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CYCLE XXXVIII

Boundary Integral Formulations for the Aeroacoustics of Deformable Bodies

Advisor

Prof. Massimo Gennaretti

Candidate

Beatrice De Rubeis

Co-Advisor

Prof. Giovanni Bernardini

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*"When everything seems to be going
against you, remember that the airplane
takes off against the wind, not with it."*

Henry Ford

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Nomenclature

a	sphere pulsation amplitude
A	amplitude of the deformation of the porous sphere
A^I	amplitude of the incident plane wave
$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$	covariant basis vectors
$\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$	contravariant basis vectors
c, c_0	local and undisturbed speed of sound
d, ϱ	observer distance from the sphere centre
$G, \hat{G}$	free-space Green's function in FFR (frame of reference fixed to the undisturbed fluid at rest) and MFR (frame of reference moving with respect to FFR)
$G_0, \hat{G}_0$	spatial term of Green's function in FFR and MFR
$J, \hat{J}$	Jacobian of transformation in FFR and MFR
J_s	Jacobian of the boundary surface
$k_1, k_2, \mathbf{w}, \mathbf{z}, \mathbf{Z}$	forcing terms of the wave equation
k	wave number of the sphere's pulsations ($= \omega/c_0$)
k_I	wave number of incident field ($= \omega_I/c_0$)
$\mathbf{M}, M$	Surface points Mach number vector and magnitude
M_n	Normal component of $\mathbf{M}$ (along direction $\mathbf{n}$)
M_r	Radial component of $\mathbf{M}$ (along direction $\mathbf{e}_r$)
M_v	number of discretization volumes
$\mathbf{M}_y$	Mach number of MFR points with respect to FFR
M_r^y	Mach number of MFR points in radial direction, $\mathbf{e}_r$
$\mathbf{n}$	outward unit normal vector of boundary surface
N_a	number of harmonics
N_p	number of BEM discretization panels
p, p_0	local and undisturbed pressure
$p', \bar{p}'$	acoustic pressure and its extension to $\mathbb{R}^3$

$\mathbf{P}$	compressive tensor
$\mathbf{r}, r$	source-observer distance vector and magnitude
r_0	average sphere radius
r_s	sphere radius
$\mathbf{R}$	rigid rotation between MFR and FFR
$\mathcal{S}$	boundary surface
t	source time
t_*	observer time
$\mathbf{T}$	Lighthill stress tensor
$\mathbf{u}, u$	fluid velocity vector and magnitude
$\mathbf{v}$	velocity of surface points with respect to FFR
$\mathbf{v}_y$	velocity of MFR points with respect to FFR
$\mathbf{v}_y^m$	boundary surface points velocity with respect to MFR
$v_s, \dot{v}_s$	velocity and acceleration of the sphere surface
$\mathbf{V}$	viscous stress tensor
$\mathbf{x}, \mathbf{y}$	source position in FFR and MFR
$\mathbf{x}_*, \mathbf{y}_*$	observer position in FFR and MFR
$\mathbf{y}_m$	mapping of boundary surface points
γ	ratio between the isobaric and isochoric specific heat capacities
ϵ_r^2	Relative L2-error
θ_x, θ_y	propagation delay from FFR- and MFR-fixed points
θ	propagation delay from boundary surface points
ξ_m	inverse of boundary surface points mapping
ρ, ρ_0	local and undisturbed density
$(\hat{\rho}, \beta, \vartheta)$	spherical coordinate system
σ	nonlinear terms of the velocity potential equation
σ^Ψ	nonlinear source terms in the wave equation
$\varphi, \bar{\varphi}$	velocity potential and its extension to $\mathbb{R}^3$
φ^I	incident velocity potential
φ^S	scattered velocity potential
χ	boundary condition
$\Psi, \bar{\Psi}$	field function and its extension to $\mathbb{R}^3$

ω	fundamental angular frequency, angular frequency of sphere pulsations
ω_d	angular frequency of porous sphere deformation
ω_I	angular frequency of the incident wave
ω_s	angular frequency of pulsating point source
Ω	boundary surface mapping domain in (ξ^1, ξ^2) -plane

Acronyms

BEM Boundary Element Method

CAA Computational Aeroacoustics

CFD Computational fluid dynamics

DP Distributed-Propulsion

FD frequency-domain

FFR fluid frame of reference

FWHE Ffowcs Willams and Hawkings equation

MFR moving frame of reference

TD time-domain

UAM Urban Air Mobility

Chapter 1

Introduction

In the last decades, the aviation industry has experienced exponential growth, sped up in recent years by the growing interest in [Urban Air Mobility \(UAM\)](#) applications. Although [UAM](#) has the potential to overcome the limits of ground transportation capacity and to mitigate congestion and pollution in urban environments, its practical implementation still faces critical challenges. Among these, the reduction of the environmental impact—both chemical and acoustic—remains a key requirement. Similar concerns also affect conventional medium- and long-range aviation, particularly during takeoff and landing phases, where noise abatement is essential. From all these considerations, it is clear that reducing the acoustic nuisance is essential to: i) facilitate the entry-into-service of [UAM](#) as well as the expansion of standard aviation; ii) guarantee the well-being of citizens in urban areas closer to the airports and aviation routes; iii) improve passenger comfort.

In response to this need, national and international committees have established quantitative goals [\[1\]](#) in line with the Flightpath 2050 vision [\[2\]](#). To achieve these targets and to deal with the ever-stringent regulations in terms of environmental sustainability, it is fundamental to provide methods and tools that enable the acoustic emissions to be characterised accurately (particularly, in case of new concept configurations). At the same time, to effectively reduce the noise emitted, acoustics should be included from the earliest vehicle’s design phases.

Significant steps forward in terms of environmental sustainability have been made by exploiting low-noise devices to reduce the noise emitted by traditional design vehicles. Despite this, an asymptotic limit has been reached, and further improvements can only be achieved by designing breakthrough concepts. Instances are [Distributed-Propulsion \(DP\)](#) layouts [\[3\]](#) based on integrating the propulsive system with the aircraft airframe, with the possibility to exploit Boundary-Layer-Ingestion (BLI) technologies [\[4\]](#).

The aeroacoustics of innovative configurations may be affected by phenomena that might be negligible in conventional concepts. For instance, the aerodynamic interactions occurring in [DP](#) systems may significantly modify the performance and noise radiation of the aircraft more than in a standard configuration [\[5, 6\]](#). Furthermore, particular attention must be paid to the effects of the body deformation, which might be significant in rotorcraft blade dynamics, as well as, in highly flexible innovative configurations [\[7, 8\]](#). These effects may play a crucial role in aerodynamic and aeroacoustic phenomena simulations and must be accurately accounted for [\[9\]](#). At the same time, the presence of the vehicle’s airframe can significantly modify the noise radiated by the main sources of noise (typically the propulsive system), making the investigation of the installation scattering

effects of primary importance [10].

While accounting for these effects is necessary to achieve a realistic description of the aeroacoustic field of complex configurations, it also greatly increases the computational cost of the analysis. This makes the development of fast and efficient prediction tools of great interest, particularly for use during preliminary design phases.

In this context, this thesis presents the development and application of boundary integral formulations for the analysis of aeroacoustic problems generated by the interaction between moving and deforming bodies and the surrounding fluid, with particular attention to sound radiation and scattering. The perturbative field is described either in terms of acoustic pressure or velocity potential, and the boundary integral formulations are applicable to solid or porous surfaces.

These formulations are employed to solve the acoustic wave equation in a framework that naturally includes radiation and scattering phenomena. In particular, when the goal is the determination of the far-field propagation of perturbations, they represent an extremely efficient approach. Their main advantage lies in the ability to provide the solution of the propagated field from the knowledge of the flow parameters on a boundary (emitting) surface, thus avoiding the computationally expensive discretization of the entire fluid domain between the emitting and receiving points [11–13].

1.1 Boundary integral formulations for aeroacoustics

Boundary integral formulations for the solution of the wave equation have traditionally been developed for fixed or rigidly moving surfaces, and in this form they can be reliably applied to many aeroacoustic problems where structural deformations are negligible. Over the years, several versions have been developed in both time and frequency domains, with applications ranging from the noise radiation of helicopter rotors and propellers to the scattering of complex geometries [14–19].

However, as mentioned above, with the transition towards innovative aircraft configurations including multi-rotor vehicles —conceived to reduce both noise impact and fuel consumption in line with the global targets of Flightpath 2050 [2]—the role of flexibility and structural dynamics becomes increasingly important. Many next-generation concepts rely on highly deformable structures, for which shape changes, local motion and variations in surface orientation can strongly affect both aerodynamic loads and the propagation of acoustic perturbations. In this context, the availability of simulation tools that explicitly account for surface deformation is essential from the earliest design phases.

To address this need, several boundary integral formulations valid for deformable surfaces have been proposed in the literature. These include extensions of the Kirchhoff equation [20] and the Farassat 1A formulation, widely used in helicopter noise and [Computational fluid dynamics \(CFD\)/Computational Aeroacoustics \(CAA\)](#) applications [21], as well as formulations for the velocity potential field around lifting bodies [22].

In line with these developments, the present work introduces a general boundary integral formulation for the solution of arbitrarily forced acoustic wave equation that includes both noise radiation and scattering analyses. The formulation is based on the free-space Green function and applies to fluid domains bounded by arbitrary solid or porous surfaces undergoing arbitrary motion and deformation. It can be considered as the extension to deforming boundaries of the formulation presented in [23] and, at the same time, as the application of the boundary integral approach introduced in [24]

to the arbitrarily forced acoustic wave equation. Although conceived for aeronautical applications, the boundary integral formulation introduced is not limited to them.

Two versions of the deformable-boundary integral formulation that solves the arbitrarily-forced acoustic wave equation problem are developed: one expressed in a frame of reference fixed with the undisturbed fluid, [fluid frame of reference \(FFR\)](#), and the second expressed in a frame of reference in arbitrary motion, [moving frame of reference \(MFR\)](#) (typically fixed to the undeformed moving body). In principle, they are theoretically equivalent, but the efficiency of their computational application may differ and depend on the specific problem under investigation.

The computational application of these formulations is carried out through the [Boundary Element Method \(BEM\)](#), which provides an efficient numerical algorithm well suited for the aerodynamic and aeroacoustic analyses of realistic configurations of aeronautical interest (see, for instance, [\[25–27\]](#)).

The boundary integral formulation can be applied in two ways. When the perturbation field (pressure or velocity potential) over the boundary surface is known from a preliminary aerodynamic solution, the formulation provides an integral representation that directly radiates the far-field perturbation (as is typically the case in applications related to rotors and propellers noise, see for instance [\[15, 17, 28, 29\]](#)). Instead, when the perturbation field over the boundary surface is unknown, first it provides a boundary integral equation to be solved to determine it, and then, it is used as an integral representation to evaluate the radiated field (as it typically occurs in potential-flow aerodynamics or noise scattering applications, see for instance [\[30–34\]](#)).

Finally, it is important to observe that most of the applications of boundary integral formulations concern problems where the contributions from the boundary surface are dominant with respect to those possibly given by the presence of source field terms (namely, problems in which the nonlinear terms are negligible, see later). Indeed, in these cases they provide particularly efficient computational solutions, while the presence of significant field terms requires the inclusion of volume contributions which decreases the computational performance of the methodology. This occurs, for instance, in the case of aerodynamic/aeroacoustic analyses of bodies in transonic flight (see, for instance, [\[35–37\]](#)).

To overcome such difficulties, the proposed boundary integral approach includes the possibility of considering domains bounded by porous (fictitious) surfaces. In fact, porous (fictitious) boundary surfaces can be introduced to enclose complex and/or nonlinear source regions, enabling the far-field propagation of acoustic perturbations to be computed efficiently from surface integrals. In this strategy, the near-field solution is typically derived through an accurate [CFD](#) solver, while the boundary integral formulation is applied as an integral representation to determine the acoustic field radiated by the porous surface. This coupling has been successfully applied to jet-noise problems [\[38–40\]](#), high-speed impulsive noise [\[17\]](#), and the noise of complex aeronautical configurations including helicopters and multi-rotor vehicles [\[41, 42\]](#). In all these cases, porous-surface formulations provide a powerful bridge between [CFD](#) and [BEM](#), combining high-fidelity near-field resolution with efficient and accurate far-field radiation predictions.

1.2 Radiation and scattering problems

The boundary integral formulation developed in this work is primarily conceived for the study of noise radiation and scattering phenomena relating to moving and deforming objects, namely for external aeroacoustics problems. These two classes of phenomena share the same mathematical modelling framework, but differ in the physical mechanism generating the acoustic field.

In noise radiation problems, the aerodynamic solution on the surface of the moving emitting body is assumed to be known. This is the typical case, for example, in aeronautical applications such as helicopters or propellers aeroacoustics, for which the pressure distribution on the blades is preliminary evaluated by an aerodynamic solver and the radiated sound is evaluated using the boundary integral formulation in the form of an integral representation. The aerodynamic loads provide the surface source terms and the boundary integral formulation propagates the corresponding perturbations to the far field (see, for instance, [17, 28, 29]).

On the other hand, scattering problems require a different application of the boundary integral formulation. Scattering formulations can be used for the efficient evaluation of the effect of the presence of secondary bodies on noise emitted by main noise sources. For instance, in aeronautical applications, these may be used to account for distortion effects due to airframe and installations on the noise emitted by the propulsive system.

Scattering formulations are based on the assumption that the perturbation signal (pressure or velocity potential) emitted by the main noise source (incident signal) is independent of the presence of the scatterer. This allows the decomposition of the total perturbation field into incident and scattered components, which can be determined separately, avoiding demanding computations solving jointly the aerodynamic field around source and the scatterer. Indeed, first, the incident field is evaluated assuming the source as it was isolated and "frozen", and then the rest of the vehicle components are considered as contributing only as scatterers. In this framework, the boundary integral formulation is first applied as an integral equation to determine the perturbation field on the scattering surface, and subsequently as an integral representation to radiate the scattered component to the far field (see, for instance, Refs. [18, 32, 34]).

A broad literature is available on acoustic scattering, mostly assuming rigid fixed or moving scatterers [43, 44]. Formulations have been developed using either the velocity potential [45, 46] or the acoustic pressure [23, 47, 48]. Scattering from deforming surfaces has also been investigated. For simple geometries such as cylinders and spheres, analytical perturbation methods have shown that kinematic nonlinearities in boundary conditions generate multi-harmonic spectra, with frequency content depending on the motion of the scatterer and on the perturbation order considered [49, 50].

A second possible source of multiple tones resides in the fluid nonlinearities, which are strongly related to the flow induced by the deformation of the scatterer. The effect of fluid nonlinearities on the scattered field was studied, for instance, in [51] for an infinitely long cylinder impinged by a plane wave, while the relative importance of the two sources of multi-harmonic scattering was examined by Mujica et al. [52]. These studies confirm that both the structural deformation and the nonlinear behaviour of the fluid may play a significant role in shaping the spectral content of the scattered sound.

Radiation and scattering formulations are not mutually exclusive but can be combined depending on the problem. For instance, in DP configurations with multiple propellers mounted on a wing, the

aeroacoustic analysis can be approached in two alternative ways. A first strategy is to compute the coupled aerodynamic solution of all components, including mutual interactions, and then radiate the noise from propellers and wing via the boundary integral representation.

A second strategy consists of solving the aerodynamics of the propellers as if they were isolated, followed by the determination of the corresponding radiated noise, which is then considered as an incident field impinging the wing (playing the role of scatterer). Solving the wing scattering problem produces the scattered contribution, which is then superimposed on the sound radiated by the propellers. In this way, the total acoustic field produced by the DP configuration is obtained at the cost of solving multiple isolated body problems instead of a single, more complex multiple body problem.

1.3 Acoustic field in terms of velocity potential or pressure

The general wave equation solved in this thesis encompasses description of the acoustic field expressed both in terms of velocity potential and pressure, thus providing a unified framework for aeroacoustic analyses. Depending on the problem under investigation, one or the other description can be more convenient.

In aeronautical applications for which the aerodynamic solution on the surface of the emitting body is available, the acoustic pressure formulation is generally the most suitable choice. In this case, the body loads given by the aerodynamic solution yield the acoustic source terms which are efficiently radiated by the boundary integral representation (see, for instance, [17, 29]).

However, a velocity potential approach for the analysis of noise radiated by helicopter rotors has also been applied in Ref. [53]. In this case, the aerodynamic solution yielding the sound perturbation sources and the acoustic radiation are both expressed in terms of velocity potential, and it is not necessary to evaluate the pressure over the emitting body. Compared to the pressure-based formulation, wake effects are considered differently, making the potential-based formulation more computationally expensive.

In scattering problems, velocity potential-based formulations are typically applied. This derives from the straightforward imposition of the boundary conditions relating to the incident perturbation wave. However, scattering formulations expressed in terms of the acoustic pressure have also been developed and applied (see, for instance, Refs. [18, 32]). Discussions of scattering formulations expressed in terms of velocity potential and acoustic pressure (the latter derived both from the Lighthill equation and the Ffowcs Williams and Hawkings equation) are presented in Refs. [23, 46, 48]. In particular, it is shown that different predictions of the signal scattered by a moving body are given by linear boundary integral formulations applied for the solution of the Ffowcs-Williams and Hawkings equation, the Lighthill equation, or the velocity potential wave equation. This occurs because the nonlinear field terms neglected in these linear formulations produce significant but different contributions in the presence of non-uniform meanflow [23, 46].

From the above, it appears that the application of pressure or velocity potential formulations for aeroacoustic analysis depends on the problem under consideration and the configurations examined.

1.4 Structure of the Thesis

This thesis is organized into six chapters and eight appendices. These provide detailed description of development and validation of the introduced boundary integral formulation for the solution of arbitrarily-forced wave equation in fluid domains bounded by deforming surfaces.

Chapter 2 recalls the equations governing the propagation of perturbation fields in fluids, and introduces the general arbitrarily-forced acoustic wave equation, which forms the basis of all subsequent developments. The mathematical structure of the problem is outlined, highlighting the dual description of the acoustic field in terms of either acoustic pressure or velocity potential. In addition, it is shown that the governing equations commonly used in aeroacoustics—namely, the velocity potential equation, Lighthill’s equation, and the [Ffowcs Willams and Hawkings equation \(FWHE\)](#)—can all be regarded as particular cases of the general formulation.

Chapter 3 is devoted to the derivation of the deformable-boundary integral formulation, which extends previous boundary integral approaches to arbitrarily moving and deforming surfaces. The solution developed in this chapter is that of the general arbitrarily-forced acoustic wave equation introduced in Chapter 2. The formulation is presented both in a fluid-fixed reference frame, [FFR](#), and in a body-fixed reference frame, [MFR](#). Furthermore, its application as an integral representation or an integral equation for the perturbation field is discussed.

Chapter 4 focuses on the validation of the proposed formulation for noise radiation problems, through a set of test cases where analytical or experimental reference solutions are available. The first case concerns the propagation of perturbations from a pulsating moving source enclosed by a porous deforming surface, where the formulation is applied in the form of an integral representation and validated by comparison with the analytical solution. The second case addresses the compressible non-circulatory flow around a translating and deforming wing, allowing a direct comparison between velocity potential and pressure formulations, as well as between [FFR](#) and [MFR](#). The third validation problem again considers the pulsating moving source enclosed by a porous deforming surface, but in this case the formulation is solved as an integral equation, providing a further verification of the methodology. The fourth case examines the far-field radiation of a pulsating solid sphere, for which the analytical solution is derived in detail in a dedicated appendix ([appendix E](#)) and compared with the predictions of the boundary integral formulation. Finally, the proposed solver is applied to a realistic problem consisting of the simulation of the noise radiated by a helicopter rotor in forward flight. In this case, numerical predictions are compared with experimental measurements, and the effect of structural deformation on the acoustic signatures is assessed and discussed.

Chapter 5 addresses the problem of noise scattering. The acoustic field is described through the deformable-boundary integral formulation, both in terms of pressure and velocity potential, and solved using time- and frequency-domain [BEM](#) approaches. Particular attention is devoted to the development of a novel frequency-domain algorithm capable of capturing the multi-harmonic content of the scattered field generated when deforming bodies are impinged by mono-harmonic incident waves. For validation purposes, analytical solutions for the scattering of a pulsating sphere are presented in [appendix G](#). While [Sec. 5.1](#) deals with the linear scattering formulation, [Sec. 5.3](#) addresses the nonlinear formulation. Indeed, a new cascade iterative algorithm has been implemented to include nonlinear terms in the velocity potential equation. This approach makes it possible to

account not only the kinematic nonlinearities in the boundary conditions introduced by the deformation of the scatterer (already taken into account in the linear formulation), but also the fluid nonlinearities, thus extending the predictive capability of the formulation and enabling the investigation of their impact on the scattered spectrum. The appendix [H](#) contains some remarks on the various iterative procedures that are applicable for solving the nonlinear scattering equation.

Finally, Chapter 6 presents the conclusions, summarizing the key findings and outcomes of this study.

Chapter 2

Acoustic wave equations

This chapter presents various formulations describing the propagation of acoustic waves. The first two are based on the acoustic analogy and are expressed in terms of acoustic pressure: we start with Lighthill's equation, and continue with Ffowcs Williams and Hawkings' equation written for solid and porous surfaces. These acoustic pressure equations differ essentially in the contribution of the nonlinear terms related to the Lighthill tensor.

Alongside these acoustic pressure formulations, the velocity potential equation is also introduced, which describes the wave propagation through the velocity potential (flow is assumed to be irrotational) and provides an alternative description of the acoustic phenomenon.

The concluding part of the chapter shows how all these different formulations – written in terms of pressure or potential – can be obtained as particular forms of a general arbitrarily-forced wave equation, which represents a unifying formulation of the problems of acoustic radiation and scattering, in domains bounded by solid or porous deforming surfaces in arbitrary motion.

2.1 Lighthill equation

Let us consider a homentropic, compressible fluid, a closed surface, $\mathcal{S}$, arbitrarily moving within it, and a frame of reference, $\mathcal{R}(\mathbf{x})$, fixed to the undisturbed fluid at rest (FFR).

By combining the continuity and Navier-Stokes equations the following Lighthill equation for the acoustic pressure, $p' = c_0^2(\rho - \rho_0)$, is obtained (acoustic analogy) [54, 55]

$$\nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = -\nabla \cdot \nabla \cdot \mathbf{T} \quad (2.1)$$

where c_0 and ρ_0 are the speed of sound and density of the undisturbed fluid, $(\rho - \rho_0)$ denotes density perturbation, whereas $\mathbf{T} = (p - c_0^2 \rho) \mathbf{I} + \rho \mathbf{u} \otimes \mathbf{u} - \mathbf{V} = \mathbf{P} - p' \mathbf{I} + \rho \mathbf{u} \otimes \mathbf{u}$ is the Lighthill tensor, where $\mathbf{P} = (p - p_0) \mathbf{I} - \mathbf{V}$ denotes the compressive stress tensor, with $\mathbf{V}, p, \mathbf{u}$ representing, respectively, the viscous stress tensor and the local fluid pressure and velocity, whereas $p_0 = c_0^2 \rho_0$ is the undisturbed fluid pressure.

This equation is assumed to be valid outside of the closed surface $\mathcal{S}$, namely for $\mathbf{x} \in \mathcal{V}_f$. To determine the solution of the boundary-value problem in Eq. (2.1) through a boundary integral formulation, it is convenient to recast the problem into an equivalent infinite-space one. To this aim, for the function $f = f(\mathbf{x}, t)$ such that $f = 0$ if $\mathbf{x} \in \mathcal{S}$, $f > 0$ if $\mathbf{x} \in \mathcal{V}_f$, $f < 0$ if $\mathbf{x} \in \mathbb{R}^3 \setminus \mathcal{V}_f$, the

definition of the acoustic pressure is extended to the entire $\mathbb{R}^3$ domain as $\bar{p}' = p' H(f)$, where H represents the Heaviside function. Then, multiplying the two sides of Eq. (2.1) by $H(f)$, considering that for a generic function $\hat{f}$, the Laplacian and time second derivative operators applied to the extension of $\hat{f}$ to the entire domain $\mathbb{R}^3$ give

$$\nabla^2(H\hat{f}) = H\nabla^2\hat{f} + \nabla \cdot (\hat{f}\nabla H) + \nabla H \cdot \nabla\hat{f} \quad (2.2)$$

and

$$\frac{\partial^2(H\hat{f})}{\partial t^2} = H \frac{\partial^2\hat{f}}{\partial t^2} + \frac{\partial}{\partial t} \left(\hat{f} \frac{\partial H}{\partial t} \right) + \frac{\partial H}{\partial t} \frac{\partial\hat{f}}{\partial t} \quad (2.3)$$

and using the definition of the extended acoustic pressure we have [23]

$$\begin{aligned} \nabla^2\bar{p}' - \frac{1}{c_0^2} \frac{\partial^2\bar{p}'}{\partial t^2} = & -H\nabla \cdot \nabla \cdot \mathbf{T} + \nabla H \cdot \nabla p' + \nabla \cdot (p'\nabla H) + \\ & -\frac{1}{c_0^2} \left[\frac{\partial H}{\partial t} \frac{\partial p'}{\partial t} + \frac{\partial}{\partial t} \left(p' \frac{\partial H}{\partial t} \right) \right] \end{aligned} \quad (2.4)$$

This equation is valid for both a solid $\mathcal{S}$ surface and a porous (fictitious) $\mathcal{S}$ surface, and provides explicit description of the effects of the presence of a fluid boundary in the Lighthill equation.

2.2 Ffowcs Williams and Hawkings equation

An alternative equation governing the propagation of acoustic perturbations within a bounded fluid is obtained by observing that

$$\nabla \cdot \nabla \cdot [H\mathbf{T}] = H\nabla \cdot \nabla \cdot \mathbf{T} + \nabla \cdot [\mathbf{T}\nabla H] + (\nabla \cdot \mathbf{T}) \cdot \nabla H \quad (2.5)$$

Indeed, combining Eq. (2.4) with Eq. (2.5) and suitably re-arranging the right-hand-side of the resulting equation [56] yields the following Ffowcs Williams and Hawkings equation in the case of a solid $\mathcal{S}$ surface [47]

$$\nabla^2\bar{p}' - \frac{1}{c_0^2} \frac{\partial^2\bar{p}'}{\partial t^2} = \nabla \cdot [\mathbf{P}\nabla H] - \frac{\partial}{\partial t} [\rho_0 \mathbf{v} \cdot \nabla H] - \nabla \cdot \nabla \cdot [H\mathbf{T}] \quad (2.6)$$

where $\mathbf{v}$ denotes the velocity of the surface $\mathcal{S}$. It is possible to derive the extension of the Ffowcs Williams and Hawkings equation to the case of a porous $\mathcal{S}$ surface. It reads [57]

$$\begin{aligned} \nabla^2\bar{p}' - \frac{1}{c_0^2} \frac{\partial^2\bar{p}'}{\partial t^2} = & \nabla \cdot [\mathbf{P}\nabla H] + \nabla \cdot [\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) \nabla H] + \\ & -\frac{\partial}{\partial t} [\rho_0 \mathbf{v} \cdot \nabla H] - \frac{\partial}{\partial t} [\rho (\mathbf{u} - \mathbf{v}) \cdot \nabla H] - \nabla \cdot \nabla \cdot [H\mathbf{T}] \end{aligned} \quad (2.7)$$

Details on the derivation of the FWHE for solid and porous surfaces from the Lighthill equation are given in appendix A.

2.3 Velocity potential equation

Next, note that under the assumption of inviscid, irrotational flow, the velocity field may be described through the velocity potential, φ , such that $\mathbf{u} = \nabla\varphi$. It is possible to show that also the propagation of velocity potential perturbations is governed by the wave equation whose solution, therefore, provides the kinematics of the flow. Specifically, in a [FFR](#), the combination of the continuity equation with Bernoulli's theorem yields

$$\nabla^2\varphi - \frac{1}{c_0^2} \frac{\partial^2\varphi}{\partial t^2} = \sigma \quad (2.8)$$

where $\sigma = [(c_0 - c)\nabla^2\varphi + 2\mathbf{u} \cdot \partial\mathbf{u}/\partial t + \mathbf{u} \cdot \nabla u^2/2]/c_0^2$ collects all the nonlinear terms, for c denoting the local speed of sound. Following an approach similar to that applied in [Sec. 2.1](#), the problem is extended to the entire $\mathbb{R}^3$ domain. Thus, multiplying the two sides of [Eq. \(2.8\)](#) by $H(f)$ and using [Eq. \(2.2\)](#) and [Eq. \(2.3\)](#), yields the following equation

$$\nabla^2\bar{\varphi} - \frac{1}{c_0^2} \frac{\partial^2\bar{\varphi}}{\partial t^2} = H\sigma + \nabla H \cdot \nabla\varphi + \nabla \cdot (\varphi \nabla H) - \frac{1}{c_0^2} \left[\frac{\partial H}{\partial t} \frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial t} \left(\varphi \frac{\partial H}{\partial t} \right) \right] \quad (2.9)$$

where $\bar{\varphi} = \varphi H(f)$. This equation is valid for both solid and porous (fictitious) boundary surfaces $\mathcal{S}$.

2.4 Arbitrarily-forced wave equation

It is interesting to note that [Eqs. \(2.4\), \(2.6\), \(2.7\) and \(2.9\)](#) govern the radiation of perturbations from solid or porous surfaces, but they can be readily extended to scattering problems (expressed in terms of acoustic pressure or velocity potential perturbations). These are problems for which an external source not affected by the surface $\mathcal{S}$ (i.e., frozen) generates perturbation (incident) waves interacting with it and, thus, are scattered. This is readily obtained by introducing a suitable external forcing term in $\mathcal{V}_f$ (see, for instance, [\[19, 23\]](#)).

Observing the structure of [Eqs. \(2.4\), \(2.6\), \(2.7\) and \(2.9\)](#) it comes out that all formulations suitable for the analysis of radiation or scattering problems can be considered as sub-cases of the following arbitrarily-forced acoustic wave equation, which is valid throughout unbounded space

$$\begin{aligned} \nabla^2\bar{\Psi} - \frac{1}{c_0^2} \frac{\partial^2\bar{\Psi}}{\partial t^2} = & \sigma^\Psi + \mathbf{z} \cdot \nabla H + \nabla \cdot (\mathbf{Z} \nabla H) + k_1 \frac{\partial H}{\partial t} + \frac{\partial}{\partial t} \left(k_2 \frac{\partial H}{\partial t} \right) + \\ & + \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) \end{aligned} \quad (2.10)$$

where, for Ψ denoting the problem perturbation variable, $\bar{\Psi} = \Psi H(f)$, whereas k_1 , k_2 , $\mathbf{w}$, $\mathbf{z}$, $\mathbf{Z}$ and σ^Ψ are forcing terms to be defined for the specific problem under examination.

In the case of acoustic pressure perturbation variable, [Eq. \(2.10\)](#) yields:

- the Lighthill equation ([Eq. \(2.4\)](#)) for $\Psi = p'$, $\mathbf{z} = \nabla p'$, $\mathbf{Z} = \mathbf{P}$, $k_1 = -1/c_0^2 (\partial p'/\partial t)$, $k_2 = -p'/c_0^2$, $\mathbf{w} = 0$ and $\sigma^\Psi = -\nabla \cdot \nabla \cdot \mathbf{T}$;
- the [FWHE](#) for solid surfaces ([Eq. \(2.6\)](#)) for $\Psi = p'$, $\mathbf{z} = 0$, $\mathbf{Z} = \mathbf{P}$, $k_1 = k_2 = 0$, $\mathbf{w} = -\rho_0 \mathbf{v}$ and $\sigma^\Psi = -\nabla \cdot \nabla \cdot (H\mathbf{T})$;

- the FWHE for porous surfaces (Eq. (2.7)) for $\Psi = p'$, $\mathbf{z} = 0$, $\mathbf{Z} = \mathbf{P} + \rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v})$, $k_1 = k_2 = 0$, $\mathbf{w} = -\rho_0 \mathbf{v} - \rho(\mathbf{u} - \mathbf{v})$ and $\sigma^\Psi = -\nabla \cdot \nabla \cdot (H\mathbf{T})$.

In the case of velocity potential perturbation variable, Eq. (2.10) yields Eq. (2.8) for $\Psi = \varphi$, $\mathbf{z} = \nabla \varphi$, $\mathbf{Z} = \varphi \mathbf{I}$, $k_1 = -1/c_0^2 (\partial \varphi / \partial t)$, $k_2 = -\varphi/c_0^2$, $\mathbf{w} = 0$ and $\sigma^\Psi = \sigma$.

In each of these formulations, the variable σ^Ψ represents nonlinear contributions, unless a scattering problem is considered. In that case, this term is suitably defined to formally include the presence of an incident field [19, 23].

The next chapter develops the deformable-boundary integral formulation that gives the solution of Eq. (2.10), and hence of any radiation/scattering or potential-aerodynamics problem defined in a domain bounded by arbitrarily moving solid or porous deforming surfaces.

Chapter 3

The deformable-boundary integral formulation for acoustic wave solution

As is well known, boundary integral methods are suitable approaches for predicting the propagation of perturbations (acoustic waves, in particular) in the field outside the perturbing object.

This is due to the use of the Green function of the wave operator to describe the effect of a perturbation source on arbitrarily distant observers, which does not require the introduction of computationally expensive discretization meshes to track the propagation of the signal (as is typical in CFD solution methods).

In this section, extending the formulation presented in [9], a boundary integral approach that solves Eq. (2.10) in a fluid domain bounded by a deforming surface is developed. It derives from the application of a boundary integral method inspired by the work in [24].

3.1 Green function and integral form of the solution of the arbitrarily-forced acoustic wave equation

The first step to determine the boundary integral formulation for the solution of Eq. (2.10) is the introduction of the free-space Green's function, G , of the wave equation. It is defined as the (fundamental) solution of the wave equation forced by time and space impulses, namely it is the solution of the following equation

$$\nabla^2 G - \frac{1}{c_0^2} \frac{\partial^2 G}{\partial t^2} = \delta(\mathbf{x} - \mathbf{x}_*) \delta(t - t_*) \quad (3.1)$$

associated with the boundary condition $G = 0$ for $\|\mathbf{x} - \mathbf{x}_*\| \rightarrow \infty$, and the time conditions $G = \dot{G} = 0$ for $t \rightarrow \infty$. It reads [58]

$$G(\mathbf{x} - \mathbf{x}_*, t - t_*) = \frac{-1}{4\pi r} \delta\left(t - t_* + \frac{r}{c_0}\right) \quad (3.2)$$

where $r = \|\mathbf{r}\| = \|\mathbf{x}_* - \mathbf{x}\|$.

Next, the application of the boundary integral equation method consists of the combination of Eq. (2.10) multiplied by the function G with Eq. (3.1) multiplied by the function $\bar{\Psi}$, followed by

the integration over the entire $\mathbb{R}^3$ domain and over time, to obtain

$$\begin{aligned}\bar{\Psi}(\mathbf{x}_*, t_*) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} G \sigma^{\Psi} dV(\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{z} \cdot \nabla H G dV(\mathbf{x}) dt \\ &+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \nabla \cdot (\mathbf{Z} \nabla H) G dV(\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_1 \frac{\partial H}{\partial t} G dV(\mathbf{x}) dt \\ &+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} \left(k_2 \frac{\partial H}{\partial t} \right) G dV(\mathbf{x}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) G dV(\mathbf{x}) dt\end{aligned}\quad (3.3)$$

With the aim of determining the general expression of the solving boundary integral formulation in an arbitrary frame of reference, let us introduce a frame of reference, $\mathcal{R}(\mathbf{y})$, in motion with respect to **FFR** (moving frame of reference - **MFR**), such that the positions of its points in the **FFR** are described as

$$\mathbf{x}(\mathbf{y}, t) = \mathbf{x}_0(t) + \mathbf{R}(t) \mathbf{y}$$

where $\mathbf{x}_0$ denotes the position of the point $\mathbf{y} = \mathbf{0}$, whereas $\mathbf{R}(t)$ represents the rigid rotation between the two frames of reference.

Note that, for an arbitrary given differentiable scalar field, $\hat{f}$, the time derivatives at fixed **FFR** and **MFR** points are related through the following expression

$$\left. \frac{\partial \hat{f}}{\partial t} \right|_x = \left. \frac{\partial \hat{f}}{\partial t} \right|_y - \mathbf{v}_y \cdot \nabla \hat{f} \quad (3.4)$$

where $\mathbf{v}_y$ denotes the velocity of a **MFR** point with respect to the **FFR**.

In addition, note also that for any two given functions $q(t)$ and $h(t)$, the application of the generalized scaling property of the Dirac delta function yields

$$\int_{-\infty}^{\infty} q(t) \delta[h(t)] dt = \sum_k \left[\frac{q(t)}{|\dot{h}(t)|} \right]_{t=t_k} = \sum_k \int_{-\infty}^{\infty} \frac{q(t)}{|\dot{h}(t)|} \delta(t - t_k) dt \quad (3.5)$$

where $\dot{h} = dh/dt$ and t_k are the roots of $h(t) = 0$. Thus, regarding the argument of the Dirac delta function in Eq. (3.2), the function h corresponds to the function $\bar{g}$ identified as $\bar{g} = t - t_* + \|\mathbf{x}(\mathbf{y}_*, t_*) - \mathbf{x}(\mathbf{y}, t)\|/c_0$ which, in a **MFR** description of the problem yields $\dot{\bar{g}} = 1 - \mathbf{r} \cdot \mathbf{v}_y/(rc_0)$. For subsonic motion $k = 1$ and the single root is $t_1 = t_* - \theta_y$, where θ_y satisfies the equation

$$\|\mathbf{x}(\mathbf{y}_*, t_*) - \mathbf{x}(\mathbf{y}, t_* - \theta_y)\| = c_0 \theta_y \quad (3.6)$$

and represents the time taken by a perturbation signal to propagate from $\mathbf{y}$ to $\mathbf{y}_*$ at the time t_* . For a surface in supersonic motion, $k > 1$ (see, for instance, [59] for the case of rotating blades). Notice also that if we choose to describe the function g in the air reference, it turns out to be $\bar{g} = t - t_* + \|\mathbf{x}_* - \mathbf{x}\|/c_0$ and its derivative in time is $\dot{\bar{g}} = 1$. Furthermore, the root is $t_1 = t_* - \theta_x$, with $\theta_x = r/c_0$ representing the delay between two fixed points in the air reference. In this way, Eq. (3.5) yields the definition of Green's function in the **FFR**.

The combination of Eq. (3.3) with Eqs. (3.4) and (3.5) gives

$$\begin{aligned}\bar{\Psi}(\mathbf{x}_*, t_*) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \hat{G} \sigma^\Psi dV(\mathbf{y}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{z} \cdot \nabla H \hat{G} dV(\mathbf{y}) dt \\ &+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \nabla \cdot (\mathbf{Z} \nabla H) \hat{G} dV(\mathbf{y}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_1 \frac{\partial H}{\partial t} \hat{G} dV(\mathbf{y}) dt \\ &+ \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} \left(k_2 \frac{\partial H}{\partial t} \right) \hat{G} dV(\mathbf{y}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) \hat{G} dV(\mathbf{y}) dt\end{aligned}\quad (3.7)$$

where $\hat{G}$ is the Green function expressed in the MFR

$$\hat{G}(\mathbf{y}, \mathbf{y}_*, t, t_*) = \hat{G}_0 \delta(t - t_* + \theta_y) = \frac{-1}{4\pi r (1 - M_r^y)} \Big|_{\theta_y} \delta(t - t_* + \theta_y) \quad (3.8)$$

In the above equation, $M_r^y = \mathbf{M}_y \cdot \mathbf{e}_r$ where $\mathbf{M}_y = \mathbf{v}_y/c_0$ and $\mathbf{e}_r = \mathbf{r}/r$, while $(...)|_{\theta_y}$ indicates evaluation at the retarded time $t = t_* - \theta_y$.

Integrating by parts of the third, fifth and sixth integral and the imposing of the condition of wave propagation away from the source without any reflection or inward radiation (Sommerfeld radiation condition [60]), we obtain

$$\int_{\mathbb{R}^3} \nabla \cdot (\mathbf{Z} \nabla H) \hat{G} dV(\mathbf{y}) dt = \left[(\mathbf{Z} \nabla H) \hat{G} \right] \Big|_{\mathcal{S}_\infty} - \int_{\mathbb{R}^3} (\mathbf{Z} \nabla H) \cdot \nabla \hat{G} dV(\mathbf{y}) \quad (3.9)$$

where $\mathcal{S}_\infty$ denotes the boundary at infinity of $\mathbb{R}^3$, where ∇H is zero because it corresponds to the Dirac delta function and is equal to one only on the surface of the body. For the other two integrals, we have

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left(k_2 \frac{\partial H}{\partial t} \right) \hat{G} dt = \left(k_2 \frac{\partial H}{\partial t} \hat{G} \right) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} k_2 \frac{\partial H}{\partial t} \frac{\partial \hat{G}}{\partial t} dt \quad (3.10)$$

and

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\mathbf{w} \cdot \nabla H) \hat{G} dt = \left(\mathbf{w} \cdot \nabla H \hat{G} \right) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mathbf{w} \cdot \nabla H \frac{\partial \hat{G}}{\partial t} dt \quad (3.11)$$

where the first term on the right in Eqs. (3.10) and (3.11) is deleted because the Dirac delta function in the Green's function is non-zero only at finite time instants.

Substituting Eqs. (3.9) to (3.11) into Eq. (3.7) and writing the time derivatives as in Eq. (3.4), we obtain

$$\begin{aligned}\bar{\Psi}(\mathbf{y}_*, t_*) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \hat{G} \sigma^\Psi dV(\mathbf{y}) dt + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{z} \cdot \nabla H \hat{G} dV(\mathbf{y}) dt \\ &- \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left[(\mathbf{Z} \nabla H) \cdot \nabla \hat{G} - k_1 \left(\dot{H} - \mathbf{v}_y \cdot \nabla H \right) \hat{G} \right] dV(\mathbf{y}) dt \\ &- \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_2 \left(\dot{H} - \mathbf{v}_y \cdot \nabla H \right) \left(\dot{\hat{G}} - \mathbf{v}_y \cdot \nabla \hat{G} \right) dV(\mathbf{y}) dt \\ &- \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla H \left(\dot{\hat{G}} - \mathbf{v}_y \cdot \nabla \hat{G} \right) dV(\mathbf{y}) dt\end{aligned}\quad (3.12)$$

where $(\dot{\cdot}) = \partial(\cdot)/\partial t|_y$.

The above equation yields the solution of Eq. (2.10) in terms of an integral formulation extended

over the entire space domain, based on the use of the free-space Green's function. In the next sections, it is suitably further developed to obtain the solution expressed in terms of deformable-boundary integral formulations written both in the [MFR](#) and the [FFR](#).

3.2 Boundary integral formulation in a moving frame of reference

The integral solution in Eq. (3.12) includes many contributions given by the space and time derivatives of generalized functions (namely, $\nabla H, \dot{H}, \nabla \hat{G}, \dot{\hat{G}}$, see Eq. (3.12)). The boundary integral formulation for the solution of the perturbation field, $\bar{\Psi}$, is obtained by suitably integrating these contributions.

To this purpose, let us introduce a time-dependent body-fitted right-handed curvilinear coordinate system, (ξ^1, ξ^2, ξ^3) , such that within an arbitrarily small thin layer around the surface $\mathcal{S}$ it is possible to define the mapping

$$\mathbf{y}_m(\xi^1, \xi^2, \xi^3, t) = \mathbf{y}_{m\mathcal{S}}(\xi^1, \xi^2, t) + \xi^3 \mathbf{n}(\xi^1, \xi^2, t) \quad (3.13)$$

where $\mathbf{y} = \mathbf{y}_{m\mathcal{S}}(\xi^1, \xi^2, t) = \mathbf{y}_m(\xi^1, \xi^2, 0, t)$ describes the location of the points of $\mathcal{S}$ in the [MFR](#), while $\mathbf{n}(\xi^1, \xi^2, t)$ denotes the outward local unit normal to $\mathcal{S}$ at the time t .

This implies that the points of the surface are mapped onto a finite domain Ω in the (ξ^1, ξ^2) -plane and that, for $\xi^\alpha = \xi_m^\alpha(\mathbf{y}, t)$ representing the inverse of the mapping in Eq. (3.13), the argument of the Heaviside function used to define $\bar{\Psi}$ can be expressed as

$$f[\mathbf{x}(\mathbf{y}, t), t] = \xi_m^3(\mathbf{y}, t) \quad (3.14)$$

If the coordinate ξ^3 is chosen to be coincident with the arclength along the unit normal to $\mathcal{S}$ we have $\nabla f = \nabla \xi_m^3 = \mathbf{n}$. Thus, observing that $H = H[\xi_m^3(\mathbf{y}, t)]$, it follows that

$$\nabla H = \delta(\xi^3) \mathbf{n} \quad (3.15)$$

and

$$\dot{H} = \left. \frac{\partial H}{\partial t} \right|_y = \delta(\xi^3) \left. \frac{\partial \xi_m^3}{\partial t} \right|_y = -\delta(\xi^3) \mathbf{v}_y^m \cdot \mathbf{n} \quad (3.16)$$

with the latter obtained by observing that, for $\partial(\dots)/\partial t|_\xi$ denoting time derivative at a (ξ^1, ξ^2, ξ^3) -fixed point

$$\left. \frac{\partial \xi_m^3}{\partial t} \right|_\xi = \left. \frac{\partial \xi_m^3}{\partial t} \right|_y + \mathbf{v}_y^m \cdot \nabla \xi_m^3 = 0$$

where $\mathbf{v}_y^m$ denotes the velocity of a (ξ^1, ξ^2, ξ^3) -fixed point with respect to the [MFR](#).

In addition, regarding the Green function's derivatives, by introducing the function $g = t - t_* + \theta_y$, and noting that $\hat{G}(\mathbf{y}, \mathbf{y}_*, t, t_*) = \hat{G}_0(\mathbf{y}, \mathbf{y}_*) \delta[g(\mathbf{y}, \mathbf{y}_*, t, t_*)]$ (see Eq. (3.8)), we have

$$\nabla \hat{G} = \nabla \hat{G}_0 \delta(g) + \hat{G}_0 \frac{\partial \delta(g)}{\partial g} \nabla \theta_y \quad (3.17)$$

and

$$\dot{\hat{G}} = \hat{G}_0 \frac{\partial \delta(g)}{\partial g} \dot{g} = \hat{G}_0 \frac{\partial \delta(g)}{\partial g} \quad (3.18)$$

The observation of Eqs. (3.15), (3.16), (3.17) and (3.18) shows that in order to integrate the Dirac delta function contributions given by $\nabla H, \dot{H}, \nabla \hat{G}, \dot{\hat{G}}$ it is convenient to introduce the following transformation of the integration variables

$$(\mathbf{y}, t) \rightarrow (\xi^1, \xi^2, \xi^3, g)$$

which is such that

$$dV(\mathbf{y}) dt = \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \quad (3.19)$$

with $\hat{J}$ denoting the Jacobian of the transformation given by

$$\hat{J} = \frac{|\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|}{|1 + \mathbf{v}_y^m \cdot \nabla \theta_y|} \quad (3.20)$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the covariant basis vectors of the curvilinear coordinate system (ξ^1, ξ^2, ξ^3) (see appendix B). Thus, by considering the transformation of the variables of integration in Eq. (3.19), and combining the corresponding form of Eq. (3.12) with Eqs. (3.15), (3.16), (3.17), (3.18), (3.20) yields

$$\begin{aligned} \bar{\Psi}(\mathbf{y}_*, t_*) = & \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \sigma^\Psi \hat{G}_0 \delta(g) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{z} \cdot \mathbf{n} \hat{G}_0 \delta(\xi^3) \delta(g) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & - \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (\mathbf{Zn}) \cdot \left(\nabla \hat{G}_0 \delta(g) + \nabla \theta_y \hat{G}_0 \frac{\partial \delta(g)}{\partial g} \right) \delta(\xi^3) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & - \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_1 (\mathbf{v}_y^m + \mathbf{v}_y) \cdot \mathbf{n} \hat{G}_0 \delta(g) \delta(\xi^3) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & - \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_2 (\mathbf{v}_y^m + \mathbf{v}_y) \cdot \mathbf{n} \mathbf{v}_y \cdot \nabla \hat{G}_0 \delta(g) \delta(\xi^3) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & + \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} k_2 (\mathbf{v}_y^m + \mathbf{v}_y) \cdot \mathbf{n} \hat{G}_0 (1 - \mathbf{v}_y \cdot \nabla \theta_y) \frac{\partial \delta(g)}{\partial g} \delta(\xi^3) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \\ & - \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \mathbf{w} \cdot \mathbf{n} \left[\hat{G}_0 (1 - \mathbf{v}_y \cdot \nabla \theta_y) \frac{\partial \delta(g)}{\partial g} - \mathbf{v}_y \cdot \nabla \hat{G}_0 \delta(g) \right] \delta(\xi^3) \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \end{aligned} \quad (3.21)$$

Note that Eq. (3.17) introduces the derivative of the Dirac delta function, $\partial \delta(g)/\partial g$. To determine its contribution, let us observe that for an arbitrary space-time function, h , after integration by parts we have

$$\int_{-\infty}^{\infty} h \frac{\partial \delta(g)}{\partial g} \Big|_{\xi} dg = - \int_{-\infty}^{\infty} \frac{\partial h}{\partial t} \Big|_{\xi} \frac{\delta(g)}{(1 + \mathbf{v}_y^m \cdot \nabla \theta_y)} dg \quad (3.22)$$

since

$$\frac{\partial h}{\partial g} \Big|_{\xi} = \frac{\partial h}{\partial t} \Big|_{\xi} \left(\frac{\partial g}{\partial t} \Big|_{\xi} \right)^{-1}$$

and, considering that $\partial \theta_y / \partial t|_{\mathbf{y}} = 0$,

$$\frac{\partial g}{\partial t} \Big|_{\xi} = 1 + \frac{\partial \theta_y}{\partial t} \Big|_{\xi} = 1 + \mathbf{v}_y^m \cdot \nabla \theta_y$$

Now, using the the Dirac delta function's properties we obtain the following general solution of the inhomogeneous wave equation in a fluid domain bounded by deforming surfaces, written in an arbitrarily moving frame of reference (see also [9])

$$\begin{aligned}
 \bar{\Psi}(\mathbf{y}_*, t_*) = & \iint_{\Omega} (\mathbf{z} - k_1 \mathbf{v}) \cdot \mathbf{n} \hat{G}_0 \hat{J} \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\
 & - \iint_{\Omega} [\mathbf{Zn} + (k_2 \mathbf{v} - \mathbf{w}) \cdot \mathbf{n} \mathbf{v}_y] \cdot \nabla \hat{G}_0 \hat{J} \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\
 & + \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} [(\mathbf{Zn}) \cdot \nabla \theta_y \hat{G}_0 \hat{J}] \right) \frac{1}{\hat{\Theta}} \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\
 & - \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} [(k_2 \mathbf{v} - \mathbf{w}) \cdot \mathbf{n} (1 - \mathbf{v}_y \cdot \nabla \theta_y) \hat{G}_0 \hat{J}] \right) \frac{1}{\hat{\Theta}} \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\
 & + \iiint_{\mathcal{V}_{\Omega}} \hat{G}_0 \sigma^{\Psi} \hat{J} \Big|_{g=0} d\xi^1 d\xi^2 d\xi^3
 \end{aligned} \tag{3.23}$$

where $\hat{\Theta} = (1 + \mathbf{v}_y^m \cdot \nabla \theta_y)$, $\mathbf{v} = \mathbf{v}_y + \mathbf{v}_y^m$ denotes the velocity with respect to the FFR of a (ξ^1, ξ^2, ξ^3) -fixed point, $(\dots)|_{\xi^3=0}$ indicates evaluation on the surface $\mathcal{S}$ (see Eq. (3.13)), and $\mathcal{V}_{\Omega}$ denotes the (ξ^1, ξ^2, ξ^3) -space where the fluid region around the surface is mapped. In addition, $(\dots)|_{g=0}$ indicates evaluation at the retarded time $t = t_* - \theta$, which is the time of emission of the signal starting from a point identified by the coordinates (ξ^1, ξ^2, ξ^3) and reaching the observer at the time t_* . Note that, the time delay θ_{ξ} satisfies the following equation

$$\|\mathbf{x}_*(\mathbf{y}_*, t_*) - \mathbf{x}[\mathbf{y}(\xi^1, \xi^2, \xi^3, t_* - \theta), t_* - \theta]\| = c_0 \theta \tag{3.24}$$

and, thus, differs from θ_y (see Eq. (3.6)): the delay θ_y is related to signals emitted from points fixed in the MFR, whereas the delay θ is related to signals emitted from points that follow the deformation of the surface $\mathcal{S}$ as indicated in Eq. (3.13) (incidentally, they coincide in the case of a non-deforming surface $\mathcal{S}$ rigidly connected to the MFR). In Eq. (3.23), θ_y has to be interpreted as the transmission delay associated with the MFR position where the point $(\xi^1, \xi^2, 0)$ of the deforming surface is passing through at the time $t_* - \theta$.

The expression in Eq. (3.23) provides a description of the solution of the wave equation in terms of integrals evaluated over the deforming boundary surface, $\mathcal{S}$, and a field integral which is typically related to nonlinear sources (the last contribution in that expression). This boundary integral formulation is particularly efficient for computational applications when the field term is absent, negligible or known a priori.

When the boundary surface forcing terms (and, eventually, those in the neighbouring field) are known (e.g., as provided by a CFD solver), Eq. (3.23) can be used as an integral representation to evaluate the radiated solution at any point in the field. This is the case, for instance, of the typical solution approach applied to rotorcraft aeroacoustics problems [53, 61, 62].

However, for a large number of problems, the forcing terms might be unknown and directly defined in terms of the perturbation field, Ψ . In the latter case, the boundary integral formulation can be applied, first, as a boundary integral equation of Fredholm of the second kind [63] to determine the solution over the boundary surface (and, eventually, in the neighbouring field), and then as an

integral representation to radiate the perturbation field at any point in the field. This is the case, for instance, of sound scattering and potential aerodynamics problems [18, 19, 31, 34, 53].

3.3 Boundary integral formulation in the fluid frame of reference

The formulation in Eq. (3.23), although general, presents some issues related to the introduction of the MFR and the associated delay θ_y .

In many applications it is convenient to describe the integral formulation directly in the FFR. Its expression can be straightforwardly derived from Eq. (3.23) by observing that, in this case: (i) $\mathbf{v}_y = \mathbf{0}$ and, thus, $\mathbf{v} = \mathbf{v}_y^m$ where $\mathbf{v}_y^m$ represents the sum of deformation and rigid-body velocity of the boundary surface; (ii) the Green function spatial term, $\hat{G}_0$, becomes $G_0 = -1/(4\pi r)$; (iii) the delay θ_y becomes $\theta_x = r/c_0$, which is the time taken by a signal to reach the observer starting from a point fixed in the FFR, and thus, $\nabla\theta_x = -\mathbf{e}_r/c_0$, with $\mathbf{e}_r$ denoting the unit vector pointing from $\mathbf{x}$ to $\mathbf{x}_*$, as in Eq. (3.8).

Combining these observations with Eq. (3.23) yields the following boundary integral solution of the arbitrarily-forced wave equation in a fluid domain bounded by deforming surfaces written in a frame of reference fixed with the undisturbed fluid (see also [9])

$$\begin{aligned} \bar{\Psi}(\mathbf{x}_*, t_*) = & \iint_{\Omega} [(\mathbf{z} - k_1 \mathbf{v}) \cdot \mathbf{n} G_0 - (\mathbf{Zn}) \cdot \nabla G_0] J \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} \left[(\mathbf{Zn}) \cdot \frac{\mathbf{e}_r}{c_0} G_0 J \right] \right) \frac{1}{\Theta} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} [(k_2 \mathbf{v} - \mathbf{w}) \cdot \mathbf{n} G_0 J] \right) \frac{1}{\Theta} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \iiint_{\mathcal{V}_{\Omega}} G_0 \sigma^{\Psi} J \Big|_{g=0} d\xi^1 d\xi^2 d\xi^3 \end{aligned} \quad (3.25)$$

where $\Theta = (1 - \mathbf{v} \cdot \mathbf{e}_r/c_0)$, and $J = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|/|\Theta|$ is the Jacobian of the transformation of integration variables in the FFR, $(\mathbf{x}, t) \rightarrow (\xi^1, \xi^2, \xi^3, g)$. The meaning of $(\dots)|_{\xi^3=0}$ and $(\dots)|_{g=0}$ is the same as that in Eq. (3.23).

The boundary integral formulations expressed in the MFR and FFR are fully equivalent. The latter is more compact and its application is simpler, especially since the delay θ_y is replaced by θ_x (see their definitions). However, the use of the MFR formulation is convenient in the case of a moving rigid boundary surface, by assuming the MFR fixed to it (indeed, in this case, $\hat{\Theta} = 1$, $\hat{J} = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|$ is time-independent and the integration surface coincides with the real boundary surface, $\hat{J} d\xi^1 d\xi^2 = d\mathcal{S}$). This is particularly true for the analysis of the potential aerodynamics of rigid lifting wings and rotors in hovering or axial flight, with prescribed wake geometry. In fact, in these cases, in the MFR the wake appears as rigidly connected to the body, and the body and wake surfaces form a rigid fixed boundary (see, for instance, [64, 65]).

3.3.1 Expanded form of the boundary integral formulation in the FFR

A more readable expression of the boundary integral solution written in the FFR is obtained by explicitly showing the result of applying the spatial and temporal differential operators acting on some of the terms that appear in it (in particular those that do not involve the forcing terms of Eq. (2.10)).

Indeed, considering the mathematical developments presented in appendix C, and neglecting the field term in which no differential operators to be explicitly applied appear, the integral formulation of Eq. (3.25) can be recast in the following form (see also [9])

$$\bar{\Psi}(\mathbf{x}_*, t_*) = I_{\mathbf{Z}}(\mathbf{x}_*, t_*) + I_{\mathbf{Z}}(\mathbf{x}_*, t_*) + I_{k_1}(\mathbf{x}_*, t_*) + I_{k_2}(\mathbf{x}_*, t_*) + I_{\mathbf{W}}(\mathbf{x}_*, t_*) \quad (3.26)$$

where

$$4\pi I_{\mathbf{Z}} = - \iint_{\Omega} \mathbf{z} \cdot \mathbf{n} \frac{J_s}{r(1-M_r)} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (3.27)$$

$$\begin{aligned} 4\pi I_{\mathbf{Z}} = & \frac{1}{c_0} \iint_{\Omega} (\mathbf{Zn}) \cdot \left(\frac{(c_0 M_r - c_0 M^2 + r \dot{\mathbf{M}} \cdot \mathbf{e}_r) \mathbf{e}_r J_s}{r^2(1-M_r)^3} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \frac{1}{c_0} \iint_{\Omega} (\mathbf{Zn}) \cdot \left(\frac{c_0 (\mathbf{e}_r - \mathbf{M}) J_s}{r^2(1-M_r)^2} + \frac{\mathbf{e}_r \dot{J}_s}{r(1-M_r)^2} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \frac{1}{c_0} \iint_{\Omega} (\dot{\mathbf{Zn}} + \mathbf{Z}\dot{\mathbf{n}}) \cdot \left(\frac{\mathbf{e}_r J_s}{r(1-M_r)^2} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \end{aligned} \quad (3.28)$$

$$4\pi I_{k_1} = \iint_{\Omega} k_1 \frac{c_0 M_n J_s}{r(1-M_r)} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (3.29)$$

$$\begin{aligned} 4\pi I_{k_2} = & \iint_{\Omega} k_2 \left(\frac{c_0 M_n (c_0 M_r - c_0 M^2 + r \dot{\mathbf{M}} \cdot \mathbf{e}_r) J_s}{r^2(1-M_r)^3} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \iint_{\Omega} k_2 \left(\frac{c_0 \dot{M}_n J_s}{r(1-M_r)^2} + \frac{c_0 M_n \dot{J}_s}{r(1-M_r)^2} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \iint_{\Omega} \dot{k}_2 \frac{c_0 M_n J_s}{r(1-M_r)^2} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \end{aligned} \quad (3.30)$$

$$\begin{aligned} 4\pi I_{\mathbf{W}} = & - \iint_{\Omega} \mathbf{w} \cdot \mathbf{n} \left(\frac{(c_0 M_r - c_0 M^2 + r \dot{\mathbf{M}} \cdot \mathbf{e}_r) J_s}{r^2(1-M_r)^3} + \frac{\dot{J}_s}{r(1-M_r)^2} \right) \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} (\dot{\mathbf{w}} \cdot \mathbf{n} + \mathbf{w} \cdot \dot{\mathbf{n}}) \frac{J_s}{r(1-M_r)^2} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \end{aligned} \quad (3.31)$$

The meaning of the symbols introduced in the above equations is explained in C and is not repeated here for the sake of brevity.

The combination of Eq. (3.26) with Eqs. (3.27)-(3.31) expresses the proposed deformable-

boundary integral formulation through a mathematical formalism typically used in aeroacoustics (see, for instance, the Farassat 1A formulation for radiation of thickness and loading noise [14], which is a special case of this combination obtained for $\mathbf{z} = \mathbf{0}$, $\mathbf{Z} = \mathbf{P}$, $k_1 = k_2 = 0$, and $\mathbf{w} = -\rho_0 \mathbf{v}$).

By following a procedure similar to that used to determine Eqs. (3.27)-(3.31), it is also possible to obtain the expansion of the deformable-boundary integral formulation expressed in MFR and given in Eq. (3.23). For reasons of brevity, this is not done here. Appendix D shows the steps needed to obtain the extended formulation in MFR. Some of these steps are shown also in [66], where the expansion is applied to the boundary integral formulation valid for rigid surfaces.

3.4 Boundary integral formulation for multiple boundary surfaces

The deformable-boundary integral formulations in Eqs. (3.23) and (3.25) may be extended to the case of N_Ω disjoint surfaces, each independently deforming and subject to arbitrary motion. In this case, for $\mathcal{D}_n$ denoting the domain enclosed by the surface $\mathcal{S}_n$, the Heaviside function, H , introduced in chapter 2 is conveniently defined as the product of N_Ω Heaviside functions, H_n , such that

$$H(\mathbf{x}) = \prod_{n=1}^{N_\Omega} H_n[f_n(\mathbf{x})]$$

where $f_n = 0$ if $\mathbf{x} \in \mathcal{S}_n$, $f_n > 0$ if $\mathbf{x} \in \mathbb{R}^3 \setminus \mathcal{D}_n$, $f_n < 0$ if $\mathbf{x} \in \mathcal{D}_n$.

The assumption of disjoint surfaces implies

$$\nabla H = \sum_{n=1}^{N_\Omega} \nabla H_n; \quad \frac{\partial H}{\partial t} = \sum_{n=1}^{N_\Omega} \frac{\partial H_n}{\partial t}$$

and, therefore, the combination of these relations with Eq. 2.10 yields the following form of the solution

$$\bar{\Psi}(\mathbf{x}_*, t_*) = \sum_{n=1}^{N_\Omega} \mathcal{I}_n(\mathbf{x}_*, t_*) \quad (3.32)$$

where $\mathcal{I}_n$ denotes the integral contribution from the n -th surface (each expressed in the most convenient, moving or fixed, frame of reference by using Eq. (3.23) or Eq. (3.25) or their expanded forms).

Chapter 4

Noise radiated by solid and porous deforming surfaces

This chapter presents the methodology used for the numerical application of the boundary integral formulation developed in the previous chapter, aimed at the prediction of the acoustic radiation in fluid domains bounded by solid or porous surfaces subject to deformation.

After introducing the spatial and temporal discretisation procedures necessary for the numerical application of the deformable-boundary integral formulation, the results of two validation cases are proposed, extending those discussed in [9]. Both of them regard the application of the boundary integral formulation as a boundary integral representation of the solution for which the surface source terms are known a priori.

The first test case concerns the evaluation of the perturbation signal outside a deforming porous surface radiated by and enclosed pulsating source, while the second test case considers the acoustic field generated by a deforming wing in translational motion, assuming the aerodynamic solution over its surface as known. In the former case the numerical solution from the present formulation is validated against the analytical one, while in the latter case the objective is to demonstrate the perfect equivalence between [FFR](#) and [MFR](#) formulations.

Next, the attention is focused on the validation of the proposed boundary integral formulation when applied as a boundary integral equation over deforming surfaces. Still considering a deforming porous surface enclosing a pulsating source, the corresponding external acoustic pressure field is determined by first evaluating the acoustic pressure over the surface by application of the boundary integral equation relating to the solution of the [FWHE](#) (with only the kinematic terms assumed to be known on the surface, see Eq. (2.7)), and then radiating it through the boundary integral representation of the solution. The numerical solution from the present formulation is validated against the analytical one directly obtained from the expression of the emitting source.

Then, the attention is turned to the problem of acoustic radiation from a solid sphere subject to large amplitude pulsations. Also in this problem, the solution (in this case expressed in terms of the velocity potential) is determined by, first, applying the boundary integral formulation as a boundary integral equation to evaluate it over the emitting surface, and then as a boundary integral representation to radiate it. This problem, originally discussed in [67], is here further developed and extended. The numerical predictions of the noise emitted by the pulsating solid sphere are validated against analytical solutions which, depending on the amplitude and frequency of the

sphere's pulsations, are partly available in the literature and partly developed originally in this doctoral work. In particular, the validation demonstrates the ability of the proposed formulation to capture the multi-harmonic content of the radiated signal due to kinematic nonlinearities.

Finally, the presented deformable-boundary integral formulation is applied to a realistic case of aeronautical interest, consisting of the noise emitted by a helicopter rotor in forward flight, for which experimental data are available in the literature, extending the analysis presented in [9].

Overall, this chapter demonstrates the ability of the presented boundary integral formulation for deforming bodies to successfully address complex acoustic radiation problems, validating its accuracy both on simple configurations for which the analytical solutions are available and on application cases of engineering relevance.

4.1 Numerical application of the boundary integral formulations

To numerically determine the perturbation field, whether expressed in terms of velocity potential or acoustic pressure, the integral formulations introduced above are rewritten in discretized form using a BEM approach. Indeed, through appropriate space and time discretisation the integral formulations are transformed into algebraic relations yielding the solution at arbitrary points in the domain of interest.

For the purposes of this chapter, the contribution relating to the generic nonlinear forcing term σ^Ψ will be neglected. It will be resumed and dealt with in detail in the second part of chapter 5.

The numerical method used to solve a boundary-integral equation problem for noise radiation consists of the following steps (this is, for instance, the case of sound radiated by a pulsating surface described in terms of velocity potential):

1. **Discretisation of the surface:** the surface of the boundary (or boundaries) is divided into a finite set of elements, generally polygonal, called panels, whose vertices lie on the original surface. This discretisation allows the integration domain to be represented approximately.
2. **Approximation of variables:** the unknown functions under the integrals (e.g. potential or acoustic pressure) are approximated as linear combinations of shape functions suitably defined on each panel. The shape functions can be constant, linear, or of higher order, depending on the required accuracy.
3. **Imposition of the integral equation:** the integral equation must be satisfied exactly at a finite set of points on the surface, called collocation points. This procedure, known as the collocation method, allows the continuous integral equation to be transformed into a linear algebraic system, whose unknowns are the coefficients of the linear combinations of the shape functions.
4. **Solving the algebraic system:** next a suitable time discretization is applied (depending on the nature of the problem under consideration, see later), and an this yield an algebraic system of equations. This system is finally solved to determine the instantaneous coefficients of the shape functions, from which the physical quantities of interest (potential or pressure) are obtained on the boundary.

5. **Solution in the field:** once the physical quantities of interest over the boundary are determined from the solution of the algebraic system, the discretised form of the integral formulation interpreted as an integral representation is applied to evaluate the radiated signal anywhere in the field.

When the quantities of interest over the boundary are known a priori like, for instance, in aeroacoustic radiation problems following a near-field aerodynamic solution, points 3 and 4 of the above numerical solution process are missing. This kind of approach is typically used in rotorcraft aeroacoustic applications.

It is worth noticing that this procedure of solution based on the boundary element method allows for effective treatment of scattering and radiation problems in external domains. Indeed, it requires the discretization of a finite boundary domain, with evident computational advantages with respect to those methods needing the discretization of the domain of propagation of the signal perturbations.

4.1.1 Space discretisation of BEM

Defining N_p as the number of panels used to discretise the surface, $\mathcal{S}$, of the boundary, in the case of a boundary-integral equation problem the generic unknown perturbation field, Ψ , is expressed as a linear combination of shape functions, $\gamma_n(\mathbf{x})$,

$$\Psi(\mathbf{x}, t) = \sum_{n=1}^{N_p} \Psi_n(t) \gamma_n(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}$$

The shape functions are defined so that their value is equal to one on the surface of the n -th panel, $\mathcal{S}_n$, and zero elsewhere, as follows (zero-th order BEM)

$$\begin{cases} \gamma_n = 1 & \mathbf{x} \in \mathcal{S}_n \\ \gamma_n = 0 & \mathbf{x} \notin \mathcal{S}_n \end{cases}$$

The $\Psi_n(t)$ time-variant coefficients of the linear combination, which play the role of problem unknowns, are assumed to be equal, at each time instant, to the value of the Ψ function at the centroid of the n -th panel, $\mathbf{x}_n^c$, namely

$$\Psi_n(t) = \Psi(\mathbf{x}_n^c, t)$$

In this way, a constant distribution of the unknown function is considered on each panel, equal to the value assumed at the centroid (zero-order boundary element method). The same type of discretisation is also applied to all forcing terms appearing in the specific problem under examination,

$$\begin{aligned} \mathbf{Z}(\mathbf{x}, t) &= \sum_{n=1}^{N_p} \mathbf{Z}_n(t) \gamma_n(\mathbf{x}) & \mathbf{z}(\mathbf{x}, t) &= \sum_{n=1}^{N_p} \mathbf{z}_n(t) \gamma_n(\mathbf{x}) & \mathbf{w}(\mathbf{x}, t) &= \sum_{n=1}^{N_p} \mathbf{w}_n(t) \gamma_n(\mathbf{x}) \\ k_1(\mathbf{x}, t) &= \sum_{n=1}^{N_p} k_{1n}(t) \gamma_n(\mathbf{x}) & k_2(\mathbf{x}, t) &= \sum_{n=1}^{N_p} k_{2n}(t) \gamma_n(\mathbf{x}) \end{aligned}$$

some of which are necessarily defined in terms of the unknown perturbation field when a boundary-integral equation problem is considered.

From the above space discretization of the problem, at each time t_* it is possible to define the following set of N_p equations for the N_p unknown functions $\Psi_n(t)$ by imposing that the integral equation be satisfied at each centroid of the discretization boundary elements, with $k = 1, \dots, N_p$

$$\begin{aligned} \lambda_k \Psi_k(t_*) &= \sum_{n=1}^{N_p} \mathbf{A}_{kn}(t_*) \cdot \mathbf{Z}_n(t_* - \theta_{kn}) + \sum_{n=1}^{N_p} \mathbf{A}_{kn}^d(t_*) \cdot \dot{\mathbf{Z}}_n(t_* - \theta_{kn}) \\ &+ \sum_{n=1}^{N_p} \mathbf{B}_{kn}(t_*) \cdot \mathbf{z}_n(t_* - \theta_{kn}) + \sum_{n=1}^{N_p} C_{kn}(t_*) k_{1n}(t_* - \theta_{kn}) \\ &+ \sum_{n=1}^{N_p} \mathbf{D}_{kn}(t_*) \cdot \mathbf{w}_n(t_* - \theta_{kn}) + \sum_{n=1}^{N_p} \mathbf{D}_{kn}^d(t_*) \cdot \dot{\mathbf{w}}_n(t_* - \theta_{kn}) \\ &+ \sum_{n=1}^{N_p} E_{kn}(t_*) k_{2n}(t_* - \theta_{kn}) + \sum_{n=1}^{N_p} E_{kn}^d(t_*) \dot{k}_{2n}(t_* - \theta_{kn}) \end{aligned} \quad (4.1)$$

where θ_{kn} denotes the signal transmission delay between the centroids of the k -th receiving panel and the n -th transmitting panel, while the expression of the coefficients depends on the applied formulation (namely, the [FFR](#) or the [MFR](#) one).

If we consider the [FFR](#) formulation, and denote the (ξ^1, ξ^2) -domain where the surface of the n -th panel is mapped as Ω_n , the coefficients $\mathbf{A}_{kn}$ and $\mathbf{A}_{kn}^d$ are second-order tensors given by the following expression

$$\begin{aligned} \mathbf{A}_{kn}(t_*) &= - \iint_{\Omega_n} \left[\mathbf{n} \otimes \nabla G_0 J + \frac{\partial}{\partial t} \Big|_{\xi} \left(\frac{\mathbf{n} \otimes \mathbf{e}_r}{c_0} G_0 J \right) \frac{1}{\Theta} \right]_{\theta_{kn}} d\xi^1 d\xi^2 \\ \mathbf{A}_{kn}^d(t_*) &= - \iint_{\Omega_n} \frac{\mathbf{n} \otimes \mathbf{e}_r}{c_0} G_0 \frac{J}{\Theta} \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \end{aligned}$$

and their scalar product with the tensors $\mathbf{Z}_n$ and $\dot{\mathbf{Z}}_n$ is defined in such a way that, for generic tensors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$. In addition, the coefficients $\mathbf{B}_{kn}$, $\mathbf{D}_{kn}$ and $\mathbf{D}_{kn}^d$ are vectors defined as

$$\begin{aligned} \mathbf{B}_{kn}(t_*) &= \iint_{\Omega_n} \mathbf{n} G_0 J \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \\ \mathbf{D}_{kn}(t_*) &= \iint_{\Omega_n} \frac{\partial}{\partial t} \Big|_{\xi} (\mathbf{n} G_0 J) \frac{1}{\Theta} \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \\ \mathbf{D}_{kn}^d(t_*) &= \iint_{\Omega_n} \mathbf{n} G_0 \frac{J}{\Theta} \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \end{aligned}$$

while the scalar coefficients E_{kn} and E_{kn}^d are expressed as

$$\begin{aligned} E_{kn}(t_*) &= - \iint_{\Omega_n} \frac{\partial}{\partial t} \Big|_{\xi} (\mathbf{v} \cdot \mathbf{n} G_0 J) \frac{1}{\Theta} \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \\ E_{kn}^d(t_*) &= - \iint_{\Omega_n} \mathbf{v} \cdot \mathbf{n} G_0 \frac{J}{\Theta} \Big|_{\theta_{kn}} d\xi^1 d\xi^2 \end{aligned}$$

In Eq. (4.1), the coefficient λ_k derives from the free term arising from the regularisation of

singular integrals that may appear when $\mathbf{x} \rightarrow \mathbf{x}_*$ in the solution procedure of a boundary integral equation ($\lambda_k = 1$ in the case of a boundary integral representation). In particular, it can be demonstrated that in the case of boundary integral equation problems solving a velocity potential field $\lambda_k = 1/2$, while $\lambda_k = 0.5/(1-M_n^2)$ in the case of boundary integral equation problems described in terms of acoustic pressure (see, for details, [18, 68]).

4.1.2 Time discretisation

Spatial discretisation of the surface alone is not sufficient for the numerical implementation of integral formulations applied to solve the wave equation: temporal discretisation must also be introduced. In other words, the evolution of the perturbation field must also be described in a discrete set of time instants in order to make the problem computable in the time domain.

A peculiar feature of integral formulations for propagation problems in compressible fluids is that the integrands must be evaluated at a delayed time, i.e. at time instants preceding the observation time, with delay depending on the distance between the source and the observation point and relative motion. Of course, the delayed time associated with each couple of emitting and receiving points is not generally a multiple of the time interval Δt used to define the discrete time instants of evaluation of the solution.

It follows that in order to calculate the delayed contributions to the integrals, it is necessary to interpolate the values of the physical quantities involved (such as, for instance, potential or acoustic pressure) from the knowledge of their values at the defined discrete time instants of solution. Thus a temporal interpolation strategy is introduced, capable of providing the physical quantities considered and their time derivatives at arbitrary time instants.

To this aim, first each propagation delay is represented in terms of the introduced time discretization interval as

$$\theta_{kn} = (m_{kn} + \alpha_{kn})\Delta t$$

with m_{kn} being an integer number and $0 < \alpha_{kn} < 1$. Assuming the discrete observation time as $t_* = n\Delta t$, a generic function at time $f(t_* - \theta_{kn})$ is re-defined as $f[(n - m_{kn} - \alpha_{kn})\Delta t]$, and its Taylor series expansion around $f[(n - m_{kn})\Delta t]$ yields

$$f[(n - m_{kn} - \alpha_{kn})\Delta t] = f[(n - m_{kn})\Delta t] + \left. \frac{\partial f}{\partial t} \right|_{(n-m_{kn})\Delta t} (-\alpha_{kn}\Delta t)$$

By defining the backward finite difference time derivative

$$\left. \frac{\partial f}{\partial t} \right|_{(n-m_{kn})\Delta t} = \frac{f[(n - m_{kn})\Delta t] - f[(n - m_{kn} - 1)\Delta t]}{\Delta t} + \mathcal{O}(\Delta t^2)$$

and combining with the above expression gives

$$f[(n - m_{kn} - \alpha_{kn})\Delta t] = (1 - \alpha_{kn}) f[(n - m_{kn})\Delta t] + \alpha_{kn} f[(n - m_{kn} - 1)\Delta t] \quad (4.2)$$

Next, let us consider the discretisation of the derivative of a delayed function $\dot{f}(t_* - \theta_{kn}) =$

$\dot{f}[(n - m_{kn} - \alpha_{kn})\Delta t]$, and apply again the backward finite difference derivative to define it

$$\dot{f}[(n - m_{kn} - \alpha_{kn})\Delta t] = \frac{f[(n - m_{kn} - \alpha_{kn})\Delta t] - f[(n - m_{kn} - 1 - \alpha_{kn})\Delta t]}{\Delta t} + \mathcal{O}(\Delta t^2)$$

Substituting Eq. (4.2) into this expression and observing that we similarly obtain

$$f[(n - m_{kn} - 1 - \alpha_{kn})\Delta t] = (1 - \alpha_{kn}) f[(n - m_{kn} - 1)\Delta t] + \alpha_{kn} f[(n - m_{kn} - 2)\Delta t]$$

the discretised form of the delayed time derivative can be expressed in the following way

$$\begin{aligned} \dot{f}[(n - m_{kn} - \alpha_{kn})\Delta t] = & \frac{1}{\Delta t} \left[(1 - \alpha_{kn}) f[(n - m_{kn})\Delta t] + (2\alpha_{kn} - 1) f[(n - m_{kn} - 1)\Delta t] \right. \\ & \left. - \alpha_{kn} f[(n - m_{kn} - 2)\Delta t] \right] \end{aligned} \quad (4.3)$$

Finally, the application of the discretised expressions of arbitrary delayed function and its time derivative given in Eqs. (4.2) and (4.3) in Eq. (4.1) provides the following set of algebraic equations with unknown the set of current values of the physical quantity to be determined, $\Psi_k(n\Delta t)$, which explicitly appear at the left-hand side and implicitly appear in the definition of some of the forcing terms at the right-hand side (where many contributions are given by terms known from the previous time history of solution).

$$\begin{aligned} \lambda_k \Psi_k(n\Delta t) = & \sum_{n=1}^{N_p} \mathbf{A}_{kn}^{(1)}(n\Delta t) \cdot \mathbf{Z}_n[(n - m_{kn})\Delta t] + \sum_{n=1}^{N_p} \mathbf{A}_{kn}^{(2)}(n\Delta t) \cdot \mathbf{Z}_n[(n - m_{kn} - 1)\Delta t] \\ & + \sum_{n=1}^{N_p} \mathbf{A}_{kn}^{(3)}(n\Delta t) \cdot \mathbf{Z}_n[(n - m_{kn} - 2)\Delta t] \\ & + \sum_{n=1}^{N_p} \mathbf{B}_{kn}^{(1)}(n\Delta t) \cdot \mathbf{z}_n[(n - m_{kn})\Delta t] + \sum_{n=1}^{N_p} \mathbf{B}_{kn}^{(2)}(n\Delta t) \cdot \mathbf{z}_n[(n - m_{kn} - 1)\Delta t] \\ & + \sum_{n=1}^{N_p} C_{kn}^{(1)}(n\Delta t) k_{1n}[(n - m_{kn})\Delta t] + \sum_{n=1}^{N_p} C_{kn}^{(2)}(n\Delta t) k_{1n}[(n - m_{kn} - 1)\Delta t] \\ & + \sum_{n=1}^{N_p} \mathbf{D}_{kn}^{(1)}(n\Delta t) \cdot \mathbf{w}_n[(n - m_{kn})\Delta t] + \sum_{n=1}^{N_p} \mathbf{D}_{kn}^{(2)}(n\Delta t) \cdot \mathbf{w}_n[(n - m_{kn} - 1)\Delta t] \\ & + \sum_{n=1}^{N_p} \mathbf{D}_{kn}^{(3)}(n\Delta t) \cdot \mathbf{w}_n[(n - m_{kn} - 2)\Delta t] \\ & + \sum_{n=1}^{N_p} E_{kn}^{(1)}(n\Delta t) k_{2n}[(n - m_{kn})\Delta t] + \sum_{n=1}^{N_p} E_{kn}^{(2)}(n\Delta t) k_{2n}[(n - m_{kn} - 1)\Delta t] \\ & + \sum_{n=1}^{N_p} E_{kn}^{(3)}(n\Delta t) k_{2n}[(n - m_{kn} - 2)\Delta t] \end{aligned}$$

In the above equation, the coefficients $\mathbf{A}_{kn}^{(i)}$, for $i = 1, 2, 3$, are given by the following expressions

$$\mathbf{A}_{kn}^{(1)} = (1 - \alpha_{kn}) \left(\mathbf{A}_{kn} + \frac{\mathbf{A}_{kn}^d}{\Delta t} \right) \quad \mathbf{A}_{kn}^{(2)} = \alpha_{kn} \mathbf{A}_{kn} + (2\alpha_{kn} - 1) \frac{\mathbf{A}_{kn}^d}{\Delta t} \quad \mathbf{A}_{kn}^{(3)} = -\alpha_{kn} \frac{\mathbf{A}_{kn}^d}{\Delta t}$$

and the same expressions define also the coefficients $\mathbf{D}_{kn}^{(i)}$ and $E_{kn}^{(i)}$ by replacing, respectively, $\mathbf{A}_{kn}$ with $\mathbf{D}_{kn}$ and E_{kn} , and $\mathbf{A}_{kn}^d$ with $\mathbf{D}_{kn}^d$ and E_{kn}^d . The coefficients $\mathbf{B}_{kn}^{(i)}$, for $i = 1, 2$, are defined as follows

$$\mathbf{B}_{kn}^{(1)} = (1 - \alpha_{kn})\mathbf{B}_{kn} \quad \mathbf{B}_{kn}^{(2)} = \alpha_{kn}\mathbf{B}_{kn}$$

while the coefficient $C_{kn}^{(i)}$ are given by the same expressions by replacing $\mathbf{B}_{kn}$ with C_{kn} .

4.2 Deformable porous sphere enclosing a pulsating source: validation of boundary integral representation

As first case of validation of the deformable-boundary integral formulation presented in this work, let us consider an inviscid fluid with a pulsating velocity potential point source moving in rectilinear uniform motion, located at the center of a co-moving translating porous (fictitious) sphere of radius r_0 .

In a translating MFR with the origin fixed to the source, the velocity potential field radiated by the source is analytically expressed as

$$\varphi_s(\mathbf{y}, t) = \frac{-1}{4\pi r_\beta} \sin[\omega_s(t - \theta_y)] \quad (4.4)$$

where $r_\beta = [(\mathbf{M}_y \cdot \mathbf{y})^2 + \beta^2 |\mathbf{y}|^2]^{1/2}$ with $\beta = (1 - \mathbf{M}_y \cdot \mathbf{M}_y)^{1/2}$, the transmission delay is given by the expression $\theta_y = (r_\beta + \mathbf{M}_y \cdot \mathbf{y})/(c_0 \beta^2)$, while ω_s represents the signal frequency (see Eq. (3.8) and, for instance, [30]).

Thus, the validation test consists of: (i) to deform dynamically the porous surface enclosing the source; (ii) to determine the potential field external to the deforming porous surface through the presented boundary integral formulation applied over that surface, where the flow parameters are assumed to be known from Eq. (4.4); (ii) to compare the velocity potential field obtained numerically with that directly provided by the analytical expression of Eq. (4.4).

Introducing the unit basis vectors, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, of a Cartesian system of coordinates, (y_1, y_2, y_3) , fixed to the MFR, let us assume $\mathbf{M}_y = -0.3\mathbf{e}_1$, $\omega_s = 10$ rad/s, $r_0 = 1$ m, and the points of the deforming sphere, $(y_1^d, y_2^d, y_3^d) = (x_d, y_d, z_d)$, given by the following expression

$$z_d(t) = \sqrt{r_0^2 - (x_d^2 + y_d^2)} + A y_d^2 \cos(\omega_d t), \quad (x_d, y_d) \in \mathcal{D}_c \quad (4.5)$$

where $\mathcal{D}_c$ represents the circular disk of radius r_0 lying on the plane (y_1, y_2) , centered at the origin of the coordinate system, while A and ω_d denote, respectively, the amplitude and frequency of deformation of the sphere.

For arbitrary values of amplitude and frequency of deformation of the sphere, $A = r_0/2$ and $\omega_d = 4$ rad/s, Fig. 4.1a shows the comparison between the velocity potential evaluated at the observer location $(y_1^o, y_2^o, y_3^o) = (-150r_0, 10r_0, 0)$ through Eq. (4.4), and that predicted by the present FFR deformable-boundary integral formulation applied to the deforming porous sphere surface which solves the wave equation for the velocity potential, Eq. (2.9). Similarly, for the same problem and still considering the parameters over the porous surface to be known from Eq. (4.4), Fig. 4.1b presents the comparison between the acoustic pressure determined analytically from Eq.

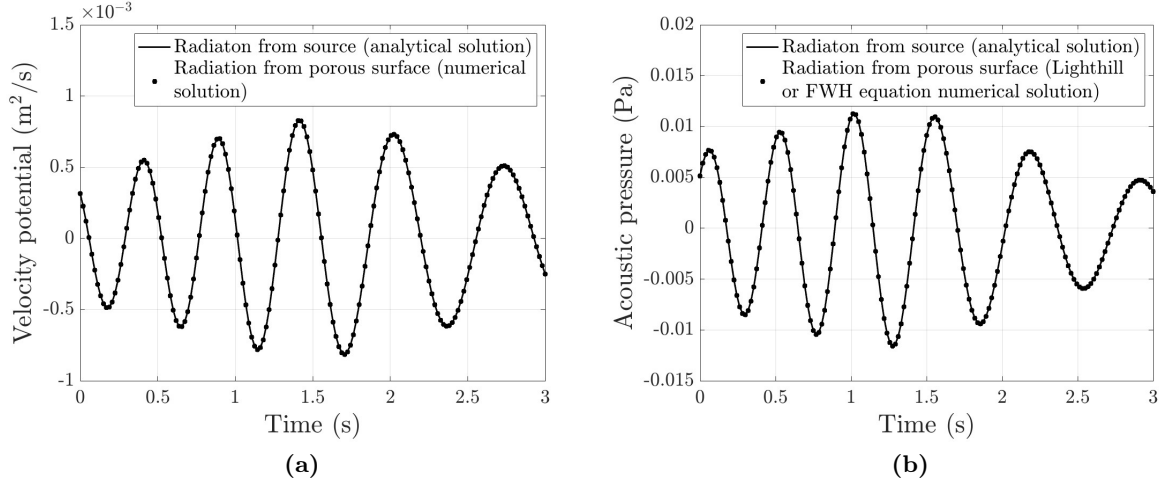


Figure 4.1: Signals emitted by a moving, pulsating source. (a): Velocity potential; (b): Acoustic pressure.

(4.4) combined with Bernoulli's theorem and that provided by the deformable-boundary integral formulations applied in the FFR over the deforming sphere to solve the Lighthill equation, Eq. (2.4). Notice that, an identical acoustic pressure signature is obtained by the deformable-boundary integral formulations applied to the Ffowcs-Williams and Hawkings equation for porous surfaces, Eq. (2.7).

Thus, these figures demonstrate that the proposed boundary integral formulation perfectly captures the radiated field from the data given over an arbitrarily deforming surface, both by solving velocity potential and acoustic pressure problems.

This is confirmed by Fig. 4.2, which depicts the comparison between the analytical acoustic pressure and that given by the deformable-boundary integral formulation applied over the deforming sphere to solve the Ffowcs-Williams and Hawkings equation (or the Lighthill equation) for porous surfaces, in the case of the presence of an additional source pulsating with a frequency equal to $\omega_\varphi/2$, also located at the origin of the coordinate system.

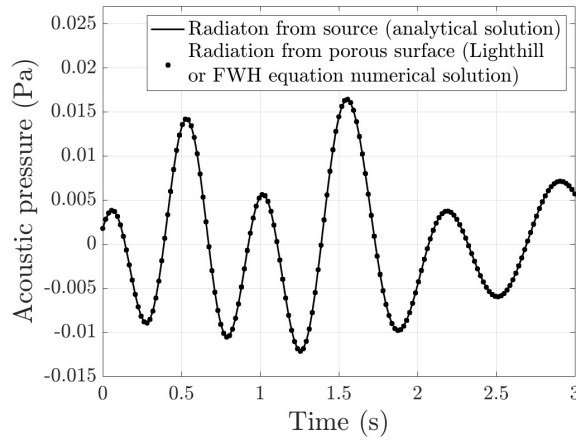


Figure 4.2: Acoustic pressure emitted by a pair of moving sources with different frequencies of pulsation.

Finally, it is worth noting that the same excellent quality of predictions is obtained by considering different deformations of the sphere, and different locations of the second pulsating source. This set

of results can be considered a validation of the proposed formulation for applications on deforming porous surfaces.

4.3 Non-circulatory flow around deforming wing

The next step of the validation process consists of proving numerically the equivalence of the deformable-boundary integral formulations expressed in the [MFR](#), Eq. (3.23), and in the [FFR](#), Eq. (3.25).

This is accomplished by examining the acoustic field radiated by a non-lifting flapping and twisting wing in uniform translation (the fluid is assumed to be perfect, the flow is assumed to remain non-circulatory and no wake is generated at the trailing edge). Specifically, we evaluate the acoustic pressure and velocity potential fields observed at a point fixed in the [FFR](#) during the wing flyover by applying, respectively, the integral solutions of Eqs. (2.6) and (2.9) both in the [FFR](#) and in the [MFR](#) translating with the wing. Note that, the signal radiation is based on the a-priori knowledge of the flow parameters over the boundary surface. These are approximately assumed to be equal to those determined through a well-assessed rigid-boundary integral formulation for the velocity potential, in which the wing deformation is only taken into account through the boundary condition of body impermeability [30].

The examined wing is rectangular, with NACA 0012 airfoil shaped cross sections, chord length equal to $c = 1$ m and span length equal to $b = 10$ m. It is assumed to fly along the x_1 -axis of a Cartesian coordinate system fixed to the [FFR](#), with the (x_1, x_3) -plane coinciding with the vertical plane of symmetry of the wing and the x_2 -axis directed starboard.

In a Cartesian coordinate system, (y_1, y_2, y_3) , fixed to the [MFR](#) and parallel to the system of coordinates (x_1, x_2, x_3) , wing bending and torsion are described by the following field of displacements

$$\mathbf{y}_d(\mathbf{y}, t) = q_b(t)\boldsymbol{\psi}_b(\mathbf{y}) + q_t(t)\boldsymbol{\psi}_t(\mathbf{y}), \quad \mathbf{y} \in \mathcal{S}_w^u$$

where $\mathcal{S}_w^u$ is the surface of the undeformed wing, $q_b(t)$ and $q_t(t)$ represent Lagrangian coordinates, while $\boldsymbol{\psi}_b$ and $\boldsymbol{\psi}_t$ are bending and torsion shape functions, respectively.

For harmonic bending and torsion wing deformations such that

$$q_b(t) = A \cos(\omega_b t); \quad q_t(t) = B \cos(\omega_t t)$$

and

$$\boldsymbol{\psi}_b(\mathbf{y}) = \begin{Bmatrix} 0 \\ 0 \\ \left(\frac{y_2}{b}\right)^2 \end{Bmatrix}; \quad \boldsymbol{\psi}_t(\mathbf{y}) = \begin{Bmatrix} y_3 \sin\left(\pi \frac{|y_2|}{b}\right) \\ 0 \\ -y_1 \sin\left(\pi \frac{|y_2|}{b}\right) \end{Bmatrix}$$

Fig. 4.3a and Fig. 4.3b present, respectively, the velocity potential and the acoustic pressure at the observer position $(x_1, x_2, x_3) = (0, -b/5, b/10)$ determined by the deformable-boundary integral formulations expressed in the [FFR](#) and [MFR](#) while the wing moves from $x_1 = -10$ m to $x_1 = 10$ m, flying at three different Mach numbers, M , with $A = 0.5$ m, $B = 0.2$ rad, $\omega_b = \pi M c_0 / 5$ rad/s, $\omega_t = 3\pi M c_0 / 10$ rad/s (i.e., such that two bending oscillations and three torsion oscillations occur during the windowed wing trajectory). These results prove the equivalence between the proposed

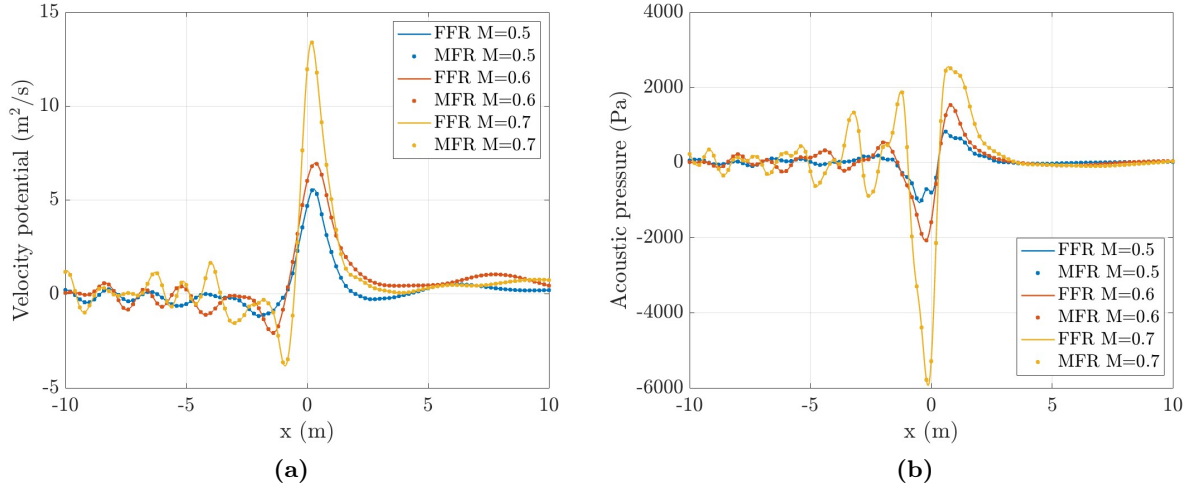


Figure 4.3: Velocity potential and acoustic pressure radiated by a translating wing subject to bending and torsion deformations. (a): Velocity potential; (b): Acoustic pressure.

integral formulations expressed in FFR and MFR, regardless of the strength of the compressibility effects and the perturbation variable that is considered to describe the propagation phenomenon, thus providing further validation of their correctness.

Next, to highlight the importance of describing the propagation phenomena through a formulation that takes into account boundary deformations (if present), Fig. 4.4a and Fig. 4.4b show the comparison between the predictions of the radiated acoustic pressure obtained by the formulation in chapter 3 with those obtained under the assumption of undeformed boundary, at two Mach numbers. It can be observed that the effects due to boundary deformation are not negligible and in particular, they become significant for the higher Mach value.

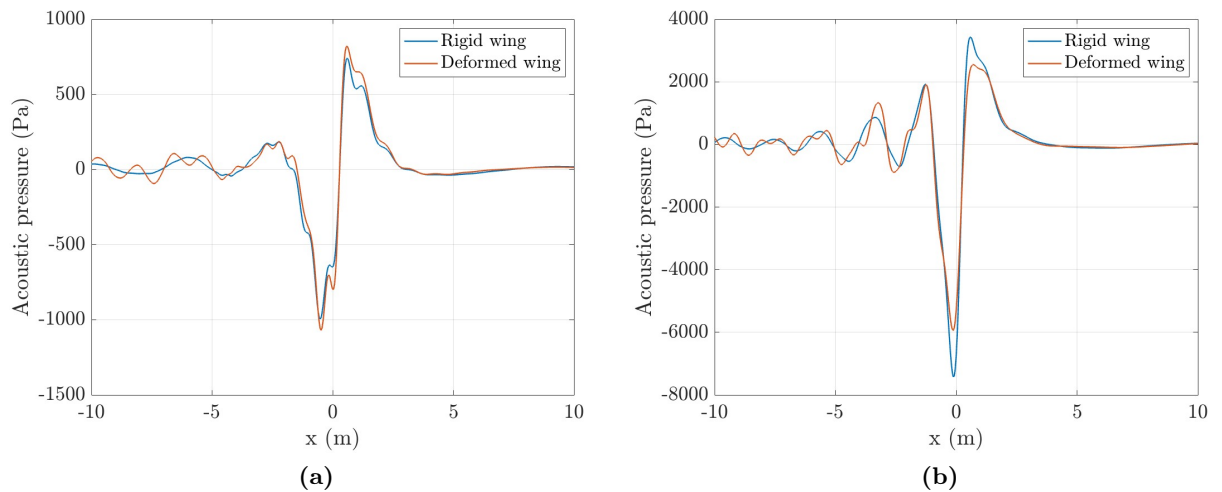


Figure 4.4: Acoustic pressure from deformed-boundary and undeformed-boundary integral formulations. (a): $M = 0.5$; (b): $M = 0.7$.

4.4 Deformable porous sphere enclosing a pulsating source: validation of boundary integral equation

Let us consider the same case of a deforming porous sphere enclosing a pulsating source examined in Sec. 4.2, but assuming both the source and the sphere at rest.

As in Sec. 4.2, the objective is to compare the pressure perturbation directly radiated by the source (determined analytically) with that predicted numerically through application of the proposed BEM tool (since the porous surface does not affect the perturbation field, the pressure propagated outward by the boundary integral solution theoretically coincides with that radiated directly by the source).

However, in this case, the boundary integral formulation is first applied as a boundary integral equation to determine the acoustic pressure over the porous surface (solution of the FWHE with kinematic terms known from imposed surface deformation and flow velocity induced by the source), and then as a boundary integral representation to radiate it in the external field.

Specifically, we consider a point source pulsating with angular frequency $\omega_s = 2\pi$ rad/s, at rest at the center of the spherical porous surface of radius $r_0 = 1$ m which deforms according to Eq. (4.5) with $A = 2r_0/5$ and $\omega_d = \sqrt{2}\omega_\varphi$ ().

For an observer located at a distance $d = 50r_0$ from the source, Fig. 4.5(a) shows the comparison between the pressure perturbation evaluated analytically with that radiated by the porous surface according to the BEM approach. Note that, although the solution from the boundary integral formulation includes the combination of several multi-harmonic contributions due to the harmonic deformation of the surface (see Fig. 4.5(b) which depicts the time-history of the pressure determined over an arbitrary point of the pulsating surface), these suitably cancel-out at the fixed observer position and thus the numerical and analytical solutions are mono-harmonic and perfectly coincide, thus validating the formulation applied here.

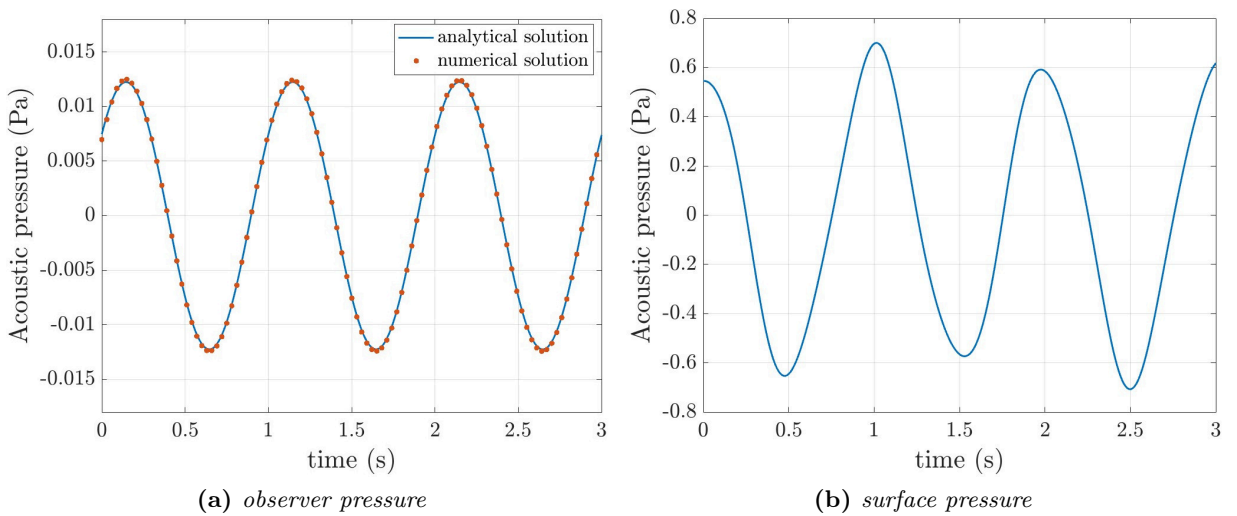


Figure 4.5: Pressure perturbation due to a pulsating source.

4.5 Sound waves emitted by large-amplitude pulsations of spheres

The objective of this section is the assessment of the accuracy of the boundary integral formulation developed in this work in predicting the acoustic field generated by bodies undergoing large deformations. In particular, it is focused on the validation of the boundary integral contributions from deforming surfaces by comparison with analytical solutions.

Thus, the problem of the linear sound waves radiated by a pulsating sphere immersed in an inviscid fluid is considered. It is solved by using the boundary integral formulation for the velocity potential field expressed in FFR, with nonlinear terms neglected, which is obtained from Eq. (3.25) by considering $\Psi = \varphi$, $\mathbf{z} = \nabla \varphi$, $\mathbf{Z} = \varphi \mathbf{I}$, $k_1 = -1/c_0^2 (\partial \varphi / \partial t)$, $k_2 = -\varphi/c_0^2$, $\mathbf{w} = 0$, $\sigma_\Psi = 0$ (null nonlinear terms), and reads

$$\begin{aligned} \varphi(\mathbf{x}_*, t_*) = & \iint_{\Omega} \left[\left(\frac{\partial \varphi}{\partial n} - M_n \mathbf{M} \cdot \nabla \varphi + \frac{M_n}{c_0} \frac{\partial \varphi}{\partial t} \right) G_0 - \varphi \frac{\partial G_0}{\partial n} \right] J \Big|_{t_* - \theta} d\xi^1 d\xi^2 \\ & + \frac{1}{c_0} \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} \left[\varphi (-\mathbf{e}_r \cdot \mathbf{n} + M_n) G_0 J \right] \right) \frac{1}{\Theta} \Big|_{t_* - \theta} d\xi^1 d\xi^2 \end{aligned} \quad (4.6)$$

where $\Theta = (1 - \mathbf{v} \cdot \mathbf{e}_r / c_0)$ and $J = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3| / |\Theta|$ (note that the contribution from nonlinear terms in Eq. (3.25) would be expressed by a field volume integral). The associated boundary conditions are: (i) the impermeability condition defined at any time over the surface, $\mathcal{S}$, of the deforming body, which reads

$$\frac{\partial \varphi}{\partial n} \Big|_{\mathcal{S}} = \mathbf{v} \cdot \mathbf{n} \quad (4.7)$$

where $\mathbf{v}$ denotes the velocity of the surface points, and $\mathbf{n}$ represents the outward unit normal vector of body surface, and (ii) the Sommerfeld condition at points that are infinitely distant from the body.

The radiated perturbation field is determined in two steps: first, Eq. (4.6) is applied as a boundary integral equation with $\mathbf{x}_* \in \Omega$ in order to evaluate the velocity potential field over the sphere surface and then, the same equation is applied as a boundary integral representation to evaluate the velocity potential at any point in the field.

The predictions from this boundary integral formulation are validated against linear analytical solutions of the sound waves emitted by the pulsating sphere, partly already available in the literature (those related to small-amplitude pulsations or large-amplitude/low-frequency pulsations [69, 70]), and partly originally developed in this work in appendix E (those related to large - amplitude/high-frequency pulsations).

It is worth remarking that, although the nonlinear forcing terms in Eq. (2.8) might considerably affect the acoustic waves radiated by large deformations of the sphere (see, for instance, Refs. [71–73]), these are neglected in this comparison because it is aimed at validating the boundary integral contribution from deforming surfaces.

Finally, note that Eq. (4.6) can be applied to develop an alternative derivation of the closed-form analytical solutions of the acoustic field radiated by a sphere undergoing small-amplitude pulsations or large amplitude pulsations with low-frequency, which are already available in the literature [69, 70]. This is accomplished in appendix F.

4.5.1 Space discretization of the velocity potential formulation

For the purpose of the numerical application of the above boundary integral formulation, and following the approach indicated in Sec. 4.1, the surface of the sphere is discretized into N_p quadrilateral panels. These panels are represented as hyperbolic paraboloid surfaces. Over these surfaces the velocity potential and its space and time derivatives are assumed to be uniformly distributed, equal to those evaluated at the centroid of the hyperbolic paraboloid (zeroth-order BEM).

When the formulation is applied as a boundary integral equation, the satisfaction of the equation is imposed at the center of each discretization panel (collocation method, see Sec. 4.1 for details). This yields the following set of time-delay equations whose solution provides the time-history of the piecewise constant velocity potential over the surfaces of the panels

$$\begin{aligned} \frac{1}{2} \varphi_k(t_*) &= \sum_{m=1}^{N_p} A_{km} \chi_m(t_* - \theta_{km}) + \sum_{m=1}^{N_p} B_{km} \varphi_m(t_* - \theta_{km}) \\ &+ \sum_{m=1}^{N_p} C_{km} \dot{\varphi}_m(t_* - \theta_{km}) + \sum_{m=1}^{N_p} F_{km} \varphi_m^t(t_* - \theta_{km}), \quad \text{for } k = 1, N_p \end{aligned} \quad (4.8)$$

where φ_m and φ_m^t denote, respectively, the velocity potential and the derivative of the velocity potential along the direction of the local body Mach number component tangent to the surface, $\mathbf{M}_t = \mathbf{M} - M_n \mathbf{n}$ evaluated at the center of the m -th panel, χ_m represents the normal derivative of the velocity potential evaluated at the same point (known from the impermeability boundary condition in Eq. (4.7)), θ_{km} denotes the signal transmission delay related to the observer at the center of the k -th panel and the emitting source at the center of the m -th panel. In addition, the influence coefficients are derived as particular cases of the general coefficients given in Sec. 4.1.1, and are defined as

$$A_{km} = \iint_{\Omega_m} [(1 - M_n^2) G_0 J] \Big|_{t_* - \theta_{km}} d\xi^1 d\xi^2 \quad (4.9)$$

$$B_{km} = \iint_{\Omega_m} \left[-\frac{\partial G_0}{\partial n} J + \frac{1}{\Theta} \frac{\partial}{\partial t} \Big|_{\xi} \left[G_0 J \left(-\frac{\mathbf{e}_r \cdot \mathbf{n}}{c_0} + \frac{M_n}{c_0} \right) \right] \right] \Big|_{t_* - \theta_{km}} d\xi^1 d\xi^2 \quad (4.10)$$

$$C_{km} = \iint_{\Omega_m} \left[\frac{M_n}{c_0} G_0 J + G_0 \frac{J}{\Theta} \left(-\frac{\mathbf{e}_r \cdot \mathbf{n}}{c_0} + \frac{M_n}{c_0} \right) \right] \Big|_{t_* - \theta_{km}} d\xi^1 d\xi^2 \quad (4.11)$$

$$F_{km}(t_*) = - \iint_{\Omega_m} M_n M_t G_0 J \Big|_{\theta_{km}} d\xi^1 d\xi^2 \quad (4.12)$$

with Ω_m denoting the (ξ^1, ξ^2) -domain where the surface of the m -th panel is mapped. The factor $1/2$ which appears at the left-hand side of Eq. (4.8) accounts for the contribution of the free terms deriving from the singularity of the kernel function $\partial G_0 / \partial n$ arising when the position of the observer tends to the limit on the emitting surface. Thus, when $k = m$ the coefficients B_{km} are intended as Cauchy principal values of the surface integral [30, 68].

Once the velocity potential over the sphere is known, it may be readily used to determine the velocity potential field radiated at external points through the discretized boundary integral representation which is formally identical to Eq. (4.8), with the factor $1/2$ at the left-hand side removed [30, 68].

4.5.2 Harmonic-balance solution algorithm

Then, let us consider a harmonically pulsating sphere with the instantaneous radius, r_s , given by

$$r_s(t) = r_0 + a \sin \omega t \quad (4.13)$$

where r_0 represents its average value, while a and ω denote, respectively, amplitude and angular frequency of the oscillations. The application of the [BEM](#) solver for the determination of the corresponding velocity potential field yields time-periodic influence coefficients in Eqs. (4.9), (4.10), (4.11) and (4.12). Indeed, they are defined over the delayed deforming surface in terms of the surface velocity, $\mathbf{v}$, and the variables r, J, θ, Θ , and thus depend on its motion. Their fundamental harmonic is equal to that of the sphere pulsations. Note that, if the surface of the sphere is assumed to be undeformed (as it is typically done in the case of small-amplitude vibration problems), these coefficients are constant over time.

Differently to general solution approach described in Sec. 4.1.2, in this case to evaluate the time history of the velocity potential perturbation it is convenient to express it in terms of its Fourier series (typically, this solution procedure is useful for any wave radiation problem that can be described as a combination of harmonics). For the piecewise constant, time-dependent variables introduced in the velocity potential problem described in Eq. (4.8) this yields

$$\begin{aligned} \varphi_m(t_* - \theta_{km}) &= \varphi_{m0} + \sum_{n=1}^{N_a} (\varphi_{mn}^c \cos [n\omega(t_* - \theta_{km})] + \varphi_{mn}^s \sin [n\omega(t_* - \theta_{km})]) \\ \dot{\varphi}_m(t_* - \theta_{km}) &= \sum_{n=1}^{N_a} n\omega (-\varphi_{mn}^c \sin [n\omega(t_* - \theta_{km})] + \varphi_{mn}^s \cos [n\omega(t_* - \theta_{km})]) \end{aligned}$$

where ω represents the fundamental harmonic, N_a is the number of harmonics included in the analysis, φ_{mn}^c and φ_{mn}^s are the cosine and sine components of the n -th harmonic of the velocity potential at the center of the m -th panel, while φ_{m0} denotes its average value. The substitution of the above expressions in Eq. (4.8) yields

$$\varphi_k(t_*) = \sum_{m=1}^{N_p} \hat{B}_{km}(t_*) \varphi_{m0} + \sum_{m=1}^{N_p} \sum_{n=1}^{N_a} D_{km}^{nc}(t_*) \varphi_{mn}^c + \sum_{m=1}^{N_p} \sum_{n=1}^{N_a} D_{km}^{ns}(t_*) \varphi_{mn}^s + f_k(t_*) \quad (4.14)$$

where

$$\begin{aligned} D_{km}^{nc}(t_*) &= [(1/2) \delta_{km} + B_{km}(t_*) + F_{kj}(t_*) G_{jm}(t_*)] \cos [n\omega(t_* - \theta_{km})] \\ &\quad - n\omega C_{km}(t_*) \sin [n\omega(t_* - \theta_{km})] \end{aligned} \quad (4.15)$$

$$\begin{aligned} D_{km}^{ns}(t_*) &= [(1/2) \delta_{km} + B_{km}(t_*) + F_{kj}(t_*) G_{jm}(t_*)] \sin [n\omega(t_* - \theta_{km})] \\ &\quad + n\omega C_{km}(t_*) \cos [n\omega(t_* - \theta_{km})] \end{aligned} \quad (4.16)$$

$$\hat{B}_{km}(t_*) = (1/2) \delta_{km} + B_{km}(t_*) + F_{kj}(t_*) G_{jm}(t_*)$$

where $G_{jm}(t_*)$ are the coefficients providing φ^t at the center of the j -th panel from the knowledge of

the velocity potentials φ_m , for $m = 1, \dots, N_p$. Since in the problem under consideration the velocity of the deforming surface is aligned with the local normal direction, the term $F_{km}(t_*)$ is zero (see Eq. (4.12)). In addition, the forcing term $f_k(t_*)$ is related to the normal derivative of the velocity potential and can be evaluated from the knowledge of the impermeability boundary conditions as

$$f_k(t_*) = \sum_{m=1}^{N_p} A_{km}(t_*) \chi_m(t_* - \theta_{km}) \quad (4.17)$$

Next, collecting the above coefficients in the $N_p \times N_p$ matrices $\mathbf{D}^{nc}$, $\mathbf{D}^{ns}$ and $\hat{\mathbf{B}}$, and the forcing terms in the N_p -element vector, $\mathbf{f}$, and introducing the following Fourier-series expansions

$$\begin{aligned} \mathbf{D}^{nc}(t_*) &= \mathbf{D}_0^{nc} + \sum_{h=1}^{N_a} [\mathbf{D}_{hc}^{nc} \cos(h\omega t_*) + \mathbf{D}_{hs}^{nc} \sin(h\omega t_*)] \\ \mathbf{D}^{ns}(t_*) &= \mathbf{D}_0^{ns} + \sum_{h=1}^{N_a} [\mathbf{D}_{hc}^{ns} \cos(h\omega t_*) + \mathbf{D}_{hs}^{ns} \sin(h\omega t_*)] \\ \hat{\mathbf{B}}(t_*) &= \hat{\mathbf{B}}_0 + \sum_{h=1}^{N_a} [\hat{\mathbf{B}}_{hc} \cos(h\omega t_*) + \hat{\mathbf{B}}_{hs} \sin(h\omega t_*)] \\ \mathbf{f}(t_*) &= \mathbf{f}_0 + \sum_{h=1}^{N_a} [\mathbf{f}_{hc} \cos(h\omega t_*) + \mathbf{f}_{hs} \sin(h\omega t_*)] \end{aligned}$$

the system of periodic-coefficient algebraic equations in Eq. (4.14) can be reformulated in the following constant-coefficient system of algebraic equations with unknowns the Fourier coefficients of the velocity potential at the centroids of the panels

$$\begin{bmatrix} \varphi_0 \\ \varphi_{1c} \\ \varphi_{1s} \\ \vdots \\ \varphi_{N_a c} \\ \varphi_{N_a s} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{B}}_0 \\ \hat{\mathbf{B}}_{1c} \\ \hat{\mathbf{B}}_{1s} \\ \vdots \\ \hat{\mathbf{B}}_{N_a c} \\ \hat{\mathbf{B}}_{N_a s} \end{bmatrix} \varphi_0 + \sum_{n=1}^{N_a} \begin{bmatrix} \mathbf{D}_0^{nc} \\ \mathbf{D}_{1c}^{nc} \\ \mathbf{D}_{1s}^{nc} \\ \vdots \\ \mathbf{D}_{N_a c}^{nc} \\ \mathbf{D}_{N_a s}^{nc} \end{bmatrix} \varphi_{nc} + \sum_{n=1}^{N_a} \begin{bmatrix} \mathbf{D}_0^{ns} \\ \mathbf{D}_{1c}^{ns} \\ \mathbf{D}_{1s}^{ns} \\ \vdots \\ \mathbf{D}_{N_a c}^{ns} \\ \mathbf{D}_{N_a s}^{ns} \end{bmatrix} \varphi_{ns} + \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_{1c} \\ \mathbf{f}_{1s} \\ \vdots \\ \mathbf{f}_{N_a c} \\ \mathbf{f}_{N_a s} \end{bmatrix} \quad (4.18)$$

where φ_{nc} and φ_{ns} are vectors collecting, respectively, the cosine and sine components of the n -th harmonic of the velocity potentials over the N_p discretization panels, whereas φ_0 is the vector collecting their average values (a similar definition is valid for the vectors $\mathbf{f}_{nc}$, $\mathbf{f}_{ns}$ and $\mathbf{f}_0$).

Equation (4.18) can be conveniently rearranged in the following equivalent expression

$$\begin{bmatrix} \varphi_0 \\ \varphi_{1c} \\ \varphi_{1s} \\ \vdots \\ \varphi_{N_a c} \\ \varphi_{N_a s} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{B}}_0 & \mathbf{D}_0^{1c} & \mathbf{D}_0^{1s} & \dots & \mathbf{D}_0^{N_a c} & \mathbf{D}_0^{N_a s} \\ \hat{\mathbf{B}}_{1c} & \mathbf{D}_{1c}^{1c} & \mathbf{D}_{1c}^{1s} & \dots & \mathbf{D}_{1c}^{N_a c} & \mathbf{D}_{1c}^{N_a s} \\ \hat{\mathbf{B}}_{1s} & \mathbf{D}_{1s}^{1c} & \mathbf{D}_{1s}^{1s} & \dots & \mathbf{D}_{1s}^{N_a c} & \mathbf{D}_{1s}^{N_a s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\mathbf{B}}_{N_a c} & \mathbf{D}_{N_a c}^{1c} & \mathbf{D}_{N_a c}^{1s} & \dots & \mathbf{D}_{N_a c}^{N_a c} & \mathbf{D}_{N_a c}^{N_a s} \\ \hat{\mathbf{B}}_{N_a s} & \mathbf{D}_{N_a s}^{1c} & \mathbf{D}_{N_a s}^{1s} & \dots & \mathbf{D}_{N_a s}^{N_a c} & \mathbf{D}_{N_a s}^{N_a s} \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_{1c} \\ \varphi_{1s} \\ \vdots \\ \varphi_{N_a c} \\ \varphi_{N_a s} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_{1c} \\ \mathbf{f}_{1s} \\ \vdots \\ \mathbf{f}_{N_a c} \\ \mathbf{f}_{N_a s} \end{bmatrix} \quad (4.19)$$

Then, denoting with $\boldsymbol{\varphi}$ the vector of $N_p(2N_a + 1)$ elements collecting the φ_0 , φ_{nc} and φ_{ns} Fourier coefficients, with $\mathbf{f}^x$ the vector of $N_p(2N_a + 1)$ elements collecting the $\mathbf{f}_0$, $\mathbf{f}_{hc}$ and $\mathbf{f}_{hs}$ Fourier coefficients, and with $\mathbf{D}$ the matrix of dimensions $N_p(2N_a + 1) \times N_p(2N_a + 1)$ that collects the Fourier coefficients of the matrices $\mathbf{D}^{nc}$, $\mathbf{D}^{ns}$ and $\hat{\mathbf{B}}$, Eq. (4.19) can be re-written in the following compact form

$$\boldsymbol{\varphi} = \mathbf{D} \boldsymbol{\varphi} + \mathbf{f}^x \quad (4.20)$$

and, thus, the Fourier coefficients of the velocity potential on each panel of the sphere surface can be determined as (see also [67])

$$\boldsymbol{\varphi} = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{f}^x \quad (4.21)$$

Note that, in general, even in the case of mono-harmonic sphere pulsations (as those considered here), the corresponding velocity potential perturbations are multi-harmonic, due to the presence of kinematic nonlinearities which imply the combination of harmonic signals. A mono-harmonic solution arises if the amplitude of pulsations is very small (see later).

As already mentioned, once the values of the velocity potential over the body's panels are evaluated, the velocity potential in the field may be readily determined through the proposed numerical algorithm applied to the integral formulation conceived as an integral representation.

Finally, it is worth highlighting that for rigid boundaries Eq. (4.19) is equivalent to the solution algorithm that can be obtained from the Fourier transform applied to Eq. (4.8), with complex variables replaced by their expressions given in terms of sine and cosine components.

4.5.3 Numerical investigation

Next, the results of the numerical investigation carried out to validate the novel deformable-boundary integral formulation are presented. Considering the velocity potential field radiated by pulsations of a sphere with small or large values of amplitude and low or high values of frequency that satisfy the kinematic and geometric constraints, namely, $ka < 1$ and $a/r_0 < 1$ (where $k = \omega/c_0$ is the wave number), the validation consists of correlations with predictions given by analytical solutions of the homogeneous wave equation discussed in appendix E (where an original analytical solution for large-amplitude, non-small frequency pulsations is introduced, and also solutions already available in the literature are presented for the sake of completeness [69, 70]).

For a physical interpretation of the results, these are presented in terms of the corresponding acoustic pressure (pressure perturbations) determined by application of the Bernoulli's theorem as $p' = -\rho_0 \dot{\varphi}$, where ρ_0 denotes the density of the undisturbed fluid. For the sake of clarity the different analytical solutions proposed here are referred to as

- solution A: valid for $a/r_0 \ll 1$ and $kr_0 \ll 1$, low-frequency, small-amplitude pulsations, Eq. (E.14);
- solution B: valid for $a/r_0 \ll 1$, small-amplitude pulsations, Eq. (E.16);
- solution C: valid for $kr_0 \ll 1$, low-frequency pulsations, see Eq. (E.18);
- solution D: valid for either large-amplitude or high-frequency pulsations, such that $ka \ll 1$, see Eq. (E.24);

- solution E (exact): valid for large-amplitude, high-frequency pulsations, such that $ka < 1$, see Eq. (E.24).

The numerical investigation concerns a sphere with average radius equal to $r_0 = 1$ m, immersed in an undisturbed fluid with density $\rho_0 = 1.22$ kg/m³, and speed of sound $c_0 = 340$ m/s. All results obtained by the BEM solver are to be considered numerically converged. Unless specifically stated, they are obtained by using 578 panels to discretize the surface, given by dividing it into 34 meridian circles and 17 parallel circles (the results of a detailed convergence analysis are given later).

Fig. 4.6 shows the acoustic pressure observed at a distance $d = 10r_0$ concerning a low-frequency, small-amplitude pulsations case with $kr_0 = 0.05$ and $a/r_0 = 0.01$ (note that, for all the results presented in the following the same values of r_0, ρ_0 and c_0 are assumed).

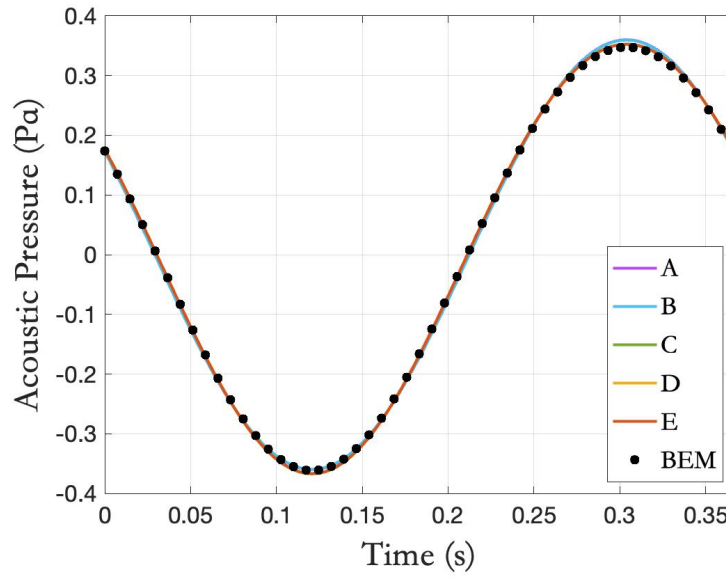


Figure 4.6: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 0.05$, $a/r_0 = 0.01$.

As expected, for this low-frequency, small-amplitude pulsations case the radiated noise given by all the proposed solutions coincides with the BEM prediction and is mono-harmonic (with the same frequency of the sphere pulsations). In contrast, when a higher-frequency case is considered, two of the proposed solutions, namely solution A and solution C which are developed under the low-frequency assumption, are of low accuracy and have discrepancies from the BEM results. This is demonstrated in Fig. 4.7 where the outcomes for $kr_0 = 0.8$ and $a/r_0 = 0.01$ are shown, with the results that are not in agreement with BEM predictions depicted by dashed lines (this type of representation to distinguish accurate from inaccurate predictions is also applied in the following figures).

The results corresponding to low-frequency, large-amplitude pulsations are depicted in Fig. 4.8, for $kr_0 = 0.05$ and $a/r_0 = 0.5$. In this case, solutions A and B present significant differences from the BEM solution (note that only one dashed curve appears since these two solutions are numerically coincident). It is worth observing that, as expected from the structure of the BEM solution in Eq. (4.21) and as discussed in the theoretical development of the analytical solutions in appendix E, large-amplitude pulsations produce multi-harmonic perturbation waves which are correctly captured by solutions C , D , and E .

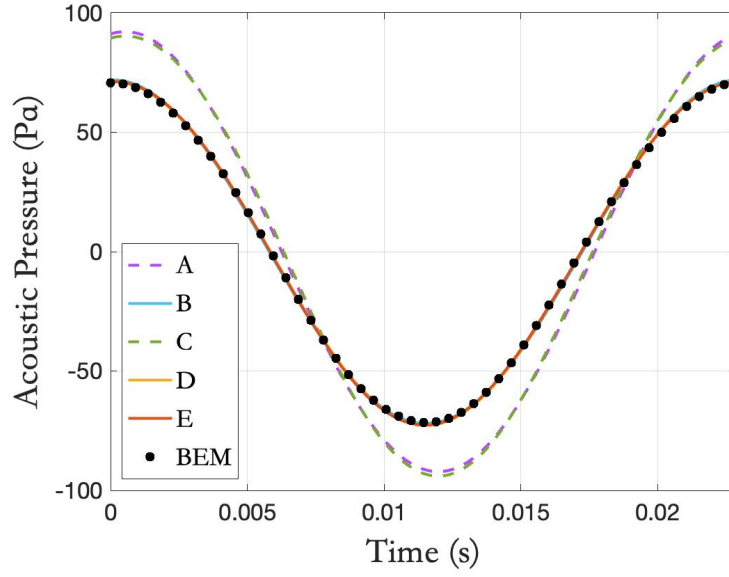


Figure 4.7: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 0.8$ and $a/r_0 = 0.01$.

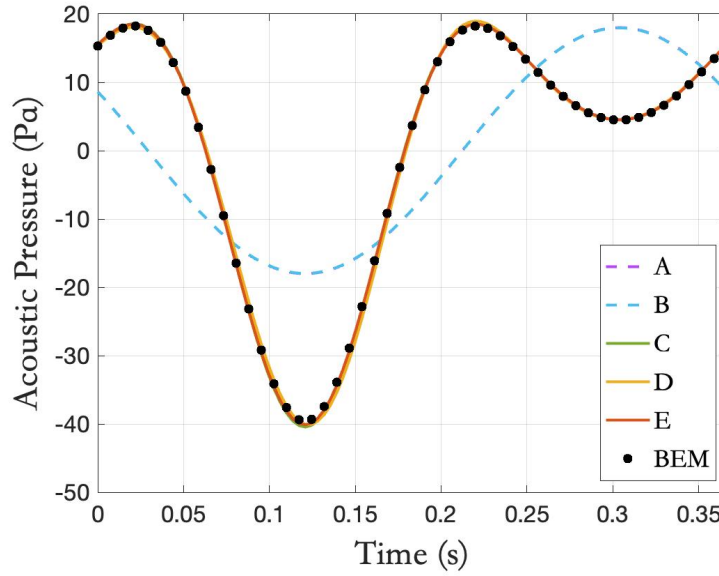


Figure 4.8: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 0.05$ and $a/r_0 = 0.5$.

As mentioned above, all results are obtained by using 578 panels to discretize the surface. This provides converged results, as proven by a dedicated numerical investigation based on the determination of the relative L2-error between the numerical prediction, p'^{num} , and the analytical solution, p'^{an} , of the acoustic pressure defined as

$$\epsilon_r^2 = \frac{\sum_{i=1}^{N_t} (p'_i{}^{\text{num}} - p'_i{}^{\text{an}})^2}{\sum_{i=1}^{N_t} (p'_i{}^{\text{an}})^2}$$

where N_t denotes the number of time intervals within a period. Indeed, for the case examined in

Fig. 4.8 regarding pulsations with $a/r_0 = 0.5$ and $kr_0 = 0.05$, Fig. 4.9 shows the decay of the relative L2-error with increasing number of discretization panels. The fourth-degree polynomial extrapolation curve (dashed line) demonstrates that correctly the asymptotic relative error is equal to zero as the number of discretization panels increases. Note that for $N_p = 17$ (which corresponds to 17 parallels and 34 meridian lines of discretization and a total number of panels equal to 578) we have a negligible relative L2-error, $\epsilon_r^2 \approx 5 \times 10^{-5}$. However, this figure also demonstrates that a very small error, $\epsilon_r^2 \approx 1 \times 10^{-3}$, is achieved by using a much smaller number of discretization panels amounting to 128. Next, we consider pulsations of higher frequency and medium amplitude.

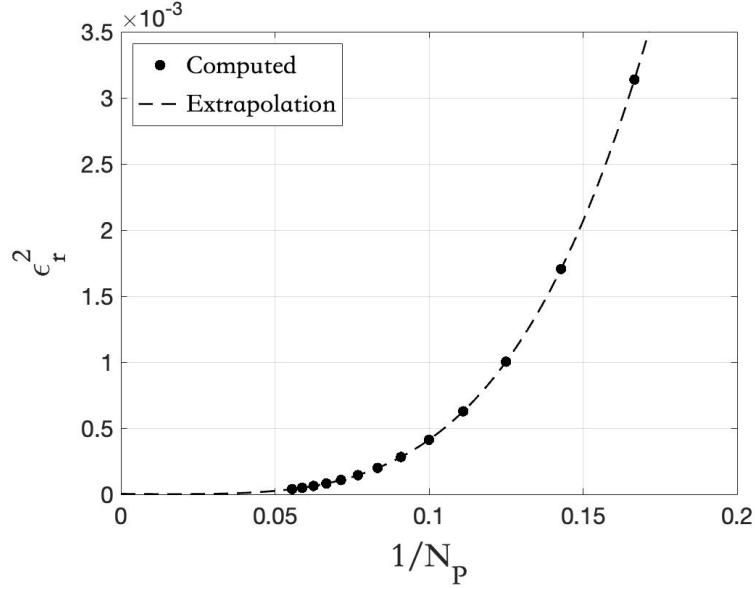


Figure 4.9: Relative L2-error between numerical prediction and analytical solution of the radiated acoustic pressure as a function of the discretization mesh, for $kr_0 = 0.05$, $a/r_0 = 0.5$, $d/r_0 = 10$ (N_p = number of parallel circles and half of the number of meridian lines).

In particular, for $kr_0 = 0.3$ and $a/r_0 = 0.1$, the results presented in Fig. 4.10 demonstrate that accurate predictions can be obtained only by solutions D and E which provide predictions well matched to those obtained by the BEM solver.

Instead, when the pulsations of the sphere are, at the same time, of large amplitude and not low frequency only the solution E is in perfect agreement with the BEM prediction. This is proven by Fig. 4.11 for $kr_0 = 1.0$ and $a/r_0 = 0.5$, where all other proposed solutions show large discrepancies from the BEM results.

The numerical investigation presented provides a successful validation of the developed BEM solver for arbitrarily deforming boundaries, with the analytical results that are perfectly in line with the expected trend in the prediction accuracy of the different solutions as frequency and amplitude of the pulsations change.

They confirm that when the amplitude of the pulsation is not small, the signal radiated by single-frequency pulsations is multi-harmonic, with the amplitude of the higher harmonics increasing with the amplitude of the pulsation amplitude. This is clearly illustrated in Fig. 4.12 which depicts the spectrum of acoustic waves propagated at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.1, 0.3, 0.5$ evaluated by the solution E and the BEM solver. In fact, this picture shows that for the lowest considered value of the amplitude, $a/r_0 = 0.1$, the signal is almost mono-harmonic, with a very small

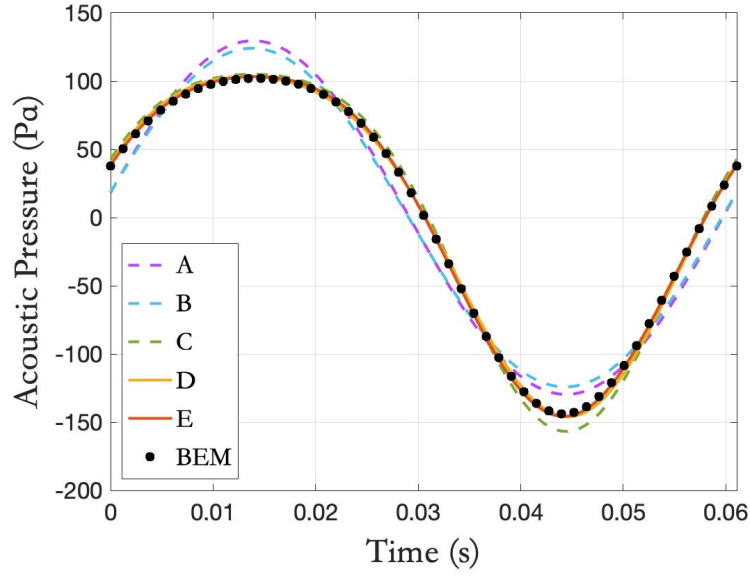


Figure 4.10: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 0.3$ and $a/r_0 = 0.1$.

contribution from the second harmonic. In contrast, for $a/r_0 = 0.3$ and $a/r_0 = 0.5$ non-negligible contributions in the spectrum of the radiated acoustic wave are observed from the first three and six harmonics, respectively.

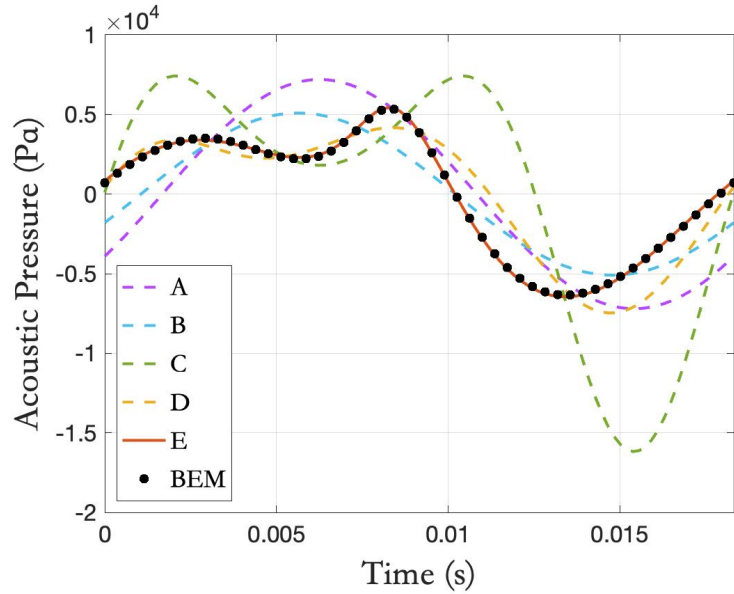


Figure 4.11: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.5$.

It is interesting to note that, when the surface deformation is significant, the terms proportional to the time derivative of the Jacobian, J , which appear in the new deformable-boundary integral formulation play a key role in the prediction of radiated waves (see Eq. (4.6)). This is proven by the results shown in Fig. 4.13 and 4.14 which are related to $kr_0 = 1.0$ and, respectively, $a/r_0 = 0.05$ and $a/r_0 = 0.5$. In fact, Fig. 4.13 demonstrates that when the sphere is slightly deformed, $a/r_0 = 0.05$, the inclusion of the terms proportional to dJ/dt in the BEM solver has a negligible effects on the prediction. To the contrary, Fig. 4.14 shows that when the sphere is subject

to large deformations, $a/r_0 = 0.5$, the contributions to the BEM solution of the terms proportional to dJ/dt are of fundamental importance, since they significantly affect the prediction of the higher harmonic content of the radiated signal due to the large amplitude of the pulsations.

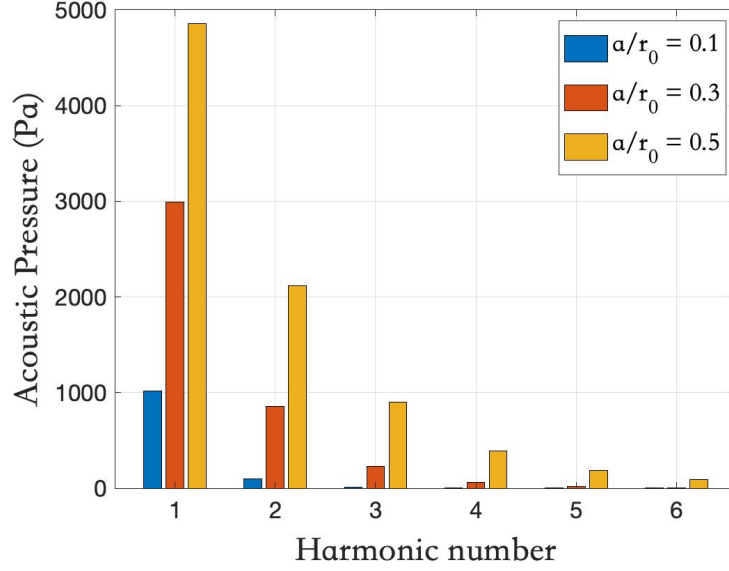


Figure 4.12: Acoustic wave spectra at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.1, 0.3, 0.5$.

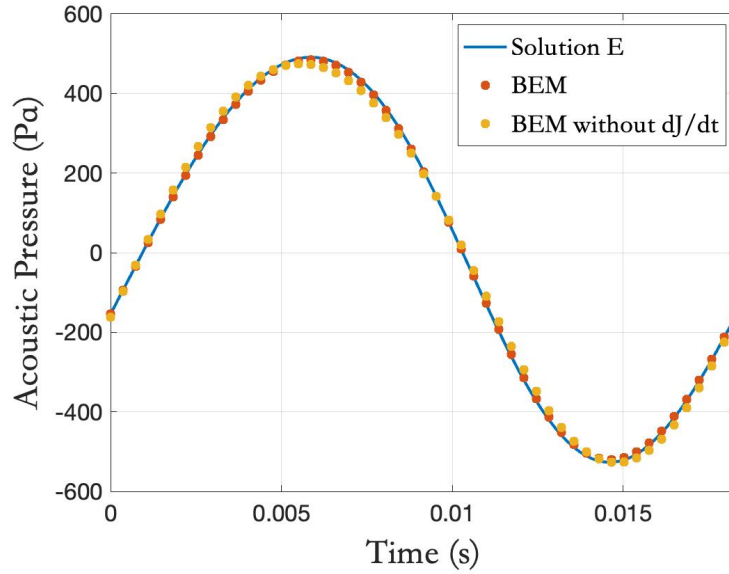


Figure 4.13: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.05$. Comparison among complete BEM solution, BEM solution without dJ/dt terms, and analytical solution E .

Finally, again considering the case of high frequency and large amplitude pulsation, $kr_0 = 1.0$ and $a/r_0 = 0.5$, the agreement between the predictions provided by the BEM solver and those provided by the E solution is confirmed whatever the distance of the observer from the centre of the sphere. This is proven by considering the results in Fig. 4.11 obtained for $d/r_0 = 10$, and those presented in Fig. 4.15 obtained for $d/r_0 = 2$ and $d/r_0 = 100$. Thus, it is possible to conclude that these two solutions coincide correctly for both near-field and far-field observers.

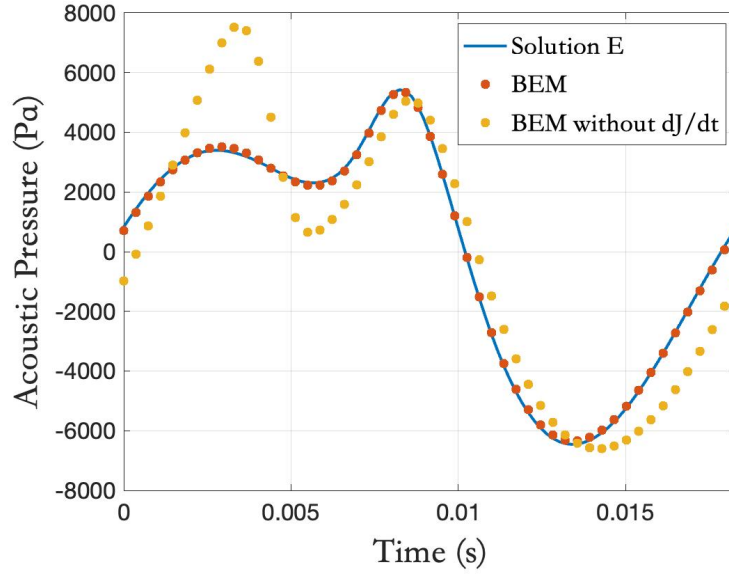


Figure 4.14: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.5$. Comparison among complete BEM solution, BEM solution without dJ/dt terms, and analytical solution E .

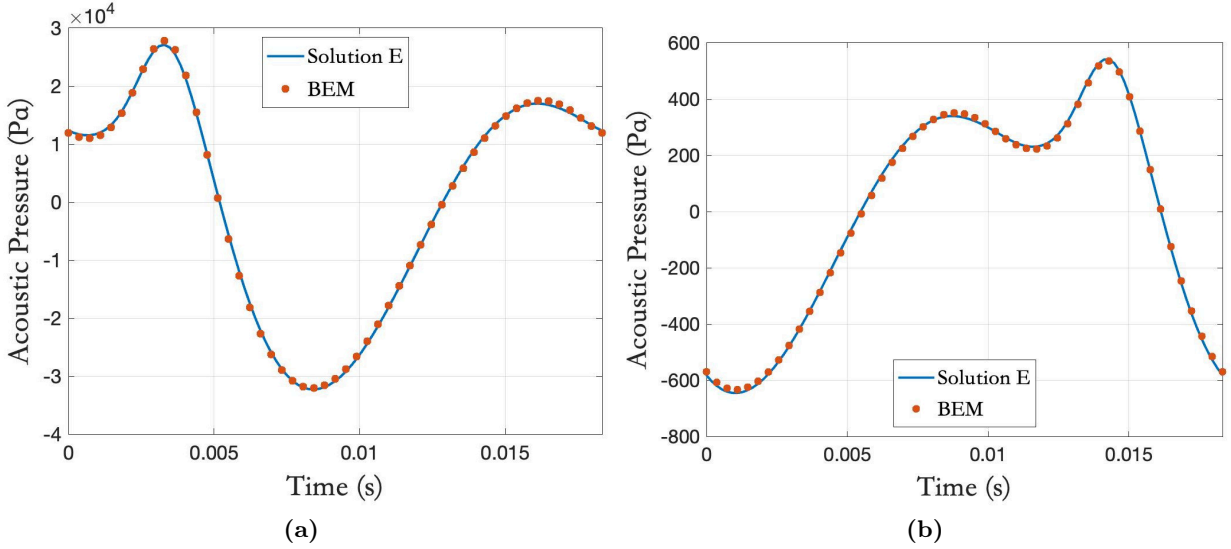


Figure 4.15: Acoustic pressure radiated by a pulsating sphere at $d/r_0 = 10$, for $kr_0 = 1.0$ and $a/r_0 = 0.5$. Comparison between BEM solution and solution E for (a) $d/r_0 = 2$, and (b) $d/r_0 = 100$.

The pulsation frequency $kr_0 = 1.0$ corresponds to a wavelength that is about three times larger than the diameter of the undeformed sphere. When the wavelength is similar to or smaller than the diameter of the sphere, the number of discretization panels needed to maintain a relative L2-error close to that of the previous results must necessarily increase.

The reason why the number of discretization panels needed to obtain accurate predictions increases with the frequency and amplitude of pulsations lies in the shape of the emitting acoustic surface on which the different contributions of the deformable-boundary integral formulation are evaluated (for a given observation time and position, the emitting acoustic surface is the geometric location where the points of the emitting surface were located at the moment they emitted the signal received by the observer).

This can be clearly understood by observing the acoustic surfaces depicted in Figs. 4.16 and 4.17. For a given time and a given position of the observer on the sphere, Fig. 4.16 shows the acoustic surface for $kr_0 = 2.0$ and $a/r_0 = 0.1$, while Fig. 4.17 shows the acoustic surface for $kr_0 = 10.0$ and $a/r_0 = 0.05$ (the color map represents the distribution of acoustic delays relative to the given observer indicated by the red dot). In the case of non-small pulsation amplitudes, as the pulsation frequencies increase, acoustic surfaces of increasingly complex shapes are generated, which, therefore, require a very large number of discretization panels to accurately capture the corresponding integral contributions.

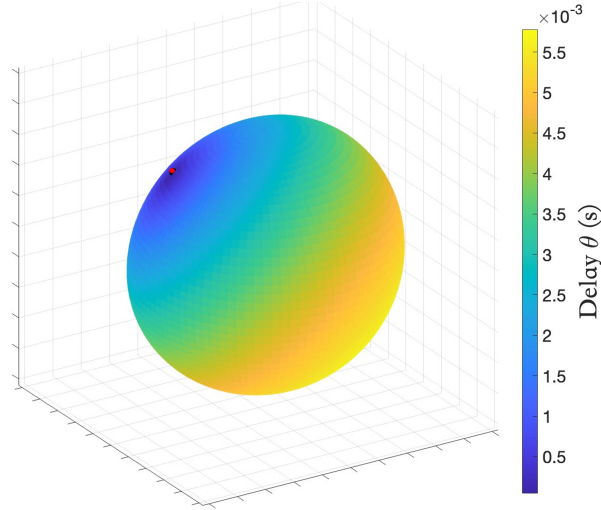


Figure 4.16: Acoustic surface for given observer time and position over the sphere; $kr_0 = 2.0$ and $a/r_0 = 0.01$.

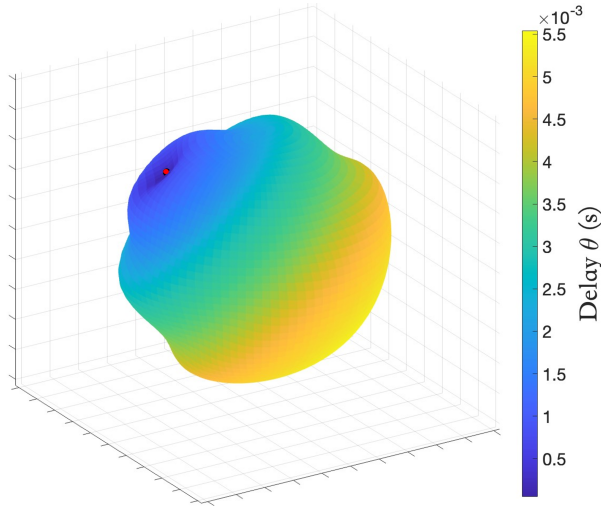


Figure 4.17: Acoustic surface for given observer time and position over the sphere; $kr_0 = 10.0$ and $a/r_0 = 0.05$.

4.6 Noise emitted by a helicopter rotor in forward flight

In this section, the proposed boundary integral formulation is applied to examine the noise emitted by the model helicopter rotor in forward flight tested in the framework of the HELINOISE project,

for which experimental measurements are available in the literature [74]. The aim of this investigation is twofold: to assess its capability to simulate realistic aeronautical configurations and to examine the influence of the deformation of the emitting surface on the emitted signal.

The rotor model corresponds to the BO-105 main rotor geometrically scaled by a scale factor of 0.4. It has a diameter $D = 4$ m, four hingeless blades with NACA 23012 airfoil shaped cross-sections, chord length $c = 0.121$ m, linear geometric twist equal to -8° and pre-coning angle equal to 2.5° . The flight condition examined here consists of a forward flight with effective tip-path-plane angle $\alpha_{TPP} = -14.63^\circ$, advance ratio $\mu = 0.148$, and hovering tip Mach number $M_{TIP} = 0.645$ (case "508" of the HELINOISE project [74]).

The evaluation of the radiated noise by the proposed boundary integral formulation follows a preliminary aeroelastic analysis of the rotor providing blade pressure and deformations. It is carried out through the methodology described and validated in [61], which is based on the application of Galerkin method, with the aerodynamic solution for potential flows determined by a boundary integral formulation for rigid boundaries, where the effect of the dynamic deformation of the blades is taken into account through the boundary conditions of impermeability.

The structural dynamics model of the blades is defined through the inertial and stiffness data given in the HART II experimental programme [75] which examines the same scaled model of the BO-105 main rotor considered in the HELINOISE project (no structural data are available in [74]). The aeroelastic solution is obtained by considering three flapping modes, three lagging modes and three torsion modes coinciding with the natural modes of vibration of a uniform, nonrotating bending-torsion beam.

Considering the blade section located at 75% of the blade radius, Fig. 4.18 compares the pressure coefficients determined numerically by the aeroelastic analysis during one revolution, with those measured experimentally [74]. It shows that the numerical prediction is in good agreement with measurements, thus providing reliable aerodynamics data for determining the aeroacoustic field.

The corresponding noise radiated by the deforming rotor blades is evaluated by the proposed deformable-boundary integral formulation applied to solve the Ffowcs-Williams and Hawkings equation for solid surfaces, Eq. (2.6). Given a Cartesian coordinate system with the origin at the rotor hub, the x -axis directed along the advancing direction pointing backward, the y -axis directed starboard advancing side, and the z -axis pointing upward, for the microphone placed at $x_m = -3.99$ m, $y_m = 1.62$ m, $z_m = -2.30$ m, Fig. 4.19 depicts the noise signatures evaluated by the proposed formulation expressed in the FFR and in a MFR following the rigid-blade motion. These are in perfect agreement, thus confirming their equivalence also in the case of rotating deforming boundaries.

Next, Fig. 4.20 presents the comparisons among noise signatures provided by measurements, those determined by the proposed deformable-boundary integral formulation, and those obtained under rigid-blade assumption at four microphones placed in the advancing side. These results show that the aeroelastic deformations scarcely affect the noise radiation predicted by boundary integral formulations, and that the numerical simulation is in overall good agreement with the experimental data: the positions of the noise signatures peaks are well captured, and the shape and amplitude of the signals are predicted quite well, with some small local discrepancies. It is worth noting that a slightly lower level of accuracy is obtained for microphones placed in the retreating side where, however, the influence of the boundary deformation on the predicted noise radiation remains negligible.

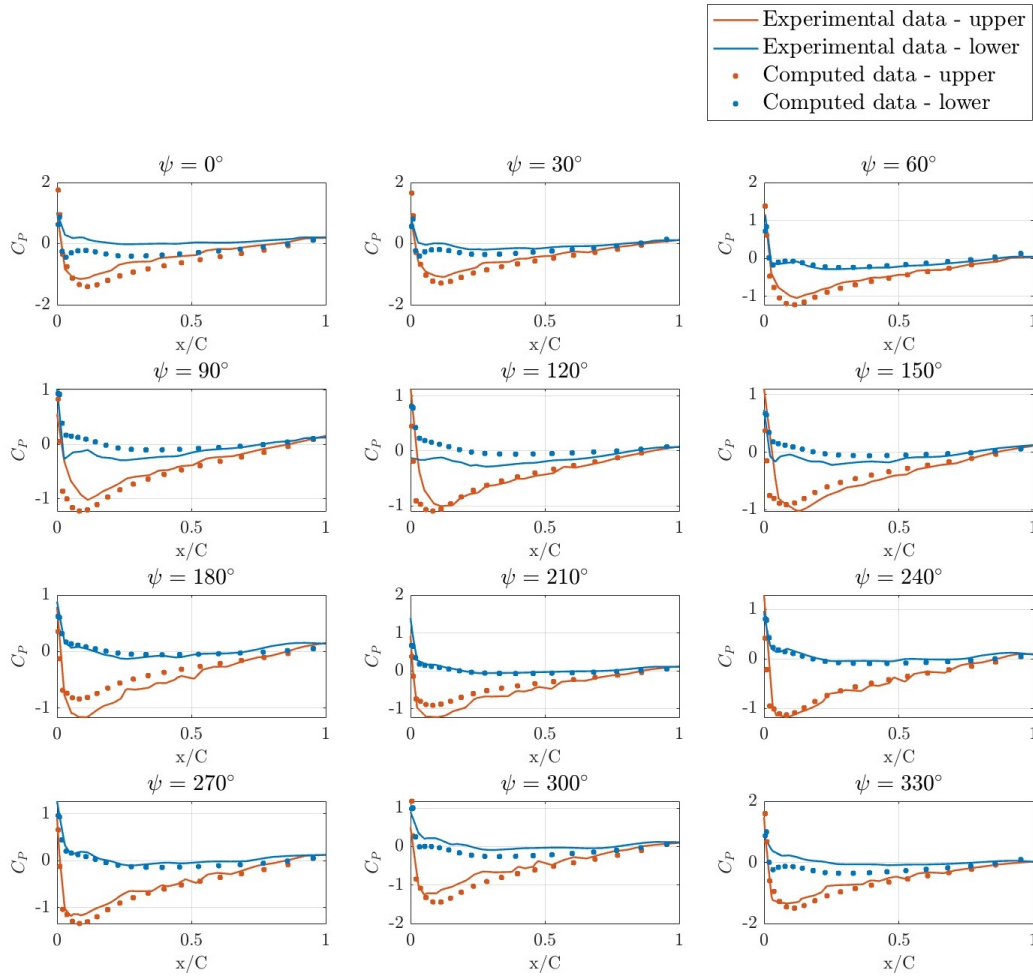


Figure 4.18: Pressure coefficient at 75% of BO-105 rotor blade radius during one revolution, $\mu = 0.148$ (case 508 of the HELINOISE project).

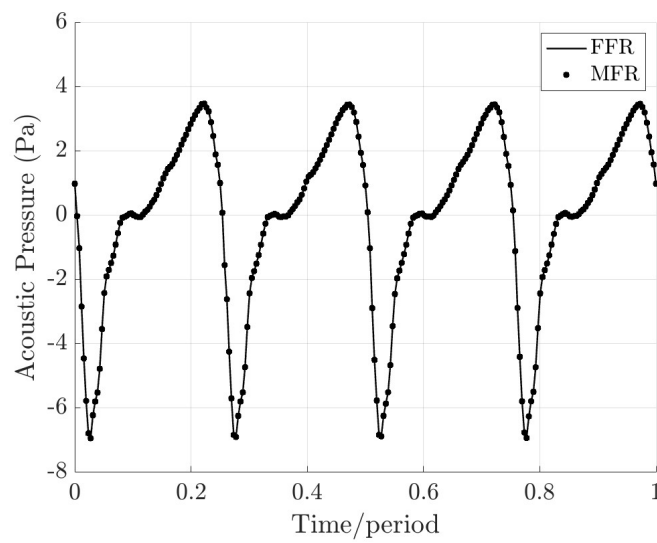


Figure 4.19: Comparison between noise signatures evaluated by the FFR and MFR deformable-boundary integral formulations solving the Ffowcs-Williams and Hawkins equation.

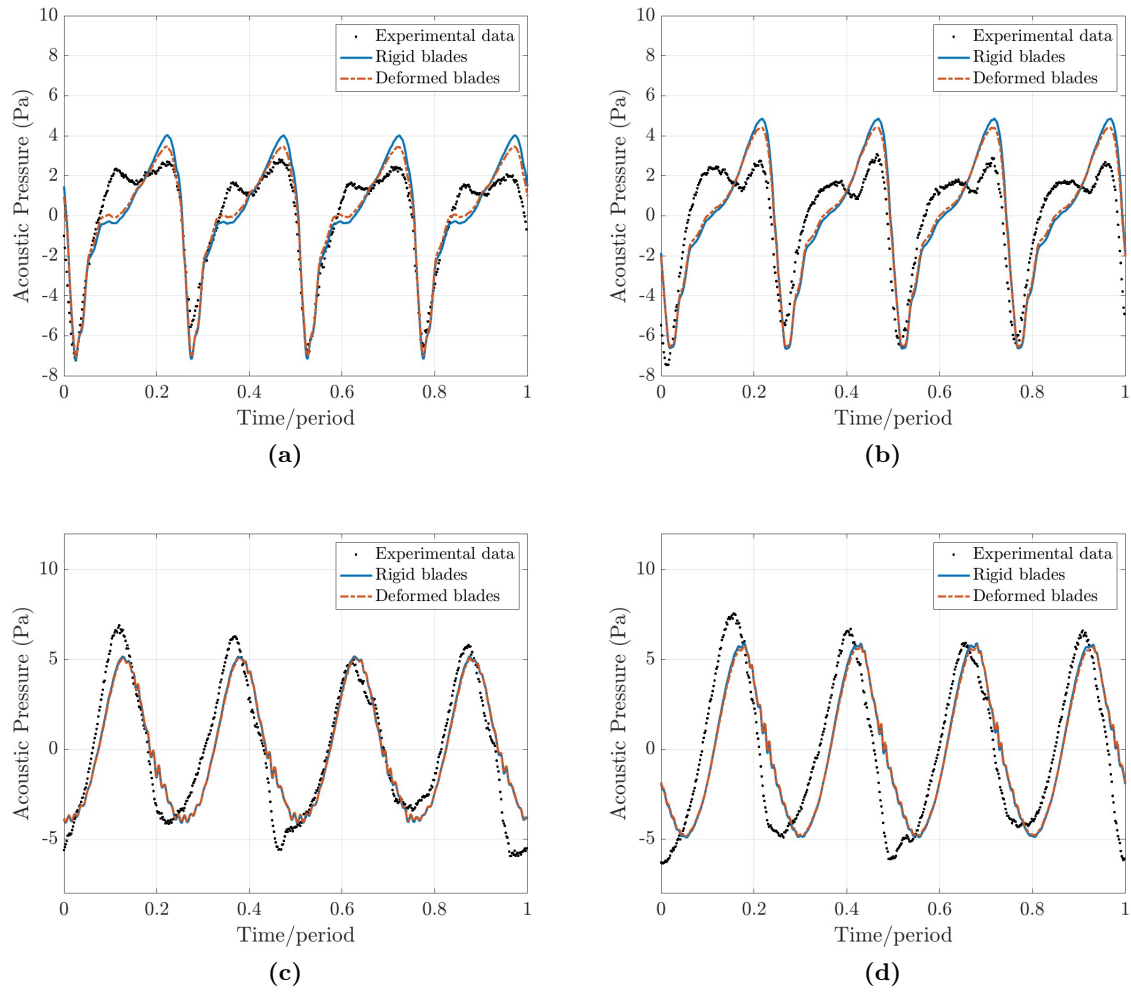


Figure 4.20: Comparison between numerical and experimental data, microphones coordinates expressed in meters. (a): $(x_m, y_m, z_m) = (-3.99, 1.62, -2.30)$; (b): $(x_m, y_m, z_m) = (-3.99, 2.70, -2.30)$; (c): $(x_m, y_m, z_m) = (2.01, 1.62, -2.30)$; (d): $(x_m, y_m, z_m) = (2.01, 2.70, -2.30)$.

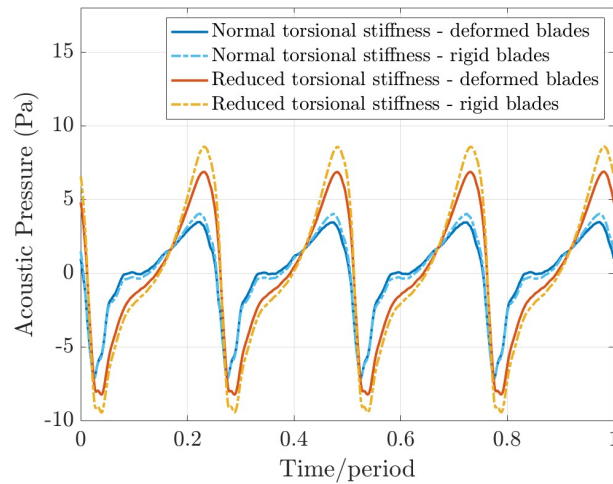


Figure 4.21: Effect of increased torsion deformation on noise signature.

Finally, in order to identify the sensitivity of the radiated noise on the deformation of the boundary, the aeroelastic and aeroacoustic analyses are repeated for rotor blades with torsional stiffness equal to 1/3 of the original one. For the microphone located at $x_m = -3.99$ m, $y_m = 1.62$ m, $z_m = -2.30$ m, Fig. 4.21 presents the comparison among the noise signatures obtained by deformable-boundary and rigid-boundary formulations, for the two different blade torsional stiffnesses, proving that when the torsional deformation increases the effect of boundary deformation on noise propagation determined by integral formulations becomes significant.

It is important to note that, in all cases analyzed in Fig. 4.21, the aeroelastic solution is obtained by considering aerodynamic loads that account for blade deformation solely through the impermeability boundary conditions applied to the undeformed blade geometry. In contrast, the effects of geometric boundary deformation are fully incorporated in the acoustic radiation integral formulation, through the corresponding variations of the local normal vector and the surface Jacobian, denoted by $\mathbf{n}$ and J_s , respectively.

To evaluate their influence on the problem under investigation, Fig. 4.22 reports the Root Mean Square (RMS) values

$$RMS^T(f) = \sqrt{\frac{1}{T} \int_0^T [f(t)]^2 dt}$$

computed over the blade surface and over one rotor revolution period, T , for the terms $f_{|\dot{\mathbf{n}}|J_s} = |\dot{\mathbf{n}}|J_s$ and $f_{|\mathbf{n}|\dot{J}_s} = |\mathbf{n}|\dot{J}_s$, with $\dot{\mathbf{n}}$ denoting the rate of change of the local normal due to deformation (rigid body motion effect is omitted). Therefore, these terms represent the contributions arising specifically from the dynamic deformation of the boundary (see, for instance, Eq. (3.26)). Figure 4.22 compares these values computed for the nominal torsional stiffness configuration (see Figs. 4.22a and 4.22c) with those obtained for the reduced torsional stiffness configuration (see Figs. 4.22b and 4.22d). As expected, the contour plots clearly indicate that both $RMS^T(f_{|\dot{\mathbf{n}}|J_s})$ and $RMS^T(f_{|\mathbf{n}|\dot{J}_s})$ attain larger magnitudes in the reduced torsional stiffness case. Moreover, the results show that the deformation effects associated with $f_{|\dot{\mathbf{n}}|J_s}$ are dominant, as evidenced by the different scales adopted in the graphical representations.

This behaviour is intrinsically related to the nature of the blade deformation, which predominantly alters the orientation of the surface and, consequently, the local normal vector, rather than its area. This represents a substantial difference with respect to the pulsating-sphere problem, in which the only non-zero contribution arising from the dynamic deformation is associated with the term $f_{|\mathbf{n}|\dot{J}_s}$ (see Sec. 4.5.3).

The significant impact of increased torsional deformation on the noise signatures is further confirmed by the results presented in Fig. 4.23 and Fig. 4.24, where the difference between the overall sound pressure level (OASPL) evaluated by the deformable-boundary and the rigid-boundary integral formulations over a hemisphere of radius $R = 4$ m centered at the rotor hub is shown, both for the nominal blade torsional stiffness (Fig. 4.23) and for the reduced blade torsional stiffness (Fig. 4.24).

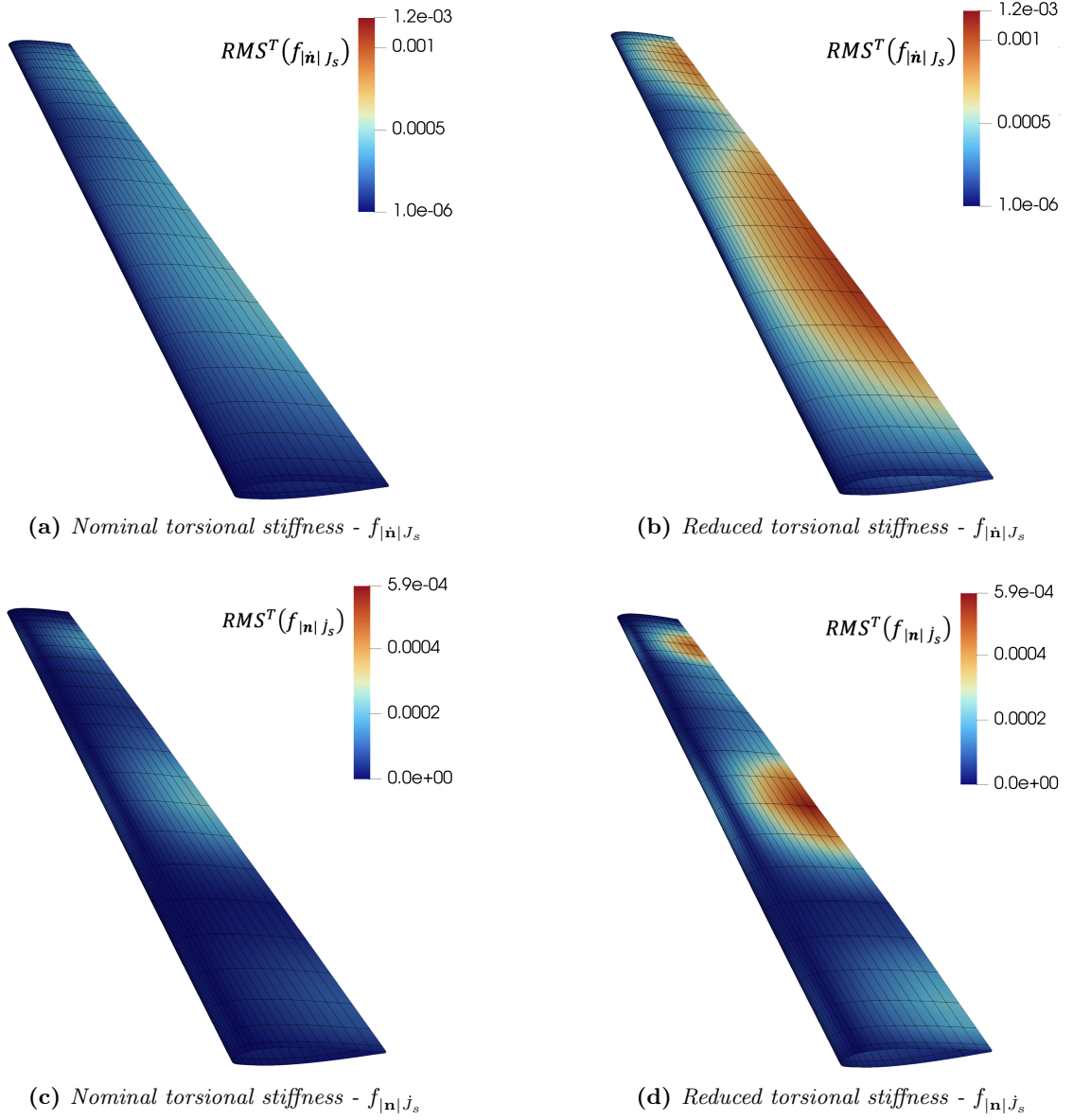


Figure 4.22: RMS^T values of the terms $f_{|\dot{n}|J_s}$ and $f_{|n|J_s}$ computed over the blade surface and over one rotor revolution period. Comparison between nominal and reduced torsional stiffness configurations.

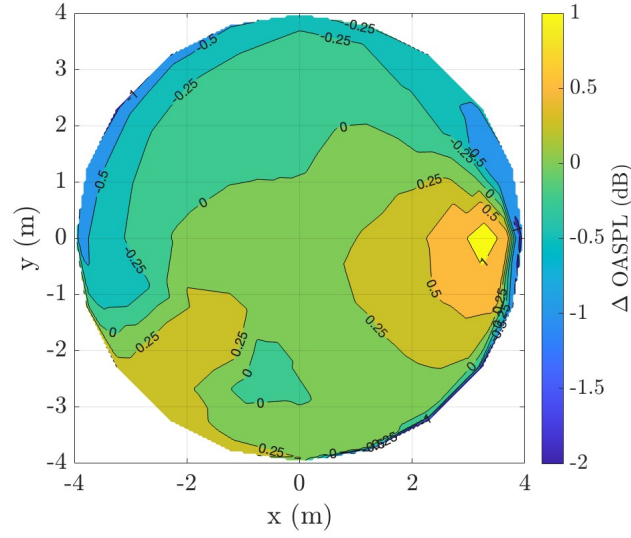


Figure 4.23: Difference between OASPL evaluated by deformable-boundary and rigid-boundary integral formulations over a hemisphere centered at the rotor hub for nominal torsional stiffness.

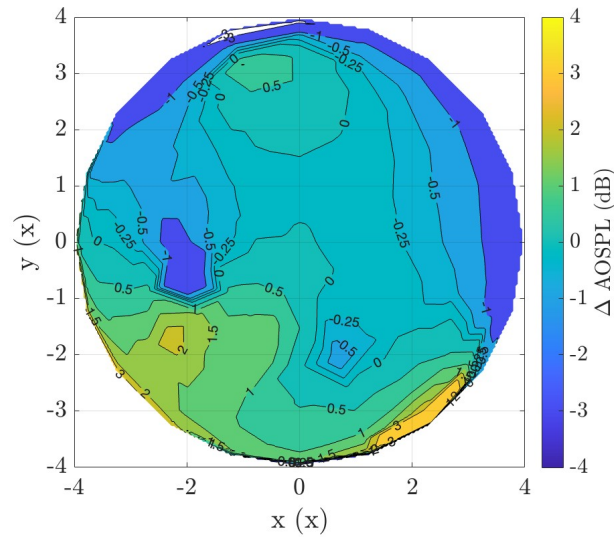


Figure 4.24: Difference between OASPL evaluated by deformable-boundary and rigid-boundary integral formulations over a hemisphere centered at the rotor hub for reduced torsional stiffness.

Chapter 5

Noise scattered by deforming bodies

This chapter deals with the theoretical and numerical analysis of sound scattering from deforming bodies. The objective is the development of advanced predictive tools, based on boundary integral formulations, capable of accurately describing noise scattered by moving surfaces subject to deformation.

In the first part of this chapter, the novel deformable-boundary integral formulation presented in the previous chapters is applied to solve the linear wave equation for the scattered perturbation field described in terms of either velocity potential or acoustic pressure, extending the analysis previously presented in [76]. As for the propagation of noise produced by a pulsating sphere, a harmonic-balance, multi-harmonic solution algorithm is applied, in order to include the nonlinear effects generated both by the kinematics of deformable surfaces and to overcome the stability issues arising when high-frequency scattering problems are solved through time-marching solution methods. The method has been successfully tested on rigid and pulsating spherical scatterers, thus providing the prediction of body deformation effects on the directivity of the scattered noise.

The second part of the chapter extends the analysis presented in [77] to include the fluid nonlinearities relating to the flow generated by the dynamic deformation of the scatterer surfaces. These appear as forcing terms in the wave equation of the velocity potential, Eq. (2.8). An innovative iterative scheme, called “velocity-potential cascade”, is introduced, capable of solving the corresponding nonlinear problem of sound scattered by deforming bodies via a harmonic-balance approach. The effectiveness of the approach is evaluated through simulations of the sound scattered by a pulsating sphere impinged by a plane wave. The convergence of the iterative procedure and the effects on the angular distribution of the scattered sound of the amplitude and frequency of the pulsation, as well as the amplitude of the incident signal, are examined and discussed.

5.1 Linear scattering formulations

In this section the linear sound scattering problem is examined through two different descriptions of the perturbation field: one in terms of the acoustic pressure governed by the FWHE and one in terms of the velocity potential.

In principle, velocity potential perturbations and acoustic pressure are scalar fields that can equivalently describe scattering acoustic phenomena. However, this is entirely true for those phenomena not significantly affected by the presence of vorticity. In this case, acoustic pressure formu-

lations provide more accurate predictions by appropriately including the nonlinear terms given by the Lighthill stress tensor. At the same time, it has been shown that when sound scattering is not significantly affected by the presence of vorticity and the scatterer is moving (as in most applications of aeronautical interest), the velocity potential formulation is convenient because it provides more accurate linear solutions than those given by the Lighthill or Ffowcs-Williams and Hawkings equations (see, for instance, Ref. [23], where it is proven that the different approaches are perfectly equivalent only if all nonlinear terms are included). For these reasons, it is appropriate to develop sound scattering solutions expressed in terms of both velocity potential and acoustic pressure.

This section is structured as follows. The boundary integral solution of the scattered field formulated in terms of the acoustic pressure is presented in Sec. 5.1.1, with a detailed description of a computationally-efficient algorithm which provides the frequency spectrum of the multi-harmonic scattered field (as shown later, it must necessarily be used for high-frequency impinging waves for which time-marching approaches present numerical inaccuracies). Next, Sec. 5.1.2 provides the boundary integral solution of the scattered field formulated in terms of the velocity potential, from which the frequency-domain solution can be expressed by straightforward application of the harmonic-balance, multi-harmonic algorithm introduced above.

5.1.1 Sound scattering described in terms of acoustic pressure

Let us consider the acoustic pressure governed by the acoustic analogy expressed by the FWHE, Eq. (2.6) [47]. Then, the integral formulation solution for the acoustic pressure field bounded by deformable surfaces can be derived from the general integral solution Eq. (3.25) suitably specialized, as given in chapter 2.

Denoting by $\mathbf{P} = (p - p_0)\mathbf{I}$ the compressive stress tensor (the viscous term appearing in the definition of $\mathbf{P}$ given in Sec. 2.1 is a negligible source of sound and is typically neglected) and by $\mathbf{v}$ and $\mathbf{u}$, respectively, the velocity of the points of the boundary surface and the local velocity of the fluid with respect to the FFR, Eq. (2.10) reproduces the FWHE for a generic porous boundary surface when $\Psi = p'$, $\sigma^\Psi = -\nabla \cdot \nabla \cdot (H\mathbf{T})$, where $\mathbf{T}$ denotes the Lighthill stress tensor, $\mathbf{Z} = \mathbf{P} + \rho\mathbf{u} \otimes (\mathbf{u} - \mathbf{v})$ and $\mathbf{w} = -\rho_0\mathbf{v} - \rho(\mathbf{u} - \mathbf{v})$, while $\mathbf{z} = 0$, $k_1 = 0$ and $k_2 = 0$ [57]. Thus, suitably re-arranging the integral formulation solution of the general wave equation in Eq. (3.25) expressed in the FFR, neglecting the nonlinear contribution of σ^Ψ , and assuming $(p - p_0) \approx p'$, the following representation of the external acoustic pressure radiated by deformable porous surfaces is readily obtained

$$\begin{aligned} p'(\mathbf{x}_*, t_*) = & - \iint_{\Omega} \left[p' \left(\mathbf{n} \cdot \nabla G_0 J + \frac{\partial}{\partial t} \Big|_{\xi} \left[\frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 J \right] \frac{1}{\Theta} \right) + p' \frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 \frac{J}{\Theta} \right] \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} \rho_0 \left[\frac{\partial}{\partial t} \Big|_{\xi} [\mathbf{v} \cdot \mathbf{n} G_0 J] \frac{1}{\Theta} \right] \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} \left[\rho \mathbf{u}^- \cdot \mathbf{n} \mathbf{u} \cdot \nabla G_0 J + \frac{\partial}{\partial t} \Big|_{\xi} \left[\rho \mathbf{u}^- \cdot \mathbf{n} \left(1 + \frac{\mathbf{u} \cdot \mathbf{e}_r}{c_0} \right) G_0 J \right] \right] \frac{1}{\Theta} \Big|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \end{aligned} \quad (5.1)$$

where $G_0 = -1/(4\pi r)$, with $r = |\mathbf{x}_* - \mathbf{x}|$ denoting the distance between the emitting point $\mathbf{x}(\xi^1, \xi^2, \xi^3, t)$ and the receiving point $\mathbf{x}_*$, whereas $(\dot{\dots}) = \partial(\dots)/\partial t|_{\xi}$. In addition, $\mathbf{u}^- = \mathbf{u} - \mathbf{v}$, $\Theta = 1 - \mathbf{M} \cdot \mathbf{e}_r$ where $\mathbf{M} = \mathbf{v}/c_0$ and $\mathbf{e}_r = \mathbf{r}/r$, while $J = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|/\Theta$ is the Jacobian of the

curvilinear co-ordinates transformation, with $\mathbf{a}_\beta$ denoting the covariant basis vectors of the curvilinear coordinate system. Furthermore, the symbol $(\dots)|_{\xi^3=0}$ indicates evaluation over the boundary surface, whereas, for $g = t - t_* + \theta$, the symbol $(\dots)|_{g=0}$ represents evaluation at the time $t = t_* - \theta$.

Following the approach outlined in Ref. [48], let us extend the acoustic pressure formulation by considering N_Ω surfaces. To simplify the notation, the first integral of Eq. (5.1) is called $\mathcal{I}_\Omega^p(p, \dot{p})$, the second integral relating to the surface velocity is called $\mathcal{I}_\Omega^v(\mathbf{v})$, and the third integral accounting for the possible porosity of the surface is called $\mathcal{I}_\Omega^u(\rho, \mathbf{u})$. This yields

$$p'(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega} \mathcal{I}_{\Omega_j}^p(p', \dot{p}') + \sum_{j=1}^{N_\Omega} \mathcal{I}_{\Omega_j}^v(\mathbf{v}) + \sum_{j=1}^{N_\Omega} \mathcal{I}_{\Omega_j}^u(\rho, \mathbf{u}) \quad (5.2)$$

For scattering applications the total acoustic pressure field (in the following named p instead of p' for the sake of symbol conciseness) is conceived as the superposition of an incident component, p^I , independent of the scatterer, and a component due to the scatterer influence, p^B . Let us call Ω^I the (eventually porous) surface that emits the incident field, which is considered independent of the presence of the solid scattering surfaces, Ω_j^S , where $\mathbf{u}^- \cdot \mathbf{n} = 0$. The incident field therefore satisfies the following equation

$$p^I(\mathbf{x}_*, t_*) = \mathcal{I}_{\Omega^I}^p(p^I, \dot{p}^I) + \mathcal{I}_{\Omega^I}^v(\mathbf{v}) + \mathcal{I}_{\Omega^I}^u(\rho^I, \mathbf{u}^I) \quad (5.3)$$

The substitution of the combination of incident and body-induced pressure fields, $p = p^I + p^B$, in Eq. (5.2) yields (note that $\mathcal{I}_{\Omega_j^S}^u(\rho, \mathbf{u}) = 0, \forall j$)

$$p^I(\mathbf{x}_*, t_*) + p^B(\mathbf{x}_*, t_*) = \mathcal{I}_{\Omega^I}^p(p^I, \dot{p}^I) + \mathcal{I}_{\Omega^I}^v(\mathbf{v}) + \mathcal{I}_{\Omega^I}^u(\rho^I, \mathbf{u}^I) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^p(p^I + p^B, \dot{p}^I + \dot{p}^B) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^v(\mathbf{v})$$

from which, considering Eq. (5.3), the following expression for the pressure component due to the presence of scatterers is obtained

$$p^B(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^p(p^I + p^B, \dot{p}^I + \dot{p}^B) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^v(\mathbf{v}) \quad (5.4)$$

Next, it is convenient to distinguish between two contributions to pressure p^B : one related to the motion of the body, independent of the incident field, p^M , and one that accounts for the scattered field p^S . Thus, for $p^B = p^M + p^S$, Eq. (5.4) gives the relation

$$p^S(\mathbf{x}_*, t_*) + p^M(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^p(p^I + p^S + p^M, \dot{p}^I + \dot{p}^M + \dot{p}^S) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^v(\mathbf{v}) \quad (5.5)$$

with p^M satisfying, by definition, the equation of pressure perturbations generated by body motion (often referred to as thickness noise in aeronautical applications)

$$p^M(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^p(p^M, \dot{p}^M) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^v(\mathbf{v}) \quad (5.6)$$

Combining Eq. (5.5) with Eq. (5.6), and recalling that the contribution of the acoustic pressure to the boundary integral terms is linear, the following expression for the scattered component of the acoustic pressure is finally obtained

$$p^S(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}^p(p^S + p^I, \dot{p}^S + \dot{p}^I)$$

From the above expression, it turns out that from the knowledge of the incident pressure field, the scattered acoustic pressure distributed over the solid surface, $\mathcal{S}$, of a single scatterer can be evaluated through solution of the following boundary integral equation derived from Eq. (5.1), with $\mathbf{x}_* \rightarrow \mathcal{S}$ (see also [48, 76])

$$\begin{aligned} [1 - \lambda(\mathbf{x}_*, t_*)]p^S(\mathbf{x}_*, t_*) &= \lambda(\mathbf{x}_*, t_*)p^I(\mathbf{x}_*, t_*) \\ &- \iint_{\Omega} \left[p^S \left(\mathbf{n} \cdot \nabla G_0 J + \frac{\partial}{\partial t} \left|_{\xi} \left[\frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 J \right] \frac{1}{\Theta} \right) + \dot{p}^S \frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 \frac{J}{\Theta} \right] \Big|_{\xi^3=0}^{g=0} d\xi^1 d\xi^2 \\ &- \iint_{\Omega} \left[p^I \left(\mathbf{n} \cdot \nabla G_0 J + \frac{\partial}{\partial t} \left|_{\xi} \left[\frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 J \right] \frac{1}{\Theta} \right) + \dot{p}^I \frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 \frac{J}{\Theta} \right] \Big|_{\xi^3=0}^{g=0} d\xi^1 d\xi^2 \end{aligned} \quad (5.7)$$

where, for $M_n = \mathbf{M} \cdot \mathbf{n}$, the contribution of the coefficient $\lambda = 0.5/(1 - M_n^2)$ represents the so-called free terms deriving from the singularity of the kernel function which arises when the position of the observer tends to the limit on the emitting surface, while the right-hand-side of the equation must be intended as the Cauchy principal value of the surface integral.

The expression of the singularity free term is demonstrated in [48] for rigid bodies. It can be demonstrated that it remains valid also for deformable surfaces (this can be straightforwardly accomplished by repeating the demonstration of [48] by applying the Prandtl-Glauert transformation related to the total local velocity Mach number).

Once Eq. (5.7) is numerically solved by a BEM algorithm, the scattered field at any point of the external domain can be readily evaluated by applying Eq. (5.1) as an integral representation that exploits the knowledge of the scattered pressure over the body.

Multi-harmonic scattering solution algorithm

The numerical solution of Eq. (5.7) can be determined by application of the BEM approach illustrated in Sec. 4.1, for which the scatterer surface is discretized into quadrilateral panels where the acoustic pressure is assumed to be uniformly distributed, and the time discretisation of the equations obtained by imposing the satisfaction of Eq. (5.7) at the centers of the panels (collocation method) yields the system of algebraic equations to be solved.

However, acoustic problems are typically represented in the frequency domain. This occurs because acoustic phenomena typically deal with high-frequency signals, for which time-domain solution algorithms are subject to the occurrence of numerical instabilities that prevent the evaluation of accurate and reliable solutions.

For rigid scatterers the frequency-domain version of the boundary integral equation for the scattered pressure is readily obtained by applying the Fourier transform to Eq. (5.7). In that case, each

harmonic of the incident pressure corresponds to a scattered pressure signal of the same harmonic, whose amplitude and phase depend on the shape of the scatterer and the position of the observer (linear, time-invariant coefficient problem). Instead, akin to the problem of sound radiated by a pulsating sphere examined in Sec. 4.5, in the case of deformable scatterers the influence coefficients derived from the BEM solution algorithm are not constant in time and each harmonic of the incident pressure corresponds to a multi-harmonic scattered pressure signal due to the combination with the harmonics of the influence coefficients (linear, time-variant problem). Thus, to determine the frequency-domain solution of sound scattered by deformable surfaces the multi-harmonic algorithm already introduced in Sec. 4.5.2 is applied to the integral formulation for scattered pressure.

To this aim, let us consider the surface of the scatterer divided into N_p panels. By imposing the satisfaction of the space-discretized version of Eq. (5.7) at the center of the k -th panel we obtain

$$\begin{aligned} (1 - \lambda_k) p_k^S(t_*) &= \lambda_k p_k^I(t_*) + \sum_{m=1}^{N_p} B_{km} p_m^I(t_* - \theta_{km}) + \sum_{m=1}^{N_p} C_{km} \dot{p}_m^I(t_* - \theta_{km}) \\ &+ \sum_{m=1}^{N_p} B_{km} p_m^S(t_* - \theta_{km}) + \sum_{m=1}^{N_p} C_{km} \dot{p}_m^S(t_* - \theta_{km}) \end{aligned} \quad (5.8)$$

where p_k^S, p_k^I denote scattered and incident acoustic pressure evaluated at the center of the k -th panel, θ_{km} denotes the delay θ related to the observer located at the center of the k -th panel and the emitting source located at the center of the m -th panel, whereas

$$B_{km} = - \iint_{\Omega_m} \left(\mathbf{n} \cdot \nabla G_0 J + \frac{\partial}{\partial t} \left|_{\xi} \left[\frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 J \right] \frac{1}{\Theta} \right) \right|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \quad (5.9)$$

and

$$C_{km} = - \iint_{\Omega_m} \left(\frac{\mathbf{n} \cdot \mathbf{e}_r}{c_0} G_0 \frac{J}{\Theta} \right) \bigg|_{\xi^3=0, g=0} d\xi^1 d\xi^2 \quad (5.10)$$

where Ω_m represents the region in the (ξ_1, ξ_2) domain where the surface of the m -th surface panel is mapped. Due to the deformation of the surface its geometrical parameters, and thus the coefficients λ_k, B_{km} , and C_{km} , are time-variant (it is worth recalling that the coefficients B_{km} are intended as Cauchy principal values of the integral when $k = m$).

To determine a frequency-domain expression of the scattered field, let us assume that the incident pressure is harmonic and that the influence coefficients given by application of the BEM approach are periodic, with their fundamental harmonic being a multiple or a sub-multiple of the harmonic of the incident signal. Note that this assumption is not too limiting or unrealistic. It is applicable, for example, in all those cases where the deformation of the scatterer is the result of the response of an elastic structure to incident pressure. In particular, when the structural dynamics operator is linear the deformation is mono-chromatic with the same harmonic as the incident pressure, whereas when nonlinear geometric contributions appear due to large deformations, these give rise to multi-harmonic contributions that are multiples or sub-multiples of the harmonic of the forcing signal.

Thus, it is useful to introduce the Fourier series expressing the scattered pressure and its time

derivative, namely

$$p_m^S(t_* - \theta_{km}) = p_{m0}^S + \sum_{n=1}^{N_a} (p_{mn}^{Sc} \cos[n\omega(t_* - \theta_{km})] + p_{mn}^{Ss} \sin[n\omega(t_* - \theta_{km})]) \quad (5.11)$$

$$\dot{p}_m^S(t_* - \theta_{km}) = \sum_{n=1}^{N_a} n\omega (-p_{mn}^{Sc} \sin[n\omega(t_* - \theta_{km})] + p_{mn}^{Ss} \cos[n\omega(t_* - \theta_{km})]) \quad (5.12)$$

where ω represents the fundamental harmonic, N_a is the number of harmonics included in the analysis, p_{mn}^{Sc} and p_{mn}^{Ss} are the cosine and sine components of the n -th harmonic of the scattered pressure at the center of the m -th panel, while p_{m0}^S denotes its average value. The substitution of the above equations in Eq. 5.8 yields

$$p_k^S(t_*) = \sum_{m=1}^{N_p} \hat{B}_{km}(t_*) p_{m0}^S + \sum_{m=1}^{N_p} \sum_{n=1}^{N_a} D_{km}^{nc}(t_*) p_{mn}^{Sc} + \sum_{m=1}^{N_p} \sum_{n=1}^{N_a} D_{km}^{ns}(t_*) p_{mn}^{Ss} + f_k^I(t_*) \quad (5.13)$$

where

$$D_{km}^{nc}(t_*) = [\delta_{km} \lambda_m(t_*) + B_{km}(t_*)] \cos[n\omega(t_* - \theta_{km})] - n\omega C_{km}(t_*) \sin[n\omega(t_* - \theta_{km})] \quad (5.14)$$

$$D_{km}^{ns}(t_*) = [\delta_{km} \lambda_m(t_*) + B_{km}(t_*)] \sin[n\omega(t_* - \theta_{km})] + n\omega C_{km}(t_*) \cos[n\omega(t_* - \theta_{km})] \quad (5.15)$$

$$\hat{B}_{km}(t_*) = \delta_{km} \lambda_m(t_*) + B_{km}(t_*) \quad (5.16)$$

and the forcing term f_k^I is related to incident pressure and thus is a known term defined as

$$f_k^I(t_*) = \lambda_k p_k^I(t_*) + \sum_{m=1}^{N_p} B_{km}(t_*) p_m^I(t_* - \theta_{km}) + \sum_{m=1}^{N_p} C_{km}(t_*) \dot{p}_m^I(t_* - \theta_{km}) \quad (5.17)$$

Next, for $\mathbf{D}^{nc}$, $\mathbf{D}^{ns}$ and $\hat{\mathbf{B}}$ denoting the $N_p \times N_p$ matrices collecting, respectively, the periodic coefficients in Eqs. (5.14), (5.15) and (5.16) whose Fourier series expansions read

$$\begin{aligned} \mathbf{D}^{nc}(t_*) &= \mathbf{D}_0^{nc} + \sum_{h=1}^{N_a} [\mathbf{D}_{hc}^{nc} \cos(h\omega t_*) + \mathbf{D}_{hs}^{nc} \sin(h\omega t_*)] \\ \mathbf{D}^{ns}(t_*) &= \mathbf{D}_0^{ns} + \sum_{h=1}^{N_a} [\mathbf{D}_{hc}^{ns} \cos(h\omega t_*) + \mathbf{D}_{hs}^{ns} \sin(h\omega t_*)] \\ \hat{\mathbf{B}}(t_*) &= \hat{\mathbf{B}}_0 + \sum_{h=1}^{N_a} [\hat{\mathbf{B}}_{hc} \cos(h\omega t_*) + \hat{\mathbf{B}}_{hs} \sin(h\omega t_*)] \end{aligned}$$

the following system of algebraic equations with unknowns the Fourier coefficients of the scattered

pressure at the centers of the panels is obtained from Eq. (5.13), applied for $k = 1, N_p$

$$\begin{bmatrix} \mathbf{p}_0^S \\ \mathbf{p}_{1c}^S \\ \mathbf{p}_{1s}^S \\ \vdots \\ \mathbf{p}_{N_{ac}}^S \\ \mathbf{p}_{N_{as}}^S \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{B}}_0 \\ \hat{\mathbf{B}}_{1c} \\ \hat{\mathbf{B}}_{1s} \\ \vdots \\ \hat{\mathbf{B}}_{N_{ac}} \\ \hat{\mathbf{B}}_{N_{as}} \end{bmatrix} \mathbf{p}_0^S + \sum_{n=1}^{N_a} \begin{bmatrix} \mathbf{D}_0^{nc} \\ \mathbf{D}_{1c}^{nc} \\ \mathbf{D}_{1s}^{nc} \\ \vdots \\ \mathbf{D}_{N_{ac}}^{nc} \\ \mathbf{D}_{N_{as}}^{nc} \end{bmatrix} \mathbf{p}_{nc}^S + \sum_{n=1}^{N_a} \begin{bmatrix} \mathbf{D}_0^{ns} \\ \mathbf{D}_{1c}^{ns} \\ \mathbf{D}_{1s}^{ns} \\ \vdots \\ \mathbf{D}_{N_{ac}}^{ns} \\ \mathbf{D}_{N_{as}}^{ns} \end{bmatrix} \mathbf{p}_{ns}^S + \begin{bmatrix} \mathbf{f}_0^I \\ \mathbf{f}_{1c}^I \\ \mathbf{f}_{1s}^I \\ \vdots \\ \mathbf{f}_{N_{ac}}^I \\ \mathbf{f}_{N_{as}}^I \end{bmatrix} \quad (5.18)$$

In the above equation, $\mathbf{p}_{nc}^S$ and $\mathbf{p}_{ns}^S$ are vectors collecting, respectively, the cosine and sine components of the n -th harmonic of the scattered pressure at the N_p panel centers, whereas $\mathbf{p}_0^S$ is the vector collecting their average values. Similarly, $\mathbf{f}_{nc}^I$ and $\mathbf{f}_{ns}^I$ are vectors collecting, respectively, the cosine and sine components of the n -th harmonic of the forcing term in Eq. (5.17) evaluated at the N_p panel centers, while $\mathbf{f}_0^I$ collects the corresponding average values.

Equation (5.18) can be conveniently rearranged in the following equivalent form

$$\begin{bmatrix} \mathbf{p}_0^S \\ \mathbf{p}_{1c}^S \\ \mathbf{p}_{1s}^S \\ \vdots \\ \mathbf{p}_{N_{ac}}^S \\ \mathbf{p}_{N_{as}}^S \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{B}}_0 & \mathbf{D}_0^{1c} & \mathbf{D}_0^{1s} & \dots & \mathbf{D}_0^{N_{ac}} & \mathbf{D}_0^{N_{as}} \\ \hat{\mathbf{B}}_{1c} & \mathbf{D}_{1c}^{1c} & \mathbf{D}_{1c}^{1s} & \dots & \mathbf{D}_{1c}^{N_{ac}} & \mathbf{D}_{1c}^{N_{as}} \\ \hat{\mathbf{B}}_{1s} & \mathbf{D}_{1s}^{1c} & \mathbf{D}_{1s}^{1s} & \dots & \mathbf{D}_{1s}^{N_{ac}} & \mathbf{D}_{1s}^{N_{as}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{\mathbf{B}}_{N_{ac}} & \mathbf{D}_{N_{ac}}^{1c} & \mathbf{D}_{N_{ac}}^{1s} & \dots & \mathbf{D}_{N_{ac}}^{N_{ac}} & \mathbf{D}_{N_{ac}}^{N_{as}} \\ \hat{\mathbf{B}}_{N_{as}} & \mathbf{D}_{N_{as}}^{1c} & \mathbf{D}_{N_{as}}^{1s} & \dots & \mathbf{D}_{N_{as}}^{N_{ac}} & \mathbf{D}_{N_{as}}^{N_{as}} \end{bmatrix} \begin{bmatrix} \mathbf{p}_0^S \\ \mathbf{p}_{1c}^S \\ \mathbf{p}_{1s}^S \\ \vdots \\ \mathbf{p}_{N_{ac}}^S \\ \mathbf{p}_{N_{as}}^S \end{bmatrix} + \begin{bmatrix} \mathbf{f}_0^I \\ \mathbf{f}_{1c}^I \\ \mathbf{f}_{1s}^I \\ \vdots \\ \mathbf{f}_{N_{ac}}^I \\ \mathbf{f}_{N_{as}}^I \end{bmatrix} \quad (5.19)$$

Then, denoting with $\mathbf{p}^S$ the vector of $N_p(2N_a + 1)$ elements collecting the $\mathbf{p}_0^S$, $\mathbf{p}_{nc}^S$ and $\mathbf{p}_{ns}^S$ Fourier coefficients, with $\mathbf{f}^I$ the vector of $N_p(2N_a + 1)$ elements collecting the $\mathbf{f}_0^I$, $\mathbf{f}_{nc}^I$ and $\mathbf{f}_{ns}^I$ Fourier coefficients, and with $\mathbf{D}$ the matrix of dimensions $N_p(2N_a + 1) \times N_p(2N_a + 1)$ that collects the Fourier coefficients of the matrices $\mathbf{D}^{nc}$, $\mathbf{D}^{ns}$ and $\hat{\mathbf{B}}$, Eq. (5.19) can be re-written in the following compact form (see also [76])

$$\mathbf{p}^S = \mathbf{D} \mathbf{p}^S + \mathbf{f}^I \quad (5.20)$$

and, thus, the Fourier coefficients of the scattered pressure on each panel of the scatterer surface are given by

$$\mathbf{p}^S = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{f}^I \quad (5.21)$$

Once the values of the scattered pressure over the body's panels are evaluated, the scattered pressure in the field may be readily determined through the proposed numerical algorithm applied to the integral formulation conceived as an integral representation. In this case, the integral solution can be formally expressed through Eq. (5.8) with $\mathbf{x}_* \notin \mathcal{S}$ and hence $\lambda_k = 0$ (no singular integrals appear). Indeed, for N_o representing the number of external observers where the scattered field is evaluated, and for $\mathbf{p}_f^S$ denoting the vector of dimensions $N_o(2N_a + 1)$ which collects the Fourier coefficients of the scattered pressure at these points we have

$$\mathbf{p}_f^S = \mathbf{D}_f \mathbf{p}^S + \mathbf{f}_f^I \quad (5.22)$$

where $\mathbf{D}_f$ is a rectangular matrix that collects the Fourier components of the BEM coefficients for

external observers, while $\mathbf{f}_f^I$ collects the Fourier components of the contribution from the incident pressure expressed by Eq. (5.17) written for the same observers.

5.1.2 Sound scattering described in terms of velocity potential

Here, following a widely-used approach to study sound scattering problems, the perturbation field is expressed in terms of the velocity potential, $\varphi(\mathbf{x}, t)$, rather than acoustic pressure.

To this aim, let us enclose by a suitable porous surface $\Omega^I = \partial\mathcal{V}_I$ the sources of the incident field. For $\mathcal{V}_I$ denoting a subdomain of the inviscid fluid region $\mathcal{V}_f$ such that the nonlinear term σ in Eq. (2.8) can be considered negligible at all points $\mathbf{x} \in \mathcal{V}_f \setminus \mathcal{V}_I$ (see Fig. 5.1), the equation governing the velocity potential field reads

$$\nabla^2 \varphi - \frac{1}{c_0^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad \mathbf{x} \in \mathcal{V}_f \setminus \mathcal{V}_I$$

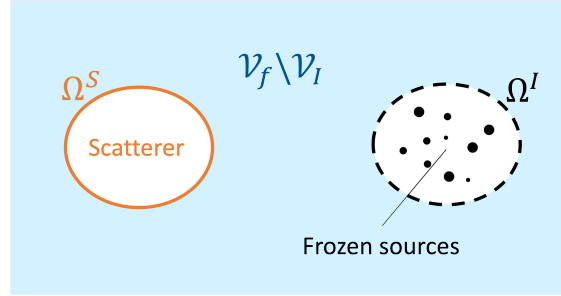


Figure 5.1: Representation of the scattered surface and incident field sources.

The integral solution of this equation is given in Eq. (4.6). Thus, considering N_Ω closed surfaces forming the boundary of the domain $\mathcal{V}_f \setminus \mathcal{V}_I$, and denoting all integrals on the right-hand side of Eq. (4.6) by $\mathcal{I}_\Omega(\varphi, \dot{\varphi})$, we can rewrite the boundary integral solution of the velocity potential in the following compact form

$$\varphi(\mathbf{x}_*, t_*) = \sum_{j=1}^{N_\Omega} \mathcal{I}_{\Omega_j}(\varphi, \dot{\varphi})$$

Observing that the incident velocity potential field satisfies the equation

$$\varphi^I(\mathbf{x}_*, t_*) = \mathcal{I}_{\Omega^I}(\varphi^I, \dot{\varphi}^I) \quad (5.23)$$

and that, since the incident velocity potential is considered frozen, i.e. independent of the presence of scatterers, also the following equation is satisfied (the surfaces of the scatterers behave like porous boundaries of the fluid domain for the incident field)

$$\varphi^I(\mathbf{x}_*, t_*) = \mathcal{I}_{\Omega^I}(\varphi^I, \dot{\varphi}^I) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}(\varphi^I, \dot{\varphi}^I) \quad (5.24)$$

for $\Omega^I \equiv \Omega_{N_\Omega}$ and Ω_j^S denoting the surface of the j -th scatterer, the combination of Eqs. (5.23) and (5.24) implies that the integral of the incident field on the surfaces of the scatterers is null,

namely

$$\sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}(\varphi^I, \dot{\varphi}^I) = 0 \quad (5.25)$$

Following the same solution procedure applied to the acoustic pressure field in the previous section, the velocity potential field is considered as the sum of a scatterer-independent contribution related to the incident field, φ^I , a contribution related to the motion of the scatterer, φ^M (if present), and finally a scattered component, φ^S . This gives

$$\varphi^I(\mathbf{x}_*, t_*) + \varphi^M(\mathbf{x}_*, t_*) + \varphi^S(\mathbf{x}_*, t_*) = \mathcal{I}_{\Omega_j^I}(\varphi^I, \dot{\varphi}^I) + \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}(\varphi^I + \varphi^M + \varphi^S, \dot{\varphi}^I + \dot{\varphi}^M + \dot{\varphi}^S)$$

Combining the above equation with Eq. (5.25), recalling that, by definition, the velocity potential field generated by the motion of the body satisfies the equation

$$\varphi^M = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}(\varphi^M, \dot{\varphi}^M)$$

with the boundary condition of body impermeability over each scatterer surface

$$\frac{\partial \varphi^M}{\partial n} = \mathbf{v} \cdot \mathbf{n}$$

and recalling that the contribution of the velocity potential to the boundary integral terms is linear, the boundary integral formulation for the scattered component of the velocity potential is expressed by the compact form

$$\varphi^S = \sum_{j=1}^{N_\Omega-1} \mathcal{I}_{\Omega_j^S}(\varphi^S, \dot{\varphi}^S) \quad (5.26)$$

which yields the following expanded form of the corresponding boundary integral equation written for a single scattering surface, $\mathcal{S}$, with $\mathbf{x}_* \in \mathcal{S}$ (see Eq. (4.6))

$$\begin{aligned} \frac{1}{2} \varphi^S(\mathbf{x}_*, t_*) = & \iint_{\Omega} \left[(1 - M_n^2) \chi^I G_0 - M_n \mathbf{M} \cdot \nabla_t \varphi^S G_0 - \varphi^S \frac{\partial G_0}{\partial n} + \frac{\partial \varphi^S}{\partial t} \Big|_{\xi} \frac{M_n}{c_0} G_0 \right] J \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \\ & + \iint_{\Omega} \frac{1}{\Theta} \frac{\partial}{\partial t} \Big|_{\xi} \left[\varphi^S G_0 J \left(-\frac{\mathbf{e}_r \cdot \mathbf{n}}{c_0} + \frac{M_n}{c_0} \right) \right] \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \end{aligned} \quad (5.27)$$

where $\nabla_t(\dots)$ denotes application of the tangential gradient operator. The definition of all terms appearing in this equation coincides with that already given for Eq. (5.7). Akin to the coefficient λ in Eq. (5.7), the coefficient $1/2$ at the left-hand side of the above equation accounts for the contribution of the free terms deriving from the singularity of the kernel function which arises when the position of the observer tends to the limit on the emitting surface, and thus the right-hand-side of the equation must be intended as the Cauchy principal value of the surface integral.

The boundary conditions to be associated with Eq. (5.27) is obtained by imposing the condition

of impermeability to the total potential which yields

$$\frac{\partial \varphi^S}{\partial n} + \frac{\partial \varphi^I}{\partial n} + \frac{\partial \varphi^M}{\partial n} = \mathbf{v} \cdot \mathbf{n} \quad \Rightarrow \quad \frac{\partial \varphi^S}{\partial n} = -\frac{\partial \varphi^I}{\partial n} = \chi^I \quad (5.28)$$

The boundary integral equation for the scattered velocity potential can be solved by application of the time-marching BEM approach described in Sec. 4.1, or through application of a the multi-harmonic solution algorithm to obtain the frequency-domain solution of Eq. (5.27). In this case, the same solution algorithm presented in Sec. 4.5.2 can be used, with the forcing vector suitably defined in terms of the boundary conditions in Eq. (5.28). Thus, the Fourier coefficients of the scattered velocity potential on each panel of the scattering surface can be determined as (see also [76])

$$\boldsymbol{\varphi}^S = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{f}^I \quad (5.29)$$

where matrix $\mathbf{D}$ is the same as that defined in Sec. 4.5.2, while $\mathbf{f}^I$ collects the Fourier coefficients of the following forcing term due to the boundary conditions

$$f_k(t_*) = \sum_{m=1}^{N_p} A_{km}(t_*) \chi_m^I(t_* - \theta_{km})$$

where A_{km} is given in Eq. (4.9).

5.2 Numerical results for linear scattering

This section considers cases of increasing complexity in acoustic scattering. Indeed, sound scattering from a rigid sphere, a pulsating sphere and a wavy deforming spherical surface is examined. This allows to highlight the ability and accuracy of the proposed solution method to study sound scattering from deforming bodies and, at the same time, to determine under which conditions significant multi-harmonic contributions and significant differences between predictions from pressure and velocity potential formulations arise.

First, the perturbation field scattered by a rigid sphere impinged by a plane wave is analysed, both in the time and frequency domains, and the results are compared with analytical solutions. It is shown that these two solution approaches are fully equivalent until too high harmonics appear in the spectrum of the scattered field, for which the time domain approach has difficulty converging and numerical inaccuracies can arise. Then, a scattering solution method based on a multi-harmonic frequency-domain approach is validated for pulsating spherical scatterers through comparison with analytical results, and the effect of body deformation on the directivity pattern of the scattered noise is examined. Two analytical linear solutions for sound scattered by pulsating spheres are derived in appendix G. First, in appendix G.1, the approximate first-order analytical solution introduced by Censor [49] for small-amplitude pulsations of the sphere is extended to second-order terms. Then, appendix G.2 presents a novel exact analytical solution valid for arbitrary subsonic pulsations.

All numerical results presented in this section are obtained through a discretization of the surface that ensures converged solutions. Specifically, as a result of a preliminary investigation of numerical convergence, the discretization panels used are $N_p = 600$, generated by dividing the sphere surface through 30 meridians and 20 parallels. The number of harmonics, N_a , considered in the multi-

harmonic solution algorithm depends on the problem examined. Converged results are obtained by increasing N_a until the significant harmonics of the output remain unchanged.

5.2.1 Sound scattered by a rigid sphere

Here, the integral formulations for sound scattering presented above are applied to the problem of a plane wave impinging a rigid sphere of radius r_0 . The the developed time-domain and frequency-domain BEM computational tools are validated against available analytical solutions.

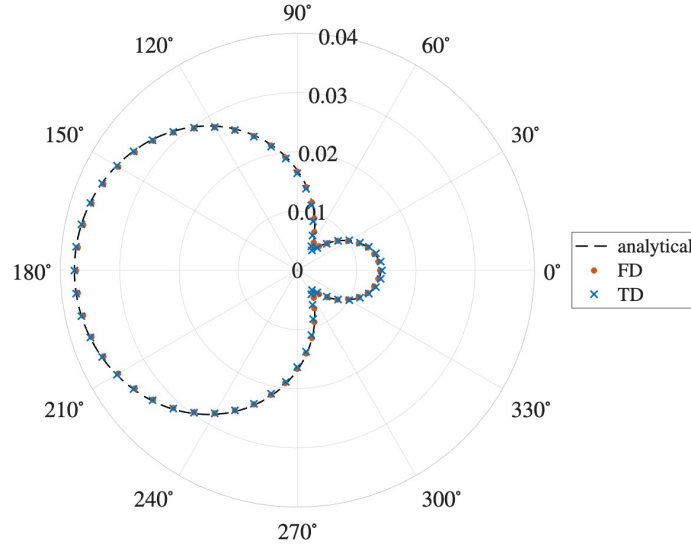


Figure 5.2: Directivity pattern of $|p^S|/|p^I|$ related to acoustic pressure scattered by a rigid sphere impinged by a plane wave, $k_I r_0 = 0.8$, $d/r_0 = 10$.

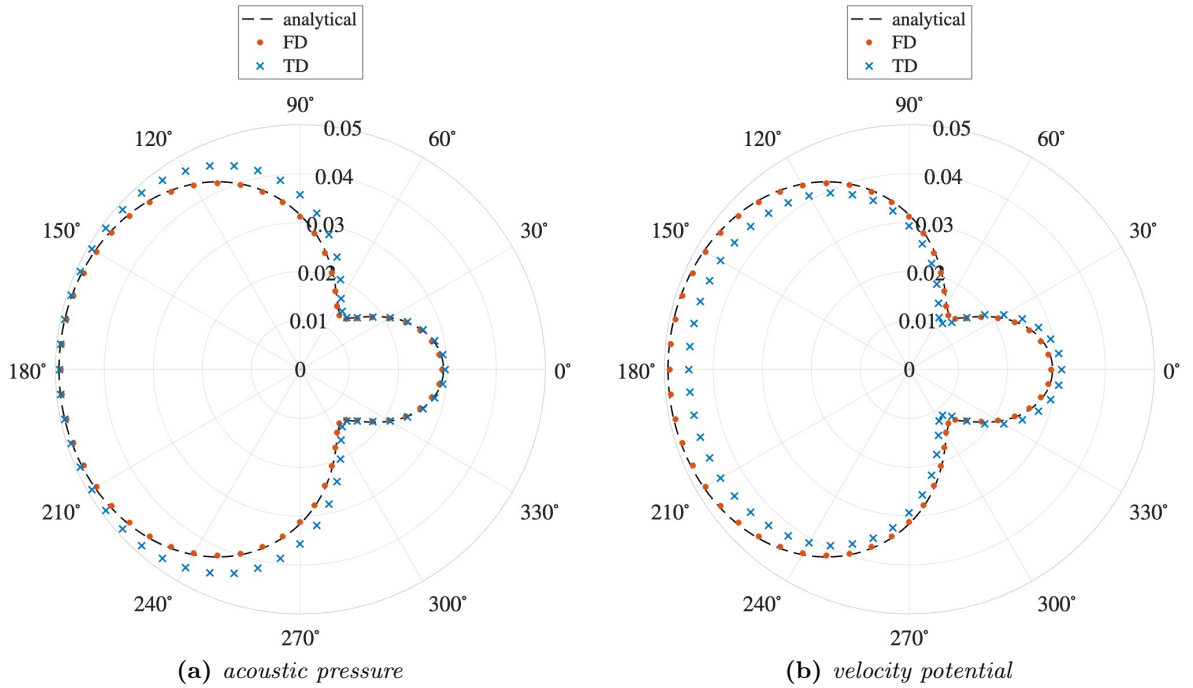


Figure 5.3: Directivity pattern of (a) $|p^S|/|p^I|$, and (b) $|\varphi^S|/|\varphi^I|$ related to acoustic pressure and velocity potential scattered by a rigid sphere impinged by a plane wave; $k_I r_0 = 1.2$, $d/r_0 = 10$.

Let us consider a set of microphones uniformly located on a circle at a distance $d = 10r_0$ from the sphere centre, and lying on a plane perpendicular to the plane of the impinging wave. For incident plane wave frequency, ω_I , such that $k_I r_0 = 0.8$ (with k_I denoting the incident wave number), Fig. 5.2 shows the comparison between the directivity patterns of the acoustic pressure scattered by the rigid sphere determined analytically [78] and numerically. The numerical results are related to the problem expressed in terms of acoustic pressure, and obtained by the [frequency-domain \(FD\)](#) formulation of Eqs. (5.21) and (5.22) and the [time-domain \(TD\)](#) solution determined by the direct application of Eq. (5.8). The agreement between the analytical solution and the two [BEM](#) solutions is excellent, thus validating the proposed boundary integral approach for both time-domain and frequency-domain acoustic scattering predictions. Note that, the result presented in Fig. 5.2 in terms of the scattered acoustic pressure is identical to that obtained in terms of the solution of the scattered velocity potential. This occurs because for the fixed, rigid sphere case the nonlinear terms neglected in the acoustic pressure and velocity potential scattering formulations discussed above are both negligible, and thus the corresponding integral solutions are fully equivalent [23].

However, similar comparisons performed for several values of the wave numbers of the impinging wave reveal that for $k_I r_0 < 1$ the level of agreement between the analytical solution and the time-domain and frequency-domain [BEM](#) predictions is comparable with that shown in Fig. 5.2, whereas for higher values of the parameter $k_I r_0$ the time-domain prediction deviates from the analytical solution due to the rise of numerical inaccuracies deriving from the approximation through which the delayed time derivative of the unknown scattered pressure is evaluated. For instance, this is shown in Fig. 5.3 for $k_I r_0 = 1.2$. From this figure we can notice that the accuracy of the time domain solution decreases with increasing frequency of the impinging wave, and that this problem affects both the solution given by the formulation expressed in terms of acoustic pressure (see Fig. 5.3a) and that given by the formulation expressed in terms of velocity potential through Eq. (5.27) or Eq. (5.29) (see Fig. Fig. 5.3b). The results in this figure are obtained by using a number of time steps per harmonic equal to that applied for the results in Fig. 5.2. The increase of this number does not lead to a converged solution, but provides less accurate predictions.

Conversely, the acoustic pressure and velocity potential solutions evaluated through the multi-harmonic approach coincide with the analytical one regardless of the wave number of the impinging wave.

5.2.2 Sound scattered by a deforming sphere

Once validated for rigid scatterers, the proposed boundary integral formulations are applied to the problem of a plane wave impinging a deforming sphere. The sphere radius is assumed to vary harmonically as follows

$$r_s(t) = r_0 + a \sin(\omega t) \quad (5.30)$$

while the impinging wave is assumed to travel along the x -axis of a Cartesian system of coordinates with the origin located at the center of the sphere, and is expressed as

$$\varphi^I(x, t) = A^I \sin(\omega_I t - k_I x) \quad (5.31)$$

First, time-domain and frequency-domain solutions are compared to confirm their perfect equivalence within the wave number range for which the time-domain algorithm is stable.

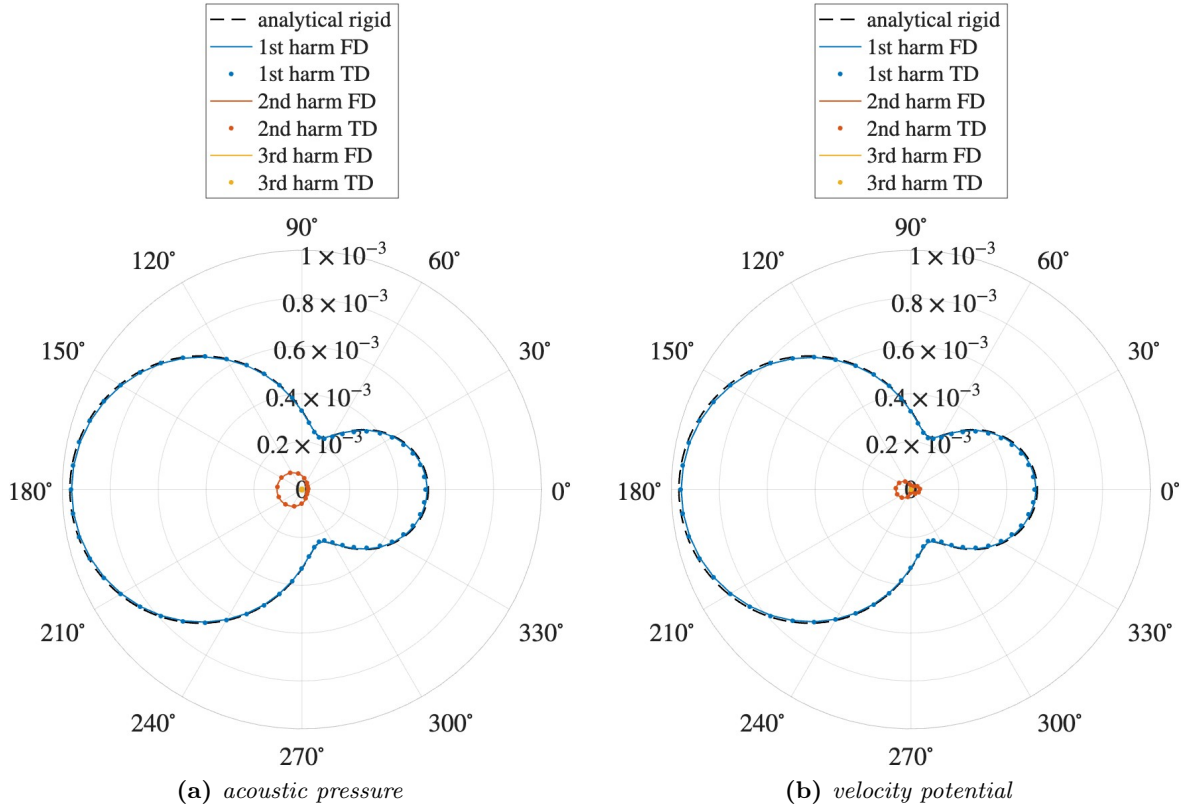


Figure 5.4: Directivity pattern of (a) $|p^S|/|p^I|$, and (b) $|\varphi^S|/|\varphi^I|$ related to acoustic pressure and velocity potential scattered by a pulsating sphere impinged by a plane wave; $k_I r_0 = 0.1$, $\omega = \omega_I$, $a = 0.03 r_0$, $d/r_0 = 10$, $N_a = 5$.

For $k_I r_0 = 0.1$, Figs. 5.4, 5.5 and 5.6 show the directivity patterns of the acoustic pressure field and velocity potential field scattered by a sphere pulsating with angular frequency, ω , equal to that of the incident wave ($\omega = \omega_I$) and amplitude, $a = 0.03 r_0$, $a = 0.1 r_0$ and $a = 0.3 r_0$, respectively (see, Eq. (4.13)). Both time-domain and frequency-domain solutions are presented, along with the analytical solution for the rigid sphere case, which is shown for comparison with the results of the conventional scattering problem.

Several considerations arise from the observations of these figures. First of all, note that the scattered signal is multi-harmonic (as already discussed in Sec. 5.1.1, even for a single-harmonic incident wave the signal scattered by a deforming body is multi-harmonic), with the amplitudes of the harmonics that are multiples of the fundamental one increasing with the increase of the sphere pulsation amplitude.

Second, these figures demonstrate that for a low-frequency scattering problem ($k_I r_0 < 1$), the solutions from time and frequency domain solvers match for all the significant harmonics, and thus are perfectly equivalent (note that, the fourth harmonics of the scattered signal are negligible for all the cases examined).

Third, in the case of small pulsation amplitudes, namely $a = 0.03 r_0$ and $a = 0.1 r_0$ in Fig. 5.4 and Fig. 5.5, the directivity pattern of the first harmonic coincides with that of the single-harmonic signal scattered by the rigid sphere. Contrarily, for higher values of the pulsation amplitude (in this case $a = 0.3 r_0$) even the first harmonic of the scattered signal significantly deviates from the rigid case solution. Note that this is due to the combination of the second harmonic of the

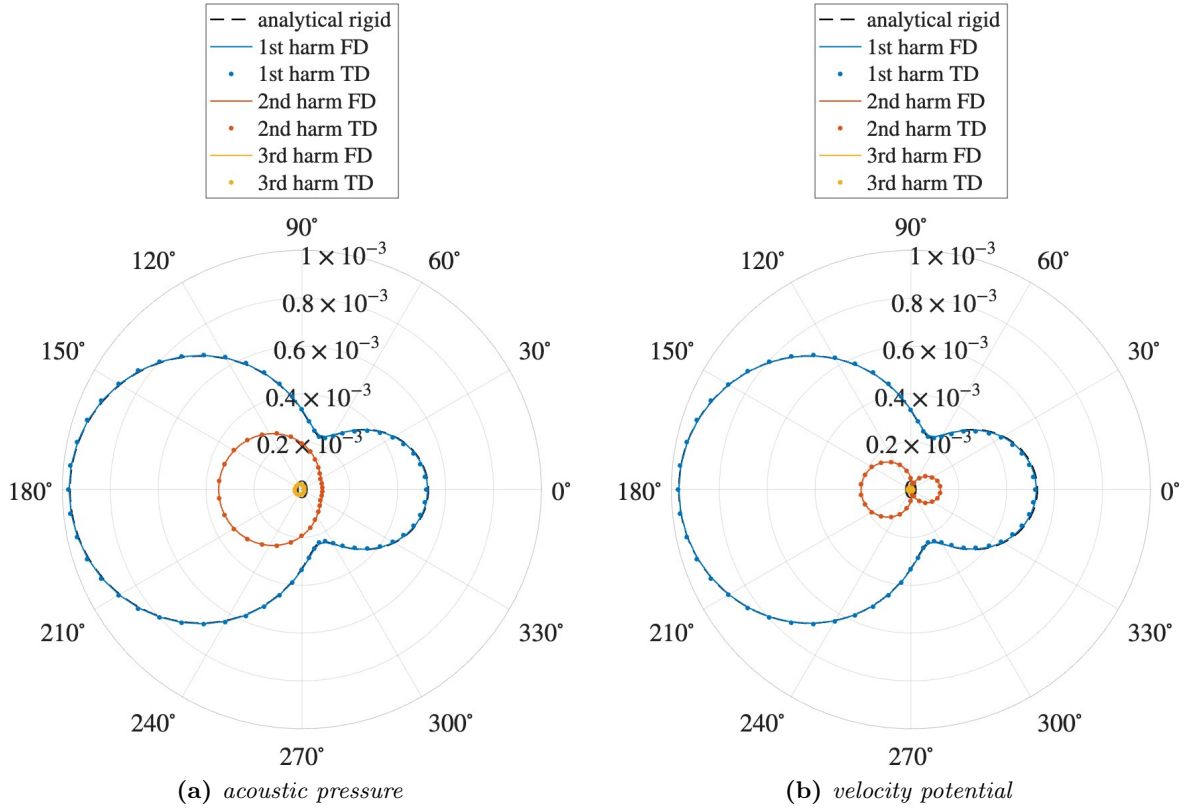


Figure 5.5: Directivity pattern of (a) $|p^S|/|p^I|$, and (b) $|\varphi^S|/|\varphi^I|$ related to acoustic pressure and velocity potential scattered by a pulsating sphere impinged by a plane wave; $k_I r_0 = 0.1$, $\omega = \omega_I$, $a = 0.1 r_0$, $d/r_0 = 10$, $N_a = 5$.

BEM coefficients with the incident signal, from which first and third harmonic contributions are generated: in fact, Fig. 5.4, Fig. 5.5 and Fig. 5.6 show that when the first harmonic scattered by the pulsating sphere deviates from that scattered by the rigid one, the amplitude of the third scattered harmonic becomes significant, while when rigid and pulsating spheres provide the same first scattered harmonic the third scattered harmonic is negligible.

Fourth, all the above observations are identically valid for the scattered problem expressed in terms of both acoustic pressure and velocity potential perturbations. However, except for the first harmonic, the directivity patterns of the scattered signal predicted by the integral formulations expressed in terms of acoustic pressure and velocity potential differ considerably. This means that when the amplitude of the sphere pulsation increases the corresponding generated fluid flow is such that the nonlinear contributions not included in Eq. (5.7) and Eq. (5.27) are not negligible, and that those related to the acoustic pressure formulation differ from those related to the velocity potential one. This outcome is in agreement with the results already available in the literature that concern the sound scattered by a moving rigid body. Indeed, in that case the nonuniform meanflow may significantly alter the scattered signal through contributions deriving from the nonlinear terms forcing the wave equation, and it was shown that these contributions in the formulation described in terms of acoustic pressure differ from those arising in the velocity potential formulation (see, for instance, [23, 46]).

Next, the effect of increasing the impinging wave frequency is examined. First of all, note that, confirming the outcomes of the rigid-body sound scattering, for higher-frequency problems

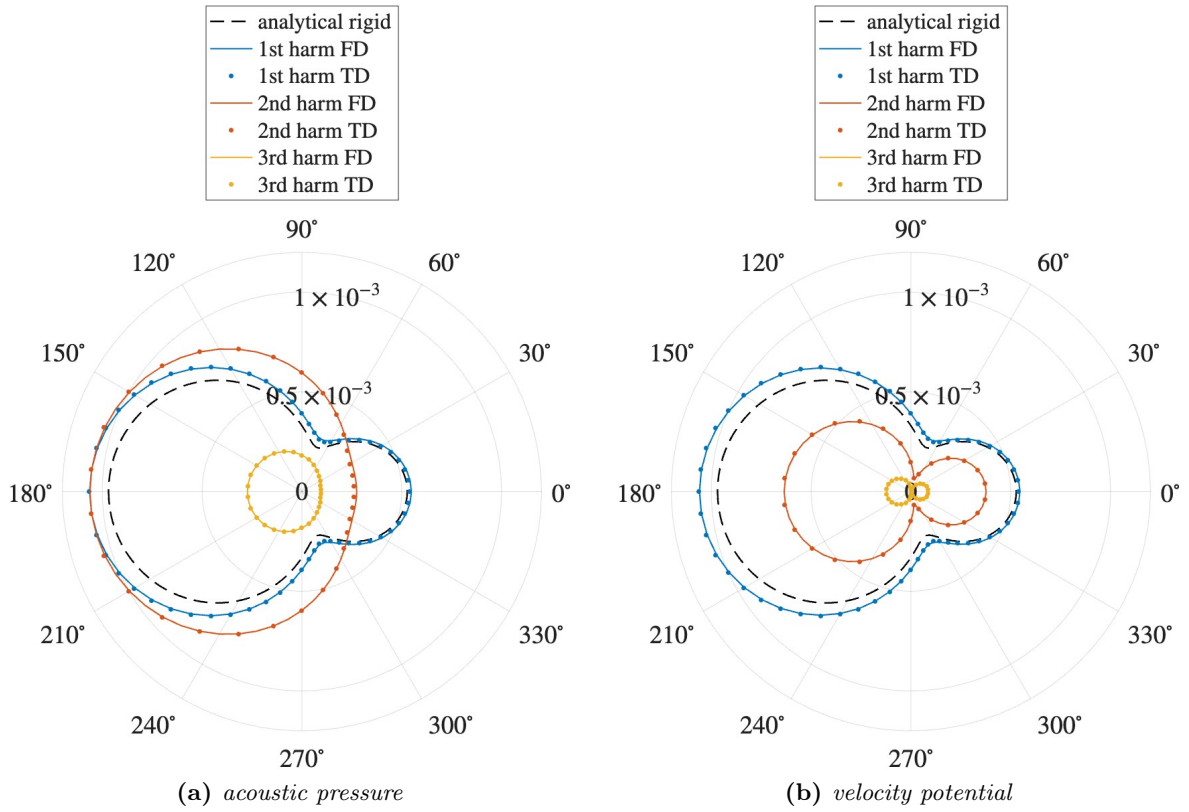


Figure 5.6: Directivity pattern of (a) $|p^S|/|p^I|$, and (b) $|\varphi^S|/|\varphi^I|$ related to acoustic pressure and velocity potential scattered by a pulsating sphere impinged by a plane wave; $k_I r_0 = 0.1$, $\omega = \omega_I$, $a = 0.3 r_0$, $d/r_0 = 10$, $N_a = 5$.

the degree of accuracy of the time-domain solver is reduced since numerical inaccuracies arise. This is shown in Fig. 5.7 which presents the predicted scattered signals for the same configuration considered in Fig. 5.5, but with an increased value of the impinging wave frequency such that $k_I r_0 = 1.2$ (the same wave number considered for the results in Fig. 5.3). Indeed, contrary to the outcomes shown in Fig. 5.5, in this higher-frequency case the convergence trend of the results from the time-domain formulation is considerably irregular. The results depicted in Fig. 5.7 are obtained by using the same time step used for the analysis of the case in Fig. 5.5 and significantly differ from those given by the frequency-domain formulation (which is based on a numerically robust algebraic solution algorithm).

This confirms that as the impinging wave frequency increases the time-domain solver becomes unsuitable for acoustic phenomena investigations, and therefore the following investigations are performed using only the frequency domain solver.

Next, the BEM solver is validated against the analytical prediction proposed by Censor [49], based on a perturbation solution method. The first-order perturbation solution is explicitly provided in Ref. [49] and is compared in Fig. 5.8a with the outcomes of the BEM solver for the same case of Fig. 5.5, related to $k_I r_0 = 0.1$, $\omega = \omega_I$, and small-amplitude pulsations, $a = 0.1 r_0$. This figure shows that the presented numerical solution captures first and second harmonics of the scattered wave, which are in excellent agreement with those given by the first-order analytical perturbation solution.

The directivity pattern of the velocity potential scattered by higher-frequency and larger-amplitude

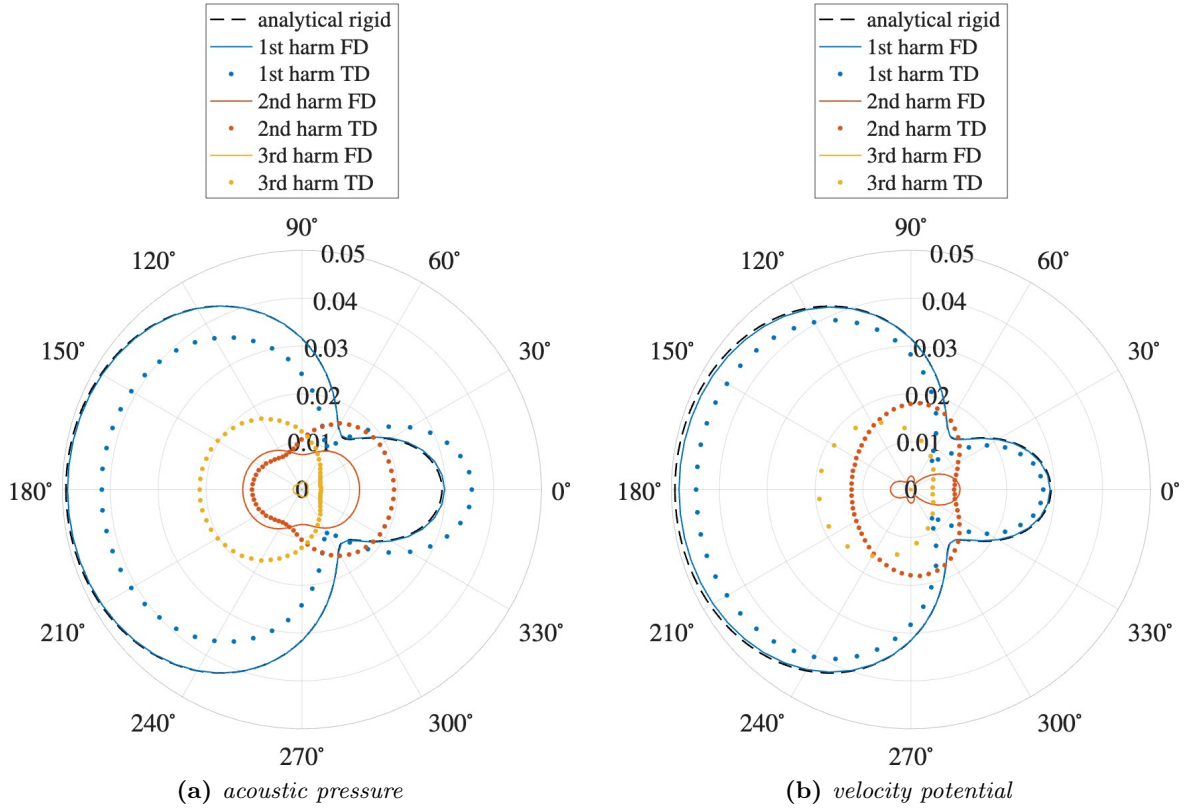


Figure 5.7: Directivity pattern of (a) $|p^S|/|p^I|$, and (b) $|\varphi^S|/|\varphi^I|$ related to acoustic pressure and velocity potential scattered by a pulsating sphere impinged by a plane wave; $k_I r_0 = 1.2$, $\omega = \omega_I$, $a = 0.1 r_0$, $d/r_0 = 10$, $N_a = 5$.

pulsations is examined in Fig. 5.8b, considering $k_I r_0 = 0.3$, $\omega = \omega_I/2$, and $a = 0.25 r_0$. In this case the spectrum of the scattered field is enriched by additional contributions from higher harmonics. These are determined by the analytical solution necessarily through inclusion of second-order perturbation terms. These are explicitly derived in appendix G.1 where, following the approach suggested by Censor [49], the extension to the analytical second-order perturbation solution is provided. Differently from the small-amplitude case of Fig. 5.8a, the second-order perturbation term (given by coefficients C_n in appendix G.1) significantly affects also the second harmonic of the scattered velocity potential which, however, is still dominated by the zeroth-order perturbation term (rigid-sphere solution, coefficients A_n in appendix G.1). Figure 5.8b confirms the excellent agreement between the BEM solution and the analytical one for the harmonics of the scattered signal that are reliably captured by the second-order analytical perturbation solution method. Notice that the same excellent agreement between numerical and analytical scattering predictions is obtained also for near-field and far-field observers. For the sound scattering case defined by $k_I r_0 = 0.1$, $\omega = \omega_I$, and $a = 0.1 r_0$, this is proven in Figs. 5.9a and 5.9b which depict the directivity patterns of sound scattered at observers placed at $d/r_0 = 2$ and $d/r_0 = 100$, respectively.

For even larger sphere pulsation amplitudes, neither the analytical solution proposed in [49] nor its extension described in appendix G.1 are capable of accurately predicting the potential scattered by the sphere. For the sake of validating the proposed multi-harmonic, boundary element formulation for the prediction of sound scattered by surfaces undergoing large deformations, a novel analytical exact linear solution is developed in appendix G.2, which is applicable to arbitrary

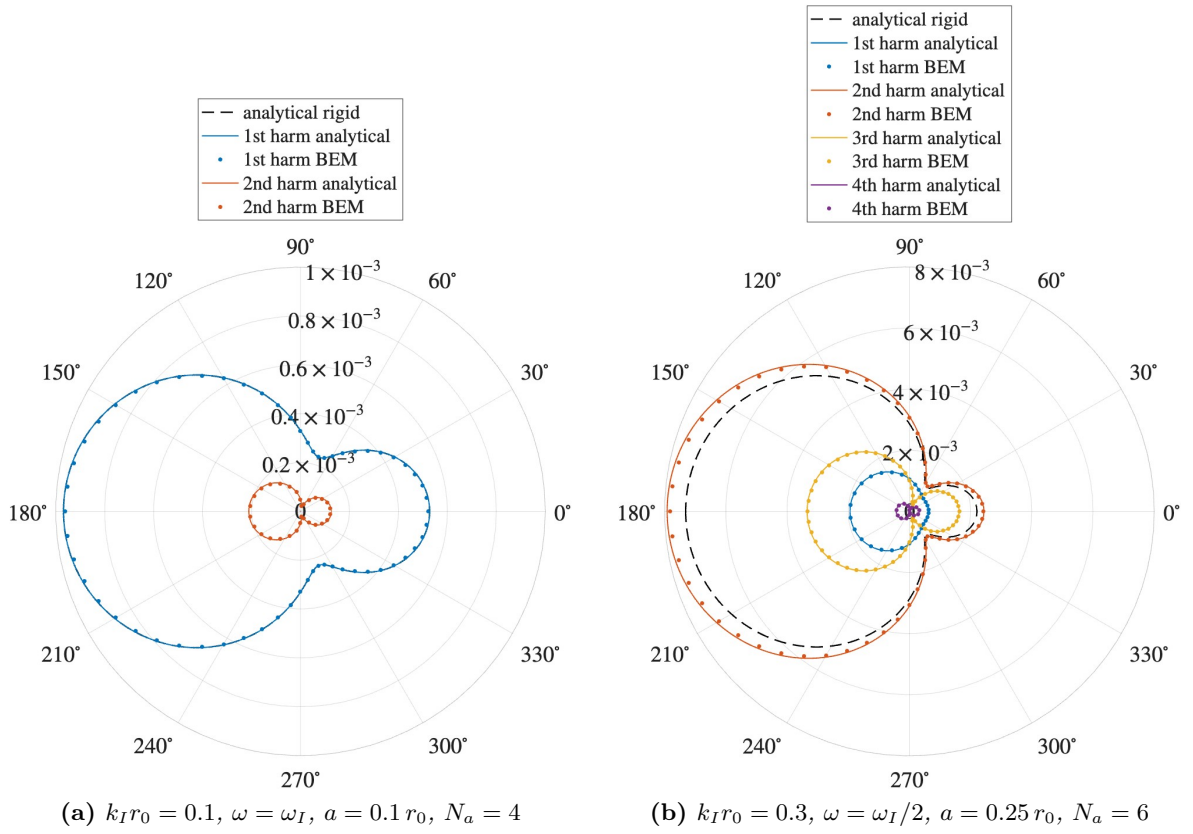


Figure 5.8: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave; $d/r_0 = 10$

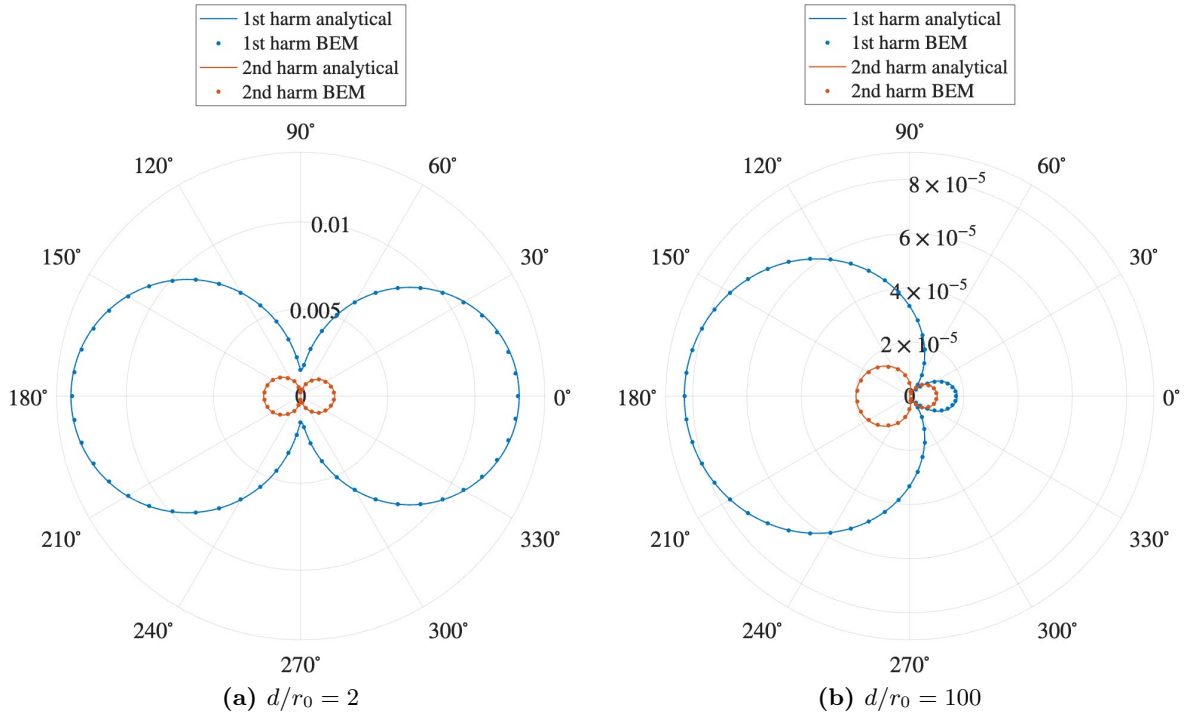


Figure 5.9: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave; $k_I r_0 = 0.1$, $\omega = \omega_I$, $a = 0.1 r_0$, $N_a = 4$.

subsonic pulsations of spheres.

For $\omega = \omega_I$ and $d/r_0 = 10$, Figs. 5.10 to 5.12 present the comparisons among the analytical solutions (the small-amplitude one, S-A an, and the exact one, E an) and those obtained by the BEM solver, respectively for $a/r_0 = 0.5$ and $k_I r_0 = 1$, $a/r_0 = 0.50$ and $k_I r_0 = 1.5$, $a/r_0 = 0.7$ and $k_I r_0 = 1$, with the surface Mach number between $ka = 0.5$ and $ka = 0.75$, and variations in the radius of the sphere up to 70%.

These results show excellent agreement between BEM predictions and exact analytical solutions, even for the extreme pulsation conditions considered. In contrast, as expected, the approximate analytical solution valid for small pulsation amplitudes gives results that show discrepancies increasing with the harmonic number, and is unable to provide predictions for higher harmonics.

An excellent agreement between the BEM predictions and the exact analytical solutions is also observed in Fig. 5.13, which presents the results for the same spherical pulsations and observers as in Fig. 5.10, but with the incident wave frequency such that $k_I r_0 = 2$. Note that, in this case, the shape of the directivity pattern of the scattered velocity potential most similar to that of the rigid sphere is that of the second harmonic, since it corresponds to the frequency of the incident wave.

The results shown in Figs. 5.8 to 5.13 definitely validate the proposed BEM solver for the analysis of sound scattered by deforming objects.

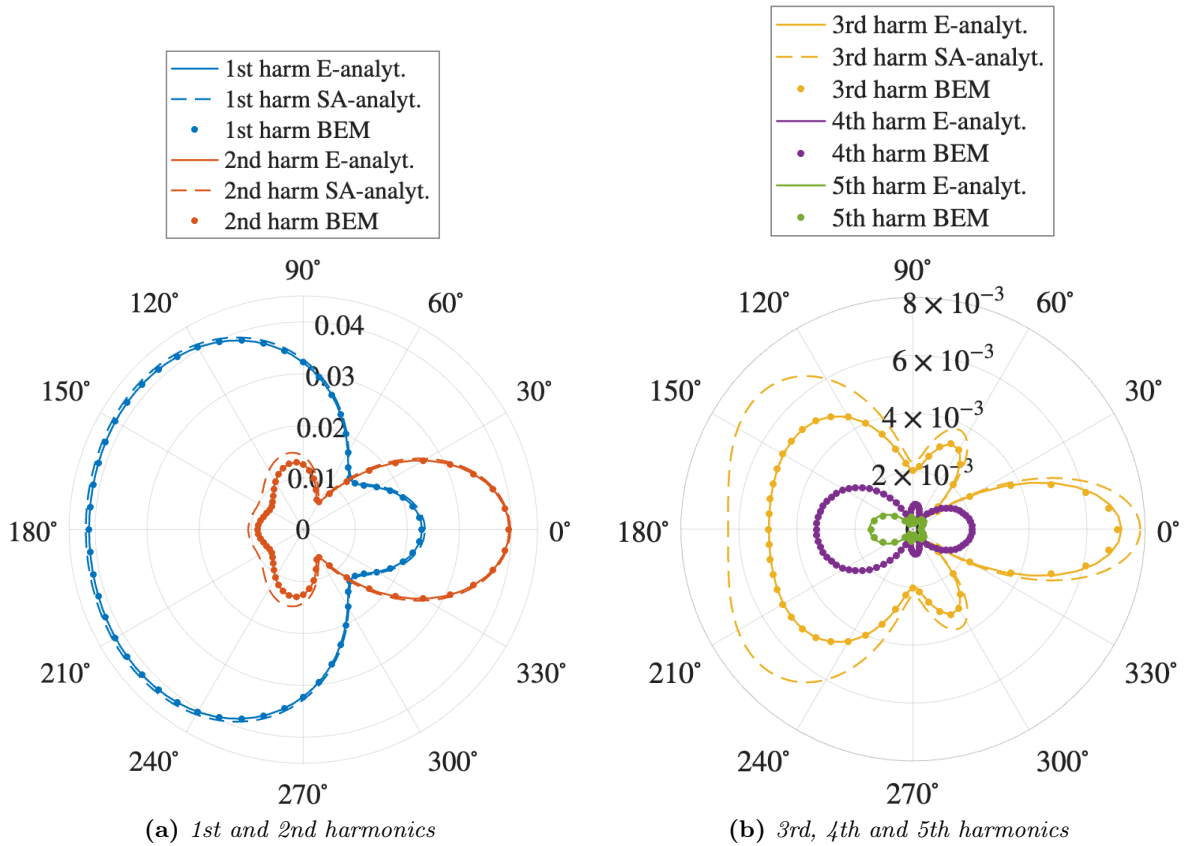


Figure 5.10: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave, comparison between numerical results (BEM), analytical solution for small pulsation amplitude (SA-analyt.) and exact analytical solution (E-analyt.); $k_I r_0 = 1$, $\omega = \omega_I$, $a = 0.5 r_0$, $d/r_0 = 10$, $N_a = 10$.

Finally, the effect of the shape of the scatterer deformation is investigated through the formu-

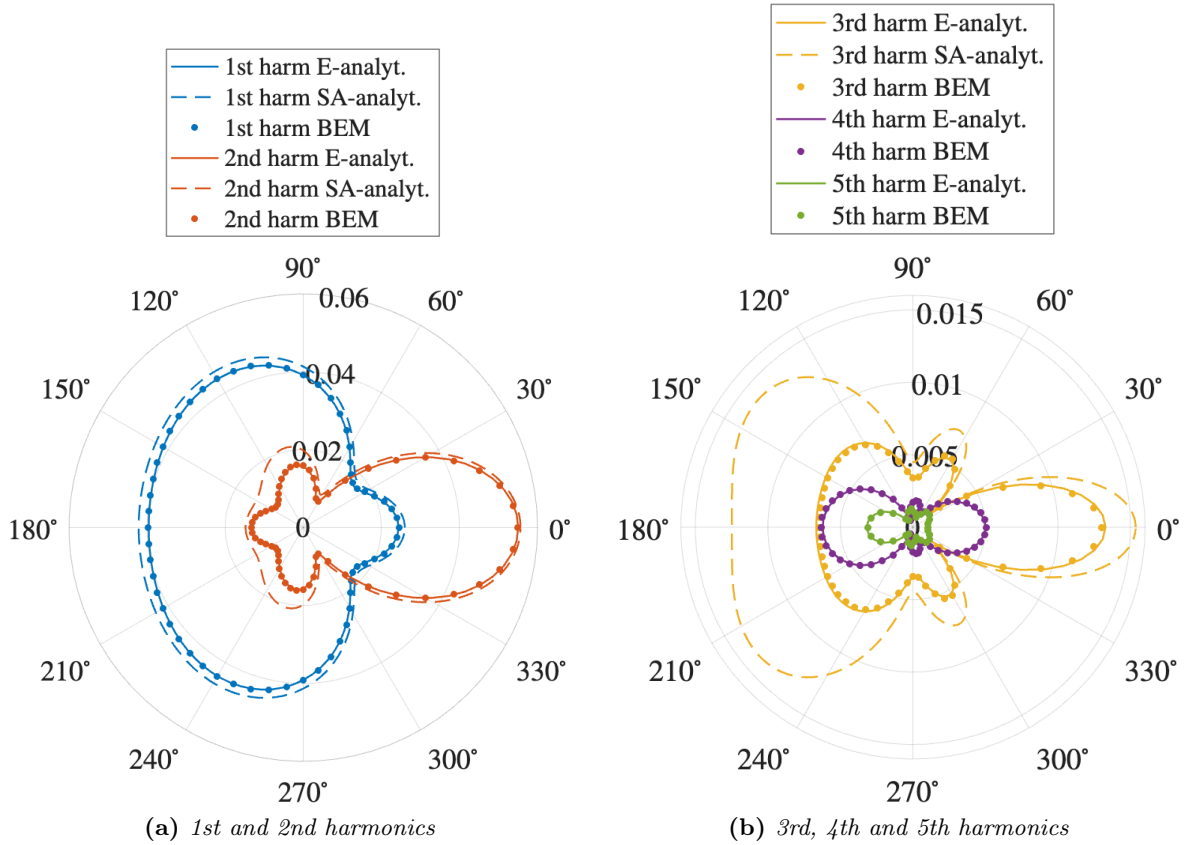


Figure 5.11: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave, comparison between numerical results (BEM), analytical solution for small pulsation amplitude (SA-analyt.) and exact analytical solution (E-analyt.); $k_I r_0 = 1.5$, $\omega = \omega_I$, $a = 0.5 r_0$, $d/r_0 = 10$, $N_a = 20$.

lation expressed in terms of the acoustic pressure presented in Sec. 5.1.1. Specifically, a pulsating wavy deformation of the sphere is considered. It is described as

$$\mathbf{x}_d = a \left(\frac{r_0 - |x|}{r_0} \right)^2 \sin(\omega t) \sin(N_w \beta) \hat{\mathbf{r}} \quad (5.32)$$

where $\mathbf{x}_d$ denotes the displacement of the points of the sphere with respect to the undeformed configuration, N_w denotes the number of deformation waves introduced along the sphere parallel circles, $\hat{\mathbf{r}}$ is the unit radial vector, and $\beta = \arctan(y/z)$. A representation of the sphere deformation is shown in Fig. 5.14 (in which colours are used only to make the shape of the sphere clearer).

The directivity patterns of the acoustic pressure scattered by the sphere deformed according to Eq. (5.32), and impinged by an incident wave propagating along the y -axis (see Fig. 5.14) are presented in Fig. 5.15 for $N_w = 5$ and $N_w = 6$. These results are obtained with $k_I r_0 = 1$, $\omega = \omega_I$, $a = 0.3 r_0$. In this case the effect of the body deformation is almost negligible with respect to that observed for the equivalent uniformly pulsating sphere case in Fig. 5.15c, with the first harmonic of the scattered signal very similar to the rigid scattering solution, and the higher harmonics having a very small amplitude. However, it is interesting to observe that when N_w is odd all harmonics provide contributions to the scattered signal (see Fig. 5.15a), while when N_w is even non-zero contributions are given only by the even harmonics of the signal (see Fig. 5.15b).

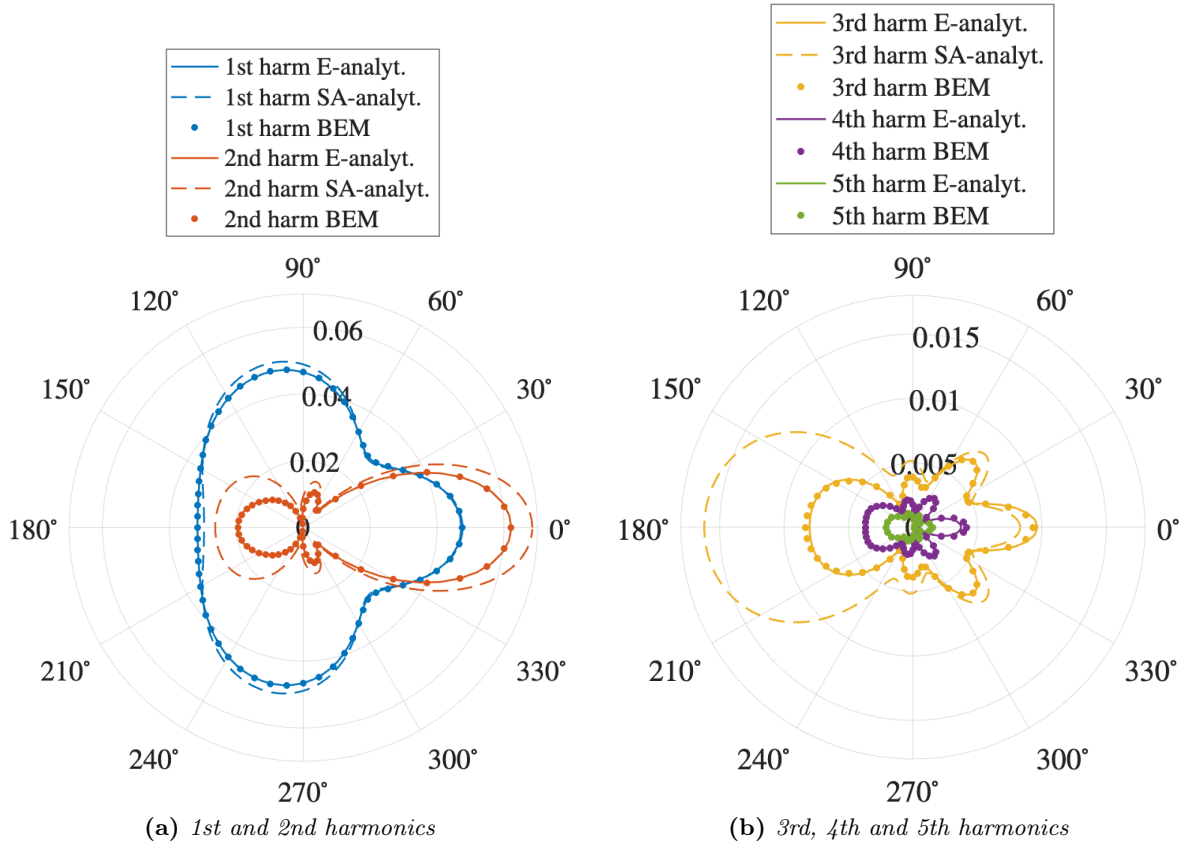


Figure 5.12: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave, comparison between numerical results (BEM), analytical solution for small pulsation amplitude (SA-analyt.) and exact analytical solution (E-analyt.); $k_I r_0 = 1$, $\omega = \omega_I$, $a = 0.7 r_0$, $d/r_0 = 10$, $N_a = 20$.

5.2.3 Remarks on the sound scattered by spheres undergoing large pulsations

It is interesting to observe that when the sphere is subject to small-amplitude pulsations, the resulting scattered field is dependent on the frequencies of sphere pulsations and incident wave, but is independent on the harmonic functions used in Eq. (5.30) and Eq. (5.31) to represent them. Instead, when the pulsation amplitude is large, the scattered field depends also on the harmonic functions used in Eq. (5.30) and Eq. (5.31).

This is illustrated in Fig. 5.16 through predictions provided by the exact analytical solution presented in appendix G.2, for $k_I r_0 = 1$ and observers located at a distance $d/r_0 = 10$ from the sphere center. Indeed, for $a/r_0 = 0.05$, Fig. 5.16a proves that the scattered velocity potential due to the incident field given in Eq. (5.31) and that resulting from the following plane wave

$$\varphi^I(x, t) = A^I \cos(\omega_I t - k_I x) \quad (5.33)$$

are practically coincident, while 5.16b shows that, for much larger pulsation amplitudes, $a/r_0 = 0.5$, the scattered fields given by the two plane wave representations considerably differ. In particular, while the 'sin' and 'cos' representations diverge at larger amplitudes, the numerical results obtained using BEM always coincide with the exact analytical solution. This confirms that both methods effectively capture the underlying physics of these scattering effects.

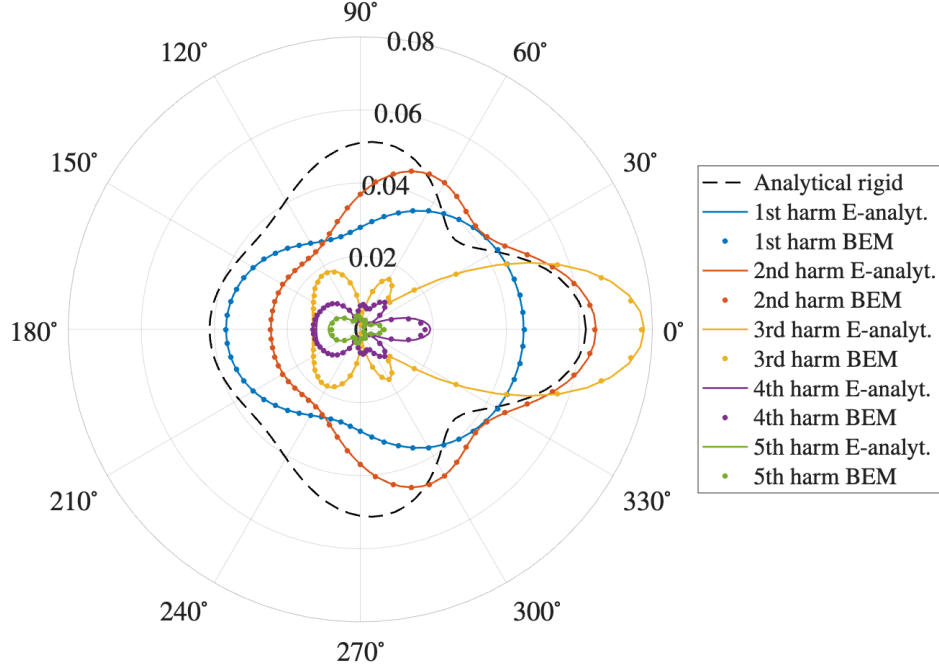


Figure 5.13: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by a plane wave, comparison between numerical results (BEM) and exact analytical solution (E-analyt.); $k_I r_0 = 2$, $\omega = \omega_I/2$, $a = 0.5 r_0$, $d/r_0 = 10$, $N_a = 20$.

5.2.4 Remarks on the proposed boundary integral formulations for sound scattering

Two issues concerning the proposed boundary integral formulations for sound scattering deserve further discussion. The first concerns the elimination of fluid nonlinearities in the wave equations for acoustic pressure and velocity potential, while the second regards the harmonic content of the signals.

The scattering formulations in Sections 5.1.1 to 5.1.2 solve linear wave propagation equations which neglect the effects due to the fluid flow produced by the motion of the scatterer. The importance of these effects depend on the kinematics of the scatterer, particularly on the normal component of the Mach number of the surface (generally given by the superposition of rigid-body motion and deformation). To account for this, it is necessary to include nonlinear terms in the wave equations. These terms would introduce additional multiple tone contributions in the scattered signal (see, for example, Ref. [51]), in addition to those already present in the numerical predictions presented in the previous figures, which are geometric in nature due to body deformation. Therefore, regardless of the ability to accurately solve the linear wave equation problem, the validity of the proposed formulations for simulating the physical phenomenon of sound scattering depends on the importance of the neglected nonlinear terms, namely on the kinematics of the scatterer.

Regarding the harmonic content of the scattered signal, recall that for the problems of sound scattered by a pulsating sphere examined above, the fundamental harmonic of the dynamic deformation, ω , was assumed to be a multiple or a submultiple of the harmonic of the incident wave, ω_I . This implies that the scattered field spectrum contains the following set of harmonics, $\omega_k = k \cdot \min[\omega, \omega_I]$.

More generally, if ω and ω_I are arbitrary, the set of harmonics of the scattered sound is given by $\omega_k = \omega_I \pm k \omega$. In this case, the fundamental harmonic, ω_f , and the number of harmonics, N_a , of the

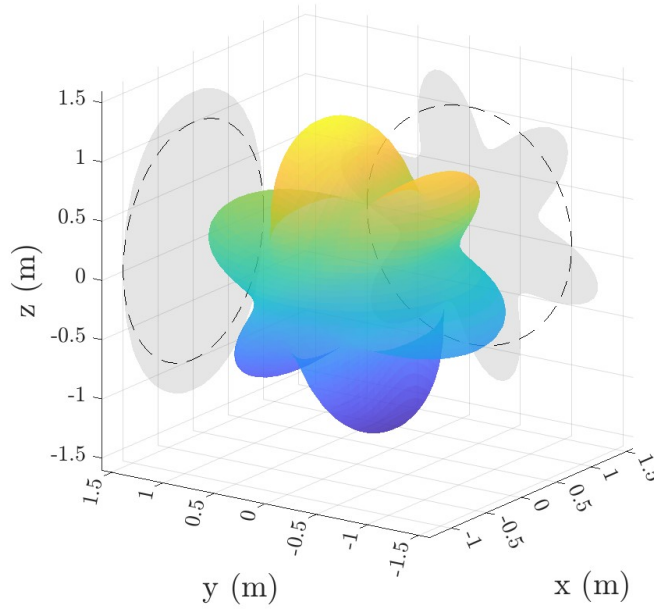


Figure 5.14: Representation of the wavy deformed sphere.

Fourier series introduced in Section 5.1.1 must be chosen so as to adequately capture the harmonics, ω_k , appearing in the resulting scattered signal (indeed, in this case, the proposed frequency solution process can be considered equivalent to the application of discrete Fourier transforms).

5.3 Nonlinear scattering formulation

The previous section has shown that, depending on the scatterer dynamic deformation, linear scattering solutions formulated in terms of velocity potential or acoustic pressure can present significant discrepancies. Indeed, the contributions of fluid nonlinearities produced by the motion of the scatterer are differently accounted for in the two formulations of the problem. Therefore, the linear formulations written in terms of velocity potential and acoustic pressure that neglect them inevitably provide two different solutions.

In this framework, this section introduces a novel harmonic-balance, velocity-potential solution approach for nonlinear sound scattering from arbitrarily shaped three-dimensional bodies impinged by an incident wave.

As highlighted above, the dynamic deformation of the scatterer generates kinematic nonlinearities through the boundary conditions that make the scattered field intrinsically multi-harmonic. The approach presented here, in addition to including these kinematic nonlinearities, also takes into account the nonlinear effects due to the flow field generated by the body deformation. The effect of fluid nonlinearities on the scattered field was studied, for instance, in Ref. [51] for an infinitely long cylinder impinged by a plane wave. The relative importance of the two sources of multi-harmonic scattering was examined by Mujica et al. [52].

In the following, the scattered noise field is obtained through a technique (here called velocity potential cascade solution mode) consisting of the superposition of incremental velocity potential solutions forced by progressively decreasing fluid nonlinearities that are given by the combination of

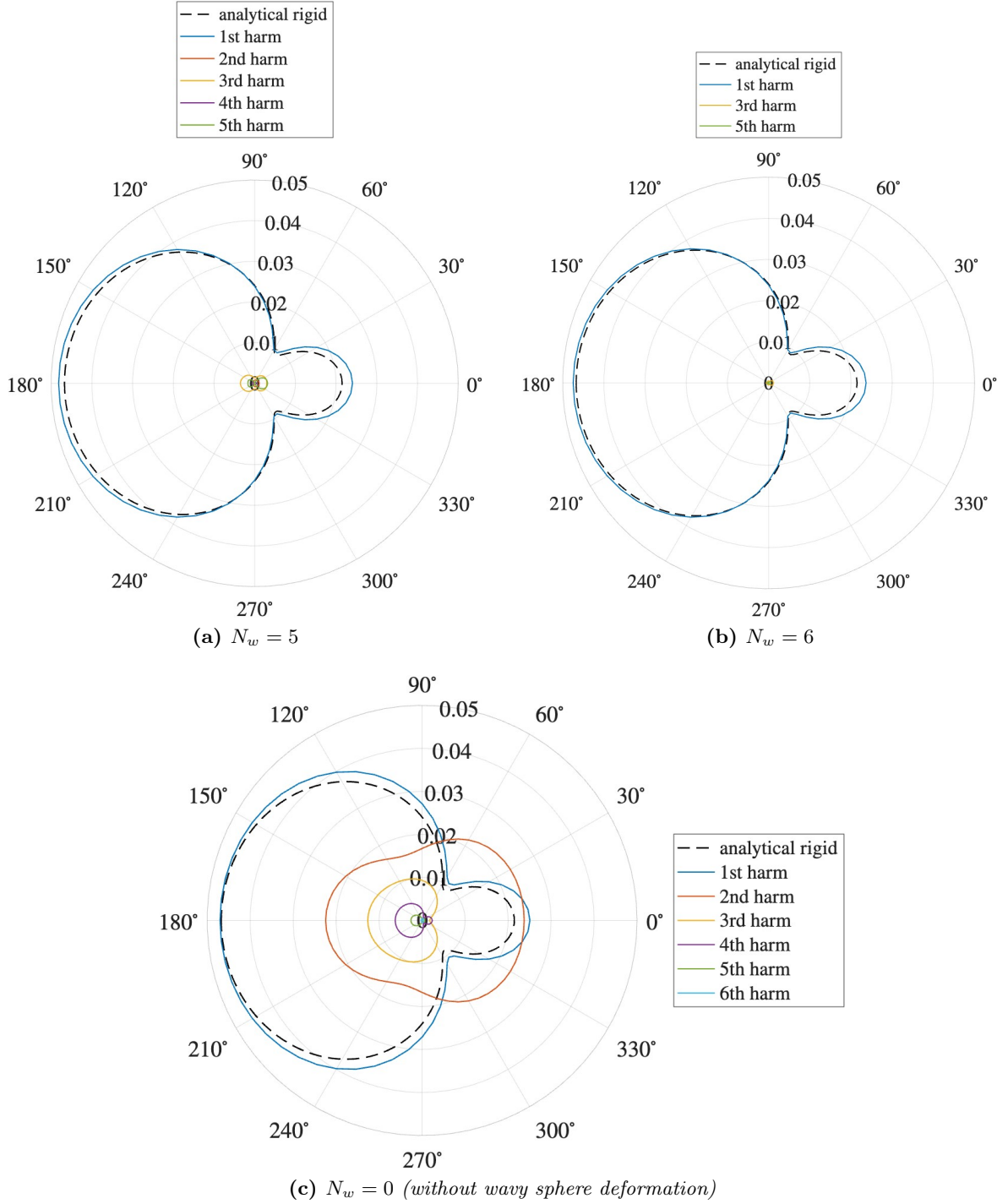


Figure 5.15: Directivity pattern of $|p^S|/|p^I|$ related to acoustic pressure scattered by a wavy deformed pulsating sphere impinged by a plane wave; $k_I r_0 = 1$, $\omega = \omega_I$, $a = 0.3 r_0$, $d/r_0 = 10$, $N_a = 7$.

incident and scattered signals with the velocity potential related to the flow-field generated by the motion of the scatterer. These incremental velocity potential solutions are determined through the application of the deformable-boundary integral formulation for solution of the velocity potential wave equation described in the previous chapters.

The velocity potential cascade solution is illustrated in Sec. 5.3.1, while the harmonic-balance/boundary-element-method numerical algorithm applied to determine the solution of the deformable-boundary integral formulation is outlined in Sec. 5.3.3. The results of a numerical investigation

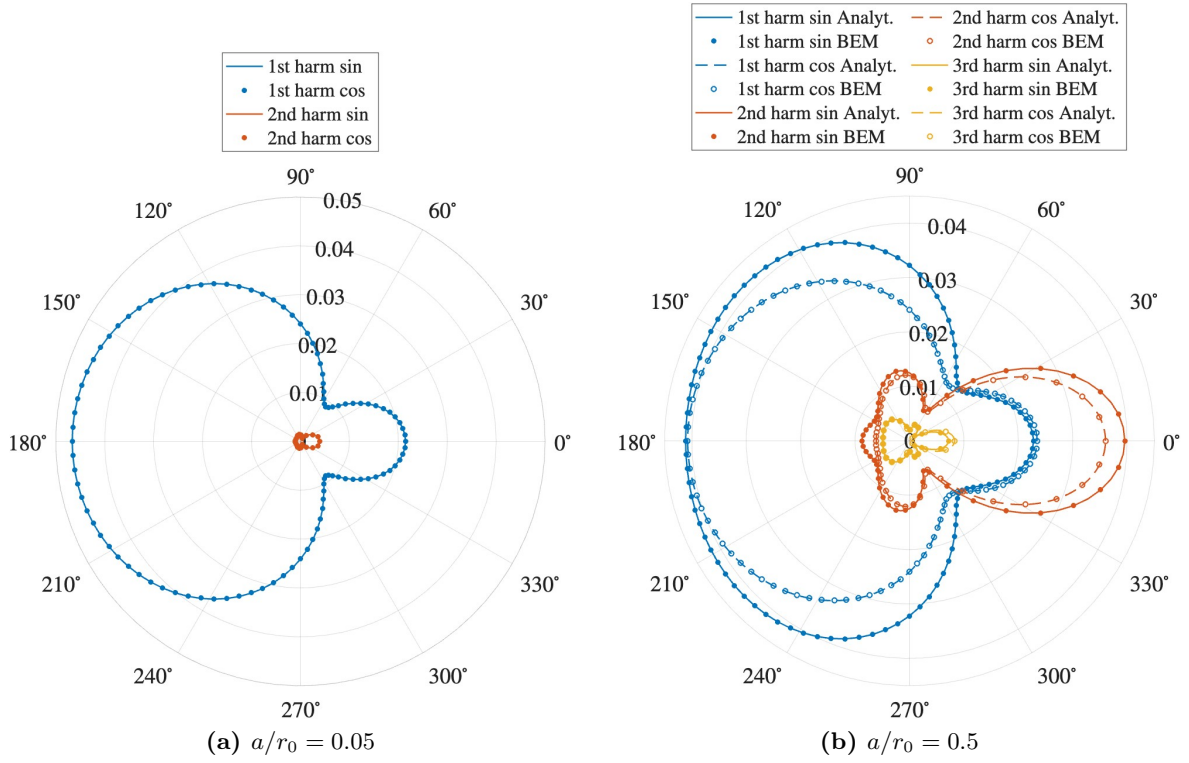


Figure 5.16: Directivity pattern of $|\varphi^S|/|\varphi^I|$ related to velocity potential scattered by a pulsating sphere impinged by the plane waves expressed in Eq. (5.31) (results indicated by ‘sin’) and in Eq. (5.33) (results indicated by ‘cos’); $k_I r_0 = 1$, $\omega = \omega_I$, $d/r_0 = 10$, $N_a = 10$.

regarding the sound scattered by a pulsating sphere impinged by a plane sound wave are discussed in Sec. 5.4. The numerical investigation examines the convergence rate of the proposed nonlinear solution algorithm and the effects of the fluid flow generated by the surface pulsations on the scattered signal. The influence of the amplitude of the incident signal on the directivity pattern of the scattered sound is also investigated.

5.3.1 Nonlinear integral formulation for sound scattered by deforming bodies

Let us consider a body moving in a perfect, inviscid gas, initially at rest, and the irrotational velocity field expressed through the velocity potential, φ , such that $\mathbf{u} = \nabla\varphi$.

Extending the governing equation of the velocity potential given in Sec. 2.3 to include the effects of the presence of arbitrary perturbation sources we have

$$-\square^2\varphi = \nabla^2\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = \sigma + \sigma_{EXT} \quad (5.34)$$

where $\square^2$ represents the d’Alembert operator, c_0 is the speed of sound of the undisturbed fluid, σ is the nonlinear forcing term that is given by

$$\sigma = [(c_0^2 - c^2)\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\dot{\varphi}] / c_0^2 + [\nabla\varphi \cdot \nabla(u^2)/2] / c_0^2 \quad (5.35)$$

with c denoting the local speed of sound

$$c^2 = c_0^2 - (\gamma - 1) \left(\frac{\partial \varphi}{\partial t} + \frac{u^2}{2} \right)$$

where γ represents the ratio between the isobaric and isochoric specific heat capacities and $(\dot{}) = \partial/\partial t$, while σ_{EXT} takes into account the effects deriving from the presence of external forcing sources independent of the body motion.

The differential problem is completed by the boundary condition of perturbation potential vanishing infinitely far from the moving body ($\varphi \rightarrow 0$ as $\mathbf{x} \rightarrow \infty$), and the imposition of impermeability of the surface, $\mathcal{S}$, of the scattering body

$$\nabla \varphi \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = \chi, \quad \mathbf{x} \in \mathcal{S} \quad (5.36)$$

where $\mathbf{n}$ is the unit vector normal to $\mathcal{S}$, and $\mathbf{v}$ represents its velocity (generally given by the superposition of rigid-body motion and deformation).

The solution of Eq. 5.34 for the velocity potential around moving and deforming bodies can be expressed through the following boundary integral formulation in the FFR (application of the formulation given in Sec. 3.3 to Eq. (5.34), see also [77])

$$\begin{aligned} \varphi(\mathbf{x}_*, t_*) = & \iint_{\Omega} [(1 - M_n^2) \chi G_0] J \Big|_{\theta} d\xi^1 d\xi^2 + \iint_{\Omega} [-M_n \mathbf{M} \cdot \nabla_t \varphi G_0] J \Big|_{\theta} d\xi^1 d\xi^2 \\ & - \iint_{\Omega} \left[\varphi \frac{\partial G_0}{\partial n} - \frac{\partial \varphi}{\partial t} \Big|_{\xi} \frac{M_n}{c_0} G_0 \right] J \Big|_{\theta} d\xi^1 d\xi^2 - \iint_{\Omega} \frac{1}{\Theta} \frac{\partial}{\partial t} \Big|_{\xi} \left[\varphi G_0 J \left(\frac{\mathbf{e}_r \cdot \mathbf{n}}{c_0} - \frac{M_n}{c_0} \right) \right] \Big|_{\theta} d\xi^1 d\xi^2 \\ & + \iiint_{\mathcal{V}_{\Omega}} (\sigma + \sigma_{EXT}) G_0 J \Big|_{\theta} d\xi^1 d\xi^2 d\xi^3 \end{aligned} \quad (5.37)$$

The difference between this solution and those applied previously is that here nonlinear terms are taken into account through the volume integral (last term of Eq. 5.37). For convenience, it is recalled that $\mathbf{x}_*$ and t_* represent, respectively, position and time of observation, (ξ^1, ξ^2, ξ^3) is a system of curvilinear coordinates defined over the body surface and the region around it, while the symbols Ω and $\mathcal{V}_{\Omega}$ denote, respectively, the domains in the (ξ^1, ξ^2) -space and the (ξ^1, ξ^2, ξ^3) -space where the body surface and the neighbouring fluid region are mapped. In addition, $G_0 = -1/(4\pi r)$ with $r = |\mathbf{x}_* - \mathbf{x}|$ denoting the distance between the emitting point $\mathbf{x}$ and the receiving point $\mathbf{x}_*$, $\mathbf{M} = \mathbf{v}/c_0$, $M_n = \mathbf{M} \cdot \mathbf{n}$, $\mathbf{e}_r = \mathbf{r}/r$, while $J = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|/|\Theta|$ is the Jacobian of the curvilinear coordinate transformation ($\mathbf{a}_k$ are the covariant base vectors related to (ξ^1, ξ^2, ξ^3)) with $\Theta = (1 - \mathbf{M} \cdot \mathbf{e}_r)$. The symbol $(\dots)|_{\theta}$ indicates that the integrands must be evaluated at time $t = t_* - \theta$, which is the instant at which the surface of the body emitted the signal that at time t_* reaches the observer.

5.3.2 Cascade solution approach of nonlinear velocity potential scattering

When the boundary integral formulation in Eq. 5.37 is applied to problems for which the nonlinear field term is not negligible, the numerical evaluation of the velocity potential over the body surface and its neighborhood requires the adoption of an iterative solution scheme. This is the case examined in this section, which considers the prediction of sound scattering in the presence of

fluid nonlinearities due to dynamic deformation of the scatterer. An iterative solution method is introduced in the following, which consists of the superposition of a sequence of velocity potential solutions, each forced by progressively decreasing nonlinear field terms.

As already done for the analysis of linear sound scattering, let us introduce an incident velocity potential field, φ^I , associated with the propagating impinging wave, which is assumed to be independent of the scatterer and governed by the equation

$$-\square^2 \varphi^I = \sigma_{EXT} = \sum_i S_i^I(t) \delta(\mathbf{x} - \mathbf{x}_i) \quad (5.38)$$

with the forcing terms consisting of an arbitrary combination of "frozen sources" of time-variant intensities, $S_i^I(t)$, located at $\mathbf{x}_i$.

Then, the total velocity potential is defined as the superposition of the incident velocity potential with the perturbation field generated by the presence of the scattering body. In turn, this is given by the superposition of the velocity potential perturbation generated by the motion of the body, φ^0 , such that

$$-\square^2 \varphi^0 = \sigma(\varphi^0) \quad (5.39)$$

and the scattered potential, φ^S , such that (see Eqs. (5.34), (5.38), and (5.39))

$$-\square^2 \varphi^S = \sigma(\varphi) - \sigma(\varphi^0) \quad (5.40)$$

where $\varphi = \varphi^0 + \varphi^S + \varphi^I$.

Previous research has demonstrated that the flow-field generated by the body motion may significantly affect the scattered field, and that its effects are taken into account by the fluid nonlinearities appearing in Eq. 5.40 [23, 45, 46]. The nonlinear scattering problem is solved by iteratively superposing incremental contributions.

First, the velocity potential perturbation due to the motion of the scatterer (reference potential), φ^0 , is solved through application of the boundary integral formulation in Eq. 5.37 with boundary conditions

$$\chi = \chi^0 = \frac{\partial \varphi^0}{\partial n} = \mathbf{v} \cdot \mathbf{n}$$

and corresponding field term

$$\sigma = \sigma^0 = \sigma(\varphi^0)$$

Next the scattered velocity potential field induced by the incident wave is decomposed into several sub-components, namely

$$\varphi^S = \sum_{j=1}^{N_S} \varphi^{S,j} \quad (5.41)$$

such that

$$\varphi^{S,1} > \varphi^{S,2} > \varphi^{S,3} > \dots$$

and with N_S denoting the order of the component still providing not negligible contribution.

It is assumed that $\varphi^{S,1}$ is obtained from the formulation in Eq. 5.37 with boundary conditions

$$\chi = \chi^1 = \frac{\partial \varphi^{S,1}}{\partial n} = -\frac{\partial \varphi^I}{\partial n}$$

and the nonlinear term given by the incremental contribution due to φ^I , namely

$$\sigma = \sigma^1 = \sigma(\varphi^0, \varphi^I) - \sigma(\varphi^0)$$

Therefore, $\varphi^{S,1}$ represents the component of the scattered field forced by the incident wave both through the impermeability boundary conditions (the dominating term) and the nonlinear field term. The contribution of $\varphi^{S,1}$ to σ^1 is omitted so as to have known field terms.

However, once $\varphi^{S,1}$ is evaluated, its incremental contribution to σ yields the second element of the scattered potential series, $\varphi^{S,2}$. Specifically, $\varphi^{S,2}$ is obtained as solution of Eq. 5.37 with boundary conditions

$$\chi = \chi^2 = \frac{\partial \varphi^{S,2}}{\partial n} = 0$$

(note that the impermeability condition is satisfied by the combination of φ^0 , φ^I and $\varphi^{S,1}$) and the nonlinear term as given by the incremental contribution

$$\sigma = \sigma^2 = \sigma(\varphi^0, \varphi^I, \varphi^{S,1}) - \sigma(\varphi^0, \varphi^I)$$

Therefore $\varphi^{S,2}$ represents the first correction to the scattered potential due to scattered potential nonlinearities.

This procedure is repeated until the j -th incremental correction of the scattered velocity potential, $\varphi^{S,j}$, obtained as solution of Eq. 5.37 with

$$\chi = \chi^j = \frac{\partial \varphi^{S,j}}{\partial n} = 0$$

and

$$\sigma = \sigma^j = \sigma(\varphi^0, \varphi^I, \varphi^{S,1} + \dots + \varphi^{S,j-1}) - \sigma(\varphi^0, \varphi^I, \varphi^{S,1} + \dots + \varphi^{S,j-2}) \quad (5.42)$$

becomes negligible. It produces a cascade of incremental scattered potential fields forced by progressively decreasing nonlinear field terms, σ^j , whose superposition yields the total scattered field, φ^S (see Eq. 5.41).

The cascade solution method is not the only one possible. Appendix H shows other methods, highlighting the pros and cons of each.

5.3.3 Velocity potential cascade BEM solution through harmonic balance

For the purpose of the numerical solution of the boundary integral formulation in Eq. 5.37 through the presented cascade solution procedure, a zeroth-order BEM is applied. The surface of the scatterer is discretized into N_p quadrilateral panels, while M_v volume elements are used to discretize the fluid domain where the field integral is evaluated. In these surface panels and volume elements the velocity potential and its space and time derivatives are assumed to be uniformly distributed, equal to those evaluated at their centroids.

When the formulation in Eq. 5.37 is applied as a boundary integral equation with unknown velocity potential over the body surface and known field contributions (as for the scattered potential cascade solution), the satisfaction of the equation is imposed at the center of each surface

discretization panel (collocation method).

This yields the following set of time-delay equations which extends that in Eq. (4.8) by inclusion of the field terms, and whose solution provides the time-history of the piecewise constant values of the velocity potential over the surfaces of the panels

$$\begin{aligned} \frac{1}{2}\varphi_k(t_*) &= \sum_{m=1}^{N_p} A_{km}\chi_m(t_* - \theta_{km}) \\ &+ \sum_{m=1}^{N_p} B_{km}\varphi_m(t_* - \theta_{km}) + \sum_{m=1}^{N_p} C_{km}\dot{\varphi}_m(t_* - \theta_{km}) \\ &+ \sum_{m=1}^{N_p} F_{km}\varphi_m^t(t_* - \theta_{km}) + \sum_{h=1}^{M_v} H_{kh}\sigma_h(t_* - \theta_{kh}), \quad \text{for } k = 1, N_p \end{aligned} \quad (5.43)$$

where φ_m, χ_m and φ_m^t denote, respectively, velocity potential, function χ and derivative of the velocity potential along the direction of the local body Mach number component tangent to the surface, $\mathbf{M}_t = \mathbf{M} - M_n \mathbf{n}$, evaluated at the center of the m -th panel. In addition, θ_{km} represents the signal transmission delay related to the observer at the center of the k -th panel and the emitting source at the center of the m -th panel, while σ_h is the nonlinear forcing term evaluated at the center of the h -th volume. The expressions of the time-varying influence coefficients defined over the body surface panels coincide with those in Eqs. (4.9) to (4.12), while the coefficients given by the field volumes are

$$H_{kh}(t_*) = \iiint_{\mathcal{V}_{\Omega_h}} G_0 J \Big|_{\theta_{kh}} d\xi^1 d\xi^2 d\xi^3$$

where $\mathcal{V}_{\Omega_h}$ denotes the (ξ^1, ξ^2, ξ^3) -domain where the surface of the m -th panel and the volume of the h -th field element are mapped. As for Eq. (4.8), the factor $1/2$ which appears at the left-hand side of Eq. (5.43) accounts for the contribution of the free terms deriving from the singularity of the kernel function $\partial G_0 / \partial n$ arising when the position of the observer tends to the limit on the emitting surface. Thus, when $k = m$ the coefficients B_{km} are intended as Cauchy principal values of the surface integral [30, 68].

Once the velocity potential over the body surface is known, it may be readily used to determine the velocity potential field radiated at external points and, in particular, to evaluate the velocity potential at the centroids of the discretization volumes for the definition of the forcing field terms in the next step of the cascade solution process. This is obtained through application of the discretized boundary integral representation, which is formally identical to Eq. (5.43) with the factor $1/2$ at the left-hand side removed [30, 68].

Multi-harmonic solution algorithm

As with linear scattering problems, also in this case it is convenient to use the multi-harmonic solution algorithm. This is developed in a similar way to that described in Sec. 4.5.2, with the only difference being the definition of the forcing terms. Indeed, in this problem, they must also include the contributions of the nonlinear volume terms as follows

$$f_k(t_*) = f_k^\chi(t_*) + f_k^\sigma(t_*)$$

where the contributions of the boundary conditions are the same as those appearing in the linear case

$$f_k^\chi(t_*) = \sum_{m=1}^{N_p} A_{km}(t_*) \chi_m(t_* - \theta_{km})$$

while those of the nonlinear field terms are given by

$$f_k^\sigma(t_*) = \sum_{h=1}^{M_v} H_{kh}(t_*) \sigma_h(t_* - \theta_{kh}) \quad (5.44)$$

As mentioned above, once the values of the scattered potential over the body's panels are obtained, it is necessary to determine the scattered potential in the field in order to evaluate the contributions σ^j and hence the forcing terms f_k^σ (see Eqs. (5.42) and (5.44)). It may be readily determined through the proposed numerical algorithm applied to the integral formulation conceived as an integral representation. In this case, the integral solution can be formally expressed through Eq. (4.14) applied at the M_v centroids of the volume elements surrounding the scatterer, with the free term contributions, $(1/2)\delta_{km}$, appearing in the definition of the coefficients in Eqs. (4.15) to (4.17) suitably removed (no singular integrals arise). Indeed, for φ_f denoting the vector of dimensions $M_v(2N_a + 1)$ which collects the Fourier coefficients of the scattered velocity potential at the centroids of the volume elements we have

$$\varphi_f = \mathbf{D}_f \varphi + \mathbf{f}_f$$

where $\mathbf{D}_f$ is the rectangular matrix of dimensions $M_v(2N_a + 1) \times N_p(2N_a + 1)$ (similar to matrix $\mathbf{D}$ in Eq. (4.19)) that collects the Fourier components of the BEM coefficients for external observers, while $\mathbf{f}_f$ collects the Fourier components of the forcing terms evaluated for the same observers.

Figure 5.17 shows a flowchart summarizing the proposed numerical procedure for the determination of sound scattered by moving and deforming bodies, where ϵ represents the limit of convergence of the iterative process.

It is worth underlying that the solution of the linear scattering formulation presented in Sec. 5.1 corresponds to the first scattering component, $\varphi^{S,1}$, under the assumption of neglecting the effects of the fluid nonlinearities.

5.4 Numerical results for nonlinear scattering

All numerical results presented in the following regard applications to a pulsating spherical scatterer. They are obtained through a boundary-field discretization that ensures converged solutions. Specifically, as a result of a preliminary investigation of numerical convergence, the discretization panels used are $N_p = 200$, generated by dividing the sphere surface through 20 meridians and 10 parallels, while the discretization volumes are $M_v = 1600$, distributed over the outer field region extending up to a distance equal to $0.4r_0$, where r_0 indicates the radius of the undeformed sphere. The number of harmonics, N_a , considered in the solution algorithm depends on the problem examined. Converged results are obtained by increasing N_a until the significant harmonics of the output remain unchanged.

First, let us consider a sphere of radius $r_0 = 1$ m subject to pulsations of amplitude $a = 0.2r_0$

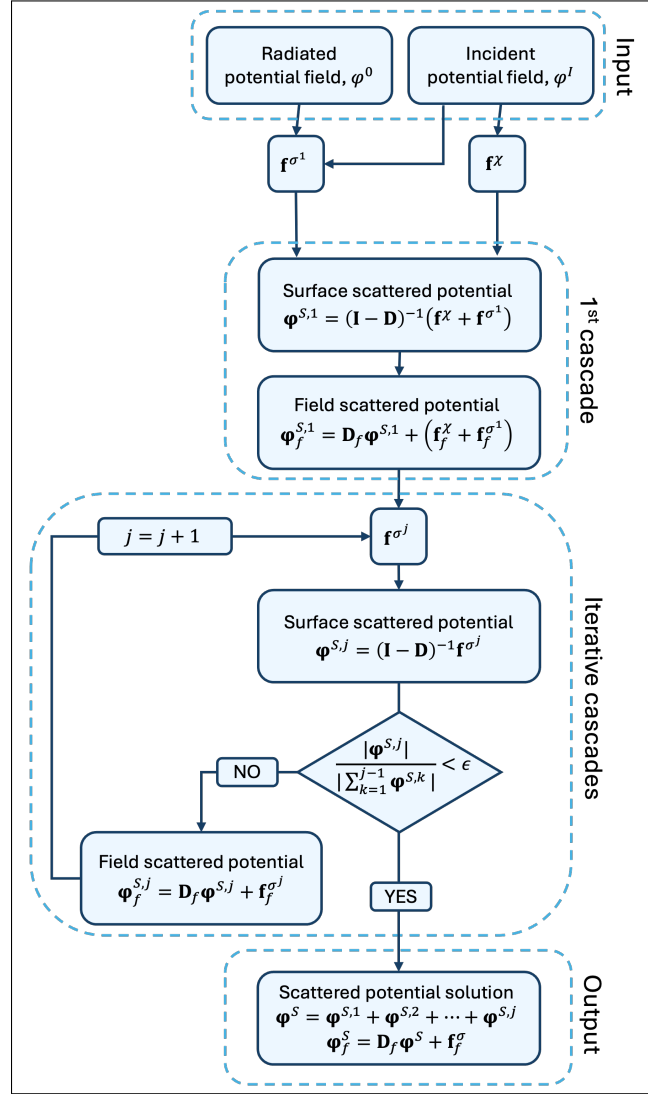


Figure 5.17: Flowchart of the proposed numerical procedure for nonlinear sound scattering analysis.

and frequency, ω , such that $kr_0 = 0.3$ (with $k = \omega/c_0$ denoting the deformation wave number), impinged by a plane wave of velocity potential, φ^I , of amplitude A^I and wave number $k_I = k$. For this problem, and for all pulsating spheres considered in the following, the reference potential φ_0 is determined analytically through the solution method presented in appendix E (for all cases examined, the nonlinear contribution $\sigma_0 = \sigma(\varphi_0)$ due to the flow induced by sphere pulsations is negligible).

Figure 5.18 presents the directivity patterns of the first three harmonics of the scattered signal evaluated at a set of observers located on a circle at a distance $d = 10r_0$ from the sphere center, lying on a plane perpendicular to the plane of the impinging wave, with $A^I = 1 \text{ m}^2/\text{s}$. In particular, it compares the results obtained by the linear formulation introduced in Sec. 5.1, with the proposed nonlinear solution which includes the distortion effects due to the flow generated by the sphere pulsations (represented by field terms involving φ_0 , φ^I and φ^S). In addition, the mono-harmonic solution for the rigid sphere is depicted. This figure proves that, as expected, the scattered signal is multi-harmonic (both due to pulsating boundary conditions and fluid nonlinearities), and that the fluid flow generated by sphere pulsations may produce significant distortion to all harmonics of the

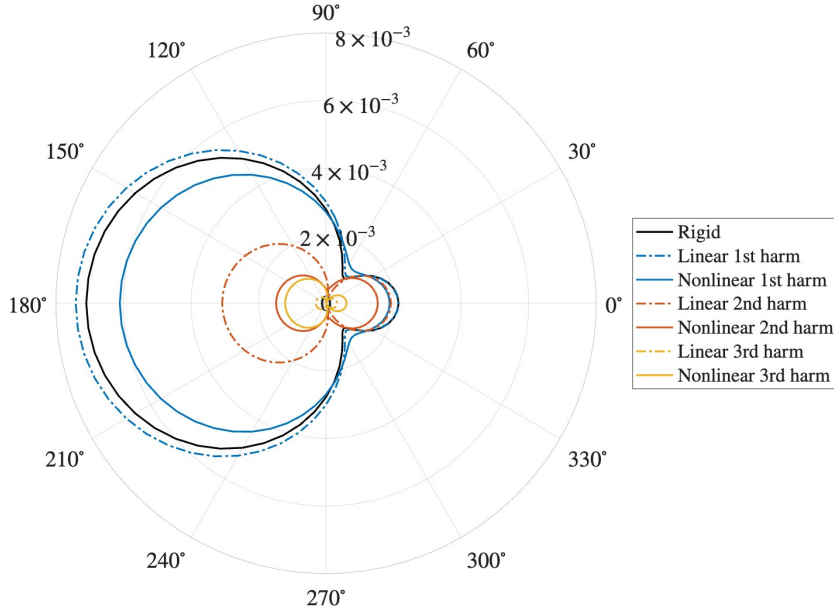


Figure 5.18: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to scattered signal provided by linear and nonlinear solutions. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

scattered signal, and thus its contribution cannot be neglected (a dedicated numerical investigation proved that nonlinear terms not involving φ_0 provide negligible contributions).

Next, Figs. 5.19-5.21 show the incremental effect of the cascade components on the directivity patterns of the first three harmonics of the scattered signal given by the nonlinear BEM solver. They demonstrate the rather rapid decrease in the contributions of the cascade components with increasing iterations, and hence the fast convergence rate of the applied algorithm for the nonlinear solution. For the examined problem, $N_S = 7$ is sufficient to obtain solutions with 2% converged tolerance (see Eq. (5.41)). This is highlighted in Fig. 5.22 which depicts, as a function of N_S , the values of the first three harmonics of the scattered potential computed at the surface point first impinged by the incident wave.

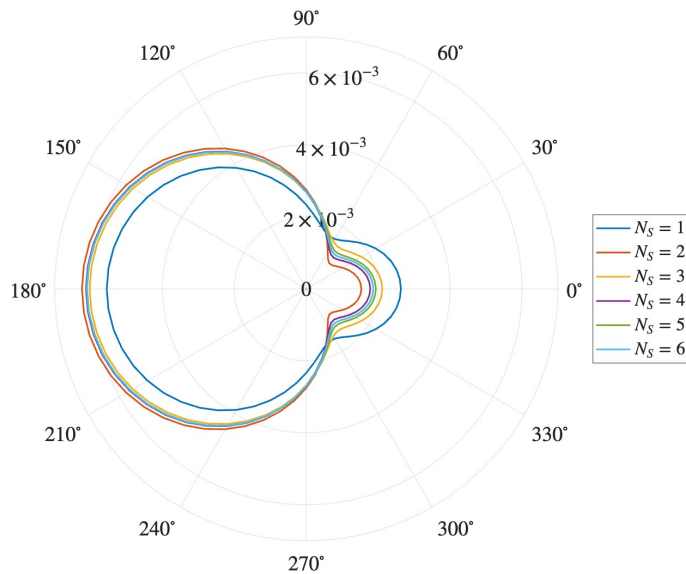


Figure 5.19: Incremental effect of $|\varphi^S|/|\varphi^I|$ related to cascade components on the directivity patterns of the first harmonic of the scattered signal. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

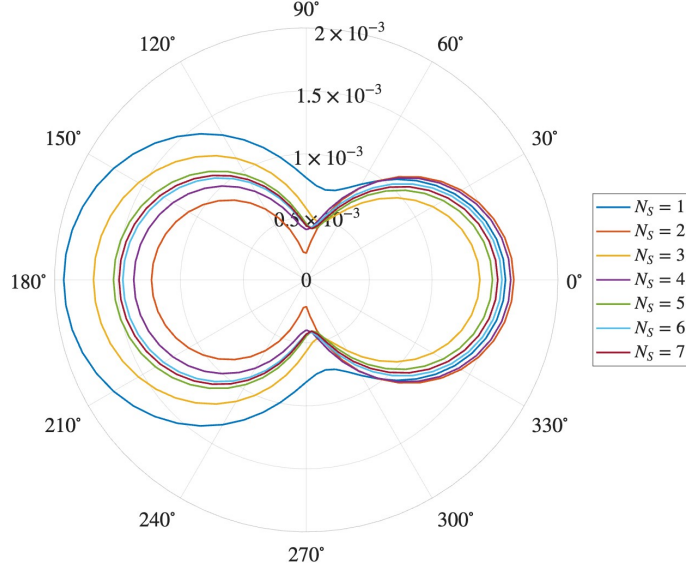


Figure 5.20: Incremental effect of $|\varphi^S|/|\varphi^I|$ related to cascade components on the directivity patterns of the second harmonic of the scattered signal. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

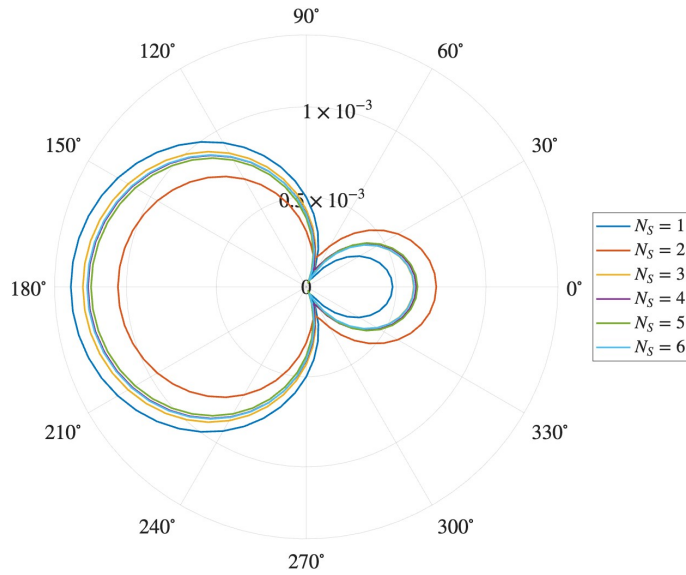


Figure 5.21: Incremental effect of $|\varphi^S|/|\varphi^I|$ related to cascade components on the directivity patterns of the third harmonic of the scattered signal. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

The comparisons among the different solutions of the scattered field considered in Fig. 5.18 are repeated for far-field and near-field observers located on a plane perpendicular to the plane of the impinging wave, still assuming $A^I = 1 \text{ m}^2/\text{s}$. These are shown, respectively, in Fig. 5.23 that presents the multi-harmonic directivity patterns of the scattered signal evaluated at a distance $d = 100 r_0$ from the sphere centre, and in Fig. 5.24 where the results regard observers much closer to the sphere centre, placed at $d = 2 r_0$.

While it is observed that, as expected, different far-field and near-field directivity patterns of scattered signal are obtained, in both cases it is confirmed that all harmonics of the scattered signal are significantly affected by the nonlinear terms associated with the fluid flow generated by the pulsations of the sphere.

Next, a case with lower pulsation frequency and larger pulsation amplitude, and a case with

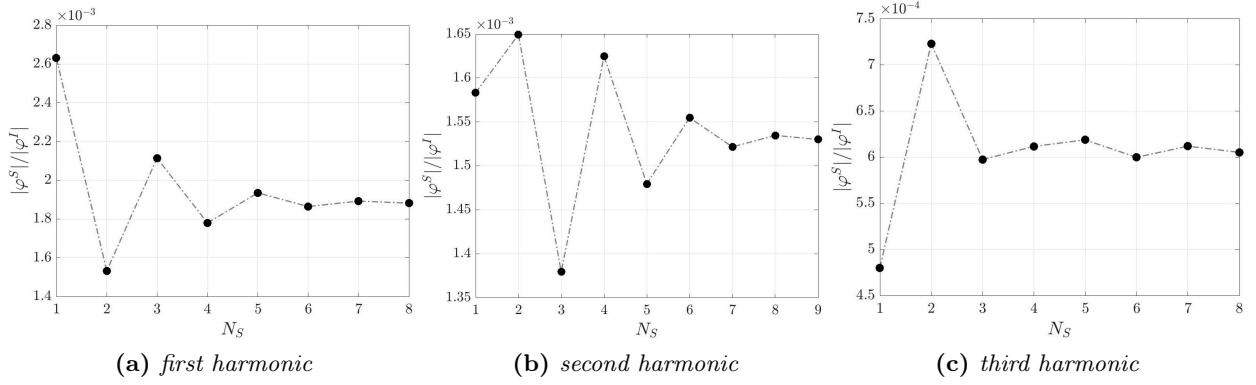


Figure 5.22: Scattered velocity potential at the observer point placed at azimuthal position $\vartheta = 0^\circ$.

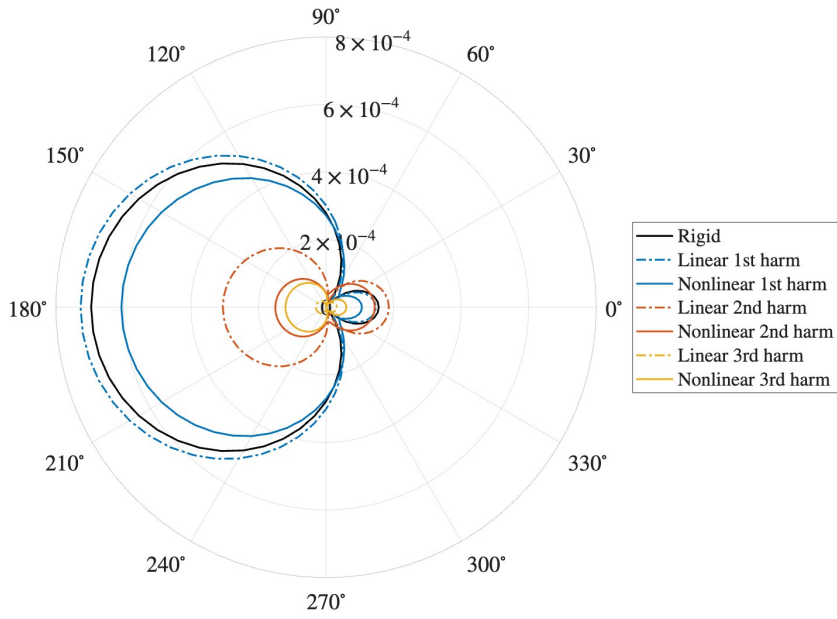


Figure 5.23: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 100$.

higher pulsation frequency and smaller pulsation amplitude are examined, with the incident planar wave still having $A^I = 1 \text{ m}^2/\text{s}$. In particular, Fig. 5.25 presents the scattered field at $d/r_0 = 10$, for $a = 0.4 r_0$ and $kr_0 = 0.1$, while, for the same positions of observers, Fig. 5.26 shows the scattered field for $a = 0.1 r_0$ and $kr_0 = 0.4$.

In both cases the amplitude of the oscillating Mach number of the sphere surface is equal to $ka = 0.04$, which is smaller than that related to the results in Fig. 5.18 ($ka = 0.06$, in that case). For this reason, linear and nonlinear predictions of the scattered signal in Figs. 5.25 and 5.26 are closer than those in Fig. 5.18. Comparing Figs. 5.25 and 5.26 it is interesting to observe that larger-amplitude pulsations produce high harmonics of the scattered signal which are more significant than those arising from the higher-frequency pulsation case (when related to the first harmonic of the signal). This derives from the kinematic nonlinearities produced by the boundary conditions applied over the pulsating sphere. At the same time, however, higher-frequency pulsations generate fluid nonlinearities which significantly affect the higher harmonics of the scattered signal. Indeed, the relative discrepancies between linear and nonlinear predictions of higher harmonics in Fig. 5.26 are

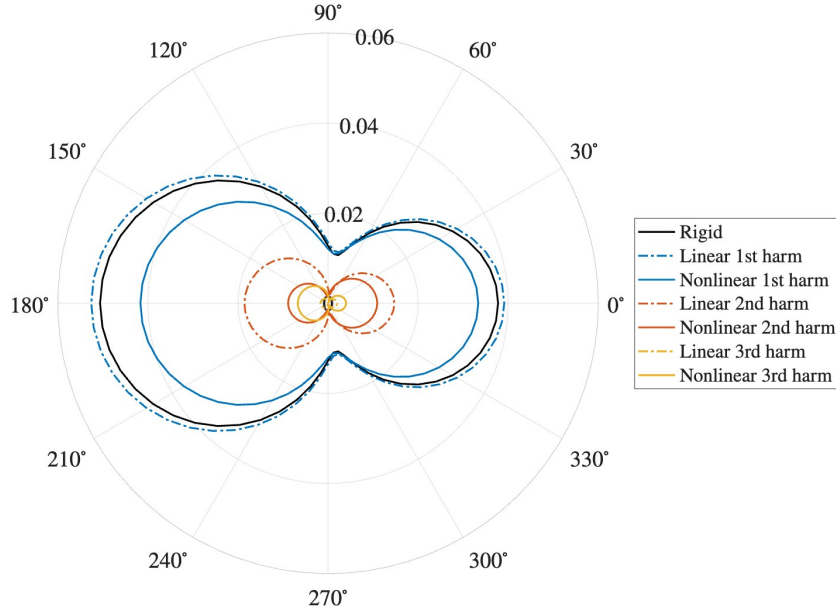


Figure 5.24: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.2 r_0$, $kr_0 = 0.3$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 2$.

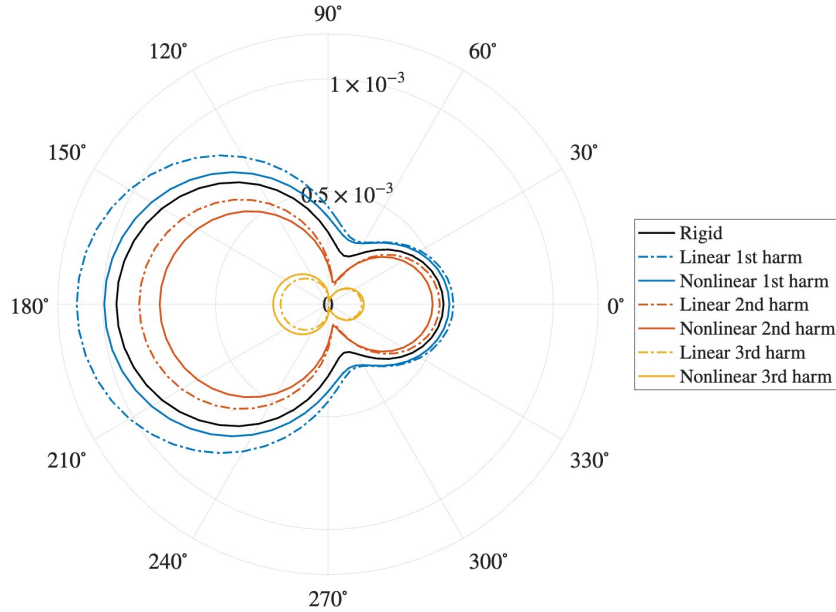


Figure 5.25: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.4 r_0$, $kr_0 = 0.1$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

considerably bigger than those observed in the larger-amplitude results of Fig. 5.25.

When linear scattering analysis is performed, the directivity patterns expressed in terms of the ratio between scattered and incident signal (as typically done and as done in this work) is invariant with respect to the amplitude of the incident signal. This is untrue when nonlinear field terms, σ , are considered. Indeed, as explained in Sec. 5.3.1, these are given by the combination of terms involving products between φ_0 , φ^I and φ^S . As the amplitude of φ^I increases, the corresponding higher-order contributions to σ become significant and the scattered signal is no longer linearly proportional to it. This is shown in Figs. 5.27, 5.28 and 5.29 for the pulsating sphere cases already examined in Figs. 5.18, 5.25 and 5.26, where the directivity patterns of the scattered velocity potential predicted for

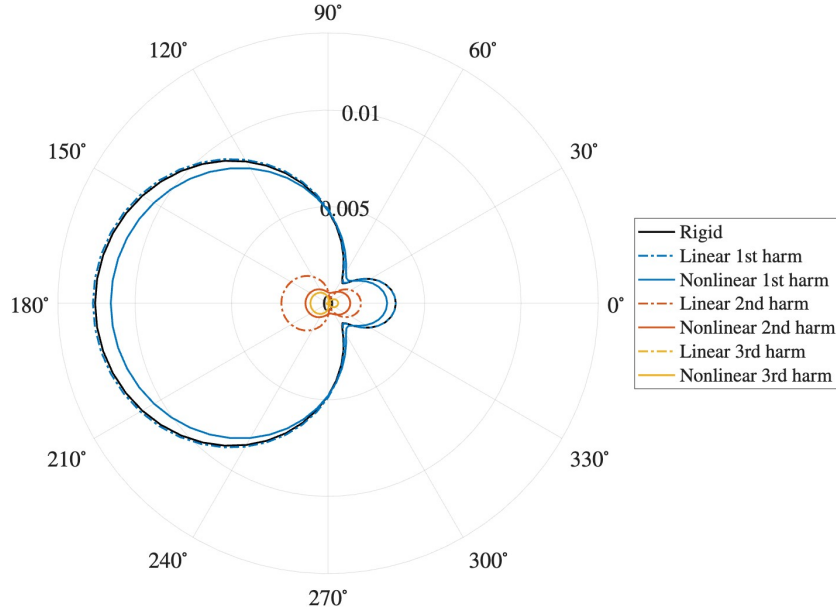


Figure 5.26: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.1 r_0$, $kr_0 = 0.4$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

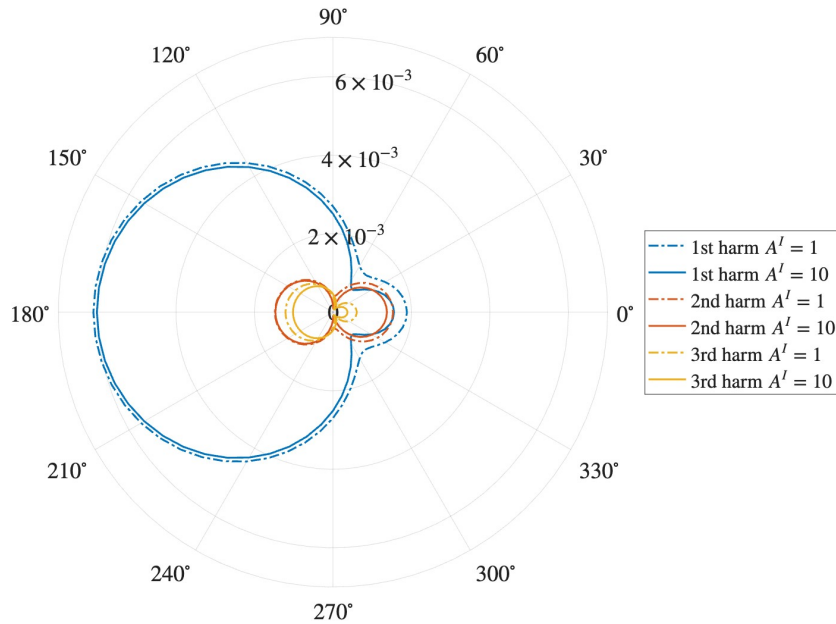


Figure 5.27: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal related to incident waves with $A^I = 1 \text{ m}^2/\text{s}$ and $A^I = 10 \text{ m}^2/\text{s}$. $a = 0.2 r_0$, $kr_0 = 0.3$, $d/r_0 = 10$ (the unit of measure of the values in the legend is m^2/s).

$A^I = 1 \text{ m}^2/\text{s}^2$ and $A^I = 10 \text{ m}^2/\text{s}^2$ are compared (still expressed in terms of the ratio $|\varphi^S|/|\varphi^I|$). The discrepancies between solid and dashed lines are representative of nonlinear field contributions due to the incident wave. It is interesting to note that when low-frequency/large-amplitude pulsations are considered, the multi-harmonic contributions to the scattered field are dominated by the kinematic nonlinear terms and the effects of increased incident wave amplitude are negligible (see Fig. 5.28). In this case, only the first harmonic is slightly affected by the amplitude of the incident signal. Instead, when higher-frequency pulsations with smaller amplitude are considered, the effects from fluid nonlinearities are dominant and the influence of the amplitude of the incident wave is stronger,

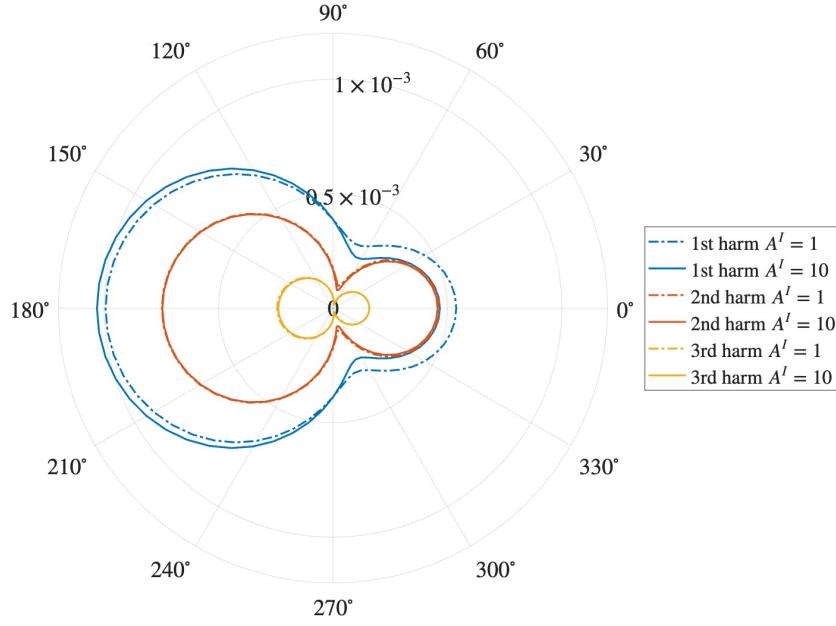


Figure 5.28: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal related to incident waves with $A^I = 1 \text{ m}^2/\text{s}$ and $A^I = 10 \text{ m}^2/\text{s}$. $a = 0.4r_0$, $kr_0 = 0.1$, $d/r_0 = 10$ the unit of measure of the values in the legend is m^2/s).

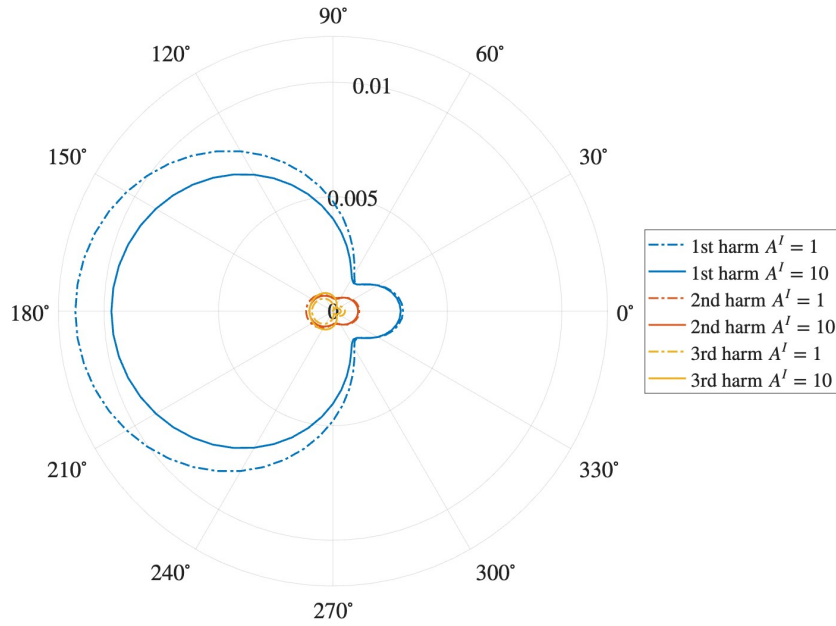


Figure 5.29: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal related to incident waves with $A^I = 1 \text{ m}^2/\text{s}$ and $A^I = 10 \text{ m}^2/\text{s}$. $a = 0.1r_0$, $kr_0 = 0.4$, $d/r_0 = 10$ the unit of measure of the values in the legend is m^2/s).

especially on the directivity pattern of the first harmonic of the scattered signal (see Fig. 5.29).

Finally, pulsation cases with either large amplitude or high frequency, but very small amplitudes of the surface Mach number are considered. Specifically, the signals scattered by pulsating spheres with $a = 0.5r_0$, $kr_0 = 0.01$ and $a = 0.01r_0$, $kr_0 = 0.5$, both corresponding to surface Mach number amplitude equal to $ka = 0.005$, are presented in Figs. 5.30 and 5.31, respectively. These results confirm that large amplitude pulsations generate higher harmonics of the scattered signal which are comparable to the first one, and that nonlinear contributions are negligible because of the very low

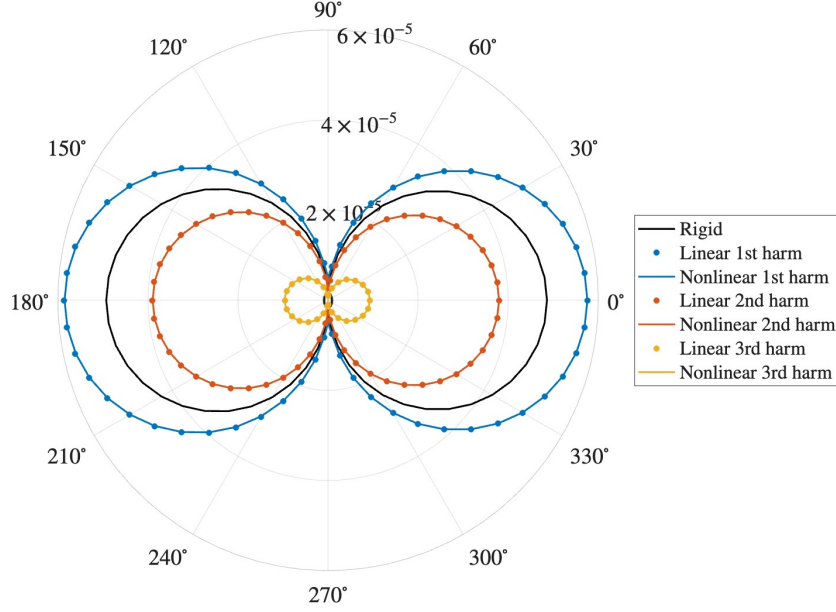


Figure 5.30: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.5 r_0$, $kr_0 = 0.01$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

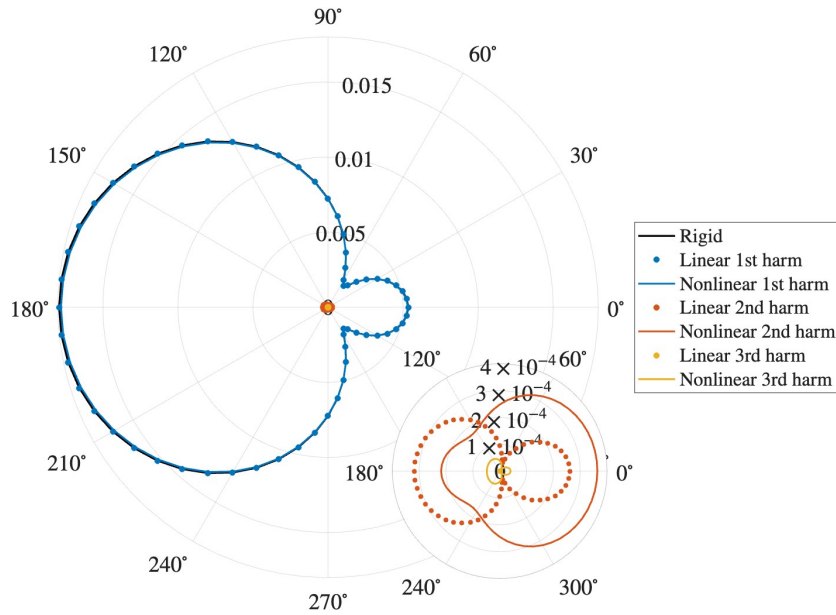


Figure 5.31: Multi-harmonic directivity patterns of $|\varphi^S|/|\varphi^I|$ related to the scattered signal provided by linear and nonlinear solutions. $a = 0.01 r_0$, $kr_0 = 0.5$, $A^I = 1 \text{ m}^2/\text{s}$, $d/r_0 = 10$.

surface Mach number and pulsation frequency (in these cases, linear and nonlinear solutions are practically identical, indeed). In addition, it is confirmed also that pulsation frequency affects fluid nonlinearities. This is highlighted by the zoom of higher harmonics in Fig. 5.31 which are very small due to the very small pulsation amplitude, but present significant discrepancy between linear and nonlinear predictions.

Chapter 6

Conclusions

The research activity carried out in this thesis was dedicated to the development and validation of a boundary integral formulation suitable for predicting acoustic and aerodynamic perturbation fields in fluid domains bounded by arbitrarily moving and deforming solid or porous surfaces. The formulation was obtained as the solution of the general arbitrarily-forced acoustic wave equation through the introduction of the free-space Green function. It allows the study of disturbance propagations problems expressed in terms of either acoustic pressure or velocity potential.

Two equivalent versions of the formulation, one expressed in the undisturbed-fluid frame and one in a frame in arbitrary motion, have been developed and implemented numerically through a zeroth-order Boundary Element Method.

The deformable-boundary integral formulation developed here can be used not only to predict radiated fields via its integral-representation form, but also to address scattering problems by solving its integral-equation form for the surface field. This dual capability allows the method to be applied to a broad spectrum of aeroacoustic scenarios, including coupled source–scatterer configurations, while offering a unified mathematical and numerical framework suitable for both linear and nonlinear problems.

The work presented in this thesis establishes a unified boundary-integral framework for predicting wave-propagation phenomena involving deforming structures, offering a foundation for the development of more advanced aeroelastic and high-fidelity aeroacoustic simulation tools.

6.1 Applications to acoustic radiation

The first set of numerical applications focused on radiation problems, in which the boundary-surface solution is prescribed and the formulation is employed as a boundary integral representation to propagate the disturbance into the far field. The chosen test cases -featuring porous and solid surfaces, both rigid and deforming- were selected to validate the method under progressively more complex physical and numerical conditions.

Specifically, by considering the propagation problems relating to a pulsating source surrounded by a deforming porous surface, a pulsating solid sphere, a non-lifting flapping and twisting translating wing and a helicopter rotor model in forward flight, the numerical investigation has proven that the proposed deformable-boundary integral formulation:

- expressed in the fixed and moving frames gives equivalent versions which provide identical

results, whatever the motion of the moving frame;

- yields predictions that are in agreement with analytical solutions when the latter are available;
- is applicable, with the same level of accuracy, to solid- and porous-boundary problems;
- is able to capture the propagation effects due to boundary deformation;
- is suitable for the analysis of aeroacoustic and potential aerodynamics problems;
- is suitable for the investigation of complex aeronautical configurations, for which the inclusion of the boundary deformation effects in the propagation of perturbations may significantly affects the simulation accuracy when the structure is subject to moderate/large elastic deflections.

In the overall, the numerical investigation on radiation problems has demonstrated that the presented deformable-boundary integral formulation provides simulations in agreement with analytical or experimental solutions whenever available, and remains robust in the presence of large structural deformations, multiphysics coupling, and significant multi-harmonic content in the radiated field. It is suitable for aeroacoustic applications involving deformable bodies, such as propellers, wings, helicopter rotors, and porous surfaces used as fictitious CFD-to-BEM interfaces.

6.2 Applications to acoustic scattering

The second part of the numerical investigation was focused on sound scattering problems for which, first, a boundary integral equation must be solved to determine the solution over the boundary surface, and, then, the boundary integral representation is applied to evaluate the scattered field in the fluid domain.

Two formulations have been developed: a linear scattering model forced by kinematic boundary conditions of body impermeability, and a fully nonlinear extension that also accounts for the forcing terms coming from the meanflow (generated by the surface motion) through fluid nonlinearities .

Both versions have been expressed in terms of either acoustic pressure or velocity potential and have been solved numerically for arbitrarily shaped deforming scatterers. Two complementary numerical strategies have been adopted: a time-domain marching scheme and a frequency-domain multi-harmonic algorithm, both based on a zeroth-order Boundary Element Method. As for the linear formulation, time-domain and frequency-domain solution strategies (related to both acoustic pressure and velocity potential fields) have been successfully validated by comparison with available analytical solutions regarding pulsating spherical scatterers, for different observers locations. However, it was noticed that the time-domain algorithm is stable and reliable for problems concerning low-frequency incident waves ($kr_0 < 1$ for the rigid-sphere scattering). Instead, for higher frequencies of the incident wave the solution given by the proposed frequency-domain multi-harmonic algorithm still remains stable and accurate, while the time-domain solution is difficult to converge to a reliable solution.

The proposed approaches for linear sound scattering analysis expressed in terms of acoustic pressure and velocity potential have been applied to examine the effect of several scatterer deformation parameters, such as amplitude, frequency, and shape, for different values of the impinging wave frequency. The main outcomes are the following:

- the combination of the incident harmonic wave with the time variation of BEM coefficients due to scatterer deformation results in significant multi-harmonic contributions to the scattered signal (which, therefore, is not mono-harmonic as in the case of the rigid scatterer);
- as the amplitude and frequency of scatterer deformation increase, the higher-harmonic content of the scattered sound also increases, becoming comparable to the scattered signal at the same frequency as the incident wave;
- the scattered sound predicted by the acoustic pressure integral formulation and the velocity pressure integral formulation may considerably differ as the amplitude and frequency of the scatterer deformation increase; this is particularly evident for the higher-harmonic content and it indicates that the nonlinear contributions neglected in both formulations are different and would provide significant contributions; this effect is due to the presence of the nonuniform fluid flow generated by the scatterer deformation and is in line with the same different accuracy of acoustic pressure-based and velocity potential-based scattering formulations already observed in the literature in the case of the presence of meanflow generated by the motion of rigid scatterers;
- it is proven that the shape of the scatterer deformation plays a crucial role; indeed, it is shown that a non-uniform wavy deformation of the sphere provides a scattering effect which is much lower than that provided by the uniform pulsation; incidentally, it is observed that considering odd or even numbers of deformation waves may cancel or not some of the harmonics of the scattered signal.

Finally, a solution strategy for nonlinear sound scattering has been applied to a pulsating sphere subjected to an incident plane sound wave. The method uses a harmonic-balance cascade algorithm in which the harmonics of the scattered velocity-potential field are iteratively coupled until convergence, while accounting for the flow field generated by the sphere's dynamic deformation.

The numerical study has demonstrated that: (i) the proposed harmonic-balance cascade algorithm yields a nonlinear scattered field that converges rapidly; (ii) when the sphere undergoes large-amplitude pulsations at non-negligible frequencies, the nonlinear solution deviates markedly from the linear prediction; (iii) for small-amplitude incident sound waves, the linearized and nonlinear solutions coincide; and (iv) as the incident-wave amplitude increases, the two solutions diverge, indicating that accurate modeling of sound scattering from deforming bodies necessitates a nonlinear approach such as the one developed in this work."

The nonlinear scattering method presented in this work represents the first application of a boundary integral formulation for deformable bodies to sound-scattering problems that account for fluid nonlinearities induced by surface deformation.

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Appendix A

From Lighthill to Ffowcs Williams and Hawkings equation

Let us consider the [FFR](#) and the Lighthill equation extended to the entire $\mathbb{R}^3$ domain as illustrated in [Sec. 2.1](#)

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = & -H \nabla \cdot \nabla \cdot \mathbf{T} + \nabla H \cdot \nabla p' + \nabla \cdot (p' \nabla H) + \\ & - \frac{1}{c_0^2} \left[\frac{\partial H}{\partial t} \frac{\partial p'}{\partial t} + \frac{\partial}{\partial t} \left(p' \frac{\partial H}{\partial t} \right) \right] \end{aligned} \quad (\text{A.1})$$

where $p' = c_0^2(\rho - \rho_0)$ denotes the acoustic pressure, and $H = H(f)$ represents the Heaviside function, with $f = f(\mathbf{x}, t)$ such that $f > 0$ outside of the boundary surface, $\mathcal{S}$, of the fluid region where the propagation of the signal is governed by [Eq. \(A.1\)](#), and $f < 0$ inside it.

Observing that the following identity holds

$$\nabla \cdot \nabla \cdot [H \mathbf{T}] = H \nabla \cdot \nabla \cdot \mathbf{T} + \nabla \cdot [\mathbf{T} \nabla H] + (\nabla \cdot \mathbf{T}) \cdot \nabla H \quad (\text{A.2})$$

[Eq. \(A.1\)](#) can be rewritten as

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = & -\nabla \cdot \nabla (H \mathbf{T}) + \nabla \cdot [(\mathbf{T} + p' \mathbf{I}) \nabla H] + \nabla \cdot (\mathbf{T} + p' \mathbf{I}) \cdot \nabla H \\ & - \frac{\partial \rho}{\partial t} \frac{\partial H}{\partial t} - \frac{\partial}{\partial t} \left(\rho \frac{\partial H}{\partial t} \right) + \rho_0 \frac{\partial^2 H}{\partial t^2} \end{aligned} \quad (\text{A.3})$$

Next, recalling that $\mathbf{T} + p' \mathbf{I} = \mathbf{P} + \rho \mathbf{u} \otimes \mathbf{u}$ (see [Sec. 2.1](#)), from the Navier-Stokes equation

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\mathbf{P} + \rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{0}$$

we obtain

$$\nabla \cdot (\mathbf{T} + p' \mathbf{I}) = -\frac{\partial}{\partial t} (\rho \mathbf{u})$$

and Eq. (A.3) becomes

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = & -\nabla \cdot \nabla(H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] + \nabla \cdot [(\rho\mathbf{u} \otimes \mathbf{u})\nabla H] - \frac{\partial}{\partial t}(\rho\mathbf{u}) \cdot \nabla H \\ & - \frac{\partial \rho}{\partial t} \frac{\partial H}{\partial t} - \frac{\partial}{\partial t} \left(\rho \frac{\partial H}{\partial t} \right) + \rho_0 \frac{\partial^2 H}{\partial t^2} \end{aligned} \quad (\text{A.4})$$

Then, considering the identity

$$\nabla \cdot [\rho(\mathbf{u} \otimes \mathbf{u})\nabla H] = \nabla \cdot (\rho\mathbf{u})(\mathbf{u} \cdot \nabla H) + \rho\mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla H)$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0 \quad (\text{A.5})$$

combined with Eq. (A.4) yields the equation

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = & -\nabla \cdot \nabla(H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] - 2\frac{\partial \rho}{\partial t} \mathbf{u} \cdot \nabla H - \rho \frac{\partial \mathbf{u}}{\partial t} \cdot \nabla H \\ & + \rho\mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla H) - 2\frac{\partial \rho}{\partial t} \frac{\partial H}{\partial t} - \rho \frac{\partial^2 H}{\partial t^2} + \rho_0 \frac{\partial^2 H}{\partial t^2} \end{aligned} \quad (\text{A.6})$$

which, suitably re-arranging the terms on the right-hand side, can be rewritten as

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = & -\nabla \cdot \nabla(H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] + \rho_0 \frac{\partial^2 H}{\partial t^2} \\ & - 2\frac{\partial \rho}{\partial t} \left(\frac{\partial H}{\partial t} + \mathbf{u} \cdot \nabla H \right) - \rho \left(\frac{\partial^2 H}{\partial t^2} + \frac{\partial \mathbf{u}}{\partial t} \nabla H - \mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla H) \right) \end{aligned} \quad (\text{A.7})$$

For $D(\dots)/Dt$ denoting the material time derivative performed following the fluid particle, we have $DH/Dt = \partial H/\partial t + \mathbf{u} \cdot \nabla H$, and thus, the last two terms of the above equation can be expressed as

$$\begin{aligned} & -2\frac{\partial \rho}{\partial t} \left(\frac{\partial H}{\partial t} + \mathbf{u} \cdot \nabla H \right) - \rho \left(\frac{\partial^2 H}{\partial t^2} + \frac{\partial \mathbf{u}}{\partial t} \nabla H - \mathbf{u} \cdot \nabla(\mathbf{u} \cdot \nabla H) \right) = \\ & = -2\frac{\partial \rho}{\partial t} \frac{DH}{Dt} - \rho \left(\frac{D^2 H}{Dt^2} - 2\mathbf{u} \cdot \nabla \frac{DH}{Dt} \right) \\ & = -2\frac{\partial}{\partial t} \left(\rho \frac{DH}{Dt} \right) + \rho \frac{D^2 H}{Dt^2} \end{aligned}$$

and Eq. (A.7) is given in the following form

$$\nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = -\nabla \cdot \nabla(H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] + \rho_0 \frac{\partial^2 H}{\partial t^2} - 2\frac{\partial}{\partial t} \left(\rho \frac{DH}{Dt} \right) + \rho \frac{D^2 H}{Dt^2} \quad (\text{A.8})$$

By suitably defining the term DH/Dt in the above equation, it is possible to derive the [FWHE](#) for solid and porous sound emitting surfaces.

First of all, let us observe that the time derivative of $H(f)$ performed by following a point of the surface $\mathcal{S}$ is equal to zero by definition, *i.e.*, for (ξ^1, ξ^2) denoting a body-fitted curvilinear coordinate

system defined over $\mathcal{S}$ and $\partial(\dots)/\partial t|_\xi$ denoting time derivative at a (ξ^1, ξ^2) -fixed point

$$\left. \frac{\partial H}{\partial t} \right|_\xi = \frac{\partial H}{\partial t} + \mathbf{v} \cdot \nabla H = 0 \quad (\text{A.9})$$

and, thus, second-order Eulerian time derivative of H can be expressed as

$$\frac{\partial^2 H}{\partial t^2} = -\frac{\partial}{\partial t} (\mathbf{v} \cdot \nabla H) \quad (\text{A.10})$$

while its material derivative is given by

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \mathbf{u} \cdot \nabla H = (\mathbf{u} - \mathbf{v}) \cdot \nabla H \quad (\text{A.11})$$

Then, considering that when the surface is solid $(\mathbf{u} - \mathbf{v}) \cdot \nabla H = 0$ on $\mathcal{S}$ (normal components of fluid and surface velocities coincide over impermeable surfaces), the combination of Eq. (A.8) with Eq. (A.10) and Eq. (A.11) yields the following **FWHE** for solid fluid boundary surfaces

$$\nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} = -\nabla \cdot \nabla (H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] - \rho_0 \frac{\partial}{\partial t} (\mathbf{v} \cdot \nabla H) \quad (\text{A.12})$$

To derive the **FWHE** for porous surfaces, the last two terms of Eq. (A.8) are combined with Eq. (A.11) and further manipulated to obtain the following expression

$$\begin{aligned} & -2 \frac{\partial}{\partial t} \left(\rho \frac{DH}{Dt} \right) + \rho \frac{D^2 H}{Dt^2} \\ &= -\frac{\partial}{\partial t} (\rho(\mathbf{u} - \mathbf{v}) \cdot \nabla H) - \frac{\partial}{\partial t} \left(\rho \frac{DH}{Dt} \right) + \rho \frac{\partial}{\partial t} \left(\frac{DH}{Dt} \right) + \rho \mathbf{u} \cdot \nabla \frac{DH}{Dt} \\ &= -\frac{\partial}{\partial t} (\rho(\mathbf{u} - \mathbf{v}) \cdot \nabla H) - \frac{\partial \rho}{\partial t} \frac{DH}{Dt} + \rho \mathbf{u} \cdot \nabla \frac{DH}{Dt} \end{aligned} \quad (\text{A.13})$$

whose last two terms are further combined with the continuity equation, Eq. (A.5), to be rewritten as

$$\begin{aligned} -\frac{\partial \rho}{\partial t} \frac{DH}{Dt} + \rho \mathbf{u} \cdot \nabla \frac{DH}{Dt} &= \nabla \cdot (\rho \mathbf{u}) \frac{DH}{Dt} + \rho \mathbf{u} \cdot \nabla \frac{DH}{Dt} = \\ &= \nabla \cdot \left(\rho \mathbf{u} \frac{DH}{Dt} \right) = \nabla \cdot (\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) \nabla H) \end{aligned} \quad (\text{A.14})$$

Finally, by combining Eq. (A.8) with Eq. (A.10), Eq. (A.13) and Eq. (A.14), the following **FWHE** for porous fluid boundary surfaces is readily obtained

$$\begin{aligned} \nabla^2 \bar{p}' - \frac{1}{c_0^2} \frac{\partial^2 \bar{p}'}{\partial t^2} &= -\nabla \cdot \nabla (H\mathbf{T}) + \nabla \cdot [\mathbf{P}\nabla H] - \rho_0 \frac{\partial}{\partial t} (\mathbf{v} \cdot \nabla H) \\ &\quad - \frac{\partial}{\partial t} (\rho(\mathbf{u} - \mathbf{v}) \cdot \nabla H) + \nabla \cdot (\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) \nabla H) \end{aligned} \quad (\text{A.15})$$

Appendix B

Jacobian of the integration variables transformation

Let us consider the transformation of integration variables

$$(\mathbf{y}, t) \rightarrow (\xi^1, \xi^2, \xi^3, g)$$

introduced in Sect. 3. The corresponding Jacobian, $\hat{J}$, such that

$$dV dt = dy_1 dy_2 dy_3 dt = \hat{J} d\xi^1 d\xi^2 d\xi^3 dg \quad (\text{B.1})$$

can be determined as $\hat{J} = \bar{J}^{-1}$ where

$$\bar{J} = \frac{\partial(\xi_m^1, \xi_m^2, \xi_m^3, g)}{\partial(y_1, y_2, y_3, t)} = \begin{vmatrix} \partial\xi_m^1/\partial y_1 & \partial\xi_m^2/\partial y_1 & \partial\xi_m^3/\partial y_1 & \partial g/\partial y_1 \\ \partial\xi_m^1/\partial y_2 & \partial\xi_m^2/\partial y_2 & \partial\xi_m^3/\partial y_2 & \partial g/\partial y_2 \\ \partial\xi_m^1/\partial y_3 & \partial\xi_m^2/\partial y_3 & \partial\xi_m^3/\partial y_3 & \partial g/\partial y_3 \\ \partial\xi_m^1/\partial t & \partial\xi_m^2/\partial t & \partial\xi_m^3/\partial t & \partial g/\partial t \end{vmatrix} \quad (\text{B.2})$$

with (y_1, y_2, y_3) denoting the coordinates of a Cartesian system fixed to the MFR, and $\xi^\alpha = \xi_m^\alpha(\mathbf{y}, t)$ representing the inverse of the mapping in Eq. (3.13).

Then, by observing that, for $\mathbf{e}_j$ representing the unit basis vectors of the Cartesian system and $\mathbf{a}^\alpha$ denoting the contravariant basis vectors of the curvilinear coordinate system

- $\partial\xi_m^\alpha/\partial y_j = \nabla\xi_m^\alpha \cdot \mathbf{e}_j = \mathbf{a}^\alpha \cdot \mathbf{e}_j, \quad j = 1, 3; \quad \alpha = 1, 3$
- $\partial g/\partial t = 1$
- $\partial g/\partial y_j = \partial\theta_y/\partial y_j = \nabla\theta_y \cdot \mathbf{e}_j, \quad j = 1, 3$
- $\partial\xi_m^\alpha/\partial t|_\xi = 0 \rightarrow \partial\xi_m^\alpha/\partial t|_y = -\mathbf{v}_m^y \cdot \nabla\xi_m^\alpha = -\mathbf{v}_y^m \cdot \mathbf{a}^\alpha, \quad \alpha = 1, 3$

the Jacobian in Eq. (B.2) can be recast in the following form

$$\bar{J} = \begin{vmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 & \nabla\theta_y \\ -\mathbf{v}_y^m \cdot \mathbf{a}^1 & -\mathbf{v}_y^m \cdot \mathbf{a}^2 & -\mathbf{v}_y^m \cdot \mathbf{a}^3 & 1 \end{vmatrix}$$

which yields

$$\begin{aligned}
 \bar{J} &= \mathbf{v}_y^m \cdot \mathbf{a}^1 \nabla \theta_y \cdot \mathbf{a}^2 \times \mathbf{a}^3 - \mathbf{v}_y^m \cdot \mathbf{a}^2 \nabla \theta_y \cdot \mathbf{a}^1 \times \mathbf{a}^3 + \mathbf{v}_y^m \cdot \mathbf{a}^3 \nabla \theta_y \cdot \mathbf{a}^1 \times \mathbf{a}^2 \\
 &+ |\mathbf{a}^1 \times \mathbf{a}^2 \cdot \mathbf{a}^3| \\
 &= |\mathbf{a}^1 \times \mathbf{a}^2 \cdot \mathbf{a}^3| \left(1 + \sum_{\alpha=1}^3 \mathbf{v}_y^m \cdot \mathbf{a}^\alpha \nabla \theta_y \cdot \mathbf{a}_\alpha \right) \\
 &= |\mathbf{a}^1 \times \mathbf{a}^2 \cdot \mathbf{a}^3| (1 + \mathbf{v}_y^m \cdot \nabla \theta_y)
 \end{aligned} \tag{B.3}$$

where $\mathbf{a}_\alpha$ denotes the covariant basis vectors of the curvilinear coordinate system. Finally, recalling that $|\mathbf{a}^1 \times \mathbf{a}^2 \cdot \mathbf{a}^3| = |\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|^{-1}$, the following expression of the Jacobian of the transformation of the integration variables is straightforwardly derived from Eq. (B.3)

$$\hat{J} = \bar{J}^{-1} = \frac{|\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3|}{|1 + \mathbf{v}_y^m \cdot \nabla \theta_y|}$$

Appendix C

Expansion of the integral formulation in the FFR

Neglecting the field term, the deformable-boundary integral formulation in Eq. (3.25) can be reformulated as the combination of the following five integral contributions

$$I_{\mathbf{z}} = \iint_{\Omega} \mathbf{z} \cdot \mathbf{n} G_0 J \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (\text{C.1})$$

$$I_{\mathbf{z}} = \iint_{\Omega} \left[-(\mathbf{Zn}) \cdot \nabla G_0 J + \left(\frac{\partial}{\partial t} \Big|_{\xi} [(\mathbf{Zn}) \cdot \nabla \theta_x G_0 J] \right) \frac{1}{\Theta} \right] \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (\text{C.2})$$

$$I_{k_1} = - \iint_{\Omega} k_1 \mathbf{v} \cdot \mathbf{n} G_0 J \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (\text{C.3})$$

$$I_{k_2} = - \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} [k_2 \mathbf{v} \cdot \mathbf{n} G_0 J] \right) \frac{1}{\Theta} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (\text{C.4})$$

$$I_{\mathbf{w}} = \iint_{\Omega} \left(\frac{\partial}{\partial t} \Big|_{\xi} [\mathbf{w} \cdot \mathbf{n} G_0 J] \right) \frac{1}{\Theta} \Big|_{\substack{\xi^3=0 \\ g=0}} d\xi^1 d\xi^2 \quad (\text{C.5})$$

Thus, noting that

$$\begin{aligned} \nabla r &= -\mathbf{e}_r \\ \frac{\partial r}{\partial t} \Big|_{\xi} &= \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r} = -\mathbf{e}_r \cdot \mathbf{v} \\ \frac{\partial \mathbf{e}_r}{\partial t} \Big|_{\xi} &= \frac{1}{r} \frac{\partial \mathbf{r}}{\partial t} \Big|_{\xi} - \frac{\mathbf{r}}{r^2} \frac{\partial r}{\partial t} \Big|_{\xi} = -\frac{\mathbf{v}}{r} + \frac{\mathbf{e}_r \mathbf{v} \cdot \mathbf{e}_r}{r} \end{aligned}$$

the following expressions are straightforwardly derived from the application of the space and time

differential operators that appear in Eqs. (C.1)-(C.5)

$$\begin{aligned}\nabla G_0 &= -\frac{\mathbf{e}_r}{4\pi r^2} \\ \frac{\partial G_0}{\partial t}\Big|_\xi &= -\frac{\mathbf{e}_r \cdot \mathbf{v}}{4\pi r^2} \\ \frac{\partial \nabla \theta}{\partial t}\Big|_\xi &= -\frac{1}{c_0} \left(\frac{\mathbf{v}}{r} - \frac{\mathbf{e}_r \cdot \mathbf{v} \mathbf{e}_r}{r} \right) \\ \frac{\partial J}{\partial t}\Big|_\xi &= \frac{\dot{J}_s}{1 - M_r} + \frac{J_s}{r(1 - M_r)^2} \left(r\dot{\mathbf{M}} \cdot \mathbf{e}_r - c_0 M^2 + c_0 M_r^2 \right)\end{aligned}$$

where $(\dot{\dots}) = \partial(\dots)/\partial t|_\xi$ represents the time derivative at a fixed surface point, $M = \|\mathbf{M}\|$, and the following symbols are conveniently introduced

- $J_s = |\mathbf{a}_1 \times \mathbf{a}_2|$: Jacobian of the boundary surface; by recalling the assumption on the ξ^3 -coordinate introduced to obtain Eq. (3.15), we note that it coincides with the Jacobian of the local transformation of space variables, i.e., $|\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3| \equiv |\mathbf{a}_1 \times \mathbf{a}_2|$, which contributes to the definition of the Jacobian, J , appearing in Eqs. (C.1)-(C.5)
- $M_r = \mathbf{M} \cdot \mathbf{e}_r = \mathbf{v} \cdot \mathbf{e}_r / c_0$: component of Mach number of surface points along source-observer direction
- $M_n = \mathbf{M} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} / c_0$: component of Mach number of surface points orthogonal to the surface.

The combination of the above expressions with Eqs. (C.1)-(C.5) yields the expanded forms of the integrals $I_{\mathbf{z}}, I_{\mathbf{Z}}, I_{k1}, I_{k2}, I_{\mathbf{w}}$ that are given in Eqs. (3.27)-(3.31).

Appendix D

Expansion of the integral formulation in the MFR

Similarly to the integral formulation expressed in [FFR](#), the one expressed in [MFR](#) can also be appropriately reformulated in expanded form.

This is done by suitably interpreting the several differential operators appearing and, in particular, the time derivatives which are always evaluated by fixing a material point, but are observed by the [MFR](#).

First of all, let us recall the expression of Green's function, time delay and Jacobian appearing in the formulation expressed in the [MFR](#) (see Sec. 3.1 and Sec. 3.2)

$$\hat{G}_0 = \frac{-1}{4\pi r(1 - M_r^y)} \Big|_{\theta_y}; \quad \theta_y = \frac{r}{c_0} \Big|_{\theta_y}; \quad \hat{J} = \frac{J_s}{|1 + \mathbf{v}_y^m \cdot \nabla \theta_y|}$$

where $J_s = |\mathbf{a}_1 \times \mathbf{a}_2|$ denotes the Jacobian of the boundary surface.

Next, let us observe that for an arbitrary delayed function, $f_\theta = f(t - \theta)$, its gradient is given by

$$\nabla f_\theta = \nabla f|_\theta - \dot{f}|_\theta \nabla \theta \quad (\text{D.1})$$

and, therefore, the gradient of θ_y is

$$\nabla \theta_y = \nabla \left(\frac{r}{c} \right) \Big|_{\theta_y} - \frac{1}{c_0} \frac{\partial r}{\partial t} \Big|_y \nabla \theta_y \quad (\text{D.2})$$

where $\nabla r = -\mathbf{e}_r$ and $\partial r / \partial t|_y = -\mathbf{v}_y \cdot \mathbf{e}_r$, and the following expression is obtained

$$\nabla \theta_y = \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \Big|_{\theta_y} \quad (\text{D.3})$$

Similarly, the gradient of Green's function yields

$$\nabla \hat{G}_0 = \nabla \left(\frac{-1}{4\pi r(1 - M_r^y)} \right) \Big|_{\theta_y} - \frac{\partial}{\partial t} \Big|_y \left(\frac{-1}{4\pi r(1 - M_r^y)} \right) \Big|_{\theta_y} \nabla \theta_y \quad (\text{D.4})$$

where

$$\nabla \left(\frac{-1}{4\pi r(1 - M_r^y)} \right) = \frac{\nabla r}{4\pi r^2(1 - M_r^y)} - \frac{\nabla(\mathbf{v}_y \cdot \mathbf{e}_r)}{4\pi r c_0(1 - M_r^y)^2} \quad (\text{D.5})$$

with

$$\nabla(\mathbf{v}_y \cdot \mathbf{e}_r) = [\nabla \mathbf{v}_y]^T \mathbf{e}_r + [\nabla \mathbf{e}_r]^T \mathbf{v}_y$$

Noting that the gradients of $\mathbf{e}_r$ and velocity $\mathbf{v}_y$ can be expressed as follows

$$\nabla \mathbf{e}_r = -\frac{\mathbf{I}}{r} + \frac{\mathbf{e}_r \otimes \mathbf{e}_r}{r}; \quad \nabla \mathbf{v}_y = \boldsymbol{\Omega}_y$$

where $\boldsymbol{\Omega}_y$ is an antisymmetric tensor such that for an arbitrary vector $\mathbf{q}$ we have $\boldsymbol{\Omega}_y \mathbf{q} = \boldsymbol{\omega}_y \times \mathbf{q}$ with $\boldsymbol{\omega}_y$ representing the angular velocity of the MFR with respect to the FFR, the gradient of $(\mathbf{v}_y \cdot \mathbf{e}_r)$ can be recast in the form

$$\nabla(\mathbf{v}_y \cdot \mathbf{e}_r) = \mathbf{e}_r \times \boldsymbol{\omega}_y - \frac{\mathbf{v}_y}{r} + \frac{\mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r}{r} \quad (\text{D.6})$$

For the second term of the gradient of Green's function, we have

$$\frac{\partial}{\partial t} \bigg|_y \left(\frac{-1}{4\pi r(1 - M_r^y)} \right) = \frac{1}{4\pi r^2(1 - M_r^y)} \frac{\partial r}{\partial t} \bigg|_y - \frac{1}{4\pi r c_0(1 - M_r^y)^2} \frac{\partial \mathbf{v}_y \cdot \mathbf{e}_r}{\partial t} \bigg|_y \quad (\text{D.7})$$

where

$$\frac{\partial r}{\partial t} \bigg|_y = -\mathbf{v}_y \cdot \mathbf{e}_r; \quad \frac{\partial (\mathbf{v}_y \cdot \mathbf{e}_r)}{\partial t} \bigg|_y = \frac{\partial \mathbf{v}_y}{\partial t} \bigg|_y \cdot \mathbf{e}_r + \mathbf{v}_y \cdot \frac{\partial \mathbf{e}_r}{\partial t} \bigg|_y \quad (\text{D.8})$$

Noting that the time derivatives of $\mathbf{e}_r$ and velocity $\mathbf{v}_y$ can be expressed as follows, if observed by the FFR

$$\frac{\partial \mathbf{e}_r}{\partial t} \bigg|_y = -\frac{\mathbf{v}_y}{r} + \frac{\mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r}{r}; \quad \frac{\partial \mathbf{v}_y}{\partial t} \bigg|_y = \dot{\mathbf{v}}_y$$

the time derivative of $(\mathbf{v}_y \cdot \mathbf{e}_r)$ can be recast in the form

$$\frac{\partial (\mathbf{v}_y \cdot \mathbf{e}_r)}{\partial t} \bigg|_y = \dot{\mathbf{v}}_y \cdot \mathbf{e}_r - \mathbf{v}_y \cdot \left(\frac{\mathbf{v}_y}{r} - \frac{\mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r}{r} \right) \quad (\text{D.9})$$

Combining Eqs. (D.4), (D.5), (D.6), (D.7), (D.8) and (D.9) gives the final expression for the gradient of the Green's function expressed in MFR

$$\begin{aligned} \nabla \hat{G}_0 = & \left[\frac{-\mathbf{e}_r}{4\pi r^2(1 - M_r^y)} - \frac{r \mathbf{e}_r \times \boldsymbol{\omega}_y - \mathbf{v}_y + \mathbf{e}_r \mathbf{v}_y \cdot \hat{\mathbf{e}}_r}{4\pi c_0 r^2(1 - M_r^y)^2} \right. \\ & \left. + \left(\frac{\mathbf{v}_y \cdot \mathbf{e}_r}{4\pi r^2(1 - M_r^y)} + \frac{r \mathbf{e}_r \cdot \dot{\mathbf{v}}_y - v_y^2 + (\mathbf{v}_y \cdot \mathbf{e}_r)^2}{4\pi c_0 r^2(1 - M_r^y)^2} \right) \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right] \bigg|_{\theta_y} \end{aligned} \quad (\text{D.10})$$

To explicit the expressions of the time derivatives of Green's function and gradient of θ_y , some considerations must be made. Both of these functions are delayed by θ_y , so if the derivative is evaluated for a point fixed to the moving reference, both $\hat{G}_0$ and $\nabla \theta_y$ no longer depend on time and it is equal to zero. However, the time derivatives that appear in Eq. (3.23) must be calculated following a point of the boundary surface. In the presence of boundary deformations, these points move with respect to the MFR, and therefore the time derivatives of $\hat{G}_0$ and $\nabla \theta_y$ are not null and

are expressed as follows

$$\left. \frac{\partial \hat{G}_0}{\partial t} \right|_{\xi} = \mathbf{v}_y^m \cdot \nabla \hat{G}_0 \quad (\text{D.11})$$

$$\left. \frac{\partial \nabla_y \theta_y}{\partial t} \right|_{\xi} = \left. \frac{\partial \nabla_y \theta_y}{\partial t} \right|_y + (\mathbf{v}_y^m \cdot \nabla_y) \nabla_y \theta_y = (\mathbf{v}_y^m \cdot \nabla_y) \nabla_y \theta_y \quad (\text{D.12})$$

where ∇_y is the gradient operator written in the MFR and the time derivative of vectors is accomplished by observing from the MFR (thus the first term on the right-hand side of Eq. (D.12) is null).

Next, combining Eqs. (D.1) and (D.3) yields

$$\nabla_y (\nabla_y \theta_y) = \nabla_y \left(\left. \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right|_{\theta_y} \right) = \nabla_y \left(\left. \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right) \right|_{\theta_y} - \left. \frac{\partial}{\partial t} \right|_y \left(\left. \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right) \right|_{\theta_y} \otimes \nabla_y \theta_y \quad (\text{D.13})$$

where, for the differential operators applied to non-delayed vector $-\mathbf{e}_r/(c_0(1 - M_r^y))$ we have

$$\nabla_y \left(\left. \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right) \right) = \frac{-\nabla_y \mathbf{e}_r}{c_0(1 - M_r^y)} - \frac{\mathbf{e}_r \otimes \nabla_y (\mathbf{v}_y \cdot \mathbf{e}_r)}{c_0^2(1 - M_r^y)^2} \quad (\text{D.14})$$

and

$$\left. \frac{\partial}{\partial t} \right|_y \left(\left. \frac{-\mathbf{e}_r}{c_0(1 - M_r^y)} \right) \right) = \frac{-1}{c_0(1 - M_r^y)} \left. \frac{\partial \mathbf{e}_r}{\partial t} \right|_y - \frac{\mathbf{e}_r}{c_0^2(1 - M_r^y)^2} \left. \frac{\partial (\mathbf{v}_y \cdot \mathbf{e}_r)}{\partial t} \right|_y \quad (\text{D.15})$$

with the time derivative of $\mathbf{e}_r$ observed by the MFR given by

$$\left. \frac{\partial \mathbf{e}_r}{\partial t} \right|_y = \frac{-\boldsymbol{\omega}_y \times \mathbf{r} - \mathbf{v}_y}{r} + \frac{\mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r}{r}$$

The combination of Eqs. (D.13), (D.14) and (D.15) yields the final expression for the gradient of the gradient of the delay θ_y

$$\begin{aligned} \nabla_y (\nabla_y \theta_y) = & \left[\frac{\mathbf{I} - \mathbf{e}_r \otimes \mathbf{e}_r}{c_0 r (1 - M_r^y)} - \frac{\mathbf{e}_r \otimes (r \mathbf{e}_r \times \boldsymbol{\omega}_y - \mathbf{v}_y + \mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r)}{r c_0^2 (1 - M_r^y)^2} \right. \\ & \left. - \left(\frac{\boldsymbol{\omega}_y \times \mathbf{r} + \mathbf{v}_y - \mathbf{e}_r \mathbf{v}_y \cdot \mathbf{e}_r}{r c_0 (1 - M_r^y)} - \frac{\mathbf{e}_r (r \mathbf{e}_r \cdot \dot{\mathbf{v}}_y - v_y^2 + (\mathbf{v}_y \cdot \mathbf{e}_r)^2)}{r c_0^2 (1 - M_r^y)^2} \right) \otimes \frac{-\mathbf{e}_r}{c_0 (1 - M_r^y)} \right] \Big|_{\theta_y} \end{aligned} \quad (\text{D.16})$$

Finally, the time derivative of the Jacobian is considered. It is straightforward to obtain the following expression

$$\left. \frac{\partial}{\partial t} \right|_{\xi} \left(\left. \frac{J_s}{|1 + \mathbf{v}_y^m \cdot \nabla \theta_y|} \right) \right) = \frac{\dot{J}_s}{|1 + \mathbf{v}_y^m \cdot \nabla \theta_y|} - \frac{\text{sign}}{(1 + \mathbf{v}_y^m \cdot \nabla \theta_y)^2} \left(\dot{\mathbf{v}}_y^m \cdot \nabla \theta_y + \mathbf{v}_y^m \cdot \left. \frac{\partial \nabla \theta_y}{\partial t} \right|_{\xi} \right) \quad (\text{D.17})$$

where the symbol *sign* denotes the sign of the function $(1 - \mathbf{v}_y^m \cdot \nabla \theta_y)$.

Substituting the expressions derived in Eqs. (D.3), (D.10), (D.11), (D.12), (D.16), (D.17) into Eq. (3.23) the expanded form of the boundary integral formulation expressed in the MFR is obtained.

Remark

The integrand terms in Eq. (3.23) are all evaluated for $g = 0$, namely at the delayed time $t_* - \theta$. However, the contributions related to $\hat{G}_0$, $\nabla \hat{G}_0$ and $\nabla \theta_y$, are, by definition, evaluated at the delayed time $t_* - \theta_y$ (see Sec. 3.1 and Sec. 3.2).

Thus, the application of the boundary integral formulation in Eq. (3.23) requires, for any boundary surface point:

- evaluation of the delay θ related to the combination of dynamic deformation and rigid-body motion of the boundary surface;
- evaluation of the delay θ_y related to the rigid-body motion of the MFR, at the position where it passes at the time $t_* - \theta$;
- evaluation of the integrands terms that depend on time, such as, for instance, the normal or velocity $\mathbf{v}_y$, at the time $t_* - \theta$, and evaluation of the quantities related to Green's function and $\nabla \theta_y$ at the time $t_* - \theta_y$.

Appendix E

Analytical solution of sound radiated by a pulsating sphere

In this appendix, an original and general analytical solution of the sound radiated by a sphere undergoing subsonic harmonic pulsations is derived.

It is introduced to provide a validation test for the novel deformable-boundary integral formulation, with particular attention on the contributions from deforming boundary surfaces (for this reason, the fluid nonlinearities giving raise to volume contributions are neglected).

The demonstration of the analytical expression of the noise radiated by a pulsating sphere is derived in term of the velocity potential, and provides a non-closed, Fourier-series expansion solution of the homogeneous form of Eq. (2.8).

This solution can be considered as the extension to arbitrary values of pulsation amplitude and frequency compatible with subsonic surface motion, of the analytical, closed-form solutions already available in the literature that are valid for small-amplitude or finite-amplitude and small-frequency pulsations of the sphere (see, for instance, [69, 70]).

E.1 Formulation of the problem

Let us consider a uniformly pulsating sphere immersed in an inviscid perfect gas, initially at rest, and assume that the instantaneous radius of the sphere, r_s , is given by

$$r_s(t_*) = r_0 + a \sin \omega t_* \tag{E.1}$$

The corresponding flowfield is governed by the wave equation in Eq. (2.8) and is necessarily spherically symmetrical, with the velocity of the particles having only the radial component, u , depending on time and distance from the centre of the sphere, *i.e.*, $\mathbf{u} = u \mathbf{i}_r$, with $\mathbf{i}_r$ representing the unit vector along the radial direction.

As mentioned above, the objective of the following developments is the determination of the analytical description of the velocity potential radiated by the pulsating sphere, under the assumption of neglected nonlinear propagation effects. Specifically, we derive the analytical solution, $\varphi(\mathbf{x}_*, t_*)$, of the homogeneous form of Eq. (2.8), with associated impermeability boundary conditions on the moving surface of the pulsating sphere.

To this purpose, let us observe that the expression of the free-space Green function of the wave equation yields the following expression for the velocity potential field linearly radiated by a source of time-varying amplitude, $A(t)$ [30, 58]

$$\varphi(\varrho, t_*) = \frac{-1}{4\pi\varrho} A(t_* - \varrho/c_0) \quad (\text{E.2})$$

where ϱ represents the distance from the source. At each time it generates a spherically symmetrical flowfield, $u = \partial\varphi/\partial\varrho$.

Therefore, the solution of the homogeneous form of Eq. (2.8) describing the flow field due to a pulsating sphere can be derived as that corresponding to the velocity potential field radiated by the source located at the centre of the sphere, whose time-varying amplitude is such that it satisfies the boundary conditions of impermeability

$$\left. \frac{\partial\varphi}{\partial\varrho} \right|_{\varrho=r_s(t_*)} = v_s(t_*) = \omega a \cos \omega t_* \quad (\text{E.3})$$

where v_s denotes the velocity of the sphere surface.

Observing that the radial derivative of the velocity potential given in Eq. (E.2) is expressed as

$$\left. \frac{\partial\varphi}{\partial\varrho} \right|_{t_*} = \frac{1}{4\pi\varrho^2} A(t_* - \varrho/c_0) + \frac{1}{4\pi\varrho c_0} \dot{A}(t_* - \varrho/c_0)$$

where $\dot{A}$ represents the derivative of the function A with respect to its argument, the combination with Eq. (E.3) yields the following periodic-coefficient, ordinary differential equation

$$\frac{r_s(t_*)}{c_0} \dot{A}[t_* - r_s(t)/c_0] + A[t_* - r_s(t)/c_0] = 4\pi r_s^2(t_*) \omega a \cos \omega t \quad (\text{E.4})$$

whose solution provides the source-amplitude function which corresponds to the condition of impermeability of the surface of the moving sphere (null relative motion between fluid and sphere surface).

Thus, once replaced in Eq. (E.2), this solution gives the velocity potential field radiated by the pulsating sphere which satisfies the homogeneous form of Eq. (2.8). It is a centrally symmetric field depending only on the radial position and time.

Through introduction of the source time variable

$$t = t_* - r_s(t_*)/c_0 = t_* - r_0/c_0 - (a/c_0) \sin \omega t_* \quad (\text{E.5})$$

Eq. (E.4) can be re-written in a more compact form in the time domain, t , as

$$\alpha(t) \dot{A}(t) + A(t) = \gamma(t) \quad (\text{E.6})$$

where

$$\alpha(t) = \frac{r_s[t_*(t)]}{c_0}, \quad \gamma(t) = 4\pi r_s^2[t_*(t)] \omega a \cos[\omega t_*(t)] \quad (\text{E.7})$$

with $t_*(t)$ denoting the inverse of the time variable transformation in Eq. (E.5). This inverse function plays a crucial role in the definition of the differential problem for the unknown A function, and is

discussed in the next section.

E.2 Time variable transformation

The complexity of the time variable transformation in Eq. (E.5) depends on the frequency and amplitude of the sphere pulsations. Indeed, taking the differential of both sides of Eq. (E.5) yields

$$dt_* = \frac{dt}{(1 - ka \cos \omega t_*)}$$

where $k = \omega/c_0$ denotes the wave number and ka is the acoustic Mach number. This expression implies that the inverse of the function in Eq. (E.5) exists locally when $ka < 1$ or, in other words, when the pulsation speed is subsonic and the causality condition is satisfied (note that the geometric constraint $a/r_0 < 1$ must be implicitly considered).

In particular, if $ka \ll 1$ we have $dt_* \approx dt$ and therefore the time variable transformation reduces to the simple delayed-time form

$$t = t_* - r_0/c_0 \quad (\text{E.8})$$

Since $ka = (kr_0)(a/r_0)$, this condition occurs for very low frequency pulsations for which $kr_0 \ll 1$, but also for pulsations of not very low frequency when the amplitude of the pulsation is suitably small, $a/r_0 \ll 1$.

If the pulsation frequency and the pulsation amplitude are such that the condition $ka \ll 1$ is not satisfied, Eq. (E.8) is not valid and the relation between time t_* and time t can be determined through the following discrete sequence of linear approximations, for $n = 0, 1, 2, \dots$

$$t_{*n+1} = t_{*n} + \frac{\Delta t}{(1 - ka \cos \omega t_{*n})}, \quad t_{n+1} = t_0 + (n+1) \Delta t \quad (\text{E.9})$$

where Δt denotes a suitably small increment of time t . Equation (E.9) corresponds to the application of the local linear Taylor series approximation and can be applied starting from an arbitrary initial value t_{*0} and the corresponding $t_0 = t_{*0} - r_0/c_0 - (a/c_0) \sin \omega t_{*0}$.

E.3 Acoustic radiation from small-amplitude pulsations

Let us consider sphere pulsation amplitudes and frequencies such that $a/r_0 \ll 1$ and $ka \ll 1$, for which Eq. (E.8) holds and $r_s \approx r_0$. This means that Eq. (E.6) can be recast in the following form

$$\begin{aligned} \dot{A}(t) + \frac{c_0}{r_0} A(t) &= 4\pi c_0 r_0 \omega a \cos \omega(t + r_0/c_0) \\ &= 4\pi (b_c \cos \omega t + b_s \sin \omega t) \end{aligned} \quad (\text{E.10})$$

where $b_c = c_0 r_0 \omega a \cos kr_0$ and $b_s = -c_0 r_0 \omega a \sin kr_0$.

It can be readily observed that the asymptotic solution of Eq. (E.10) (with the vanishing transient solution) provides the following expression for the velocity potential field generated by the pulsating sphere, see Eq. (E.2)

$$\varphi(\varrho, t_*) = \frac{-1}{\varrho} [a_c \cos \omega(t_* - \varrho/c_0) + a_s \sin \omega(t_* - \varrho/c_0)]$$

where

$$a_c = \left(\frac{b_c}{\omega} + \frac{r_0 b_s}{c_0} \right) / \left(\frac{1}{kr_0} + kr_0 \right) \quad (\text{E.11})$$

$$a_s = \frac{b_c}{\omega} - \left(\frac{b_c}{\omega} - \frac{r_0 b_s}{c_0} \right) / [1 + (kr_0)^2] \quad (\text{E.12})$$

Including the assumption of small-frequency pulsations, $kr_0 \ll 1$, the combination of the expressions of b_c and b_s with the above equations yields $a_c \approx \omega r_0^2 a$ and $a_s \approx 0$. Thus, the corresponding radiated velocity potential reads

$$\varphi(\varrho, t_*) = -\frac{r_0^2}{\varrho} \omega a \cos \omega(t_* - \varrho/c_0) = -\frac{r_0^2}{\varrho} v_s \Big|_{t_* - \varrho/c_0} \quad (\text{E.13})$$

with the associated perturbation pressure field given by (solution *A*)

$$p'(\varrho, t_*) = -\rho_0 \frac{r_0^2}{\varrho} \omega^2 a \sin \omega(t_* - \varrho/c_0) = \rho_0 \frac{r_0^2}{\varrho} \dot{v}_s \Big|_{t_* - \varrho/c_0} \quad (\text{E.14})$$

with $\dot{v}_s$ denoting the acceleration of the surface points, while $(\dots)|_{t_* - \varrho/c_0}$ indicates evaluation at the retarded time, $t_* - \varrho/c_0$. The acoustic field expressed in Eq. (E.14) coincides with the classical solution presented in the literature for the same conditions of small pulsation frequency and amplitude (see, for instance, Refs. [70, 79]).

Instead, when the assumption $kr_0 \ll 1$ is removed the combination of the expressions of b_c and b_s with Eqs. (E.11) and (E.12) followed by appropriate mathematical manipulations yield the velocity potential field radiated by the non-low frequency pulsating sphere expressed by

$$\begin{aligned} \varphi(\varrho, t_*) &= -\frac{r_0^2}{\varrho} \omega a [1 + (kr_0)^2]^{-1/2} \cos \omega(t_* - \bar{r}/c_0) \\ &= -\frac{r_0^2}{\varrho} [1 + (kr_0)^2]^{-1/2} v_s \Big|_{t_* - \bar{r}/c_0} \end{aligned} \quad (\text{E.15})$$

where $\bar{r} = \varrho - r_0 + (1/k) \tan^{-1} kr_0$, which corresponds to the following description of the perturbation pressure field (solution *B*)

$$\begin{aligned} p'(\varrho, t_*) &= -\rho_0 \frac{r_0^2}{\varrho} \omega^2 a [1 + (kr_0)^2]^{-1/2} \sin \omega(t_* - \bar{r}/c_0) \\ &= \rho_0 \frac{r_0^2}{\varrho} [1 + (kr_0)^2]^{-1/2} \dot{v}_s \Big|_{t_* - \bar{r}/c_0} \end{aligned} \quad (\text{E.16})$$

The acoustic field expressed in Eq. (E.16) coincides with the classical solution presented in the literature for the same only condition of small pulsation amplitude (see, for instance, Ref. [70]).

Note that in the above equations t_* denotes the time variable through which the evolution of the radiated acoustic field is described. This evolution corresponds to the sphere pulsation described through time t which, due to the imposition of the impermeability boundary conditions, is shifted and distorted with respect to t_* (see Eqs. (E.4) and (E.5)).

Finally, it is worth noting that the solution in Eq. (E.13) can be straightforwardly derived from

Eq. (E.10) through algebraic manipulations. Indeed, for low-frequency pulsations of the sphere ($kr_0 \ll 1$) the contribution of the term $\dot{A}$ which is proportional to kr_0 is negligible and, at the same time, $b_c \approx c_0 r_0 \omega a$ and $b_s \approx 0$. Thus, deriving the function $A(t)$ from the corresponding approximated expression of Eq. (E.10) the solution in Eq. (E.13) is directly obtained by combining it with Eq. (E.2). This solution procedure is applied in the following section to determine the acoustic field for small- frequency, large-amplitude pulsations of the sphere.

E.4 Acoustic radiation from non-small pulsation amplitudes

Let us consider arbitrary sphere pulsation amplitudes which satisfy the condition $a/r_0 < 1$. Since the acoustic Mach number, ka , is assumed to be subsonic, $ka < 1$, the upper limit of the admissible pulsation frequency parameter, kr_0 , is correspondingly readily defined. Three different classes of solutions are distinguished: those valid for pulsations with low frequency and low acoustic Mach number (both conditions $kr_0 \ll 1$ and $ka \ll 1$ are satisfied), those valid for low acoustic Mach number (only the condition $ka \ll 1$ is satisfied), and those valid for large-amplitude, high-frequency pulsation cases (only the condition $ka < 1$ is satisfied).

E.4.1 Solution for low frequency and low acoustic Mach number

As mentioned at the end of appendix E.3, this assumption implies that the contribution of the term $\dot{A}$ in Eq. (E.4) is negligible and that $ka \ll 1$, thus yielding $t = t_* - r_0/c_0$ (see Eq. (E.8)). However, since the pulsation amplitude can be large and hence $r_s \not\approx r_0$, the corresponding approximated form of Eq. (E.4) yields

$$A(t) = 4\pi [r_0 + a \sin(\omega t + kr_0)]^2 \omega a \cos(\omega t + kr_0)$$

Then, developing the trigonometric terms at the right-hand-side of the above equation, observing that for $kr_0 \ll 1$ we may assume $\cos mkr_0 \approx 1$ and $\sin mkr_0 \approx mkr_0$ for $m = 1, 2, 3$, and neglecting the terms proportional to kr_0 provide the following solution for the radiated velocity potential field

$$\begin{aligned} \varphi(\varrho, t_*) = & -\omega a \frac{r_0^2}{\varrho} \left[\left(1 + \frac{1}{4} \frac{a^2}{r_0^2} \right) \cos \omega(t_* - \varrho/c_0) + \frac{a}{r_0} \sin 2\omega(t_* - \varrho/c_0) \right. \\ & \left. - \frac{1}{4} \frac{a^2}{r_0^2} \cos 3\omega(t_* - \varrho/c_0) \right] \end{aligned} \quad (\text{E.17})$$

with the acoustic pressure given by (solution C)

$$\begin{aligned} p'(\varrho, t_*) = & -\rho_0 \omega^2 a \frac{r_0^2}{\varrho} \left[\left(1 + \frac{1}{4} \frac{a^2}{r_0^2} \right) \sin \omega(t_* - \varrho/c_0) - 2 \frac{a}{r_0} \cos 2\omega(t_* - \varrho/c_0) \right. \\ & \left. - \frac{3}{4} \frac{a^2}{r_0^2} \cos 3\omega(t_* - \varrho/c_0) \right] \end{aligned} \quad (\text{E.18})$$

As expected, for $a/r_0 \rightarrow 0$ (small-amplitude pulsations) Eqs. (E.17) and (E.18) tend to coincide with Eqs. (E.13) and (E.14). Instead, in the case of large amplitude pulsations they show that the spectrum of the radiated perturbation wave contains contributions from harmonics that are multiples of that of the surface motion (thus confirming the outcomes of Ref. [69]). This multi-

harmonic structure of the radiated acoustic signal is a characteristics of all large-amplitude sphere pulsations cases, as it will be demonstrated in the following sections.

E.4.2 Solution for low acoustic Mach number

Assuming that $ka \ll 1$, this case considers sphere pulsations such that $t = t_* - r_0/c_0$, without further specific limitations on their amplitude and frequency (it includes, for instance, pulsations that at the same time have non-small amplitude and non-low frequency). Thus, Eq. (E.4) can be rewritten as, for $\eta(t) = [r_0 + a \sin(\omega t + kr_0)]/c_0$,

$$\eta(t)\dot{A}(t) + A(t) = 4\pi c_0^2 \eta(t)^2 \omega a \cos(\omega t + kr_0) \quad (\text{E.19})$$

which is a differential equation with periodic coefficients, subject to a periodic forcing term.

The solution of Eq. (E.19) is not available in closed form. It is conveniently obtained by expanding the unknown function in Fourier series and determining the corresponding coefficients through a harmonic balance between the left-hand-side and the right-hand-side terms.

To this purpose, let us represent the function $A(t)$ as

$$A(t) = A_0 + \sum_{n=1}^{N_a} A_n^c \cos n\omega t + A_n^s \sin n\omega t \quad (\text{E.20})$$

with N_a denoting the order of the highest harmonic included in the analysis, and replace this expression in Eq. (E.19). Combining the harmonics deriving from the different contributions, and then equating the coefficients appearing at the left- and right-hand sides for each pair of homologous harmonic terms (harmonic balance) the following algebraic system is obtained

$$\mathbf{P} \mathbf{z} = \mathbf{f} \quad (\text{E.21})$$

where $\mathbf{z}$ collects the coefficients in Eq. (E.20), $\mathbf{z}^T = \{A_0 \ A_1^c \ A_1^s \ A_2^c \ A_2^s \dots A_{N_a}^c \ A_{N_a}^s\}$, the vector $\mathbf{f}$ collects the Fourier coefficients of the right-hand side in Eq. (E.19)

$$\mathbf{f} = 4\pi\omega a \left\{ \begin{array}{c} 0 \\ r_0^2 \cos kr_0 + \frac{1}{4}a^2 (2 \cos kr_0 - \cos 2kr_0 \cos kr_0 - \sin 2kr_0 \sin kr_0) \\ -r_0^2 \sin kr_0 - \frac{1}{4}a^2 (2 \sin kr_0 + \cos 2kr_0 \sin kr_0 - \sin 2kr_0 \cos kr_0) \\ a r_0 \sin 2kr_0 \\ a r_0 \cos 2kr_0 \\ \frac{1}{4}a^2 (\sin 2kr_0 \sin kr_0 - \cos 2kr_0 \cos kr_0) \\ \frac{1}{4}a^2 (\cos 2kr_0 \sin kr_0 + \sin 2kr_0 \cos kr_0) \\ 0 \\ \vdots \\ 0 \end{array} \right\}$$

while the matrix $\mathbf{P}$ of dimensions $(1 + 2N_a) \times (1 + 2N_a)$ is expressed as

$$\mathbf{P} = \begin{bmatrix} \mathbf{C}_{011} & \mathbf{R}_{011}\mathbf{R}_{012} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}_{012} & \mathbf{C}_1 & \mathbf{R}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}_{021} & & & & & & & \\ \mathbf{0} & \mathbf{L}_2 & \mathbf{C}_2 & \mathbf{R}_2 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{L}_{N-1} & \mathbf{C}_{N-1} & \mathbf{R}_{N-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{L}_N & \mathbf{C}_N \end{bmatrix} \quad (\text{E.22})$$

where

$$\mathbf{C}_n = \begin{bmatrix} 1 & nkr_0 \\ -nkr_0 & 1 \end{bmatrix}; \quad \mathbf{L}_n = \frac{1}{2}(n+1)ka \begin{bmatrix} \cos kr_0 & \sin kr_0 \\ -\sin kr_0 & \cos kr_0 \end{bmatrix}$$

and

$$\mathbf{R}_n = -\frac{1}{2}(n+1)ka \begin{bmatrix} \cos kr_0 & -\sin kr_0 \\ \sin kr_0 & \cos kr_0 \end{bmatrix}$$

Solving the algebraic problem in Eq. (E.21) yields the Fourier coefficients of Eq. (E.20), from which the radiated velocity potential field is obtained as

$$\varphi(\varrho, t_*) = -\frac{1}{4\pi\varrho} \left[A_0 + \sum_{n=1}^{N_a} A_n^c \cos n\omega(t_* - \varrho/c_0) + A_n^s \sin n\omega(t_* - \varrho/c_0) \right] \quad (\text{E.23})$$

with the acoustic pressure given by (solution D)

$$p'(\varrho, t_*) = -\frac{\rho_0\omega}{4\pi\varrho} \left[\sum_{n=1}^{N_a} n (A_n^c \sin n\omega(t_* - \varrho/c_0) - A_n^s \cos n\omega(t_* - \varrho/c_0)) \right] \quad (\text{E.24})$$

It is worth noting that, as the pulsation amplitudes decrease, the off-diagonal $\mathbf{L}_n$ and $\mathbf{R}_n$ matrices in Eq. (E.22) tend to become negligible with respect to the diagonal block matrices $\mathbf{C}_n$, and the higher-harmonic contributions in the forcing vector, $\mathbf{f}$, tend to become negligible with respect to the fundamental one (related to the sphere pulsation frequency). This clearly highlights the role of the amplitude of a single-harmonic pulsation in generating multi-harmonic wave propagation.

E.4.3 Solution for large-amplitude, high-frequency pulsations

This is the most general solution case for which arbitrary pulsation amplitudes such that $a/r_0 < 1$, and pulsation frequencies such that $ka < 1$ can be considered. Therefore, the unknown function $A(t)$ is governed by the complete version of the differential equation in Eq. (E.6), with the coefficient $\alpha(t)$ and the forcing term $\gamma(t)$ being multi-harmonic periodic functions depending on the pulsating motion of the sphere given in Eq. (E.1).

Also in this case, the solution is not available in closed form and can be conveniently determined by the Fourier series expansion of the unknown function, followed by the application of the harmonic-balance solution method already discussed in appendix E.4.2 to evaluate the corresponding coefficients. However in this case the solution process is more complicated since time t_* to be used in Eq. (E.1) to determine the sphere motion and hence the time histories of the functions α

and γ in the domain t has to be evaluated from Eq. (E.5). This is accomplished by application of the discrete scheme of identification of the inverse function $t_* = t_*(t)$ given in Eq. (E.9).

The solution procedure begins with the introduction of the Fourier series expansion of $A(t)$ expressed in Eq. (E.20), which is then replaced in Eq. (E.6) to yield the following equation

$$A_0 + \sum_{n=1}^{N_a} A_n^c b_n^c(t) + A_n^s b_n^s(t) = \gamma(t) \quad (\text{E.25})$$

where

$$b_n^c(t) = \cos n\omega t - n\omega \alpha(t) \sin n\omega t; \quad b_n^s(t) = \sin n\omega t + n\omega \alpha(t) \cos n\omega t$$

with $\alpha(t)$ given in Eq. (E.7). Then, performing also the Fourier series expansions of $b_n^c(t)$, $b_n^s(t)$, and the forcing term $\gamma(t)$

$$\begin{aligned} b_n^c(t) &= b_{n0}^c + \sum_{m=1}^{N_a} b_{nm}^{cc} \cos m\omega t + b_{nm}^{cs} \sin m\omega t \\ b_n^s(t) &= b_{n0}^s + \sum_{m=1}^{N_a} b_{nm}^{sc} \cos m\omega t + b_{nm}^{ss} \sin m\omega t \\ \gamma(t) &= \gamma_0 + \sum_{m=1}^{N_a} \gamma_m^c \cos m\omega t + \gamma_m^s \sin m\omega t \end{aligned}$$

Eq. (E.25) is recast in the following form

$$\begin{aligned} A_0 &+ \sum_{n=1}^{N_a} A_n^c b_{n0}^c + A_n^s b_{n0}^s \\ &+ \sum_{m=1}^{N_a} \left(\sum_{n=1}^{N_a} A_n^c b_{nm}^{cc} + A_n^s b_{nm}^{sc} \right) \cos m\omega t + \sum_{m=1}^{N_a} \left(\sum_{n=1}^{N_a} A_n^c b_{nm}^{cs} + A_n^s b_{nm}^{ss} \right) \sin m\omega t \\ &= \gamma_0 + \sum_{m=1}^{N_a} \gamma_m^c \cos m\omega t + \gamma_m^s \sin m\omega t \end{aligned}$$

Balancing the harmonics at the right- and left-hand sides provides an algebraic system of the type of that in Eq. (E.21) with the vector $\mathbf{z}$ coinciding with that given in Eq. (E.21), the forcing vector, $\mathbf{f}$, such that $\mathbf{f}^T = \{\gamma_0 \ \gamma_1^c \ \gamma_1^s \ \gamma_2^c \ \gamma_2^s \ \dots \ \gamma_{N_a}^c \ \gamma_{N_a}^s\}$, and the matrix $\mathbf{P}$ of dimensions $(1 + 2N_a) \times (1 + 2N_a)$ expressed as

$$\mathbf{P} = \begin{bmatrix} 1 & b_{10}^c & b_{10}^s & b_{20}^c & b_{20}^s & \dots & b_{N0}^c & b_{N0}^s \\ 0 & b_{11}^{cc} & b_{11}^{sc} & b_{21}^{cc} & b_{21}^{sc} & \dots & b_{N1}^{cc} & b_{N1}^{sc} \\ 0 & b_{11}^{cs} & b_{11}^{ss} & b_{21}^{cs} & b_{21}^{ss} & \dots & b_{N1}^{cs} & b_{N1}^{ss} \\ 0 & b_{12}^{cc} & b_{12}^{sc} & b_{22}^{cc} & b_{22}^{sc} & \dots & b_{N2}^{cc} & b_{N2}^{sc} \\ 0 & b_{12}^{cs} & b_{12}^{ss} & b_{22}^{cs} & b_{22}^{ss} & \dots & b_{N2}^{cs} & b_{N2}^{ss} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & b_{1N}^{cc} & b_{1N}^{sc} & b_{2N}^{cc} & b_{2N}^{sc} & \dots & b_{NN}^{cc} & b_{NN}^{sc} \\ 0 & b_{1N}^{cs} & b_{1N}^{ss} & b_{2N}^{cs} & b_{2N}^{ss} & \dots & b_{NN}^{cs} & b_{NN}^{ss} \end{bmatrix} \quad (\text{E.26})$$

Once vector $\mathbf{z}$ is evaluated from the solution of the algebraic problem in Eq. (E.21) with matrix

$\mathbf{P}$ and forcing vector $\mathbf{f}$ given above, the radiated velocity potential and acoustic pressure can be determined through the expressions given in Eqs. (E.23) and (E.24) (solution E).

E.5 Assessment of the validity limit of the approximate analytical solutions

Next, for a clear definition of the validity limits of the different approximate analytical solutions of the homogeneous wave equation of the velocity potential proposed in this appendix, the results of a numerical investigation addressed to the evaluation of the error as a function of the frequency parameter, kr_0 , and nondimensional amplitude, a/r_0 , of the pulsations of the sphere are presented.

Specifically, for the observer located at the distance $d/r_0 = 10$ from the centre of the sphere, the relative L2-errors, ϵ_r^2 , between the predictions given by the BEM solver, p'^{BEM} , and the acoustic pressures, p'^{AN} , obtained by solutions A , B , C and D are evaluated for $0 \leq kr_0 \leq 1.5$ and $0 \leq a/r_0 \leq 0.5$ (the error of solution E is not considered since this solution always coincides with the BEM prediction, see Sec. 4.5.3). The relative L2-error is defined as

$$\epsilon_r^2 = \frac{\sum_{i=1}^{N_T} (p_i'^{\text{AN}} - p_i'^{\text{BEM}})^2}{\sum_{i=1}^{N_T} (p_i'^{\text{BEM}})^2}$$

where N_T denotes the number of samples within a period.

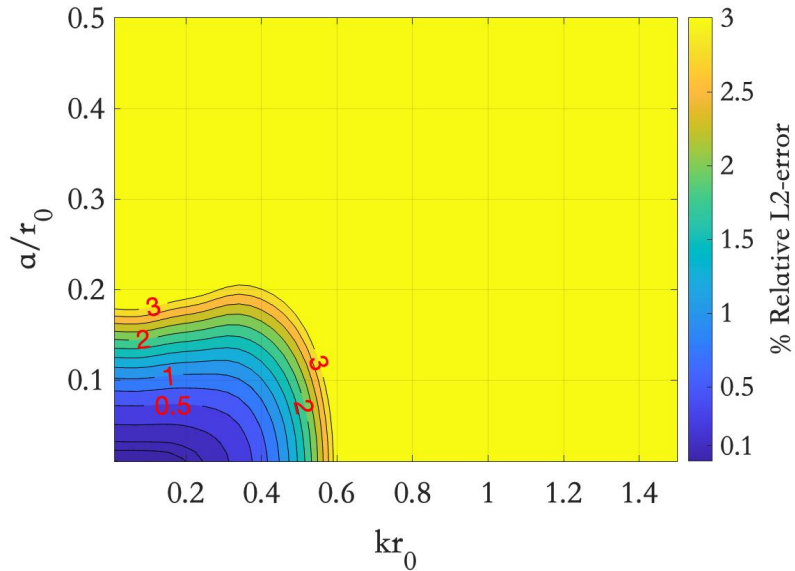


Figure E.1: Relative L2-error between solution A and BEM predictions, $d/r_0 = 10$.

The results of this investigation are presented in Figs. E.1-E.4. These figures present regions of good accuracy of the various solutions, which are compatible with the assumptions their derivations are based on. They provide a clear numerical view of the validity limits of these solutions. By assuming the maximum acceptable relative L2-error equal to 3%, the boundaries of the yellow regions of Figs. E.1-E.4 represent the limit of validity of the examined solutions (note that a

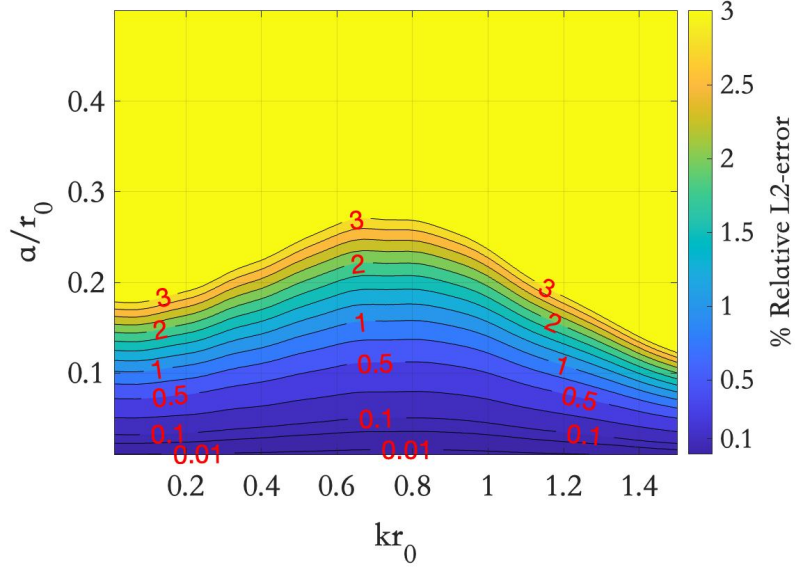


Figure E.2: Relative L2-error between solution B and BEM predictions, $d/r_0 = 10$.

relative L2-error equal to 3% is like that of solutions A and B in Fig. 4.10).

In particular, in Fig. E.1 the prediction accuracy grows as both pulsation frequency and amplitude tends to be small (see the blue region), as expected from the assumptions applied to determine solution A . Similarly, again based on the assumptions considered to define solutions B and C , in Figs. E.2 and E.3 the solution accuracy grows, respectively, as the pulsation amplitude decreases (with no limitation on the pulsation frequency), and as the pulsation frequency decreases (with no limitation on the pulsation amplitude except the geometric one, $a/r_0 < 1$). On the other hand, in Fig. E.4, which is related to the solution D , the iso-accuracy lines are hyperbolic since this solution was determined under the $ka \ll 1$ assumption, with no restrictions on the individual characteristic parameters kr_0 and a/r_0 . Therefore, the blue region of highly accurate predictions includes the cases of small amplitude/low frequency pulsations, small amplitude/high frequency pulsations, and large amplitude/low frequency pulsations.

It is important to highlight that these results are obtained for $d/r_0 = 10$, but it has been proven that they are exactly of the same quality for any distance between observer and centre of the sphere.

E.6 Assessment of the validity limit of the exact analytical linear solution

Finally, a numerical investigation is performed to estimate the validity limits of the exact analytical solution of the homogeneous wave equation for the velocity potential. To this purpose, as the amplitude and frequency of the pulsations vary, the magnitude of the corresponding neglected nonlinear forcing term of the wave equation for the velocity potential, σ , is evaluated (see Eq. (2.8)). Indeed, this value measures the importance of the neglected contributions and thus the suitability of the applied linear propagation model to provide acceptable solutions from the point of view of the physics of the problem.

First of all, we note that for any given pair of pulsation frequency parameter, kr_0 , and nondi-

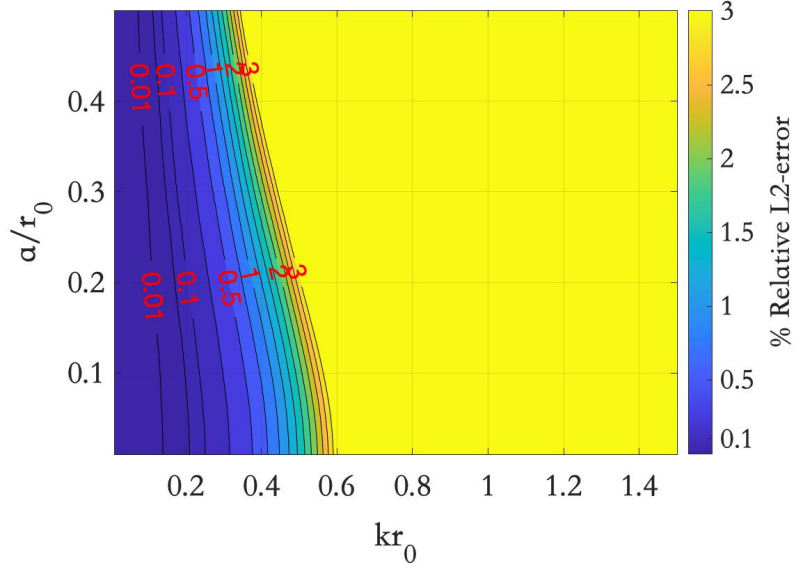


Figure E.3: Relative L2-error between solution C and BEM predictions, $d/r_0 = 10$.

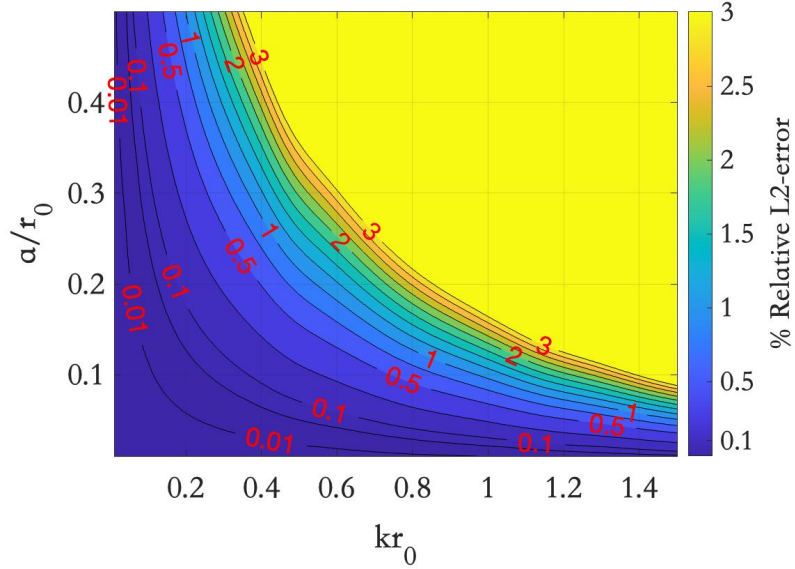


Figure E.4: Relative L2-error between solution D and BEM predictions, $d/r_0 = 10$.

mensional amplitude, a/r_0 , the exact linear solution yields a multi-harmonic σ , with the second harmonic being the main one. As an example, Fig. E.5 shows the magnitude of the several harmonics contributing to nondimensional $\hat{\sigma} = \sigma/(c_0/r_0)$ evaluated on the sphere surface, for $kr_0 = 0.8$ and $a/r_0 = 0.2$.

Then, Fig. E.6 illustrates the distribution of the second harmonic of the nondimensional non-linear term, $\hat{\sigma}_2$, evaluated on the sphere surface as a function of kr_0 and a/r_0 . From this figure it is possible to have an idea of the sphere pulsation conditions for which the proposed analytical solution provides a reliable simulation of the physical sound radiation phenomenon.

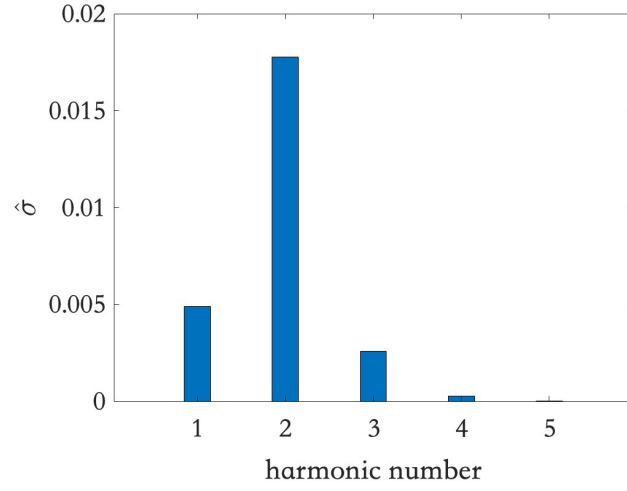


Figure E.5: Harmonic content of $\hat{\sigma}$ at $r = r_s$, for $kr_0 = 0.8$ and $a/r_0 = 0.2$.

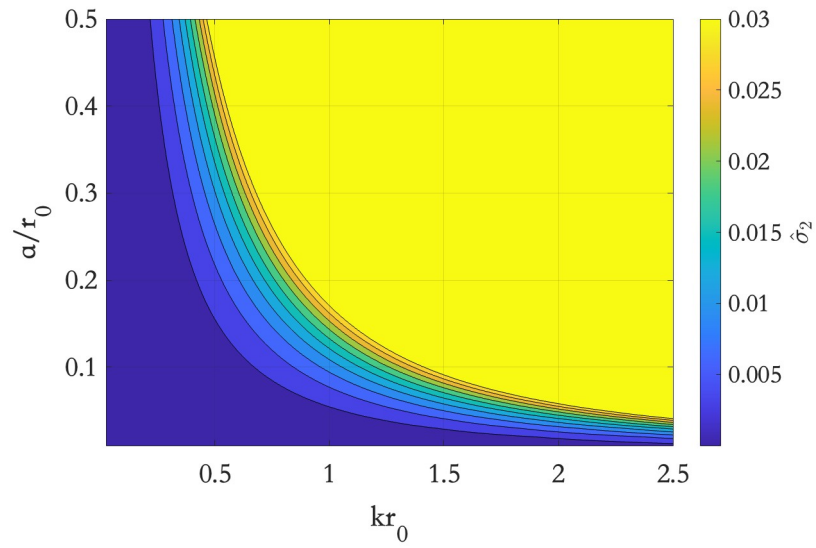


Figure E.6: Second harmonic of $\hat{\sigma}$ at $r = r_s$.

Appendix F

Velocity potential radiated by a pulsating sphere: analytical closed-form solution from a boundary-integral formulation

Considering the pulsating sphere problem discussed in appendix E, let us assume that the sphere pulsation is such that $ka \ll 1$, namely the pulsation frequency is suitably low and/or the pulsation amplitude is suitably small.

In this case, since $\mathbf{M} = ka \cos \omega t \mathbf{i}_r$, all terms in Eq. (4.6) that are proportional to the Mach number can be neglected. Thus, observing that the above assumption implies also that the time derivative of the Jacobian, J , is negligible (note that $\dot{J}/c_0$ is proportional to ka), Eq. (4.6) can be reduced to the following simpler form

$$\varphi(\varrho, t_*) = \iint_{\mathcal{S}_{r/c_0}} \left(v_s G_0 - \varphi \frac{\partial G_0}{\partial n} + \frac{1}{c_0} \frac{\partial \varphi}{\partial t} \Big|_{\xi} \frac{\partial r}{\partial n} G_0 \right) \Big|_{t_* - r/c_0} d\mathcal{S}(\mathbf{x}) \quad (\text{F.1})$$

where $\mathcal{S}_{r/c_0}$ denotes the sphere surface at the time $t_* - r/c_0$, whose radius is equal to $r_s|_{t_* - r/c_0} = r_0 + a \sin \omega(t_* - r/c_0)$. In addition, in Eq. (F.1) the impermeability boundary condition $\partial \varphi / \partial n = v_s$ is imposed and the equivalence $-\mathbf{e}_r \cdot \mathbf{n} = \partial r / \partial n$ is applied.

Since the velocity of the sphere is uniformly distributed, the velocity potential is necessarily spherically symmetrical and, therefore, the velocity potential field is a function only of the distance of the observer from the centre of the sphere, as expressed in Eq. (F.1).

Note that if the additional condition of small amplitude of pulsations is introduced ($a/r_0 \ll 1$), the distance r can be assumed to be time-invariant for any given pair of observer point and source point over the average surface of the sphere. This implies that the Fourier transform applied to Eq.

(F.1) yields

$$\begin{aligned}
 \tilde{\varphi}(\varrho) &= \tilde{v}_s \iint_{\mathcal{S}} G_0 e^{-ikr} d\mathcal{S}(\mathbf{x}) - \tilde{\varphi}_s \iint_{\mathcal{S}} \frac{\partial G_0}{\partial \hat{\varrho}} e^{-ikr} d\mathcal{S}(\mathbf{x}) \\
 &+ ik \tilde{\varphi}_s \iint_{\mathcal{S}} \frac{\partial r}{\partial \hat{\varrho}} G_0 e^{-ikr} d\mathcal{S}(\mathbf{x}) \\
 &= \tilde{v}_s \iint_{\mathcal{S}} G_0 e^{-ikr} d\mathcal{S}(\mathbf{x}) - \tilde{\varphi}_s \iint_{\mathcal{S}} \frac{\partial(G_0 e^{-ikr})}{\partial \hat{\varrho}} d\mathcal{S}(\mathbf{x})
 \end{aligned} \tag{F.2}$$

where $\mathcal{S}$ denotes the average surface of the sphere having radius r_0 , φ_s is the uniform velocity potential distribution over the sphere surface, and, for $(\hat{\varrho}, \beta, \vartheta)$ representing a spherical coordinate system with the origin at the centre of the sphere, the equivalence $\partial(\dots)/\partial n = \partial(\dots)/\partial \hat{\varrho}$ is used.

F.1 Low-frequency, small-amplitude, far-field solution

Considering the spherical coordinate system introduced above, for arbitrary source and observer positions we have (see Fig. F.1)

$$\begin{aligned}
 r(\varrho, \hat{\varrho}, \beta, \vartheta) &= (\varrho^2 + \hat{\varrho}^2 - 2\varrho\hat{\varrho}\sin\beta)^{1/2} \\
 &= \varrho [1 - 2(\hat{\varrho}/\varrho)\sin\beta + (\hat{\varrho}/\varrho)^2]^{1/2}
 \end{aligned} \tag{F.3}$$

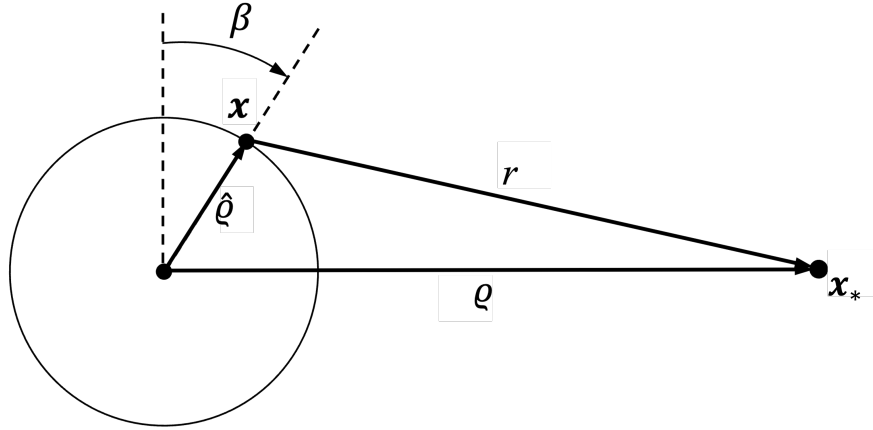


Figure F.1: Sketch of sphere-observer geometry on a ϑ -constant, meridian plane.

Thus, for the source points located over the sphere, under the assumption of small-amplitude pulsations ($a/r_0 \ll 1$ which implies $r_s \approx r_0$) and far-field observers ($r_0/\varrho \ll 1$), it is possible to consider the following linear approximation obtained by truncation of the radial Maclaurin series of distance r (see Eq. (F.3))

$$r(\varrho, r_0, \beta, \vartheta) \approx \varrho \left[1 - \left(\frac{r_0}{\varrho} \right) \sin\beta \right] \tag{F.4}$$

From this relation, by introducing also the low-frequency assumption, we have

$$e^{-ikr} \approx e^{-ik\varrho} e^{ikr_0 \sin\beta} \approx e^{-ik\varrho}$$

and the velocity potential field is described by Eq. (F.2) which can be recast in the following form

$$\tilde{\varphi}(\varrho) = \tilde{v}_s e^{-ik\varrho} \iint_{\mathcal{S}} G_0 d\mathcal{S}(\mathbf{x}) - \tilde{\varphi}_s e^{-ik\varrho} \iint_{\mathcal{S}} \frac{\partial G_0}{\partial \hat{\varrho}} d\mathcal{S}(\mathbf{x}) \quad (\text{F.5})$$

where $\mathcal{S}$ denotes the average surface of the sphere.

The first integral can be evaluated by expressing it in the spherical coordinate system, and then introducing the variable transformation $r_{r_0} = r|_{\hat{\varrho}=r_0} = (\varrho^2 + r_0^2 - 2\varrho r_0 \sin \beta)^{1/2}$ such that $dr_{r_0} = -(\varrho r_0 / r_{r_0}) \cos \beta d\beta$. This yields

$$\iint_{\mathcal{S}} G_0 d\mathcal{S} = \frac{-r_0^2}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos \beta}{r_{r_0}} d\beta d\vartheta = \frac{r_0}{2\varrho} \int_{\varrho+r_0}^{\varrho-r_0} dr_{r_0} = -\frac{r_0^2}{\varrho} \quad (\text{F.6})$$

For the evaluation of the second integral we observe that

$$\left. \frac{\partial}{\partial \hat{\varrho}} \left(\frac{1}{r} \right) \right|_{\hat{\varrho}=r_0} = -\frac{r_0}{r_{r_0}^3} + \frac{\varrho \sin \beta}{r_{r_0}^3}$$

and thus, by introducing the same variable transformation, $r_{r_0} = r|_{\hat{\varrho}=r_0}$, considered in Eq. (F.6) we obtain

$$\iint_{\mathcal{S}} \frac{\partial G_0}{\partial \hat{\varrho}} d\mathcal{S} = \frac{1}{2\varrho^2} \left[(r_0^2 - \varrho^2) \int_{\varrho+r_0}^{\varrho-r_0} \frac{1}{r_{r_0}^2} dr_{r_0} + \int_{\varrho+r_0}^{\varrho-r_0} dr_{r_0} \right] = 0 \quad (\text{F.7})$$

As expected, the second integral in Eq. (F.5) vanishes since it coincides with the solid angle through which the sphere surface is observed from $\mathbf{x}_*$ (see, for instance, Ref. [68]).

Finally, combining Eqs. (F.6) and (F.7) with Eq. (F.5) the following expression for the radiated velocity potential field is obtained

$$\tilde{\varphi}(\varrho) = -\frac{r_0^2}{\varrho} \tilde{v}_s e^{-ik\varrho} \quad (\text{F.8})$$

The inverse Fourier transform applied to the above equation followed by application of the Bernoulli theorem yields

$$p'(\varrho, t_*) = -\rho_0 \frac{r_0^2}{\varrho} \omega^2 a \sin \omega(t_* - \varrho/c_0) = \rho_0 \frac{r_0^2}{\varrho} \dot{v}_s \Big|_{t_* - \varrho/c_0} \quad (\text{F.9})$$

with $\dot{v}_s$ denoting the acceleration of the surface. This expression coincides with solution *A* in appendix E and with the widely-known solution available in the literature for the same conditions of small pulsation frequency and amplitude [70, 79].

F.2 Low-frequency, large-amplitude, far-field solution

For large-amplitude, low-frequency pulsations the radiated velocity potential is described by the boundary integral representation in Eq. (F.1). However, for far-field observers it is possible to assume that the time delay to be considered in evaluating the integrand terms in Eq. (F.1) is

uniformly distributed over the sphere surface and is such that $\theta = \varrho/c_0$ (see Eq. (F.4)). This yields

$$\begin{aligned}\varphi(\varrho, t_*) &= v_s|_{t_*-\theta} \iint_{\mathcal{S}_\theta} G_0 d\mathcal{S} - \varphi_s|_{t_*-\theta} \iint_{\mathcal{S}_\theta} \frac{\partial G_0}{\partial \hat{\varrho}} d\mathcal{S} \\ &+ \frac{1}{c_0} \frac{\partial \varphi_s}{\partial t} \Big|_{t_*-\theta} \iint_{\mathcal{S}_\theta} \frac{\partial r}{\partial \hat{\varrho}} G_0 d\mathcal{S}\end{aligned}$$

where $\mathcal{S}_\theta$ denotes the sphere surface at the time $t_* - \theta$, i.e., having radius $r_s|_{t_*-\theta} = r_0 + a \sin \omega(t_* - \theta)$. For the third integral, once expressed in spherical coordinates, we observe that

$$\frac{\partial r}{\partial \hat{\varrho}} = \frac{\hat{\varrho}}{r} - \frac{\varrho}{r} \sin \beta \quad (\text{F.10})$$

and, thus, introducing the variable transformation $r_{r_s} = r|_{\hat{\varrho}=r_s} = (\varrho^2 + r_s^2 - 2\varrho r_s \sin \beta)^{1/2}$, we have, at any time

$$\begin{aligned}\iint_{\mathcal{S}_\theta} \frac{\partial r}{\partial \hat{\varrho}} G_0 d\mathcal{S} &= \frac{1}{4\varrho} \left[\int_{\varrho+r_s}^{\varrho-r_s} \left(\frac{r_s^2 - \varrho^2}{r_{r_s}} + r_{r_s} \right) dr_{r_s} \right] \Big|_{t_*-\theta} \\ &= -\frac{r_s}{2} \left(\frac{r_s}{\varrho} \right)^2 \Big|_{t_*-\theta}\end{aligned} \quad (\text{F.11})$$

Then, combining Eq. (F.11) with Eqs. (F.6) and (F.7) applied to the sphere of instantaneous radius $r_s|_{t_*-\theta}$, recalling the assumptions of low-frequency pulsations and far-field observer for which the contribution from the time derivative of φ_s is negligible with respect to the contribution from v_s , the following expression for the radiated velocity potential field is obtained

$$\begin{aligned}\varphi(\varrho, t_*) &= -\frac{r_s^2}{\varrho} v_s \Big|_{t_*-\theta} \\ &= -\frac{\omega a}{\varrho} [r_0 + a \sin \omega(t_* - \varrho/c_0)]^2 \cos \omega(t_* - \varrho/c_0) \\ &= -\omega a \frac{r_0^2}{\varrho} \left[\left(1 + \frac{1}{4} \frac{a^2}{r_0^2} \right) \cos \omega(t_* - \varrho/c_0) + \frac{a}{r_0} \sin 2\omega(t_* - \varrho/c_0) \right. \\ &\quad \left. - \frac{1}{4} \frac{a^2}{r_0^2} \cos 3\omega(t_* - \varrho/c_0) \right]\end{aligned}$$

which coincides with the solution C of appendix E written in terms of velocity potential, and with the solution obtained in Ref. [69] through a different approach.

F.3 General solution for small-amplitude pulsations

Removing the assumptions of low-frequency pulsations and far-field observers, but restoring the assumption of small-amplitude pulsations, the determination of the closed-form velocity potential solution from its boundary integral formulation requires the evaluation of the two integrals appearing in Eq. (F.2), which include the exponential delay contributions.

Considering the spherical coordinate system, the expression in Eq. (F.3), and the variable transformation applied in Eq. (F.6) the source integral has the form (where $\mathcal{S}$ denotes the surface

of the sphere of average radius, r_0)

$$\begin{aligned} \iint_{\mathcal{S}} G_0 e^{-ikr} d\mathcal{S} &= -\frac{r_0^2}{2} \int_{-\pi/2}^{\pi/2} \frac{e^{-ikr_0}}{r_0} \cos \beta d\beta \\ &= -\frac{r_0^2}{2} \int_{\varrho+r_0}^{\varrho-r_0} e^{-ikr_0} dr_0 = -\frac{r_0^2}{\varrho} \frac{\sin kr_0}{kr_0} e^{-ik\varrho} \end{aligned} \quad (\text{F.12})$$

Regarding the evaluation of the integral of doublets in Eq. (F.2), it is possible to demonstrate that the application of the spherical coordinates combined with Eq. (F.10), and lengthy mathematical manipulations based on the variable transformation, $r_{r_0} = r|_{\hat{\varrho}=r_0}$, provide

$$\iint_{\mathcal{S}} \frac{\partial(G_0 e^{-ikr})}{\partial \hat{\varrho}} d\mathcal{S} = \frac{r_0}{\varrho} \left(\cos kr_0 - \frac{\sin kr_0}{kr_0} \right) e^{-ik\varrho} \quad (\text{F.13})$$

The combination of Eq. (F.2) with Eqs. (F.12) and (F.13) yields the following expression for the velocity potential field

$$\tilde{\varphi}(\varrho) = -\tilde{v}_s \frac{r_0^2}{\varrho} \frac{\sin kr_0}{kr_0} e^{-ik\varrho} + \tilde{\varphi}_s \frac{r_0}{\varrho} \left(\cos kr_0 - \frac{\sin kr_0}{kr_0} \right) e^{-ik\varrho} \quad (\text{F.14})$$

where the potential over the sphere surface can be related to the velocity potential at any field point by the expression

$$\tilde{\varphi}_s = \tilde{\varphi}(r_0) = \frac{\varrho}{r_0} e^{-ik(r_0-\varrho)} \tilde{\varphi}(\varrho) \quad (\text{F.15})$$

which is readily derived from Eq. (E.2).

Substituting Eq. (F.15) in Eq. (F.14) we obtain

$$\tilde{\varphi}(\varrho) \left[\frac{\sin kr_0}{kr_0} + i \sin kr_0 \right] e^{-ikr_0} = -\tilde{v}_s \frac{r_0^2}{r} \frac{\sin kr_0}{kr_0} e^{-ikr} \quad (\text{F.16})$$

and, then, observing that

$$\frac{\sin kr_0}{kr_0} + i \sin kr_0 = [1 + (kr_0)^2]^{\frac{1}{2}} \frac{\sin kr_0}{kr_0} e^{i \tan^{-1}(kr_0)}$$

Eq. (F.16) provides the following form of the solution

$$\tilde{\varphi}(\varrho) = -\tilde{v}_s \frac{r_0^2}{\varrho} [1 + (kr_0)^2]^{-\frac{1}{2}} e^{-ik[\varrho-r_0+(1/k) \tan^{-1}(kr_0)]}$$

which coincides with the solution B in appendix E written in terms of velocity potential and, after application of the Bernoulli's theorem, yields the analytical acoustic pressure solution already available in the literature for small-amplitude pulsations of the sphere (see, for instance, Ref. [70]).

Finally, note that introducing the assumption of low-frequency pulsations, $kr_0 \ll 1$, the above equation reproduces the solution in Eq. (F.8) proving, therefore, that the latter is also valid for near field observers.

Appendix G

Analytical linear solution of sound scattered by a pulsating sphere

This appendix presents two analytical linear formulations for the predictions of the sound scattered by a pulsating sphere impinged by a plane wave. The first one is valid for small amplitudes of sphere pulsations and is an extension to the second order amplitudes of the first-order perturbation solution technique developed in Ref. [49]. The second analytical formulation yields the exact solution of the linear wave equation for the scattered field, and is valid for any amplitude and frequency of pulsation that guarantee subsonic motion of the sphere surface.

G.1 Analytical solution for small-amplitude pulsations

Following Censor's work in Ref. [49], let us consider an acoustical (velocity potential) plane wave of frequency ω_I impinging on a sphere pulsating with frequency ω .

In spherical coordinates, (r, ϑ, ϕ) , the impinging plane wave propagating in the direction $\vartheta = 0$ (polar axis) can be expressed through the following Bauer's formula [49, 78, 80]

$$\varphi^I(t) = \sum_{n=0}^{\infty} i^n (2n+1) j_n \left(\frac{\omega_I r}{c_0} \right) P_n(\cos \vartheta) e^{-i\omega_I t} \quad (\text{G.1})$$

where i denotes the imaginary unit, j_n are the spherical Bessel functions, and P_n are the Legendre functions.

Then, let us observe that the scattered field that satisfies the Helmholtz equation can be expressed in the following general form [49, 78]

$$\varphi^S(t) = \int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} i^n (2n+1) a_n(\nu) h_n \left(\frac{\nu r}{c_0} \right) P_n(\cos \vartheta) \right] e^{-i\nu t} d\nu \quad (\text{G.2})$$

where h_n are the spherical Hankel functions of the first kind, ν denotes the arbitrary frequency, and $a_n(\nu)$ are coefficient to be determined by the relevant geometry and kinematics. Note that this solution is for linear acoustics, namely it is valid when the nonlinear field terms in the governing wave differential equation are neglected, like in Sec. 5.1.2 where the contribution from σ has been eliminated.

Introducing the perturbation parameter, ε , the radius of the sphere is expressed as $r_s(t) = r_0[1 + \varepsilon \sin(\omega t)]$, where r_0 denotes the unperturbed radius and the pulsation amplitude is $a = \varepsilon r_0$. Thus, the unknown coefficients in Eq. (G.2) that provide the solution of the scattered field can be obtained as those that satisfy the impermeability boundary condition over the pulsating surface [49, 78]

$$\left. \frac{\partial \varphi^I}{\partial r} \right|_{r=r_s(t)} + \left. \frac{\partial \varphi^S}{\partial r} \right|_{r=r_s(t)} = 0, \quad \forall t \quad (\text{G.3})$$

To this aim, note that (see Eqs. (G.1) and (G.2))

$$\left. \frac{\partial \varphi^I}{\partial r} \right|_{r=r_s(t)} = \sum_{n=0}^{\infty} i^n (2n+1) \frac{\omega_I}{c_0} j'_n \left[\frac{\omega_I r_0}{c_0} (1 + \varepsilon \sin \omega t) \right] P_n(\cos \vartheta) e^{-i\omega_I t} \quad (\text{G.4})$$

$$\left. \frac{\partial \varphi^S}{\partial r} \right|_{r=r_s(t)} = \int_{-\infty}^{\infty} \left[\frac{\nu}{c_0} \sum_{n=0}^{\infty} i^n (2n+1) a_n(\nu) h'_n \left[\frac{\nu r_0}{c_0} (1 + \varepsilon \sin \omega t) \right] P_n(\cos \vartheta) \right] e^{-i\nu t} d\nu \quad (\text{G.5})$$

Then, considering the following second-order perturbation approximations of the derivatives of the spherical Bessel and Hankel functions appearing in the above expressions

$$j'_n \left(\frac{\omega_I r_s}{c_0} \right) \approx j'_n \left(\frac{\omega_I r_0}{c_0} \right) + j''_n \left(\frac{\omega_I r_0}{c_0} \right) \left[\frac{\omega_I r_0}{c_0} \varepsilon \sin(\omega t) \right] + \frac{1}{2} j'''_n \left(\frac{\omega_I r_0}{c_0} \right) \left[\frac{\omega_I r_0}{c_0} \varepsilon \sin(\omega t) \right]^2 \quad (\text{G.6})$$

$$h'_n \left(\frac{\nu r_s}{c_0} \right) \approx h'_n \left(\frac{\nu r_0}{c_0} \right) + h''_n \left(\frac{\nu r_0}{c_0} \right) \left[\frac{\nu r_0}{c_0} \varepsilon \sin(\omega t) \right] + \frac{1}{2} h'''_n \left(\frac{\nu r_0}{c_0} \right) \left[\frac{\nu r_0}{c_0} \varepsilon \sin(\omega t) \right]^2 \quad (\text{G.7})$$

and of the unknown $a_n(\nu)$ coefficients,

$$a_n(\nu) \approx A_n(\nu) + \varepsilon B_n(\nu) + \varepsilon^2 C_n(\nu) \quad (\text{G.8})$$

the combination of Eq. (G.3) with Eqs. (G.4), (G.5), (G.6), (G.7) and (G.8) gives the following equation for each index n in the summation defining the scattered velocity potential

$$\begin{aligned} & \frac{\omega_I}{c_0} \left\{ j'_n \left(\frac{\omega_I r_0}{c_0} \right) + j''_n \left(\frac{\omega_I r_0}{c_0} \right) \left[\frac{\omega_I r_0}{c_0} \varepsilon \sin(\omega t) \right] + \frac{1}{2} j'''_n \left(\frac{\omega_I r_0}{c_0} \right) \left[\frac{\omega_I r_0}{c_0} \varepsilon \sin(\omega t) \right]^2 \right\} e^{-i\omega_I t} + \\ & + \int_{-\infty}^{\infty} \frac{\nu}{c_0} [A_n(\nu) + \varepsilon B_n(\nu) + \varepsilon^2 C_n(\nu)] \left\{ h'_n \left(\frac{\nu r_0}{c_0} \right) + h''_n \left(\frac{\nu r_0}{c_0} \right) \left[\frac{\nu r_0}{c_0} \varepsilon \sin(\omega t) \right] + \right. \\ & \left. + \frac{1}{2} h'''_n \left(\frac{\nu r_0}{c_0} \right) \left[\frac{\nu r_0}{c_0} \varepsilon \sin(\omega t) \right]^2 \right\} e^{-i\nu t} d\nu = 0 \end{aligned}$$

Then, balancing the terms of the same power in ε in the above equation yields the expressions providing the coefficients in Eq. (G.8). Specifically:

- balancing the zero-th order terms yields

$$\omega_I j'_n \left(\frac{\omega_I r_0}{c_0} \right) e^{-i\omega_I t} + \int_{-\infty}^{\infty} \nu A_n(\nu) h'_n \left(\frac{\nu r_0}{c_0} \right) e^{-i\nu t} d\nu = 0 \quad (\text{G.9})$$

which is satisfied only when $\nu = \omega_I$ and gives the expression of $A_n(\nu)$;

- balancing the first order terms yields

$$\begin{aligned} \omega_I j_n'' \left(\frac{\omega_I r_0}{c_0} \right) \frac{\omega_I r_0}{c_0} \sin(\omega t) e^{-i\omega_I t} + \int_{-\infty}^{\infty} \nu A_n(\nu) h_n'' \left(\frac{\nu r_0}{c_0} \right) \frac{\nu r_0}{c_0} \sin(\omega t) e^{-i\nu t} d\nu \\ + \int_{-\infty}^{\infty} \nu B_n(\nu) h_n' \left(\frac{\nu r_0}{c_0} \right) e^{-i\nu t} d\nu = 0 \end{aligned} \quad (\text{G.10})$$

from which the coefficient $B_n(\nu)$ is obtained once the expression of $A_n(\nu)$ given by Eq. (G.9) is suitably replaced, observing that

$$\sin(\omega t) e^{-i\omega_I t} = \frac{e^{-i(\omega_I - \omega)t} - e^{-i(\omega_I + \omega)t}}{2i}$$

and noting that Eq. (G.10) is satisfied only when in the last integral $\nu = \omega_I - \omega$ or $\nu = \omega_I + \omega$;

- balancing the second order terms yields

$$\begin{aligned} \frac{\omega_I}{2} j_n''' \left(\frac{\omega_I r_0}{c_0} \right) \left[\frac{\omega_I r_0}{c_0} \right]^2 \sin^2(\omega t) e^{-i\omega_I t} \\ + \int_{-\infty}^{\infty} \frac{\nu}{2} A_n(\nu) h_n''' \left(\frac{\nu r_0}{c_0} \right) \left[\frac{\nu r_0}{c_0} \right]^2 \sin^2(\omega t) e^{-i\nu t} d\nu \\ + \int_{-\infty}^{\infty} \nu B_n(\nu) h_n'' \left(\frac{\nu r_0}{c_0} \right) \frac{\nu r_0}{c_0} \sin(\omega t) e^{-i\nu t} d\nu \\ + \int_{-\infty}^{\infty} \nu C_n(\nu) h_n' \left(\frac{\nu r_0}{c_0} \right) e^{-i\nu t} d\nu = 0 \end{aligned} \quad (\text{G.11})$$

from which the coefficient $C_n(\nu)$ is obtained once the expression of $A_n(\nu)$ and $B_n(\nu)$ given by Eqs. (G.9) and (G.10) are suitably replaced, observing that

$$\sin^2(\omega t) e^{-i\omega_I t} = \frac{e^{-i\omega_I t}}{2} - \frac{e^{-i(\omega_I - 2\omega)t} + e^{-i(\omega_I + 2\omega)t}}{4}$$

and noting that Eq. (G.11) is satisfied only when in the last integral $\nu = \omega_I$, or $\nu = \omega_I - 2\omega$, or $\nu = \omega_I + 2\omega$.

Thus, the satisfaction of Eqs. (G.9), (G.10) and (G.11) yields the following expressions for the coefficients in Eq. (G.8)

$$\left\{ \begin{aligned} A_n(\nu) &= \frac{-1/\nu}{h_n'(\nu r_0/c_0)} \alpha_n \delta(\nu - \omega_I) \\ B_n(\nu) &= \frac{-1/\nu}{h_n'(\nu r_0/c_0)} \beta_n [\delta(\nu - \omega_I + \omega) - \delta(\nu - \omega_I - \omega)] \\ C_n(\nu) &= \frac{-1/\nu}{h_n'(\nu r_0/c_0)} \left[\left(\frac{1}{4} \gamma_n - \gamma_n^- - \gamma_n^+ \right) \delta(\nu - \omega_I) - \left(\frac{1}{8} \gamma_n + \gamma_n^- \right) \delta(\nu - \omega_I + 2\omega) \right. \\ &\quad \left. - \left(\frac{1}{8} \gamma_n + \gamma_n^+ \right) \delta(\nu - \omega_I - 2\omega) \right] \end{aligned} \right. \quad (\text{G.12})$$

where

$$\alpha_n = \omega_I j_n'(\rho), \quad \beta_n = \omega_I \frac{\rho}{2i} \left[j_n''(\rho) - \frac{\alpha_n h_n''(\rho)}{\omega_I h_n'(\rho)} \right], \quad \gamma_n = \omega_I \rho^2 \left[j_n'''(\rho) - \frac{\alpha_n h_n'''(\rho)}{\omega_I h_n'(\rho)} \right]$$

$$\gamma_n^- = \frac{\rho^-}{2i} \frac{\beta_n h_n''(\rho^-)}{h_n'(\rho^-)}, \quad \gamma_n^+ = \frac{\rho^+}{2i} \frac{\beta_n h_n''(\rho^+)}{h_n'(\rho^+)}$$

with $\rho = \omega_I r_0 / c_0$, $\rho^+ = (\omega_I + \omega) r_0 / c_0$ and $\rho^- = (\omega_I - \omega) r_0 / c_0$. Note that in some cases, for example, if $\omega = \omega_I$, it is necessary to use the asymptotic expression for the definition of spherical Hankel functions evaluated at zero, from which the following expression is obtained

$$\lim_{z \rightarrow 0} h_n(z) = -i \frac{(2n+1)!!}{z^{n+1}} \Rightarrow \lim_{z \rightarrow 0} \frac{z h_n''(z)}{h_n'(z)} = -(n+2)$$

which can be used to calculate γ_n^- .

Replacing Eq. (G.8) in Eq. (G.2) with the A_n, B_n and C_n coefficients given by Eq. (G.12) provides the second-order perturbation analytical solution of the velocity potential field scattered by the pulsating sphere. This can be regarded as a second-order extension of the first-order solution presented by Censor [49], which makes it possible to identify higher-order harmonics such as those appearing, for example, in Fig. 5.8(b). Furthermore, while the above expression of the coefficient A_n coincides with the one given in Ref. [49], the expression of the coefficient B_n corrects the corresponding one in Ref. [49], which is affected by an error.

G.2 Exact analytical solution

Let us now consider a plane wave of frequency ω_I impinging on a sphere harmonically pulsating with frequency ω , and arbitrary amplitude, a . The incident field is expressed again through Bauer's formula rewritten as (for $k_I = \omega_I / c_0$)

$$\varphi^I(r, \vartheta, t) = \sum_{n=0}^{\infty} \left[(2n+1) j_n(k_I r) e^{-i(\omega_I t - \alpha_n)} \right] P_n(\cos \vartheta)$$

where the phase angle α_n represents the contribution of the factor i^n appearing in Eq. (G.1), and is defined as follows

$$\alpha_n = \begin{cases} 0 & \text{if } i^n = 1 \\ \pi & \text{if } i^n = -1 \\ \pi/2 & \text{if } i^n = i \\ -\pi/2 & \text{if } i^n = -i \end{cases}$$

Let us assume that harmonic of the sphere pulsations is a multiple or a submultiple of the harmonic of the incident signal, and introduce the fundamental frequency of the problem, $\omega_f = \min(\omega_I, \omega)$.

Then, for $\omega_m = m \omega_f$ denoting the m -th harmonic of the resulting scattered signal, similarly to the formulation given in appendix G.1, the solution is suitably expressed in the following general

form [49, 78]

$$\varphi^S(r, \vartheta, t) = \sum_{n=0}^{\infty} \left[a_{n0} y_n^a(r) + \sum_{m=1}^{\infty} a_{nm} h_n(k_m r) e^{-i\omega_m t} \right] P_n(\cos \vartheta) \quad (\text{G.13})$$

where, $k_m = \omega_m/c_0$, h_n are the spherical Hankel functions of the first kind, P_n are Legendre polynomials, whereas a_{n0} and a_{nm} are, respectively, real and complex coefficients to be determined based on the geometry and kinematics of the scatterer and the characteristics of the incident wave.

Note that, the above representation of the scattered field includes the presence of mean values. Typically, in acoustic problems stationary contributions are not of interest and are not considered. However, since in our problem the mean values are functions of the radial position, the imposition of the boundary conditions over the pulsating sphere surface as expressed in Eq. (G.3) yields an expression in which they provide non-stationary contributions which couples with the harmonics of the problem, and thus cannot be neglected.

In the expression of the mean values, $a_{n0} y_n^a(r)$, the functions $y_n^a(r)$ represent the asymptotic expressions of Neumann's functions for their argument tending to zero. Indeed, for $r \rightarrow 0$ the spherical Bessel equation reads [81]

$$r^2 \frac{d^2 w}{dr^2} + 2r \frac{dw}{dr} - n(n+1)w = 0 \quad (\text{G.14})$$

which coincides with the equation for the radial factor of the wave equation solution when $k = \omega/c_0 = 0$ (stationary solution) [78]. Therefore, the asymptotic solutions of the spherical Bessel equation for $t \rightarrow 0$ coincide with the stationary solutions of the radial factor equation which are consequently used to define the mean values in Eq. (G.13).

Both spherical Bessel's functions and Neumann's functions are solutions of the spherical Bessel equation. However, the former tend to vanish for the argument tending to zero and, therefore, the mean values in Eq. (G.13) are expressed through the following asymptotic form of the Neumann's functions [78, 81]

$$y_n^a(r) = -\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{r^{n+1}}$$

Akin to the solution process applied in appendix G.1, the unknown coefficients in Eq. (G.13) are obtained as those that satisfy the impermeability boundary condition on the surface of the pulsating sphere expressed by Eq. (G.3). It is worth noting that, the impermeability boundary condition is completed by the radial derivative of the velocity potential component induced by the velocity of the surface. For linear problems it is uncoupled from the scattered one, and for uniformly pulsating spheres, it uniformly radiates to the far field.

Next, observing that

$$\frac{\partial \varphi^I}{\partial r} = \sum_{n=0}^{\infty} \left[(2n+1) k_I j_n'(k_I r) e^{-i(\omega_I t - \alpha_n)} \right] P_n(\cos \vartheta)$$

and that

$$\frac{\partial \varphi^S}{\partial r} = \sum_{n=0}^{\infty} \left[a_{n0} y_n^{a'}(r) + \sum_{m=1}^{\infty} a_{nm} k_m h_n'(k_m r) e^{-i\omega_m t} \right] P_n(\cos \vartheta)$$

where $()'$ denotes derivation with respect to the argument of the function, and recalling that the Legendre polynomials are linearly independent, it is possible to conclude that the boundary condi-

tion in Eq. (G.3) is satisfied if, for $r_s(t)$ denoting the time-periodic variations of the radius of the sphere, the equation

$$a_{n0} y_n^{a'}(r_s(t)) + \sum_{m=1}^{\infty} a_{nm} k_m h'_n(k_m r_s(t)) e^{-i\omega_m t} = -(2n+1) k_I j'_n(k_I r_s(t)) e^{-i(\omega_I t - \alpha_n)} \quad (\text{G.15})$$

is satisfied for each index n , at each time. Therefore, the unknown coefficients of the scattered potential are determined as those that, for each time and n index considered, assure the satisfaction of Eq. (G.15).

To this aim, let us express the complex coefficients, a_{nm} , and the derivative of the Hankel functions, $h'_n(k_m r)$, in terms of their magnitude and phase, namely

$$a_{nm} = |a_{nm}| e^{i\gamma_{nm}}; \quad h'_n(k_m r) = |h'_n(k_m r)| e^{i\psi_n(k_m r)} = |h'_{nm}(r)| e^{i\psi_{nm}(r)}$$

where the notation $h'_{nm}(r) \equiv h'_n(k_m r)$ is introduced. Then, once these expressions are substituted in Eq. (G.15), we get

$$\begin{aligned} a_{n0} y_n^{a'}(r_s(t)) &+ \sum_{m=1}^{\infty} k_m |a_{nm}| |h'_{nm}(r_s(t))| e^{-i[\omega_m t - \psi_{nm}(r_s(t)) - \gamma_{nm}]} \\ &= -(2n+1) k_I j'_n(k_I r_s(t)) e^{-i(\omega_I t - \alpha_n)} \end{aligned}$$

whose real domain representation yields

$$\begin{aligned} a_{n0} y_n^{a'}(r_s(t)) &+ \sum_{m=1}^{\infty} k_m |a_{nm}| |h'_{nm}(r_s(t))| \cos[\omega_m t - \psi_{nm}(r_s(t)) - \gamma_{nm}] \\ &= -(2n+1) k_I j'_n(k_I r_s(t)) \cos(\omega_I t - \alpha_n) \end{aligned}$$

or, equivalently,

$$\begin{aligned} a_{n0} y_n^{a'}(r_s(t)) &+ \sum_{m=1}^{\infty} k_m |h'_{nm}(r_s(t))| \left\{ a_{nm}^c \cos[\omega_m t - \psi_{nm}(r_s(t))] + a_{nm}^s \sin[\omega_m t - \psi_{nm}(r_s(t))] \right\} \\ &= -(2n+1) k_I j'_n(k_I r_s(t)) \cos(\omega_I t - \alpha_n) \end{aligned} \quad (\text{G.16})$$

with $a_{n0}, a_{nm}^c = |a_{nm}| \cos \gamma_{nm}, a_{nm}^s = |a_{nm}| \sin \gamma_{nm}$ representing the set of coefficients to be determined to obtain the scattered field solution. The number of unknowns depends on the number of Legendre polynomials used to describe the spatial distribution of the signal, and the number of harmonics that are necessary to describe the time evolution of the signal.

Given the function describing the sphere pulsations, $r_s(t)$, the left-hand side of Eq. (G.16) is a multi-harmonic function of time depending on the scattering unknown coefficients, while the right-hand side of the same equation is a known multi-harmonic function of time depending on the combination of the incident wave harmonic with the harmonic pulsations of the sphere appearing in the argument of the radial derivative of the spherical Bessel function.

Balancing the harmonics of the left- and right-hand side of Eq. (G.16) yields the algebraic system whose solution provides the scattered potential coefficients. To this aim, first the following

time-periodic functions are introduced

$$\begin{cases} q_{n0}(t) = y_n^{a'}(r_s(t)) \\ f_{nm}(t) = k_m |h'_{nm}(r_s(t))| \cos(\omega_m t - \psi_{nm}(r_s(t))) \\ g_{nm}(t) = k_m |h'_{nm}(r_s(t))| \sin(\omega_m t - \psi_{nm}(r_s(t))) \\ z_n(t) = -(2n+1) k_I j'_n(k_I r_s(t)) \cos(\omega_I t - \alpha_n) \end{cases} \quad (\text{G.17})$$

such that Eq. (G.16) is recast in the following form

$$q_{n0}(t) a_{n0} + \sum_{m=1}^{\infty} [f_{nm}(t) a_{nm}^c + g_{nm}(t) a_{nm}^s] = z_n(t) \quad (\text{G.18})$$

Then, the Fourier-series expansions of the functions in Eq. (G.17) are accomplished

$$\begin{cases} q_{n0}(t) = q_{n0}^0 + \sum_{j=1}^{\infty} [q_{n0}^{jc} \cos(\omega_j t) + q_{n0}^{js} \sin(\omega_j t)] \\ f_{nm}(t) = f_{nm}^0 + \sum_{j=1}^{\infty} [f_{nm}^{jc} \cos(\omega_j t) + f_{nm}^{js} \sin(\omega_j t)] \\ g_{nm}(t) = g_{nm}^0 + \sum_{j=1}^{\infty} [g_{nm}^{jc} \cos(\omega_j t) + g_{nm}^{js} \sin(\omega_j t)] \\ z_n(t) = z_n^0 + \sum_{j=1}^{\infty} [z_n^{jc} \cos(\omega_j t) + z_n^{js} \sin(\omega_j t)] \end{cases} \quad (\text{G.19})$$

with $\omega_j = j\omega_f$, and are combined with Eq. (G.18). Thus the harmonic balance of left- and right-hand sides of the resulting expression yields the following algebraic system with unknowns coefficients, $a_{n0}, a_{nm}^c, a_{nm}^s$

$$\begin{bmatrix} q_{n0}^0 & f_{n1}^0 & g_{n1}^0 & \cdots & f_{nN_a}^0 & g_{nN_a}^0 \\ q_{n0}^{1c} & f_{n1}^{1c} & g_{n1}^{1c} & \cdots & f_{nN_a}^{1c} & g_{nN_a}^{1c} \\ q_{n0}^{1s} & f_{n1}^{1s} & g_{n1}^{1s} & \cdots & f_{nN_a}^{1s} & g_{nN_a}^{1s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n0}^{N_a c} & f_{n1}^{N_a c} & g_{n1}^{N_a c} & \cdots & f_{nN_a}^{N_a c} & g_{nN_a}^{N_a c} \\ q_{n0}^{N_a s} & f_{n1}^{N_a s} & g_{n1}^{N_a s} & \cdots & f_{nN_a}^{N_a s} & g_{nN_a}^{N_a s} \end{bmatrix} \begin{bmatrix} a_{n0} \\ a_{n1}^c \\ a_{n1}^s \\ \vdots \\ a_{nN_a}^c \\ a_{nN_a}^s \end{bmatrix} = \begin{bmatrix} z_n^0 \\ z_n^{1c} \\ z_n^{1s} \\ \vdots \\ z_n^{N_a c} \\ z_n^{N_a s} \end{bmatrix} \quad (\text{G.20})$$

where N_a denotes the finite number of harmonics used to represent both the scattered signal and the Fourier-series expansions in Eq. (G.19).

Once the algebraic system in Eq. (G.20) is solved for all $n = 1, \dots, N_L$, with N_L representing the finite number of Legendre polynomials used to describe incident and scattered velocity potential fields, the scattered potential solution is analytically expressed as

$$\varphi^S(r, \vartheta, t) = \sum_{n=0}^{N_L} \left[a_{n0} y_n^a(r) + \sum_{m=1}^{N_a} |a_{nm}| |h_{nm}(r)| e^{-i[\omega_m t - \chi_{nm}(r) - \gamma_{mn}]} \right] P_n(\cos \vartheta)$$

where

$$\begin{aligned} |a_{nm}| &= \sqrt{a_{nm}^c{}^2 + a_{nm}^s{}^2} \\ \gamma_{nm} &= \arctan(a_{nm}^s/a_{nm}^c) \end{aligned}$$

are determined through the solution of Eq. (G.20), whereas $|h_{nm}(r)|$ and $\chi_{nm}(r)$ are, respectively, magnitude and phase of the Hankel function $h_n(k_m r)$.

The exact linear solution of the scattered velocity potential field presented in this section is valid for any subsonic harmonic pulsation of the sphere, $ka < 1$, arbitrarily phased with respect to the incident harmonic plane wave, assuming that the harmonic of the incident wave is a multiple or submultiple of the harmonic pulsation of the sphere.

However, note that for the general case of arbitrary values of ω and ω_I , considerations similar to those made in Sec. 5.2.4 regarding the BEM solution of linear scattering from pulsating spheres apply. In fact, in this case, the fundamental harmonic, ω_f , and the number of harmonics, N_a , of the Fourier series expansions introduced in the solution process must be chosen so as to adequately capture the harmonics that appear in the resulting scattered signal, and the solution obtained can be considered to be expressed in terms of its discrete Fourier transform.

Appendix H

Remarks on the iterative procedure for the nonlinear solution of the velocity potential scattered by a moving and deforming body

In Sec. 5.3.2 the cascade method for the solution of the nonlinear problem associated to the sound scattered by moving and deforming bodies is presented.

However, this is not the only possible approach. In general, to solve a nonlinear problem, it is possible to use the Newton-Raphson method or a simple iterative algorithm (possibly relaxed).

In the following, a comparison among these solution procedures is accomplished, highlighting the advantages and disadvantages of each one.

For the sake of simplicity, let us consider the nonlinear problem consisting of the steady motion of a body in a compressible perfect fluid, impinged by the flow emitted by a steady source co-moving with it. In this case, φ^0 denotes the velocity potential field associated with the steady motion (whose corresponding nonlinear contributions are assumed to be negligible, $\sigma(\varphi^0) \approx 0$), while φ^I and φ^S denote, respectively, the incident velocity potential field generated by the moving source, and the scattered one. The corresponding boundary integral equation for the scattered velocity potential solution, discretized through the presented BEM approach, in the case of not negligible fluid nonlinearities reads

$$\frac{1}{2}\varphi_k^S = \sum_{m=1}^{N_p} A_{km}\chi_m^I + \sum_{m=1}^{N_p} B_{km}\varphi_m^S + \sum_{h=1}^{M_v} H_{kh}\sigma_h \quad \text{for } k = 1, N_p \quad (\text{H.1})$$

with N_p and M_v cells denoting, respectively, the number of surface discretization panels and number of volume cells. At the centroids of the volume cells the discretized integral representation yields

$$\varphi_i^S = \sum_{m=1}^{N_p} A_{im}\chi_m^I + \sum_{m=1}^{N_p} B_{im}\varphi_m^S + \sum_{h=1}^{M_v} H_{ih}\sigma_h \quad \text{for } i = 1, M_v \quad (\text{H.2})$$

where χ_m^I coincides with the opposite of the normal derivative of the incident velocity potential at the centroid of the m -th panel, derived from the impermeability boundary conditions (see Sec. 5.1.2).

H.1 Solution through Newton-Raphson method

Let us rewrite the system composed of Eqs. (H.1) and (H.2) in the following homogeneous, matrix compact form

$$\left(\frac{1}{2}\mathbf{I} - \mathbf{B}\right) \boldsymbol{\varphi}^S - \mathbf{A}\boldsymbol{\chi}^I - \mathbf{f}(\boldsymbol{\varphi}^S, \boldsymbol{\varphi}^0, \boldsymbol{\varphi}^I) = \mathbf{0}$$

where the vector $\boldsymbol{\varphi}^S$, of dimension $N_p + M_v$, collects the values of the velocity potential at the centroids of all surface panels and volume cells, while the vector $\mathbf{f}$, also of dimension $N_p + M_v$, collects the nonlinear contributions

$$\mathbf{f}(\boldsymbol{\varphi}^S, \boldsymbol{\varphi}^0, \boldsymbol{\varphi}^I) = \mathbf{H}\boldsymbol{\sigma}(\boldsymbol{\varphi}^S, \boldsymbol{\varphi}^0, \boldsymbol{\varphi}^I)$$

(with the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{H}$ suitably defined from Eqs. (H.1) and (H.2)).

The application of the Newton-Raphson method yields the solution of the system as the vector $\boldsymbol{\varphi}_k^S$ for which the following iterative scheme converges

$$\boldsymbol{\varphi}_{j+1}^S = \boldsymbol{\varphi}_j^S - \mathbf{J}_j^{-1} \left[\left(\frac{1}{2}\mathbf{I} - \mathbf{B}\right) \boldsymbol{\varphi}_j^S - \mathbf{A}\boldsymbol{\chi}^I - \mathbf{f}(\boldsymbol{\varphi}_j^S, \boldsymbol{\varphi}^0, \boldsymbol{\varphi}^I) \right] \quad (\text{H.3})$$

namely, it is such that $\|\boldsymbol{\varphi}_{k+1}^S - \boldsymbol{\varphi}_k^S\| < \epsilon$, with ϵ denoting a prescribed error tolerance.

In Eq. (H.3) the elements of the Jacobian matrix, $\mathbf{J}$, are defined as

$$J_{km} = \frac{1}{2}\delta_{km} - B_{km} - \frac{\partial f_k}{\partial \varphi_m} \Big|_{\varphi_m = \varphi_m^0 + \varphi_{m_j}^S + \varphi_m^I} \quad (\text{H.4})$$

where

$$\frac{\partial f_k}{\partial \varphi_m} \Big|_{\varphi_m = \varphi_m^0 + \varphi_{m_j}^S + \varphi_m^I} = H_{kn} \frac{\partial \sigma_n}{\partial \varphi_m} \Big|_{\varphi_m = \varphi_m^0 + \varphi_{m_j}^S + \varphi_m^I} = H_{kn} \hat{H}_{nm} \Big|_{\varphi_m = \varphi_m^0 + \varphi_{m_j}^S + \varphi_m^I}$$

and, thus, in compact form Eq. (H.4) is rewritten as follows

$$\mathbf{J}_j = \frac{1}{2}\mathbf{I} - \mathbf{B} - \mathbf{H}\hat{\mathbf{H}}_j$$

where $\hat{\mathbf{H}}_j$ is the matrix with elements given by $\hat{H}_{nm} \Big|_{\varphi_m = \varphi_m^0 + \varphi_{m_j}^S + \varphi_m^I}$.

The advantage of this method is the presence of the Jacobian matrix which improves the convergence rate of the iterative process. The disadvantage is that the Jacobian matrix must be evaluated and inverted at each iteration, and a high computational cost could be required (this is especially true, for instance, in the case of an unsteady problem formulated through the multi-harmonic approach).

H.1.1 Solution of the linearized system

A non iterative solution can be obtained by introducing the following linear approximation of the components of the vector $\mathbf{f}$ of nonlinear contributions around the velocity potential field associated

with the steady motion

$$f_k(\varphi_j^S, \varphi^0, \varphi^I) \simeq \frac{\partial f_k}{\partial \varphi_m} \bigg|_{\varphi_m = \varphi_m^0} (\varphi_m^S + \varphi_m^I) = H_{kn} \hat{H}_{nm} |_{\varphi_m = \varphi_m^0} (\varphi_m^S + \varphi_m^I)$$

Indeed, the substitution of this linearisation into Eq. (H.3) gives

$$\varphi_{j+1}^S = \varphi_j^S - \mathbf{J}_0^{-1} \left[\left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right) \varphi_j^S - \mathbf{A} \chi^I - \mathbf{H} \hat{\mathbf{H}}_0 \varphi_j^S - \mathbf{H} \hat{\mathbf{H}}_0 \varphi^I \right] \quad (\text{H.5})$$

where the Jacobian $\mathbf{J}_0$ is defined as

$$\mathbf{J}_0 = \frac{1}{2} \mathbf{I} - \mathbf{B} - \mathbf{H} \hat{\mathbf{H}}_0$$

with $\hat{\mathbf{H}}_0$ denoting the compact form of $\hat{H}_{nm} |_{\varphi_m = \varphi_m^0}$. Then, rearranging Eq. (H.5) we obtain the expression

$$\varphi_{j+1}^S = \varphi_j^S - \mathbf{J}_0^{-1} \mathbf{J}_0 \varphi_j^S - \mathbf{J}_0^{-1} \left[-\mathbf{A} \chi^I - \mathbf{H} \hat{\mathbf{H}}_0 \varphi^I \right]$$

from it is straightforward to determine the following non-iterative linearized solution of the problem in the following form, by noting that the initial iterative scheme collapses into a direct definition of the output vector

$$\varphi^S = \mathbf{J}_0^{-1} \left[\mathbf{A} \chi^I + \mathbf{H} \hat{\mathbf{H}}_0 \varphi^I \right]$$

The computational cost of this linearized approximate solution is significantly lower than that of the complete nonlinear one.

H.2 Solution form simple iterative method

A widely used approach for evaluating the solution of nonlinear systems is the simple iterative method. Starting from a reasonably “close” solution (for instance, the solution to the linear problem obtained by neglecting the non-linear terms), it consists of deriving an updated scattered velocity potential as the solution to an updated system in which the nonlinear terms are evaluated as those given by the known scattered velocity potential. This process is represented by the following iterative scheme

$$\varphi_{j+1}^S = \left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right)^{-1} \left[\mathbf{A} \chi^I + \mathbf{f}(\varphi_j^S, \varphi^I, \varphi^0) \right] \quad (\text{H.6})$$

which converges to a solution if, for a given iterative index k , the condition $\|\varphi_{k+1}^S - \varphi_k^S\| < \epsilon$, with ϵ denoting a prescribed error tolerance, is achieved.

This solution scheme can be considered a simplification of the Newton-Raphson method. It does not require the evaluation of the Jacobian matrix, which is the most computationally expensive part of the Newton-Raphson method. However, the disadvantage of this approach is that the convergence quality of the iterative solution scheme is significantly lower than that of the scheme derived from the Newton-Raphson method. Therefore, in order to increase the quality of convergence, the iterative scheme in Eq. (H.6) can be modified by including a “relaxation” in the expression that produces the updated solution. This is achieved by replacing φ_j^S on the right-hand side with a combination with the previous one, *i.e.*, $(1 - \alpha) \varphi_j^S + \alpha \varphi_{j+1}^S$, where $0 \leq \alpha \leq 1$ represents the relaxation factor.

H.3 Solution form cascade method

The cascade algorithm used in Sec. 5.3 to solve the nonlinear scattering problem is equivalent to the simple iterative method. The difference between the two approaches is the definition of the j -th iterative solution: while in the simple iterative method each iteration gives an updated solution, in the cascade method each iteration provides an incremental value of the solution which, therefore, is obtained by adding up the incremental terms evaluated at all iterations.

Denoting with $\varphi_{c_j}^S$ the j -th incremental value of the cascade solution, the first iteration of the cascade algorithm yields

$$\varphi_{c_1}^S = \left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right)^{-1} [\mathbf{A} \chi^I + \mathbf{f}(\varphi^I, \varphi^0)]$$

while second iteration is given by

$$\varphi_{c_2}^S = \left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right)^{-1} [\mathbf{f}(\varphi_{c_1}^S, \varphi^I, \varphi^0) - \mathbf{f}(\varphi^I, \varphi^0)]$$

and a generic iteration is expressed as

$$\varphi_{c_{j+1}}^S = \left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right)^{-1} [\mathbf{f}(\varphi_{c_1}^S + \dots + \varphi_{c_j}^S, \varphi^I, \varphi^0) - \mathbf{f}(\varphi_{c_1}^S + \dots + \varphi_{c_{j-1}}^S, \varphi^I, \varphi^0)]$$

Then, it is readily verified that the sum of the results from the first two steps of the cascade yields exactly the result of the second iteration of the simple iterative method, namely,

$$\varphi_{c_1}^S + \varphi_{c_2}^S = \left(\frac{1}{2} \mathbf{I} - \mathbf{B} \right)^{-1} [\mathbf{A} \chi^I + \mathbf{f}(\varphi^I, \varphi^0) + \mathbf{f}(\varphi_{c_1}^S, \varphi^I, \varphi^0) - \mathbf{f}(\varphi^I, \varphi^0)] = \varphi_2^S$$

This proves that the cascade method and the simple iterative method are equivalent. From a numerical point of view, it may be convenient to use the cascade algorithm to work with incremental quantities of solution, which progressively decrease with the iterations.