



Wavelet-based entropy rate measure for characterizing stochastic, deterministic, and chaotic dynamics: An application on cryptocurrencies

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ABSTRACT

In this article, we introduce the Wavelet-Based Entropy-Rate Measure, designed to characterize the complexity and predictability of time series by combining wavelet decomposition with ordinal-pattern and entropy-rate estimation. The measure weights scale-specific entropy-rate contributions according to the relative wavelet energy, thus jointly accounting for local dynamical structure and global multiscale organization. We show that WERM effectively distinguishes among purely stochastic processes, persistent and anti-persistent noises, periodic deterministic components, and chaotic dynamics. In particular, the measure captures systematic differences between random, regular, and nonlinear deterministic regimes, while remaining sensitive to the interaction between stochastic memory and deterministic structure. We then apply the proposed framework to cryptocurrency. The empirical results indicate that price levels are mainly associated with persistent stochastic dynamics, whereas returns are better characterized by short-memory or anti-persistent regimes combined with chaotic components. WERM offers a useful and flexible tool for the analysis of complex multiscale signals in financial systems and other applications.

1. Introduction

In recent decades, the analysis of complex signals has revealed that many real-world systems arise from a nontrivial interplay between stochastic and deterministic components. Across a wide range of fields, from physics to financial systems and, more recently, cryptocurrency markets, observed time series are rarely purely random or fully deterministic. Instead, they typically exhibit multiscale structures in which noise, long-range memory, periodicity, and chaotic dynamics coexist and interact.

From a theoretical perspective, noise represents the simplest yet most subtle component. Different types of noise, such as white, pink, brown, and black noise, are characterized by distinct spectral properties and varying degrees of persistence or anti-persistence, as extensively studied within the framework of power-law noise and $1/f$ processes (e.g., [1]). In particular, processes with long-range memory introduce temporal correlations that fundamentally differentiate them from uncorrelated noise, as formalized in fractional Brownian motion and fractional Gaussian noise models. These approaches are closely related to scaling analyses such as Detrended Fluctuation Analysis [2,3], which has become a standard tool for detecting long-range dependence in empirical time series, and to

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classical measures such as the Hurst exponent [4,5]. This distinction is crucial, as empirically observed structures may be mistakenly attributed to deterministic mechanisms when they actually emerge from correlated stochastic processes.

At the same time, deterministic systems can display markedly different behaviors. On one hand, periodic or quasi-periodic signals introduce regular and predictable patterns, often modeled through harmonic and trigonometric polynomial representations [6]. On the other hand, chaotic systems generate trajectories that appear random despite being governed by nonlinear deterministic laws, as described by classical systems such as the Lorenz system [7], or the logistic map [8]. Deterministic chaos thus occupies an intermediate position between order and randomness: it is highly complex, non-periodic, yet not reducible to stochastic noise. The characterization of such dynamics has been widely addressed using tools such as Lyapunov exponents [9,10] and Kolmogorov–Sinai entropy [11,12], which quantify sensitivity to initial conditions and dynamical complexity. Discriminating among these different sources of complexity: structured noise, periodic dynamics, and chaos, remains a central challenge in time series analysis.

Over the last decades, several methodologies have been developed to distinguish deterministic dynamics from stochastic processes. Early approaches include the direct tests for determinism proposed by Kaplan and Glass [13] and by Wayland et al. [14], which exploit geometrical properties of reconstructed phase spaces to identify deterministic structure in observed time series. Subsequently, chaos-specific methodologies such as the 0–1 test for chaos [15] provided computationally efficient tools for discriminating chaotic deterministic dynamics from regular motion. Other approaches have relied on symbolic and combinatorial representations of time series, including ordinal-pattern methods for detecting determinism in noisy environments [16].

More recently, recurrence-based methodologies and statistical hypothesis-testing frameworks have been proposed to distinguish deterministic, stochastic, and nonlinear stochastic systems [17,18]. Similarly, Zhou et al. [19] introduced a methodology based on missing dispersion patterns to identify determinism and nonlinearity through symbolic representations of time series. While these methods are primarily designed as tests for the presence of determinism or chaos, the objective of the present work is different. Rather than providing a binary decision between deterministic and stochastic behavior, we propose a multiscale entropy-rate framework that quantifies the relative contribution of stochastic persistence, periodic structure, and chaotic dynamics across scales.

Entropy-based approaches have proven particularly effective in addressing this problem, as they provide quantitative measures of unpredictability and structural complexity. Classical measures such as Shannon entropy [20] and Kolmogorov complexity [21] provide foundational frameworks, although their direct estimation in empirical settings is often challenging. In this context, the permutation entropy of Bandt and Pompe [22] has gained widespread use due to its ability to capture the ordinal structure of time series while remaining robust to noise and invariant under monotonic transformations (see e.g. [23,24]). Extensions such as sample entropy [25] and multiscale entropy [26] have further emphasized the importance of capturing complexity across multiple temporal scales. However, when applied directly to raw signals, such measures may fail to fully capture scale-dependent features and multiscale organization.

Beyond its original formulation, Shannon entropy has inspired a large family of information-theoretic measures for the study of complex systems. In particular, generalized entropy frameworks, as Rényi [27,28], Tsallis [29,30] and Sharma Mittal entropy [31, 32], statistical complexity measures, transfer entropy, and entropy-rate approaches have been widely employed to characterize nonequilibrium dynamics, information production, and emergent organization in physical and socio-economic systems [33,34]. Within statistical mechanics, entropy-based methodologies have become fundamental tools for investigating phase transitions, long-range interactions, self-organization, and the interplay between order and disorder in complex systems. These developments have reinforced the idea that complexity cannot be fully described by randomness alone, but requires the simultaneous assessment of information generation, temporal dependence, and structural organization across scales.

In parallel, wavelet-based methods have become increasingly important in statistical mechanics and complex systems analysis due to their ability to represent signals simultaneously in time and scale [35,36]. Wavelet decompositions have been successfully applied to the study of turbulence, anomalous diffusion, critical phenomena, non-equilibrium fluctuations, and systems exhibiting multiscale correlations. Their capability to separate local dynamics occurring at different frequencies makes them particularly suitable for investigating processes in which stochastic and deterministic mechanisms coexist. In this context, wavelet entropy emerged as a natural extension of information-theoretic approaches, providing a quantitative description of energy organization across scales.

To overcome these limitations, multiresolution approaches based on wavelet theory have been introduced. In particular, wavelet entropy [31,37,38] provides a natural way to quantify the distribution of energy across scales, and has been successfully applied in various contexts involving nonstationary signals (e.g. [39–42]). Building on this line of research, we introduce the Wavelet-Based Entropy-Rate Measure (WERM), which combines wavelet decomposition with permutation entropy computed on wavelet coefficients across multiple scales. The key idea is to weight the entropic contribution of each scale by its relative energy, thereby jointly accounting for the signal's energy distribution and its local complexity. This multiscale framework enables a more refined characterization of signals, enhancing sensitivity to both deterministic structures and stochastic properties.

The aim of this work is to systematically investigate how WERM responds to signals of different nature, including purely random processes, signals with long-range dependence, periodic components, and deterministic chaotic dynamics. In particular, we analyze the WERM distributions simulated on the basis of the type of noise and the chaotic or periodic deterministic component, considering different combinations of the elements, to assess whether WERM distribution differs in a way to be suitable for generating stochastic process identification. This perspective is closely related to previous efforts aimed at separating stochastic and deterministic components, such as surrogate data testing [43].

Finally, to demonstrate its practical relevance, we apply WERM to real-world time series from cryptocurrency markets. These systems provide a natural test due to their high volatility, strong noise components, possible cyclic behaviors, and potentially nonlinear and chaotic dynamics. Previous studies have highlighted the presence of scaling laws, long memory, and complex

dynamics in financial and crypto markets using tools such as scaling analysis in finance [44], multiscaling approaches [45], ordinal networks [46], wavelet-based analysis in cryptocurrencies [47], and DFA-based efficiency studies [48]. In this setting, a measure capable of disentangling stochastic and deterministic contributions may offer valuable insights into the underlying complexity of financial markets.

The article is structured as follows. In Section 2 we present the methodology. Then, in Section 3, we show the numerical experiment and applications of our methodology on simulated and real data, in particular on cryptocurrencies. Finally, in Section 4 we provide conclusions.

2. Methodology: a Wavelet-Based Entropy-Rate Measure (WERM)

In this section, we introduce a multiscale predictability measure that combines the discrete wavelet transform with an ordinal-pattern-based approximation of the entropy rate. The main idea is to quantify, at each wavelet scale, how much the corresponding dynamics deviates from a maximally random (white-noise-like) process, and then to aggregate information across scales by weighting each contribution according to its wavelet energy.

2.1. Wavelet decomposition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued time series such that $f \in L^2(\mathbb{R})$, i.e.

$$L^2(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} |f(x)|^2 dx < \infty \right\}.$$

This assumption ensures that f has finite energy and allows for a representation in terms of an orthonormal wavelet basis, see for details [49,50].

Let $\psi \in L^2(\mathbb{R})$ be a mother wavelet satisfying the usual admissibility and regularity conditions, and define the family of dilated and translated wavelets

$$\psi_{j,n}(x) = 2^{-j/2} \psi(2^{-j}x - n), \quad j \in \mathbb{Z}, n \in \mathbb{Z},$$

where j is called the scale parameter and n the translation parameter.

The normalization factor $2^{-j/2}$ ensures that $\|\psi_{j,n}\|_{L^2} = \|\psi\|_{L^2}$, so that energy is preserved across scales.

The discrete wavelet transform of f , truncated at level J , is defined through the collection of detail coefficients

$$\{w_j(t) = \langle f, \psi_{j,n} \rangle\}_{j=1}^J,$$

where the inner product in $L^2(\mathbb{R})$ is given by

$$\langle f, \psi_{j,n} \rangle = \int_{\mathbb{R}} f(x) \overline{\psi_{j,n}(x)} dx.$$

In particular, if $f \in L^2(\mathbb{R})$, it has the unconditionally convergent expansion [51]

$$f = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \psi_{j,n} \rangle \psi_{j,n}.$$

Specifically, $\{w_j(t)\}_{j=1}^J$ denote the collection of wavelet (detail) coefficients obtained via the DWT up to level J .

The coefficients $\{w_j(t)\}$ capture fluctuations of $f(t)$ at scale 2^j :

- $j = 1$ corresponds to high-frequency components (rapid variations),
- increasing j corresponds to progressively lower-frequency information,
- $j = J$ represents slow and large-scale structures of the signal.

This decomposition separates the signal dynamics into a collection of processes $w_j(t)$, each interpretable as the contribution of $f(t)$ at scale j , [49,52].

In the present construction, we consider only the wavelet detail coefficients $\{w_j(t)\}_{j=1}^J$, since they contain the variability and dynamical information that is relevant for entropy and predictability analysis.

2.2. Ordinal patterns and permutation entropy

To characterize the dynamical structure of each process $w_j(t)$, we adopt the ordinal-pattern approach introduced by Bandt and Pompe [22].

In ordinal analysis, the embedding dimension m and the embedding delay τ determine the temporal resolution at which the dynamics is observed. Throughout this study, we set $m = 4$ and $\tau = 1$ for all simulations and empirical applications. This choice represents a compromise between dynamical resolution and statistical reliability. Since the number of admissible ordinal patterns grows factorially as $m!$, larger values of m require substantially longer time series to obtain stable probability estimates. The selected configuration is widely adopted in the permutation entropy literature and allows robust estimation while preserving sensitivity to short- and medium-range temporal dependencies [22,53].

Fixed an embedding dimension $m \geq 2$ and considered, for each scale j , the vector windows

$$\mathbf{w}_j^{(t)} = [w_j(t), w_j(t + \tau), \dots, w_j(t + (m-1)\tau)]. \quad (1)$$

Each window $\mathbf{w}_j^{(t)}$ is associated with a permutation $\pi \in S_m$ that orders its components in increasing order. This procedure defines a symbolic representation that assigns to each time t one of the $m!$ possible ordinal patterns.

The empirical distribution of ordinal patterns at scale j is then given by

$$P_j(\pi) = \frac{\#\{t : \mathbf{w}_j^{(t)} \text{ induces pattern } \pi\}}{N_j}, \quad \pi \in S_m, \quad (2)$$

where N_j denotes the total number of available windows. This distribution encodes the ordinal (dynamical) behavior of the process $w_j(t)$.

The permutation entropy of order m at scale j is defined as

$$H_m^{(j)} = - \sum_{\pi \in S_m} P_j(\pi) \log P_j(\pi). \quad (3)$$

Permutation entropy provides a nonparametric, robust, and monotonic-invariant measure of dynamical complexity at scale j .

2.3. Entropy rate and local estimation

The entropy rate of a stochastic process quantifies the average amount of information generated per unit time [54,55], and is classically defined as

$$h_\mu = \lim_{m \rightarrow \infty} (H_m - H_{m-1}), \quad (4)$$

where H_m denotes the Shannon entropy of length- m blocks.

Since ordinal-pattern analysis relies on a finite embedding dimension m , we define, for each wavelet scale j , the finite- m entropy-rate estimator

$$h_\mu^{(j,m)} = H_m^{(j)} - H_{m-1}^{(j)}. \quad (5)$$

This provides an entropy-rate estimate of the ordinal symbolic process, which approximates the dynamical complexity of the original signal.

For sufficiently large m , the quantity $h_\mu^{(j,m)}$ typically stabilizes and provides an approximation of the entropy rate at scale j . In practice, convergence can be assessed numerically by verifying that

$$|h_\mu^{(j,m)} - h_\mu^{(j,m-1)}| < \varepsilon,$$

for a small tolerance ε .

For notational simplicity, whenever the dependence on m is clear from the context and m is chosen in the stable regime, we write

$$h_\mu^{(j)} \approx h_\mu^{(j,m)}.$$

The quantity $h_\mu^{(j)}$ approximates the conditional complexity of the process $w_j(t)$: values close to the maximal entropy rate indicate nearly random, unpredictable dynamics, while significantly smaller values signal temporal dependencies and predictive structure.

To normalize predictability across scales, we introduce a white-noise reference entropy rate. For an ideal white-noise process, all ordinal patterns are equiprobable, yielding the maximal permutation entropy

$$H_m^{\max} = \ln(m!), \quad (6)$$

The corresponding entropy-rate increment is therefore

$$h_\mu^{\text{wn}} = H_m^{\max} - H_{m-1}^{\max} = \ln(m!) - \ln((m-1)!) = \ln m. \quad (7)$$

This quantity serves as a theoretical benchmark for maximal unpredictability and represents the entropy-rate growth of a white-noise process in the ordinal framework.

2.4. Wavelet-based Entropy-Rate Measure

Based on the local entropy-rate estimate, we define a normalized predictability index at scale j as

$$\text{WERM}_\mu^{(j)} = 1 - \exp(h_\mu^{(j)} - h_\mu^{\text{wn}}) = 1 - \exp(h_\mu^{(j)} - \ln m). \quad (8)$$

This transformation satisfies two key properties:

- $\text{WERM}_\mu^{(j)} = 0$ if and only if $h_\mu^{(j)} = \ln m$, i.e., the dynamics at scale j is compatible with white noise and hence maximally unpredictable.
- $\text{WERM}_\mu^{(j)} \rightarrow 1$ when $h_\mu^{(j)} \ll \ln m$, indicating strong temporal structure and high predictability at scale j .

The embedding dimension m controls the ordinal resolution of the entropy-rate estimate, while normalization by $\ln m$ ensures comparability across scales.

The overall measure should reflect not only the predictability of each scale but also its relative importance in the signal. We therefore define the wavelet energy at scale j as

$$E_j = \sum_t w_j(t)^2, \quad (9)$$

and normalize it to obtain the weights

$$P_j = \frac{E_j}{\sum_{k=1}^J E_k}, \quad \sum_{j=1}^J P_j = 1. \quad (10)$$

Each P_j represents the fraction of total signal energy carried by scale j [56].

The *Wavelet-based Entropy-Rate Measure* (WERM) is defined as the energy-weighted average of the scale-dependent predictability indices:

$$\text{WERM} = \sum_{j=1}^J P_j \text{WERM}_\mu^{(j)}. \quad (11)$$

Explicitly,

$$\text{WERM} = \sum_{j=1}^J \left(\frac{\sum_t w_j(t)^2}{\sum_{k=1}^J \sum_t w_k(t)^2} \right) \left[1 - \exp\left(H_m^{(j)} - H_{m-1}^{(j)} - \ln m\right) \right]. \quad (12)$$

The resulting measure provides a global assessment of the predictability of $f(t)$ by jointly integrating multiscale information (via wavelets), ordinal dynamics (via ordinal patterns), and information-theoretic complexity (via entropy rate).

2.5. Relationship with classical wavelet entropy

The proposed measure is related to, but conceptually distinct from, classical wavelet entropy (WE) [37,56] and related generalized wavelet entropy measures [39,41]. In particular, standard wavelet entropy is defined as the Shannon entropy of the normalized wavelet energy distribution,

$$WE = - \sum_{j=1}^J P_j \log P_j,$$

where P_j denotes the relative wavelet energy at scale j . Consequently, WE quantifies how energy is distributed across scales and reaches its maximum when energy is uniformly spread among them.

WERM uses the same multiresolution energy decomposition, but it does not compute entropy on the energy distribution itself. Instead, for each scale j , it evaluates the entropy rate of the ordinal dynamics associated with the wavelet coefficients $w_j(t)$. The normalized energies P_j are then employed only as weights to aggregate the scale-dependent predictability estimates.

Therefore, while classical WE measures multiscale energy dispersion, WERM measures multiscale predictability. Two signals may exhibit similar wavelet entropy values because they share a comparable energy distribution across scales, yet display different WERM values if their scale-specific temporal organization differs. This distinction is particularly relevant when comparing stochastic, periodic, and chaotic dynamics, which may possess similar spectral energy allocations but substantially different entropy-rate structures.

2.6. Asymptotic properties of WERM

Proposition 1 (Consistency of WERM). *Let $\{f(t)\}$ a stationary and ergodic process, with finite second moment, defined in a probability space $(\Omega, \mathcal{F}, P)$. Let $m < \infty$ an arbitrary embedding dimension and $J < \infty$ the number of wavelet scales. We assume that the marginal distribution of wavelet coefficients is continuous, such that $P(w_j(t + r\tau) = w_j(t + s\tau)) = 0$ for each $r \neq s$. For each scale j , let $w_j(t)$ be the DWT coefficients, $\hat{P}_T^{(j)}(\pi)$ the empirical ordinal frequencies, $\hat{H}_m^{(j)}$ the plug-in entropies, and $\widehat{\text{WERM}}_T$ the estimator of WERM. For $T \rightarrow \infty$:*

$$\widehat{\text{WERM}}_T \xrightarrow{a.s.} \text{WERM}.$$

Proof. The proof is pronounced into five sequential steps

- (i) Wavelet layer: For a discrete-time realization of the process $f(t)$, the wavelet coefficients $w_j(t)$ introduced in Section 2.1 admit the equivalent filter representation

$$w_j(t) = \sum_{\ell} g_{j,\ell} f(t - \ell).$$

where $\{g_{j,\ell}\}$ denotes the wavelet filter at scale j . Under standard assumptions on the wavelet filter, such as finite support (e.g. Daubechies wavelets) or, more generally, square-summability,

$$\sum_{\ell} g_{j,\ell}^2 < \infty,$$

the filtering operation is linear and stable. Consequently, if $\{f(t)\}$ is stationary and ergodic, then the wavelet coefficient process [57].

(ii) Ordinal layer: For each scale j , the empirical frequency of the ordinal pattern π on a sample of size N_T is given by:

$$\hat{P}_T^{(j)}(\pi) = \frac{1}{N_T} \sum_{i=1}^{N_T} P_j(\pi).$$

Since the wavelet coefficient process $\{w_j(t)\}$ is stationary and ergodic, and the mapping defining the ordinal pattern is a local and measurable function based on m consecutive elements, the induced symbolic process is also stationary and ergodic [22,58]. The hypothesis of continuity of the distribution of $w_j(t)$ guarantees the P -almost certain uniqueness of the permutation π , eliminating the ambiguity of ties. Consequently, for any stationary and ergodic stochastic process, the standard ergodic theorem implies that the sample averages of the indicator functions converge almost surely to their theoretical probabilities:

$$\hat{P}_T^{(j)}(\pi) \xrightarrow{a.s.} P^{(j)}(\pi).$$

(iii) Entropy layer: Since the alphabet of permutations is finite and has a cardinality equal to $m!$ states, the Shannon entropy function $H(P) = -\sum_{\pi} P(\pi) \log P(\pi)$ is a smooth and continuous function on the interior of the $m!$ -simplex. Following [59], since the empirical distribution $\hat{P}_T^{(j)}(\pi)$ converges almost surely to the true discrete probability measure $P^{(j)}(\pi)$, the standard properties of the empirical maximum likelihood framework guarantee the asymptotic consistency of the plug-in estimator for ordinal entropy:

$$\hat{H}_m^{(j)} \xrightarrow{a.s.} H_m^{(j)}.$$

We define a finite-order ordinal entropy-rate estimator for each scale j as:

$$h_{\mu,m}^{(j)} := H_m^{(j)} - H_{m-1}^{(j)}.$$

This quantity should be interpreted as a finite-order surrogate of the entropy rate, based on ordinal block entropies. By the almost sure convergence of the plug-in estimators for ordinal block entropies, and by linearity of the limit, we have:

$$\hat{h}_{\mu,m}^{(j)} := \hat{H}_m^{(j)} - \hat{H}_{m-1}^{(j)} \xrightarrow{a.s.} H_m^{(j)} - H_{m-1}^{(j)} = h_{\mu,m}^{(j)}.$$

(iv) Energy layer: Assuming that the original process $\{f(t)\}$ has a finite second moment, the filtered wavelet process $\{w_j(t)\}$ preserves this property, yielding a well-defined variance at each scale [57]. Under stationarity, ergodicity, and finite second moment assumptions, the ergodic theorem implies convergence of the empirical wavelet energy to its expectation:

$$\frac{1}{T} \sum_t w_j(t)^2 \xrightarrow{a.s.} \mathbb{E}[w_j^2].$$

Following the framework of wavelet variance analysis for time series [57], these scale energies are normalized to define the relative energy distribution. Due to the algebraic continuity of the normalization mapping on the positive orthant, the almost certain convergence of the individual scale energies ensures convergence of the normalized energy vector via the continuous mapping theorem, since all components converge almost surely and the number of scales is finite:

$$\hat{P}_j = \frac{\hat{E}_j}{\sum_k \hat{E}_k} \xrightarrow{a.s.} P_j.$$

(v) Continuous Mapping: Finally, define the global functional operator $\Lambda(P, h) = \sum_j P_j \left[1 - e^{h_j - \ln m} \right]$. Since Λ is a combination of products and exponential functions defined on compact domains, it is continuous on the positive orthant and the domain is finite-dimensional, the continuous mapping theorem applies:

$$\widehat{\text{WERM}}_T = \Lambda(\hat{P}, \hat{h}) \xrightarrow{a.s.} \Lambda(P, h) = \text{WERM}.$$

Therefore, the estimator is consistent for the WERM functional defined in terms of finite-order ordinal entropy rates and wavelet energy distributions. $\square$

Remark (Extension to Non-Stationary Dynamics). The previous proposition assumes global stationarity and ergodicity of the process $\{f(t)\}$. However, many physical and financial systems of interest, do not satisfy these restrictive assumptions. In such complex environments, a global asymptotic framework ($T \rightarrow \infty$) is often not meaningful in practice. To preserve its operational validity, WERM can be evaluated locally on finite moving windows. For a window of length L , defined as $W_t = [t, t + L - 1]$, we compute the time-varying local energy:

$$E_j(t) = \sum_{s \in W_t} w_j(s)^2$$

and the local entropy rate $\hat{h}_{\mu}^{(j)}(t)$ using exclusively the observations falling within the temporal boundaries of W_t . This time-localized formulation is supported by several results in the literature:

- **Wavelet Localization:** The Discrete Wavelet Transform is intrinsically localized in the time-frequency plane. The wavelet atoms provide a localized representation in time and scale, enabling the analysis of transient energy variations without requiring asymptotic limits [57].
- **Local Stationarity:** Processes with slowly time-varying structure can be locally approximated by stationary processes within sufficiently small neighborhoods under the framework of locally stationary processes [60].
- **Ordinal Pattern Stability:** As shown in [61], ordinal pattern methods are nonparametric and do not rely on linearity or Gaussian assumptions. We additionally assume that within sufficiently small analysis windows the ordinal distribution can be treated as approximately stable, i.e., without abrupt structural changes.
- **Entropy plug-in behavior:** The plug-in estimator for discrete entropy, studied in [59], is applied here in a localized manner over sliding windows. In each window, it acts as a non-parametric estimator of the empirical distribution over a finite alphabet of $m!$ ordinal patterns. This local estimation introduces a statistical trade-off in the choice of the window length L :
 1. Lower bound: A sufficiently large window size is required to reduce finite-sample bias in the estimation of the empirical distribution and to ensure adequate coverage of the permutation space.
 2. Upper bound: The window length must remain small relative to the time scale of structural changes in the underlying process in order to preserve the validity of the local stationarity approximation [60].

Under local stationarity or piecewise ergodicity assumptions, the plug-in estimator can be interpreted as a consistent empirical estimator within each fixed window, in the sense that it inherits the same finite-sample properties as in the stationary case applied locally.

Consequently, WERM can be interpreted as a time-varying multiscale functional that provides local estimates of ordinal entropy-based complexity across scales.

3. Numerical experiment

3.1. Discrete-time stochastic processes simulation

In order to monitor in detail the WERM behavior, we define a simulation design, introducing stochastic, multiscale deterministic, and chaotic behaviors. In particular, for the stochastic part, we consider white noise, pink noise, violet noise, brown noise (random walk) and a combination of brown and pink noise, to simulate the case of coexistence of persistence and long memory (black noise). To these we add a linear (autoregressive) short-memory process and a persistent process with a higher order of integration than brown noise. This discrete-time stochastic processes represent different order of temporal dependence and scale structures, allowing investigating the effect of memory and autocorrelation on the WERM value.

For each noise, B independent realization of length n are simulated, according to a Monte Carlo scheme. Formally, for each process $X_t^{(b)}$, with $b = 1, \dots, B$

$$X_t^{(b)} = \varepsilon_t^{(b)},$$

where $\varepsilon_t^{(b)}$ is the stochastic dynamics examined.

A further step, to introduce a controlled deterministic component, a trigonometric polynomial element is added to each stochastic process

$$X_t^{(b)} = \varepsilon_t^{(b)} + \alpha t + \sin\left(\frac{2\pi t}{\omega}\right),$$

where α is arbitrarily small to avoid trend dominance, and ω is the periodicity.

Alternatively, the deterministic component is a chaotic logistic map

$$X_{t+1}^{(b)} = rX_t^{(b)}(1 - X_t^{(b)}) + \varepsilon_t^{(b)},$$

where r is chosen in the chaotic regime.

This construction highlights the effects of noise and deterministic structure on entropy measure. The goal is to evaluate how WERM reacts to different nature of noise; presence of periodic or chaotic deterministic component; how stochastic and deterministic components interact each other. In particular, we aim to test whether WERM is able to distinguish between purely random signals, signals with long-range memory, and signals containing chaos-deterministic components. In particular, in trigonometric polynomial case, determine the impact of a cycle-trend on the WERM value compared to pure noise and in chaotic case, test whether WERM can distinguish chaotic signals from stochastic signals with structure, highlighting its ability to detect non-periodic deterministic complexity.

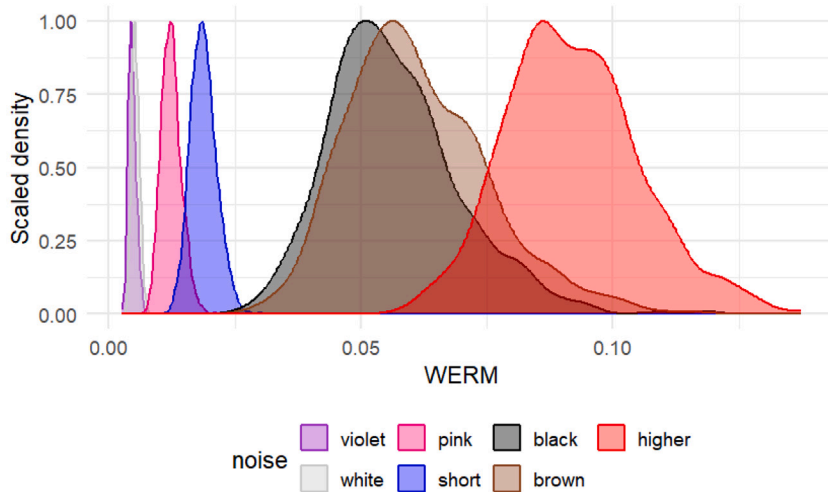
Expected results on WERM sensitivity is the following:

- **Noise:** Relatively low values of WERM for all the noises, with the following ordering: violet, white, pink, brown, black. In other words, the greater the persistence, the higher the WERM value, with lower white noise values in case of anti-persistence.
- **Periodic component:** systematic increase in WERM values for each noise due to the introduction of a recognizable pattern.
- **Chaotic:** high WERM, but lower than periodic component, reflecting the complexity of chaotic deterministic dynamics.

Table 1

Description of simulated noise and deterministic components.

Noise type	Memory (d)	Autoregression (ϕ)	Description
White	$d = 0$	$\phi = 0$	IID Gaussian
Violet	$d = -0.3$	$\phi = 0$	Anti-persistence
Pink	$d = 0.3$	$\phi = 0$	Long-memory
Short	$d = 0$	$\phi = 0.6$	Mean reverting autoregressive
Brown	$d = 1$	$\phi = 0$	Random Walk
Black	$d = 1.3$	$\phi = 0$	Persistent long-memory
Higher	$d = 2$	$\phi = 0$	Second-order persistence
Signal Type	Parameters	Equation	Description
Trigonometric polynomial	$\alpha = 0.05, \omega = 50$	$0.05t + \sin\left(2\pi\frac{t}{50}\right)$	Periodic Component
Chaotic	$r = 3.9$	$x_{t+1} = 3.9x_t(1 - x_t)$	Chaotic logistic map

**Fig. 1.** WERM distributions of simulated pure noise for $B = 1000$ replications of 10,000 observation, according to the parameters defined in Table 1.

This simulation framework allows systematically assessing how WERM reacts to signals of increasing complexity, distinguishing between pure randomness, periodic patterns, and deterministic chaos.

Table 1 describes the simulated noise and deterministic components used to investigate WERM behavior

To further explore the role of noise and the deterministic component, we simulate the series as weighted sums of the components, where $\beta_d = 0$ indicates pure noise and $\beta_d = 1$ indicates pure signal. This allows for performing a sensitivity analysis by varying the weights. For simplicity, we consider a total of eleven cases $\beta_d = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

WERM distributions are computed for $B = 1000$ replications of 10,000 observations of simulated time series, according to all the combinations of signal and noise, with permutation dimension $m = 4$. Fig. 1 shows the results of the pure noise simulations.

Fig. 1 highlights a clear monotonic trend, with WERM values increasing as the degree of temporal organization and predictability of the noise grows. In particular, white noise exhibits values concentrated near zero, as expected from its maximal entropy-rate and lack of temporal correlations. Similarly, violet and pink noise remain in the low-WERM regime, indicating that spectral coloration alone does not imply predictability, especially when correlations are weak or dominated by high-frequency components. Processes with short-range correlations display a noticeable shift toward higher WERM values, reflecting the presence of local temporal structure. This trend becomes more pronounced for signals with stronger persistence, such as black and brown noise, whose energy is concentrated at larger scales and thus contributes more significantly to the weighted predictability index. Finally, highly structured signals exhibit the largest WERM values, confirming that the proposed measure effectively captures strong deviations from randomness and highlights multiscale temporal organization. Overall, these results demonstrate that WERM provides a consistent and interpretable quantification of signal predictability, integrating scale-dependent entropy-rate estimates with energy-based weighting. The measure successfully discriminates between different stochastic regimes, ranging from fully uncorrelated noise to highly structured dynamics, while preserving sensitivity to both short- and long-range correlations.

Fig. 2 shows the mixture of noise and deterministic component defined by the trigonometric polynomial.

In particular, Fig. 2 exhibits a common pattern across all noise types, confirming that the measure effectively captures the transition from stochastic to deterministic dynamics. However, the rate and magnitude of this increase strongly depend on the underlying noise. For (a) white and (b) violet noise, which are characterized by minimal or high-frequency-dominated correlations,

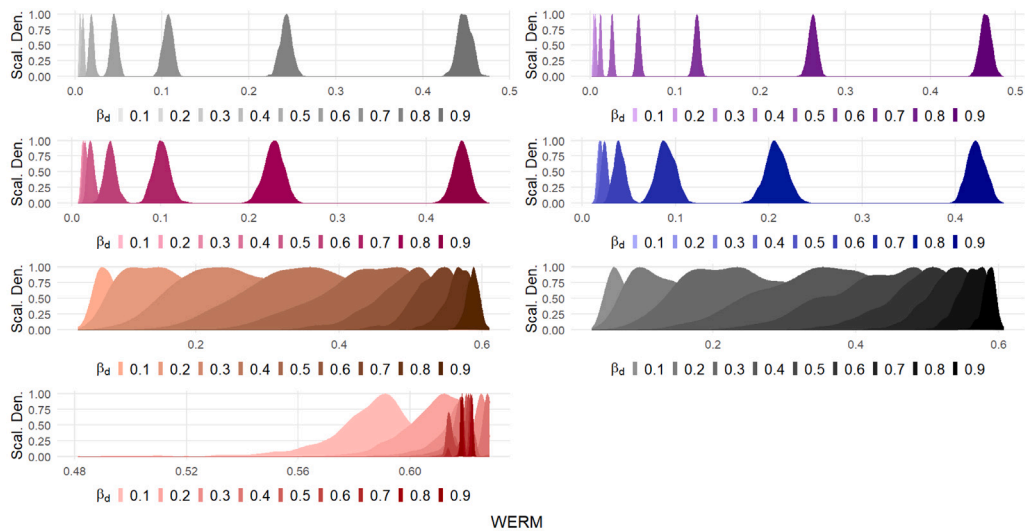


Fig. 2. WERM distributions of mixture of noise and trigonometric polynomial. From the bottom left: (a) white, (b) violet, (c) pink, (d) short, (e) brown, (f) black, and (g) higher-order noise, for $B = 1000$ replications of 10,000 observations, according to the parameters defined in Table 1.

WERM exhibits a pronounced and well-separated shift toward higher values as β_d increases, maintaining a symmetric distribution that suggests a uniform interaction between noise and signal. This indicates high sensitivity to exogenous structure. In contrast, for (c) pink and (d) short processes, the increase is more gradual and defined by broader distributions, reflecting the coexistence of intrinsic correlations and deterministic components. However, also in these cases, the shape of the distribution remains symmetric, with noticeable differences only in the variability. For persistent processes such as (e) brown, (f) black and (g) higher-order noise, WERM values are already elevated at $\beta_d = 0$, and the addition of deterministic structure leads to a comparatively smaller relative increase. This saturation effect highlights a reduced sensitivity of the measure in regimes where the signal is already highly predictable. A notable feature in the distributions for brown, black, and higher-order processes is relationship between β_d and the distribution skewness. Unlike the symmetric, quasi-Gaussian profiles observed for mean-reverting and long-range processes, these distributions exhibit a tail extending toward lower WERM values, while the bulk of the probability mass concentrates near the upper boundary. This asymmetry suggests that for strongly correlated or high-order processes, the deterministic component does not merely shift the average complexity, but acts as a statistical attractor, moving WERM estimates away from the most likely value. Overall, these findings demonstrate that WERM captures both the intrinsic predictability of the background process and the additional structure introduced by deterministic components. The measure exhibits a nonlinear response to β_d , modulated by the spectral and correlation properties of the underlying noise, thus providing a nuanced and multiscale characterization of signal complexity.

Fig. 3 shows the behavior of WERM in the presence of deterministic chaotic dynamics by combining stochastic noise with a signal generated by the logistic map.

Specifically, Fig. 3 highlights a distinct entropic behavior compared to other deterministic processes. When β_d approaches 1, the chaotic component exhibits a stable and relatively low WERM value, confirming that the measure identifies the underlying deterministic rules despite the apparent randomness of the time series. For (a) white and (b) violet noise, the WERM distributions are characterized by high symmetry and narrow dispersion. In these regimes, the high-frequency stochastic components do not interfere with the chaotic attractor topology, and the linear, well-separated shift of the density peaks suggests that the chaotic signature emerges with high resolution, as these noise types act as a uniform background. As a temporal relationship is introduced, such as in (c) pink and (d) short processes, the WERM distributions start to widen, reflecting a competition between the noise intrinsic structure and the non-linear dynamics of the chaos. In highly persistent regimes, specifically for (e) brown and (f) black noise, the behavior of the measure changes qualitatively; a significant negative skewness emerges alongside a marked saturation effect. Even at low β_d , the WERM values are elevated due to the persistent structure of the noise, and while the addition of the chaotic signal pushes the system toward the deterministic target, the long-term memory of the noise creates “shadows” of lower perceived complexity, resulting in characteristic heavy tails toward the left. The case of (g) higher-order correlated noise represents the most complex interaction, showing the maximum degree of asymmetry and the highest base complexity. Here, the deterministic component acts as a statistical attractor, but the convergence is hindered by the non-linear correlations of the noise. This transition from symmetric quasi-Gaussian profiles to highly asymmetric left-skewed distributions highlights that WERM sensitivity is fundamentally modulated by the spectral properties of the background. The emergence of negative skewness in persistent noise suggests that the chaotic component acts as a limiting bound, where the deterministic structure forces the system toward a specific entropic footprint while residual noise correlations introduce intermittent biases. Ultimately, the measure exhibits a non-linear response to β_d , with sensitivity being inversely proportional to the noise persistence, demonstrating that WERM effectively captures the subtle interplay between stochastic memory and deterministic order.

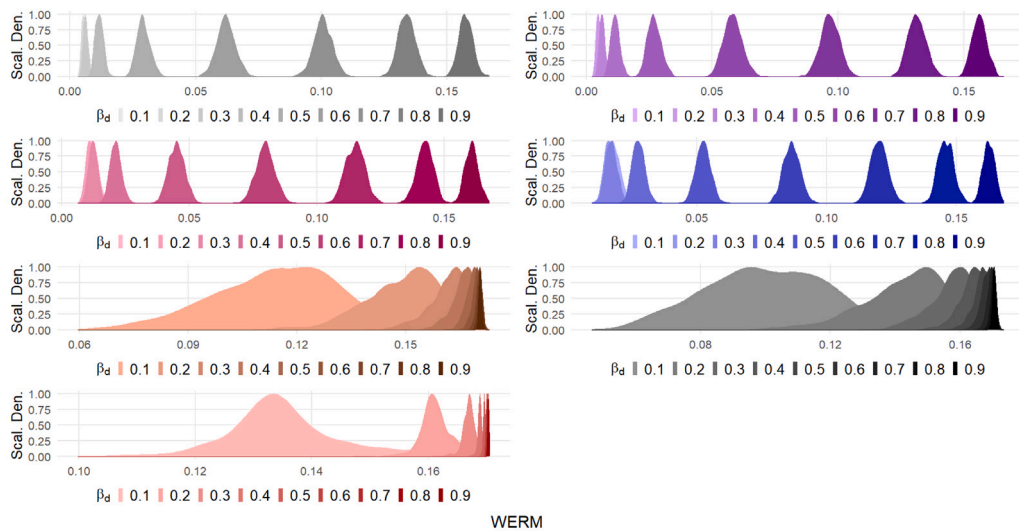


Fig. 3. WERM distributions of mixture of noise and chaotic logistic map. From the bottom left: (a) white, (b) violet, (c) pink, (d) short, (e) brown, (f) black, and (g) higher-order noise, for $B = 1000$ replications of 10,000 observations, according to the parameters defined in Table 1.

Comparing Figs. 2 and 3 reveals fundamental differences in how WERM characterizes deterministic structures. The most striking divergence lies in the limit values at $\beta_d = 1$: while the polynomial signal converges toward high complexity estimates, approaching the theoretical maximum, the chaotic logistic map stabilizes at noticeably lower values. This suggests that WERM does not merely measure randomness but distinguishes between types of determinism, identifying the recursive, self-similar nature of the chaotic attractor as a more ordered or compressible state compared to the high-order polynomial fluctuations.

The interaction with the stochastic background also follows distinct trajectories. In (a) white and (b) violet noise regimes, both signals show high sensitivity and symmetric shifts; however, the chaotic signal exhibits a more rapid collapse of variance, reaching its characteristic footprint even at moderate β_d . Conversely, the polynomial signal requires a higher deterministic weight to overcome the stochastic spread, indicating that its entropic signature is more easily masked by high-frequency noise.

The phenomenon of negative skewness in persistent noise (e) brown, (f) black, and (g) higher-order) serves as a common denominator, yet its physical interpretation differs. For the polynomial signal, the negative skewness represents a drag effect where the noise persistence prevents the ascent toward higher complexity. In the chaotic case, the skewness reflects a containment effect: the chaotic attractor acts as a lower complexity bound that the persistent noise attempts to inflate.

Finally, the saturation effect observed in (e) brown and (f) black noise is more pronounced for the chaotic signal. Because the chaotic WERM value is lower than the baseline complexity of persistent noise, the transition from $\beta_d = 0$ to $\beta_d = 1$ actually involves a “complexity inversion” or a compression of the state space. This highlights that WERM sensitivity is not absolute but relative to the spectral distance between the noise and the signal: the measure is most effective when the deterministic component shows a spectral signature that contrasts sharply with the background, whether that involves moving toward higher (polynomial) or lower (chaotic) entropic regimes.

3.2. Real data evidence

The proposed methodological framework aims to develop a data-driven decision protocol that uses the WERM distribution as a dynamic descriptor to identify the operational regime of a real-world time series $f(t)$, subsequently validating this identification through empirical predictive performance. The process begins with the characterization of a mapping space $S = \{M_1, M_2, \dots, M_n\}$, representing the set of simulated distributions derived from various combinations of weighted noise and deterministic components. For each regime M_i , we define a characteristic vector $\Phi_{M_i} = [\mu_i, \sigma_i, \gamma_i, \kappa_i]$ composed of the four statistical moments, i.e. mean, standard deviation, skewness, and kurtosis, of the generated WERM density. This mapping is a reference system to define the best entropic case for real data.

The mapping of observed data, is performed by calculating the corresponding moment vector Φ_{obs} from the WERM distribution obtained from the real-world time series $f(t)$. The matching follows a minimization criterion where the observed data is associated with the simulated regime M^* that minimizes the Mahalanobis distance $d_M(\Phi_{obs}, \Phi_{i,j}) = \sqrt{(\Phi_{obs} - \Phi_{i,j})^T S_{i,j}^{-1} (\Phi_{obs} - \Phi_{i,j})}$. The Mahalanobis distance was preferred over the other metrics to account for the heterogeneous scales of the statistical moments and their inherent cross-correlations. By incorporating the precision matrix $S_{i,j}^{-1}$, this metric ensures that higher-order moments, such as kurtosis, do not disproportionately bias the matching process, while simultaneously respecting the elliptical morphology of the reference dynamical clusters in the four-dimensional feature space.

Finally, the robustness of the identified mapping is assessed through an out-of-sample validation procedure. The observed time series is partitioned into training and validation sets, and a set of candidate forecasting models is evaluated within each inferred regime M^* . Rather than assuming a deterministic correspondence between regime identification and predictive superiority, we evaluate whether regime-consistent models exhibit statistically lower forecast errors on average compared to non-consistent alternatives. This allows for the quantification of the extent to which WERM-based classification provides informative priors for model selection in complex, non-stationary financial time series.

To complement the descriptive analysis of data dynamics provided by WERM, we introduce a surrogate-based inferential procedure to assess whether the observed multiscale structure of a time series can be explained by a linear stochastic process, according to the procedure proposed by [58]. Let $\{f(t)\}_{t=1}^T$ the observed log-return series and let $W_{\text{obs}} = \text{WERM}(\{f(t)\})$. We consider the null hypothesis

H_0 : the multiscale structure of $\{f(t)\}$, as quantified by WERM,
is compatible with that generated by a linear stochastic process,

against the alternative

H_1 : the multiscale structure of $\{f(t)\}$, as quantified by WERM,
significantly deviates from that generated under the linear stochastic benchmark.

To approximate the distribution of W_{obs} under H_0 , we generate B Iterative Amplitude Adjusted Fourier Transform surrogate series, which preserve the empirical distribution and linear autocorrelation structure of the original data while removing nonlinear dependencies. For each surrogate series $\{f(t)^{(b)}\}$, we compute

$$W_{\text{sur}}^{(b)} = \text{WERM}(\{f(t)^{(b)}\}), \quad b = 1, \dots, B.$$

An empirical p -value is then obtained as

$$p = \frac{1 + \sum_{b=1}^B \mathbf{1}(W_{\text{sur}}^{(b)} \geq W_{\text{obs}})}{B + 1}.$$

Small values of p indicate that the observed WERM value is unlikely under the surrogate distribution, providing statistical evidence against the null hypothesis of linear stochastic dynamics and supporting the presence of nonlinear structure.

To do this, we explore the cryptocurrency market, selecting sixteen series, focusing of different market predictability and micro-structures. In particular, cryptocurrencies can be broadly classified into different categories based on liquidity, market capitalization, volatility, and degree of speculation. However, it is important to note that such classifications are inherently approximate and may evolve over time.

Bitcoin (BTC) and Ethereum (ETH) constitute the core “blue-chip” segment of the cryptocurrency market. Due to their high liquidity, widespread adoption, and large market capitalization, they tend to exhibit relatively more stable and structured price dynamics compared to other digital assets. While still subject to significant volatility, these assets are generally considered more suitable for long-term investment strategies and portfolio diversification.

A second group includes large- and mid-cap altcoins such as Binance Coin (BNB), Cardano (ADA), Solana (SOL), Avalanche (AVAX), Polkadot (DOT), Chainlink (LINK), Uniswap (UNI), Aave (AAVE), and NEAR Protocol (NEAR). These cryptocurrencies are typically characterized by lower liquidity than blue-chip assets and higher volatility. Their price dynamics are often influenced by technological developments, ecosystem growth, and market narratives, making them attractive for both medium-term investment strategies and more active trading approaches.

Finally, a third category consists of highly speculative and community- or narrative-driven tokens, including Dogecoin (DOGE), Shiba Inu (SHIB), and Pepe (PEPE). These assets are generally associated with strong social media influence and exhibit highly volatile and less predictable price movements. Although they may generate substantial short-term returns, they also involve a significantly higher level of risk.

Additionally, it is worth noting that some projects, such as Render (RNDR) and Bittensor (TAO), are primarily driven by emerging technological narratives (e.g., distributed computing and artificial intelligence) rather than purely speculative dynamics. As such, they do not fit neatly into a single category and may display hybrid characteristics depending on market conditions. All cryptocurrencies are analyzed hourly from July 12, 2024, to April 1, 2026, for a total of 17,534 observations. To evaluate the WERM effectiveness, we split the sample into a training and validation set with a 70/30 ratio. Furthermore, we implement a Moving Block Bootstrap procedure along time dimension. For a time block of length L , let $\{f(t)\}_{t=1}^T$ denote the underlying log-price series. For each bootstrap replication $b = 1, \dots, B$, we construct a pseudo-sample by randomly drawing with replacement $\lceil T/L \rceil$ from the original series and concatenating them until reaching length T . For each bootstrap sample we recompute the WERM to calculate its distribution and moment vectors ϕ_{obs} . Since we deal with hourly data, we define $L = 512$, corresponding to a trading month, for a total of approximately 10,000 blocks, to calculate $B = 1000$ bootstrap replicates.

To control for outlier bias in the WERM calculation, we apply a winsorization to all the log-return series for the left and right tails corresponding to probabilities 0.001 and 0.999. This approach eliminates extreme values without excessively reducing volatility and preserving the continuity of the series for calculating the block bootstrap.

Table 2 summarizes the main characteristics of the considered cryptocurrencies.

Fig. 4 shows the log-prices of the selected cryptocurrencies. In particular, AAVE shows an initial accumulation phase followed by a strong uptrend culminating in a local double top, signaling liquidity distribution before a final retracement that nonetheless

Table 2
Overview of selected cryptocurrencies.

Symbol	Name	Category	Liquidity	Volatility	Description
BTC	Bitcoin	Blue-chip	Very High	Moderate	Largest market capitalization; benchmark asset with relatively stable and structured price dynamics
ETH	Ethereum	Blue-chip	Very High	Moderate/High	Leading smart contract platform; strong ecosystem (DeFi, NFTs) and high market participation
BNB	Binance Coin	Large-cap altcoin	High	Moderate	Exchange-related token; influenced by Binance ecosystem developments
ADA	Cardano	Large-cap altcoin	Medium	Medium	Proof-of-stake blockchain with strong academic foundation and moderate adoption
SOL	Solana	Large-cap altcoin	High	High	High-performance blockchain; characterized by significant volatility and active trading
AVAX	Avalanche	Large-cap altcoin	Medium	High	Scalable blockchain with DeFi applications; sensitive to market cycles
DOT	Polkadot	Large-cap altcoin	Medium	Medium/High	Interoperability-focused protocol connecting multiple blockchains
LINK	Chainlink	Mid-cap altcoin	Medium	Medium/High	Decentralized oracle network; price influenced by adoption and partnerships
UNI	Uniswap	Mid-cap altcoin	Medium	Medium	Governance token of a major decentralized exchange; linked to DeFi activity
AAVE	Aave	Mid-cap altcoin	Medium	Medium	DeFi lending protocol; sensitive to liquidity flows and market stress
NEAR	NEAR Protocol	Mid-cap altcoin	Medium/Low	High	Emerging smart contract platform with relatively high volatility
RENDER	Render	Narrative-driven	Medium/Low	High	Token related to decentralized GPU rendering; driven by AI and computing narratives
TAO	Bittensor	Narrative-driven	Low/Medium	Very High	Blockchain-based AI network; highly sensitive to technological trends and speculation
DOGE	Dogecoin	Meme/speculative	Medium	High	Community-driven cryptocurrency; strongly influenced by social media sentiment
SHIB	Shiba Inu	Meme/speculative	Medium	High	Highly speculative token with strong retail participation
PEPE	Pepe	Meme/speculative	Variable	Very High	Recent meme token characterized by extreme volatility and speculative dynamics

preserves some of the gains. ADA displays a weaker morphology, characterized by a less confident rise and a subsequent steady decline that brings prices back close to the opening lows, almost entirely erasing the positive momentum. AVAX displays nervous volatility, peaking earlier than the rest of the market, suggesting an early exhaustion of buying pressure. BNB stands out for its superior structural resilience, where the final negative slope appears less pronounced, indicating strong institutional support. BTC serves as the center of gravity of the entire dataset, displaying a complete expansion–contraction cycle that dictates the timeline for all other assets. DOGE displays extreme reactivity with a very pronounced peak and a subsequent steep fall, typical of assets driven by speculative momentum. DOT shows a marked structural deterioration, with a sequence of lower highs culminating in a breakdown below its starting levels. ETH replicates BTC trajectory but with broader swings, confirming its role as a high-beta asset among the majors. LINK shows a series of rebound attempts that are systematically rejected, creating a “sell the rally” dynamic that depresses its value in the final part of the session. NEAR follows a nearly uninterrupted downward trajectory after the first third of the chart, highlighting a capital flight to more defensive assets. PEPE is characterized by extremely high stochastic noise, with frequent micro-oscillations reflecting frenetic algorithmic trading activity over a short time horizon. RENDER shows one of the deepest corrections in percentage terms, with a final negative slope suggesting a strong divestment in the decentralized computing services sector. SHIB follows a trajectory mirroring PEPE, but with a more markedly bearish underlying trend starting midway through the observation period. SOL shows notable relative strength in the middle phase, but undergoes a particularly sharp final correction that tests the strength of its previous lows. TAO exhibits erratic price action with sudden price swings that indicate thinner liquidity and extreme sensitivity to sentiment fluctuations. Finally, UNI mirrors the DeFi sector, with unconvincing technical rebounds that foreshadow a session close in negative territory. From their characteristics, it is possible to distinguish among pilot assets (BTC, ETH, BNB, SOL),

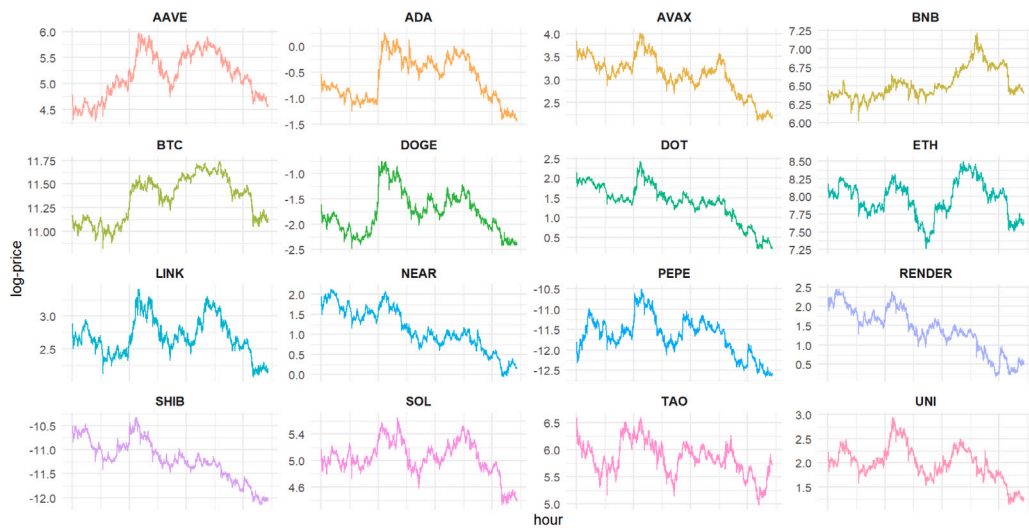


Fig. 4. Cryptocurrencies log-prices.

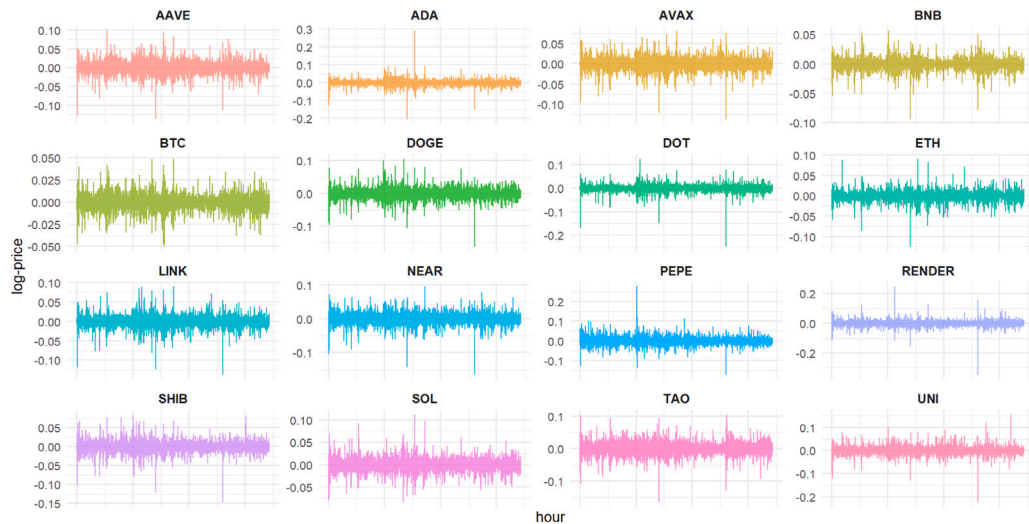


Fig. 5. Cryptocurrencies log-returns.

defining the macro direction of the market and demonstrates a greater capacity to absorb selling volumes; Deteriorating Assets (ADA, DOT, NEAR, RENDER), with relative structural weakness, failing to hold support levels during BTC consolidation phases; Speculative Proxies (DOGE, SHIB, PEPE), for their granular and noisy nature of its charts, where intraday volatility prevails over the persistence of the long-term trend; Finally, the Platforms and Services (LINK, UNI, TAO, AAVE), highlighting a sector correlation based on technological risk appetite, acting as multipliers of market leaders movements in both expansionary and corrective phases.

Fig. 5 shows the log-returns, highlighting volatility patterns and microstructural features that are less evident in price levels. In particular, AAVE exhibits a phase of elevated volatility in the initial period, consistent with a strong upward trend, followed by a gradual stabilization. The distribution of returns appears relatively symmetric in the later phase, suggesting a more balanced market condition during consolidation. ADA shows a pronounced negative asymmetry, with downward spikes occurring more frequently and with greater magnitude than upward movements. This pattern is indicative of persistent downward pressure and structural weakness. AVAX is characterized by pronounced volatility clustering, with irregular bursts of high variance. This suggests a market environment marked by rapid adjustments to short-term information and potentially fragmented liquidity. BNB has stable dynamics, with returns confined within a narrower range. This behavior is consistent with a more resilient structure and lower sensitivity to short-term noise relative to other assets. BTC serves as a benchmark, with returns densely distributed around zero and relatively stable variance. Its dynamics appear to sustain broader market movements, as large shocks tend to propagate across other series. DOGE shows a high level of stochastic variability, with a visually noisy return structure. This reflects a dynamics dominated by short-term trading activity

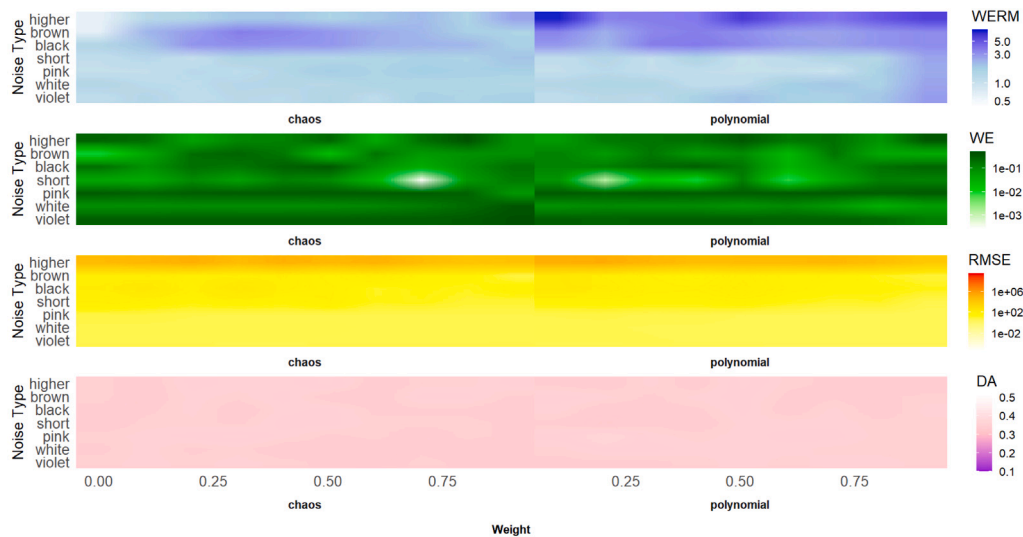


Fig. 6. Heatmaps for (from top to bottom): average WERM distribution distance, Wavelet Entropy, Root Mean Square Error and Mean Directional Accuracy of forecasts according to the noise and the non-random component (chaos on the left and polynomial on the right). All metrics are evaluated on cryptocurrency log-price levels using a fixed sample length and identical forecasting and estimation procedure across scenarios.

and limited persistence in directional movements. DOT shows recurrent negative spikes that deviate significantly from the mean, indicating structural fragility and limited support during market downturns. ETH confirms its role as a market-aggressive asset, with return dynamics closely aligned with BTC but characterized by higher variance. This suggests strong co-movement with amplified responses to market shocks. LINK has a pattern in which positive returns are frequently offset by subsequent negative movements of similar magnitude, limiting the formation of sustained trends. NEAR shows a gradual reduction in return variability toward the end of the time horizon, consistent with declining volatility and potentially reduced market participation. PEPE shows extremely high-frequency fluctuations, with return dynamics dominated by noise. This indicates intense trading activity and a weak signal-to-noise ratio. RENDER presents elevated variance, with large deviations from the mean. This reflects heightened uncertainty and sensitivity to evolving market narratives. SHIB is similar to PEPE, but with a more pronounced negative drift in the latter part of the sample, as evidenced by clusters of negative returns. SOL has high but structured volatility, with significant movements in both directions. This suggests an asset actively traded in both bullish and bearish conditions. TAO is characterized by isolated and large spikes within an otherwise low-activity context, indicating sensitivity to discrete information shocks rather than continuous dynamics. UNI reflects sector-specific dynamics, with upward movements often followed by corrections, resulting in limited persistence of positive returns.

Overall, the sixteen return series highlights several common features. First, a strong synchronization of risk emerges across assets. Periods of market stress are characterized by simultaneous negative spikes, suggesting that systemic factors dominate idiosyncratic dynamics during downturns. Second, a clear hierarchy in return variance can be observed. Large-cap assets such as BTC and BNB exhibit more stable distributions, whereas smaller or more speculative assets display heavier tails and higher variability. Third, differences in resilience are evident. Assets associated with weaker performance exhibit more persistent volatility, with slower reversion to the mean following adverse shocks, compared to leading assets.

To evaluate the effectiveness of WERM indices, Figs. 6–7 shows the heatmaps of average WERM distribution distance (WERM), the average difference of Wavelet Entropy (WE), the average Root Mean Square Error (RMSE) and the average Mean Directional Accuracy (DA) of the model, isolated the effects of noise type and deterministic component, considering also the different levels of weights, for prices and log-returns of the sixteen cryptocurrency series. Notes that the indices are estimated out-of-sample, using the validation set, left out in block bootstrap procedure.

Fig. 6 shows the heatmaps of WERM (first from top), WE (second), RMSE (third), and MDA (bottom) for cryptocurrency log-price series. The upper heatmap highlights the ability of the WERM measure to discriminate among alternative data-generating processes. For pure stochastic specifications, persistent noise structures show the lowest distribution distances, indicating the best fitting performance relative to all other configurations, including those incorporating chaotic or periodic deterministic components. This result suggests that persistence represents a dominant feature in price dynamics. In contrast, the introduction of a polynomial deterministic component systematically worsens the fit, as reflected by the increase in WERM distance. This indicates that periodicity acts as an alternative, rather than complementary, mechanism to stochastic persistence, and does not adequately capture the intrinsic structure of the data. The interaction between stochastic persistence and chaotic dynamics provides further insights. Configurations combining chaos and persistent noise yield good performance only when the chaotic component receives a relatively higher weight. Conversely, when the stochastic component dominates, anti-persistent noise specifications become preferable, suggesting that the chaotic contribution tends to overlap with the effects of persistence. More generally, chaotic specifications consistently outperform periodic deterministic models across all weight configurations, confirming that non-linear, aperiodic structures provide a more

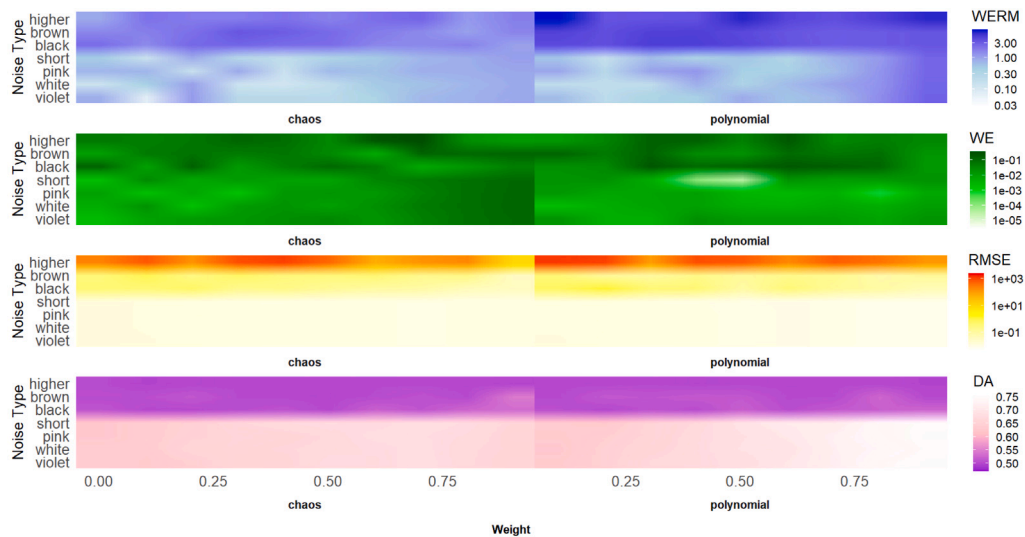


Fig. 7. Heatmaps for (from top to bottom): average WERM distribution distance, Wavelet Entropy, Root Mean Square Error and Mean Directional Accuracy of forecasts according to the noise and the non-random component (chaos on the left and polynomial on the right). All metrics are evaluated on cryptocurrency log-returns using a fixed sample length and identical forecasting and estimation procedure across scenarios.

suitable representation of the underlying price dynamics than regular, repetitive patterns. The second heatmap, reporting WE, appears much more “fragmented” and localized, with isolated light-colored peaks, e.g. corresponding to short or brown noise at certain weights. This suggests that it struggles to provide a homogeneous and coherent reading of the noise type, reacting in a more unstable or less linear manner to systematic variations in the model. Furthermore, the difference between chaos and polynomial regimes is much less marked. The structural patterns do not change macroscopically between the two regimes; the metric seems dominated by local fluctuations rather than the macro-dynamics of the underlying deterministic component. The third heatmap, representing RMSE, shows a relatively homogeneous pattern across specifications, indicating limited discriminative power of this metric. Although slight variations emerge across noise types and weights, RMSE primarily serves to exclude poorly performing models rather than to identify the most structurally appropriate specification. Similarly, the bottom heatmap, reporting DA, displays a largely uniform distribution across configurations, suggesting that directional predictive performance is only weakly affected by the choice of noise structure or deterministic component. Overall, these results confirm that while RMSE and DA provide useful measures of predictive performance, and WE captures specific local characteristics, they are insufficient for model selection in isolation. The WERM distance, by contrast, captures structural differences across competing specifications, allowing for a more informed identification of the data-generating process underlying cryptocurrency price dynamics.

Fig. 7 shows the heatmaps of the WERM (first from top), WE (second), RMSE (third), and MDA (bottom), according to the type of noise and the non-random component, calculated on all the cryptocurrency log-returns, distinguishing between the effect of chaos and polynomial trend. In particular, the upper heatmap represents WERM ability to identify cryptocurrency characteristics in terms of chaos and persistence/anti-persistence. It shows a marked concentration of minimum values in correspondence with short-memory or anti-persistent noise models. Furthermore, the chaos deterministic component has a better WERM distribution fitting with respect to polynomial deterministic component, suggesting that there is a noticeable presence of complex, non-repetitive and non-random behavior. The second heatmap, representing WE, appears almost completely saturated on dark tones, due to the limited scale of values, which indicates a general flatness of the metric. The only real exception is found in the polynomial regime corresponding to the short noise, where a conspicuous white spot around the weights 0.40 – 0.60 can be observed. This isolation suggests that the WE does not read the global structure of the noise, but reacts to specific local characteristics. More specifically, in the chaos regime, the WE is completely blind to weight variations, remaining a uniform dark green expanse. In the polynomial regime, the emergence of a single clear anomaly on the short noise shows that the metric detects a strong entropy loss, but fails to map the continuous trends of the other types of noise. The third heatmap represents the forecast accuracy of the noise and deterministic components, measured by RMSE. Although less discriminative than the WERM distance in the stationary noise range, the RMSE accurately identifies the unsuitability of persistent models. The stability of the minimum values in the lower rows confirms that the models selected by WERM are also those that minimize the mean squared deviation from actual returns. However, the large homogeneity zone suggests the inadequacy of the RMSE as an isolated metric for model selection, as it does not distinguish between different dynamic “textures” with similar errors. The lower heatmap represents directional predictive power, measured by DA. The emergence of a large high-accuracy region at the intermediate weights of the chaos and anti-persistent noises is a notable finding. This metric confirms that the integration of non-linearly dependent chaotic components significantly improves the models’ ability to predict the sign of the next move, overcoming the market efficiency barrier. The integrated analysis of the four heatmaps suggests that the WERM approach is not merely complementary, but fundamental to correct model specification. While RMSE and DA provide

Table 3

p – values of the surrogate-based WERM test for log-prices and log-returns across cryptocurrencies.

Symbol	Log-prices	Log-returns
AAVE	0.0198	0.0198
ADA	0.0099	0.0198
AVAX	0.0495	0.0099
BNB	0.2970	0.0099
BTC	0.0099	0.0099
DOGE	0.0198	0.0495
DOT	0.0297	0.0099
ETH	0.0495	0.0099
LINK	0.1980	0.0198
NEAR	0.0297	0.0297
PEPE	0.1190	0.0396
RENDER	0.2180	0.0099
SHIB	0.0396	0.0297
SOL	0.0396	0.0099
TAO	0.2080	0.0099
UNI	0.0495	0.1089

extrinsic performance measures, i.e. how closely the model fits the data, WERM distance provides a measure of intrinsic fidelity, i.e. how closely the model resembles the data. It suggests that the optimal configuration for the analyzed series lies in a stochastic anti-persistence regime corrected by chaotic determinism. The spatial likeness of the “good” zones among the four maps validates the hypothesis that a crypto financial returns is better described as a complex system with short memory rather than a persistence or long-term models. In this context, the WE exhibits a highly localized and fragmented response, acting merely as an indicator of isolated structural anomalies or resonant frequencies rather than a continuous descriptor of the dynamic complexity. Conversely, adding WERM to the set of traditional metrics solves the problem of model identification under uncertainty, where DA shows a “plateau” of apparently uniform performance, WERM acts as a fine-tuner, isolating the configuration that guarantees maximum structural consistency at different time scales. This reduces the risk of directional overfitting, ensuring that the observed predictive power is rooted in the true dynamic nature of the asset.

Figs. 6 and 7 highlight that WERM distance shows markedly different patterns across levels and returns. For level series, the minimum distances are predominantly associated with persistent stochastic processes, indicating that long-memory dynamics represent the most suitable characterization of price evolution. In this setting, non-chaotic deterministic components do not improve the structural fit and, in most cases, worsen it. Chaotic components provide some improvement only when sufficiently dominant, suggesting that non-linear dynamics can partially replace persistence, but it is preferable model the long-memory behavior in a stochastic sense. In contrast, the WERM distance for returns identifies short-memory and anti-persistent noise structures as the best-fitting specifications, with a clear preference for configurations that incorporate chaotic dynamics rather than deterministic periodic components. This indicates that, once prices are transformed into returns, the dominant persistence disappears and is replaced by a more complex combination of short-range dependence and non-linear, aperiodic behavior. Crucially, while the standard WE remains largely uniform or pixelated across these shifts, failing to track the overarching transition between the two series representations, the WERM distance uncovers this scale-dependent transition. While prices are well described by persistent stochastic processes, returns are better characterized as short-memory systems linked to chaotic dynamics. This finding is consistent with the interpretation of financial time series as integrated processes, where persistence at the level of prices translates into more intricate and less predictable structures in returns. The comparison of RMSE heatmaps further reinforces this interpretation. In both cases, RMSE displays a relatively homogeneous pattern, with limited ability to discriminate among alternative specifications. For prices, it does not clearly favor any particular combination beyond excluding poorly performing models. For returns, it correctly penalizes persistent specifications but still exhibits a broad plateau of similarly performing models. This confirms that RMSE alone is insufficient to identify the true data-generating process, as it captures predictive accuracy but not structural adequacy. A similar limitation emerges from the analysis of directional accuracy. For price series, DA is largely uniform across specifications, indicating that directional predictability is weakly sensitive to the underlying model structure. For returns, a more structured pattern emerges, with higher accuracy concentrated in regions combining anti-persistent noise and intermediate levels of chaotic contribution. The integrated evidence highlights the complementary roles of the three metrics. While RMSE and DA provide extrinsic measures of predictive performance, they fail to distinguish between models with fundamentally different dynamic properties. Similarly, WE acts primarily as a localized detector of isolated characteristics, missing the global evolutionary path of the data-generating process. WERM distance, by contrast, acts as an intrinsic criterion, capturing the structural similarity between simulated and observed processes. Its ability to identify consistent “optimal regions”, playing a crucial role in preventing model misspecification.

The surrogate-based WERM test is used on cryptocurrencies log-prices and log-returns as an inferential complement to the descriptive analysis of their multiscale structure. Table 3 shows the empirical p – values calculated via $B = 1000$ replications

Table 3 shows a significant difference between log-prices and log-returns. In particular, log-prices p – values are in many cases higher than log-returns and only BTC and ADA show a strong deviation from linearity in their p – values. This result is consistent with the idea that prices are dominated by a persistence regime, which determines a predominantly linear long-term dynamic. For return

logs, p -values are generally lower and more concentrated toward significant values, suggesting that return dynamics contain a more clearly detectable component of nonlinear structure than levels. Overall, the comparison highlights that WERM is more sensitive to the dynamic structure of returns than levels. Log-returns amplify local dependence and complex components, making deviations from behavior consistent with linear processes more evident. Conversely, log-prices incorporate a greater component of trend and global persistence, which may attenuate or make differences less identifiable than linear surrogates.

4. Conclusions

In this work, we propose the Wavelet-Based Entropy-Rate Measure, a multiscale indicator designed to quantify the degree of predictability and structural characteristics of time series by combining wavelet decomposition, ordinal-pattern analysis, and entropy-rate estimation. The proposed framework was developed to address the discrimination between stochastic dynamics, deterministic regularities, and chaotic behavior when these components coexist across different temporal scales.

The simulation results show that WERM provides a consistent and interpretable representation of a broad spectrum of processes, highlighting the ability to distinguish among different nature of determinism and noise. Furthermore, response of WERM is not linear, but depends on the interaction between the deterministic component and noise. In this sense, the measure provides a richer representation of complexity than approaches based solely on global entropy or forecasting errors.

The empirical application to cryptocurrency markets further supports the usefulness of the proposed methodology. For price levels, the WERM-based matching procedure identifies persistent stochastic dynamics as the dominant structural feature, with only limited improvement from deterministic components. For returns, the picture changes noticeably, with the best-fitting configurations associated with short-memory or anti-persistent stochastic regimes combined with chaotic deterministic contributions. Across both levels and returns, chaotic specifications consistently outperform periodic deterministic alternatives, indicating that the non-random component of cryptocurrency dynamics is more likely irregular and nonlinear than cyclic and repetitive. These results are consistent with the literature relating to cryptocurrency forecasting.

Overall, the findings indicate that WERM offers a flexible and robust framework for studying multiscale complexity in both simulated and empirical time series. Its main strength is the ability to integrate local ordinal information with the distribution of energy across scales, thereby capturing both temporal dependence and scale-specific organization.

A limitation concerns the choice of the ordinal-analysis parameters. Although the present study adopts a fixed embedding dimension ($m = 4$) and delay ($\tau = 1$), future research should investigate the robustness of WERM under alternative parameter configurations and explore adaptive parameter selection strategies.

Furthermore, WERM is primarily investigated as a descriptive multiscale measure of memory, persistence, and predictability, with emphasis on its empirical ability to differentiate between short-memory stochastic processes, long-memory structures, and prototypical chaotic dynamics in controlled simulations. This exploratory analysis suggests that WERM captures meaningful differences across dynamical regimes, but does not yet provide a full statistical decision theory for classification between them. A possible direction for future work is the construction of a formal hypothesis testing framework distinguishing ordered from chaotic dynamics. Such a framework could be based either on asymptotic distributions of WERM under representative high- and low-entropy limiting regimes, or on appropriately designed surrogate procedures targeting specific null classes beyond linear stochasticity. In particular, surrogate schemes tailored to preserve deterministic structure while disrupting stochastic components, or vice versa, could enable refined tests for dynamical classification. Achieving this goal would require a systematic study of WERM under a broader family of nonlinear deterministic systems, including chaotic maps and low-dimensional deterministic attractors, as well as a more detailed analysis of its sensitivity to different forms of dependence. These developments would extend the current work from a primarily descriptive and diagnostic framework toward a full inferential theory of multiscale predictability.

CRedit authorship contribution statement

Maria Carannante: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Oday Masoudi:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Alessandro Mazzocchi:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.physa.2026.131935>.

Data availability

Data will be made available on request.

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