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Isolated Diophantine Numbers

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Abstract—In this note, we discuss the topology of Diophantine numbers, giving simple explicit examples of Diophantine isolated numbers (among those with the same Diophantine constants), showing that *Diophantine sets are not always Cantor sets*.
General properties of isolated Diophantine numbers are also briefly discussed.

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1. INTRODUCTION

Diophantine numbers are irrational numbers poorly approximated by rationals, namely, real numbers ξ satisfying, for some $\gamma, \tau > 0$,

$$|\xi q - p| \geqslant \frac{\gamma}{q^\tau}, \qquad \forall\, p \in \mathbb{Z},\ q \in \mathbb{N} = \{1, 2, \ldots\}. \tag{1.1}$$

Such numbers, which form a set of full Lebesgue measure in[1)] $\mathbb{R}$, arise naturally in number theory and in small divisor problems in dynamics. Indeed, since the seminal works of C. L. Siegel, in the context of linearization of holomorphic diffeomorphisms around a fixed point [11], and of A. N. Kolmogorov, in the context of Hamiltonian systems [7], Diophantine conditions as in (1.1) (or higher-dimensional analogs) are ubiquitous in perturbative Hamiltonian dynamics both in finite and infinite dimensions. Let us denote by $D_{\gamma,\tau}$ the Diophantine set of all real numbers satisfying condition (1.1) with fixed[2)] $0 < \gamma < 1/2$ and $\tau \geqslant 1$. Clearly, $D_{\gamma,\tau}$ is a closed and nowhere dense set. It is, therefore, natural to ask whether $D_{\gamma,\tau}$ is actually a Cantor set[3)], i. e., if it is also a perfect set (no isolated points). Such a question, in view of the Cantor – Bendixson theorem[4)], is equivalent to asking whether the discrete set of Diophantine sets is empty. An (authoritative) place where the term Cantor set appears in association with Diophantine sets is Chapter III of the fundamental and beautiful book, *Lectures on Celestial Mechanics*, by C. L. Siegel and J. K. Moser [12]. In § 32–36

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[1)] See, e. g., [12], end of § 25.
[2)] Notice that $D_{\gamma,\tau} = \emptyset$ whenever $\gamma \geqslant 1/2$ (trivially, taking $q = 1$ and the minimum over p) or, (by Dirichlet's Theorem), when $\tau < 1$.
[3)] We recall that a general Cantor set is a closed set consisting entirely of boundary points.
[4)] I. e., *Any closed subset of a Euclidean space can be written as the (disjoint) union of a discrete set and a perfect set.*

of [12], Moser, extending the previous text of Siegel[5)], includes a proof, in the analytic case, of his theorem on the persistence of invariant curves for area-preserving twist diffeomorphisms of the annulus [10]. As is well known, one of the main hypotheses is that $\omega/2\pi$ belongs to $D_{\gamma,\tau}$ for given γ, τ, where ω denotes the rotation number of the unperturbed invariant curve. Moser calls such numbers *admissible* and, on p. 245, writes[6)]: "*the set of admissible values for ω form a Cantor set of positive measure*". Although it does not appear a formal statement about the sets $D_{\gamma,\tau}$, reading p. 245 of [12], one might be led to the belief that Diophantine sets are Cantor sets.

However, it turns out that, in general, this is *not* the case: In Section 1, we show that the quadratic numbers $\alpha := (n + \sqrt{n^2+4})/2 = [n, n, n, \ldots] = [\bar{n}]$ (in continued fraction expansion) are, for any $n \geqslant 2$, isolated in $D_{\gamma,\tau}$ with $\gamma := 1/\alpha$ and $\tau := \log\alpha/\log n$. In Section 2, we briefly review some general properties of isolated Diophantine points, proven in [1, 2], which show, in particular, that isolated Diophantine points are not that rare; Section 3 contains concluding remarks.

2. ELEMENTARY EXAMPLES OF ISOLATED DIOPHANTINE NUMBERS

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$ and define*

$$\alpha := \frac{n + \sqrt{n^2+4}}{2}\ , \qquad \gamma := \frac{1}{\alpha}\ , \qquad \tau := \frac{\log\alpha}{\log n}. \tag{2.1}$$

Then, α is an isolated point of $D_{\gamma,\tau}$.

Remark 1. (*i*) By definition of $D_{\gamma,\tau}$, it follows immediately that

$$I_{\gamma,\tau}(p,q) := \left\{\xi \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < \frac{\gamma}{q^{\tau+1}}\right\} \subset \mathbb{R}\backslash D_{\gamma,\tau}\ , \quad \forall q \in \mathbb{N}, \forall p \in \mathbb{Z}\ . \tag{2.2}$$

(*ii*) $D_{\gamma,\tau}$ is invariant by translations by integers, as, for $k \in \mathbb{Z}$, $\xi \in D_{\gamma,\tau} \iff \xi + k \in D_{\gamma,\tau}$. Therefore, $\alpha - n = \frac{\sqrt{n^2+4}-n}{2} = [0, \bar{n}] \in (0,1)$ is an isolated point of $D_{\gamma,\tau} \cap [0,1]$.

As one may expect, proofs make use of the theory of continued fractions; see [6] for general information. Let $\gamma > 0$, $\tau \geqslant 1$; let ξ be an irrational number and let

$$\xi = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

be its continued fraction expansion, $p_k/q_k = [a_0, a_1, \ldots, a_k]$ its k^{th} convergent, and $a'_k := [a_k, a_{k+1}, \ldots]$ its k^{th} complete quotient.

Lemma 1. *A number ξ belongs to $D_{\gamma,\tau}$ if and only if*

$$\frac{q_{k+1}}{q_k^\tau} + \frac{1}{a'_{k+2} q_k^{\tau-1}} \leqslant \frac{1}{\gamma}\ , \qquad \forall k \geqslant 0\ . \tag{2.3}$$

Proof. From continued fraction theory, one knows that[7)]

$$\xi \in D_{\gamma,\tau} \quad \iff \quad \left|\xi - \frac{p_k}{q_k}\right| \geqslant \frac{\gamma}{q_k^{\tau+1}}\ , \qquad \forall k \geqslant 0. \tag{2.4}$$

Then[8)],

$$\left|\xi - \frac{p_k}{q_k}\right| = \frac{1}{q_k(a'_{k+1}q_k + q_{k-1})} = \frac{1}{q_k{}^{\tau+1}}\ \frac{q_k^\tau}{a'_{k+1}q_k + q_{k-1}}$$

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5) Compare with the 1971 Preface to the English Edition of [12].

6) Obviously, Moser considers Diophantine sets with exponent $\tau > 1$ (see p. 242), as it is well known that for $\tau = 1$, $D_{\gamma,\tau}$ is not, in general, a Cantor set (see, e. g., [3]).

7) For a proof, see, e. g., Lemma 1, Appendix 8, p. 122 of [5].

8) $p_{-1} := 1$, $q_{-1} := 0$; for the first equality, see [6, §10.7].

$$= \frac{1}{{q_k}^{\tau+1}} \; \frac{q_k^\tau}{a_{k+1}q_k + q_{k-1} + \frac{q_k}{a'_{k+2}}}$$
$$= \frac{1}{{q_k}^{\tau+1}} \; \frac{q_k^\tau}{q_{k+1} + \frac{q_k}{a'_{k+2}}}$$
$$= \frac{1}{{q_k}^{\tau+1}} \left(\frac{q_{k+1}}{q_k^\tau} + \frac{1}{a'_{k+2} q_k^{\tau-1}} \right)^{-1} ,$$

and the claim follows from (2.4). □

Proof (*of Theorem* 1). One immediately verifies that

$$\begin{cases} \alpha = n + \dfrac{1}{\alpha} \,, \qquad n^\tau = \alpha \,, \\ \alpha = [n, n, n, n, \dots] \,, \\ p_0 = n,\; q_0 = 1,\; p_1 = n^2+1,\; q_1 = n,\; a'_k = \alpha,\; q_{k+1} = p_k \;(\forall k \geqslant 0) \,. \end{cases} \tag{2.5}$$

Thus, for $k = 0$ we have

$$\left| \alpha - \frac{p_0}{q_0} \right| \overset{(2.5)}{=} \alpha - n \overset{(2.5)}{=} \frac{1}{\alpha} \overset{(2.1)}{=} \gamma. \tag{2.6}$$

For $k \geqslant 1$, using (2.5) and the facts that $p_k/q_k \leqslant p_1/q_1$ and $q_k \geqslant q_1$, one finds

$$\begin{aligned} \frac{q_{k+1}}{q_k^\tau} + \frac{1}{a'_{k+2} q_k^{\tau-1}} &= \frac{p_k}{q_k} \frac{1}{q_k^{\tau-1}} + \frac{1}{\alpha q_k^{\tau-1}} \\ &\leqslant \frac{p_1}{q_1} \frac{1}{q_1^{\tau-1}} + \frac{1}{\alpha q_1^{\tau-1}} = \frac{n^2+1}{n^\tau} + \frac{1}{n^{\tau-1}\alpha} \\ &= \frac{n^2+1}{\alpha} + \frac{n}{\alpha^2} = \frac{1}{\alpha}\Big(n^2 + 1 + \frac{n}{\alpha} \Big) \\ &= \frac{1}{\alpha}\,(\alpha n + 1) = n + \frac{1}{\alpha} = \alpha \\ &= \frac{1}{\gamma} \,, \end{aligned}$$

which, together with (2.6) and the lemma, shows that $\alpha \in D_{\gamma,\tau}$.
Next, because of (2.5),

$$\begin{aligned} \left| \alpha - \frac{p_1}{q_1} \right| &= \frac{p_1}{q_1} - \alpha = \frac{n^2+1}{n} - \alpha = \frac{1}{n} + n - \alpha \\ &= \frac{1}{n} - \frac{1}{\alpha} = \frac{1}{n\alpha^2} = \frac{1}{\alpha} \; \frac{1}{q_1 n^\tau} = \frac{1}{\alpha q_1^{\tau+1}} \\ &= \frac{\gamma}{q_1^{\tau+1}}. \end{aligned}$$

Such a relation, together with (2.6), shows that α separates the two intervals[9] $I_{\gamma,\tau}(p_0, q_0)$ and $I_{\gamma,\tau}(p_1, q_1)$, and, therefore, α is an *isolated point* of $D_{\gamma,\tau}$. □

3. GENERAL PROPERTIES OF ISOLATED DIOPHANTINE NUMBERS

General properties of isolated Diophantine numbers have been investigated in [1, 2], where proofs may be found. Let us briefly report here the main results in [1, 2].

[9] Recall the definition of the open intervals $I_{\gamma,\tau}(p, q)$ in (2.2).

The first result in [1] shows that isolated points are not that rare: Indeed, *any* Diophantine number has at least one equivalent representative[10)], which is isolated in some Diophantine set:

Theorem 2 ([1, Theorem B]). *Fix* $\gamma \in (0, \frac{1}{2})$, $\tau \geqslant 1$, $\alpha \in D_{\gamma,\tau}$, *and let* $m := \left[\frac{3\cdot 2^\tau}{\gamma}\right]$. *Then, the equivalent Diophantine number* $\alpha' := \dfrac{m\alpha+1}{(2m+1)\alpha+2}$ *is an isolated point of* $D_{\gamma_\alpha,\tau_\alpha}$ *for suitable* $\tau_\alpha > \tau$ *and* $\gamma_\alpha > 0$.

Actually, it can happen that a Diophantine number is *simultaneously isolated for infinitely many Diophantine sets*. More precisely[11)]:

Theorem 3 ([1, Theorem A]). *For all* $\tau \geqslant 1$, *there exist* $\gamma > 0$ *and* $\alpha \in D_{\gamma,\tau}$ *such that* α *is an isolated point for* D_{γ_n,τ_n}, *for suitable sequences* $\tau_n \searrow \tau$, $\gamma_n \searrow \gamma$.

Even though these two theorems show the existence of many isolated Diophantine numbers, from the metric point of view, the typical situation seems to be that Diophantine sets are Cantor sets:

Theorem 4 ([2]). *Let* $\tau > \tau_0 := \frac{3+\sqrt{17}}{2} = [\overline{3,1,1}]$. *Then, for almost all* $\gamma \in (0, 1/2)$, $D_{\gamma,\tau}$ *is a Cantor set.*

4. REMARKS

(*i*) The Diophantine exponent $\tau_0 := \frac{3+\sqrt{17}}{2}$ in Theorem 4 is certainly not optimal, and it would not be difficult to improve it. On the other hand, it is not so obvious what is the *optimal* τ_0, for which the statement of Theorem 4 holds.

(*ii*) Diophantine sets, as pointed out in the Introduction, play a fundamental role in dynamics, e. g., in the theory of exact symplectic twist diffeomorphisms. Arithmetic properties of the rotation number of an invariant curve of a twist diffeomorphism are, in particular, relevant for the renormalization point of view; compare [8]. Now, even though Theorem 2 above indicates that the property of being isolated for Diophantine numbers may not be a stable property under renormalization, it would be interesting to see if such a property does have a counter part in dynamics. For example,

Does there exists a C^r *exact symplectic twist diffeomorphism* f, $r \geqslant 2$, *having an isolated invariant curve of rotation number* α *that is not of bounded type, with* α *isolated point of a suitable Diophantine set?*

(*iii*) We point out that, modifying suitably the definition of the set of Diophantine numbers, one obtains right away Cantor sets; compare Lemma 2.3 in [9].

(*iv*) A final comment on higher-dimensional Diophantine sets.
Let $n \geqslant 1$, $\gamma, \tau > 0$, an define

$$D^n_{\gamma,\tau} := \{\alpha \in \mathbb{R}^n : |q \cdot \alpha - p| \geqslant \frac{\gamma}{|q|^\tau}, \forall q \in \mathbb{Z}^n \backslash \{0\}, p \in \mathbb{Z}\}.$$

The analogous problem discussed in this note is[12)]:

For $n \geqslant 2$, *do there exist* $\gamma, \tau > 0$ *such that* $D^n_{\gamma,\tau}$ *is not a Cantor set?*

Clearly, such a question may be more difficult to analyze due to the lack of the beautiful and powerful theory of continued fractions.

[10)] Recall that two irrational numbers ξ and ξ' are *equivalent* if and only if $\xi' = \frac{a\xi+b}{c\xi+d}$ with integers a, b, c, d satisfying $ad - bc = \pm 1$, and that happens if and only if the continued fractions of ξ and ξ' differ only by a finite number of terms; compare [6, §10.11].

[11)] In [1] something stronger is proven, in the sense that the sequence τ_n in Theorem 3 can be assigned arbitrarily up to small errors.

[12)] For related questions on *homogeneous* Diophantine sets $\mathbb{R}^n_{\gamma,\tau} := \{\omega \in \mathbb{R}^n : |\omega \cdot k| \geqslant \gamma/|k|^\tau,\ \ \forall k \in \mathbb{Z}^n, k \neq 0\}$, compare [4].

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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