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RESEARCH PAPER

Biproportional allocation of parliamentary seats via vector apportionment and minimum paths

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Abstract

The Biproportional Apportionment Problem arises in countries which allocate electoral seats at two different territorial levels by a proportional rule. Mathematically, given the constituency/party vote matrix V, vectors c of national party seats and r of constituency seats, the problem is to compute an integer matrix $Aśś$ proportional to V whose column- and row- totals correspond to c and r, respectively. To pursue proportionality, one can first find a fractional matrix Q satisfying the above requirements, and then search for an integer matrix A that minimizes the distance from Q. Starting from the Italian electoral law for the Chamber of Deputies, in this paper, we develop a two-phase procedure for BAP which exploits Hare Quota plus Largest Remainders for the seat allocation in the first phase, while, in the second phase, it performs seat transfers by shortest paths on a suitable network. We test our method on real elections of different countries and compare our results with the corresponding institutional allocations. The quality of an apportionment is measured by its distance to the fractional Fair-Share matrix, a benchmark for biproportionality introduced by Balinski and Demange in 1989. The empirical results show that, even if the proposed methods are heuristics, they are able to output high quality solutions. The application to elections in Italy and in other European countries confirms both the effectiveness of the methods and their general applicability.

Keywords Biproportional seat apportionment · Alternating row-column apportionment · Seat transfers · Shortest paths · Electoral systems

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1 Introduction

There is no other democratic country in the world in which the electoral system for the national Parliament has been modified as frequently as it has been in Italy over the past 30 years. In all successive reforms, from 1993 to the present day, the electoral system has preserved its main underlying logic based on a proportional method,[1] while supplementing it, either by awarding a "majority bonus" or by assigning a share of seats through a majoritarian system in single-member districts. In all these reforms, in compliance with the constitutional principles, the method of allocating seats across multiple territorial levels has always been maintained; however, there has never been the willingness or the capability to solve a chronic technical problem that affects the two-dimensional seat allocation procedure prescribed by Italian law. The same problem still persists in the latest proposal presented in April 2026.[2]

In Italy, as in other countries in the world, the national territory is subdivided into electoral constituencies (or districts) and seat allocation is performed both at a national level, by assigning seats to parties on the basis of their total vote, and at a constituency level, where seats are allocated to parties proportionally to the vote cast in the constituency. Typically, the allocation is conceived as a doubly-constrained problem: the first constraint is given by the number of seats at stake in the constituencies[3]; the second is given by the total number of seats assigned to each party proportionally to their national vote. The allocation of seats at the national level is clearly meant to fulfill the principle of *political representation* nationwide. Similarly, proportional allocation within the constituencies applies the same principle at a local level. However, since national votes are computed as the sum of citizens' votes from the different constituencies, the previous two allocations are not independent. Actually, when proportionality in two dimensions is pursued, the final seat allocation should satisfy the following: i) in each constituency, seats are assigned to parties proportionally to their local vote; ii) each party receives its total seats in the different constituencies proportionally to the votes cast there. The last requirement embodies the principle of *territorial representation*.

The above doubly-constrained and doubly-proportional seat allocation problem is known in the literature as the *Biproportional Apportionment Problem* (BAP, for short). For a complete reference on the subject, the reader is referred to Pukelsheim (2017). Over the last 30 years, governments of other European and non-European countries have faced the same problem by proposing different solutions; however, it has only rarely been possible to incorporate a formal method for the biproportional allocation of seats into electoral legislation. The only case in which a rigorous allocation pro-

[1] Originally it was a pure proportional system.

[2] Modifiche al testo unico di cui al decreto del Presidente della Repubblica 30 marzo 1957, n. 361, in materia di elezione della Camera dei deputati, al testo unico di cui al decreto legislativo 20 dicembre 1993, n. 533 e in materia di elezione del Senato della Repubblica.

[3] These numbers are computed before the election proportionally to the resident population in the constituencies.

cedure has been concretely adopted is the electoral law of the Canton of Zurich for the election of the cantonal parliament that in 2006 has implemented the procedure provided in Pukelsheim (2004); Pukelsheim and Schuhmacher (2004). This example was subsequently followed in later years by other Swiss cantons who decided to adopt the same procedure.

In Italy, the current law for the Chamber of Deputies applies the well-known Hare Quota (HQ) method plus the Largest Remainder (LR) rule (Grilli Di Cortona et al. 1999; Pukelsheim 2017) to assign proportional seats. The problem with the law is that allocations in the different constituencies are performed independently, with the consequence that it is not guaranteed that the sum of the seats obtained by a party in the constituencies is equal to the prefixed number of party's national seats. Actually, some parties in total receive more seats than their national number (*surplus parties*), while others receive less (*deficit parties*). In the successive electoral reforms in Italy a sequence of transfers was introduced in order to re-balance the column totals of the surplus and deficit parties in the apportionment matrix. Unfortunately, in spite of the many efforts for finding suitable ways for regulating these transfers, the operations prescribed by the law are unable to guarantee the final convergence towards a feasible solution of the doubly-constrained problem. Several cases can be found in the Italian past elections in which, to match party national seats, the law produces a final allocation that does not satisfy the constituency totals. This implies that, in the final distribution of seats, some constituencies will have been allocated more seats than were available, while others will have received fewer. This Italian phenomenon, known as "shifting seats", was widely analyzed in the literature by authors of different countries (Lari et al. 2014; Maier et al. 2010; Pennisi 2006; Pennisi et al. 2007, 2008). We point out that the law governing elections to the Italian Chamber of Deputies has been affected by the same technical problem since the adoption of the first electoral law of the Italian Republic (Presidential Decree No. 361 of 1957). The same problem also affects the Italian electoral law for the European Parliament, as well as several of the laws governing the election of Regional Councils. [4]

For the Italian case, which is the focus of the present paper, we propose a procedure that follows a two-phase approach similar to the one analyzed in Maier et al. (2010), but exploiting Hare Quota plus Largest Remainders for the bidimensional allocation in the first phase, and performing, in the second phase, seat transfers by the solution of shortest path optimization problems on a suitable network. The main advantage of the proposed method is that it is close to the procedure of the Italian electoral law for the Chamber of Deputies, and therefore it could be considered as a solution to fix the technical problem affecting that law. Unfortunately, differently from the Zurich case, in Italy a fully automated optimization procedure is still not admitted in the law; on the other hand, we believe that the law can acknowledge at least the idea of "controlled" seat transfers between a surplus and a deficit party performed by means of a "path". Clearly, the problem of making each step of the seat allocation algorithm understandable by everyone is not completely solved, but we believe that, potentially in the law, paths can be explained in a technical note where tables can be exploited to describe how seat transfer operates concretely (see Sect. 3). Even if the difficulty

[4] This is due to the fact that these laws adopt the same rules established by Decree No. 361/1957.

of rephrasing the text of the law in algorithmic terms remains, in view of the positive feedback that we received by the experts on electoral systems from the Italian Chamber of Deputies, we believe that our proposed procedure can be considered a concrete step towards the solution of the problem.

The paper is organized as follows. Section 2 states the basic principles and requirements of BAP, providing a brief review of the papers on BAP and the related solution procedures. Different methods have been proposed for the solution of BAP of specific countries; not surprisingly, many refer to Italy. Section 3 formalizes BAP in mathematical terms, while Sect. 4 presents our proposed algorithms. Section 5 presents the results of the application to the parliamentary and European elections in Italy, as well as to other European countries. Finally, the concluding section provides some further thoughts and final remarks.

2 Biproportional apportionment problem: general definition and a brief literature review

Assume that the national territory is divided into a given number of electoral constituencies and that the total number of seats to be allocated in each constituency (*magnitude*) is fixed a priori proportional to the population of the constituency. At the same time, the law prescribes that the number of seats to be assigned to each party at a national level is given by a (rounded) proportion of their national votes. This produces two super-apportionments for the total number of seats, one among the constituencies, and another one among the parties. The final (disaggregated) allocation of seats to parties inside the constituencies must be performed without altering these two super-apportionments. Data are usually organized into a matrix whose rows and columns correspond to constituencies and parties, respectively. The problem input is the matrix of votes in each constituency for each party and the prefixed constituency magnitudes (row-totals), as well as party national seats (column-totals). BAP consists in finding a matrix of seats such that:

(1) The number of seats assigned to each party in each constituency is integral and nonnegative;
(2) Each row-sum (or constituency-sum) is equal to the total number of seats at stake in the corresponding constituency (constituency magnitude);
(3) Each column-sum (or party-sum) is equal to the total number of seats of the corresponding party at a national level (nationwide party seats);
(4) A party cannot obtain seats in a constituency where it received no votes;
(5) The number of seats assigned to each party in each constituency is "as proportional as possible" to the corresponding number of votes.

Conditions (1)–(4) can be easily formulated as mathematical constraints, while the principle of biproportionality stated in (5) cannot be formulated in a unique and unambiguous way. From a mathematical point of view, generalizing proportionality from one dimension to two is not trivial. As discussed in Maier et al. (2010), a unique formal definition of proportionality in two dimensions is not straightforward, and, in the literature, double proportionality is usually evaluated in terms of *fairness* of the allocation

procedure and *quality* of the final seat apportionment. Following Balinski and Young (2010) and Maier et al. (2010), fairness means to equalize voters' political representation, and involves "how proportional the seat numbers are to the vote numbers" On the other hand, quality is a concrete measure of "how well fairness is achieved" and can be operationally computed for any given apportionment, for example, by means of distance measures from a fairness benchmark matrix. In the mathematical formulations of BAP suggested in the literature, double proportionality is treated in different ways, and the benchmarks can also be different fractional quota matrices (see Maier et al. (2010), Sect. 4). In the following, we report a brief review of the main papers in the wide literature dealing with this problem.

Balinski and Demange (1989a, b) provide a method to find a biproportional allocation satisfying a specified set of proportionality axioms. First a fractional matrix is computed, called (fractional) Fair-Share matrix (F); then, the corresponding integer allocation matrix is obtained by an ordinary rounding to the closest integer of each fraction different from 0.5, and by a special—ad hoc—rounding procedure for each value equal to 0.5, if any. The Fair-Share matrix has the property that it is always possible to obtain a seat allocation satisfying (1)–(4) by rounding each element of the matrix either to its floor or to its ceiling. Gaffke and Pukelsheim (2008a, b) suggest an optimization model that can be reduced to a minimum cost flow problem with a convex and separable objective function. Furthermore, they show that Balinski and Demange's procedure can be seen as a dual algorithm for their model. Balinski and Young (2010) provides a comprehensive reference on proportional seat apportionment, its mathematical aspects, and its history. It is well known that the existence of an allocation satisfying conditions (1)–(4) can be decided via the solution of a maximum flow problem on a suitable network (Ahuja et al. 1993). This result is exploited in many papers of the optimization literature concerning biproportional apportionment (see, for example, (Cox and Ernst 1982; Pukelsheim et al. 2012; Rote and Zachariasen 2007)) because it guarantees that the problem can be solved in polynomial time. In Pennisi et al. (2007) and in Serafini and Simeone (2012) BAP is modeled starting from a target (fractional) matrix, that is assumed to be the most proportional one with respect to the vote matrix, through optimization models in which constraints (1)–(4) are formulated as linear constraints and the objective function—to be minimized—corresponds to a measure of the total deviation between the target matrix and the seat allocation solution matrix. The target matrix can be for example, the fractional Fair-Share, as well as any other quota matrix representing a good yardstick for biproportionality, as the matrix of natural quotas computed in countries adopting district-wise apportionment methods. For a review of the main procedures for BAP based on network flows, the reader is referred to Pukelsheim et al. (2012).

The biproportional seat allocation was also formulated in a *reduced* version in Lari et al. (2014) after the assumption that parties have already received in each constituency a number of seats equal to the floor of their natural quotas in the constituency. After this first partial allocation, BAP reduces to the assignment of the remaining seats in each constituency (with reduced row-sums) to fulfill the seats that each party must still receive (reduced column-sums). The reduced BAP can be still formulated as a network flow problem, but with arc capacities that can be only 0 or 1. When constraints (1)–(3) must be guaranteed, this problem is known in the literature as the *Matrix Feasibility*

Problem, which was first studied by Gale (1957) and Ryser (1957) for matrices with positive cells. They independently suggested a simple method essentially based on a greedy procedure to fill in the matrix of seats (see also Ryser (1963)). Under the additional constraint (4), a similar but more general algorithm was developed in Lari et al. (2014), providing sufficient conditions for feasibility in presence of zero-cells in the vote matrix. It is based on a central result of Gale and Ryser and can be applied when the vote matrix has special zero-cell configurations.

As already observed, biproportional apportionment is prescribed by the laws of different countries in the world. In 2003, the Swiss Canton of Zurich adopted a doubly proportional system based on a biproportional divisor method with standard rounding (see Pukelsheim and Schuhmacher (2004, 2011) and Balinski and Pukelsheim (2006)). Pennisi (2006); Pennisi et al. (2007, 2008) studied the problem for Italy, and Zachariassen and Zachariassen (2006) for the Faroe Islands. Balinski and Ramírez (1996, 1997, 1999) analyzed Mexico, while Ramírez studied Spain in Márquez and Ramírez (1998) and Ramirez et al. (2008); Oelbermann and Pukelsheim (2011) also studied BAP for the European Union Elections. Among the examples listed, only the Canton of Zurich has actually solved the problem using the algorithm proposed in Pukelsheim and Schuhmacher (2004), which was formally incorporated into the law. Here, and in all the other Swiss Cantons who have adopted the same double proportionality method, the law has been subjected to the citizen's approval by plebiscites, and then the formal algorithm was implemented in the systems of the electoral offices that execute the law procedure in practice. Some other countries have acknowledged the issue and made attempts to amend their legislation in order to solve the problem, although not all have succeeded. For example, Faroe Islands solved the problem by their electoral reform in 2008, when they passed from a district-wise allocation of seats to a single national district. On the other hand, Mexico, which has the territorial division in constituencies, does not adopt a single unified BAP procedure. Rather, the law first prescribes assigning seats within constituencies and then adjusting the outcome to comply with the constitutional constraints at the national-level. For the Italian case, in the recent years, a fruitful collaboration between academia and experts from the Research Department of the Chamber of Deputies[5] has started with the objective of developing new technical proposals for a sound electoral law able to definitively solve the problem (Ricca and Scozzari 2019).

3 Doubly-constrained biproportional seat apportionment

Let H be the *house size* of a national parliament. When BAP is formalized as a doubly-constrained problem, the H parliamentary seats are first apportioned among the m electoral constituencies proportionally to population counts, giving r_i seats to the constituency $i \in M = \{1, \ldots, m\}$. Secondly, the same H seats are distributed among the n parties, proportionally to their national votes, assigning c_j seats to party j, $j \in N = \{1, \ldots, n\}$. Clearly, one has $\sum_{i \in M} r_i = \sum_{j \in N} c_j = H$. Throughout the

[5] Servizio Studi Camera dei deputati.

paper, we always assume that this condition holds, and we refer to it as the *general house size assumption*.

Let v_{ij} be the number of votes in district i for party j. The vote counts are the input data and form an $m \times n$ matrix V. The seat numbers a_{ij} form an integer $m \times n$ matrix A. Let $Z = \{(i, j) : v_{ij} = 0\}$ be the zero-pattern of V, that is, the set of the structural zeros of V.

Formally, an $m \times n$ nonnegative matrix $A = [a_{ij}]$ of seats is an *apportionment* if it satisfies the following constraints:

(1) a_{ij} integer;
(2) $\sum_{j \in N} a_{ij} = r_i, \quad i \in M$ (district sum);
(3) $\sum_{i \in M} a_{ij} = c_j, \quad j \in N$ (party sum);
(4) $a_{ij} = 0, \qquad (i, j) \in Z$ (zero-vote zero-seat).

We denote by $\mathcal{A}$ the set of apportionments and by $\mathcal{F}$ the set of *fractional apportionments*, i.e., real matrices satisfying (2), (3) and (4). Conditions (2) and (3) impose that row- and column- sums must be equal to the pre-specified totals $r_i, i \in M$, and $c_j, j \in N$, respectively. Condition (4) establishes that a party j that does not receive votes in a district i is not given any seat in that district. BAP can be formulated as follows: *given the vote matrix V and the vectors r and c, find an apportionment $A \in \mathcal{A}$ as proportional as possible to V.*

It is not obvious to turn the proportionality requirement into an operational concept. The Italian electoral law for the election of the Chamber of Deputies adopts the HQ method plus the LR rule, which is applied independently in each constituency, actually performing m separated row vector apportionments. First, a constituency divisor is computed by:

$$\gamma_i = \frac{\sum_{\ell \in N} v_{i\ell}}{r_i}, \quad i = 1, \ldots, m \tag{1}$$

and then party *exact* shares are obtained by scaling the row of votes in the constituency as follows:

$$q_{ij} = \frac{v_{ij}}{\gamma_i}, \quad i = 1, \ldots, m. \tag{2}$$

Exact shares, also known as *natural quotas* (or simply *quotas*), correspond to perfect (row-) proportionality, but they are usually not integers. One way to obtain an apportionment is to find a "suitable" rounding of the natural quotas that satisfies constraints (1)–(4). The Italian biproportional allocation procedure relies on the underlying assumption that one can always get an apportionment by rounding up or down the ideal shares. Unfortunately, realistic examples can be exhibited in which no up- or down-rounding of the ideal shares satisfies both the district- and the party-sum constraints (Pennisi 2006; Pennisi et al. 2009). The Italian procedure tries to solve the doubly-constrained apportionment problem in the wrong way, the consequence being that the matrix of seats produced by such procedure may fail to satisfy simultaneously both the row-sum or the column-sum constraints.

Indeed, this has happened systematically in the history of political elections for the Italian Chamber of Deputies, with the unavoidable consequence that the number of seats in some constituencies after the elections was different from the original magnitudes, thereby creating an imbalance in the voting power of citizens residing in different regions. For example, in the 2006 political elections, the Trentino-Alto Adige district got 11 seats instead of the 10 granted by the Constitutional Law, while the Molise got 2 seats instead of 3. This means that in Trentino-Alto Adige 85,456 votes were necessary to get a seat, while in Molise one needed 160,300 votes per seat, with the clear violation of the 'one-man-one-vote' principle for Molise's citizens w.r.t. people voting in Trentino-Alto Adige. The same phenomenon happened in previous and following Italian political elections for the Chamber of Deputies.

Besides the problem of satisfying constraints (2) and (3) simultaneously, the Italian electoral procedure is also affected by the fact that seat allocation is performed considering only row-proportionality, while in biproportional apportionment both row- and column-proportionality should be pursued.

Motivated by the study of the Italian case, in the following section, we present a two-phase heuristic procedure for the doubly-constrained BAP based on the proportional HQ method plus the LR rule that is able to output apportionments of good quality with respect to the fractional Fair-Share benchmark. This conclusion is supported by the empirical results presented in Sect. 5.

4 Alternating vector apportionment algorithms

The main problem in the Italian electoral law is that it fails in controlling row- and column-sums, so that constraints (2) are not always satisfied by the final seat allocation. Actually, after the computation of the natural quotas, the Largest Remainders rule is applied in each constituency, satisfying all row-sums (2), but not necessarily (3), which, in fact, are usually violated for some parties. Therefore, the law prescribes a series of seat transfers from one party to another with the aim of balancing column-sums. The reason why it does not succeed in its objective lies in the fact that the transfers are implemented by looking at local adjustments and not in a doubly-constrained approach involving the whole matrix. In addition, while trying to obtain feasibility, this process unavoidably loses the previously obtained (row) proportionality. This fact has frequently implied that a decree of the President of the Republic had to be emanated to approve the (infeasible) institutional Parliament seat allocation. To fix this bug, in this section, we present a procedure in two phases that first pursues double-proportionality by alternating row- and column- vector apportionment; in the second phase, a series of *controlled* transfers is performed to re-balance deficit and surplus parties and obtain the final allocation.

4.1 Alternating vector apportionment

Consider the $m \times n$ matrix V, the national seats of the parties, c_j, $j = 1, \ldots, n$, and the seats at stake in each constituency r_i, $i = 1, \ldots, m$. The Alternating Vector

Apportionment (AVA) algorithm performs vector apportionments by applying the HQ method iteratively to rows and columns. The input of this first step is the vote matrix V. In each constituency i, the *row* exact share of each party j is computed by the usual formula that scales the number of votes of j by the constituency divisor $\gamma_i = \frac{\sum_{\ell \in N} v_{i\ell}}{r_i}$:

$$q^r_{ij}(1) = \frac{v_{ij}}{\gamma_i}, \quad i = 1, \ldots, m. \tag{3}$$

This produces a fractional matrix of row quotas at iteration 1, that we denote by $Q^r(1)$. Note that $q^r_{ij}(1) = q_{ij}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. A number of seats equal to $\lfloor q^r_{ij}(1) \rfloor$ is assigned to party j in constituency i. The LR rule is then applied to assign to parties the row residual seats. The output of the first iteration is an integer matrix $A^r(1)$ that satisfies constraints (1), (2) (row-sums) and (4), but generally it does not satisfy constraints (3) (column-sums). Matrix $A^r(1)$ is the first row apportionment of the AVA procedure and we observe that it corresponds to what is currently prescribed by the Italian law for the Chamber of Deputies. In a second step (of the same iteration), the algorithm applies the HQ method to generate the matrix of column fractional shares. It is obtained starting from matrix $Q^r(1)$ which, in this column step, is considered as the input matrix. For each party j, the *column* exact shares are computed by scaling row quotas $q^r_{ij}(1)$ by the related divisor $\gamma_j = \frac{\sum_{\ell \in M} q^r_{\ell j}(1)}{c_j}$:

$$q^c_{ij}(1) = \frac{q^r_{ij}(1)}{\gamma_j}, \quad j = 1, \ldots, n \tag{4}$$

producing the fractional matrix of column quotas at iteration 1, that we denote by $Q^c(1)$.

The first AVA iteration is completed with this second step in which no integer apportionment is computed, since only the fractional matrix Q^c is necessary to go to the next iteration. The procedure is repeated for successive iterations, each performing a row and a column step. In a generic iteration, in *odd steps*, the HQ method is applied to the rows to obtain the fractional matrix Q^r and then, by the LR rule, the corresponding integer matrix A. In *even steps*, only HQ is applied to the columns of Q^r, obtaining the fractional matrix Q^c. The procedure ends when, in two successive iterations, one gets the same Q^r. In our implementation, it stops when the maximum difference in a cell of the two matrices is smaller than 10^{-5}. Our AVA algorithm always returns a final integer allocation matrix A after an odd (row) step, so that, at the end of phase-1 the allocation matrix satisfies conditions (1), (2), (4), but not necessarily (3). This last set of constraints will be satisfied after the second phase of the algorithm (see, Sect. 4.2). In the following, we provide the pseudo-code that describes the formal steps of the AVA algorithm.

The sequence of alternating row and column vector scaling steps corresponds to a well-known procedure, called Iterative Proportional Fitting (IPF),[6] which is applied for solving the fractional version of BAP[7] and consists in finding a scaling of a given input matrix of non negative numbers such that the output matrix achieves row- and

[6] also known as the RAS method.

[7] known in the literature as *Biproportional fitting*.

Algorithm 1

procedure AVA ALGORITHM
  Let V be the $m \times n$ vote matrix;
  Let $r_i, i = 1, \ldots, m$ and $c_j, j = 1, \ldots, n$ be the row and column seat totals
  Set $Q := V$
  Set $A := 0$
  Set $Q_{it-prec} := V$
  Set $Q_{it} = \mathbf{0}$
  while $Q_{it-prec} \neq Q_{it}$ **do**
    Set $Q_{it-prec} = Q_{it}$
    Odd row step (HQ+LR)
    for $i = 1, 2, \ldots, m$ **do**
      Let q_i^r be the vector of natural quotas of row i of Q
      Apply the HQ plus the LR method to row i of Q
      Let h_i be the resulting (row) apportionment vector
    Let Q^r be the $m \times n$ fractional matrix formed by the vectors $q_i^r, i = 1, 2, \ldots, m$
    Update A to be the $m \times n$ integer matrix formed by the m row-vectors $h_i, i = 1, 2, \ldots, m$
    Set $Q = Q^r$, $Q_{it} = Q^r$

    Even column step (HQ)
    for $j = 1, 2, \ldots, n$ **do**
      Let q_j^c be the natural quotas vector of column j of Q
    Let Q^c be the $m \times n$ fractional matrix formed by the vectors $q_j^c, j = 1, 2, \ldots, n$
    Set $Q = Q^c$
  Return matrix A satisfying constraints (1), (2) and (4) (phase-1 output matrix).

column- sums equal to pre-specified row- and column-marginals. When the entries of the final output matrix are required to be integer we have the Biproportional Apportionment Problem. The continuous and discrete versions of the problem were studied by Pukelsheim and Simeone (2009); Pukelsheim (2014) for the electoral problem of Parliament seat allocation in countries which adopt a doubly-proportional electoral system. The integer problem can be solved using the Alternating Scaling (AS) algorithm, which is proved to converge in almost all cases, but in few special ones (see, Gaffke and Pukelsheim (2008a, b); Pukelsheim (2014)). In fact, AS is the discrete version of the IPF algorithm; it iteratively solves vector apportionment problems by a proportional divisor method applied to the rows and the columns of the given matrix. It alternates scaling of rows (so that row marginals are met) followed by scaling of columns (column marginals are met). The procedure terminates as soon as row and column marginals are met simultaneously, implying that the current matrix is the doubly-proportional seat apportionment sought (see, Balinski and Pukelsheim (2006); Maier (2006); Maier et al. (2010); Pukelsheim (2017)). Algorithm AVA works similar to AS, but uses Hare Quota plus Largest Remainders in place of a divisor method. AVA scales the fractional matrix Q in the successive row (Q^r) and column (Q^c) steps, in fact, as the RAS procedure. It performs the rounding only in the odd step of each iteration so that, when the stopping condition is met, it is guaranteed that the final allocation satisfies row totals. In view of the stopping rule implemented in AVA, and after the asymptotic convergence of RAS Bacharach (1965, 1970); Kalantari et al. (2008); Speed (2005); Zachariasen (2005), in our empirical tests AVA always stops after a finite number of steps (ranging between 6 and 12).

	Party 1 (surplus)	Party 2 (deficit)	Party 3 (balanced)		Party j (balanced)		Party n (balanced)	Constituency Total
Constituency 1	−1		+1					=
Constituency 2								
Constituency i			−1		+1			=
Constituency m		+1			−1			=
Party total	+1	−1	=		=		=	

Fig. 1 Example of a controlled transfer of 1 seat from a surplus party P_1 (cedes a seat in Constituency 1) to a deficit party P_2 (receives a seat in Constituency m). In the transfer from party P_1 to P_2, some balanced parties are involved

The AVA procedure represents the first phase of our algorithm to solve the Biproportional Apportionment Problem. Since AVA outputs a matrix that does not necessarily satisfy column-sum constraints, a further phase is added to find an apportionment.

4.2 Seat-transfer for row- and column-feasible allocations

The second phase of our algorithm is a corrective procedure of the phase-1 output matrix: it performs a sequence of seat transfers to satisfy constraints (3) without loosing feasibility with respect to the other constraints. Seat transfers of this type are called *controlled*, since each of them reduces by 1 the imbalance between the column sums of the current matrix and the prefixed totals, without affecting the feasible row sums.

To understand how a controlled seat transfer works, we visualize in Fig. 1 an example of the "path" that a seat may follow through the cells of matrix A in order to transfer a seat from a surplus party to one in deficit. Parties having a total number of seats at the national level equal to the sum of the seats received in all constituencies are called *balanced* and in Fig. 1 they are denoted by a '=' in the last cell of their column. Similarly, a '=' in the last cell of a row i means that the corresponding row-sum constraint in constituency i is satisfied. On the other hand, a '$+1$' and a '-1' in the last cell of a column denote that the corresponding party is in surplus or in deficit of one seat, respectively. Differently, a '-1' in an entry (i, j) of the table means that party j is ceding one seat in constituency i; a '$+1$' in an entry (i, j) means that party j is receiving one seat in constituency i.

The structure of the path in Fig. 1 is the most general one and shows that it could be necessary to involve many cells of the matrix to transfer the seat, but, clearly, paths may be shorter. For example, Figs. 2 shows simpler cases in which only two constituencies are involved in the transfer (Fig. 2a) or even only one (Fig. 2b). Under the idea of changing the phase-2 input allocation matrix as little as possible, transfers on short paths are preferred, because they involve a small number of parties and constituencies. In fact, this is one of the two optimization strategies that we implement in our algorithm (see Sect. 4.3).

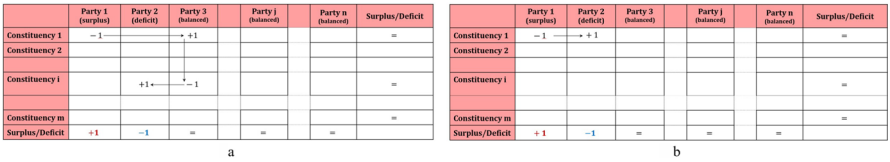


Fig. 2 Example of controlled transfers of 1 seat from a surplus party P_1 to a deficit party P_2. In 2a, party P_3 (balanced) is involved in the transfer of the seat from party P_1 to party P_2 (the national number of seats of P_3 does not change, but its seat allocation is modified in 2 constituencies); in 2b the simplest possible transfer is shown, performed between parties P_1 and P_2 within the same constituency, without involving any other party

In the following, we present a formal description of phase-2 in which controlled seat transfers are implemented by means of paths on a suitable graph that we call *transfer network*. We first illustrate how this network is constructed and then provide a pseudocode that describes the steps of the phase-2 procedure.

Consider the vote matrix V and the integer allocation matrix A returned by the AVA algorithm at the end of phase-1. Allocation A guarantees (1), (2) and (4), but, since not all column-sums (3) are satisfied, there are at least two unbalanced columns in A. In this section, we will assume that all parties receive votes in all constituencies, that is, $v_{ij} > 0 \ \forall ij$. This avoids zero-cells in V, that is, $v_{ij} = 0$ for some constituency i and party j.
We introduce a set of coefficients $u_{ij}, i \in M, j \in N$ to classify the cells of matrix A according to the different combination of values of a_{ij} and v_{ij}. Indeed, we define a cell (i, j) *active*, *passive*, or *neutral* as follows:

- *Active* if $a_{ij} > 0$
- *Passive* if $a_{ij} = 0$ and $v_{ij} > 0$
- *Neutral* if $v_{ij} = 0$.

In active and passive cells (i, j) we have $v_{ij} > 0$, meaning that party j can receive a seat in constituency i; on the other hand, since in a passive cells (i, j) we have $a_{ij} = 0$, party j has no seats to transfer in i; on the contrary, it can cede seats in active cells. Coefficients u_{ij} are then defined as follows:

$$u_{ij} = \begin{cases} 2 & \text{if cell}(i, j)\text{is active} \\ 1 & \text{if cell}(i, j)\text{is passive} \\ 0 & \text{if cell}(i, j)\text{is neutral} \end{cases} \qquad i = 1, \ldots, m \quad j = 1, \ldots, n. \tag{5}$$

The second set of coefficients refers to the columns of A and distinguishes balanced columns (those for which the column-sums are satisfied) and unbalanced columns (for which we have either $a_{.j} > c_j$ or $a_{.j} < c_j$. The coefficients are denoted by δ_j, $j = 1, \ldots, n$, and they are defined as follows:

$$\delta_j = \begin{cases} 1 & \text{if } a_{.j} > c_j \\ 0 & \text{if } a_{.j} = c_j \\ -1 & \text{if} a_{.j} < c_j \end{cases} \qquad j = 1, \ldots, n. \tag{6}$$

Using the above coefficients for a given allocation matrix A, we can compute the number of unbalanced parties in A (surplus or deficit) by $\Delta = \sum_{j=1,\ldots,n} |\delta_j|$. We also have:

- Number of active cells: $\alpha = |\{(i, j) : u_{ij} = 2\}|$
- Number of passive cells: $\pi = |\{(i, j) : u_{ij} = 1\}|$
- Number of neutral cells: $\nu = |\{(i, j) : u_{ij} = 0\}|$

and $\alpha + \pi + \nu = mn$ holds.
We construct the transfer network G, whose nodes correspond to constituencies and parties, such that a controlled transfer from a surplus party P_h to a deficit party P_k can be represented by a directed path from node h to node k. For a given matrix A with m rows and n columns, the network is, in fact, a flow network with $m + n + 2$ nodes, the last 2 being a source s and a sink t, that we include as fictitious nodes.
The arcs of G are defined as follows:

- (s, j), for each party P_j such that $\delta_j = 1$;
- (j, t), for each party P_j such that $\delta_j = -1$;
- (i, j) and (j, i), for a pair C_i and P_j such that $u_{ij} = 2$;
- (i, j), for a pair C_i and P_j such that $u_{ij} = 1$.

In total, in G we have $\Delta + 2\alpha + \pi$ arcs. Figure 3 shows the possible structure of a network G corresponding to a generic phase-1 output matrix A. The source node s is connected only to surplus parties (green nodes), while the sink node t is connected only to deficit parties (red nodes).
When a cell (i, j) of A is active, we have $a_{ij} > 0$ and this means two things: i) party j has received some seats in constituency i and, therefore, it can potentially cede one of them; ii) party j has some votes in constituency i, and this motivates the possibility of receiving an additional seat. This implies that both arcs (i, j) and (j, i) must be included in G: the first, directed from constituency-node i to party-node j, corresponds to party P_j receiving a seat in that constituency; the second, directed from party-node j to constituency-node i means that party P_j can cede one of its seats in constituency C_i (in Fig. 3 this case is represented by a red double arrow).
When a cell (i, j) of A is passive, we have $a_{ij} = 0$, but $v_{ij} > 0$ and, therefore, party j can only receive seats in constituency i; for this reason, only one arc, directed from i to j can be inserted in G (in Fig. 3 this is represented by a blue arrow).

Remark 1 Given a phase-1 output matrix A, for a surplus party j both arcs (i, j) and (j, i) exist in the transfer network for at least one constituency i, since j has necessarily received some seats at least in one constituency. The same holds for a deficit party that has received at least one seat nationwide.

Remark 2 Given matrix A, a neutral cell (i, j) corresponds to a party j that has no votes in constituency i, and, therefore, it has no seats, too. This means that party j can never be involved in any seat transfer in constituency i, and implies that no link between i and j exits in G.

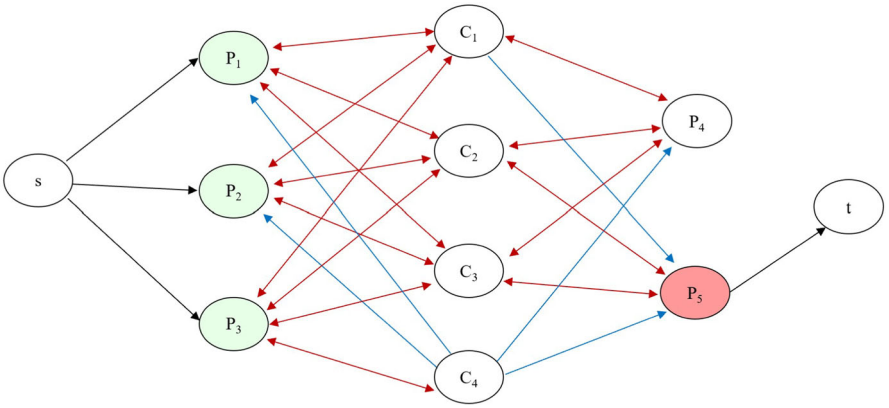


Fig. 3 Example of a transfer network G with 5 parties and 4 constituencies: a pair of red arcs exists between a constituency-node C_i and a party-node P_j if the cell (i, j) of A is active; only one blue arc exists from a constituency C_i to a party-node P_j if the cell (i, j) of A is passive

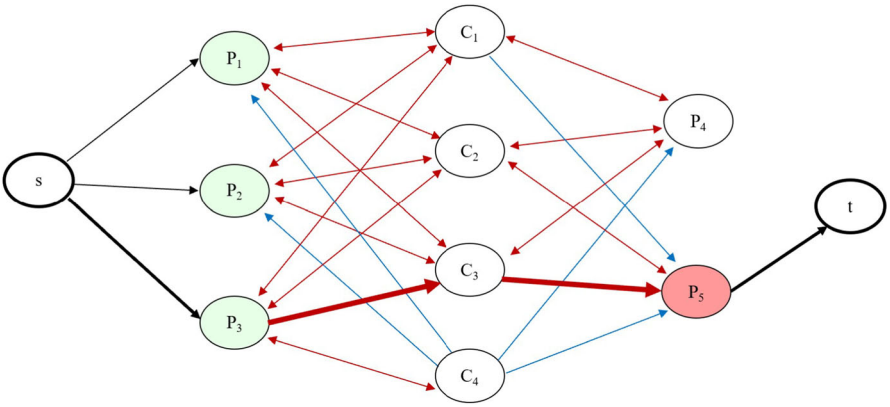


Fig. 4 Example of a seat transfer from the surplus party P_3 to the deficit party P_5 in which only one constituency is involved (C_3)

Any possible transfer of a seat from a surplus party to a deficit one can be traced by following a path from the source node s to the sink t. For example, Fig. 4 shows (in bold) one path in which a seat of the surplus party P_3 is transferred to the deficit party P_5. The path can be interpreted as follows. The first arc of the path, (s, P_3), means that the transfer starts from the surplus party P_3 which, through arcs (P_3, C_3) and (C_3, P_5), cedes one seat in C_3 to the deficit party P_5, so that P_3 and P_5 reduce by 1 their respective surplus and deficit.

Fig. 5 shows a transfer of one seat from P_3 to P_5, but performed along a path in G with a larger number of arcs with respect to the one in Fig. 4. In this case, also a balanced party P_4 is involved. As before, the first arc of the path, (s, P_3), means that the transfer starts from the surplus party P_3 and, through arcs (P_3, C_4) and (C_4, P_4) we know that P_3 cedes a seat to P_4 in C_4. Then arcs (P_4, C_2) and (C_2, P_5) trace the fact that P_4

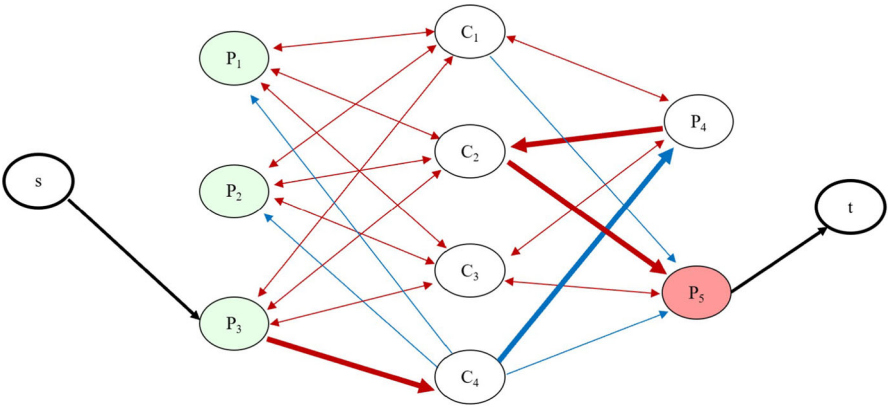


Fig. 5 Example of a seat transfer from the surplus party P_3 to the deficit party P_5 in which the balanced party P_4 is involved, since it receives a seat in constituency C_4 from party P_3 but then cedes another seat in constituency C_2 to the deficit party P_5

cedes one of its seats in C_2 to the deficit party P_5 which is definitively rewarded by one additional seat in this constituency, thus decreasing its deficit by 1.
Even if the path in Fig. 5 is longer than the one in Fig. 4, and involves more parties and constituencies, it should be clear that, both correspond to a single-seat transfer. In the end, only the number of seats of P_3 and P_5 are affected by this transfer and, as before, the number of seats of the surplus party P_3 decreases by 1, while the number of seats of the deficit party P_5 increases by 1. The numebr of seats of the balanced party P_4 does not change. Clearly, in 5 the transfer involves more constituencies and parties than in Fig. 4, implying that the distribution of seats of the balanced party P_4 is modified by the transfer, in C_4, where P_4 obtains one more seat, and in C_2, where, to equalize its national seats, it must cede one.

We point out that this kind of effect could be criticized by the local candidate of P_4 who (personally) looses the seat in C_2 in favor of another candidate (of his/her party) who receives a (unexpected) seat in C_4. If, on the one hand, this is reasonable under a political viewpoint, transfers like the one in Fig. 5 are technically possible and can be performed, if this is useful to pursue feasibility and/or biproportionality.

Therefore, after each transfer, the distribution of seats to parties within a constituency may change, but the number of seats allocated in each constituency never changes. This guarantees that, after a transfer, constraints (2) are preserved, while the balance in national political representation of parties always improves by 1.
Let S and D denote the sets of labels of surplus and deficit parties, respectively, and $d_j = a_{\cdot j} - c_j$ be the excess of seats of a surplus party $j \in S$. The seat-transfer procedure of our phase-2 performs $\sum_{j \in S} d_j$ one-seat transfers: at each iteration, a path $\mathcal{P}$ from s to t is selected in G and the seat transfer modifies the values of the elements of the matrix A in the cells corresponding to arcs in $\mathcal{P}$. After each seat transfer, also G must be updated by removing an arc (j, i) in A if cell (i, j) is active and becomes passive. In addition, arc (s, h) has to be removed as soon as the surplus party h becomes balanced; the same holds for arc (k, t) as soon as the deficit party k

becomes balanced. Note that, after a transfer, it may also happen that a passive cell (i, j) of A becomes active, namely, when party j, with $v_{ij} > 0$, that previously had no seat in constituency i receives there one transferred seat; in this case an arc (j, i) must be added to G after the transfer.
The pseudo-code of the phase-2 procedure that generates the sequence of controlled transfers is reported in the following.

Algorithm 2

```
1: procedure CONTROLLED SEAT TRANSFERS
2:     Let A be an integer matrix satisfying constraints (1),(2), but not (3)
3:     Construct the transfer network G associated to A
4:     repeat
5:         select in G a path P from s to t
6:         perform a seat transfer along P
7:         update a_ij for all cells (i, j) corresponding to arcs in P
8:         update G
9:     until no paths P from s to t exist in G (or, no surplus parties exist)
10:    Return the updated matrix A* satisfying constraints (1),(2),(3).
```

Proposition 1 *If all parties receive votes in all constituencies, that is, $v_{ij} > 0\ \forall ij$, for any given integer matrix A satisfying constraints (1) and (2), but not constraint (3), in the corresponding transfer network G there exists at least a path from a surplus party P_h to a deficit party P_k such that one seat can be transferred from node P_h to node P_k.*

Proof Since A does not satisfy constraint (3), there exists at least one surplus party P_h and at least one active cell in column h of A, say in row i, corresponding to a red double arrow in G, implying that party P_h can cede a seat in constituency i. Under the general house size assumption, $\sum_{i \in M} r_i = \sum_{j \in N} c_j$, a deficit party P_k must exist, too. In addition, since $v_{ij} > 0\ \forall ij$, for the deficit party P_k all (blue) arcs (i, k) exist in G, meaning that party P_k can receive seats from any constituency, $C_i, i \in M$. This implies that a one-seat transfer can be done from P_h to P_k in constituency i. It follows that in G there exists a path from the source node s, passing through the surplus party P_h and the deficit party P_k and ending in the sink node t, that is, the path given by the sequence of nodes $s - P_h - C_i - P_k - t$. □ □

Proposition 2 *Under the assumption that all parties receive votes in all constituencies, that is, $v_{ij} > 0\ \forall ij$, given an integer matrix A satisfying constraints (1),(2), but not (3), the procedure CONTROLLED SEAT TRANSFERS outputs an integer matrix A^* satisfying constraints (1),(2) and (3) in $\sum_{j \in S} d_j$ iterations.*

Proof Given an integer matrix A satisfying constraints (1),(2), but not (3), at each iteration of the algorithm, we have at least a pair given by a surplus party P_h and a deficit party P_k (otherwise the algorithm stops) and, by Proposition 2, a path from s to t exists where one seat of the surplus party P_h can be transferred to the deficit party P_k. This reduces by 1 both the surplus of P_h and the deficit of P_k. In the updated

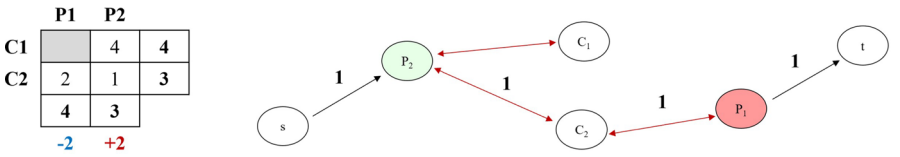


Fig. 6 A infeasible example with zero-cells in matrix V. First iteration of phase-2: one directed path from s to t exists for performing a one-seat transfer from P_2 to P_1

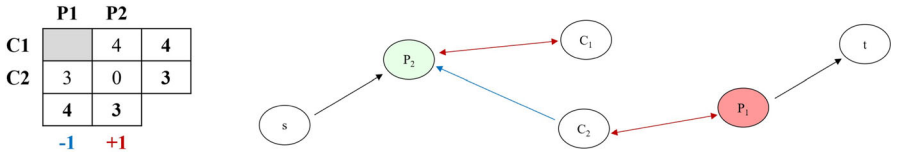


Fig. 7 A infeasible example with zero-cells in matrix V. Second iteration of phase-2: no directed paths exist from s to t

graph G, arc (s, P_h) disappears as soon as a surplus party P_h has ceded all its surplus seats, and the same happens to arc (P_k, t) as soon as the deficit party P_k has received all its deficit seats. Therefore, after $\sum_{j \in S} d_j$ iterations, the algorithm stops, since all nodes in G corresponding to surplus parties are disconnected from s and all nodes corresponding to deficit parties are disconnected from t. Since no surplus and deficit nodes remain, s and t are disconnected from the rest of the graph and no paths from s to t exist any more. □ □

The above result is grounded on the existence of a path from s to t given by Proposition 1, which is based on the assumption that the vote matrix V has no zero-cells. To understand the importance of this assumption, in the following we show a small example with two constituencies and two parties, and where $v_{11} = 0$.

Example 1 Consider the simple case of 2 parties and 2 constituencies in Fig. 6 where the vote matrix V has a structural zero-cell $v_{11} = 0$. Assume that the output matrix A of phase-1 and the prefixed row and column totals are those given in Fig. 6, with the corresponding transfer network G. A satisfies constraints (1) and (2), but not (3) since P_1 has a deficit of 2 seats, that, in fact, corresponds to the surplus of party P_2.
In the network, there exists a path from s to t, and a seat can be transferred to reduce by 1 the imbalance. After the transfer, the allocation matrix and the related network are updated, obtaining those shown in Fig. 7. Here, potentially P_2 can cede a seat in C_1, but this is useless since arc (1, 1) does not exist in G, due to the fact that $v_{11} = 0$, and this implies that P_1 cannot receive seats in C_1, while P_2 has already ceded to P_1 all its seats in C_2.

The above example illustrates an infeasible instance of BAP in which an integer matrix satisfying row and column totals cannot be found, due to the presence of the zero-cell v_{11}. In fact, in this case, there is a set of constituencies (C_1) such that the total number of seats in those is larger than the number of seats allocated on a national level to all parties which have any votes in those constituencies (the number of seats allocated to P_2 in C_1 is $4 > 3$, 3 being the national seats entitled to P_2), and no biproportional

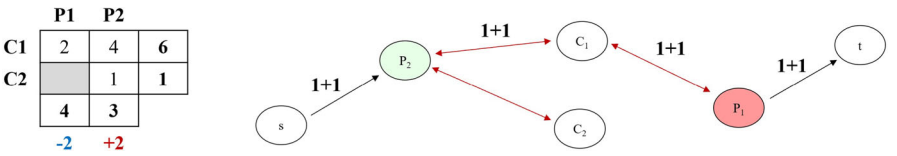


Fig. 8 *A* feasible example with zero-cells in matrix V: 2 seats can be transferred from P_2 to P_1 on the given directed path

apportionment can be found since it is impossible to transfer seats from P_2 to P_1 in constituency C_1.

It is worth mentioning that also feasible instances exist with structural zero-cells; an example is given in Fig. 8 where the zero-cell is located in row 2, column 1. Clearly, this case is feasible, since party P_2 can cede 2 seats in C_1 and provide a BAP feasible solution.

We conclude that the presence of zero-cells in the vote matrix is not a condition implying infeasibility for BAP, but the problem instance can be infeasible depending on how many zero-cells are present in V, and where they are located. On the contrary, when there are no structural zero-cells, Proposition 2 guarantees that procedure CONTROLLED SEAT TRANSFERS is always able to solve the doubly-constrained BAP.

We point out that, in real elections, the vote matrix rarely contains structural zeros. Indeed, such cells occur only when a party runs in a constituency, but receives no votes there (a situation that is practically impossible), or if a party deliberately chooses not to field candidates in certain constituencies, as was the case for the Lega Nord party in Italy during the 2000 s.

The procedure CONTROLLED SEAT TRANSFERS presented in this section does not specify any selection rule for the next path $\mathcal{P}$ from s to t. In the following sections, we suggest two different strategies to implement for this choice.

4.3 Minimum cardinality and shortest paths for biproportional allocations

In our approach, the phase-1 output matrix is considered as the benchmark for BAP, since it differs from the fractional Fair-Share matrix F of the last AVA row step only because the Largest Remainders rule is applied (row-wise) to obtain an integer allocation. The objective of phase-2 is then to modify A as least as possible, in order to satisfy constraints (3), but maintaining feasibility w.r.t. constraints (1) and (2). In this common view, two different principles are pursued to select a path in G, each embodying a different idea of *least possible modification* of A.

Our first path selection strategy is guided by the principle of minimizing as much as possible the number of parties and constituencies which are involved in the corresponding seat transfer. In fact, as shown before in Fig. 4 and Fig. 5, a transfer of one seat can be performed on different paths of G; since all values a_{ij} corresponding to an arc (i, j) along the path are modified by this transfer, the idea is that modify fewer is better than modify more. According to this, at each iteration, the algorithm selects the path with the smallest number of edges. We call this version of the algorithm Alternating Vector Apportionment with *Minimum Cardinality Paths* (AVA-MCP).

A different path selection strategy is motivated by the idea of performing the transfer that minimizes the deviation of the new obtained A from the fractional Fair-Share matrix F, measured by a suitable matrix distance. In this case, we introduce non negative weights associated to the arcs of the network, and, at each iteration, phase-2 selects in G the path from s to t with minimum weight[8] (*shortest path*). We call this second version of the algorithm Alternating Vector Apportionment with *Shortest Paths* (AVA-SP).

Consider the Fair-Share F and the phase-1 output matrix A. We measure the deviation of A from F as the sum of the absolute deviations computed cell by cell. The formula is the following:

$$D(A, F) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left| a_{ij} - f_{ij} \right|. \tag{7}$$

For any given pair (i, j) we consider the corresponding term in the above sum $\left| a_{ij} - f_{ij} \right|$ and define the weight of the corresponding arc (i, j) in G as follows:

$$w_{ij} = \left| (a_{ij} + 1) - f_{ij} \right| - \left| a_{ij} - f_{ij} \right| \qquad \text{for directed arcs } (i, j) \tag{8}$$

$$w_{ij} = \left| (a_{ij} - 1) - f_{ij} \right| - \left| a_{ij} - f_{ij} \right| \qquad \text{for directed arcs } (j, i). \tag{9}$$

The above weights measure the change in the objective function (7) when one seat is transferred along an arc. More precisely, formula (8) applies when party j is (locally) receiving one seat in constituency i, while formula (9) is used when party j is (locally) ceding one seat in i. It follows that, after a seat transfer along a path from a surplus party P_h to a deficit party P_k, the value of the above deviation measure (7) can be updated by adding the length of the corresponding path that is a non negative value. As our experimental results show (see next Sect. 5), using the above weights in phase-2 to select shortest paths moves the focus from A to F, and guides the AVA-SP search towards a final apportionment that is close to it.

The above two path selection rules implement operations that are only locally best. Since, clearly, the greedy approach does not solve the general problem of finding the matrix that minimizes the distance from F, our algorithms follow a heuristic approach. In spite of this, in the paper section dedicated to the experimental results we show that the performance of our algorithms is very good by computing for the solutions a set of quality measures given in Maier et al. (2010).

We point out that, in phase-2, both path selection rules are applied to an integer matrix A and a transfer network G that changes at each next iteration. In fact, after each transfer, the graph may loose some arcs, and, in AVA-SP, also the weights of the arcs along the path may change.

Proposition 3 *Assume that $v_{ij} > 0$, for all i and j. Let A be an integer matrix satisfying constraints (1),(2), but not (3), and G the corresponding transfer network, with weights w_{ij} associated to each arc (i, j). If $O(\mathcal{C}_{\mathcal{SP}})$ is the time needed to find a shortest path*

[8] as usual, the weight of a path is the sum of the arc weights of its edges.

from s to t in G, the procedure CONTROLLED SEAT TRANSFERS is polynomial, and requires $O(\mathcal{C}_{\mathcal{SP}} \sum_{j \in S} d_j)$ time to find the final seat allocation.

The above result implies that both AVA-SP and AVA-MCP (the latter solving a shortest path problem with arc weights all equal to 1) algorithms solve the BAP in polynomial time.

5 Application and results

In this section we present the results of the application of our AVA algorithms to the Italian elections of the Chamber of Deputies and of the European Parliament. As in Maier et al. (2010), we also consider some other European countries adopting biproportional apportionment. Data are both from recent and past elections and provide a variety of instances with different numbers of parties and house/district sizes. We evaluate the performance of our algorithms by the L_∞, L_1, and L_2 distances computed between any apportionment A and the corresponding fractional Fair-Share matrix F. We also consider the quality measure presented in Maier et al. (2010) by computing the number of *quota breaches*, i.e., the number of elements in A for which $|a_{ij} - f_{ij}| > 1$. For each dataset, we compare the apportionment returned by our AVA-MCP and AVA-SP methods with the institutional seat allocation obtained by the procedure prescribed in the law. Our empirical results emphasize the different characteristics of our algorithms, certifying that AVA-SP performs better that AVA-MCP in terms of biproportionality.

5.1 Italian elections for the National and the European parliaments

We consider the data from the most recent Italian elections of the Chamber of Deputies (2018 and 2022) and from the Italian elections for the representatives in the European Parliament in 2009 and 2018,[9] in which the vote is expressed in five macro-constituencies.

It is worth mentioning that the latest Italian electoral reform of 2017[10] introduced only minor changes to the seat allocation procedure for the Chamber of Deputies in order to address the problem of balancing surplus and deficit parties. In addition, since 2018 these changes have been incorporated also in the Italian electoral law governing elections to the European Parliament. In spite of this, it is important to clarify that, although no shifting seats have occurred in the subsequent elections, the amendments only partially address the issue and do not fully resolve the underlying technical problem. In fact, the shifting seat phenomenon can still arise under different vote scenarios producing infeasible allocations. In addition, the law re-balancing procedure for surplus and deficit parties does not follow any proportionality criteria. The aim of our experiments is therefore to show that our proposed algorithms are able to solve correctly the doubly-constrained BAP and, in particular, that by AVA-SP one can obtain

[9] Data have been downloaded by the institutional Italian electoral data repository Eligendo (https://elezioni.interno.gov.it/) and then elaborated in the matrix form.

[10] Law 165/2017.

Table 1 Italian elections of the Chamber of Deputies 2018

Quality evaluation w.r.t matrix F—CD 2018				
Method	L_∞	L_1	L_2	n. breaches
Institutional	0.95985	30.20	3.4414	0
AVA-SP	**0.72698**	**29.7479**	**3.375**	0
AVA-MCP	5.3286	38.9905	7.3062	3

Table 2 Italian elections of the Chamber of Deputies 2022

Quality evaluation w.r.t matrix F—CD 2022				
Method	L_∞	L_1	L_2	n. breaches
Institutional	0.82062	31.8026	3.6634	0
AVA-SP	**0.77191**	**29.7367**	**3.3697**	0
AVA-MCP	1.2413	31.7506	3.7217	1

an apportionment with a high degree of proportionality in two dimensions measured with respect to a chosen quota matrix benchmark.

Tables 1 and 2 present the results of the Italian elections for the Chamber of Deputies held in 2018 and 2022 provided by algorithms AVA-SP and AVA-MCP, compared to the seat allocation obtained by applying the law procedure (Institutional). The columns report the values of the L_∞, L_1 and L_2 for the distances between each allocation matrix and the fractional Fair-Share; the smallest distance in each column is emphasized in bold. The last column shows the number of breaches of each matrix.

In both tables, the best allocation is the one given by the AVA-SP algorithm, always with 0 breaches. As expected, AVA-MCP outputs allocations with a high distance from the Fair-Share matrix, and this is clearly due to the fact that this method does not control proportionality loss when selecting paths for seat transfers. Sometimes AVA-MCP apportionment distance from the Fair-Share is worse than the institutional allocation, but, referring to Italy, this solution is always feasible, while the same is not true for the institutional one.[11]

For AVA-MCP the number of breaches is different from zero. This confirms that, by pursuing a strategy based on transfers along minimum-cardinality paths, the algorithm tends to transfer seats within the same constituency. In doing so, it disregards the resulting effect of assigning a number of seats that may deviate substantially from the corresponding quota. Note that, for the Chamber of Deputies, the number of breaches in the institutional allocation is equal to 0. However, as we will see in the analysis of the 2024 European Parliament elections, this is not always guaranteed, due to the (uncontrolled) seat tranfer procedure adopted by the law.

Tables 3 and 4 present the results for the elections of the Italian representatives in the European Parliament 2009 and 2024. The law for these elections (Law 18/1979) is similar to the one for the National Parliament and allocates seats among parties at

[11] In this view, despite its inferiority in terms of double proportionality, AVA-MCP could be still a valid proposal for politicians who can easily understand the minimum cardinality strategy and, as discussed before, do not like the idea that their party could loose a seat in a constituency (for example in the south of Italy) that is compensated by one received in another constituency (north of Italy).

Table 3 Italian elections for the European Parliament 2009

Quality evaluation w.r.t matrix F—EU Parliament 2009				
Method	L_∞	L_1	L_2	n. breaches
Institutional	1.6377	11.7202	3.1985	4
AVA-SP	**0.95766**	**6.6128**	**1.6617**	0
AVA-MCP	1.23073	7.9637	2.1385	1

Table 4 Italian elections for the European Parliament 2024

Quality evaluation w.r.t matrix F—EU Parliament 2024				
Method	L_∞	L_1	L_2	n. breaches
Institutional	0.92139	9.8231	2.1124	0
AVA-SP	**0.68376**	**8.267**	**1.7048**	0
AVA-MCP	0.81644	9.15203	1.9471	0

Table 5 The institutional allocation in the Italian elections for the European Parliament 2009

EU Institutional allocation 2009							
Constituencies	Seats at stake	PDL	LN	PD	IdV	UDC	Seats assigned
Italia nord-occidentale	19	8	5	5	2	1	21
Italia nord-orientale	13	5	3	5	1	1	15
Italia centrale	14	6	1	6	1	1	15
Italia meridionale	18	8	0	4	2	1	15
Italia insulare	8	2	0	2	1	1	6
National seats	–	29	9	22	7	5	–

the national level first, but then requires to distribute them to parties within territorial constituencies with the HQ plus LR method. There are 5 macro-constituencies, *Italia nord-occidentale*, *Italia nord-orientale*, *Italia centrale*, *Italia meridionale*, *Italia insulare*, and each has a pre-specified number of seats at stake.

As before, the empirical results confirm that AVA-SP apportionments are the best, while those provided by AVA-MCP are structurally less proportional. In spite of this, here AVA-MCP outperforms the institutional allocation. The bad quality of the 2009 institutional allocation derives from its particular configuration, in which row constraints (2) have been violated in *all* constituencies as shown in Table 5.[12] This also accounts for the high number of breaches that is equal to 4 (see Table 3).

We exploit data of the Italian elections for the European Parliament in 2009 to describe our AVA-SP algorithm in detail, providing all matrices and illustrating all steps of the two-phase algorithm. Table 6 reports the vote matrix, while Table 7 shows the fractional Fair-Share (computed in the last step of the phase-1 procedure).

[12] Data are taken from "Ufficio elettorale nazionale presso la Corte suprema di Cassazione—Verbale delle operazioni dell'Ufficio elettorale nazionale (2009)"

Table 6 Italian elections for the European parliament 2009: vote matrix

EU vote matrix 2009 Constituencies	PDL	LN	PD	IdV	UDC
Italia nord-occidentale	2935126	1684842	2002790	663495	460487
Italia nord-orientale	1780429	1205587	1919116	456182	354131
Italia centrale	2344306	186988	2030062	483471	341612
Italia meridionale	2869765	39521	1575928	688368	582421
Italia insulare	900583	9243	615089	186326	256370

Table 7 Italian elections for the European parliament 2009: fair-share matrix

EU fair-share matrix 2009 Constituencies	Seats at stake	PDL	LN	PD	IdV	UDC
Italia nord-occidentale	19	6.7346	4.9249	4.7693	1.6135	0.9577
Italia nord-orientale	13	3.7866	3.2665	4.2360	1.0283	0.6826
Italia centrale	14	5.9550	0.6051	5.3518	1.3016	0.7865
Italia meridionale	18	8.8861	0.1559	5.0644	2.2590	1.6346
Italia insulare	8	3.6377	0.0476	2.5785	0.7977	0.9386
National seats	–	29	9	22	7	5

Table 8 European elections 2009: phase-1 allocation with surplus and deficit parties

EU Allocation 2009 at the end of phase-1 of AVA-SP Constituencies	Seats at stake	PDL	LN	PD	IdV	UDC	Seats assigned
Italia nord-occidentale	19	7	5	5	**1**	**1**	19
Italia nord-orientale	13	4	3	4	1	1	13
Italia centrale	14	6	1	5	1	1	14
Italia meridionale	18	9	0	5	2	2	18
Italia insulare	8	**4**	0	**2**	1	1	8
Total seats	–	**30**	**9**	**21**	**6**	**6**	–
National seats	–	**29**	**9**	**22**	**7**	**5**	–
Total-National seats	–	**1**	**0**	**−1**	**−1**	**1**	–

Table 8 reports the allocation A obtained after phase-1, that is the input matrix for phase-2. In this allocation, 2 out of the 5 parties are in surplus and other 2 are in deficit (of one seat each).

In phase-2 the shortest path procedure performs two controlled transfers: 1 seat is transferred in constituency "Italia insulare" from PDL to Pd; 1 seat is ceded by UDC to IDV in constituency Italia "nord-occidentale". Table 9 reports the final seat allocation produced by AVA-SP.

Shifting seats occurred several times in the history of these elections, typically with shifting from south to north Italy. In view of this, in 2018 the legislator modified the

Table 9 European elections 2009: AVA-SP apportionment

EU AVA-SP apportionment 2009 Constituencies	Seats at stake	PDL	LN	PD	IdV	UDC	Seats assigned
Italia nord-occidentale	19	7	5	5	**2**	**0**	19
Italia nord-orientale	13	4	3	4	1	1	13
Italia centrale	14	6	1	5	1	1	14
Italia meridionale	18	9	0	5	2	2	18
Italia insulare	8	**3**	0	**3**	1	1	8
Total seats	–	**29**	**9**	**22**	**7**	**5**	–
National seats	–	**29**	**9**	**22**	**7**	**5**	–

Table 10 European elections 2009: cell-to-cell differences between the AVA-SP and the institutional solution

Differences AVA-SP—institutional, EU 2009 Constituencies	PDL	LN	PD	IdV	UDC
Italia nord-occidentale	−1	0	0	0	−1
Italia nord-orientale	−1	0	−1	0	0
Italia centrale	0	0	−1	0	0
Italia meridionale	1	0	1	0	1
Italia insulare	1	0	1	0	1

law for the European elections acknowledging the same transfer procedure prescribed in the law for the Chamber of Deputies,[13] an intervention that failed to resolve the issue.

In Table 10 we compute the cell-to-cell difference matrix between the (feasible) AVA-SP apportionment and the (infeasible) Institutional allocation.

To conclude, we provide additional empirical evidence in favor of the good performance of AVA-SP. Since it is a heuristic algorithm to minimize the L_1 distance between the allocation matrix and the Fair-Share F, we evaluate the quality of its solutions by a comparison with an apportionment with minimum L_1 distance from F.

Consider the optimal solution of the Mixed Integer Linear Program (MILP) that finds the apportionment that minimizes function (7) under the standard formulation of BAP feasibility conditions as the set of constraints of a transportation problem (see, for example, Pukelsheim et al. (2012)). Clearly, the optimal apportionment found by this model is the best according to norm L_1. As shown in Table 11, The L_1 value for AVA-SP is always very close to the optimum, being exactly the same for the European elections.

5.2 Results for other European countries

In this section, we test our methods on data from the elections in other European countries adopting double proportionality. We consider the cantonal elections of the

[13] (Law 165/2017)

Table 11 Comparison between MILP and AVA-SP on the Italian elections

Quality evaluation of AVA-SP heuristics				
Method	2018	2022	2009	2024
MILP	29.699	29.6967	6.6128	8.267
AVA-SP	29.7479	29.7367	6.6128	8.267

Table 12 Zurich Cantonal elections 2019

Quality evaluation w.r.t matrix F—Zurich Canton 2019				
Method	L_∞	L_1	L_2	n. breaches
Institutional	1.1276	44.477	4.1703	1
AVA-SP	**0.8034**	**43.995**	**4.0809**	0
AVA-MCP	1.7503	46.9187	4.607	2

Table 13 Aargau Cantonal elections 2020

Quality evaluation w.r.t matrix F—Aargau Canton 2020				
Method	L_∞	L_1	L_2	n. breaches
Institutional	1.2608	24.2563	3.3284	1
AVA-SP	**0.89036**	**21.6828**	**2.8255**	0
AVA-MCP	1.0521	23.2217	3.1026	1

Swiss Cantons of Zurich and Aargau, where a divisor method is currently adopted, considering data of 2019 and 2020, respectively.[14].

Tables 12 and 13 report the results for the Zurich and Aargau cantons, respectively. The results confirm that AVA-SP has the best performance in terms of distance to the Fair-Share, measured by different norms. As before, AVA-MCP algorithm tends to produce solutions with breaches, but, in these tests, this happens also for the institutional seat allocation that, in both elections, has one.

This is clearly due to the fact that the double proportional seat allocation method applied in these countries does not follow the principle of minimizing the L_1 distance between the apportionment and the Fair-Share. In fact, as already pointed out in Maier et al. (2010), quota breaches are possible and may arise depending on the particular configuration of the vote matrix. This fact brings the attention to how much important is for a country to avoid breaches. When a divisor method is applied, clearly this is not a main issue; on the contrary, in countries, like Italy, applying HQ plus LR, a 1-seat breach for a party in a constituency is seen as a double award for a fraction of votes that is not sufficient to gain one seat. To corroborate the above result, Table 14 reports the L_1 distance from the Fair-Share for AVA-SP and the optimal solution of the MILP. As before, AVA-SP values are very close to the optimal ones.

In view of its optimization strategy, AVA-SP can be seen as an effective procedure that minimizes the number of breaches in the final apportionment, and, therefore, a good proposal for reforming the Italian electoral laws analyzed in this paper.

[14] Data for these experiments have been downloaded from the repository of the University of Augsurg[15]

Table 14 Comparison between MILP and AVA-SP on Zurich and Aargau elections

Quality evaluation of AVA-SP heuristics		
Method	Zurich 2019	Aargau 2020
MILP	43.1976	21.2786
AVA-SP	43.995	21.6828

The empirical results presented in this section support a positive assessment of the proposed two-phase procedure for the doubly-constrained BAP. The choice between the two variants ultimately depends on the desired trade-off between simplicity and performance. AVA-MCP is conceptually simpler, but generally less effective, whereas AVA-SP consistently achieves superior performance at the cost of increased mathematical sophistication.

6 Conclusions

The Alternating Vector Apportionment algorithms proposed in this paper can be used in electoral systems adopting doubly-constrained biproportional seat allocation. The paper focuses on the Italian case, but the method could be used in any country allocating seats in multi-member constituencies by Hare Quota plus Largest Remainders. The main innovation in our algorithms is phase-2 for controlled seat transfers, that, in fact, can be applied after our phase-1, but, more generally, can be used to obtain a feasible apportionment from any allocation matrix that does not satisfy only one between row- or column-sum constraints. For example, bypassing phase-1, our phase-2 can be combined with the first allocation prescribed in the Italian electoral law, after the constituency-wise proportional seat allocation. We emphasize that the idea of transferring seats is already present in the Italian law, so that our proposal can be seen as an adjusted version of the current seat transfer procedure that, by controlled transfers, is able to balance column-sums, without damaging row-ones. This was conceived with the aim of eliminating the phenomenon of shifting seats through the smallest possible change of the law procedure.

In the paper, we propose two algorithms, AVA-MCP and AVA-SP, and provide empirical evidence of their ability to solve the problem efficiently. Although these are heuristic procedures, the results obtained from their application to elections in Italy and in other European countries confirm both the effectiveness of the methods and their general applicability. The quality of the solutions in terms of proportionality is supported by the values of the performance indices computed as the distance between the allocated seat matrix and the corresponding fractional Fair-Share F. In particular, the solutions produced by the AVA-SP algorithm achieve quality values that are very close to those of the optimal allocation obtained through the solution of a MILP that minimizes the L_1 distance from F, and, in some cases, they are exactly the same.

We point out that stopping phase-1 of our algorithm after the first iteration is equivalent to apply the Hare Quota plus Largest Remainders rule independently in each constituency. This can be viewed as a special case of our phase-1 algorithm, repro-

ducing the procedure currently prescribed by the Italian law for the election of the Chamber of Deputies. In this version, our two-phase algorithm could be immediately integrated in the law to solve the problem of shifting seats that are not compliant with the constituency magnitude fixed by the Constitutional provisions.
Even if the elections of the Italian Chamber of Deputies have been our first motivation, the proposed algorithms can be used also for the electoral law of the European Parliament, that in Italy reproduces the previous one,[16] as well as in the great majority of the laws for the election of the Italian Regional Parliaments[17] that follow the same rule and are affected by the same technical defect. In conclusion, the proposed algorithms are general, formally correct, and able to find high quality solutions for the doubly-constrained BAP.

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Declarations

Conflict of Interest The authors declare none.

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