Physarum Can Compute Shortest Paths*

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Abstract

Physarum Polycephalum is a slime mold that is apparently able to solve shortest path problems. A mathematical model has been proposed by Tero, Kobayashi and Nakagaki [Journal of Theoretical Biology, 244, 2007, pp. 553–564] to describe the feedback mechanism used by the slime mold to adapt its tubular channels while foraging two food sources s_0 and s_1. We prove that, under this model, the mass of the mold will eventually converge to the shortest s_0-s_1 path of the network that the mold lies on, independently of the structure of the network or of the initial mass distribution.

This matches the experimental observations by Tero et al. and can be seen as an example of a "natural algorithm", that is, an algorithm developed by evolution over millions of years.

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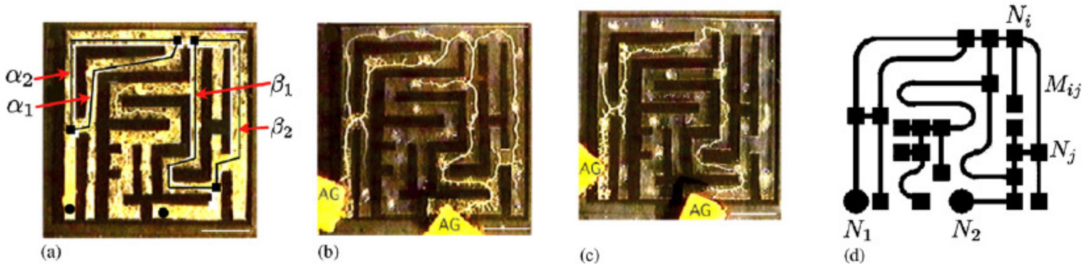

Figure 1: The experiment in [14] (reprinted from there): (a) shows the maze uniformly covered by Physarum; the yellow color indicates the presence of Physarum. Food (oatmeal) is provided at the locations labelled AG. After a while, the mold retracts to the shortest path connecting the food sources as shown in (b) and (c). (d) shows the underlying abstract graph. The video [17] shows the experiment.

1 Introduction

Physarum Polycephalum is a slime mold in the Mycetozoa group [2] that is apparently able to solve shortest path problems. Nakagaki, Yamada, and Tóth [14] report on the following experiment, see Figure 1: They built a maze, covered it with pieces of Physarum (the slime can be cut into pieces that will reunite if brought into vicinity), and then fed the slime with oatmeal at two locations. After a few hours, the slime retracted to a path that follows the shortest path connecting the food sources in the maze. The authors report that they repeated the experiment with different mazes; in all experiments, Physarum retracted to the shortest path. There are several videos available on the web that show the mold in action [17].

Tero, Kobayashi and Nakagaki [15] propose a mathematical model for the behavior of the mold and argue extensively that the model is adequate. We will not repeat the discussion here, but only introduce the model. Physarum is modeled as a tube network traversed by liquid flow, with the flow satisfying the standard Poiseuille assumption from fluid mechanics. In the following, we use terminology from the theory of electrical networks, relying on the well-known fact that the equations for electrical flow and Poiseuille flow are the same [9].

We have an undirected graph $G = (N, E)$ with distinguished nodes s_0 and s_1; the edges of the graph model the tubular channels of the Physarum, while s_0 and s_1 model the food sources. Each edge $e \in E$ has a positive length L_e and a positive diameter (or conductivity[1]) $D_e(t)$; L_e is fixed, while D_e is a function of time. The resistance $R_e(t)$ of e is $R_e(t) = L_e/D_e(t)$. A current of value 1 is forced from s_0 to s_1. Let $Q_e(t)$ be the resulting current over any edge $e = (u, v)$, where (u, v) is an arbitrary orientation of the edge; the current models the protoplasmic flow across tubes. The diameter of edge e evolves according to the equation

$$\dot{D}_e(t) = |Q_e(t)| - D_e(t), \tag{1}$$

where $\dot{D}_e$ is the derivative of D_e with respect to time.[2] In equilibrium ($\dot{D}_e = 0$ for all e), the flow through any edge is equal to its diameter. In non-equilibrium, the diameter grows or

[1]From a dimensional point of view, the value $D_e(t)$ is indeed a conductivity which is proportional to the fourth power of the actual diameter of the tubular channel; we prefer to use the term diameter, to avoid confusion with the notion of conductance from electrical networks.

[2]Tero et al. define the dynamics more generally as $\dot{D}_e(t) = f(|Q_e(t)|) - D_e(t)$ where f is increasing and then specialize among others to $f(x) = x$ for all x.

shrinks if the absolute value of the flow is larger or smaller than the diameter, respectively. In the sequel, we will mostly drop the argument t as is customary in the treatment of dynamical systems. We also observe that because of the presence of the absolute value in Equation (1), any inversion in the direction of the flow (that is, exchanging the source and sink) would not bear any effect on the dynamics of the D_e values.

The model is readily turned into a computer simulation. In an electrical network, every vertex v has a potential p_v (in the Physarum, this models hydrostatic pressure); p_v is a function of time. We may fix p_{s_1} to zero. For an edge $e = (u, v)$, the flow across e is given by $(p_u - p_v)/R_e$. We have flow conservation in every vertex except for s_0 and s_1; we inject one unit at s_0 and remove one unit at s_1. Thus,

$$\sum_{v \in \delta(u)} \frac{p_u - p_v}{R_{uv}} = b(u) \quad \text{for all } u \in N. \tag{2}$$

where $\delta(u)$ is the set of nodes adjacent to u in G, $b(s_0) = 1$, $b(s_1) = -1$, and $b(u) = 0$ otherwise. The linear system (2) and the convention $p_{s_1} = 0$ determine the node potentials uniquely. They can be computed by solving the linear system either directly or indirectly. Tero, Kobayashi and Nakagaki [15] were the first to perform simulations of the model. They report that the dynamics (1) always converge to the shortest s_0-s_1 path, that is, the diameters of the edges on the shortest path converge to one, and the diameters on the edges outside the shortest path converge to zero. This holds true for any initial condition and assumes the uniqueness of the shortest path.

Miyaji and Ohnishi [11, 12] initiated the analytical investigation of the model. They argued convergence against the shortest path if G is a planar graph and s_0 and s_1 lie on the same face in some planar embedding of G.

Our main result is a convergence proof for all graphs. For a network $G = (N, E, s_0, s_1, L)$, where $(L_e)_{e \in E}$ is a positive length function on the edges of G, we use $G_0 = (N, E_0)$ to denote the subgraph of all shortest source-sink paths, L^* to denote the length of a shortest source-sink path, and $\mathcal{E}^*$ to denote the set of all source-sink flows of value one in G_0. If we define the cost of flow Q as $\sum_e L_e Q_e$, then $\mathcal{E}^*$ is the set of minimum cost source-sink flows of value one. If the shortest source-sink path is unique, $\mathcal{E}^*$ is a singleton. The dynamics are *attracted* by a set $A \subseteq \mathbb{R}^E$ if the distance (measured in any L_p-norm) between $D(t)$ and A converges to zero over time.

Theorem [Theorem 2 in Section 6] *Let $G = (N, E, s_0, s_1, L)$ be an undirected network with positive length function $(L_e)_{e \in E}$. Let $D_e(0) > 0$ be the diameter of edge e at time zero. The dynamics (1) are attracted to $\mathcal{E}^*$. If the shortest source-sink path is unique, the dynamics converge to the flow of value one along the shortest source-sink path.*

When the shortest source-sink path is not unique, we conjecture that the dynamics converge to an element of $\mathcal{E}^*$, though we only show attraction to $\mathcal{E}^*$. A key part of our proof is to show that the function

$$V = \frac{1}{\min_{S \in \mathcal{C}} C_S} \sum_{e \in E} L_e D_e + (C_{\{s_0\}} - 1)^2 \tag{3}$$

decreases along all trajectories that start in a non-equilibrium configuration. Here, $\mathcal{C}$ is the set of all s_0-s_1 cuts, that is, the set of all $S \subseteq N$ with $s_0 \in S$ and $s_1 \notin S$; $C_S = \sum_{e \in \delta(S)} D_e$ is the total diameter of the cut S or equivalently, the capacity of the cut S when the capacity of edge

e is set to D_e (the expression $\delta(S)$ denotes the set of edges with exactly one endpoint in S); and $\min_{S \in \mathcal{C}} C_S$ (also abbreviated by C) is the capacity of the minimum cut. The first term in the definition of V is the normalized "hardware" cost; for any edge, the product of its length and its diameter may be interpreted as the hardware cost of the edge; the normalization is by the capacity of the minimum cut. The first term decreases except when $|Q_e| = \lambda \cdot D_e$ for all $e \in E$ and some $\lambda \geq 0$. The second term decreases as long as the capacity of the cut defined by s_0 is different from 1. We show that the capacity of the minimum cut converges to one and that V is decreasing. Since V is non-negative, this implies that the derivative of V must converge to zero. We then bound the quantity $\sum_e (D_e/C - |Q_e|)^2$ in terms of the absolute value of the derivative of V; this allows us to conclude that $|D_e - |Q_e||$ converges to zero for all $e \in E$. In the next step, we show that the potential difference $\Delta = p_{s_0} - p_{s_1}$ between source and sink converges to the length L^* of a shortest-source sink path. We use this to conclude that D_e and Q_e converge to zero for any edge $e \notin E_0$. Finally, we show that the dynamics are attracted by $\mathcal{E}^*$.

We found the function V by analytical investigation of a network of parallel links (see Section 4), extensive computer simulations, and guessing. Functions decreasing along all trajectories are called Lyapunov functions in dynamical systems theory [7]. The fact that the right-hand side of system (1) is not continuously differentiable and that the function V is not differentiable everywhere introduces some technical difficulties.

The direction of the flow across an edge depends on the initial conditions and time. We do not know whether flow directions can change infinitely often or whether they become ultimately fixed. Under the assumption that flow directions stabilize, we can characterize the (late stages of the) convergence process. An edge $e = \{u, v\}$ becomes *horizontal* if $\lim_{t\to\infty} |p_u - p_v| = 0$, and it becomes *directed* from u to v (directed from v to u) if $p_u > p_v$ for all large t ($p_v > p_u$ for all large t). An edge *stabilizes* if it either becomes horizontal or directed, and a network *stabilizes* if all its edges stabilize. If a network stabilizes, we partition its edges into a set E_h of horizontal edges and a set $\overrightarrow{E}$ of directed edges. If $\{u, v\}$ becomes directed from u to v, then $(u, v) \in \overrightarrow{E}$.

We introduce the notion of a *decay rate*. Let $r \leq 0$. A *quantity $D(t)$ decays with rate at least r* if for every $\varepsilon > 0$ there is a constant A such that $\ln D(t) \leq A + (r + \varepsilon)t$ for all t. A *quantity $D(t)$ decays with rate at most r* if for every $\varepsilon > 0$ there is a constant a such that $\ln D(t) \geq a + (r - \varepsilon)t$ for all t. A quantity $D(t)$ decays with rate r if it decays with rate at least and at most r.

Lemma [Lemma 20 in Section 7] *For $e \in E_h$, D_e decays with rate -1 and $|Q_e|$ decays with rate at least -1.*

We define a decomposition of G into paths P_0 to P_k, an orientation of these paths, a slope $f(P_i)$ for each P_i, a vertex labelling p^*, and an edge labelling r. P_0 is a[3] shortest s_0-s_1 path in G, $f(P_0) = 1$, $r_e = f(P_0) - 1$ for all $e \in P_0$, and $p^*_v = \text{dist}(v, s_1)$ for all $v \in P_0$, where $\text{dist}(v, s_1)$ is the shortest path distance from v to s_1. For $1 \leq i \leq k$, we have[4] $P_i = \text{argmax}_{P \in \mathcal{P}} f(P)$, where $\mathcal{P}$ is the set of all paths P in G with the following properties: (1) the startpoint a and the endpoint b of P lie on $P_0 \cup \ldots \cup P_{i-1}$, $p^*_a \geq p^*_b$, and $f(P) = (p^*_a - p^*_b)/L(P)$; (2) no interior vertex of P lies on $P_0 \cup \ldots \cup P_{i-1}$; and (3) no edge of P belongs to $P_0 \cup \ldots \cup P_{i-1}$. If $p^*_a > p^*_b$, we direct P_i from a to b. If $p^*_a = p^*_b$, we leave the edges in P_i undirected. We set

[3] We assume that P_0 is unique.

[4] We assume that P_i is unique except if $f(P_i) = 0$.

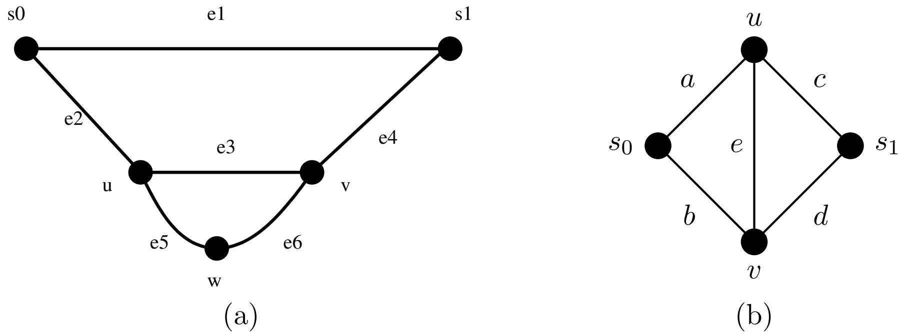

Figure 2: Part (a) illustrates the path decomposition. All edges are assumed to have length 1; $P_0 = (e_1)$, $P_1 = (e_2, e_3, e_4)$, $P_2 = (e_5, e_6)$, $p^*_{s_0} = 1$, $p^*_{s_1} = 0$, $p^*_v = 1/3$, $p^*_u = 2/3$, $p^*_w = 1/2$, $f(P_1) = 1/3$, and $f(P_2) = 1/6$.
Part (b) shows the Wheatstone graph. The direction of the flow on edge $\{u, v\}$ may change over time; the flow on all other edges is always from left to right.

$r_e = f(P_i) - 1$ for all edges of P_i, and $p^*_v = p^*_b + f(P_i)\,\mathrm{dist}_{P_i}(v, b)$ for every interior vertex v of P_i. Figure 2(a) illustrates the path decomposition.

Lemma [Lemma 21 in Section 7] *There is an $i_0 \le k$ such that*

$$f(P_0) > f(P_1) > \ldots > f(P_{i_0}) > 0 = f(P_{i_0+1}) = \ldots = f(P_k).$$

Theorem [Theorem 3 in Section 7] *If a network stabilizes, $\overrightarrow{E} = \cup_{i \le i_0} E(P_i)$, the orientation of any edge $e \in \overrightarrow{E}$ agrees with the orientation induced by the path decomposition, and $E_h = \cup_{i > i_0} E(P_i)$. The potential of each node v converges to p^*_v. The diameter of each edge $e \in E \setminus P_0$ decays with rate r_e.*

We cannot prove that flow directions stabilize in general. For all series-parallel graphs, flow directions trivially stabilize. The Wheatstone graph, shown in Figure 2(b), is the simplest graph in which flow directions may change over time.

Theorem [Theorem 6 in Section 8] *The Wheatstone graph stabilizes.*

Finally, we remark that for a more general excess vector $(b_v)_{v \in V}$ (cf. Eq. (2)), it is possible to extend the techniques of the current article to show that the model converges to an optimal solution of the so-called transportation problem; see article [4] for details.

The remainder of this article is organized as follows: In Section 2, we discuss related work, and in Section 3, we put our results into the context of natural algorithms and state open problems. The technical part of the paper starts in Section 4. We first treat a network of parallel links; this situation is simple enough to allow a direct analytical treatment. In Section 5, we review basic facts about electrical networks and prove some simple facts about the dynamics of Physarum. In Section 6, we prove our main result, the convergence for general graphs. In Section 7, we prove exponential convergence under the assumption that flow directions stabilize, and finally, in Section 8, we show that the Wheatstone network stabilizes.

2 Related Work

Miyaji and Ohnishi [11, 12] initiated the analytical investigation of the model. They argued convergence against the shortest path if G is a planar graph and s_0 and s_1 lie on the same face in some embedding of G. Ito et al. [8] study the dynamics (1) in a *directed* graph $G = (N, E)$; they do not claim that the model is justified on biological grounds. Each directed edge e has a diameter D_e. The node potentials are again defined by the equations

$$\sum_{v \in \delta(u)} \frac{p_u - p_v}{R_{uv}} = b(u) \quad \text{for all } u \in N.$$

The summation on the right-hand side is over all neighbors u of v; edge directions do not matter in this equation. If there is an edge from u to v and an edge from v to u, u occurs twice in the summation, once for each edge. The dynamics for the diameter of the directed edge (u, v) are then $\dot{D}_{uv} = Q_{uv} - D_{uv}$, where $Q_{uv} = D_{uv}(p_u - p_v)/L_{uv}$. The dynamics of this model are very different from the dynamics of the model studied in this article. For example, assume that there is an edge (v, u), no edge (u, v), and $p_u > p_v$ always. Then $Q_{vu} < 0$ always and hence D_{vu} will vanish at least with rate -1. The model is simpler to analyze than our model. Ito et al. prove that the directed model is able to solve transportation problems and that the D_e's converge exponentially to their limit values.

3 Discussion and Open Problems

Physarum may be seen as an example of a natural computer, that is, a computer developed by evolution over millions of years. It can apparently do more than compute shortest paths and solve transportation problems. In an article by Tero et al. [16], the computational capabilities of Physarum are applied to network design, and it is shown in lab and computer experiments that Physarum can compute approximately minimum Steiner trees. No theoretical analysis is available. The book [1] and the tutorial [13] contain many illustrative examples of the computational power of this slime mold.

Chazelle [5] advocates the study of natural algorithms; i.e., "algorithms developed by evolution over millions of years", using computer science techniques. Traditionally, the analysis of such algorithms belonged to the domain of biology, systems theory, and physics. Computer science brings new tools. For example, in our analysis, we crucially use the max-flow min-cut theorem.

We have only started the theoretical investigation of Physarum computation, and so many interesting questions are open. We prove convergence for the dynamics $\dot{D}_e = f(|Q_e|) - D_e$, where f is the identity function. The literature also suggests the use of $f(x) = x^\gamma/(1 + x^\gamma)$ for some parameter γ. Can one prove convergence for other functions f? We prove that flow directions stabilize in the Wheatstone graph. Do they stabilize in general? We prove, but only for stabilizing networks, that the diameters of edges that are not on the shortest path converge to zero exponentially for large t. What can be said about the initial stages of the process? The Physarum computation is fully distributed; node potentials depend only on the potentials of the neighbors, currents are determined by potential differences of edge endpoints, and the update rule for edge diameters is local. Can the Physarum computation be used as the basis for an efficient distributed shortest path algorithm? What other problems can be provably solved with Physarum computations?

4 Parallel Links

We discovered the Lyapunov function used in the proof of our main theorem through experimentation. The experimentation was guided by the analysis of a network of parallel links. In such a network, there are vertices s_0 and s_1 connected with m edges of lengths $L_1 < L_2 < \ldots < L_m$. Let D_i be the diameter of the i-th link, and let $D = \sum_i D_i$. Let $\Delta = p_{s_0} - p_{s_1}$ be the potential difference between source and sink. Then, $Q_i = \Delta/R_i = D_i\Delta/L_i$. Since $\sum_i Q_i = 1$, we have $\Delta = 1/\sum_i D_i/L_i$.

Lemma 1 *The equilibrium points are precisely the single links.*

Proof: In an equilibrium point, $Q_i = D_i$ for all i. Since $Q_i = D_i\Delta/L_i$, this implies $\Delta = L_i$ whenever $Q_i \neq 0$. Thus, in an equilibrium there is exactly one i with $Q_i \neq 0$. Then, $Q_i = 1$. ∎

Lemma 2 *Let $D = \sum_i D_i$. Then, D converges to 1.*

Proof: We have $\dot{D} = \sum_i \dot{D}_i = \sum_i Q_i - \sum_i D_i = 1 - D$. The claim follows by directly solving the differential equation: $D(t) = 1 + (D(0) - 1)\exp(-t)$. ∎

For networks of parallel links, there are many Lyapunov functions.

Lemma 3 *Let $D = \sum_i D_i$, $x_i = D_i/D$, and let L be such that $1/L = \sum_j x_j/L_j$. The quantities*

$$\sum_{i\geq 2} D_i/D,\ \sum_i x_i L_i,\ L,\ \sum_i Q_i L_i,\ \Delta\sum_i D_i L_i,\ \text{and}\ \sum_{i\geq 2}(L_i \ln D_i - L_1 \ln D_1)$$

decrease along all trajectories, starting in non-equilibrium points.

Proof: Clearly, $\sum_j x_j = 1$ and $\Delta = L/D$. The derivative $\dot{x}_i$ of x_i computes as:

$$\dot{x}_i = \frac{\dot{D}_i D - D_i \dot{D}}{D^2} = \frac{(D_i\Delta/L_i - D_i)D - D_i(1-D)}{D^2} = \left(\frac{L}{L_i D} - \frac{1}{D}\right) x_i = \frac{1}{D}\left(\frac{L}{L_i} - 1\right) x_i.$$

We have $L > L_1$ iff $\sum_{j\geq 2} x_j > 0$. Thus, the derivative of x_1 is zero if $x_1 = 1$ and positive if $x_1 < 1$. Thus, $\sum_{i\geq 2} x_i$ decreases along all trajectories, starting in non-equilibrium points.

Let $V = \sum_i x_i L_i$. Then,

$$\dot{V} = \sum_i \frac{1}{D}\left(\frac{L}{L_i} - 1\right) x_i L_i = \frac{1}{D}\sum_i (L - L_i) x_i.$$

So, it suffices to show $\sum_i L_i x_i \geq L = 1/\sum_i x_i/L_i$, or equivalently, $(\sum_i L_i x_i)(\sum_i x_i/L_i) \geq 1$. This is an immediate consequence of the Cauchy-Schwarz inequality. Namely,

$$1 = \left(\sum_i \sqrt{x_i L_i}\sqrt{x_i/L_i}\right)^2 \leq \left(\sum_i (\sqrt{x_i L_i})^2\right) \cdot \left(\sum_i (\sqrt{x_i/L_i})^2\right).$$

Now, let $V = 1/L = \sum_j x_j/L_j$. We show that V is increasing. We have

$$\dot{V} = \sum_i \frac{\dot{x_i}}{L_i} = \frac{1}{D}\sum_i \left(\frac{L}{L_i} - 1\right)\frac{x_i}{L_i} = \frac{1}{D}\sum_i \left(\frac{Lx_i}{L_i}\frac{1}{L_i} - \frac{x_i}{L_i}\right).$$

Let $z_i = Lx_i/L_i$. Then, $z_i \ge x_i$ if $L \ge L_i$, and $z_i \le x_i$ if $L \le L_i$. Also $\sum_i z_i = 1$. Thus,

$$D \cdot \dot{V} = \sum_i \frac{z_i - x_i}{L_i} = \sum_{i:L \ge L_i} \frac{z_i - x_i}{L_i} + \sum_{i:L<L_i} \frac{z_i - x_i}{L_i} \ge \sum_{i:L\ge L_i} \frac{z_i - x_i}{L} + \sum_{i:L<L_i} \frac{z_i - x_i}{L} = 0.$$

Moreover, $\dot{V} = 0$ if and only if $z_i = x_i$ for all i if and only if x is a unit vector.

Consider next the function $\sum_i Q_i L_i$. Then,

$$\sum_i Q_i L_i = \sum_i \Delta \frac{D_i}{L_i} L_i = \Delta D = \frac{D}{\sum_i \frac{D_i}{L_i}} = \frac{1}{\sum_i \frac{x_i}{L_i}} = L;$$

hence, $\sum_i Q_i L_i$ is decreasing.

The function $\Delta \sum_i D_i L_i = L \cdot \sum_i x_i L_i$ is the product of positive decreasing functions and hence decreasing.

Finally, let $V = \sum_{i\ge2}(L_i \ln D_i - L_1 \ln D_1)$. Then

$$\begin{aligned}\dot{V} &= \sum_{i\ge2}\left(L_i \frac{\dot{D}_i}{D_i} - L_1\frac{\dot{D}_1}{D_1}\right) = \sum_{i\ge2}\left(L_i\frac{Q_i - D_i}{D_i} - L_1\frac{Q_1 - D_1}{D_1}\right)\\ &= \sum_{i\ge2}\left(L_i\frac{D_i\Delta/L_i - D_i}{D_i} - L_1\frac{D_1\Delta/L_1 - D_1}{D_1}\right) = \sum_{i\ge2}(L_1 - L_i) < 0.\end{aligned}$$

∎

The Lyapunov function $\sum_{i\ge2}(L_i \ln D_i - L_1 \ln D_1)$ was already considered in [11].

Theorem 1 (Miyashi-Ohnishi [11]) *For a network of parallel links, the dynamics converge against $D_1 = 1$ and $D_i = 0$ for $i \ge 2$.*

Proof: $x_1 = D_1/D$ is monotonically increasing and bounded by 1. Hence, it converges. Assume that the limit x_1^* is less than one. Clearly, $x_1^* > 0$. For $x_1 \le x_1^*$, we have $1/L = \sum_i x_i/L_i \le x_1^*/L_1 + (1 - x_1^*)/L_2$. Moreover, for large enough t, $x_1 \ge x_1^*/2$ and $D \le 2$ (Lemma 2), and hence, $\dot{x_1} \ge \varepsilon$ for some $\varepsilon > 0$. Thus, $x_1^* < 1$ is impossible. ∎

Some of the Lyapunov functions have natural interpretations: $\sum_i Q_i L_i$ is the total cost of the flow; $(\sum_i D_i L_i)/\sum_i D_i$ is the total hardware cost normalized by the total diameter, where a link of length L and diameter D has cost DL; and $\Delta \sum_i D_i L_i$ is the potential difference between source and sink multiplied by total hardware cost. These functions are readily generalized to general networks by interpreting the summations as summations over all edges of the network. Our computer simulations showed that none of these functions is a Lyapunov function for general networks.

However, $\sum_i D_i$ can also be interpreted as the total diameter of a source-sink cut. With this interpretation, $(\sum_i D_i L_i)/\sum_i D_i$ becomes

$$\frac{\sum_e D_e L_e}{\min_{S\in\mathcal{C}} C_S},$$

where $\mathcal{C}$ is the set of all s_0-s_1 cuts and C_S is the total diameter of the cut C. Our computer simulations suggested that this function may serve as a Lyapunov function for general graphs. We will see below that a slight modification is actually a Lyapunov function.

5 Graphs, Electrical Networks and Simple Facts

In this section we establish some more notation, review basic properties of graphs and electrical networks, and prove some simple facts.

Each node v of the graph G has a potential p_v that is a function of time. A potential difference Δ_e between the endpoints of an edge e induces a flow on the edge. For $e = (u, v)$,

$$Q_e = D_e \Delta_e / L_e = D_e(p_u - p_v)/L_e = (p_u - p_v)/R_e \tag{4}$$

is the flow across e in the direction from u to v. If $Q_e < 0$, the flow is in the reverse direction. The potentials are such that there is flow conservation in every vertex except for s_0 and s_1 and such that the net flow from s_0 to s_1 is one, that is, for every vertex u, we have

$$\sum_{v\in\delta(u)} Q_{uv} = b(u), \tag{5}$$

where $\delta(u)$ is the set of neighbors of u, $b(s_0) = 1 = -b(s_1)$ and $b(u) = 0$ for all other vertices u. After fixing one potential to an arbitrary value, say $p_{s_1} = 0$, the other potentials are readily determined by solving a linear system. This means that each Q_e can be expressed as a function of the vector R only.

For the main convergence proof, we will use some fundamental principles from the theory of graphs and electrical networks (for a complete treatment, see for example [3, Chapters II, III, IX]).

Basic definitions. For any $e \in E$, we also call the value D_e the *capacity* of edge e. A *cut (separating s_0 from s_1)* is a subset $S \subseteq N$ such that $s_0 \in S$, $s_1 \notin S$. The set of edges with exactly one endpoint in S is denoted by $\delta(S)$. The set of all cuts separating s_0 from s_1 is denoted by $\mathcal{C}$. The *capacity C_S of the cut S* is the total capacity of the edges in $\delta(S)$: $C_S = \sum_{e\in\delta(S)} D_e$. A *flow* x is a function $x : E \to \mathbb{R}$ such that

$$\sum_{v\in\delta(u)} x_{uv} = 0 \qquad \text{for any } u \neq s_0, s_1. \tag{6}$$

The *value* of flow x is the quantity $\mathrm{val}(x) = \sum_{v\in\delta(s_0)} x_{s_0 v}$. A *maximum flow* F is a flow having maximum value subject to $|F_e| \le D_e$ for all $e \in E$.

Max-Flow Min-Cut Theorem. The value of a maximum flow from s_0 to s_1 is equal to the minimum of the capacities of cuts separating s_0 from s_1. Equivalently, if F is a maximum flow,

$$\mathrm{val}(F) = \min_{S\in\mathcal{C}} C_S. \tag{7}$$

Thomson's Principle. The flow Q is uniquely determined as a feasible flow of unit value that minimizes the total energy dissipation $\sum_e R_e Q_e^2$, with $R_e = L_e/D_e$. In other words, for any flow x satisfying (6) such that $\mathrm{val}(x) = 1$,

$$\sum_e R_e Q_e^2 \le \sum_e R_e x_e^2. \tag{8}$$

Kirchhoff's Theorem. For a graph $G = (N, E)$ and an oriented edge $e = (u, v) \in E$, let

- Sp be the set of all spanning trees of G, and let
- $\mathrm{Sp}(u, v)$ be the set of all spanning trees T of G, for which the oriented edge (u, v) lies on the unique path from s_0 to s_1 in T.

For a set of trees S, define $\Gamma(S) = \sum_{T\in S} \prod_{e\in T} D_e/L_e$. Then, the current through the edge e is

$$Q_{uv} = \frac{\Gamma(\mathrm{Sp}(u, v)) - \Gamma(\mathrm{Sp}(v, u))}{\Gamma(\mathrm{Sp})}. \tag{9}$$

Gronwall's Lemma. Let $\alpha, \beta \in \mathbb{R}$ and let x be a continuous differentiable real function on $[0, \infty)$. If $\alpha x(t) \le \dot{x}(t) \le \beta x(t)$ for all $t \ge 0$, then

$$x(0)\, e^{\alpha t} \le x(t) \le x(0)\, e^{\beta t} \quad \text{for all } t \ge 0.$$

Proof:

$$\frac{d}{dt}\frac{x}{e^{\beta t}} = \frac{\dot{x}e^{\beta t} - \beta x e^{\beta t}}{e^{2\beta t}} \le 0 \Rightarrow \frac{x(t)}{e^{\beta t}} \le \frac{x(0)}{e^{\beta 0}} = x(0).$$

A similar calculation establishes $x(t) \ge x(0)e^{\alpha t}$. ∎

The next lemma gives some properties that are easily derived from (1), (4), and (5). Recall that $\mathcal{C}$ is the set of s_0-s_1 cuts and $C_S = \sum_{e\in\delta(S)} D_e$. Also, let $L_{\min} = \min_e L_e$, $L_{\max} = \max_e L_e$, $n = |N|$, and $m = |E|$.

Lemma 4 *The following hold for any edge $e \in E$ and any cut $S \in \mathcal{C}$:*

(i) $|Q_e| \le 1$.

(ii) $\sum_{e\in\delta(\{s_0\})} |Q_e| = 1$.

(iii) $D_e(t) \ge D_e(0) \exp(-t)$ *for all t,*

(iv) $D_e(t) \le 1 + (D_e(0) - 1) \exp(-t)$ *for all t.*

(v) $R_e \ge L_{\min}/2$ *for all sufficiently large t.*

(vi) $C_S(t) \ge 1 + (C_S(0) - 1) \exp(-t)$ *for all t, with equality if $S = \{s_0\}$.*

(vii) $C_{\{s_0\}} \to 1$ *as $t \to \infty$.*

(viii) Orient the edges according to the direction of the flow. For sufficiently large t, there is a directed source-sink path in which all edges have diameter at least $1/2m$.

(ix) $|\Delta_e| \le 2nmL_{\max}$ *for all sufficiently large* t.

(x) $\dot{D}_e/D_e \in [-1, 2nmL_{\max}/L_{\min}]$ *for all sufficiently large* t.

Proof:

(i) Since Q is a flow, it can be decomposed into s_0-s_1 flow paths and cycles. If $|Q_e| > 1$, since $b(s_0) = 1$, there exists a positive cycle in this decomposition, a contradiction to the existence of potential values at the nodes. The claim is also an immediate consequence of (9).

(ii) It follows from equations (4) and (5) that $p_{s_0} = \max_v p_v$, so $Q_{s_0,v} \ge 0$ for all $\{s_0, v\} \in E$, and $\sum_{e\in\delta(\{s_0\})} |Q_e| = \sum_{e\in\delta(\{s_0\})} Q_e = 1$.

(iii) From the evolution equation (1), $\dot{D}_e \ge -D_e$. The claim follows by Gronwall's Lemma.

(iv) $|Q_e| \le 1$ for any edge e, so $\dot{D}_e \le 1 - D_e$ from (1), and the claim follows as before.

(v) From (iv), $D_e \le 2$ for all sufficiently large t, so $R_e = L_e/D_e \ge L_{\min}/2$ for the same t's.

(vi) $\dot{C}_S = \sum_{e\in\delta(S)} \dot{D}_e = \sum_{e\in\delta(S)}(|Q_e| - D_e) \ge 1 - C_S$, with equality if $S = \{s_0\}$.

(vii) Follows by noting that the inequality in (vi) becomes tight for the cut $\{s_0\}$, due to (ii).

(viii) From (vi), eventually $C_S \ge 1/2$ for all $S \in \mathcal{C}$, so there is an edge of diameter at least $1/2m$ in every cut. Thus, there is a s_0-s_1 path in which every edge has diameter at least $1/2m$.

(ix) Consider a source-sink path in which every edge has diameter at least $1/2m$. By (4) the total potential drop $p_{s_0} - p_{s_1}$ is at most $2nmL_{\max}$.

(x) $\dot{D}_e/D_e = (|Q_e| - D_e)/D_e = |\Delta_e|/L_e - 1$, and the bound follows from (ix). ∎

6 Convergence

We will prove convergence for general graphs. Throughout this section, we will assume that t is large enough for all the claims of Lemma 4 requiring a sufficiently large t to hold.

6.1 Properties of Equilibrium Points.

Recall that $D \in \mathbb{R}_+^E$ is an *equilibrium point*, when $\dot{D}_e = 0$ for all $e \in E$, which by (1) is equivalent to $D_e = |Q_e|$ for all $e \in E$.

Lemma 5 *At an equilibrium point,* $\min_{S\in\mathcal{C}} C_S = C_{\{s_0\}} = 1$.

Proof:

$$1 \le \min_{S\in\mathcal{C}} \sum_{e\in\delta(S)} |Q_e| = \min_{S\in\mathcal{C}} C_S \le C_{\{s_0\}} = \sum_{e\in\delta(\{s_0\})} |Q_e| = 1.$$

∎

Lemma 6 *The equilibria are precisely the flows of value 1, in which all source-sink paths have the same length. If no two source-sink paths have the same length, the equilibria are precisely the simple source-sink paths.*

Proof: Let Q be a flow of value 1, in which all source-sink paths have the same length. We orient the edges such that $Q_e \geq 0$ for all e and show that $D = Q$ is an equilibrium point. Let E_1 be the set of edges carrying positive flow, and let V_1 be the set of vertices lying on a source-sink path consisting of edges in E_1. For $v \in V_1$, set its potential to the length of the paths from v to s_1 in (V_1, E_1); observe that all such paths have the same length by assumption. Let Q' be the electrical flow induced by the potentials and edge diameters. For any edge $e = (u, v) \in E_1$, we have $Q'_e = D_e \Delta_e / L_e = D_e = Q_e$. Thus, $Q' = Q$. For any edge $e \notin E_1$, we have $Q_e = 0 = D_e$. We conclude that D is an equilibrium point.

Let D be an equilibrium point and let Q_e be the corresponding current along edge e, where we orient the edges so that $Q_e \geq 0$ for all $e \in E$. Whenever $D_e > 0$, we have $\Delta_e = Q_e L_e / D_e = L_e$ because of the equilibrium condition. Since all directed s_0-s_1 paths span the same potential difference, all directed paths from s_0 to s_1 in $\{e \in E : D_e > 0\}$ have the same length. Moreover, by Lemma 5, $\min_S C_S = 1$. Thus, D is a flow of value 1. ∎

Let $\mathcal{E}^*$ be the set of flows of value one in the network of shortest source-sink paths. If the shortest source-sink path is unique, $\mathcal{E}^*$ is a singleton, namely the flow of value one along the shortest source-sink path.

6.2 The Convergence Process

The following functions play a crucial role in the convergence proof. Let $C = \min_{S \in \mathcal{C}} C_S$, and

$$\begin{aligned} V_S &= \frac{1}{C_S} \sum_{e \in E} L_e D_e \text{ for each } S \in \mathcal{C}, \\ V &= \max_{S \in \mathcal{C}} V_S + W, \text{ and} \\ h &= -\frac{1}{C} \sum_{e \in E} R_e |Q_e| D_e + \frac{1}{C^2} \sum_{e \in E} R_e D_e^2. \end{aligned}$$

We will first prove that V is decreasing (Lemma 10); more precisely, we show $\dot{V}(t) \leq -h(t) - 2W(t) \leq 0$. Lemma 7 to 9 pave the way for Lemma 10. Since V is nonnegative we conclude that $h(t)$ must converge to zero. We next bound $\sum_e (D_e/C - |Q_e|)^2$ in terms of h and derive that $|D_e - |Q_e||$ converges to zero for all e (Lemma 12). In the next step (Lemma 13), we show that the potential difference $\Delta = p_{s_0} - p_{s_1}$ between source and sink converges to the length L^* of a shortest-source sink path. We use this to conclude (Lemma 14) that D_e and Q_e converge to zero for every edge $e \notin E_0$. Finally (Theorem 2), we show that the dynamics are attracted by $\mathcal{E}^*$.

Lemma 7 *Let S be a minimum capacity cut at time t. Then, $\dot{V}_S(t) \leq -h(t)$.*

Proof: Let X be the characteristic vector of $\delta(S)$, that is, $X_e = 1$ if $e \in \delta(S)$ and 0 otherwise. Observe that $C_S = C$ since S is a minimum capacity cut. We have

$$\begin{aligned}
\dot{V}_S &= \sum_e \frac{\partial V_S}{\partial D_e} \dot{D}_e \\
&= \sum_e \frac{1}{C^2} \left(L_e C - \sum_{e'} L_{e'} D_{e'} X_e \right) (|Q_e| - D_e) \\
&= \frac{1}{C} \sum_e L_e |Q_e| - \frac{1}{C^2} \left(\sum_{e'} L_{e'} D_{e'} \right) \left(\sum_e X_e |Q_e| \right) + \\
&\qquad - \frac{1}{C} \sum_e L_e D_e + \frac{1}{C^2} \left(\sum_{e'} L_{e'} D_{e'} \right) \left(\sum_e X_e D_e \right) \\
&\leq \frac{1}{C} \sum_e R_e |Q_e| D_e - \frac{1}{C^2} \sum_e R_e D_e^2 - \frac{1}{C} \sum_e L_e D_e + \frac{1}{C} \sum_e L_e D_e \\
&= -h.
\end{aligned}$$

The only inequality follows from $L_e = R_e D_e$ and $\sum_e X_e |Q_e| \geq 1$, which holds because at least one unit current must cross S. ∎

Lemma 8 *Let $W = (C_{\{s_0\}} - 1)^2$. Then, $\dot{W} = -2W \leq 0$, with equality iff $C_{\{s_0\}} = 1$.*

Proof: Let $C_0 = C_{\{s_0\}}$ for short. Then, since $\sum_{e \in \delta(\{s_0\})} |Q_e| = 1$,

$$\dot{W} = 2(C_0 - 1) \sum_{e \in \delta(\{s_0\})} (|Q_e| - D_e) = 2(C_0 - 1)(1 - C_0) = -2(C_0 - 1)^2 \leq 0.$$

∎

The next lemma is a necessary technicality.

Lemma 9 *Let $f(t) = \max_{S \in \mathcal{C}} f_S(t)$, where each f_S is continuous and differentiable. If $\dot{f}(t)$ exists, then there is $S \in \mathcal{C}$ such that $f(t) = f_S(t)$ and $\dot{f}(t) = \dot{f}_S(t)$.*

Proof: Since $\mathcal{C}$ is finite, there is at least one $S \in \mathcal{C}$ such that for each fixed $\delta > 0$, $f(t+\varepsilon) = f_S(t+\varepsilon)$ for infinitely many ε with $|\varepsilon| \leq \delta$. By continuity of f and f_S, this implies $f(t) = f_S(t)$. Moreover, since

$$\lim_{\varepsilon \to 0} \frac{\max_{S'} f_{S'}(t+\varepsilon) - \max_{S'} f_{S'}(t)}{\varepsilon}$$

exists and is equal to $\dot{f}(t)$, any sequence $\varepsilon_1, \varepsilon_2, \ldots$ converging to zero has the property that

$$\frac{\max_{S'} f_{S'}(t+\varepsilon_i) - \max_{S'} f_{S'}(t)}{\varepsilon_i} \to \dot{f}(t) \qquad \text{for } i \to \infty.$$

Taking $(\varepsilon_i)_{i=1}^{\infty}$ to be a sequence converging to zero such that $f(t+\varepsilon_i) = f_S(t+\varepsilon_i)$ for all i, we obtain

$$\dot{f}(t) = \lim_{i \to \infty} \frac{f_S(t+\varepsilon_i) - f_S(t)}{\varepsilon_i} = \dot{f}_S(t).$$

∎

Lemma 10 *$\dot{V}$ exists almost everywhere. If $\dot{V}(t)$ exists, then $\dot{V}(t) \le -h(t) - 2W(t) \le 0$, and $\dot{V}(t) = 0$ if and only if $\dot{D}_e(t) = 0$ for all e.*

Proof: V is Lipschitz-continuous since it is the maximum of a finite set of continuously differentiable functions. Since V is Lipschitz-continuous, the set of t's where $\dot{V}(t)$ does not exist has zero Lebesgue measure (see for example [6, Ch. 3], [10, Ch. 3]). When $\dot{V}(t)$ exists, we have $\dot{V}(t) = \dot{W}(t) + \dot{V}_S(t)$ for some S of minimum capacity (Lemma 9). Then, $\dot{V}(t) \le -h(t) - 2W(t)$ by Lemmas 8 and 7.

The fact that $W \ge 0$ is clear. We now show that $h \ge 0$. To this end, let F represent a maximum s_0-s_1 flow in an auxiliary network, having the same structure as G, and where the capacity on edge e is set equal to D_e. In other words, F is an s_0-s_1 flow satisfying $|F_e| \le D_e$ for all $e \in E$ and having maximum value. By the max-flow min-cut theorem (Equation (7)), this maximum value is equal to $C = \min_{S \in \mathcal{C}} C_S$. But then,

$$\begin{aligned}
-h &= \frac{1}{C}\sum_e R_e|Q_e|D_e - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\le \frac{1}{C}\left(\sum_e R_e Q_e^2\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\le \frac{1}{C}\left(\sum_e R_e \frac{F_e^2}{C^2}\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&\le \frac{1}{C^2}\left(\sum_e R_e D_e^2\right)^{1/2}\left(\sum_e R_e D_e^2\right)^{1/2} - \frac{1}{C^2}\sum_e R_e D_e^2 \\
&= 0,
\end{aligned}$$

where we used the following inequalities:

- the Cauchy-Schwarz inequality $\sum_e (R_e^{1/2}|Q_e|)(R_e^{1/2}D_e) \le (\sum_e R_e Q_e^2)^{1/2}(\sum_e R_e D_e^2)^{1/2}$;
- Thomson's Principle (8) applied to the unit-value flows Q and F/C: Q is a minimum energy flow of unit value, while F/C is a feasible flow of unit value;
- the fact that $|F_e| \le D_e$ for all $e \in E$.

Finally, one can have $h = 0$ if and only if all the above inequalities are equalities, which implies that $|Q_e| = |F_e|/C = D_e/C$ for all e. And, $W = 0$ iff $\sum_{e\in\delta(\{s_0\})} D_e = 1 = \sum_{e\in\delta(\{s_0\})} |Q_e|$. So, $h = W = 0$ iff $|Q_e| = D_e$ for all e. ∎

The next lemma is a necessary technicality.

Lemma 11 *The function $t \mapsto h(t)$ is Lipschitz-continuous.*

Proof: Since $\dot{D}_e$ is continuous and bounded (by (1)), D_e is Lipschitz-continuous. Thus, it is enough to show that Q_e is Lipschitz-continuous for all e.

First, we claim that $D_e(t+\varepsilon) \le (1+2K\varepsilon)D_e$ for all $\varepsilon \le 1/4K$, where $K = 2nmL_{\max}/L_{\min}$. For if not, take

$$\varepsilon = \inf\{\delta \le 1/4K : D_e(t+\delta) > (1+2K\delta)D_e(t)\},$$

then $\varepsilon > 0$ (since $\dot{D}_e(t) \leq KD_e(t)$ by Lemma 4) and, by continuity, $D_e(t+\varepsilon) \geq (1+2K\varepsilon)D_e(t)$. There must be $t' \in [t, t+\varepsilon]$ such that $\dot{D}_e(t') = 2KD_e(t)$. On the other hand,

$$\begin{aligned} \dot{D}_e(t') &\leq KD_e(t') \leq K(1+2K\varepsilon)D_e(t) \\ &\leq K(1+2K/4K)D_e(t) < 2KD_e(t), \end{aligned}$$

which is a contradiction. Thus, $D_e(t+\varepsilon) \leq (1+2K\varepsilon)D_e$ for all $\varepsilon \leq 1/4K$. Similarly, $D_e(t+\varepsilon) \geq (1-2K\varepsilon)D_e$.

Consider now a spanning tree T of G. Let $\gamma_T = \prod_{e\in T} D_e/L_e$. Then $\gamma_T(t+\varepsilon) \leq (1+2K\varepsilon)^n\gamma_T(t) \leq (1+4nK\varepsilon)\gamma_T(t)$ for sufficiently small ε. Similarly, $\gamma_T(t+\varepsilon) \geq (1-4nK\varepsilon)\gamma_T(t)$.

By Kirchhoff's Theorem,

$$Q_{uv} = \frac{\sum_{T\in \mathrm{Sp}(u,v)} \gamma_T - \sum_{T\in \mathrm{Sp}(v,u)} \gamma_T}{\sum_{T\in \mathrm{Sp}} \gamma_T},$$

and plugging the bounds for $\gamma_T(t+\varepsilon)/\gamma_T(t)$ shows that $Q_e(t+\varepsilon) = Q_e(t)(1+O(\varepsilon))$, where the constant implicit in the $O(\cdot)$ notation does not depend on t. Since $|Q_e| \leq 1$, we obtain that $|Q_e(t+\varepsilon) - Q_e(t)| \leq O(1)\cdot\varepsilon$, that is, Q_e is Lipschitz-continuous, and this in turn implies the Lipschitz-continuity of h. ∎

Lemma 12 $|D_e - |Q_e||$ *converges to zero for all* $e \in E$.

Proof: Consider again the function h. We claim $h \to 0$ as $t \to \infty$. If not, there is $\varepsilon > 0$ and an infinite unbounded sequence $t_1, t_2, \ldots$ such that $h(t_i) \geq \varepsilon$ for all i. Since h is Lipschitz-continuous (Lemma 11), there is δ such that $h(t_i + \delta') \geq h(t_i) - \varepsilon/2 \geq \varepsilon/2$ for all $\delta' \in [0, \delta]$ and all i. So by Lemma 10, $\dot{V}(t) \leq -h(t) \leq -\varepsilon/2$ for every t in $[t_i, t_i + \delta]$ (except possibly a zero measure set), meaning that V decreases by at least $\varepsilon\delta/2$ infinitely many times. But this is impossible since V is positive and non-increasing.

Thus, for any $\varepsilon > 0$, there is t_0 such that $h(t) \leq \varepsilon$ for all $t \geq t_0$. Then, recalling that $R_e \geq L_{\min}/2$ for all sufficiently large t (Lemma 4.v), we find

$$\begin{aligned} \sum_e \frac{L_{\min}}{2}\left(\frac{D_e}{C} - |Q_e|\right)^2 &\leq \sum_e R_e\left(\frac{D_e}{C} - |Q_e|\right)^2 \\ &= \frac{1}{C^2}\sum_e R_e D_e^2 + \sum_e R_e Q_e^2 - \frac{2}{C}\sum_e R_e|Q_e|D_e \\ &\leq \frac{2}{C^2}\sum_e R_e D_e^2 - \frac{2}{C}\sum_e R_e|Q_e|D_e \\ &= 2h \leq 2\varepsilon, \end{aligned}$$

where we used once more the inequality $\sum_e R_e Q_e^2 \leq \sum_e R_e D_e^2/C^2$, which was proved in Lemma 10. This implies that for each e, $D_e/C - |Q_e| \to 0$ as $t \to \infty$. Summing across $e \in \delta(\{s_0\})$ and using Lemma 4.ii, we obtain $C_{\{s_0\}}/C - 1 \to 0$ as $t \to \infty$. From Lemma 4, $C_{\{s_0\}} \to 1$ as $t \to \infty$, so $C \to 1$ as well.

To conclude, we show that $D_e/C - |Q_e| \to 0$ and $C \to 1$ together imply $D_e - |Q_e| \to 0$. Let $\varepsilon > 0$ be arbitrary. For all sufficiently large t, $|D_e/C - |Q_e|| \leq \varepsilon$, $|1 - C| \leq \varepsilon$, $D_e \leq 2$, and $C \geq 1/2$. Thus,

$$|D_e - |Q_e|| \leq |D_e - D_e/C| + |D_e/C - |Q_e|| \leq D_e\frac{|C-1|}{C} + |D_e/C - |Q_e|| \leq 5\varepsilon. \quad ∎$$

Lemma 13 *Let $\Delta = p_{s_0} - p_{s_1}$ be the potential difference between source and sink. Δ converges to the length L^* of a shortest source-sink path.*

Proof: Let $\mathcal{L}$ be the set of lengths of simple source-sink paths. We first show that Δ converges to a point in $\mathcal{L}$ and then show convergence to L^*.

Orient edges according to the direction of the flow. By Lemma 4.viii, there is a directed source-sink path P of edges of diameter at least $1/2m$. Let $\varepsilon > 0$ be arbitrary. We will show $|\Delta - L_P| \le \varepsilon$. For this, it suffices to show $|\Delta_e - L_e| \le \varepsilon/n$ for any edge e of P, where Δ_e is the potential drop on e. By Ohm's law, the potential drop on e is $\Delta_e = (Q_e/D_e)L_e$, and hence, $|\Delta_e - L_e| = |Q_e/D_e - 1|L_e = |(Q_e - D_e)/D_e|L_e \le 2mL_{\max}|Q_e - D_e|$. The claim follows since $|Q_e - D_e|$ converges to zero.

The set $\mathcal{L}$ is finite. Let ε be positive and smaller than half the minimal distance between two elements in $\mathcal{L}$. By the preceeding paragraph, there is for all sufficiently large t a path P_t such that $|\Delta - L_{P_t}| \le \varepsilon$. Since Δ is a continuous function of time, L_{P_t} must become constant. We have now shown that Δ converges to an element in $\mathcal{L}$.

We will next show that Δ converges to L^*. Assume otherwise, and let P' be a shortest undirected source-sink path. Let $W_{P'} = \sum_{e \in P'} L_e \ln D_e$. This function was already used by Miyaji and Ohnishi [12]. We have

$$\dot{W}_{P'} = \sum_{e \in P'} \frac{L_e}{D_e}(|Q_e| - D_e) = \sum_{e \in P'} |\Delta_e| - \sum_{e \in P'} L_e \ge p_{s_0} - p_{s_1} - L_{P'} = \Delta - L^*.$$

Let $\delta > 0$ be such that there is no source-sink path with length in the open interval $(L^*, L^* + 2\delta)$. Then, $\Delta - L^* \ge \delta$ for all sufficently large t, and hence, $\dot{W}_{P'} \ge \delta$ for all sufficiently large t. Thus, $W_{P'}$ goes to $+\infty$. However, $W_{P'} \le nL_{\max}$ for all sufficiently large t since $D_e \le 2$ for all e and t large enough. This is a contradiction. Thus, Δ converges to L^*. ∎

Lemma 14 *Let e be any edge that does not lie on a shortest source-sink path. Then, D_e and Q_e converge to zero.*

Proof: Since $|D_e - |Q_e||$ converges to zero, it suffices to prove that Q_e converges to zero. Assume otherwise. Then, there is a $\delta > 0$ such that $|Q_e| \ge \delta$ for arbitrarily large t.

Consider any such t and orient the edges according to the direction of the flow at time t. Let $e = (u, v)$. Because of flow conservation, there must be an edge into u and an edge out of v carrying flow at least Q_e/n. Continuing in this way, we obtain a source-sink path P in which every edge carries flow at least $Q_e/n^n \ge \delta/n^n$; P depends on time and $L_P > L^*$ always. We will show $|\Delta - L_P| \le (L_P - L^*)/4$ for sufficiently large t, a contradiction to the fact that Δ converges to L^*. For this, it suffices to show $|\Delta_g - L_g| \le (L_P - L^*)/(4n)$ for any edge g of P, where Δ_g is the potential drop on g. By Ohm's law, the potential drop on g is $\Delta_g = (Q_g/D_g)L_g$, and hence, $|\Delta_g - L_g| = |Q_g/D_g - 1|L_g = |(Q_g - D_g)/D_g|L_g \le L_{\max}|Q_g - D_g|/D_g$. For large enough t, $|Q_g - D_g| \le \min(\delta/(2n^n), \delta(L_P - L^*)/(8n^{n+1}L_{\max}))$. Then, $D_g \ge Q_g - |Q_g - D_g| \ge \delta/(2n^n)$, and hence, $L_{\max}|Q_g - D_g|/D_g \le (L_P - L^*)/(4n)$. ∎

Theorem 2 *The dynamics are attracted by $\mathcal{E}^*$. If the shortest source-sink path is unique, the dynamics converge against a flow of value 1 on the shortest source sink path.*

Proof: Q is a source-sink flow of value one at all times. We show first that Q is attracted to $\mathcal{E}^*$. Orient the edges in the direction of the flow. We can decompose Q into flowpaths. For an oriented path P, let 1_P be the unit flow along P. We can write $Q = \sum_P x_p 1_P$, where x_P is the flow along the path P. This decomposition is not unique. We group the flowpath into two sets, the paths running inside G_0 and the paths using an edge outside G_0, i.e.,

$$Q = Q_0 + Q_1, \text{ where } Q_0 = \sum_{P \text{ is a path in } G_0} x_P 1_P.$$

Q_0 is a flow in G_0, and each flowpath in Q_1 is a non-shortest source-sink path.[5] We show that the value of Q_0 converges to one.

Assume otherwise. Then, there is a $\delta > 0$ such that the value of Q_1 is at least δ for arbitrarily large times t. At any such time, there is an edge $e \notin E_0$ carrying flow at least δ/m; this holds since source-sink cuts contain at most m edges. Since there are only finitely many edges, there must be an edge $e \notin E_0$ for which Q_e does not converge to zero, a contradiction to Lemma 14.

We have now shown that the distance between Q and $\mathcal{E}^*$ converges to zero. By Lemma 12, $|D_e - |Q_e||$ converges to zero for all e, and hence, the distance between Q and D converges to zero. Thus, D is attracted by $\mathcal{E}^*$.

Finally, if the shortest source-sink path is unique, $\mathcal{E}^*$ is a singleton, and hence, D converges to the flow of value one along the shortest source-sink path. ∎

Lemma 15 *If the shortest source-sink path is unique, p_v converges to $\operatorname{dist}(v, s_1)$ for each node v on the shortest source-sink path, where $\operatorname{dist}(v, s_1)$ is the shortest path distance from v to s_1.*

Proof: Let P_0 be the shortest source-sink path. For any $e \in P$, D_e converges to one and $|D_e - Q_e|$ converges to zero. Thus, Δ_e converges to L_e. ∎

We believe that Theorem 2 can be strenghtened. The dynamics are not only attracted to $\mathcal{E}^*$ but to an element in $\mathcal{E}^*$, i.e., the dynamics converge to some flow of value one in the network of shortest paths.

6.3 More on the Lyapunov Function V

In this section, we study $V = \sum_e L_e D_e / C + (C_{\{s_0\}} - 1)^2$ as a function of D. Recall that $C = C(D) = \min_{S \in \mathcal{C}} C_S$, where $C_S = \sum_{e \in \delta(S)} D_e$.

Lemma 16 *Let D^0 and D^1 be two equilibrium points. Define*

$$D^\lambda = (1 - \lambda) D^0 + \lambda D^1, \qquad \lambda \in [0, 1].$$

If $V(D^0) < V(D^1)$, then $V(D^\lambda)$ is a linear, increasing function of λ.

[5] The decomposition into Q_0 and Q_1 can be constructed as follows: Initialize Q_0 to Q and Q_1 to the empty flow. Consider any edge $e \notin E_0$ carrying positive flow in Q_0, say ε. Let P be an oriented source-sink path carrying ε units of flow and using e. Add $\varepsilon 1_P$ to Q_1 and subtract it from Q_0. Continue until Q_0 is a flow in G_0.

Proof: By Lemma 5, $C(D^0) = C(D^1) = 1$, and $C_{\{s_0\}}(D^0) = C_{\{s_0\}}(D^1) = 1$. Since $C_S(D)$ is linear in D for any fixed cut S, one has $C_S(D^0) \geq 1$ and $C_S(D^1) \geq 1$, so $C_S(D^\lambda) \geq 1$ for all S. Thus, $C(D^\lambda) \geq 1$. On the other hand, $C_{\{s_0\}}(D^\lambda) = 1$. Thus, $C(D^\lambda) = 1$, and $V(D^\lambda) = \sum_e L_e D_e^\lambda$, that is, $V(D^\lambda)$ is a linear function of D^λ. ∎

Lemma 17 *The problem of minimizing $V(D)$ for $D \in \mathbb{R}_+^E$ is equivalent to the shortest path problem.*

Proof: By introducing an additional variable $C = \min_S C_S > 0$, the problem of minimizing $V(D)$ is equivalently formulated as

$$\begin{aligned}
&\min \frac{1}{C} \sum_e L_e D_e + \left(\sum_{e \in \delta(\{s_0\})} D_e - 1 \right)^2 \\
&\text{s.t. } C_S \geq C \qquad \forall S \in \mathcal{C} \\
&\quad\;\; C > 0 \\
&\quad\;\; D \geq 0.
\end{aligned}$$

Substituting $x_e = D_e / C$, we obtain

$$\begin{aligned}
&\min \sum_e L_e x_e + C^{1/2} \left(\sum_{e \in \delta(\{s_0\})} x_e - \frac{1}{C} \right)^2 \\
&\text{s.t. } \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \in \mathcal{C} \\
&\quad\;\; x \geq 0, C > 0,
\end{aligned}$$

which is easily seen to be equivalent to the (fractional) shortest path problem. ∎

7 Rate of Convergence for Stable Flow Directions

The direction of the flow across an edge depends on the initial conditions and time. We do not know whether flow directions can change infinitely often or whether they become ultimately fixed. In this section, we assume that flow directions stabilize and explore the consequences of this assumption. We will be able to make quite precise statements about the convergence of the system. We assume uniqueness of the shortest source-sink path and add more non-degeneracy assumptions as we go along.

An edge $e = \{u, v\}$ becomes *horizontal* if $\lim_{t \to \infty} |p_u - p_v| = 0$, and it becomes *directed* from u to v (directed from v to u) if $p_u > p_v$ for all large t ($p_v > p_u$ for all large t). An edge *stabilizes* if it either becomes horizontal or directed, and a network *stabilizes* if all its edges stabilize. If a network stabilizes, we partition its edges into a set E_h of horizonal edges and a set $\overrightarrow{E}$ of directed edges. If $\{u, v\}$ becomes directed from u to v, then $(u, v) \in \overrightarrow{E}$.

We already know that the diameters of the edges on the shortest source-sink path (we assume uniqueness in this section) converge to one. The diameters of the edges outside G_0

converge to zero. The potential of a vertex $v \in G_0$ converges to $\mathrm{dist}(v, s_1)$. For stabilizing networks, we can prove a lot more. In particular, we can predict the decay rates of edges, the limit potentials of the vertices, and for each edge the direction in which the flow will stabilize.

Definition 1 (Decay Rate) *Let $r \le 0$.*

A quantity $D(t)$ decays with rate at least *r if for every $\varepsilon > 0$ there is a constant $A > 0$ such that for all t*

$$D(t) \le Ae^{(r+\varepsilon)t}, \quad \textit{or equivalently,} \quad \ln D(t) \le (\ln A) + (r+\varepsilon)t.$$

A quantity $D(t)$ decays with rate at most *r if for every $\varepsilon > 0$ there is a constant $a > 0$ such that for all t*

$$D(t) \ge ae^{(r-\varepsilon)t}, \quad \textit{or equivalently,} \quad \ln D(t) \ge (\ln a) + (r-\varepsilon)t.$$

A quantity $D(t)$ decays with rate *r if it decays with rate at least and at most r.*

We first establish a simple Lemma that, for any edge, connects the decay rate of the flow across the edge and the diameter of the edge.

Lemma 18 *Let $-1 \le a < 0$ and let $e, g \in E$. If Q_e decays with rate at least a, then so does D_e. D_e decays with rate at most -1. If $||Q_e| - |Q_g||$ decays with rate at least a, then $|D_e - D_g|$ decays with rate at least a.*

Proof: Assume first that Q_e decays with rate at least a, where $-1 \le a < 0$. Then, for any $\varepsilon > 0$, there is an $A > 0$ such that $Q_e \le Ae^{(a+\varepsilon)t}$ for all t. Consider f with $\dot{f} = Ae^{(a+\varepsilon)t} - f$. This has solution $f = f_0 e^{-t} + \alpha e^{(a+\varepsilon)t}$, where $\alpha = A/(1 + a + \varepsilon)$ and f_0 is determined by the value of f at zero, namely, $f(0) = f_0 + \alpha$. Consider $D_e - f$. Then,

$$\frac{d}{dt}(D_e - f) = |Q_e| - D_e - (Ae^{(a+\varepsilon)t} - f) \le -(D_e - f).$$

Thus, $D_e - f \le C' e^{-t}$ for some constant C' by Gronwall's Lemma, and hence,

$$D_e \le (f_0 + C')e^{-t} + \alpha e^{(a+\varepsilon)t} \le C'' e^{(a+\varepsilon)t}$$

for some constant C''. Thus, D_e decays with rate at least a.

$\dot{D}_e = |Q_e| - D_e \ge -D_e$. Thus, D_e decays with rate at most -1 by Gronwall's Lemma.

Finally, assume that $||Q_e| - |Q_f||$ decays with rate at least a. Then,

$$\frac{d}{dt}(D_e - D_g) = |Q_e| - |Q_f| - (D_e - D_g) \le ||Q_e| - |Q_f|| - (D_e - D_g),$$

and therefore, $D_e - D_g$ decays with rate at least $-a$. The same argument applies to $D_g - D_e$. ∎

For a path P, let $W(P) := \sum_{e \in P} L_e \ln D_e$ be its weighted sum of log diameters, and let $\Delta(P) = p_a - p_b$ be the potential difference between its endpoints. The function $W(P)$ was introduced by Miyaji and Ohnishi [11, 12].

Lemma 19 *Let P be an arbitrary path, let $\Delta(P)$ be the potential drop along P, and let $W(P) = \sum_{e \in P} L_e \ln D_e$. Then,*

$$\dot{W}(P) = \Delta(P) - L(P) + 2 \sum_{e \in P:\ \Delta(e) < 0} |\Delta(e)|.$$

If $\Delta(P) \le \Delta$ and $\Delta(e) \ge -\delta$ for some $\delta \ge 0$, all $e \in P$ and for all sufficiently large t, then

$$W(P)(t) \le C + (\Delta - L(P) + 2n\delta)t$$

for some constant C and all t. If $\Delta(P) \ge \Delta$ for all sufficiently large t, then

$$W(P)(t) \ge C + (\Delta - L(P))t$$

for some constant C and all t.

Proof: The first claim follows immediately from the dynamics of the system.

$$\dot{W}(P) = \sum_{e \in P} |\Delta(e)| - L(P) = \Delta(P) - L(P) + 2 \sum_{e \in P:\ \Delta(e) < 0} |\Delta(e)|.$$

Let t_0 be such that $\Delta(P) \le \Delta$ and $\Delta(e) \ge -\delta$ for all $t \ge t_0$. We integrate the equality from t_0 to t and obtain

$$W(P)(t) - W(P)(t_0) = \int_{t_0}^{t} \dot{W}(P) dt \le (\Delta - L(P) + 2n\delta)(t - t_0).$$

This establishes the claim for $t \ge t_0$. Choosing C sufficiently large extends the claim to all t.

Let t_0 be such that $\Delta(P) \ge \Delta$. We integrate the equality from t_0 to t and obtain

$$W(P)(t) - W(P)(t_0) = \int_{t_0}^{t} \dot{W}(P) dt \ge (\Delta - L(P))(t - t_0).$$

This establishes the claim for $t \ge t_0$. Choosing C sufficiently large extends the claim to all t. ∎

Edges that do not lie on a source-sink path never carry any flow, and hence, their diameter evolves as $D_e(0)\exp(-t)$. From now on, we may therefore assume that every edge of G lies on a source-sink path.

Lemma 20 *For $e \in E_h$, D_e decays with rate -1, and $|Q_e|$ decays with rate at least -1.*

Proof: We certainly have $D_e \le 2$ for all large t. Let $e = \{u, v\}$, and let $\varepsilon > 0$ be arbitrary. Then, $|p_u - p_v| \le \varepsilon L_e$ for all large t, and hence, $|Q_e| = (D_e/L_e)|p_u - p_v| \le \varepsilon D_e$ for all large t. Thus, $\dot{D}_e \le (\varepsilon - 1)D_e$ for all large t, and hence, $(d/dt) \ln D_e \le -1 + \varepsilon$. Thus, D_e decays with rate at least -1. Since $\dot{D}_e \ge -D_e$, D_e decays with rate at most -1.

$|Q_e| = (D_e/L_e)|p_u - p_v| \le AD_e$ for some constant A. Thus, $|Q_e|$ decays with rate at least -1. ∎

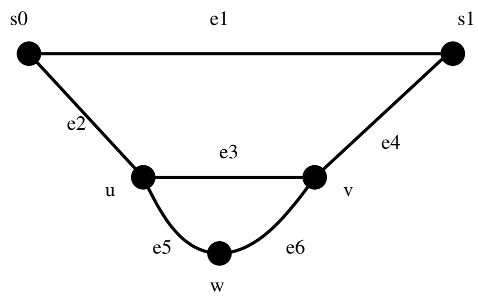

Figure 3: All edges are assumed to have length 1; $P_0 = (e_1)$, $P_1 = (e_2, e_3, e_4)$, $P_2 = (e_5, e_6)$, $p^*_{s_0} = 1$, $p^*_{s_1} = 0$, $p^*_v = 1/3$, $p^*_u = 2/3$, $p^*_w = 1/2$, $f(P_1) = 1/3$, and $f(P_2) = 1/6$.
The path (e_2, e_5, e_6, e_4) has f-value 1/4.

We define a decomposition of G into paths P_0 to P_k, an orientation of these paths, a slope $f(P_i)$ for each P_i, a vertex labelling p^*, and an edge labelling r. P_0 is a[6] shortest s_0-s_1 path in G, $f(P_0) = 1$, $r_e = f(P_0) - 1$ for all $e \in P_0$, and $p^*_v = \text{dist}(v, s_1)$ for all $v \in P_0$, where $\text{dist}(v, s_1)$ is the shortest path distance from v to s_1. For $1 \le i \le k$, we have[7]

$$P_i = \operatorname*{argmax}_{P \in \mathcal{P}} f(P),$$

where $\mathcal{P}$ is the set of all paths P in G with the following properties:

- the startpoint a and the endpoint b of P lie on $P_0 \cup \ldots \cup P_{i-1}$, $p^*_a \ge p^*_b$, and $f(P) = (p^*_a - p^*_b)/L(P)$;
- no interior vertex of P lies on $P_0 \cup \ldots \cup P_{i-1}$; and
- no edge of P belongs to $P_0 \cup \ldots \cup P_{i-1}$.

If $p^*_a > p^*_b$, we direct P_i from a to b. If $p^*_a = p^*_b$, we leave the edges in P_i undirected. We set $r_e = f(P_i) - 1$ for all edges of P_i, and $p^*_v = p^*_b + f(P_i)\text{dist}_{P_i}(v, b)$ for every interior vertex v of P_i. Here, $\text{dist}_{P_i}(v, b)$ is the distance from v to b along path P_i. Figure 3 illustrates the path decomposition.

Lemma 21 *There is an $i_0 \le k$ such that*

$$f(P_0) > f(P_1) > \ldots > f(P_{i_0}) > 0 = f(P_{i_0+1}) = \ldots = f(P_k).$$

Proof: It suffices to show: if there is an i such that $f(P_{i+1}) \ge f(P_i)$, then $f(P_i) = f(P_{i+1}) = 0$. If no endpoint of P_{i+1} is an internal vertex of P_i, then $f(P_{i+1}) = f(P_i)$; otherwise P_{i+1} would have been chosen instead of P_i. By assumption, equality is only possible if the f-values are zero. So we may assume that at least one endpoint of P_{i+1} is an internal node of P_i; call it c and assume w.l.o.g. that it is the startpoint of P_{i+1}. Split P_i at c into P_i^1 and P_i^2, and let d be the other endpoint of P_{i+1}; d may lay on P_i.

[6] We assume that P_0 is unique.
[7] We assume that P_i is unique except if $f(P_i) = 0$.

Assume first that d does not lie on P_i and consider the path $P_i^1 P_{i+1}$. The f-value of this path is

$$\frac{p_a^* - p_d^*}{L(P_i^1) + L(P_{i+1})} = \frac{p_a^* - p_c^* + p_c^* - p_d^*}{L(P_i^1) + L(P_{i+1})}.$$

Next, observe that $(p_a^* - p_c^*)/L(P_i^1) = f(P_i)$ since p_c^* is defined by linear interpolation and $(p_c^* - p_d^*)/L(P_{i+1}) = f(P_{i+1}) \geq f(P_i)$. In case of inequality, $P_i^1 P_{i+1}$ is chosen instead of P_i. In case of equality, there are two paths with the same f-value. By assumption, this is only possible if the f-values are zero.

Assume next that d also lies on P_i. We then split P_i into three paths P_i^1, P_i^2, and P_i^3 and consider the path $P_i^1 P_{i+1} P_i^3$. We then argue as in the preceding paragraph. ∎

Theorem 3 *If a network stabilizes, then* $\overrightarrow{E} = \cup_{i \leq i_0} E(P_i)$*, the orientation of any edge* $e \in \overrightarrow{E}$ *agrees with the orientation induced by the path decomposition, and* $E_h = \cup_{i > i_0} E(P_i)$*. The potential of each node* v *converges to* p_v^**. The diameter of each edge* $e \in E \setminus P_0$ *decays with rate* r_e*.*

Proof: We use induction on i to prove:

- for every vertex $v \in P_0 \cup \ldots \cup P_i$, the node potential p_v converges to p_v^*;
- for every edge $e \in P_0 \cup \ldots \cup P_{\min(i,i_0)}$, the flow stabilizes in the direction of the path P_j containing e;
- for every edge $e \in P_1 \cup \ldots \cup P_i$, the diameter converges to zero with rate r_e, and the flow converges to zero with rate at least[8] r_e. If $e \in P_i$ and $i \leq i_0$, the flow converges to zero with rate r_e.

Lemma 15 establishes the base of the induction, the case $i = 0$. Assume now that the induction hypothesis holds for $i - 1$; we establish it for i. Let $P_{\leq i-1} = P_0 \cup \ldots \cup P_{i-1}$.

For $e \in E \setminus P_{\leq i-1}$, let

$$f_e = \max \left\{ \frac{p_a^* - p_b^*}{L(P')} \ ; \ P' \in \mathcal{P}_e \right\},$$

where $\mathcal{P}_e$ is the set of paths P' in $G \setminus P_{\leq i-1}$ from some $a \in P_{\leq i-1}$ to some $b \in P_{\leq i-1}$ with $p_a^* \geq p_b^*$ and containing e. Then, $\max_{e \notin P_{\leq i-1}} f_e = f(P_i)$. For $i \leq i_0$, we have further $f(P_i) > \max_{e \notin P_{\leq i}} f_e \geq f(P_{i+1})$. In general, the last inequality may be strict; see Figure 3.

Lemma 22 *For* $e \in E \setminus P_{\leq i-1}$*,* $|Q_e|$ *and* D_e *decay with rate at least* $f_e - 1$*.*

Proof: According to Lemma 18, it suffices to prove the decay of $|Q_e|$. Let $e \in E \setminus P_{\leq i-1}$ and let $\varepsilon > 0$ be arbitrary. We need to show

$$\ln |Q_e(t)| \leq C + (f_e + \varepsilon - 1)t$$

for some constant C and all sufficiently large t.

If $Q_e(t) = 0$, the inequality holds for any value of C. So assume $Q_e(t) \neq 0$ and also assume that the flow across $e = \{u, v\}$ is in the direction from u to v. We construct a path

[8] If for an edge $e = \{u, v\}$, $p_u - p_v = 0$ always, then $Q_e = 0$ always. Thus, for horizontal edges, Q_e may converge to zero faster than with rate -1.

$R(t)$ containing uv. For every vertex, except for source and sink, we have flow conservation. Hence there is an edge (v, w) carrying a flow of at least Q_e/n in the direction from v to w. Similarly, there is an edge (x, v) carrying a flow of at least Q_e/n in the direction from x to v. Continuing in this way, we reach vertices in $P_{\le i-1}$. Any edge on the path $R(t)$ carries a flow of at least Q_e/n^n.

Since potential differences are bounded by $B := 2nmL_{\max}$ (Lemma 4.ix), any edge e' on $R(t)$ must have a diameter of at least $Q_e L_e/(n^n B) \ge (L_{\min}/(n^n B))Q_e$. Let $c = L_{\min}/(n^n B)$. Then,

$$W(R(t)) = \sum_{e' \in R(t)} L_{e'} \ln D_{e'} \ge L(R(t))(\ln c + \ln |Q_e(t)|).$$

The path $R(t)$ depends on time. Let $a(t)$ and $b(t)$ be the endpoints of $R(t)$. Since e does not belong to $P_{\le i-1}$,

$$f(R(t)) = \frac{p^*_{a(t)} - p^*_{b(t)}}{L(R(t))} \le f_e.$$

For large enough t, we have $\Delta(R(t)) \le \Delta^*(R(t)) + \varepsilon L(R)/2$. Every edge $e \in R(t)$ either belongs to $\overrightarrow{E}$ or to E_h due to the assumption that the network stabilizes. In the former case, R must use e in the direction fixed in $\overrightarrow{E}$, in the latter case, the potential difference across e converges to zero. We now invoke Lemma 19 with $\delta = \varepsilon L(R)/(4n)$. It guarantees the existence of a constant C_1 such that

$$W(R(t))(t) \le C_1 + (\Delta^*(R(t)) + \varepsilon L(R)/2 - L(R) + \varepsilon L(R)/2)t$$

for all t. The constant C_1 depends on the path $R(t)$. Since there are only finitely many different paths $R(t)$, we may use the same constant C_1 for all paths $R(t)$.

Combining the estimates, we obtain, for all sufficiently large t,

$$L(R(t))(\ln c + \ln |Q_e(t)|) \le C_1 + (\Delta^*(R(t)) + \varepsilon L(R(t)) - L(R(t)))t,$$

and hence,

$$\ln |Q_e(t)| \le C_1/L(R(t)) - \ln c + (f_e + \varepsilon - 1)t.$$

∎

Corollary 4 *For $e \in E \setminus P_{\le i-1}$, $|Q_e|$ and D_e decay with rate at least $f(P_i) - 1$. If $i \le i_0$, then for any $e \in E \setminus P_{\le i}$, $|Q_e|$ and D_e decay with rate at least $f(P_i) - \delta - 1$ for some $\delta > 0$.*

Proof: If $i \le i_0$, and hence, $f(P_i) > 0$, $f_e < f(P_i)$ for any edge $e \in E \setminus P_{\le i}$. The claim follows. ∎

Lemma 23 *Let $e \in P_i$. Then, D_e decays with rate $f(P_i) - 1$. If $i \le i_0$, then $|Q_e|$ decays with rate $f(P_i) - 1$.*

Proof: We distinguish the cases $f(P_i) = 0$ and $f(P_i) > 0$. If $f(P_i) = 0$, the diameter of all edges $e \in P_i$ decays with rate at least -1 (Lemma 19). No diameter decays with a rate faster than -1.

We turn to the case $f := f(P_i) > 0$. The flows across the edges in $E \setminus P_{<i}$ decay with rate at least $f - 1$, and the flows across the edges edges in $E \setminus P_{\le i}$ decay faster, say with rate at least $f - \delta - 1$ for some positive δ (Corollary 4). We first show

$$W(P_i) \le C + L(P_i) \cdot \max(\ln D_e, (f - \delta - 1)t) \tag{10}$$

for sufficiently large t and some constant C. If P_i consists of a single edge e, $W(P_i) = L_e \ln D_e(t)$ and (10) holds. Assume next that $P_i = e_1 \dots e_k$ with $k > 1$. Consider any interior node u of the path. The flow into u is equal to the flow out of u, and u has two incident edges[9] in P_i. The flow on the other edges incident to u decays with rate at least $f - \delta - 1$. Thus for any two consecutive edges on P_i, $| |Q_{e_j}| - |Q_{e_{j+1}}| |$ decays with rate at least $f - \delta - 1$. By Lemma 18, this implies that $|D_{e_j} - D_{e_{j+1}}|$ decays with rate at least $f - \delta - 1$. Thus, we have $D_{e_j} = D_e + g_{e_j}$, where $|g_{e_j}| \le C_1 e^{(f-\delta-1)t}$ for some constant C_1 and all j. Plugging into the definition of $W(P_i)$ yields

$$\begin{aligned} W(P_i) &\le \sum_{e_j \in P_i} L_{e_j} \ln \big(2 \max(D_e, g_{e_j})\big) \\ &\le L(P_i) \ln 2 + L(P_i) \max(\ln D_e, \ln C_1 e^{(f-\delta-1)t}), \end{aligned}$$

and we have established (10).

Let t_0 be large enough such that $|\Delta(P_i) - \Delta^*(P_i)| \le \delta L(P_i)/2$ for all $t \ge t_0$. Then, by Lemma 19,

$$W(P_i) \ge A + L(P_i)(f - \delta/2 - 1)t \tag{11}$$

for some constant A and all t.

Combining (10) and (11) yields

$$A + L(P_i)(f - \delta/2 - 1)t \le C + L(P_i) \cdot \max(\ln D_e, (f - \delta - 1)t).$$

Thus, for every t we have either

$$A + L(P_i)(f - \delta/2 - 1)t \le C + L(P_i) \cdot \ln D_e$$

or

$$A + L(P_i)(f - \delta/2 - 1)t \le C + L(P_i) \cdot (f - \delta - 1)t.$$

The latter inequality does not hold for any sufficiently large t. Thus, the former inequality holds for all sufficiently large t, and hence, D_e decays with rate at most $f(P_i) - 1$. By Lemma 18, $|Q_e|$ cannot decay at a faster rate if $f(P_i) > 0$. ∎

Lemma 24 *For $v \in P_i$, the potentials converge to p_v^*. For $e \in P_i$ and $i \le i_0$, the flow direction stabilizes in the direction of P_i.*

Proof: Assume $i \le i_0$ first. Let $P_i = e_1 \dots e_k$. The flows and the diameters of the edges in P_i decay with rate $f(P_i) - 1$ (Lemma 23). The flows and diameters of the edges incident to the interior vertices of P_i and not on P_i decay faster, say with rate at least $f(P_i) - \delta - 1$, where $\delta > 0$. For large t and any interior vertex of P_i, one edge of P_i must, therefore, carry flow into

[9] Here, we need uniqueness of P_i. Otherwise we would have a network of paths with the same slope.

the vertex, and the other edge incident to the vertex must carry it out of the vertex. Thus, the edges in P_i must either all be directed in the direction of P_i or in the opposite direction. As current flows from higher to lower potential, they must be directed in the direction of P_i.

Because the flow and the diameters of the edges not on P_i and incident to interior vertices decay faster, we have for any $\varepsilon > 0$ and sufficiently large t

$$Q_{e_j} = Q_{e_1}(1+\varepsilon_j) \quad \text{and} \quad D_{e_j} = D_{e_1}(1+\varepsilon'_j),$$

where $|\varepsilon_j|, |\varepsilon'_j| \le \varepsilon$. The potential drop Δ_{e_j} on edge e_j is equal to

$$\Delta_{e_j} = \frac{Q_{e_j} L_{e_j}}{D_{e_j}} = \frac{Q_{e_1}(1+\varepsilon'_j)}{D_{e_1}(1+\varepsilon_j)} L_{e_j},$$

and hence, the potential drop along the path is

$$p_a - p_b = \sum_j \Delta_{e_j} = \frac{Q_{e_1}}{D_{e_1}} L(P_i)(1+\varepsilon''),$$

where ε'' goes to zero with ε. The potential drop along the path converges to $p^*_a - p^*_b$. Thus, Q_{e_1}/D_{e_1} converges to $f(P_i)$, and therefore, the potential of any interior vertex v of P_i converges to p^*_v.

We turn to the case $i > i_0$. The potentials of the endpoints of P_i converge to the same value. Thus, the potentials of all interior vertices of P_i converge to the common potential of the endpoints. ∎

We have now completed the induction step. ∎

8 The Wheatstone Graph

Do edge directions stabilize? We do not know. We know one graph class for which edge directions are unique, namely series-parallel graphs. The simplest graph which is not series-parallel is the Wheatstone graph shown in Figure 4. We show that the Wheatstone graph stabilizes.

We use the following notation: We have edges a to e as shown in the figure. For an edge x, $R_x = L_x/D_x$ denotes its resistance and $C_x = D_x/L_x$ denotes its conductance.[10] For edges a, b, c, and d, the direction of the flow is always downwards. For the edge e, the direction of the flow depends on the conductances. We have an example where the direction of the flow across e changes twice.

A shortest path from source to sink may have two essentially different shapes. It either uses e, or it does not. If e lies on a shortest path, Lemma 19 suffices to prove convergence as observed by [12]. If (a, e, d) is a shortest path[11], let $P = (a, e)$ and $P' = (b)$. Then,

$$\frac{d}{dt}(W(P) - W(P')) \ge \Delta(P) - L(P) - (\Delta(P') - L(P')) = L(P') - L(P) > 0.$$

Since $W(P)$ is bounded, this implies $W(P') \to -\infty$. Thus, D_b converges to zero. Similarly, D_d must converge to zero. More precisely, $W(P')$ goes to $-\infty$ linearly, and hence, D_b and similarly D_d decay exponentially.

[10] Observe that we use the letter C with a different meaning than in preceding sections.

[11] For simplicity, we assume uniqueness of the shortest path in this section.

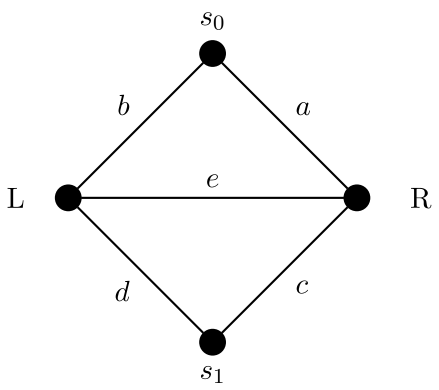

Figure 4: The Wheatstone graph.

The non-trivial case is that the shortest path does not use e. We may assume w.l.o.g. that the shortest path uses the edges a and c. The ratio

$$x_a = \frac{R_a}{R_a + R_c} = \frac{1}{1 + R_c/R_a} = \frac{1}{1 + C_a/C_c} = \frac{C_c}{C_a + C_c}$$

is the ratio of the resistance of a to the total resistance of the right path; define x_b, x_c, and x_d analogously. Observe $x_a + x_c = 1$ and $x_b + x_d = 1$. Let

$$x_a^* = \frac{L_a}{L_a + L_c};$$

define x_b^*, x_c^*, and x_d^* analogously. Without edge e, the potential drop on the edge a is x_a times the potential difference between source and sink. If $D_a = D_c$, which we expect in the limit, $x_a = x_a^*$.

Lemma 25 *Let* $S = C_aC_b(C_c + C_d) + (C_a + C_b)C_cC_d + (C_a + C_b)(C_c + C_d)C_e$. *Then,*

$$\dot{x_a} = \frac{C_aC_c}{SL_aL_c(C_a + C_c)^2}\left((C_b + C_d + C_e)(L_a + L_c)(C_a + C_c)(x_a^* - x_a) + C_eC_bL_c\left(\frac{x_a^*}{x_c^*} - \frac{x_b}{x_d}\right)\right)$$
$$\dot{x_b} = \frac{C_bC_d}{SL_bL_d(C_b + C_d)^2}\left((C_a + C_c + C_e)(L_b + L_d)(C_b + C_d)(x_b^* - x_b) + C_eC_aL_d\left(\frac{x_b^*}{x_d^*} - \frac{x_a}{x_c}\right)\right).$$

Proof: The derivatives of C_a to C_e were computed by Miyaji and Ohnishi [11]:

$$\dot{C}_a = \frac{C_a}{SL_a}(C_bC_c + C_cC_d + C_cC_e + C_dC_e) - C_a$$
$$\dot{C}_c = \frac{C_c}{SL_c}(C_aC_d + C_aC_b + C_aC_e + C_bC_e) - C_c.$$

The derivatives of C_b and C_d can be obtained from the above by symmetry (exchange a with

b and c with d). We now compute $\dot{x_a}$:

$$\begin{aligned}
\frac{d}{dt}\frac{C_c}{C_a+C_c} &= \frac{-(\dot{C}_a C_c - C_a \dot{C}_c)}{(C_a+C_c)^2} \\
&= \frac{-\left(\frac{C_a}{SL_a}(C_bC_c + C_cC_d + C_cC_e + C_dC_e) - C_a\right)C_c}{(C_a+C_c)^2} + \\
&\quad + \frac{C_a\left(\frac{C_c}{SL_c}(C_aC_d + C_aC_b + C_aC_e + C_bC_e) - C_c\right)}{(C_a+C_c)^2} \\
&= \frac{C_aC_c}{S(C_a+C_c)^2}\left(\frac{C_aC_d + C_aC_b + C_aC_e + C_bC_e}{L_c} - \frac{C_bC_c + C_cC_d + C_cC_e + C_dC_e}{L_a}\right) \\
&= \frac{C_aC_c}{S(C_a+C_c)^2}\left((C_b + C_d + C_e)\left(\frac{C_a}{L_c} - \frac{C_c}{L_a}\right) + C_e\left(\frac{C_b}{L_c} - \frac{C_d}{L_a}\right)\right) \\
&= \frac{C_aC_c}{SL_aL_c(C_a+C_c)^2}\left((C_b + C_d + C_e)(D_a - D_c) + C_e(C_bL_a - C_dL_c)\right) \\
&= \frac{C_aC_c}{SL_aL_c(C_a+C_c)^2}\left((C_b + C_d + C_e)(D_a - D_c) + C_eC_bL_c\left(\frac{L_a}{L_c} - \frac{L_b/D_b}{L_d/D_d}\right)\right).
\end{aligned}$$

Finally, observe

$$x_a^* - x_a = \frac{L_a}{L_a+L_c} - \frac{C_c}{C_a+C_c} = \frac{L_a(C_a+C_c) - C_c(L_a+L_c)}{(L_a+L_c)(C_a+C_c)} = \frac{D_a - D_c}{(L_a+L_c)(C_a+C_c)}.$$

∎

We draw the following conclusions:

- if $C_e = 0$, then $\text{sign}(\dot{x_a}) = \text{sign}(D_a - D_c) = \text{sign}(x_a^* - x_a)$. Thus, x_a converges monotonically against x_a^*.
- From $x_b + x_d = 1$ and $x_a^* + x_c^* = 1$, we conclude

$$\text{sign}\left(\frac{x_a^*}{x_c^*} - \frac{x_b}{x_d}\right) = \text{sign}(x_a^* - x_b).$$

- if $s = \text{sign}(x_a^* - x_b) = \text{sign}(x_a^* - x_a)$, then $\text{sign}(\dot{x_a}) = s$.
- if $x_a, x_b > x_a^*$, then x_a decreases.
- if $x_a, x_b < x_a^*$, then x_a increases.
- if $x_d, x_c > x_d^*$, then x_d decreases (equivalent to: if $x_a, x_b < x_b^*$, then x_b increases).
- if $x_d, x_c < x_d^*$, then x_d increases (equivalent to: if $x_a, x_b > x_b^*$, then x_b decreases).

Theorem 5 *Assume $x_a^* < x_b^*$, that is, $L_a/L_c < L_b/L_d$. Then,*

1. *The regime $x_a, x_b > x_b^*$ cannot be entered. By symmetry, the regime $x_a, x_b < x_a^*$ cannot be entered.*

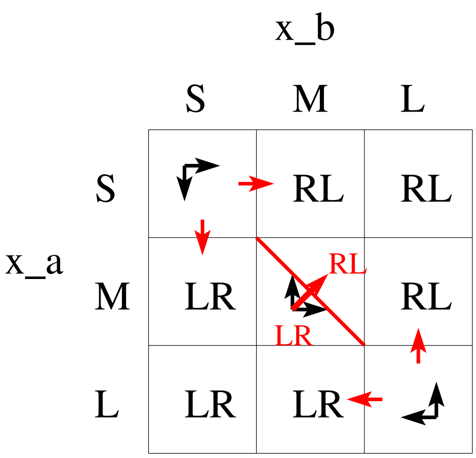

Figure 5: The transition diagram under the assumption $x_a^* < x_b^*$.

2. *In the regime $x_a, x_b \in [x_a^*, x_b^*]$, x_a decreases and x_b increases. Hence, in this regime, the direction of the middle edge e can change at most once.*

3. *If the dynamics stay in the regime $x_a, x_b \geq x_b^*$ forever, x_a and x_b converge.*

4. *If the dynamics stay in the regime $x_a, x_b \leq x_a^*$ forever, x_a and x_b converge.*

Proof: At (1): In the regime $x_a, x_b > x_b^*$, x_a and x_b both decrease, and hence, the dynamics cannot enter the regime from the outside. More precisely, we consider two cases: $x_b \geq x_b^*$ and $x_a = x_b^*$, or $x_a > x_b^*$ and $x_b = x_b^*$.

If $x_b \geq x_b^*$ and $x_a = x_b^*$, x_a is non-increasing, and hence, we cannot enter the regime.

If $x_a > x_b^*$ and $x_b = x_b^*$, x_b is non-increasing, and hence, we cannot enter the regime.

At (2): Obvious from the equations.

At (3): Then, x_a and x_b are monotonically decreasing and hence converging. The derivative of x_b clearly goes to zero if x_b and x_a converge to x_b^*.

At (4): Symmetrically to (3). ∎

In Figure 5, we use S, M, and L to denote the three ranges: $S = [0, x_a^*]$, $M = [x_a^*, x_b^*]$, and $L = [x_b^*, 1]$. The box $M \times M$ is divided into the triangles $x_a < x_b$ and $x_a > x_b$. The figure also shows that the boxes $S \times S$ and $L \times L$ cannot be entered and that the latter triangle cannot be entered from the former.

We conclude the following dynamics: Either the process stays in $S \times S$ or $L \times L$ forever or it does not do so. If it leaves these sets of states, it cannot return. Moreover, there is no transition from the set of states RL to the set of states LR. Thus, if the process does not stay in $S \times S$ or $L \times L$ forever, the direction of the middle edge stabilizes.

Assume now that the dynamics stay forever in $S \times S$, or in $L \times L$. Then, x_a and x_b converge. Let x_a^∞ and x_b^∞ be the limit values. If the limit values are distinct, the direction of the middle edge stabilizes. If the limit values are the same, the edge is horizontal and hence stabilizes. We summarize the discussion.

Theorem 6 *The dynamics of the Wheatstone graph stabilize.*

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