{co}(\mathcal{S}_K) \subset \mathbb{R}^{n+1}$ with the plane

$$\Pi = \langle (v_1, \dots, v_n, 0), (0, \dots, 0, 1) \rangle \subset \mathbb{R}^{n+1}.$$

So, if $\mathrm{co}(\mathcal{S}_G)$ is not closed, then $\mathrm{co}(\mathcal{S}_K)$ is not closed either, and this is enough to show that neither $\mathbb{C}[\mathbf{x}, \mathbf{y}]^G$ nor $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$ are finitely generated.

We want to show that the convex cone $\mathrm{co}(\mathcal{S}_G)$ spanned by the support semigroup

$$\mathcal{S}_G = \{(d, m) \mid [I(mZ_{\mathbf{v}})]_d \neq 0\} \subset \mathbb{Z}^2.$$

is not closed for suitable $\mathbf{v}$. Set $\delta = \sqrt{\sum_{j=1}^n v_j^2}$. The symbolic powers $I(mZ_{\mathbf{v}})$ form a *multiplicative filtration*, i.e.,

$$I(mZ_{\mathbf{v}}) I(m' Z_{\mathbf{v}}) \subseteq I((m + m') Z_{\mathbf{v}}) \tag{8}$$

in particular $(I(mZ_{\mathbf{v}}))^\ell \subseteq I(\ell m Z_{\mathbf{v}})$.

Example 1.5. If $p_1, \dots, p_{10} \in \mathbb{P}^2$ are the 10 nodes of an irreducible nodal rational sextic, then for $Z = p_1 + \cdots + p_{10}$, one has $I(Z)_3 = 0$, hence $I(Z)_3 I(Z)_3 = 0$; but $I(2Z)_6 \neq 0$, thus $(I(Z))^2 \subsetneq I(2Z)$.

For any homogeneous ideal I in $\mathbb{C}[\mathbb{P}^2]$, let $\alpha(I) = \min\{t \mid I_t \neq 0\}$.

Lemma 1.6. *Suppose that for every $m \geq 1$ it is $\alpha(I(mZ_{\mathbf{v}})) > m\delta$. Then for every $m \geq 1$ there is $\ell > 1$ such that $(I(mZ_{\mathbf{v}}))^\ell \subsetneq I(\ell m Z_{\mathbf{v}})$.*

Proof. By (8), $\alpha(I(mZ_{\mathbf{v}}))$ is a subadditive sequence, hence, by the Fekete Lemma, the limit $\lim\limits_{m \to \infty} \frac{\alpha(I(mZ_{\mathbf{v}}))}{m}$ exists, and it equals

$$\hat{\alpha}(I(Z_{\mathbf{v}})) = \inf \left\{ \frac{\alpha(I(mZ_{\mathbf{v}}))}{m} \,\middle|\, m > 0 \right\},$$

which is called the *Waldschmidt constant* of $I(Z_{\mathbf{v}})$. Since

$$\dim [I(Z_{\mathbf{v}})]_d \geq \frac{d^2 - m^2\delta^2}{2} + \cdots,$$

where the dots denote lower degree terms (see Harbourne's notes [31]), we have $\lim\limits_{m \to \infty} \frac{\alpha(I(mZ))}{m} \leq \delta$. On one hand, by hypothesis $\frac{\alpha(I(mZ))}{m} > \delta$ for all positive integers m. Hence,

$$\lim_{\ell \to \infty} \frac{\alpha(I(\ell m Z))}{\ell m} = \lim_{m \to \infty} \frac{\alpha(I(mZ))}{m} = \delta.$$

On the other hand, $\alpha\left((I(mZ))^\ell\right) = \ell \alpha(I(mZ))$ for every ℓ,

$$\frac{\alpha\left((I(mZ))^\ell\right)}{\ell m} = \frac{\alpha(I(mZ))}{m} > \delta$$

from which we conclude that for some large ℓ (depending on m) $\alpha((I(mZ))^\ell) > \alpha(I(\ell m Z))$ and the claim follows. $\square$

Exercise 1.2. Let $Z_{\mathbf{v}} = v_1 p_1 + \cdots + v_n p_n$ be a nonzero fat point subscheme of $\mathbb{P}^2$. Show that $1 \leq \widehat{\alpha}(I(Z_{\mathbf{v}})) \leq \delta$.
Hint: look at $[I(kmZ)]_{kd}$ where d/m is rational and close to but bigger than δ and $k \gg 0$. (See also Exercise 1.3.6 in Harbourne's notes [31]).

1.3 Ciliberto-Miranda's proof for Nagata's theorem

The Severi variety and degenerations

By assigning the multiplicities $\mathbf{v}$ to any choice of n points of $\mathbb{P}^2$, one gets a scheme $Z_{\mathbf{v}} = v_1p_1 + \cdots + v_np_n$ as above. The ideal $I(Z)$ of course depends on the choice of the points. Nagata's theorem deals with $\mathbf{v} = (1, \ldots, 1)$ and a square number of *very general* points, i.e., outside of a countable union of proper closed subsets of $(\mathbb{P}^2)^n$.

We will follow the usual convention that, when a claim is made for *general* points, it is meant that that claim is satisfied for every choice of the points outside a proper closed subset of $(\mathbb{P}^2)^n$. Similarly when dealing with a collection of objects (e.g., valuations) parameterized by some variety X, claiming a fact for general (resp. very general) objects will mean that all objects parameterized by a Zariski open subset of X (resp. a countable intersection of Zariski opens) satisfy the claim.

Theorem 1.7 (Nagata [45])**.** *Let $\delta \geq 4$ be an integer. If $p_1, \ldots, p_{\delta^2}$ are very general points in $\mathbb{P}^2$, and $Z = p_1 + \cdots + p_{\delta^2}$, then $\alpha(I(mZ)) > \delta m$ for all $m \geq 1$.*

Assigning a point p of multiplicity m to a homogeneous polynomial of fixed degree d corresponds to a set of $\binom{m+1}{2}$ linear equations on the coefficients of the polynomial. As the position of the assigned point varies, the coefficients determining these linear equations vary polynomially in the coordinates of the point. Thus, for each d and $\mathbf{v} = (v_1, \ldots, v_n)$ there are $\sum \binom{v_i+1}{2}$ equations determining a (possibly empty) "Severi variety"

$$V_{\mathbf{v},d} \subset (\mathbb{P}^2)^n \times \mathbb{P}(\mathbb{C}[w_1, w_2, w_3]_d)$$

formed by the closure of the set of the tuples $(p_1, \ldots, p_n, F)$ such that F has multiplicity at least v_i at p_i, i.e., the fibres of $V_{\mathbf{v},d}$ for the projection to $(\mathbb{P}^2)^n$ are the (projectivized) degree d pieces of the ideals $I(Z_{\mathbf{v}})$ as the points in $Z_{\mathbf{v}}$ vary.

Since the Severi variety is Zariski-closed and the projection to $(\mathbb{P}^2)^n$ is a projective map, general fibers of the Severi variety $V_{\mathbf{v},d}$ are nonempty exactly when the image of $V_{\mathbf{v},d}$ is the whole $(\mathbb{P}^2)^n$. Moreover, if we set $\alpha_{\mathrm{gen}}(\mathbf{v})$ the value of $\alpha(I(Z_{\mathbf{v}}))$ for general p_i, then for every $0 < d < \alpha_{\mathrm{gen}}(\mathbf{v})$ the image of $V_{\mathbf{v},d}$ on $(\mathbb{P}^2)^n$ is a closed proper subset, and therefore $\alpha(I(Z_{\mathbf{v}})) = \alpha_{\mathrm{gen}}(\mathbf{v})$ for all choices of points p_i off these (finitely many) closed subsets. This allows for *specialization* and *degeneration* arguments: if there is some position of the points such that $[I(Z_{\mathbf{v}})]_d = 0$, then the same holds for general points and so $\alpha_{\mathrm{gen}}(\mathbf{v}) > d$. Thus Theorem 1.7 is equivalent to:

Theorem 1.8. *Let $\delta \geq 4$, $m \geq 1$ and $d \geq 1$ be integers with $d \leq \delta m$. If $p_1, \ldots, p_{\delta^2}$ are general points in $\mathbb{P}^2$, and $Z = p_1 + \cdots + p_{\delta^2}$, then $[I(mZ)]_d = 0$.*

A semicontinuity argument was used by Nagata to prove his theorem, and this is also the route we shall follow here, adapting a plane curves degeneration argument of Ciliberto and Miranda [15], to prove it.

Consider $\pi : Y \to \mathbb{D}$ the family obtained by blowing up the trivial family $\mathbb{D} \times \mathbb{P}^2 \to \mathbb{D}$ over a disc $\mathbb{D}$ at a point in the central fiber. The general fibre Y_u for $u \neq 0$ is a $\mathbb{P}^2$, and the central fibre Y_0 is the union of two surfaces $\mathbb{P} \cup \mathbb{F}$, where $\mathbb{P} \cong \mathbb{P}^2$ is the exceptional divisor and $\mathbb{F} \cong \mathbb{F}_1$ is the original central fibre blown up at a point. The surfaces $\mathbb{P}$ and $\mathbb{F}$ meet transversally along a rational curve E that is the negative section on $\mathbb{F}$ and a line on $\mathbb{P}$.

One can split n as a sum $n = a + b - 1$, and choose a points $q_1, q_2, \ldots, q_a \in \mathbb{P} \setminus E$, and $b - 1$ points $q_{a+1}, q_{a+2}, \ldots, q_n \in \mathbb{F} \setminus E$. Consider these n points as limits of n general points in the general fibre Y_u, i.e., fix n sections $\sigma_1, \ldots, \sigma_n$ of $Y \to \mathbb{D}$ going through the chosen points. These sections determine a map $\mathbb{D} \setminus \{0\} \to (\mathbb{P}^2)^n$. Consider the scheme $Z_{\mathbf{v}} = v_1 p_1 + \cdots + v_n p_n$, if $[I(Z_{\mathbf{v}})]_d$ is nonempty for a general choice of points, then (pulling back from the Severi variety) there is a family of curves $C \subset (\mathbb{D} \setminus \{0\}) \times \mathbb{P}^2$ of degree d such that the fiber C_u over every $u \neq 0$ has multiplicity at least v_i at the point $\sigma_i(u) \in \mathbb{P}^2_u$. The closure $\bar{C} \subset Y$ of C in Y has a "central fiber" C_0 which is the union of a curve in each component of Y_0, $C_0 = C_{\mathbb{F}} + C_{\mathbb{P}}$, and has multiplicity at least v_i at each p_i (because p_i is a smooth point of Y_0 and of Y, so that the section σ_i meets Y_0 transversely at p_i). More explicitly, $C_{\mathbb{F}}$ is the proper transform in $\mathbb{F}$ of a curve of degree d, with some multiplicity e at the blown up point and multiplicities $(v_{a+1}, ..., v_n)$ at the $b-1$ points $Z_{\mathbb{F}}$ in $\mathbb{F}$, and $C_{\mathbb{P}}$ is the proper transform in $\mathbb{P}$ of a curve of degree e and multiplicities $(v_1, ..., v_a)$ at the other chosen points $Z_{\mathbb{P}}$. Moreover, the two curves have the same intersection with the rational curve E, that is

$$C_{\mathbb{F}} \cap E = C_{\mathbb{P}} \cap E. \tag{9}$$

In other words, we have a family of curves $\bar{C}$ that fits in the diagram

$$\begin{tikzcd} \bar{C} \arrow[r] \arrow[d] & Y = \mathrm{Bl}(\mathbb{D} \times \mathbb{P}^2) \arrow[d, "\pi"] \\ \mathbb{D} \arrow[r, "="] & \mathbb{D} \end{tikzcd} \tag{10}$$

in which the specialized curve over $0 \in \mathbb{D}$ splits with the splitting of the surface $Y_0 = \mathbb{P} \cup \mathbb{F}$ in the central fiber of the family of surfaces. The scheme of points contained in the general curve also splits with the curve in the central fiber of $\bar{C}$.

The preceding discussion can be summarized by saying that the limit of a family of Cartier divisors is a union of divisors matching their intersections on E. We refer to [14], [15] and [13] for more on these particular degenerations.

Theorem 1.8 will follow from the following two lemmas, which will be proved in the next subsection.

Lemma 1.9. *Let $\delta \geq 4$ and $m \geq 1$ be integers and assume that for $p_1, \ldots, p_{\delta^2}$ general points in $\mathbb{P}^2_{\mathbb{C}}$, and $Z = p_1 + \cdots + p_{\delta^2}$, one has $\alpha(I(mZ)) > \delta m$. Then if $p_1, \ldots, p_{(\delta+1)^2}$ are general points in $\mathbb{P}^2_{\mathbb{C}}$, and $Z' = p_1 + \cdots + p_{(\delta+1)^2}$, one has $\alpha(I(mZ')) > (\delta+1)m$.*

Lemma 1.10. *Let $m \geq 1$ be an integer. If $p_1, \ldots, p_{16}$ are general points in $\mathbb{P}^2_{\mathbb{C}}$, and $Z = p_1 + \cdots + p_{16}$, then $\alpha(I(mZ)) > 4m$.*

Cremona maps

For the proof of Lemma 1.9, it will be useful to exploit some particular Cremona transformations. Computations are very explicit and will be left as exercises. The general theory of Cremona maps (birational maps of $\mathbb{P}^2$) including the description of their effect on plane curves, can be found in [1].

Recall that, given three points p, q, r in $\mathbb{P}^2$, not on a line, $\dim[I(p+q+r)]_2 = 3$, and three independent quadratic forms vanishing at p, q, r define a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, called a *standard Cremona map*. This map is defined everywhere except at p, q, r and contracts the line $p \wedge q$ to a point r', the line $q \wedge r$ to p' and the line $p \wedge r$ to q'. The standard Cremona map based at p', q', r' is the inverse of the previous map, i.e., the composition of both maps is the identity on the complement of the triangle determined by p, q, r.

We say that a collection of $n \geq 3$ points in $\mathbb{P}^2$ is in *linear general position* if no subset of 3 points is contained in a line; in particular, the points are all distinct. Note that, given n points in linear general position, we can perform the standard Cremona transformation on any subset of 3 points among the n points. This gives a different collection of n points in $\mathbb{P}^2$, that need not be in linear general position.
We say that a collection of n points is *in Cremona general position* if they are in linear general position and this remains true after any finite sequence of standard Cremona transformations on subsets of 3 points.

Exercise 1.3 (Transforming curves by standard Cremona maps)**.** Taking projective coordinates x, y, z with vertices at p, q, r, the Cremona map based at p, q, r is given by $(x : y : z) \mapsto (yz : xz : xy)$. Check that the points p', q', r' coincide with p, q, r and this map is its own inverse. Therefore, direct image and proper preimage of curves (i.e., disregarding components supported on the coordinate triangle) of curves under this standard Cremona map coincide. Show that a curve of degree d with multiplicities m_p, m_q, m_r at the three given points is mapped by the Cremona map to a curve of degree $d + c$ with multiplicities $m_{p'} = m_p + c$, $m_{q'} = m_q + c$, $m_{r'} = m_r + c$ at the three distinguished points in the image, where $c = d - m_p - m_q - m_r$. Any singularity off the triangle with vertices p, q, r is preserved because the Cremona map acts as an isomorphism there.
Hint: Plugging the expression of the Cremona map into the equation of the curve shows that the preimage curve has degree $2d$; check that its equation contains the factor x exactly m_p times, y exactly m_q times and z exactly m_r times, to obtain the proper preimage.

Exercise 1.4 (Openness conditions for collections of points in $(\mathbb{P}^2)^n$)**.** Show that, for every positive integer δ, the locus in $(\mathbb{P}^2)^n$ of n-tuples of points that are in linear general position, and such that this remains true after a sequence of $k \leq \delta$ standard Cremona transformations on subsets of 3 points, is Zariski open.
Show that the locus in $(\mathbb{P}^2)^n$ of n-tuples of points in Cremona general position is the intersection of at most countably many Zariski-open subsets of $(\mathbb{P}^2)^n$.

Remark 1.11. By [1, Theorem 5.7.3] (a result apparently first stated by H. P. Hudson and proved by P. Du Val), the locus in $(\mathbb{P}^2)^n$ of n-tuples of points in Cremona general position is Zariski-open if and only if $n \leq 8$. See also [33, Example V.4.2.3 and Exercise V.4.15].

Exercise 1.5. Let δ, m and e be positive integers with $e \geq \delta m$, and let $d = (\delta + 1)m$, $\Delta = e - \delta m$ and $n = 2\delta + 1$. Pick points $p_1, \dots, p_n \in \mathbb{P}^2$ in general position. Show that a plane curve of degree d, with multiplicity e at p_1 and multiplicity m at each of $p_2, \dots, p_r$ can be transformed by a sequence of standard Cremona maps into a curve of degree $m - \Delta\delta$ with a point of multiplicity m.

Proof of Lemma 1.9. We argue by contradiction. Assume that $\alpha(I(mZ')) \leq (\delta+1)m$, which means that $[I(mZ')]_{(\delta+1)m} \neq 0$, and consider the degeneration (10), where the $(\delta+1)^2$ general points in the general fiber will degenerate to $(\delta+1)^2$ points in the special fiber, $a = \delta^2$ which can be assumed to be general on the surface $\mathbb{P}$, and $b - 1 = 2\delta + 1$ which can be assumed to be general on the surface $\mathbb{F}$. Since $[I(mZ')]_{(\delta+1)m} \neq 0$ for general Z', we obtain a family of curves and a central curve, as in (9), consisting of a curve $C_{\mathbb{F}}$ of degree $t = (\delta + 1)m$, with some multiplicity e at the blown up point and multiplicity m at the $b - 1 = 2\delta + 1$ points chosen in $\mathbb{F}$, plus a curve $C_{\mathbb{P}}$ of degree e and multiplicity m at the $a = \delta^2$ general points. By hypothesis, if $e \leq \delta m$ such a curve does not exist in $\mathbb{P}$, so it will be enough to prove that the claimed curve in $\mathbb{F}$ does not exist for $e > \delta m$.

Let $\Delta = e - \delta m$, it is positive by hypothesis. It was seen in the preceding exercise that after a sequence of δ Cremona transformations centered at the three biggest multiplicities, a curve of degree $(\delta+1)m$ with multiplicity e at a general point and multiplicity m at further $t-1 = 2\delta+1$ points would give a curve of degree $m - \Delta\delta$ with a point of multiplicity m. Obviously this is impossible, as $\Delta > 0$. □

Proof of Lemma 1.10. We try to apply the same degeneration argument to a general curve of degree $4m$ with 16 assigned points. In this case the only output is that a possible central curve (9) would consist of a curve $C_{\mathbb{F}}$ of degree $4m$, with multiplicity $e = 3m$ at the blown up point and multiplicity m at the 7 points chosen in $\mathbb{F}$, plus a curve $C_{\mathbb{P}}$ of degree $3m$ and multiplicity m at the 9 general points. Such curves do exist. In this case the key point is that they cannot match on E for general points.

Indeed, $C_{\mathbb{P}}$ can only be the unique cubic through the 9 general points taken m times, whereas $C_{\mathbb{F}}$ consists of m curves in the pencil of quartics with a triple point and 7 simple points (this follows from the Cremona transformations as in the previous lemma). These can only match on E if the curve in $\mathbb{F}$ consists of m times one single curve in the pencil of quartics, that matches the cubic of $\mathbb{P}$, i.e., it meets E at the same three points. Now, the pencil of quartics induces a pencil of degree 3 on E, i.e. a non-complete linear series of degree 3. Choose 3 points on E that do not belong to this pencil, choose a cubic $C \subset \mathbb{P}$ through these 3 points, and choose the 9 points on $\mathbb{P}$ as general points of C. Then the matching is not possible; therefore it is not possible for general points either. □

1.4 Generalization to an arbitrary number of points

The Ciliberto-Miranda method works more generally to yield the following.

Theorem 1.12 (Ciliberto-Harbourne-Miranda-Roé [13])**.** *For every $n \geq 10$ there exist multiplicities $\mathbf{v} = (v_1, \dots, v_n)$ such that, if $p_1, \dots, p_n \in \mathbb{P}^2$ are very general points, $Z_{\mathbf{v}} = v_1p_1 + \cdots + v_np_n$ and $\delta = \sqrt{\sum v_i^2}$, then $\alpha(I(mZ_{\mathbf{v}})) > \delta m$ for all $m > 1$. In particular, $\widehat{\alpha}(I(Z_{\mathbf{v}})) = \delta$.*

The method of proof is essentially the same as for Nagata's theorem: there are three initial cases $n = 10, 11, 12$ and an induction step. The $n = 10$ case is slightly more difficult, and we refer the reader to [13] for the complete proof, that uses the same basic principle with a modified degeneration obtained by blowing up the central fiber of $Y \to \mathbb{D}$ along a suitable rational curve. The vector of multiplicities in this case is $\mathbf{v} = (5, 4^9)$ (so that $\delta = 13$), and for very general points $p_1, \dots, p_{10} \in \mathbb{P}^2$, and $Z_{\mathbf{v}} = 5p_1 + 4(p_2 + \cdots + p_{10})$, the inequality $\alpha(I(mZ_{\mathbf{v}})) > \delta m$ holds for all $m \geq 1$. The initial cases $n = 11, 12$ and the induction step are left as exercises:

Exercise 1.6. ($n = 12$) Prove that for $\mathbf{v} = (2^8, 1^4)$ (so that $\delta = 6$), and $Z_{\mathbf{v}} = 2(p_1 + \cdots + p_8) + p_9 + \cdots + p_{12}$ where $p_1, \dots, p_{12} \in \mathbb{P}^2$ are very general points, $\alpha(I(mZ_{\mathbf{v}})) > \delta m$ for all $m \geq 1$.
Hint: use the Ciliberto-Miranda method of the first section, with all four m-fold points on $\mathbb{P}$ and all eight $2m$-fold points on $\mathbb{F}$.

Exercise 1.7. ($n = 11$) Prove that for $\mathbf{v} = (3, 2^{10})$ (so that $\delta = 7$), and $Z = 3p_1 + 2(p_2 + \cdots + p_{11})$ where $p_1, \dots, p_{11} \in \mathbb{P}^2$ are very general points, $\alpha(I(mZ_{\mathbf{v}})) > \delta m$ for all $m \geq 1$.
Hint: use the Ciliberto-Miranda method of the first section, with four of the $2m$-fold points on $\mathbb{P}$ and the rest on $\mathbb{F}$.

Exercise 1.8. (induction step) Let a, b, d be positive integers, and assume the multiplicity vectors $\mathbf{m} = (m_1, \ldots, m_a)$, $\mathbf{n} = (n_1, \ldots, n_b)$ and $\mu = (\mu_1, \ldots, \mu_b)$ satisfy:

1. For $p_1, \ldots, p_a$ very general points in $\mathbb{P}^2$, and letting $Z_{\mathbf{m}} = m_1 p_1 + \cdots + m_a p_a$, then for each $m \geq 1$, one has $\alpha(I(mZ_{\mathbf{m}})) > dm$.
2. For $p_1, \ldots, p_b$ very general points in $\mathbb{P}^2$, and letting $Z_\mu = \mu_1 p_1 + \cdots + \mu_b p_b$, one has $\widehat{\alpha}(I(Z_\mu)) = \delta = \sqrt{\sum \mu_i^2}$.
3. $\sum \mu_i n_i \geq \delta m_1$.

Show that in this case the multiple point scheme $Z_{\mathbf{m}\sharp\mathbf{n}}$ determined by the multiplicity vector $\mathbf{m}\sharp\mathbf{n} = (n_1, \ldots, n_b, m_2, \ldots, m_a)$ at $n = a + b - 1$ very general points satisfies that for each $m \geq 1$, it is $\alpha(I(mZ_{\mathbf{m}\sharp\mathbf{n}})) > dm$.
Hint: Show, using the ideas of Exercise 1.2, that if $\alpha(I_{Z_{\mathbf{n}}}) < m_1$, with $\sum \mu_i n_i \geq \delta m_1$, then $\widehat{\alpha}(I_{Z_\mu}) < \delta$. See also Exercise 2.3.

How general need the points be?

Nagata's theorem can be rephrased in terms of *semi-effective divisors* [31, Section 1.2]. Consider the blow-up $\pi : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_n$, denote by L the pull-back to X of the class of a line, by E_i the class of the exceptional divisor above p_i. A divisor class D is called semi-effective if for some positive integer m one has $H^0(X, \mathcal{O}_X(mD)) \neq 0$, i.e., if for some $m > 0$, the divisor mD is linearly equivalent to an effective divisor. Nagata's theorem is equivalent to the fact that, if $n = \delta^2$ and $p_1, \ldots, p_n$ are very general points, then the divisor $D_\delta = \delta L - E_1 - \cdots - E_n$ is not semi-effective.

Clearly, without some generality assumption, the divisor D_δ can be semi-effective (in fact, it can be effective: it suffices to choose δ^2 points on a curve of degree δ; this shows that the Severi variety $V_{1,\delta}$ is nonempty). Harbourne's notes [31] raise the question of how large the least integer m such that $H^0(X, \mathcal{O}_X(mD)) \neq 0$ can be, when a divisor D is semi-effective; let us consider the case $D = D_\delta$. Looking at the Severi varieties $V_{m,\delta m}$ corresponding to multiples of D_δ, one easily sees that

$$V_{1,\delta} \subset V_{m,\delta m} \subset V_{km,k\delta m}$$

for all m and k (in particular they are nonempty). On the other hand, the naïve expectation (which is a lower bound) for the dimension of the Severi varieties obtained by counting equations is

$$\dim V_{m,\delta m} \geq 2n + \binom{\delta m + 2}{2} - n\binom{m+1}{2} = \frac{(3-\delta)\delta}{2} m + 2\delta^2 + 1,$$

that is a strictly decreasing function of m if $\delta \geq 4$. So one naïvely would *not* expect $V_{km,k\delta m}$ to be strictly larger than $V_{m,\delta m}$ for large m; in other words, it is conceivable that the answer to the following problem is positive:

Open Problem 1.13. *Is there any bound $m_0 = m_0(\delta)$ such that if the points p_i are chosen so that mD_δ is not effective for all $m \leq m_0$, then D is not semi-effective?*

If the answer to this question is positive, then the set of n-tuples of points for which Nagata's theorem is true would be a Zariski open set. One among many consequences that would follow is, for example, that there would be sets of points with coordinates in $\mathbb{Q}$ (or any infinite field) with the Nagata property. Note that Totaro's work [53] shows that there exist sets of 9 points in $\mathbb{P}^2_{\mathbb{Q}}$ with non finitely generated multigraded Rees algebra, but the support in this case spans a closed cone.

1.5 Nagata's conjecture

Conjecture 1.14 (Nagata, 1959)**.** *Let $n \geq 10$ be an integer. If $p_1, \dots, p_n$ are generic points in $\mathbb{P}^2$, and $Z = p_1 + \cdots + p_n$, then $\alpha(I(mZ)) > m\sqrt{n}$ for all $m \geq 1$.*

This statement holds true for n a square, by Nagata's theorem, but it remains open for all other values of n.

Remark 1.15. For nonsquare n the equality $\alpha(I(mZ)) = m\sqrt{n}$ is impossible. Therefore, to prove Nagata's conjecture it is enough to compute the Waldschmidt constants: $\widehat{\alpha}(I(Z)) = \sqrt{n}$ for all $n \geq 10$.

Remark 1.16. Some bounds are known for $\widehat{\alpha}(I(Z))$ that approximate the square root of the number of points. For instance, [32] gives

$$\sqrt{n} \leq \widehat{\alpha}(I(Z)) \leq \sqrt{n}\sqrt{1 + \frac{2}{n^2 - 5n\sqrt{n} - 2}},$$

if $Z = p_1 + \cdots + p_n$ consists of general points. Observe that the upper bound is a worst case estimate, as all known methods for obtaining such bounds give in fact rational numbers.

One can also look at the question in terms of Seshadri constants [19]. The *Seshadri constant* of a set of points $p_1, \dots, p_n$ is defined as

$$\varepsilon(p_1, \dots, p_n) = \inf \left\{ \frac{\deg C}{\sum \operatorname{mult}_{p_i} C} \right\},$$

where the infimum is taken with respect to all plane curves passing through at least one of the points p_i. Equivalently, and denoting as before $\pi : X \to \mathbb{P}^2$ the blow-up at the n points, L the pull-back to X of the class of a line, E_i the class of the exceptional divisor above p_i,

$$\varepsilon(p_1, \dots, p_n) = \sup \{ t \in \mathbb{R} \,|\, L - t(E_1 + \cdots + E_n) \text{ is nef} \}$$

Exercise 1.9. For all choices of $p_1, \dots, p_n \in \mathbb{P}^2$, prove that $\varepsilon(p_1, \dots, p_n) \leq 1/\sqrt{n}$.
Hint: Use Exercise 1.2.

In fact, we will see below that $\varepsilon(p_1, \dots, p_n) = \widehat{\alpha}(I(Z))^{-1}$ for $Z = p_1 + \cdots + p_n$. Therefore, Nagata's conjecture is equivalent to the claim that, for very general points $p_1, \dots, p_n \in \mathbb{P}^2$,

$$\varepsilon(p_1, \dots, p_n) = 1/\sqrt{n}.$$

Following this approach, one can formulate an analogous conjecture for arbitrary surfaces:

Conjecture 1.17 (Biran–Szemberg [52], [38, Remark 5.1.24])**.** *Let X be a smooth projective surface and L be a nef divisor on X. Then there is a positive integer n_0 such that for every $n \geq n_0$, one has $\epsilon(n; X, L) = \sqrt{L^2/n}$.*

Remark 1.18. If there exists a smooth curve of positive genus in the linear system $|kL|$ then $n_0 = k^2 L^2$ is expected to work in Conjecture 1.17.

Summarizing, Nagata's approach to showing that the k-algebra A is not finitely generated has three steps:

1. A is isomorphic to a bigraded ring $\oplus_{m,d} A_{m,d}$ (in fact with $A_{m,d} = [I(mZ)]_d$),
2. the *support* $\mathcal{S} = \{(m,d) \mid A_{m,d} \neq 0\}$ is not finitely generated as a semigroup, because
3. the cone $\mathrm{co}(\mathcal{S}) \subset \mathbb{R}^2$ is not closed.

In these notes, we call *Nagata-type statement* a theorem or conjecture stating that, for a given multigraded algebra A with some geometric meaning (most often a Rees algebra) the cone C in $\mathbb{R}^n$ spanned by the support semigroup of A is not closed. Of course, this implies that A is not finitely generated. In the same spirit, we say that a ray $R = co(v) \subset \mathbb{R}^n$ is a Nagata-type ray if $R \subset \overline{C} \setminus C$.

Exercise 1.10. Show that, if C is a closed cone in $\mathbb{R}^2$, then it is finitely generated. Give an example of a closed cone in $\mathbb{R}^3$ that is not finitely generated. Is it true that if $S \subset \mathbb{Z}^2$ spans a closed cone, then S itself is finitely generated?

2 Conjectures on the cone of curves

In this section we review some conjectures that generalize and strengthen Nagata's conjecture, following ideas from [13]. We fix for most of the section the hypothesis that $p_1, \dots, p_n$ are n very general points of $\mathbb{P}^2$ and consider the blow-up $f : X = X_n \to \mathbb{P}^2$ of the plane at the points $p_1, \dots, p_n$. The object of interest is the geometry of X, from the point of view of Mori theory. More precisely, we shall see what Nagata's conjecture and its generalizations tell about the shape of the Mori cone of X.

2.1 Total coordinate ring of the blown up plane

The *Picard group* $\mathrm{Pic}(X)$ of the blown-up plane $f : X \to \mathbb{P}^2$ is the abelian group freely generated by:

- the *line class*, i.e., the pullback $L = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$;
- the classes of the *exceptional divisors* $E_1, \dots, E_n$ that are contracted to $p_1, \dots, p_n$.

So, $\mathrm{Pic}\, X \cong \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_n \cong \mathbb{Z}^{n+1}$. More generally we consider $\mathbb{Q}$ and $\mathbb{R}$-divisor classes and work in $N^1(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{n+1}$, viewed as a real vector space with its standard Euclidean topology. The real cones spanned by effective (or ample, or nef, etc) divisors are objects of great interest to understand the geometry of X. This approach was pioneered by Kleiman in [36] and is explained in detail in [38, 1.4.C].

A class $\xi \in N^1(X)$ is *integral* (respectively *rational*) if it sits in $\mathrm{Pic}(X)$ (respectively in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$). A ray $\mathrm{co}(\xi)$ in $N^1(X)$ is *rational* if it is generated by a rational class. A rational ray in $N^1(X)$ is *effective* if it is generated by an effective class.

We will use the notation $\mathcal{L} = (d; m_1, \dots, m_n)$ for the complete linear system

$$\mathcal{L} = \left| dL - \sum_{i=1}^n m_i E_i \right| = \mathbb{P}\left(H^0\left(X, \mathcal{O}_X\left(dL - \sum m_i E_i\right)\right)\right)$$

on X. With this convention the integers $d, m_1, \ldots, m_n$ are the components with respect to the ordered basis $(L, -E_1, \ldots, -E_n)$ of $N^1(X)$, and the intersection form on $N^1(X)$ can we written as follows:

$$(d; m_1, \ldots, m_n) \cdot (d'; m'_1, \ldots, m'_n) = dd' - m_1 m'_1 - \cdots - m_n m'_n.$$

We also use the shorthand exponent notation, so that m^k denotes k-fold repetition of the integer m. Thus the *canonical class* on X is $K_n = (-3; -1^n)$ (called K if there is no danger of confusion).

The pull-back by the blow-up map f induces a natural isomorphism for each $(d, \mathbf{m}) \in \mathbb{Z}^{n+1}$

$$H^0\left(X, \mathcal{O}_X\left(dL - \sum m_i E_i\right)\right) \cong [I(Z_{\mathbf{m}})]_d\,, \tag{11}$$

where $Z_{\mathbf{m}} = m_1 p_1 + \cdots + m_n p_n$ as usual. Therefore, there is an isomorphism between the *total coordinate ring* (also called Cox ring) of X

$$\mathcal{TC}(X) = \bigoplus_{L \in \operatorname{Pic} X} H^0(X; L)$$

and the multigraded Rees algebra

$$\bigoplus_{\mathbf{m} \in \mathbb{Z}^n, d \geq 0} [I(Z_{\mathbf{m}})]_d,$$

that by Lemma 1.3 can be identified with the ring of invariants of the unipotent action. The isomorphisms (11) are compatible with product operations on each side, so $\mathcal{TC}(X)$ and $\oplus_{\mathbf{m},d}[I(Z_{\mathbf{m}})]_d$ are isomorphic as graded algebras.

The support semigroup of $\mathcal{TC}(X)$ is by definition $\operatorname{Eff} X$, the semi-group in $\operatorname{Pic} X$ of effective classes on X, that is,

$$\mathcal{S}_K = Supp\left(\bigoplus_{\mathbf{m} \in \mathbb{Z}^n, d \in \mathbb{Z}} I(Z_{\mathbf{m}})\right) \cong Supp(\mathcal{TC}(X)) = \operatorname{Eff} X = \left\{L \in \operatorname{Pic} X | H^0(X; L) \neq 0\right\}.$$

As usual, the algebra $\mathcal{TC}(X)$ is not finitely generated if its support $\operatorname{Eff} X$ is not so as semi-group.

The *Mori cone* $\overline{\mathrm{NE}}(X)$ is the topological closure in $N^1(X) \cong \mathbb{R}^{n+1}$ of the cone

$$\mathrm{NE}(X) = \operatorname{co}(\operatorname{Eff} X)$$

of all effective rays, and it is the dual of the *nef cone* $\operatorname{Nef}(X)$, that is the closed cone described by all nef rays. We have

$$\operatorname{co}(\mathcal{S}_K) \cong \mathrm{NE}(X) = \operatorname{co}(\operatorname{Eff} X) \subset N_1(X).$$

In this language, Nagata's theorem and the generalizations seen in the first section provide a non-closedness result for $\mathrm{NE}(X)$:

Corollary 2.1. *For $n \geq 10$, there exists a rational ray $\operatorname{co}(\xi)$ in $\overline{\mathrm{NE}}(X)$ that is not contained in $\mathrm{NE}(X)$.*

Proof. By Theorem 1.12, for every $n \geq 10$ there exist multiplicities $\mathbf{v} = (v_1, \ldots, v_n)$ such that, if $\delta = \sqrt{\sum v_i^2}$, then $\alpha(I(mZ_{\mathbf{v}})) > \delta m$ for all $m > 1$ and $\widehat{\alpha}(I(Z_{\mathbf{v}})) = \delta$. By the isomorphisms (11), this implies that the ray spanned by $\xi_{\mathbf{v}} = (\delta; v_1, \ldots, v_n)$ lies in $\overline{\mathrm{NE}}(X) \setminus \mathrm{NE}(X)$. □

On the other hand, Nagata's conjecture is equivalent to the following:

Conjecture 2.2. *For $n \geq 10$, the ray spanned by the class $(\sqrt{n}; 1^n)$ is not contained in* $\mathrm{NE}(X)$*.*

We call $\nu_n = \mathrm{co}(\sqrt{n}; 1^n)$ the *Nagata ray.*

Remark 2.3. Corollary 2.1 obviously implies that Eff X is not finitely generated, but it is much stronger. We will see in the next section that, for $n = 9$, the cone $\mathrm{NE}(X) = \overline{\mathrm{NE}}(X)$ is closed, but still Eff X is not finitely generated.

Corollary 2.1 also implies that the ray spanned by $\xi_{\mathbf{v}} = (\delta; v_1, \ldots, v_n)$ is extremal in $\overline{\mathrm{NE}}(X)$. The existence of extremal rays of selfintersection zero on every X_n, $n \geq 9$ was first proved by F. Monserrat in [39]. The following set of exercises leads to a proof of extremality of $\xi_{\mathbf{v}}$.

Let

$$\mathcal{Q} = \mathcal{Q}_n = \left\{ \xi \in N^1(X) \text{ such that } \xi \cdot L \geq 0 \text{ and } \xi^2 \geq 0 \right\} \subset N^1(X) \tag{12}$$

be the *nonnegative cone.* Clearly the ray $\mathrm{co}(\xi_{\mathbf{v}})$ lies on the boundary $\partial\mathcal{Q}$ of $\mathcal{Q}$.

Exercise 2.1. Show that $\mathcal{Q} \subseteq \overline{\mathrm{NE}}(X)$.
Hint: use Exercise 1.2, applied to classes $\xi = (d; m_1, \ldots, m_n)$ with $\xi^2 > 0$ via (11).

Exercise 2.2. Show that if C is an irreducible curve on a surface X with $C^2 < 0$, then $[C]$ belongs to every system of generators of Eff X and the ray $\mathrm{co}([C])$ is extremal in $\overline{\mathrm{NE}}(X)$.
Hint: For every positive d, the unique effective divisor in the complete linear system $|dC|$ is dC.

Exercise 2.3. Show that if a rational class $\xi \in \partial\mathcal{Q}$ and a class $\eta \in \mathrm{NE}(X) \setminus \mathcal{Q}_n$ satisfy $\xi \cdot \eta < 0$ then $\xi \in \mathrm{NE}(X)$.
Hint: Show that there exists an irreducible curve C with $C^2 < 0$ and $C \cdot \xi < 0$. Deduce that for suitable integers d, m, the class $d\xi - m[C]$ lies in the interior of $\mathcal{Q}_n$ and hence in $\mathrm{NE}(X)$. Compare with Exercise 1.8

Exercise 2.4. Show that the class $\xi_{\mathbf{v}} = (\delta; v_1, \ldots, v_n)$ of corollary 2.1 is nef. Deduce that $\xi_{\mathbf{v}}$ is extremal in $\overline{\mathrm{NE}}(X)$.
Hint: If $\xi_{\mathbf{v}}$ were not extremal, then it could be written as a finite sum $\sum a_i[C_i]$ with a_i nonnegative rational numbers and C_i irreducible curves with $C_i^2 < 0$ and $C_i \cdot \xi_{\mathbf{v}} = 0$.

2.2 Mori's cone theorem and consequences

Let $K = K_n = (-3, -1^n)$ be the canonical divisor on X. For any subset $S \subset N^1(X)$, let $S^{\succcurlyeq}$ (respectively $S^{\preccurlyeq}$, $S^{\succ}$ and $S^{\prec}$) be the subset of S consisting of the nonzero classes ξ such that $\xi \cdot K_n \geq 0$ (respectively, $\xi \cdot K_n \leq 0$, $\xi \cdot K_n > 0$ and $\xi \cdot K_n < 0$). Rays in $N^1(X)^{\preccurlyeq}$ spanned by rational curves play a special role in Mori's theory (see theorem 2.4 below). The cone spanned by them is called

$$R_n = \mathrm{co}\left(\{[E] \,|\, E \text{ rational smooth curve with } 0 \leq -E \cdot K_n\}\right) \subseteq \overline{\mathrm{NE}}(X_n)^{\preccurlyeq},$$

or simply $R = R_n$ if there is no danger of confusion. A particularly important case is that of (-1)-*rays* in $N^1(X)$, namely those spanned by the class of a (-1)-*curve*, i.e., a smooth, irreducible, rational curve E with $E^2 = -1$ (hence $E \cdot K_n = -1$ by adjunction and so

$[E] \subset R_n$). Every (-1)-ray $\mathrm{co}([E])$ is effective and extremal in $\overline{\mathrm{NE}}(X)$ (Exercise 2.2). On X_n, $n \geq 2$ the cone R_n is in fact spanned by (-1)-rays:

$$R_n \ = \mathrm{co}\left(\{[E] \,|\, E \text{ a } (-1)\text{-curve}\}\right) \subseteq \ \overline{\mathrm{NE}}(X_n)^{\preccurlyeq}. \tag{13}$$

We now state Mori's cone theorem in the form that is most useful for the rational surface X_n:

Theorem 2.4 (Mori [40], see [38, 1.5F])**.**

$$\overline{\mathrm{NE}}(X_n) = \overline{\mathrm{NE}}(X_n)^{\succcurlyeq} + R.$$

The *negative part* R in the Mori decomposition of $\overline{\mathrm{NE}}(X)$ is well understood, whereas known descriptions of $\overline{\mathrm{NE}}(X)^{\succcurlyeq}$ are only conjectural if $n \geq 10$. We summarize a few facts known about (-1)-curves that appropriately describe R.

Recall from Exercise 1.3 that applying a standard Cremona map based at the points p_1, p_2, p_3 to curve of degree d with multiplicity m_i at the point p_i transforms it into a curve of degree $2d - m_1 - m_2 - m_3$ with multiplicity $d + m_i - m_1 - m_2 - m_3$ at the point p_i. The *arithmetic Cremona transformation* based at the points p_i, p_j, p_k is the automorphism $N^1(X) \longrightarrow N^1(X)$ that maps the class $(d; m_1, \dots, m_n)$ to $(d'; m'_1, \dots, m'_n)$, where

$$\begin{aligned} &d' = 2d - m_i - m_j - m_k, && m'_\ell = m_\ell \ \forall \ell \notin \{i,j,k\} \\ &m'_i = d - m_j - m_k, && m'_j = d - m_i - m_k, && m'_k = d - m_i - m_j. \end{aligned}$$

A divisor D is called 1*–connected* (classically, *virtually connected*) if it is effective and, for every decomposition $D = D_1 + D_2$ where D_1 and D_2 are effective, $D_1 \cdot D_2 > 0$ (cf. [3, Chapter II.12]).

Theorem 2.5 (Hudson-Nagata)**.** *Assume $n \geq 3$. Let $\xi = (d; m_1, \dots, m_n) \in \operatorname{Pic} X$ be a class with $\xi^2 = K \cdot \xi = -1$, with $d \geq 0$ and $m_i \geq 0$. The following are equivalent:*

1. *ξ is the class of a (-1)-curve E.*
2. *ξ is the class of a 1–connected divisor D.*
3. *Recursively applying arithmetic Cremona transformations based at the three points with largest multiplicities, the degree d decreases at each step and the final class is a permutation of the multiplicities in $(0; -1, 0^{n-1})$.*

We refer to [46] and [1, Chapter 5] for proofs. The third equivalent condition in theorem 2.5 is known as "Hudson's test" and can be effectively used to find all (-1) curves of a given degree in X_n.

Exercise 2.5. Show that, if $n \leq 8$, there are finitely many (-1)-curves on X and finitely many Cremona maps whose indeterminacy locus is contained in $\{p_1, \dots, p_n\}$. Specifically, justify the numbers in this table (where $\#(-1)$ denotes the number of (-1)-curves in X_n):

n	3	4	5	6	7	8
$\#(-1)$	6	10	16	27	56	240
$\max \deg(-1)$	1	1	2	2	3	5

Deduce that the locus of sets of $n \leq 8$ points in Cremona-general position is a Zariski open set.
Hint: Start with the class $(0; -1, 0^{n-1})$ and perform all possible arithmetic Cremona transformations that increase the degree. Compute the number of permutations of each class obtained in this way.

It is not hard to see (use the Nakai-Moishezon Criterion, [33, V.1.10]) that for $n \leq 8$, the divisor $-K_n$ is ample, hence $\overline{\mathrm{NE}}(X) \subseteq \overline{\mathrm{NE}}(X)^{\prec}$ and so $\overline{\mathrm{NE}}(X) = R_n$ by Mori's theorem. These are *Del Pezzo* surfaces. As there are only finitely many (-1)-curves on X, the cone $\overline{\mathrm{NE}}(X)$ is polyhedral. If $\kappa_n = \mathrm{co}(3; 1^n)$ is the *anticanonical ray*, then κ_n is in the interior of the nonnegative cone $\mathcal{Q}_n$.

Exercise 2.6. Show that, if $n \geq 9$, there are infinitely many (-1)-curves on X and infinitely many Cremona maps whose indeterminacy locus is contained in $\{p_1, \dots, p_n\}$. Deduce that the locus of sets of points in Cremona-general position is dense but not Zariski open in $(\mathbb{P}^2)^n$.

When $n = 9$, the anticanonical divisor $-K$ is an irreducible curve with self-intersection 0. Hence κ is nef, sits on $\partial\mathcal{Q}$, and the tangent hyperplane to $\partial\mathcal{Q}$ at κ is the hyperplane $\kappa^{\perp}$ of classes ξ such that $\xi \cdot K = 0$. Then $\overline{\mathrm{NE}}(X)^{\succcurlyeq} = \kappa$ and $\overline{\mathrm{NE}}(X) = \kappa + R \subseteq \overline{\mathrm{NE}}(X)^{\preccurlyeq}$. The infinitely many (-1)-curves on X determine infinitely many (-1)-rays, and κ is the only limit ray of the (-1)-rays. The anticanonical ray κ_9 coincides with the Nagata ray ν_9.

Since the classes of all (-1) curves must belong to every system of generators of $\mathrm{Eff}\, X$, it follows that $\mathrm{Eff}\, X$ and the total coordinate ring $\mathcal{TC}(X)$ are not finitely generated as soon as $n \geq 9$. In this case, $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$ is a counterexample to Hilbert's 14-th problem. In fact, it is possible to exhibit explicit configurations of points whose blowup contains infinitely many (-1)-curves. The interested reader can find details in Mukai [43] and Totaro [53] (see also Exercise 2.7 below). Mukai shows, more generally, that when the points $p_1, \dots, p_n$ are sufficiently general in $\mathbb{P}^{r-1}$, the inequality $n \geq \frac{r^2}{r-2}$ is a sufficient condition for the existence of infinitely many negative divisors, and hence for non finite generation of $\mathbb{C}[\mathbf{x}, \mathbf{y}]^K$. Totaro gives specific sets of points that work over every field (including finite fields).

For $n \geq 10$, the shape of $\overline{\mathrm{NE}}(X)$ is not well known. $-K_n$ is not effective, and has negative self-intersection $9 - n$. Hence κ_n lies off the nonnegative cone $\mathcal{Q}_n$, which in turn has nonempty intersection with both $\overline{\mathrm{NE}}(X)^{\succ}$ and $\overline{\mathrm{NE}}(X)^{\prec}$. The rays spanned by the infinitely many (-1)-curves on X lie in $\overline{\mathrm{NE}}(X)^{\prec}$ and their limit rays lie at the intersection of $\partial\mathcal{Q}$ with the hyperplane $\kappa^{\perp}$. The Nagata ray ν sits on $\partial\mathcal{Q}^{\succ}$. The plane joining the rays κ and ν is the *homogeneous slice*, formed by the classes of *homogeneous linear systems* of the form $(d; m^n)$, with $d \geq 0$.

2.3 Nagata-type statements for extremal rays

Nagata's conjecture states a necessary condition for the linear system $\mathcal{L} = (d; m_1, \dots, m_n)$ to be nonempty. In fact there is a stronger conjecture that posits what the dimension of $\mathcal{L}$ should be, and that has also been open for decades, namely the Segre-Harbourne-Gimigliano-Hirschowitz (or SHGH) conjecture (see [50], [29], [27], [34] [12], quoted in chronological order).

Conjecture 2.6 (SHGH Conjecture)**.** *Let $d \geq 0, m_i \geq 0$ be such that $(d; m_1, \dots, m_n) \cdot E \geq 0$ for every (-1)-curve E. Then*

$$\dim \left| dL - \sum_{i=1}^{n} m_i E_i \right| = \max \left\{ -1, \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2} \right\}. \tag{14}$$

This conjecture expresses the expectation that the conditions imposed by the multiple points should be independent, except when the system meets negatively a (-1)-curve, the only known case in which the conditions become dependent. One can compute which linear systems are expected to be nonempty according to the SHGH conjecture, and obtain the following conjecture, first proposed by T. de Fernex in [18].

Conjecture 2.7 (De Fernex conjecture)**.** *If $n \geq 10$, then*

$$\overline{\mathrm{NE}}(X) = \mathcal{Q}^{\succcurlyeq} \ + \ R, \tag{15}$$

where $\mathcal{Q}$ is the nonnegative cone (12) *and R is the negative part in the Mori decomposition of $\overline{\mathrm{NE}}(X)$ (as in* (13)*).*

Let

$$D_n = (\sqrt{n-1}, 1^n) \in N^1(X)$$

be the *de Fernex* class and $\delta_n = \mathrm{co}(D_n)$ the corresponding ray. One has $D_n^2 = -1$, and $D_n \cdot K_n = n - 3\sqrt{n-1} = \frac{n^2-9n+9}{n+3\sqrt{n-1}} > 0$ for $n \geq 8$ and, if $n = 10$, one has $D_n = -K_n$. Set

$$\begin{aligned} \Delta_n^{\succcurlyeq} &= \left\{\xi \in N^1(X) \text{such that } \xi \cdot D_n \geq 0\right\} \\ \Delta_n^{\preccurlyeq} &= \left\{\xi \in N^1(X) \text{such that } \xi \cdot D_n \leq 0\right\}. \end{aligned} \tag{16}$$

Theorem 2.8 (de Fernex [18])**.** *If $n \geq 10$ then:*

(i) all (-1)-rays lie in the cone $\mathcal{D}_n := \mathcal{Q}_n - \delta_n$;

(ii) if $n = 10$, all (-1)-rays lie on the boundary of the cone $\mathcal{D}_n$;

(iii) if $n > 10$, all (-1)-rays lie in the complement of the cone $\kappa_n := \mathcal{Q}_n - \kappa_n$;

(iv) $\overline{\mathrm{NE}}(X) \subseteq \overline{\kappa + R}$;

(v) if Conjecture 2.7 holds, then

$$\overline{\mathrm{NE}}(X) \cap \Delta_n^{\preccurlyeq} = \mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}. \tag{17}$$

Remark 2.9. As noted in [18], Conjecture 2.7 does not imply that $\overline{\mathrm{NE}}(X)^{\succcurlyeq} = \mathcal{Q}^{\succcurlyeq}$, unless $n = 10$, in which case this is exactly what it says (see Theorem 2.8(v)). Conjecture 2.7 does imply Nagata's conjecture, and is consistent with what is known about the boundary of $\overline{\mathrm{NE}}(X)$, like Theorem 1.12.

Proof. All statements except (iv) are computations that can be done as exercises; [18] contains all the details. For (iv), de Fernex uses a specialization argument, showing that the claim holdes when the points are very general on an irreducible cubic curve; if there exists an effective integral class outside $\kappa + R$ for very general position of the points, then it also exists for points on a cubic curve. Note that an irreducible cubic through all points has class $(3; 1^n) = -K_n$, that is fixed by arithmetic Cremona transformations, and it follows from this that very general choices of points on an irreducible cubic are Cremona general (see Exercise 2.7 below); so R_n stays the same before and after specializing to the cubic.

For points on an irreducible cubic, the strict transform of this cubic is the only irreducible curve C with $[C] \in \overline{\mathrm{NE}}(X)^{\succ}$. On the other hand, there are no irreducible curves D with

$D\cdot K_n = D\cdot C = 0$, because such a curve would have class $(d; m_1,\dots,m_n)$ with $3d=\sum m_i$, so
$$\mathcal{O}_X(D)|_C = \mathcal{O}_C(dL|_C - m_1p_1 - \cdots - m_np_n)$$
would be an effective line bundle of degree 0, that by the choice of the $n\geq 10$ points would be general in $\operatorname{Pic} C$, therefore non-effective (C is of genus 1), a contradiction.

Thus $\mathrm{NE}(X)\subseteq \kappa_n + R_n$ and the claim follows. □

Exercise 2.7. Show that if C is an irreducible cubic curve with multiplicity 1 at each of n points $p_1,\dots,p_n$, then its transform by a standard Cremona map based at any three of the n points is again an irreducible cubic with multiplicity 1 at each of the resulting n points. Deduce that for every n there are subsets of n points in C that are Cremona general.
Hint: The Cremona transform of a line through 3 points cuts on C an effective divisor of degree 0.

Conjecture 2.10 (Δ-conjecture, [13])**.** *Let $\mathcal{Q}_n$ be the nonnegative cone* (12) *and $\Delta_n^{\preccurlyeq}$ as in* (16)*. If $n\geq 10$ then*
$$\partial\mathcal{Q}_n\cap\Delta_n^{\preccurlyeq}\subset \mathrm{Nef}(X). \tag{18}$$

Proposition 2.11. *If the Δ-conjecture holds, then*
$$\overline{\mathrm{NE}}(X)\cap\Delta_n^{\preccurlyeq} = \mathrm{Nef}(X)\cap\Delta_n^{\preccurlyeq} = \mathcal{Q}_n\cap\Delta_n^{\preccurlyeq}. \tag{19}$$

Proof. By (18) and by convexity of $\mathrm{Nef}(X)$ one has
$$\mathcal{Q}_n\cap\Delta_n^{\preccurlyeq}\subseteq \mathrm{Nef}(X)\cap\Delta_n^{\preccurlyeq}.$$
Moreover $\mathrm{Nef}(X)\cap\Delta_n^{\preccurlyeq}\subseteq\overline{\mathrm{NE}}(X)\cap\Delta_n^{\preccurlyeq}$. Finally (18) implies (17) because $\overline{\mathrm{NE}}(X)$ is dual to $\mathrm{Nef}(X)$. □

The following proposition shows that Nagata-type conjectures we are discussing here can be interpreted as asymptotic forms of the SHGH conjecture.

Proposition 2.12. *Let $n\geq 10$. If the Δ-conjecture holds, then all classes in $\mathcal{Q}_n\cap\Delta_n^{\preccurlyeq}-\partial\mathcal{Q}_n\cap\Delta_n^{\preccurlyeq}$ are ample and therefore, if integral, there is an integer y such that for all nonnegative integers $x\geq y$ the dimension of $(xd; xm_1,\dots,xm_n)$ is given by* (14)*.*

Proof. It follows from Proposition 2.11 and the fact that the ample cone is the interior of the nef cone (by Kleiman's theorem, see [36]). □

One can give a stronger form of the Δ-conjecture.

Lemma 2.13. *Any rational, non-effective ray in $\partial\mathcal{Q}_n$ is nef and it is extremal for both $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$. Moreover it lies in $\partial\mathcal{Q}_n^{\succcurlyeq}$.*

Proof. That such a ray is nef and extremal for $\overline{\mathrm{NE}}(X)$ was proved in Exercise 2.4. The duality between $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ shows that it is also extremal in $\mathrm{Nef}(X)$. The final assertion follows by Mori's cone theorem. □

A rational, non-effective ray in $\partial\mathcal{Q}_n$ will be called a *good ray.* An irrational, nef ray in $\partial\mathcal{Q}_n$ will be called a *wonderful ray.* No wonderful ray has been detected so far.

The following conjecture implies the Δ-conjecture.

Conjecture 2.14 (Strong Δ-conjecture). *If $n > 10$, all rational rays in $\partial\mathcal{Q}_n \cap \Delta_n^{\preccurlyeq}$ are non-effective. If $n = 10$, a rational ray in $\mathcal{Q}_{10} \cap \Delta_{10}^{\preccurlyeq} = \mathcal{Q}_{10}^{\succcurlyeq}$ is non–effective, unless it is generated by a Cremona transform of the curve with class $(3; 1^9, 0)$.*

Proposition 2.15. *For $n = 10$, the strong Δ-conjecture is equivalent to the following statement ("Strong Nagata conjecture"): If C is an irreducible curve of genus $g > 0$ on X, then $C^2 > 0$ unless $n \geq 9$, $g = 1$ and C is a Cremona transform of the curve with class $(3; 1^9, 0^{n-9})$, in which case $C^2 = 0$.*

Proof. If the strong Δ-conjecture holds, then clearly the Strong Nagata conjecture holds. Conversely, consider a rational effective ray in $\partial\mathcal{Q}_{10}^{\succcurlyeq}$ and let C be an effective divisor in the ray. Then $C = n_1C_1 + \cdots + n_hC_h$, with $C_1, \ldots, C_h$ distinct irreducible curves and $n_1, \ldots, n_h$ positive integers. One has $C_i \cdot C_j \geq 0$, hence $C_i \cdot C_j = 0$ for all $1 \leq i \leq j \leq h$. This clearly implies $h = 1$, hence the assertion. $\square$

By the proof of Proposition 2.11, any good ray gives a constraint on $\overline{\mathrm{NE}}(X)$, so it is useful to find good rays. Even better would be to find wonderful rays.

Example 2.16. Consider the family of linear systems

$$\mathcal{B} = \{B_{q,p} := (9q^2 + p^2; 9q^2 - p^2, (2qp)^9) : (q, p) \in \mathbb{N}^2, q \leq p\}$$

generating rays in $\partial\mathcal{Q}_{10}^{\succcurlyeq}$. Take a sequence $\{(q_n, p_n)\}_{n\in\mathbb{N}}$ such that $\lim_n \frac{p_n+q_n}{p_n} = \sqrt{10}$. For instance take $\frac{p_n+q_n}{p_n}$ to be the convergents of the periodic continued fraction expansion of $\sqrt{10} = [3; \overline{6}]$, so that

$$p_1 = 2,\ p_2 = 13,\ p_3 = 80, \ldots \quad q_1 = 1,\ q_2 = 6,\ q_3 = 37, \ldots.$$

The sequence of rays $\{[B_{q_n,p_n}]\}_{n\in\mathbb{N}}$ converges to the Nagata ray ν_{10}. If we knew that the rays of this sequence are good, this would imply Nagata's conjecture for $n = 10$.

2.4 When does finite generation hold?

We have seen that the blow-up $f : X = X_n \to \mathbb{P}^2$ of the plane at very general points $p_1, \ldots, p_n$ has finitely generated (i.e., polyhedral) Mori cone $\overline{\mathrm{NE}}(X)$ if and only if $n \leq 8$. Although the main focus of these notes is on Nagata type rays, i.e., on non finitely generated cases, it should be mentioned that characterizing the sets of points $p_1, \ldots, p_n \in \mathbb{P}^2$ such that the Mori cone (respectively, the effective semigroup $\mathrm{Eff}\, X$, the Cox ring $\mathcal{TC}(X)$) of the blow-up is finitely generated, and studying these particular surfaces, is an important and active area of research. With no attempt at being comprehensive, we now review a few results in this area. In this section we drop the assumption that the points p_i are general.

On any blowup of $n \leq 9$ points, the anticanonical divisor $-K = (3; 1^n)$ is effective. This puts great restrictions on curves C with negative selfintersection; namely, by adjunction we have that the genus g of such a curve satisfies

$$C^2 + KC = 2g - 2 \geq -2,$$

so the inequality $(-K) \cdot C \geq 0$ (which holds unless C is a fixed component of $|-K|$) implies that $C^2 \geq -2$ and every curve with negative selfintersection is rational. Observe that this will continue to hold for $n \geq 10$ points, as long as the anticanonical divisor is effective, i.e., the points lie on a (possibly reducible) cubic curve. Note that this is essentially the same idea used in the proof of Theorem 2.8, and it will be thoroughly exploited in the third section.

If $n \leq 8$, or more generally, if $\dim |-K| > 0$, every curve C that is not a fixed component of $|-K|$ must have $(-K) \cdot C > 0$. In this case the only curves with negative selfintersection are the fixed components of $|-K|$ and the (-1)-curves.

Using these facts, it is not hard to prove the following:

Proposition 2.17. *The blow-up X of $\mathbb{P}^2$ at an arbitrary set of $n \leq 8$ points or at a set of $n \geq 9$ points lying on a conic has finitely generated* $\operatorname{Eff} X$ *and* $\overline{\mathrm{NE}}(X)$*.*

In fact under the conditions of the proposition more can be said: B. Harbourne computed the dimension of all linear systems $(d; m_1, \ldots, m_n)$, i.e., the Hilbert functions of all $I(Z_{\mathbf{m}})$, and even their graded free resolutions, in [30].

In the cases when $-K$ is effective (or some multiple $-mK$ is effective, i.e., on a Coble surface) but fixed, there is to the best of our knowledge no complete characterization of the sets of points that give finitely generated Mori cones; see however [2], [9], [11], [25] and references therein. A few of these works care also about the finite generation of the total coordinate ring $\mathcal{TC}(X)$; this is in itself an interesting problem, and it turns out that there are special blow-ups of $\mathbb{P}^2$ where no multiple of $-K$ is effective and yet the total coordinate ring is finitely generated [25].

Finally, let us also mention that by a result of Nikulin [47], the surfaces with polyhedral Mori cone whose generating curves have bounded degree and genus can be classified.

3 Conjectures on valuations

3.1 Valuations and good rays

We now move to a slightly different setting, namely blowups of $\mathbb{P}^2$ determined by some particular valuations, and finite generation questions on them. We refer to the references O. Zariski–P. Samuel [56, Chapter VI. and Appendix 5.] and E. Casas–Alvero [10, Chapter 8] for the general theory of valuations and complete ideals on surfaces.

A rank 1 valuation on a domain R is a map

$$v : R \to \mathbb{R} \cup \{\infty\}$$

satisfying

$$v(fg) = v(f) + v(g), \qquad v(f+g) \geq \min(v(f), v(g)), \qquad v(f) = \infty \Leftrightarrow f = 0, \tag{20}$$

for all $f, g \in R$. Note that a valuation on a domain R determines a unique valuation on its quotient field K by setting $v(f/g) = v(f) - v(g)$, and conversely a valuation on a field K restricts to a valuation on any subring $R \subset K$. The *value group* of v is $v(K^*) \subset \mathbb{R}$, a subgroup of the additive group of $\mathbb{R}$. We will be mostly interested in the case $K = \mathbb{C}(x, y)$, and we only consider valuations with trivial restriction to $\mathbb{C}$, that is $v(w) = 0 \ \forall w \in \mathbb{C}$.

Given a valuation $v : K \to \mathbb{R} \cup \{\infty\}$, the set of elements $f \in K$ with $v(f) \geq 0$ is a subring $R_v \subset K$ called the *valuation ring* of v. Valuation rings are characterized as those subrings $S \subset K$ such that, for every $f \in K$, either $f \in S$ or $f^{-1} \in S$. Every valuation ring R_v is a local ring, with maximal ideal $\mathfrak{m}_v$ consisting of those elements with positive value. Except when the value group is discrete (i.e., there is $a \in \mathbb{R}$ such that $v(K^*) = \mathbb{Z}a$), valuation rings are not noetherian.

Valuation ideals and volume

We are interested in valuations on the field $\mathbb{C}(x,y)$ of rational functions on $\mathbb{P}^2$. Choose homogeneous coordinates w_1,w_2,w_3 on $\mathbb{P}^2$, in such a way that $x=w_2/w_1, y=w_3/w_1$. Given a valuation v on $\mathbb{C}(x,y)$ it is possible to extend it to a nonnegative valuation v on the ring $R=\mathbb{C}[w_1,w_2,w_3]$ as follows. If $v(x)\geq 0$, $v(y)\geq 0$, then one simply sets $v(F_d(w_1,w_2,w_3)) = v(F_d(1,x,y))$. Otherwise let $v_{\min}=\min(v(x),v(y))<0$, and set

$$v(F_d(w_1,w_2,w_3)) = v(F_d(1,x,y)) - dv_{\min}.$$

In particular for instance $v(w_1)=-v_{\min}$ and $\min\{v(w_1),v(w_2),v(w_3)\}=0$. Then the definition extends to nonhomogeneous polynomials as $v(F)=\min\big\{v(F_d)\big\}$ for any $F=\sum F_d$ in R, where F_d is the homogeneous degree d part of F. For every non-negative $m\in\mathbb{R}$, the homogeneous ideals

$$I_m = \{F\in R\,|\,v(F)\geq m\},\quad\text{and}\quad I_m^+ = \{F\in R\,|\,v(F)>m\}$$

are called *valuation ideals.* They form multiplicative filtrations, that is $I_m^+\subset I_m\subset I_{m'}^+\subset I_{m'}$ whenever $m'>m$, moreover $I_mI_{m'}\subset I_{m+m'}$, and $I_mI_{m'}^+\subset I_{m+m'}^+$. Recall from previous sections the notation $\alpha(I)=\min\{d|I_d\neq 0\}$ whenever I is a graded ideal, and consider the number

$$\mu_d(v) = \max\{m\in\mathbb{Z}\,|\,[I_m]_d\neq 0\} = \max\{m\in\mathbb{Z}\,|\,\alpha(I_m)\leq d\}.$$

Exercise 3.1. The limits

$$\widehat{\alpha}(v)=\lim_{m\to\infty}\frac{\alpha(I_m)}{m},\qquad \widehat{\alpha}^+(v)=\lim_{m\to\infty}\frac{\alpha(I_m^+)}{m},\qquad \widehat{\mu}(v)=\lim_{d\to\infty}\frac{\mu_d(v)}{d}$$

exist and $\widehat{\alpha}(v)=\widehat{\alpha}^+(v)=\widehat{\mu}(v)^{-1}$. The number $\widehat{\alpha}(v)$ is called the *Waldschmidt constant* of v Hint: Look at Exercise 1.3.3(d) in Harbourne's notes [31].

The description above of valuation ideals on $\mathbb{P}^2$ is a particular instance of a more general construction. Let X be a projective algebraic variety and $v: K(X)\to\mathbb{R}\cup\{\infty\}$ a rank 1 valuation on the field of rational functions of X. Then, for every nonnegative $m\in\mathbb{R}$ one has valuation ideal sheaves

$$\mathcal{I}_m = \left(f\in\mathcal{O}_X\,|\,v(f)\geq m\right),\quad\text{and}\quad \mathcal{I}_m^+ = \left(f\in\mathcal{O}_X\,|\,v(f)>m\right),$$

and for every divisor class D, graded ideals I_m, I_m^+ in the graded ring $\oplus_{k\geq 0}H^0(X,kD)$. The definitions of $\widehat{\alpha}$, $\widehat{\alpha}^+$ and $\widehat{\mu}$ also carry over to this setting.

Exercise 3.2. Work out the details of the previous sheaf-theoretic definitions. More precisely:

1. For every affine open set $U\subset X$, let $R_U=\Gamma(\mathcal{O}_X,U)$. Check that the valuation v restricts to a valuation of R_U.

2. If there is $f\in R_U$ with $v(f)<0$ then set $I_{U,m}=I_{U,m}^+=R_U$ for every $m\geq 0$; otherwise the set of elements in R_U with value greater than or equal to (respectively, greater than) m is an ideal $I_{U,m}$ (respectively, $I_{U,m}^+$) of R_U.

3. Gluing: the data $U\mapsto I_{U,m}$ (respectively $U\mapsto I_{U,m}^+$) define a subsheaf $\mathcal{I}_m$ (respectively $\mathcal{I}_m^+$) of the structure sheaf $\mathcal{O}_X$.

4. If there is no $f \in R_U$ with $v(f) < 0$ then $I^+_{U,0}$ is a proper prime ideal of R_U.

By part 4 of Exercise 3.2, the sheaf $\mathcal{I}^+_0$ determines an irreducible proper subvariety of X, called *the center* of the valuation v on X, and denoted by $\mathrm{center}_X(v) = \mathrm{center}(v)$. Set $R_v = \{f \in K(X) \,|\, v(f) \geq 0\}$ the valuation ring of v, the generic point $\eta = \eta_{\mathrm{center}_X(v)}$ of the center is the image of the closed point of R_v under the unique map $\operatorname{Spec} R_v \to X$ that exists by the valuative criterion of properness [33, II.4.7]. Therefore $\mathrm{center}(X)$ is nonempty, and v is nonnegative on the local ring $\mathcal{O}_{X,\eta}$.

Exercise 3.3. Work out the details of the graded valuation ideals, and check that for the plane $\mathbb{P}^2$ they agree with the former definitions. More precisely:

1. Every trivializing open subset $U \subset X$ for $\mathcal{O}_X(D)$ is also trivializing for $\mathcal{O}_X(kD)$.

2. Via the induced maps $\Gamma(\mathcal{O}_X(kD), U) \xrightarrow{\sim} \Gamma(\mathcal{O}_X, U)$, the valuation v determines a valuation $v_{D,U}$ on $R(D) = \oplus_{k\geq 0} H^0(X, kD)$.

3. If U is a neighborhood of $\mathrm{center}(v)$, then the valuation $v_{D,U}$ is nonnegative on $R(D)$, independent on the choice of U. Denote it by v_D.

4. For every m, the spaces $I_{k,m} = H^0(X, \mathcal{I}_m \otimes \mathcal{O}_X(kD))$ are the graded pieces of an ideal $I_m = \oplus_{k\geq 0} I_{k,m} \subset R(D)$, and
$$I_m = \left\{ s \in \bigoplus_{k\geq 0} H^0(X, kD) \,|\, v_D(s) \geq m \right\}.$$

In particular, by the preceding exercise, given any Cartier divisor D on X its valuation $v(D)$ is well defined: it equals the valuation of any local equation of D on a neighborhood of $\mathrm{center}_X(v)$. We use this fact without further mention in the sequel.

For a valuation v with zero-dimensional center on an n-dimensional variety X, the volume was defined in [21] as
$$\mathrm{vol}(v) := \lim_{m\to\infty} \frac{\dim_{\mathbb{C}}(\mathcal{O}_X/\mathcal{I}_m)}{m^n/n!}$$
(note that $\mathcal{O}_X/\mathcal{I}_m$ is an artinian $\mathbb{C}$-algebra supported at the center of the valuation). On the other hand, the volume of a divisor class D on X is defined as
$$\mathrm{vol}(D) := \limsup_{k\to\infty} \frac{h^0(S, kD)}{k^n/n!}.$$

Boucksom-Küronya-MacLean-Szemberg [7] show that the limit
$$\widehat{\alpha}_D(v) = \lim_{m\to\infty} \frac{\min\{k \in \mathbb{Z} \,|\, I_{k,m} \neq 0\}}{m}$$
exists (generalizing Exercise 3.1) and can be bounded in terms of volumes:

Proposition 3.1 ([7, Proposition 2.9]). *Let D be a big divisor and v a real valuation centered at a point $p \in X$. Then*
$$\widehat{\alpha}_D(v) \leq \sqrt[n]{\mathrm{vol}(v)/\,\mathrm{vol}(D)}\ .$$

When D is ample this bound is equivalent to $\widehat{\alpha}_D(v) \leq \sqrt[n]{\mathrm{vol}(v)/D^n}$. The interested reader will find in section 3.4 below a hint (Exercise 3.10) for the proof of this result in the particular cases of interest to us. Valuations satisfying the equality in Proposition 3.1 will be called *maximal*.

We are especially interested in finding maximal valuations with respect to a line $D = L \subset X = \mathbb{P}^2$. Analogously to the previous sections, we may consider the support semigroup

$$Supp_v(\bigoplus H^0(X, kD)) = \{(k, m) \in \mathbb{Z}^2 \mid I_{k,m} \neq 0\} \subset \mathbb{Z}^2 \subset \mathbb{R}^2$$

and the cone spanned by it:

$$\mathrm{co}(v(D)) = \mathrm{co}\left(Supp_v\left(\bigoplus H^0(X, kD)\right)\right) \subset \mathbb{R}^2 \ .$$

As a planar cone, $\mathrm{co}(v(D))$ has two boundary rays: $\mathrm{co}(1, 0)$ and $\mathrm{co}(\widehat{\alpha}_D(v), 1)$. If the valuation v is maximal, the latter may be a *good ray*, that is, it may happen that

$$\mathrm{co}(\widehat{\alpha}_D(v), 1) \subset \overline{\mathrm{co}(v(D))} \setminus \mathrm{co}(v(D)),$$

and in that case $v(s) < k/\widehat{\alpha}_D(v)$ for all $s \in H^0(X, kD)$, i.e., a Nagata-type statement holds. Hence our interest in maximal valuations on the projective plane.

3.2 The space of valuations with given center

If X is a surface and v is a valuation on $K(X)$, whose center is not a closed point, then either $\mathrm{center}(v) = X$, in which case v is the trivial valuation ($v(f) = 1\ \forall f \neq 0$) or $\mathrm{center}(v) = C$ is a curve. In the latter case, let $p \in C \subset X$ be any point on C and assume $f \in \mathcal{O}_{X,p}$ is a germ of equation for C. Then v is non-negative on $\mathcal{O}_{X,\eta}$, where η is the generic point of C, and hence on $\mathcal{O}_{X,p} \subset \mathcal{O}_{X,\eta}$, and therefore $v(u) = 0$ for every invertible element u of $\mathcal{O}_{X,p}$. For any $g \in \mathcal{O}_{X,p}$ one can write $g = g' f^s$ for some g' invertible in $\mathcal{O}_{X,p}$ and some non-negative integer s, and therefore $v(g) = sv(f)$. Thus, whenever $\mathrm{center}(v)$ is a curve C, the valuation v is (up to a constant $c = v(f) \in \mathbb{R}$) the order of vanishing along C; i.e., for every divisor D, one has $v(D) = c \cdot \mathrm{ord}_C D = c \cdot \max\{k \mid D - kC \geq 0\}$. These are called *divisorial valuations.*

Henceforth we focus in the case that the center of v is a closed point $p \in X$. Such valuations are non-negative on the local ring $\mathcal{O}_{X,p}$, i.e., they restrict to maps $v : \mathcal{O}_{X,p} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying (20). The minimal strictly positive value of v on $\mathcal{O}_{X,p}$ is called the *value* of v at p, $v(p)$; it is the common value of *general* elements in the maximal ideal $\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ [10, 8.1]. An example of a valuation with zero-dimensional center is the order of vanishing at p, that can be also obtained blowing up X at p, and considering the divisorial valuation centered on the exceptional divisor.

Example 3.2 (Monomial valuations)**.** Fix affine coordinates (x, y) near $\mathrm{center}(v) = p = (0, 0) \in \mathbb{A}^2 = \mathrm{Spec}\,\mathbb{C}[x, y] \subset \mathbb{P}^2 = \mathrm{Proj}\,\mathbb{C}[w_1, w_2, w_3]$, with $x = w_2/w_1, y = w_3/w_1$. Given two nonnegative real numbers s, t, we can define a valuation on $\mathbb{C}[x, y]$ by setting

$$v_{s,t}\left(\sum_{\substack{i,j\geq 0\\ i+j\leq d}} a_{ij}x^iy^j\right) = \min\{si + tj | a_{ij} \neq 0\}.$$

As particular cases we obtain that $v_{0,0}$ is the trivial valuation; $v_{s,0}$ is s times the divisorial valuation centered on the line $x = 0$; while $v_{0,t}$ is t times the divisorial valuation centered on the line $y = 0$; and $v_{1,1}$ is the order of vanishing at p. Whenever $s \cdot t > 0$, the center of $v_{s,t}$ is $p = (0, 0)$.

Remark that for every $\lambda > 0$, one has $v_{\lambda s,\lambda t} = \lambda v_{s,t}$, hence there is an equality of valuation rings $R_{v_{\lambda s,\lambda t}} = R_{v_{s,t}}$. Two valuations v, v' with the same valuation ring are called *equivalent*.

Example 3.3 (Quasimonomial valuations)**.** Let $u, w \in \mathbb{C}[x, y]$ be a system of parameters for p, i.e.,

$$(u, w)\mathcal{O}_{X,p} = \mathfrak{m}_{X,p},$$

or in other words, the curves $\{u = 0\}$ and $\{w = 0\}$ meet transversely at $p = (0, 0)$. Then, every element f in $\mathcal{O}_{X,p}$ (or in its completion $\widehat{\mathcal{O}_{X,p}}$, or in the polynomial ring $\mathbb{C}[x, y]$) has a Taylor expansion

$$f = \sum_{i,j\geq 0} a_{ij} u^i w^j,$$

and we can define $v^{u,w}_{s,t}(f) = \min\{si + tj | a_{ij} \neq 0\}$. Again one obtains as extreme cases the divisorial valuations associated to the curves $\{u = 0\}$ and $\{w = 0\}$, and for positive parameters the valuations obtained have center at p. Note that this construction is possible on every smooth point of a surface X.

Exercise 3.4. Assume $s, t > 0$ and let $\mathcal{I}_m$ be the valuation ideal with respect to the valuation $v^{u,w}_{s,t}$. Show that $\mathcal{I}_m$ has cosupport at the point p. Let $I_{m,p}$ be the stalk at p of $\mathcal{I}_m$. Show that the set of classes $\{[u^i w^j]\}_{si+tj<m}$ form a basis of $\mathcal{O}_{X,p}/I_{m,p}$ as a $\mathbb{C}$-vector space. Deduce that $\operatorname{vol} v^{u,w}_{s,t} = 1/st$.

Proposition 3.4. *Let $u_1, u_2, w_1, w_2 \in \mathcal{O}_{X,p}$ and $t > s > 0$. Assume that*

1. *$(u_1, w_1) = (u_1, w_2) = (u_2, w_1) = (u_2, w_2) = \mathfrak{m}_{X,p}$, i.e., each pair (u_i, w_j) is a system of parameters;*
2. *$\dim_{\mathbb{C}} \mathcal{O}_{X,p}/(w_1, w_2) \geq t/s$.*

Then $v^{u_1,w_1}_{s,t} = v^{u_1,w_2}_{s,t} = v^{u_2,w_1}_{s,t} = v^{u_2,w_2}_{s,t}$.

Note that the second hypothesis means that the intersection multiplicity of $\{w_1 = 0\}$ and $\{w_2 = 0\}$ at p is at least t/s; given that both $\{w_1 = 0\}$ and $\{w_2 = 0\}$ are smooth germs of curves at p by the first hypothesis, this is equivalent to saying that the $\lceil t/s\rceil$-jets of w_1 and w_2 coincide, i.e., $w_1 - w_2 \in \mathfrak{m}^{\lceil t/s\rceil}_{X,p}$. Thus Proposition 3.4 says that whenever $t > s$, the valuation $v^{u,w}_{s,t}$ does not depend on the choice of u, and it only depends on the $\lceil t/s\rceil$-jet of w.

Proof of Proposition 3.4. By the first hypothesis, there is a series $h(w_1) = \sum_{i\geq 1} a_i w_1^i$ with $u_2 = u_1 + h(w_1)$. Since $t > s$, then $v^{u_1,w_1}_{s,t}(h(w_1)) \geq v^{u_1,w_1}_{s,t}(w_1) > v^{u_1,w_1}_{s,t}(u_1)$, and

$$v^{u_1,w_1}_{s,t}(u_1) = v^{u_1,w_1}_{s,t}(u_1 + h(w_1)) = v^{u_1,w_1}_{s,t}(u_2).$$

Therefore, for every $f \in \mathcal{O}_{X,p}$, the Taylor expansions of f with respect to (u_1, w_1) and (u_2, w_1) are related by

$$f = \sum_{i,j\geq 0} a_{ij} u_2^i w_1^j = \sum_{i,j\geq 0} a_{ij}(u_1 + h(w_1))^i w_1^j = \sum_{i,j\geq 0} a_{ij} u_1^i w_1^j + \text{ terms with higher } v^{u_1,w_1}_{s,t}.$$

By the definition of quasimonomial valuations, it follows that $v^{u_1,w_1}_{s,t} = v^{u_2,w_1}_{s,t}$ and so also $v^{u_1,w_2}_{s,t} = v^{u_2,w_2}_{s,t}$.

On the other hand, the second hypothesis implies that there is some series $h(u_1) = \sum_{i\geq\lceil t/s\rceil} a_i u_1^i$ with $w_2 = w_1 + h(u_1)$. As before, this implies that $v_{s,t}^{u_1,v_1}(w_1) = v_{s,t}^{u_1,v_1}(w_2)$, and plugging $w_2 = w_1 + h(u_1)$ into the Taylor series of any f, the equality $v_{s,t}^{u_1,w_1} = v_{s,t}^{u_1,w_2}$. We leave the details to the reader. $\square$

Valuative trees

Our next goal is to describe the space of all equivalence classes of quasimonomial valuations of $\mathcal{O}_{X,p}$, in the spirit of [23]. In order to avoid dealing with equivalent valuations, we *normalize* them in such a way that the minimum strictly positive value of $f \in \mathcal{O}_{X,p}$ is 1 (i.e., the value of v at p, $v(p) = 1$). For $v_{s,t}^{u,w}$, this minimal value is $\min\{s, t\}$. Fix a system of parameters $(x, y) \in \mathcal{O}_{X,p}$ (for $X = \mathbb{P}^2$, we set x, y to be local affine coordinates). Set

$$\begin{aligned}
\mathcal{Q} &= \{\text{ quasimonomial valuations centered at } p\}/\text{equiv} \\
\mathcal{Q}_x &= \{v \in \mathcal{Q} \text{ such that } v(x) = v(p)\} \\
\mathcal{Q}_y &= \{v \in \mathcal{Q} \text{ such that } v(y) = v(p)\}\,.
\end{aligned} \tag{21}$$

Note that $v(p) = \min\{v(x), v(y)\}$, so $\mathcal{Q} = \mathcal{Q}_x \cup \mathcal{Q}_y$.

Exercise 3.5. Let $\xi = \xi(x)$ be a formal power series in x and define, for every $f \in \mathcal{O}_{X,p}$,

$$v_{\xi,t}(f) := \operatorname{ord}_x(f(x, \xi(x) + \theta x^t))\ , \tag{22}$$

where the symbol θ is transcendental over $\mathbb{C}$. Show that $v_{\xi,t}$ is a valuation of $\mathcal{O}_{X,p}$.

Let $w \in \mathcal{O}_{X,p}$ be such that $w = 0$ is not tangent to $x = 0$ at the point $p = (0, 0)$. Expand w as a Taylor series or polynomial, $w = w(x, y)$. By the implicit function theorem, there is a convergent power series $\xi(x)$ such that $w(x, \xi(x)) = 0$. Show that for this ξ, one has $v_{\xi,t} = v_{1,t}^{x,w}$.

Theorem 3.5. *Fix a system of parameters $(x, y) \in \mathcal{O}_{X,p}$.*

1. *For every $\xi = \xi(x)$ a formal power series in x, the map*

$$\begin{aligned}
\mathbb{R}_{\geq 1} &\xrightarrow{v_\xi} \mathcal{Q}_x \\
t &\longmapsto v_{\xi,t}
\end{aligned}$$

   is injective, and for every $f \in \mathcal{O}_{X,p}$ the map $t \mapsto v_{\xi,t}(f)$ is continuous.

2. *$v_{\xi_1,t_1} = v_{\xi_2,t_2}$ if and only if $t_1 = t_2$ and $\operatorname{ord}_x(\xi_1 - \xi_2) \geq t_1$.*

3. *For every $v \in \mathcal{Q}_x$, there exist ξ and t such that $v = v_{\xi,t}$.*

Proof. We will show that v_ξ is injective in the interval $[1, n]$ for every positive integer n. It follows that it is injective in the whole half line. Let $\xi(x) = \sum_{i=1}^{\infty} a_i x^i$ and consider

$$\omega_n = y - \sum_{i=1}^{n} a_i x^i \in \mathcal{O}_{X,p}.$$

An elementary computation shows that $v_{\xi,t}(\omega_n) = t$ for $t \in [1, n]$, so the claimed injectivity follows. The rest of the claims are immediate consequences of previous results. $\square$

Remark 3.6. For a fixed $f \in \mathcal{O}_{X,p}$, the map $t \mapsto v_{\xi,t}(f)$ is continuous, concave, piecewise linear with integer coefficients (i.e., a tropical polynomial function). To see this, let

$$\omega = y - \sum_{i=1}^{\infty} a_i x^i \in \widehat{\mathcal{O}_{X,p}},$$

and expand f as a power series

$$f = \sum_{i,j \geq 0} a_{ij} x^i w^j \in \widehat{\mathcal{O}_{X,p}}\,.$$

Let

$$S(f) = \operatorname{conv}\left(\left\{(i,j) \in \mathbb{N}^2 \,|\, a_{ij} \neq 0\right\}\right)$$

be the convex hull of the support of f. Its lower left boundary is called the *Newton polygon* of f (in the formal coordinates (x,w)), and denoted by

$$N(f) = \partial\left(S(f) + (\mathbb{R}_{\geq 0})^2\right).$$

The Newton polygon $N(f)$ consists of a vertical half line followed by a finite sequence of segments with increasing negative (rational) slopes and a horizontal half line. Let $\Gamma_1, \ldots, \Gamma_k$ be the segments with slopes ≥ -1, and call these slopes $-1 \leq \gamma_1 \leq \cdots \leq \gamma_k$. Let also $V_1, \ldots, V_{k+1}$ be the vertices, so that $\Gamma_{\ell-1} \cap \Gamma_\ell = V_\ell$.

By Exercise 3.5 we know that $v_{\xi,t}(f) = \min\{i + tj \,|\, a_{ij} \neq 0\}$, and clearly this minimum is attained at at a monomial $a_{ij}x^i w^j$ with $(i,j) \in N(f)$. Moreover the monomial is unique, with (i,j) one of the vertices V_ℓ, unless $-t^{-1}$ is the slope of one of the segments Γ_ℓ. More precisely, for all $t \in [-\gamma_{\ell-1}^{-1}, -\gamma_\ell^{-1}]$, the minimum is attained at $(i,j) = V_\ell$ (and $v_{\xi,t}(f) = i+tj$ in this interval, which is linear with integer slope).

In convex geometry the function $t \mapsto v_{\xi,t}(t)$ obtained in this way is usually called the *Legendre transform* of the Newton polygon.

We endow $\mathcal{Q}_x$ with the final topology with respect to all maps v_ξ. Because of the second statement in Theorem 3.5, each of these maps becomes an homeomorphism of the half-line $\mathbb{R}_{\geq 1}$ with its image, and the intersection

$$v_{\xi_1}(\mathbb{R}_{\geq 1}) \cap v_{\xi_2}(\mathbb{R}_{\geq 1})$$

is homeomorphic to the segment $[1, \operatorname{ord}_x(\xi_1 - \xi_2)]$. It is easy to see that the topologies induced by $\mathcal{Q}_x$ and $\mathcal{Q}_y$in $\mathcal{Q}_x \cap \mathcal{Q}_y$ agree, endowing the whole set $\mathcal{Q}$ with a topology that makes it into a *profinite* $\mathbb{R}$-*tree*, rooted at the valuation $v_{\xi,1}$ (which is the 'order at p' valuation) with maximal branches of the tree corresponding to the series ξ, two branches separating at the points (of integer parameter t) corresponding to $\operatorname{ord}_x(\xi_1 - \xi_2)$.

Remark 3.7. The tree of quasimonomial valuations just constructed is a subset of the *valuative tree* $\mathcal{T}$ of all classes of rank 1 valuations centered at p introduced by Favre and Jonsson. To build the whole $\mathcal{T}$ one proceeds in essentially the same way, observing that in (22) one may allow formal series $\xi(x) = \sum_{j\geq 1} a_j x^{\beta_j}$ whose exponents β_j form an arbitrary increasing sequence of rational numbers, and one still obtains valuations $v_{\xi,t}$ (no longer quasimonomial). Unless the series defines an *algebraic* function, i.e., unless it vanishes identically on some curve $C \subset X$, it is also possible to allow $t = \infty$. The precise statement and proof of Theorem 3.5 then becomes technically more involved, see [23, Chapter 4] and [10, 8.2] for details. The resulting tree in that case has branching points at all rational values of the parameter t (not just at the integers) and also branches of finite length.

The topology we just described on $\mathcal{Q}$ (and on $\mathcal{T}$) is sometimes called *the strong topology* in contrast with a second (weaker) natural topology on $\mathcal{Q}$ and $\mathcal{T}$, namely the coarsest such that for all $f \in K(X)$, the map $v \mapsto v(f)$ is a continuous map $\mathcal{T} \to \mathbb{R}$.

3.3 The Waldschmidt constant as a function on $\mathcal{Q}$

In certain cases, the invariant $\widehat{\alpha}$ of valuations centered at a point p of the plane is known. We now review, following [20], what is known for quasimonomial valuations, referring to [26] for an overview and extension of the results to arbitrary valuations centerd at p. Fix again affine coordinates (x, y) near center$(v) = p = (0, 0) \in \mathbb{A}^2$; for simplicity, given a series $\xi(x)$, write

$$\alpha(\xi, t, m) = \alpha(I_{v_{\xi,t},m}) \ , \quad \widehat{\alpha}(\xi, t) = \widehat{\alpha}(v_{\xi,t}) \ , \quad \text{and} \quad \widehat{\mu}(\xi, t) = \widehat{\mu}(v_{\xi,t}) \ .$$

Recall from Exercise 3.1 that $\widehat{\mu}(v_{\xi,t}) = \widehat{\alpha}(v_{\xi,t})^{-1}$. In this section we consider $\widehat{\alpha}$ and $\widehat{\mu}$ as functions of ξ and t; it will turn out that $\widehat{\mu}$ is simpler, as a function of t, than $\widehat{\alpha}$, and we shall focus on the former.

Proposition 3.8. *For every $\xi(x)$, the function $t \mapsto \widehat{\mu}(\xi, t)$, for $t \in [1, \infty)$, is Lipschitz continuous with Lipschitz constant 1.*

Proof. For every $f \in \mathbb{C}[x, y]$, the function $t \mapsto v_{\xi,t}(f)$ is a tropical polynomial function of degree at most $\deg(f)$, as explained in remark 3.6. Therefore, the scaled function $\mu_f : t \mapsto v_{\xi,t}(f)/\deg(f)$ is continuous concave and piecewise affine linear with slopes in $\{0, 1/\deg(f), 2/\deg(f), \ldots, 1\}$ (compare with [6, Corollary C]). In particular, it is Lipschitz continuous with Lipschitz constant at most 1.

The function $t \mapsto \widehat{\mu}(\xi, t)$ in the claim is $\sup_{f\in\mathbb{C}[x,y]}\{\mu_f\}$; therefore it is also Lipschitz continuous with Lipschitz constant at most 1 (and it is not hard to see that it is actually equal to 1). □

It is immediate to extend the definition of μ and $\widehat{\mu}$ to the tree $\mathcal{T}$ of all valuations centered at p. The continuity properties of the resulting function $\widehat{\mu} : \mathcal{T} \to \mathbb{R}$ —which we shall not need— are summarized as follows:

Theorem 3.9 (Dumnicki-Harborune-Küronya-Roé-Szemberg, [20])**.** *The function $\widehat{\mu} : \mathcal{T} \to \mathbb{R}$ is lower semicontinuous for the weak topology and continuous for the strong topology.*

If the series $\xi(x)$ is chosen with coefficients general enough (see [20] for details), one obtains a function $\widehat{\mu}(\xi, t)$ that is minimal for all values of t:

$$\widehat{\mu}(\xi_{\text{general}}, t) = \min\{\widehat{\mu}(\xi, t) \,|\, \xi \in \mathbb{C}[[x]]\}.$$

This minimal function, that is the same for every sufficiently general choice, will be denoted by $\widehat{\mu}(t) = \widehat{\mu}(\xi_{\text{general}}, t)$.

Corollary 3.10 (of proposition 3.8)**.** *The function $\widehat{\mu}(t)$ is Lipschitz continuous with Lipschitz constant 1.*

The behaviour of the function $\widehat{\mu}(t)$ is known for small values and also for square integer values of t, by [20]. Let $F_{-1} = 1$, $F_0 = 0$ and $F_{i+1} = F_i + F_{i-1}$ be the Fibonacci numbers, and $\phi = (1 + \sqrt{5})/2 = \lim F_{i+1}/F_i$ the "golden ratio".

$t \in \left[1, 7 + \frac{1}{9}\right]$	$i \geq 1$ odd: $t \in \left[\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}}{F_{i-2}}\right]$	$\widehat{\mu}(t) = \frac{F_{i-2}}{F_i}\, t$
	$i \geq 1$ odd: $t \in \left[\frac{F_{i+2}}{F_{i-2}}, \frac{F_{i+2}^2}{F_i^2}\right]$	$\widehat{\mu}(t) = \frac{F_{i+2}}{F_i}$
	$t \in [\phi^4, 7]$	$\widehat{\mu}(t) = \frac{1+t}{3}$
	$t \in \left[7, \left(\frac{8}{3}\right)^2\right]$	$\widehat{\mu}(t) = \frac{8}{3}$
$t \sim 7 + \frac{1}{8}$	$t \in \left[\left(\frac{24+\sqrt{457}}{17}\right)^2, 7 + \frac{1}{8}\right]$	$\widehat{\mu}(t) = \frac{7+17\,t}{48}$
	$t \in \left[7 + \frac{1}{8}, \left(24 - \sqrt{455}\right)^2\right]$	$\widehat{\mu}(t) = \frac{121+t}{48}$
$t \sim 7 + \frac{1}{7+1/2}$	$t \in \left[\left(\frac{16+\sqrt{179}}{11}\right)^2, 7 + \frac{1}{7+1/2}\right]$	$\widehat{\mu}(t) = \frac{7+11\,t}{32}$
	$t \in \left[7 + \frac{1}{7+1/2}, \left(\frac{32-\sqrt{177}}{7}\right)^2\right]$	$\widehat{\mu}(t) = \frac{121+7\,t}{64}$
$t \sim 7 + \frac{1}{7}$	$t \in \left[\left(\frac{6+\sqrt{22}}{4}\right)^2, 7 + \frac{1}{7}\right]$	$\widehat{\mu}(t) = \frac{7+8\,t}{24}$
	$t \in \left[7 + \frac{1}{7}, \left(12 - \sqrt{87}\right)^2\right]$	$\widehat{\mu}(t) = \frac{57+t}{24}$
$t \sim 7 + \frac{1}{6+1/2}$	$t \in \left[\left(\frac{20+\sqrt{218}}{13}\right)^2, 7 + \frac{1}{6+1/2}\right]$	$\widehat{\mu}(t) = \frac{14+13\,t}{40}$
	$t \in \left[7 + \frac{1}{6+1/2}, \left(\frac{107}{40}\right)^2\right]$	$\widehat{\mu}(t) = \frac{107}{40}$
$t \sim 7 + \frac{1}{5}$	$t \in \left[\left(\frac{8+\sqrt{29}}{5}\right)^2, 7 + \frac{1}{5}\right]$	$\widehat{\mu}(t) = \frac{7+5\,t}{16}$
	$t \in \left[7 + \frac{1}{5}, \left(\frac{43}{16}\right)^2\right]$	$\widehat{\mu}(t) = \frac{43}{16}$
$t \sim 7 + \frac{1}{4}$	$t \in \left[\left(\frac{35}{13}\right)^2, 7 + \frac{1}{4}\right]$	$\widehat{\mu}(t) = \frac{13\,t}{35}$
	$t \in \left[7 + \frac{1}{4}, \left(\frac{35-\sqrt{877}}{2}\right)^2\right]$	$\widehat{\mu}(t) = \frac{87+t}{35}$
$t \sim 7 + \frac{1}{2}$	$t \in \left[\left(\frac{4+\sqrt{2}}{2}\right)^2, 7 + \frac{1}{2}\right]$	$\widehat{\mu}(t) = \frac{7+2\,t}{8}$
	$t \in \left[7 + \frac{1}{2}, \left(\frac{22}{8}\right)^2\right]$	$\widehat{\mu}(t) = \frac{22}{8}$
$t \sim 8$	$t \in \left[\left(\frac{3+\sqrt{7}}{2}\right)^2, 8\right]$	$\widehat{\mu}(t) = \frac{1+2\,t}{6}$
	$t \in \left[8, \left(\frac{17}{6}\right)^2\right]$	$\widehat{\mu}(t) = \frac{17}{6}$
	$t = n^2, n$ an integer	$\widehat{\mu}(n^2) = n$

Table 1: Piecewise linear function that agrees with $\widehat{\mu}$ on each interval.

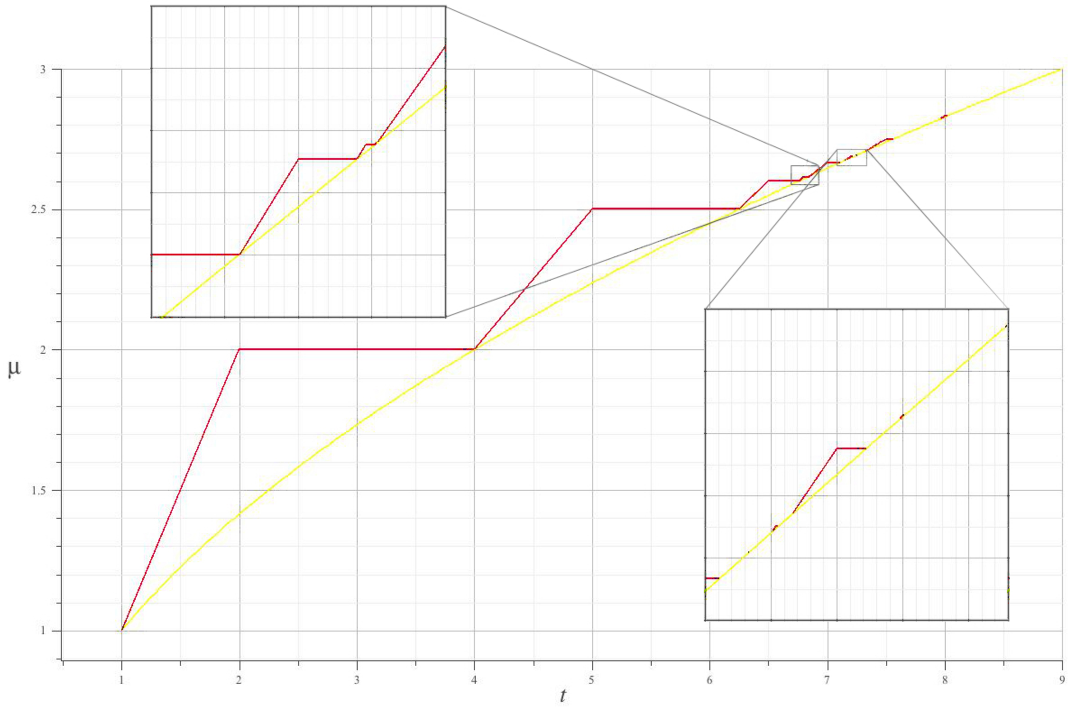

Figure 1: In red, the known behaviour of $\hat{\mu}(t)$ for $t \leq 9$; in yellow, the lower bound $\sqrt{t}$.

Theorem 3.11 (Dumnicki-Harbourne-Küronya-Roé-Szemberg)**.** *The continuous piecewise linear function defined in table 1 agrees with $\widehat{\mu}(t)$ in its domain.*

It may be informative to look at the graphical representation of the known behaviour of $\widehat{\mu}(t)$ for $t \leq 9$ in figure 1.

Remark 3.12*.* At the lower endpoints of the intervals included in the left column of table 1, and at the upper endpoints of the intervals in the right column, one has $\widehat{\mu}(t) = \sqrt{t}$. Note that all such endpoints given in table 1 are squares in $\mathbb{Q}$ or in the quadratic field to which they belong.

In particular there is a sequence of *rational* squares $t < 8$ with $\widehat{\mu}(t) = \sqrt{t}$, with an accumulation point at ϕ^4; we suspect that $\widehat{\mu}(t)$ can be computed for at least some rational squares $t > 9$ by existing techniques, that by continuity of $\widehat{\mu}$ would allow to compute $\widehat{\mu}(t)$ for some nonsquare t.

In the next section we will sketch the proof of Theorem 3.11, and show the relationship between the partial knowledge we have on the function $\widehat{\mu}$ and the partial knowledge we have on the Mori cone of the blown up $\mathbb{P}^2$. At this point we already see the first analogy to Nagata's onjecture, as the last row of table 1 tells us that for a sufficiently general choice of ξ, and every integer square $t = n^2$, the valuation $v_{\xi,t}$ is maximal. We will see that the connection is in fact stronger than an analogy: the following conjecture, put forward in [20] implies Nagata's conjecture:

Conjecture 3.13. *For a sufficiently general choice of ξ, and every $t \geq 8+1/36$, the valuation $v_{\xi,t}$ is maximal.*

3.4 The cluster of centers of a valuation

Next we are going to introduce some geometric structures attached to valuations that allow to study $\widehat{\alpha}$ and $\widehat{\mu}$, to prove Theorem 3.11, and motivate the new extension of Nagata's conjecture.

Each valuation with center at a closed point of a surface X naturally determines a *cluster of centers*, as follows. To begin with, let $p_1 = \mathrm{center}(v) \in X$. Consider the blowup $\pi_1 : X_1 \to X$ centered at p_1 and let E_1 be the corresponding exceptional divisor. The center of v on X_1 satisfies $\pi_1(\mathrm{center}_{X_1}(v)) = p_1$, so it may only be E_1 or a closed point $p_2 \in E_1$.

As long as the center is a closed point, the process can be iterated: blowing up the centers $p_1, p_2, \ldots$ of v either ends with a model where the center of v is an exceptional divisor E_n, in which case

$$v(f) \;=\; c \cdot \mathrm{ord}_{E_n} f$$

for some constant c and for every $f \in K(X)$, and v is (still) called a divisorial valuation, or the sequence of blowing up centers goes on indefinitely. For each center p_i of v, let $v_i = v(p_i) = v(E_i)$ be the value of general curves through p_i

Following [10, Chapter 4], we call the collection of points with weights

$$K = (p_1^{v_1}, p_2^{v_2}, \ldots),$$

the *weighted cluster of points* associated to the valuation v. The cluster K completely determines v, because for every effective divisor $D \subset X$,

$$v(D) = \sum_i v_i \cdot \mathrm{mult}_{p_i} \widetilde{D}_i, \tag{23}$$

where $\widetilde{D}_i$ is the proper transform at X_i. The sum may be infinite, but $\widetilde{D}$ can have positive multiplicity at only a finite number of centers [10, 8.2] (this property is not satisfied by higher rank valuations, as explained in [10, 8], but we don't consider such valuations in these notes).

Exercise 3.6. Prove the equality (23).
Hint: If $\bar{D}$ is the total transform (pullback) after blowing up p_1, then $v(D) = v(\bar{D})$, and

$$\bar{D} = \mathrm{mult}_{p_1}(D)E_1 + \tilde{D}.$$

Sometimes we shall say that a divisor goes through an infinitely near point to mean that its proper transform on the appropriate surface goes through it.

Example 3.14. Let $C \subset \mathbb{P}^2$ be a curve smooth at $p = (0,0)$, and $t = n \geq 1$ a natural number. Let $\xi \in \mathbb{C}[[x]]$ be the Taylor power series locally parameterizing C. The cluster of centers associated to the quasimonomial valuation $v_{\xi,t}$ is $K = (p_1^1, \ldots, p_n^1)$, i.e., it consists of n points with weights $v_i = 1$, and the points are determined by the fact that $p_1 = p$ and C goes through each p_i.

Definition 3.15. With notation as above, given indices $j < i$, the center p_i is called *proximate to* p_j ($p_i \succ p_j$) if p_i belongs to the proper transform $\widetilde{E}_j$ of the exceptional divisor of p_j. Each p_i with $i > 0$ is proximate to p_{i-1} and to at most one additional center p_j, with $j < i-1$; in this case $p_i = \widetilde{E}_j \cap E_{i-1}$ and p_i is called a *satellite point*. A point that is not a satellite point is called *free*.

Remark 3.16. The irreducible components of exceptional divisors can be computed, writing proper transforms as combinations of total transforms, if the proximity relations are known: $\tilde{E}_j = E_j - \sum_{p_i \succ p_j} E_i$.

Remark 3.17. For every valuation v, and every center p_i such that v is not the divisorial valuation associated to p_i, equation (23) applied to $D = E_j$ gives rise to the so-called *proximity equality*

$$v_j = \sum_{p_i \succ p_j} v_i \ .$$

For effective divisors D on X, the intersection number $\widetilde{D} \cdot \widetilde{E}_j \geq 0$ together with remark 3.16 yield the proximity inequality

$$\mathrm{mult}_{p_j}(\widetilde{D}_j) \geq \sum_{p_i \succ p_j} \mathrm{mult}_{p_i}(\widetilde{D}_i) \ .$$

Example 3.18. Let $C \subset \mathbb{P}^2$ be a curve smooth at $p = (0,0)$, and let $t = 3/2$. Let $\xi \in \mathbb{C}[[x]]$ be the Taylor power series locally parameterizing C. The cluster of centers associated to the quasimonomial valuation $v_{\xi,t}$ is $K = (p_1^1, p_2^{1/2}, p_3^{1/2})$, where

- $p_1 = p$,
- $p_2 = E_1 \cap \tilde{C} \subset X_1$,
- $p_3 = E_2 \cap \tilde{E}_1 \subset X_2$.

In other words, the associated cluster consists of three points, the second of which is determined by the degree 1 coefficient of ξ, and the third is a satellite.

Indeed, by definition of $v_{\xi,t}$, one has $v_{\xi,t}(p_1) = 1$ and $v_{\xi,t}(C) = t$. For $t > 1$, (23) applied to C means that $\mathrm{mult}_{p_2} \tilde{C}_1 > 0$, hence the point p_2 is as claimed, and in fact for $t = 3/2$ one has

$$1 = v(E_1) > v(\tilde{C}_1) = t - 1 = 1/2,$$

hence $v(p_2) = 1/2$. The determination of p_3 with its value follows, applying (23) to E_1.

Valuation divisors and valuation ideals

Assume now that $v = \mathrm{ord}_{E_s}$ is the divisorial valuation with associated cluster $K = (p_1^{v_1}, \ldots, p_s^{v_s})$, and let $\pi_K : X_K \to X$ be the composition of the blowups of all points of K (in this case, $v_s = 1$). Then, for every $m > 0$, the valuation ideal sheaf $\mathcal{I}_m$ can be described as

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(-mE_s)) \ .$$

Remark 3.19. As soon as $s > 1$, the negative intersection number $-mE_s \cdot \widetilde{E}_{s-1} = -m$ implies that all global sections of $\mathcal{O}_{X_K}(-mE_s)$ vanish along $\widetilde{E}_{s-1}$, and therefore

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(-mE_s - \widetilde{E}_{s-1})) = (\pi_K)_*(\mathcal{O}_{X_K}(-E_{s-1} - (m-1)E_s)) \ .$$

This *unloads* a unit of multiplicity from p_s to p_{s-1}. The finite process of subtracting all exceptional components that are met negatively, (i.e., starting from a divisor $D_0 = -mE_s$ and successively replacing D_i by $D_i - \widetilde{E}_j$, starting with $i = 0$, whenever $D_i \cdot \widetilde{E}_j < 0$ for some j, until one obtains a D_i such that $D_i \cdot \widetilde{E}_j \geq 0$ for all j) is classically called *unloading the weights of the cluster*. The final uniquely determined system of weights $\bar{m}_i$ satisfies a relative nefness property; a divisor is said to be *nef relative to a morphism* f when it intersects nonnegatively every curve mapping to a point by f [38, 1.7.11]. Then

$$D_m = -\sum \bar{m}_i E_i \quad \text{is nef relative to } \pi_K \ .$$

Moreover,

$$\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(D_m))$$

and in fact general sections of $\mathcal{I}_m$ have multiplicity exactly $\bar{m}_i$ at p_i, and no other singularity. More precisely, for any ample divisor class A on X, the complete system $|k(\pi_K)^*A + D_m|$ for $k \gg 0$ is base-point-free, its general members are smooth and they meet each E_j transversely at $\bar{m}_j - \sum_{p_i \succ p_j} \bar{m}_i$ distinct points. D_m will be called *valuation divisor* because of its link with the valuation ideal sheaf. Note that relative nefness of D_m is equivalent to the proximity inequality $\bar{m}_j \geq \sum_{p_i \succ p_j} \bar{m}_i$.

It follows using (23) that the valuation of an effective divisor D on X can be computed as a local intersection multiplicity

$$v(D) = I_{p_1}(D, C)$$

where C is the image in X of a general element of $|k(\pi_K)^*A + D_m|$.

Exercise 3.7. Let $v = \mathrm{ord}_{E_s}$ be the divisorial valuation whose associated cluster is $K = (p_1^{v_1}, \ldots, p_s^{v_s})$, where $v_s = 1$, and set $m_0 = \sum v_i^2$. For every $m > 0$ let $D_m = -\sum \bar{m}_i E_i$ be the unique nef divisor relative to π_K with $\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(D_m))$. Then

$$D_m \leq -\frac{m}{m_0} \sum v_i E_i,$$

and equality holds when the right hand side is an integer divisor.

Exercise 3.8. Any divisor D on X_K supported on $E_1, \ldots, E_s$ may be uniquely written in terms of the exceptional components:

$$D = \sum c_i \tilde{E}_i.$$

The round down of such a divisor is defined as $\lfloor D \rfloor = \sum \lfloor c_i \rfloor \tilde{E}_i$. Show that in the previous exercise one has $D_m = \left\lfloor -\frac{m}{m_0} \sum v_i E_i \right\rfloor$.

The preceding results for $v = \mathrm{ord}_{E_s}$ readily extend to rational valuations to give the following theorem. To state it, let us say that a divisor on X_K is *contracted* if it is supported on the exceptional divisors $E_1, \ldots, E_s$.

Theorem 3.20. *Let v be a rational quasimonomial valuation (i.e., assume $v(K(X)) \subset \mathbb{Q}$). Then the associated cluster $K = (p_1^{v_1}, \ldots, p_s^{v_s})$ is finite and has rational weights v_i. For every $m \geq 0$ there is a unique contracted divisor D_m on X_K, nef relative to π_K and with $\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(D_m))$. Moreover the contracted $\mathbb{Q}$-divisor D_v on X_K determined by the equalities $D_v \cdot \tilde{E}_i = 0$ for all $i = 1, \ldots, s-1$ and $D_v \cdot E_s = \frac{v_s}{\sum v_i^2}$ (in particular D_v is nef relative to π_K) satisfies*

$$D_m \leq m D_v,$$

and equality holds when the right hand side is an integer divisor.

Proof. We refer to [10, 8.2] for the finiteness of the associated cluster K. Then K differs from the cluster associated to the divisorial valuation v_{E_s} in the multiplicative constant v_s for all values, and the claims follow from the discussion above. □

Exercise 3.9. Let v be a divisorial valuation with associated cluster $K = (p_1^{v_1}, \dots, p_s^{v_s})$. Then

$$\operatorname{vol}(v) = \left(\sum v_i^2\right)^{-1} .$$

Hint: Use the *codimension formula* [10, 4.7.1].

Consider the group of numerical equivalence classes of $\mathbb{R}$-divisors $N_1(X_K)$, and the Mori cone $\overline{\mathrm{NE}}(X_K) \subset N_1(X_K)$. Theorem 3.20 allows to rephrase the definition of $\widehat{\alpha}$ and $\widehat{\mu}$ as follows. Assume that v is a rational valuation on the projective smooth surface X. Then clearly

$$\widehat{\alpha}_D(v) = \max\{\delta \in \mathbb{R} \mid \delta\pi_K^*(D) + D_v \in \overline{\mathrm{NE}}(X_K)\}, \tag{24}$$

$$\widehat{\mu}_D(v) = \min\{\epsilon \in \mathbb{R} \mid \pi_K^*(D) + \epsilon D_v \in \overline{\mathrm{NE}}(X_K)\}. \tag{25}$$

Exercise 3.10. Prove Proposition 3.1 for rational valuations on surfaces, using (24).

In cases when $\overline{\mathrm{NE}}(X_K)$ is a rational polyhedral cone, (24) yields that $\widehat{\mu}_D(v)$ is a rational number, and therefore v can be maximal only if $\sqrt{D^2/\operatorname{vol}(v)}$ is rational. In fact, all examples known of divisorial maximal valuations correspond to rational values of $\sqrt{D^2/\operatorname{vol}(v)}$, even for nonpolyhedral $\overline{\mathrm{NE}}(X_K)$. For some examples of non-divisorial maximal valuations, see Remark 3.12.

Quasimonomial valuations are exactly the valuations whose associated cluster consists of a few free points followed by satellites, that may be finite or infinite in number, but not infinitely many proximate to the same center. We are interested in *very general quasimonomial valuations* on $\mathbb{P}^2$ (see [20] and also [26]); we linked the genericity condition to the coefficients of the series ξ used to define the quasimonomial valuations, but it can be translated by saying that the free center points of the associated cluster are general in the exceptional divisors where they belong.

Remark 3.21. [10] The cluster K of centers of $v_{\xi,t}$ can be easily described from the continued fraction expansion

$$t = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}} .$$

K consists of $s = \sum n_i$ centers; if $t = n_1$ then they all lie on the proper transform of the germ

$$\Gamma\colon \{y = \xi(x)\} ,$$

otherwise the first $n_1 + 1$ lie on Γ and the rest are satellites: starting from p_{n_1+1} there are $n_2 + 1$ points proximate to p_{n_1}, the last of which starts a sequence of $n_3 + 1$ points proximate to $p_{n_1+n_2}$ and so on. If the continued fraction is finite, with r terms, then the last n_r points (not $n_r + 1$) are proximate to $p_{n_1+\cdots+n_{r-1}}$. The weights are

$$v_i = \begin{cases} 1 & \text{if } 1 \le i \le n_1, \\ t - n_1 & \text{if } n_1 + 1 \le i \le n_1 + n_2, \\ v_{n_1+\cdots+n_{j-1}} - n_j v_{n_1+\cdots+n_j} & \text{if } n_1 + \cdots + n_j + 1 \le i \le n_1 + \cdots + n_{j+1} \end{cases} .$$

If t is rational, there are only finitely many coefficients $n_1, \dots, n_r$, so the associated cluster is finite with rational weights and the valuation is divisorial. If t is irrational, then the sequence of centers is infinite, the group of values has rational rank 2, and there is no surface X_K.

Exercise 3.11. If $t = n^2$ is the square of an integer, then a very general quasimonomial valuation $v_{\xi,t}$ is maximal.
Hint: by the generality assumption, it is enough to prove maximality for some choice of ξ. Consider a smooth curve of degree n and its Taylor series.

Submaximal curves

We end this section by showing how to prove Theorem 3.11. For all values of t where $\widehat{\mu}(t) > \sqrt{t}$ there must exist some curve C with $m = v_{\xi,t}(C) > d\sqrt{t}$. In other words, $\alpha(I_m)$ is smaller than expected because of the equation $f \in I_m$ of C. The curve C is said to be *submaximal.*

Lemma 3.22. *If there is an irreducible polynomial $f \in \mathbb{C}[x,y]$ with*

$$v_{\xi,t}(f) > \frac{1}{\sqrt{\operatorname{vol}(v_{\xi,t})}} \deg(f) \ ,$$

then $v_{\xi,t}(f) = \widehat{\mu}(\xi,t)\deg(f)$.
Moreover, if $\widehat{\mu}(\xi,t) > \frac{1}{\sqrt{\operatorname{vol}(v_{\xi,t})}}$, then there is such an irreducible polynomial f.

In the case above we say that f (or the curve $\{f = 0\}$) computes $\widehat{\mu}(\xi,t)$. Since for any given f, the function $v_{\xi,t}(f)$ is concave and piecewise linear, the subset of $t \in \mathbb{R}$ such that $v_{\xi,t}(f) = \widehat{\mu}(\xi,t)\deg(f)$ is always a closed interval, and if nonempty (i.e., if $f = 0$ is a submaximal curve for some value of t), each endpoint of this interval corresponds to a maximal valuation $v_{\xi,t}$. Each pair of linear functions in table 1 is determined by submaximal curve that is submaximal in the union of the corresponding pair of intervals.

Proof of Lemma3.22. By continuity of $\widehat{\mu}(\xi,t)$ as a function of t, it is enough to consider the case $t \in \mathbb{Q}$. Let $v = v_{\xi,t}$.

Let f be as in the claim, and $d = \deg f$. It will be enough to prove that, for every polynomial g with degree e and $v(g) = w > \frac{e}{\sqrt{\operatorname{vol}(v)}}$, the polynomial f divides g. So assume by way of contradiction that f does not divide g, and compute the local intersection multiplicity

$$I_p(f,g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{P}^2,p}}{(f,g)}.$$

Choose an integer k such that $kw \in \mathbb{N}$ is an integer multiple of t, and consider the ideal

$$I_{kw} = \{h \in \mathbb{C}[x,y] \,|\, v(h) \geq kw\}.$$

Since obviously $g^k \in I$, the computation in Exercise 3.12 below shows that

$$I_p(g^k, f) \geq kwv(f) > \frac{kwd}{\sqrt{t}}$$

$$I_p(g,f) > \frac{wd}{\sqrt{t}} = dw\sqrt{\operatorname{vol}(v)} > de,$$

so f is a component of g.

Now assume $\hat{\mu}(v) > \frac{1}{\sqrt{\mathrm{vol}(v)}}$. So there is a polynomial $g \in \mathbb{C}[x,y]$ of degree e with $v(g) > \frac{e}{\sqrt{\mathrm{vol}(v)}}$. Since $v(f_1 \cdot f_2) = v(f_1) + v(f_2)$, it follows that at least one irreducible component f of g, satisfies $v(f) > \frac{\deg f}{\sqrt{\mathrm{vol}(v)}}$. □

Exercise 3.12. Let $f \in I_m$ and $g \in I_n$. Then

$$I_p(f,g) \geq \frac{\sum_{i=1}^{s}(mv_i)(nv_i)}{\left(\sum_{i=1}^{s} v_i^2\right)^2}.$$

Hint: Using linearity with respect to m and n and Theorem 3.20, reduce to the case when $\mathcal{I}_m = (\pi_K)_*(\mathcal{O}_{X_K}(mD_v))$. Then by Exercise 3.7 the claim is equivalent to Exercises 4.13, 4.14 of [10].

The existence of the submaximal curves needed to prove Theorem 3.11 is due to Orevkov [48]. For the intervals $[1,2]$ and $[2,4]$ (i.e., $i=1$ in the first two rows of table 1) the curve is just the line tangent to $\{y = \xi(x)\}$. For the intervals corresponding to $i=3$, the curve is a conic. In general, the submaximal curve that gives the i-th pair of linear functions, described in Proposition 3.23 below, is built by applying a sequence of Cremona transformations of degree 8 to the line tangent to $\{y = \xi(x)\}$.

Exercise 3.13. Given any power series $\xi(x) = \sum_{i\geq 1} a_i x^i$, let C be the line tangent to $\{y = \xi(x)\}$, namely $C : \{y - a_1 x = 0\}$. Show that C is submaximal for $v_{\xi,t}$ with $t \in (1,4)$ and compute $\widehat{\mu}(\xi,t)$ in this range.

Proposition 3.23. *Assume a power series $\xi(x) = \sum_{i\geq 1} a_i x^i$ is given with the coefficients $a_1,\ldots,a_6$ very general. For each odd $i \geq 1$, there is a rational curve C_i with the following properties:*

1. $\deg C_i = F_i$.

2. *C_i has a single cuspidal singularity at p.*

3. *The Newton polygon of its equation (with respect to coordinates (x,w) as in remark 3.6) consists of a unique segment, with vertices $(0, F_{i-2})$ and $(F_{i+2}, 0)$.*

Let K_i be the weighted cluster associated to the valuation v_{ξ,t_i} with $t_i = F_{i+2}/F_{i-2}$, and let $\pi_i : X_{K_i} \to \mathbb{P}^2$ be the blow up of all points of K. Then:

4. *π_i is an embedded resolution of C_i.*

5. *The strict transform $\tilde{C}_i \subset X_{K_i}$ is a (-1)-curve.*

6. *C_i is submaximal for t in the interval $\left(\frac{F_i^2}{F_{i-2}^2}, \frac{F_{i+2}^2}{F_i^2}\right)$.*

4 Cones of b–divisors

At the end of the preceding section it became clear that Nagata-type statements for valuations and for extremal rays of the Mori cone are connected, beyond simple analogy. However, from the perspective of Mori cones, the values taken by the Waldschmidt function $\widehat{\mu}$ on different parameters t (or different valuations $v_{\xi,t}$ in the valuative tree) appear to be unrelated, as they correspond to different blown up surfaces. In particular, the piecewise linear nature of the known parts of the Waldschmidt function, and the quadratic nature of its conjectural parts, show striking analogies with the (known and conjectural) shape of the Mori cones, with no satisfactory explanation at this point. This last section is an attempt at giving such an explanation for the existing deep connection, in Shokurov's language of b–divisors. This is joint work in progress of the first author with S. Urbinati [49].

4.1 Zariski–Riemann space and b–divisors

Birational divisors, or simply b–divisors, were introduced by V. V. Shokurov in the context of the Minimal Model Program, see [51]. We next review some basic facts about them, addressing the reader to [5], [35], [16] for details.

Given a normal projective variety X (for our purposes, $X = \mathbb{P}^2$) consider the set

$$\{\pi : X_\pi \to X \text{ birational morphism}\} / \cong$$

of isomorphism classes ($\cong$ denotes isomorphisms $X_\pi \cong X_{\pi'}$ commuting with π and π') of *birational models* of X. This set is partially ordered by setting $\pi_1 \geq \pi_2$ if π_1 factors through π_2. This order is inductive, i.e. any two proper birational morphisms to X can be dominated by a third one. The *Riemann-Zariski space of* X is the projective limit

$$\mathfrak{X} = \lim_{\leftarrow}\{X_\pi \to X \text{ birational morphism}\} / \cong$$

in the category of locally ringed topological spaces, each X_π being viewed as a scheme with its Zariski topology and structure sheaf $\mathcal{O}_{X_\pi}$. As a topological space $\mathfrak{X}$ is quasi-compact.

By a well known theorem of Zariski [56, VI,§17] the stalks of the structure sheaf of $\mathfrak{X}$ are exactly the valuation rings of $K(X)$ containing $\mathbb{C}$, so there is a natural bijection

$$\mathfrak{X} \longleftrightarrow \{\text{valuations on } K(X) \text{ trivial on } \mathbb{C}\}$$

The topology induced on the set of valuations admits as a basis of open sets the subsets of the form

$$U_{f_1,\dots,f_k} = \{v \text{ valuation such that } v(f_i) \geq 0\, \forall i\}, \qquad \text{where } f_i \in K(X),$$

or in other words, the subsets consisting of those valutations whose valuation rings contain a given finite subset of $K(X)$. The locally ringed space structure is given by assigning to any open subset the intersection of the valuation rings of the valuations of the subset.

A Weil divisor $\overline{W}$ on $\mathfrak{X}$ is defined to be a collection of divisors $W_\pi \in \mathrm{Div}(X_\pi)$, one on each birational model $\pi : X_\pi \to X$, compatible under push-forward, that is, $\mu_* W_\pi = W_{\pi'}$ if $\pi = \mu \circ \pi'$. The element W_π of the collection $\overline{W}$ is called *trace* of $\overline{W}$ on X_π. The group of Weil divisors on $\mathfrak{X}$ is therefore

$$\mathrm{Div}(\mathfrak{X}) = \lim_{\leftarrow}\{\mathrm{Div}(X_\pi)\}$$

where the arrow refers to push–forwards of the divisors.

On the other hand, a Cartier divisor $\overline{D}$ on $\mathfrak{X}$ is a Weil divisor for which there is a model X_0 such that for every other model X_π dominating X_0, the trace D_π of $\overline{D}$ on X_π is the pull-back of the trace D_0. Thus the group of Cartier divisors $\mathfrak{X}$ is also a limit of groups of Cartier divisors, but under pullbacks rather than pushforwards.

$$\mathrm{CDiv}(\mathfrak{X}) = \lim_{\rightarrow}\{\mathrm{CDiv}(X_n)\}.$$

In particular, a Cartier divisor D_π on a model X_π defines a Cartier divisor $\overline{D}$ on $\mathfrak{X}$, by pulling back D_π on all models dominating X_π and pushing forward on all other models. D_π is called a *determination* of $\overline{D}$. In other words, there is an injection $CDiv(\mathfrak{X}) \hookrightarrow Div(\mathfrak{X})$, due to the fact that $\pi_*\pi^*(D) = D$ when π is a birational map. Cartier and Weil divisors on $\mathfrak{X}$ are called *b–divisors* of X, to recall that they are divisors up to birational equivalence. We set $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}) = \mathrm{Div}(\mathfrak{X}) \otimes \mathbb{R}$ and $\mathrm{CDiv}_{\mathbb{R}}(\mathfrak{X}) = \mathrm{CDiv}(\mathfrak{X}) \otimes \mathbb{R}$ the $\mathbb{R}$–Weil b–divisors and the $\mathbb{R}$–Cartier b–divisors respectively.

Since nefness and bigness are stable under pullbacks by birational morphisms, we can refer to nefness and bigness of Cartier b–divisors. Since the valuation of a divisor is preserved by pullback, $v(\overline{D})$ is well defined for every valuation v and every Cartier b–divisor $\overline{D}$. In the case of a divisorial valuation v, one can even define the valuation of a Weil b–divisor, as follows. Let X_π be a model in which there is a prime divisor $E \in \mathrm{CDiv}(X)$ with $v = t \cdot \mathrm{ord}_E$ for some $t \in \mathbb{R}$; then for every b–divisor $\overline{W}$, set $v(\overline{W}) = (t \cdot \mathrm{ord}_E)(\overline{W})$ to be equal to t times the coefficient of E in W_π. A b–divisor can therefore be interpreted as a function $v \mapsto v(\overline{W})$ on the set $\mathcal{V}$ of divisorial valuations of X. Since distinct b–divisors clearly give distinct functions, we obtain an immersion

$$\mathrm{Div}_{\mathbb{R}}(\mathfrak{X}) \hookrightarrow \mathbb{R}^{\mathcal{V}} = \Pi_{v \in \mathcal{V}} \mathbb{R} = \mathrm{func}(\mathcal{V}, \mathbb{R}),$$

that is then used to endow the set of b–divisors with the topology induced by the topology of pointwise convergence on $\mathbb{R}^{\mathcal{V}}$; this is called the *topology of coefficent–wise convergence* on $\mathrm{Div}_{\mathbb{R}}(\mathfrak{X})$, for which $\lim_j \overline{W}_j = \overline{W}$ if and only if $\lim_j v_{E_\pi}(W_j)_\pi = v_{E_\pi}(W_\pi)$ for each prime divisor E_π on the model $X_\pi \to X$.

One can also consider the group of Cartier b–divisors modulo numerical equivalence, defining the Neron–Severi space

$$N^1(\mathfrak{X}) \otimes \mathbb{R} = N^1(\mathfrak{X})_{\mathbb{R}} = \lim_{\to} N^1(X_\pi)$$

where the maps defining the projective limit are given by pulling back: a class is determined by the class of a Cartier divisor in some blow up of X.

Assume from now on that X is a surface. In that case, the group of 1–dimensional numerical classes of $\mathfrak{X}$ is

$$N_1(\mathfrak{X}) = \lim_{\leftarrow} N^1(X_\pi),$$

here the maps are given by push–forward (on arbitrary dimension, the $(n-1)$–dimensional numerical classes are defined by the projective limit on the smooth models). The limit topology on $N^1(\mathfrak{X})$ and $N_1(\mathfrak{X})$ is compatible with the topology of coefficient-wise convergence defined above for $\mathrm{Div}_{\mathbb{R}} \mathfrak{X}$.

There is a natural injection $N^1(\mathfrak{X}) \hookrightarrow N_1(\mathfrak{X})$, by identifying a class $\beta \in N^1(\mathfrak{X})$ to the class $\overline{\beta} \in N_1(\mathfrak{X})$ determined by pulling back β on all higher models: by definition, the injection is continuous in the projective limit topology and $N^1(\mathfrak{X})$ is dense in $N_1(\mathfrak{X})$ [5, 1.9].

Relative Zariski decomposition

Zariski —in what can be considered a foundational work of the asymptotic theory of linear systems— showed in [57] that any effective divisor D on a smooth surface can be decomposed as a sum of a *positive* (nef, accountable for all sections in $H^0(X, mD)$) and an effective *negative* part (whose multiples are a fixed part in all multiples of the linear series D) with the following properties:

Theorem 4.1 (Zariski decomposition). *Every pseudoeffective $\mathbb{Q}$-divisor D on a smooth surface X admits a unique decomposition $D = P + N$, where P is a nef $\mathbb{Q}$-divisor, N is an effective $\mathbb{Q}$-divisor, and if N is nonzero then the components N_i of N have negative definite intersection matrix, and $P \cdot N_i = 0$.*

The generalization to pseudoeffective $\mathbb{Q}$-divisors is due to Fujita [24]. We refer to [8] and to the more recent and nice [4] for a proof. The Zariski decomposition is a most powerful tool; from our viewpoint, since $\oplus_{m\geq 0} H^0(X, mD) = \oplus_{m\geq 0} H^0(X, \lfloor mP \rfloor)$, Zariski decomposition allows us to reduce the question of finite generation of $\oplus_{m\geq 0} H^0(X, mD)$ to the case where D is nef. On the other hand, in the previous section, the notion of nef divisor relative to a morphism (more specifically, relative to the blow up morphism of the points of a cluster) became important to study the valuation ideals of a rank 1 valuation. The notion of relative nefness naturally leads to a notion of *relative* Zariski decomposition, that has been considered in the literature [41], [17] in more general settings and for different purposes. The version most useful for us is the following.

Theorem 4.2 (Relative Zariski decomposition). *Let $\pi : X_\pi \to X$ be a birational morphism of smooth surfaces. Every $\mathbb{Q}$-divisor D on X_π admits a unique decomposition $D = P_\pi + N_\pi$, where P_π is a $\mathbb{Q}$-divisor nef relative to π, N_π is an effective $\mathbb{Q}$-divisor with $\pi_*(N_\pi) = 0$, and if N_π is nonzero then the components N_i of N_π have negative definite intersection matrix, and $P \cdot N_i = 0$.*

If D is pseudoeffective, then the relative $D = P_\pi + N_\pi$ and absolute $D = P + N$ Zariski decompositions are related; N (respectively N_π) is the smallest effective $\mathbb{Q}$–divisor such that $P = D - N$ (respectively $P_\pi = D - N_\pi$) is nef (respectively nef relative to π); therefore $N_\pi \leq N$. However, the relative version is far easier to prove!

Exercise 4.1. Prove Theorem 4.2.
Hint: Let $E_1, \dots, E_n$ be the finite set of curves contracted by π. You can use the well-known fact that the intersection matrix of $(E_1, \dots, E_n)$ is negative definite to solve for the coefficients of $N_\pi = a_1 E_1 + \cdots + a_n E_n$.

Example 4.3. Let $X = \mathbb{P}^2$, and $L \subset \mathbb{P}^2$ a line. Given be a divisorial valuation v on $K(X)$, with associated weighted cluster $K = (p_1^{v_1}, \dots, p_s^{v_s})$, let $\pi_K : X_K \to \mathbb{P}^2$ be the blowup of all points in K, and let $E_1, \dots, E_s$ be the (total transforms of the) exceptional divisors. Then positive part of the Zariski decomposition of $-E_s$ is the divisor D_v of Theorem 3.20. Therefore, for every δ, the Zariski decomposition of $\delta\pi_K^*(L) - mE_s$ has positive part $\delta\pi_K^*(L) + mD_v$

Zariski decompositions are preserved by pullbacks, because nefness is, so it is natural to ask about a Zariski-type decomposition for b–divisors. A b–divisor $\overline{P}$ on X is b–nef if there is a determination P_π of $\overline{P}$ on a model $\pi : X_\pi \to X$ such that P_π is nef. This question has been addressed by A. Küronya and C. Maclean in [37], showing that such a decomposition exists for b–divisors on (normal) varieties of arbitrary dimension.

Theorem 4.4 (Zariski decomposition for b–divisors, [37]). *Let X be a smooth projective surface, D an effective $\mathbb{Q}$–b–divisor on X. There is a unique decomposition $D = P + N$, where P, N are effective $\mathbb{Q}$–b–divisors, such that $H^0(X, \lfloor mD \rfloor) = H^0(X, \lfloor mP_D \rfloor)$, and P_D is a limit of b-nef b–divisors on every proper birational model $Y \to X$, and for any nef b–divisor $P' \leq D$ the inequality $P' \leq P_D$ holds.*

Remark 4.5. Given a b–divisor $\overline{D}$, the associated b-divisorial sheaf $\mathcal{O}_X(\overline{D})$ is defined on an open subset U by $\Gamma(U, \mathcal{O}_X(\overline{D})) = \{\varphi \in K(X) | (div_X \varphi + \overline{D})|_U \geq 0\}$ (see [37] or [6]). It is not a coherent sheaf, but there is a natural inclusion $H^0(X, \mathcal{O}_X(\overline{D})) \hookrightarrow H^0(X, \mathcal{O}_X(D))$ thus $H^0(X, \mathcal{O}_X(\overline{D}))$ is finite-dimensional.

The positive part of $\mathbb{Q}$-b-divisor $\overline{D}$ on X in the theorem is

$$\overline{P}_D = \max\{\overline{P} | \overline{P} \text{ a nef } \mathbb{Q}\text{–b–divisor }, \overline{P} \leq \overline{D}\}.$$

4.2 Waldschmidt function through cones in $N^1(\mathfrak{X})_{\mathbb{R}}$

Fix the origin point p_1 on $\mathbb{P}^2$, and affine coordinates x, y around it. For every power series $\xi \in \mathbb{C}[[x]]$ and every real number $t \geq 1$ consider the valuation $v_{\xi,t}$ defined in section 3.2. Whenever $t \in \mathbb{Q}$, Theorem 3.20 provides an associated cluster $K = (p_1^{v_1}, \ldots, p_s^{v_s})$ and a relatively nef divisor $D_{v_{\xi,t}}$, which we have shown to be the positive part of the relative Zariski decomposition of $-v_s E_s$. Set $D_{\xi,t}$ the Cartier b–divisor on $\mathfrak{X}$ defined by $D_{v_{\xi,t}}$.

We are now ready to give an alternative proof of Proposition 3.8, which relates the continuity of the Waldschmidt function with the closed convex nature of the Mori cone.

We will consider the map div extended by continuity to $[0, \infty) \times [1, \infty)$.

Proposition 4.6. *Let L be the class in $N^1(\mathfrak{X})_{\mathbb{R}}$ of a line in $\mathbb{P}^2$, and fix a series $\xi \in \mathbb{C}[[x]]$. For every rational t, set $D_{\xi,t}$ the associated Cartier b–divisor. The function*

$$\begin{aligned} \mathrm{div} : \mathbb{R}_{\geq 0} \times \mathbb{Q}_{\geq 1} &\longrightarrow N^1(\mathfrak{X})_{\mathbb{R}} \\ (a, t) &\longmapsto aL - D_{\xi,t} \end{aligned}$$

is continuous.

Proof. The topology of $N^1(\mathfrak{X})_{\mathbb{R}}$ is induced by the topology of coefficientwise convergence, so it is enough to observe that for every prime divisor D_π on every model X_π the map $t \mapsto v_{\xi,t}(D_\pi)$ is continuous. $\square$

Alternative proof of Theorem 3.8 (sketch). We will prove that $\widehat{\alpha}$ is continuous (for the strong topology).

The Mori cone $\overline{\mathrm{NE}}(\mathfrak{X})_{\mathbb{R}}$ in $N^1(\mathfrak{X})_{\mathbb{R}}$ is a closed convex cone. Because it is closed, and div is continuous, its preimage $\mathrm{div}^{-1}(\overline{\mathrm{NE}}(\mathfrak{X})_{\mathbb{R}})$ is a closed set. Therefore

$$\widehat{\alpha}(\xi, t) = \min\{a \mid \mathrm{div}(a, t) \in \overline{\mathrm{NE}}(\mathfrak{X})_{\mathbb{R}}\}$$

is lower semicontinuous as a function of t. Then using that $\overline{\mathrm{NE}}(\mathfrak{X})_{\mathbb{R}}$ is convex, it follows that $\widehat{\alpha}$ is actually continuous. $\square$

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