

## On a front evolution problem for the multidimensional East model

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### Abstract

We consider a natural front evolution problem for the East process on  $\mathbb{Z}^d$ ,  $d \geq 2$ , a well studied kinetically constrained model for which the facilitation mechanism is oriented along the coordinate directions, as the equilibrium density  $q$  of the facilitating vertices vanishes. Starting with a unique unconstrained vertex at the origin, let  $S(t)$  consist of those vertices which became unconstrained within time  $t$  and, for an arbitrary positive direction  $\mathbf{x}$ , let  $v_{\max}(\mathbf{x})$ ,  $v_{\min}(\mathbf{x})$  be the maximal/minimal velocities at which  $S(t)$  grows in that direction. If  $\mathbf{x}$  is independent of  $q$ , we prove that  $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{(1+o(1))} = \gamma_d^{(1+o(1))}$  as  $q \rightarrow 0$ , where  $\gamma_d$  is the spectral gap of the process on  $\mathbb{Z}^d$ . We also analyse the case in which  $\mathbf{x}$  depends on  $q$  and some of its coordinates vanish as  $q \rightarrow 0$ . In particular, for  $d = 2$  we prove that if  $\mathbf{x}$  approaches one of the two coordinate directions fast enough, then  $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{(1+o(1))} = \gamma_1^{(1+o(1))} = \gamma_d^{d(1+o(1))}$ , i.e. the growth of  $S(t)$  close to the coordinate directions is much slower than the growth in the bulk and it is dictated by the *one dimensional* process. As a result the region  $S(t)$  becomes extremely elongated inside  $\mathbb{Z}_+^d$ . We also establish mixing time cutoff for the chain in finite boxes with minimal boundary conditions. A key ingredient of our analysis is the renormalisation technique of [12] to estimate the spectral gap of the East process. A main novelty here is the extension of this technique to get the main asymptotic as  $q \rightarrow 0$  of a suitable principal Dirichlet eigenvalue of the process.

**Keywords:** kinetically constrained models, East model; front evolution; interacting particle systems; cutoff phenomenon, renormalization.

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## 1 Introduction

The East<sup>1</sup> process on  $\mathbb{Z}^d$  (see [1], [15] and references therein for  $d = 1$ , and [12, 11, 19] for  $d \geq 2$ ), is a keynote example of the class of *facilitated interacting particle*

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<sup>1</sup>The nickname “East” here is only to keep up with the tradition. In two dimension “South-or-West” would be more appropriate.

systems or *kinetically constrained models* (KCM) which play an important role in several qualitative and quantitative approaches to describe the complex behaviour of glassy dynamics (see e.g. [17] and references therein). It is the interacting particle system with state space  $\Omega = \{0, 1\}^{\mathbb{Z}^d}$  (a continuous time Markov chain on  $\{0, 1\}^\Lambda$  if restricted to a finite  $\Lambda \subset \mathbb{Z}^d$ ) which is informally described as follows. Each vertex  $x \in \mathbb{Z}^d$ , with rate one and independently across  $\mathbb{Z}^d$ , is resampled from  $\{0, 1\}$  according to the Bernoulli( $p$ )-measure,  $p = 1 - q$ , iff the current state carries at least one vacancy (i.e. a state “0”) among the neighbours of  $x$  of the form  $y = x - e$ ,  $e \in \mathcal{B}$ , where  $\mathcal{B} = (e^{(1)}, \dots, e^{(d)})$  is the canonical basis of  $\mathbb{Z}^d$ . The product Bernoulli( $p$ ) measure on  $\Omega$  is a reversible measure for this process and the parameter  $q$  is the equilibrium density of the vacancies, i.e. of the facilitating vertices. In the physical applications  $q \simeq e^{-\beta}$ , where  $\beta$  is the inverse temperature.

Thanks to the oriented character of its kinetic constraint (i.e. the requirement that has to be fulfilled in order to permit the update of a vertex), the East process is one of the few KCM for which a rigorous analysis of the actual evolution of the process with some arbitrary initial distribution has been accessible for any value of  $q \in (0, 1)$  [6, 9, 10, 12, 11, 14, 20, 19]. In this paper, building in particular on [12, 11], we make some progress in the analysis of a natural front evolution problem in  $\mathbb{Z}_+^d = \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_i \geq 0\}$  for  $q \ll 1$  (i.e. low temperature) and  $d \geq 2$ . We refer the reader to Section 2 for a precise formulation of the problem and of the main results.

### 1.1 Notation

- Let  $\mathbb{R}_+^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0\}$  and for any  $x \in \mathbb{R}_+^d$  let  $\lfloor x \rfloor \in \mathbb{Z}_+^d$  be such that  $\lfloor x \rfloor_i = \lfloor x_i \rfloor \forall i$ . Unit vectors of  $\mathbb{R}_+^d$  will be written in bold. Given  $x, y \in \mathbb{Z}_+^d$  we will write  $x \prec y$  iff  $x_i \leq y_i \forall i$ ,  $x \prec V, V \subset \mathbb{Z}_+^d$ , if  $x \prec y \forall y \in V$ , and  $\|x - y\|_1 := \sum_i |x_i - y_i|$  for their  $\ell_1$ -distance. We shall also write  $x = 0$  to denote the origin of  $\mathbb{Z}_+^d$ .
- For any  $\Lambda \subset \mathbb{Z}_+^d$  we define its *oriented boundary*  $\partial_\downarrow \Lambda$  as  $\partial_\downarrow \Lambda \triangleq \{x \in \mathbb{Z}_+^d \setminus \Lambda : x + e \in \Lambda \text{ for some } e \in \mathcal{B}\}$ . Notice that vertices of  $\mathbb{Z}^d \setminus \mathbb{Z}_+^d$  are *not* part of the oriented boundary.
- $\Omega_\Lambda$  will denote for the product space  $\{0, 1\}^\Lambda$  endowed with the product topology. If  $\Lambda = \mathbb{Z}_+^d$  we simply write  $\Omega$ . We will write  $\omega_x \in \{0, 1\}$  for the state at  $x \in \Lambda$  of the configuration  $\omega \in \Omega_\Lambda$  and we will refer to the vertices of  $\Lambda$  where  $\omega \in \Omega_\Lambda$  is equal to one (zero) as the *particles* (*vacancies*) of  $\omega$ . If  $V \subset \Lambda$  we will write  $\omega \upharpoonright_V$  for the restriction of  $\omega \in \Omega_\Lambda$  to  $V$ . In particular we will write  $\omega \upharpoonright_V = 1$  if  $\omega(x) = 1 \forall x \in V$ .
- For any  $\Lambda \subset \mathbb{Z}_+^d$ , a configuration  $\sigma \in \Omega_{\partial_\downarrow \Lambda}$  will be referred to as a *boundary condition* for  $\Lambda$ . If  $\sigma$  contains no particles it will be referred to as *maximal boundary condition*. Finally, for any given boundary condition  $\sigma \in \Omega_{\partial_\downarrow \Lambda}$  and  $\omega \in \Omega_\Lambda$ , we will write  $\sigma \cdot \omega \in \Omega_{\partial_\downarrow \Lambda \cup \Lambda}$  for the configuration equal to  $\sigma$  on  $\partial_\downarrow \Lambda$  and to  $\omega$  on  $\Lambda$ .
- Given  $\Lambda \subset \mathbb{Z}_+^d$  we will write  $\mu_\Lambda$  for the product Bernoulli( $p$ ) measure on  $\Omega_\Lambda$  and  $\mu_\Lambda(f), \text{Var}_\Lambda(f)$  for the average and variance of  $f : \Omega_\Lambda \mapsto \mathbb{R}$  w.r.t.  $\mu_\Lambda$ . As for  $\Omega_\Lambda$ , if  $\Lambda = \mathbb{Z}_+^d$  we omit the subscript  $\Lambda$  from the notation.

### 1.2 The $d$ -dimensional East process

Given  $\Lambda \subset \mathbb{Z}_+^d$ ,  $\sigma \in \Omega_{\partial_\downarrow \Lambda}$  and  $\omega \in \Omega_\Lambda$ , define the *constraint*  $c_x^{\Lambda, \sigma}(\omega)$  at  $x \in \Lambda$  as

$$c_x^{\Lambda, \sigma}(\omega) = \begin{cases} 1 & \text{if either } x = 0 \text{ or } \exists e \in \mathcal{B} : x - e \in \partial_\downarrow \Lambda \cup \Lambda \text{ and } (\sigma \cdot \omega)(x - e) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 1.1.** Notice that the origin is *unconstrained*.

The infinitesimal generator  $\mathcal{L}_\Lambda^\sigma$  of the East process in  $\Lambda$  with vacancy density parameter  $q \in (0, 1)$  and boundary configuration  $\sigma$  has the form

$$\begin{aligned} \mathcal{L}_\Lambda^\sigma f(\omega) &= \sum_{x \in \Lambda} c_x^{\Lambda, \sigma}(\omega) [\omega_x q + (1 - \omega_x) p] \cdot [f(\omega^x) - f(\omega)] \\ &= \sum_{x \in \Lambda} c_x^{\Lambda, \sigma}(\omega) [\mu_x(f) - f](\omega), \end{aligned} \tag{1.1}$$

where  $\omega^x$  is the configuration in  $\Omega_\Lambda$  obtained from  $\omega$  by flipping its value at  $x$ . We refer the reader to [8]. As the local constraint  $c_x^{\Lambda, \sigma}(\cdot)$  does not depend on the state of the process at  $x$ ,  $\mu_\Lambda$  is a reversible measure. Actually, thanks to the orientation of the constraints a stronger property of local stationarity holds [11, Proposition 3.1] together with local exponential ergodicity (see [11, Theorem 4.1] and [19, Theorem 2.2]). When the initial law of the process is  $\nu$  we will write  $\mathbb{P}_\nu^{\Lambda, \sigma}(\cdot)$ ,  $\mathbb{E}_\nu^{\Lambda, \sigma}(\cdot)$  for the law and the associated expectation of the process. When  $\nu$  is the Dirac mass at one configuration  $\omega$  we will simply write  $\mathbb{P}_\omega^{\Lambda, \sigma}(\cdot)$  and  $\mathbb{E}_\omega^{\Lambda, \sigma}(\cdot)$ . The superscript  $\Lambda$  will be dropped from the notation if  $\Lambda = \mathbb{Z}_+^d$ . Similarly for the superscript  $\sigma$  if  $\partial_\downarrow \Lambda = \emptyset$ . Finally,  $\mathcal{D}_\Lambda^\sigma(f)$ ,  $f : \Omega_\Lambda \mapsto \mathbb{R}$  denotes the Dirichlet form of the process (i.e. the quadratic form of  $-\mathcal{L}_\Lambda^\sigma$ ). By construction,  $\mathcal{D}_\Lambda^\sigma(f) = \sum_{x \in \Lambda} \mu_\Lambda(c_x^{\Lambda, \sigma} \text{Var}_x(f))$ .

**Remark 1.2.** For  $d \geq 2$  and any integer  $d' \in [1, d - 1]$  the projection of the East process on  $\mathbb{Z}_+^d$  onto  $\mathbb{Z}_+^{d'} = \{x \in \mathbb{Z}_+^d : x_j = 0 \ \forall j > d'\}$  coincides with the East process on  $\mathbb{Z}_+^{d'}$ . Similarly, for any finite  $V \subset \mathbb{Z}_+^d$  and any box  $\Lambda \supset V$  the projection of the East process on  $\mathbb{Z}_+^d$  onto  $V$  coincides with the same projection of the East chain on  $\Lambda$ .

### 1.3 Structure of the paper

- In Section 2 we formulate the front evolution problem on the positive quadrant of  $\mathbb{Z}^d$  and state our main result as  $q \rightarrow 0$  on smallest/largest front velocity in a given direction (cf. Theorem 1). In turn, Theorem 1 implies the main result on the local equilibrium behind the front (cf. Theorem 2) together with the mixing time cutoff for the East chain on a box with sides along the coordinate axes (cf. Theorem 3).
- In Section 3 we develop the two main technical tools needed for the proof of the main results, namely a sharp lower bound on a suitable Dirichlet eigenvalue of the Markov generator (cf. section 3.1) and a bottleneck result (cf. Section 3.2).
- Section 4 is devoted to the proof of the three main theorems, while Section 5 contains the proof of Proposition 3.6, the key technical result from Section 3.
- Finally the Appendix contains the proof of a couple lemmas.

## 2 The front evolution problem and main result

Let  $\omega^* \in \Omega$  be the configuration identically to one and write  $\tau_x, x \in \mathbb{R}_+^d$ , for the hitting time of the set  $\{\omega : \omega_{\lfloor x} = 0\}$ . Sometimes we will refer to  $\tau_x$  as the *infection time* of  $x$ . More generally, for any  $A \subset \mathbb{Z}_+^d$  we will write  $\tau_A$  for the hitting time of the set  $\{\omega : \omega \upharpoonright_A \neq 1\}$ . Given a unit vector  $\mathbf{x} \in \mathbb{R}_+^d$ , it is known [11, Theorem 5.1] that for any  $q \in (0, 1)$

$$\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}}) = \Theta(n), \quad \text{as } n \rightarrow +\infty, \tag{2.1}$$

and that the mixing time of the East chain in  $\{0, \dots, n - 1\}^d$  is  $\Theta(n)$ . It is then natural to define

$$\frac{1}{v_{\max}(\mathbf{x})} = \liminf_{n \rightarrow \infty} \frac{\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})}{n}, \quad \frac{1}{v_{\min}(\mathbf{x})} = \limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})}{n},$$

and denote them as the maximal and minimal front velocity in the direction of  $x$  respectively. Using (2.1)  $0 < v_{\min}(\mathbf{x}) \leq v_{\max}(\mathbf{x}) < +\infty$  for all  $\mathbf{x}$ .

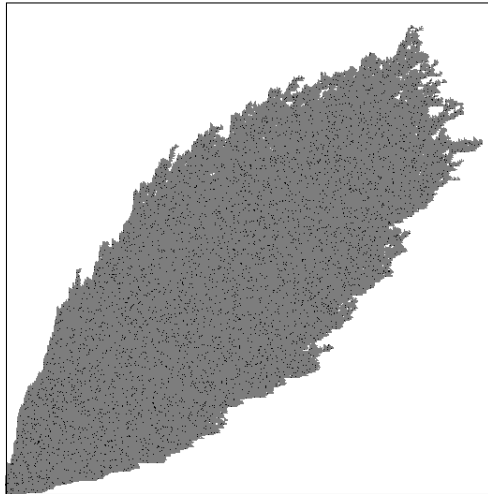


Figure 1: A simulation of the random set  $S(t)$  for  $q = 0.04$  suggesting the existence of a limit shape. The grey region corresponds to vertices that have been updated at least once before time  $t$ , while the black dots denote the actual infected sites at time  $t$ .

**Remark 2.1.** Using the strong Markov property and subadditivity, it is not difficult to see that  $\hat{v}(\mathbf{x})^{-1} := \lim_{n \rightarrow \infty} \max_{\omega} \mathbb{E}_{\omega}(\tau_{n\mathbf{x}})/n$  exists. Clearly  $v_{\min}(\mathbf{x}) \geq \hat{v}(\mathbf{x})$ .

In analogy with the classic *shape theorem* for e.g. first passage percolation (see e.g. [5]) we conjecture that  $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x}) := v(\mathbf{x})$  and in that case  $v(\mathbf{x})$  represents the *front velocity in the direction  $\mathbf{x}$* . Similarly, for any  $t > 0$  we could define the random set (see Fig. 1)

$$S(t) = \{x \in \mathbb{R}_+^d : \tau_x \leq t\},$$

and conjecture that there exists a compact subset  $\hat{S} \subset \mathbb{R}_+^d$  such that

$$\forall \epsilon > 0 \quad \lim_{t \rightarrow \infty} \mathbb{P}_{\omega^*}((1 - \epsilon)t\hat{S} \subseteq S(t) \subseteq (1 + \epsilon)t\hat{S}) = 1.$$

**Remark 2.2.** Using coupling arguments, it has been proved for  $d = 1$  [6] that  $\forall q \in (0, 1)$  the position  $\xi_t$  of the rightmost vacancy for the process started from  $\omega^*$  obeys a law of large numbers  $\lim_{t \rightarrow \infty} \xi_t/t = v$  a.s. and that the law of the East process to the left of  $\xi_t$  converges exponentially fast to a limiting law. A precise CLT for  $\xi_t$  was later proved in [16] together with a cutoff result for the mixing time in a finite interval. In particular, for  $d = 1$  both conjectures are known to be true. For  $d \geq 2$ , Remark 1.2 together with the law of large numbers in  $d = 1$  imply that  $v_{\max}(\mathbf{e}) = v_{\min}(\mathbf{e}) = v \forall \mathbf{e} \in \mathcal{B}$ . For all other directions both conjectures are still widely open.

In this paper, for any  $d \geq 2$  we provide a contribution towards the understanding of the front evolution problem as the vacancies equilibrium density  $q \rightarrow 0$ . Specifically, our main result concerns the small  $q$  behaviour of  $v_{\max}(\mathbf{x}), v_{\min}(\mathbf{x})$  as a function of  $\mathbf{x} \in \mathbb{R}_+^d$ . We will distinguish between the case in which the direction  $\mathbf{x}$  is fixed independent of  $q$  and all its coordinates are positive, and the case in which  $\mathbf{x} = \mathbf{x}(q)$  and  $\min_i x_i \rightarrow 0$  as  $q \rightarrow 0$ . In the first case we will say that  $\mathbf{x}$  points towards the *bulk of  $\mathbb{R}_+^d$* , while in the second case  $\mathbf{x}$  points to the *boundary of  $\mathbb{R}_+^d$* . In the sequel  $\theta_q := |\log_2 q|$  will be the relevant parameter.

**Theorem 1.** Fix  $d \geq 2$ .

(A) Let  $\mathbf{x} \in \mathbb{R}_+^d$  be a unit vector with  $\min_i x_i > 0$ . Then

$$\lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x})) = \lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x})) = \frac{1}{d}.$$

(B) Let  $0 < \beta < 1, \kappa \geq 1$  and let  $\{\mathbf{x}(q)\}_{q \in (0,1)}$  be a family of unit vectors in  $\mathbb{R}_+^d$  such that  $\max_{i,j} x_i(q)/x_j(q) \leq \kappa 2^{\beta \theta_q}$ . Then

$$\limsup_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) < 1.$$

(C) Assume  $d = 2$  and let  $\alpha > 0$ . Let  $\{\mathbf{x}(q)\}_{q \in (0,1)}$  be a family of unit vectors in  $\mathbb{R}_+^2$  such that  $\max_{i,j} x_i(q)/x_j(q) \geq 2^{\alpha \theta_q^2}$ . Then

$$\liminf_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x}(q))) \geq \frac{(1 + 4\alpha) \wedge 2}{2}.$$

Moreover, if  $\alpha > 1/4$  then

$$\lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x}(q))) = \lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x}(q))) = 1.$$

The same results apply to  $\hat{v}(\mathbf{x})$  defined in Remark 2.1.

**Remark 2.3.** Part (C) is presented here only for  $d = 2$  for simplicity. Remark 1.2 and the same proof ideas give similar, although more involved, results also for  $d \geq 3$ .

By combining (A) above together with Remark 1.2 we immediately get

**Corollary 1.** Fix  $d \geq 2$  and let  $\mathbf{x} \in \mathbb{R}_+^d$  be a unit vector such that  $\min_i x_i = 0$ . Then

$$\lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\max}(\mathbf{x})) = \lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(v_{\min}(\mathbf{x})) = \frac{1}{d(\mathbf{x})},$$

where  $d(\mathbf{x}) := \#\{i \in [d] : x_i > 0\}^2$ .

**Remark 2.4.** In order to better understand Theorem 1, let us recall a key feature of the East process on the *full lattice*  $\mathbb{Z}^d, d \geq 1$ . It is a reversible process with a positive spectral gap  $\gamma_d$  satisfying (see [1, 8] for  $d = 1$  and [12] for  $d \geq 2$ ):

$$\lim_{q \rightarrow 0} -\frac{2}{\theta_q^2} \log_2(\gamma_d) = 1/d.$$

Notice that  $\gamma_{d+1} = \gamma_d^{(1+o(1))d/(d+1)}$ . Then the three statements of the theorem can be interpreted respectively as follows:

- (A) if the direction  $\mathbf{x}$  points towards the *bulk* of  $\mathbb{R}_+^d$ , then  $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = \gamma_d^{1+o(1)}$ ;
- (B) if  $\mathbf{x} = \mathbf{x}(q)$  points to the boundary of  $\mathbb{R}_+^d$  *slowly enough* as  $q \rightarrow 0$ , then  $v_{\min}(\mathbf{x})$  is much larger than the velocity  $v(\mathbf{e}), \mathbf{e} \in \mathcal{B}$ , in any coordinate direction;
- (C) for  $d = 2$  if  $\mathbf{x} = \mathbf{x}(q)$  points to the boundary of  $\mathbb{R}_+^2$  *fast enough*, then  $v_{\max}(\mathbf{x})$  is much smaller than the minimal velocity associated to any direction pointing to the bulk of  $\mathbb{R}_+^2$  and, if sufficiently fast then  $v_{\max}(\mathbf{x}) = v_{\min}(\mathbf{x})^{1+o(1)} = v(\mathbf{e})^{1+o(1)}, \mathbf{e} \in \mathcal{B}$ .

<sup>2</sup>Here  $[n] := \{1, 2, \dots, n\}$  for any positive integer  $n$

**Remark 2.5.** Theorem 1 has been largely motivated by [12, Theorem 3]. There the authors considered  $\Lambda = \{0, \dots, L\}^d, \mathbb{N} \ni L \leq 2^{\theta_q/d}$ , and, using capacity methods combined with a sophisticated combinatorial analysis, analysed the asymptotic behaviour as  $q \rightarrow 0$  of the mean hitting time  $\mathbb{E}_{\omega^*}(\tau_x)$  for two special vertices:  $x_\Lambda = (L, \dots, L)$  and  $x'_\Lambda = (L, 0, \dots, 0)$ . One of the main outcomes was that for  $L = 2^{\theta_q/d}$  and as  $q \rightarrow 0$   $\mathbb{E}_{\omega^*}(\tau_{x'_\Lambda}) = \mathbb{E}_{\omega^*}(\tau_{x_\Lambda})^{d(1+o(1))}$ . In other words, for  $q$  small enough and at the length scale  $2^{\theta_q/d}$ , there is a big time scale separation between the two mean hitting times. The restriction  $L \leq 2^{\theta_q/d}$  was dictated by the need of having at equilibrium a constant number of vacancies in the box  $\Lambda$  and it was basically unavoidable.

Extending the analysis of the mean hitting time  $\mathbb{E}_{\omega^*}(\tau_x)$  to vertices  $x$  of the form  $x = nx$ , where  $x$  is any direction of  $\mathbb{R}_+^d$  and  $n \in \mathbb{N}$  is arbitrary, using capacity methods as in [12] seems prohibitive. Therefore, in order to prove Theorem 1 we must to appeal to large deviations combined with a fine analysis of certain principal Dirichlet eigenvalues of the process using the renormalization group ideas developed in [12]. The latter technique is illustrated in Section 3.1.

The second result analyses the law at time  $t \gg 0$  of the East process with initial condition  $\omega^*$ . It proves that for  $q$  small enough the region of  $\mathbb{Z}_+^d$  where the East process at time  $t$  has relaxed to the reversible measure  $\mu$  is extremely elongated in the bulk of  $\mathbb{Z}_+^d$  (see Fig. 1).

**Theorem 2.** Fix  $d \geq 2, 0 \leq \delta < 1$  and  $\varepsilon > 0$ . Let

$$\Lambda(\delta, \varepsilon, t) = \{x \in \mathbb{Z}_+^d : \min_{i,j} x_i/x_j \geq \delta \text{ and } \|x\|_1 \leq 2^{-\frac{\theta_q^2}{2d}(1+\varepsilon)} \times t\}, \quad t > 0,$$

and let  $\nu_t^{\delta, \varepsilon}$  be the marginal on  $\Omega_{\Lambda(\delta, \varepsilon, t)}$  of the law of the East process at time  $t$  with initial condition  $\omega^*$ . Then,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{q \rightarrow 0} \limsup_{t \rightarrow \infty} \|\nu_t^{\delta, \varepsilon} - \mu_{\Lambda(\delta, \varepsilon, t)}\|_{TV} = 0 \quad \text{if } \delta > 0, \tag{2.2}$$

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{q \rightarrow 0} \liminf_{t \rightarrow \infty} \|\nu_t^{\delta, \varepsilon} - \mu_{\Lambda(\delta, \varepsilon, t)}\|_{TV} = 1 \quad \text{if } \delta = 0. \tag{2.3}$$

**Remark 2.6.** A slightly more refined formulation of Theorem 2 avoiding the  $\limsup$  on  $\varepsilon, q$  would have been possible. However, we opted for the present version for simplicity.

Finally we analyse the *mixing time* (see e.g. [18]) of the East chain on the sequence of boxes  $\Lambda_n = \{0, \dots, n\}^d, d \geq 2$ . For  $q$  small enough and any  $n$  large enough we prove total variation cutoff – i.e. a sharp transition in mixing (see [3, 13] and references therein) – around the time

$$T_n = n/v, \tag{2.4}$$

where  $v$  is the front velocity along any coordinate direction  $e \in \mathcal{B}$  (see Remark 2.2). More precisely, let  $d_n(t) = \max_{\omega \in \Omega_{\Lambda_n}} \|\mathbb{P}_\omega^t(\cdot) - \mu_{\Lambda_n}\|_{TV}$ , where  $\mathbb{P}_\omega^t(\cdot)$  denotes the law at time  $t$  of the East process on  $\Lambda_n$  with initial condition  $\omega$ .

**Theorem 3.** There exists  $q_0 \in (0, 1)$  such that for any  $0 < q \leq q_0$

$$\lim_{\alpha \rightarrow \infty} \liminf_{n \rightarrow +\infty} d_n(T_n - \alpha\sqrt{n}) = 1 \tag{2.5}$$

$$\limsup_{n \rightarrow +\infty} d_n(T_n + n^{2/3}) = 0 \tag{2.6}$$

**Remark 2.7.** Above we didn't try to optimise the cutoff window size. Using [16, Theorem 2]  $T_n$  is the mixing time of the standard one dimensional East chain on the interval  $\{0, \dots, n\}$ . Hence, in a very precise sense, the one dimensional evolution along the coordinate axes dominates the mixing process of the multidimensional East chain in  $\Lambda_n$ .

Theorem 3 may look a bit surprising given that we don't know the existence of the front velocity in any direction  $\mathbf{x}$ . However, here we exploit the geometry of the boxes  $\Lambda_n$  together with the chosen boundary conditions for the East chain (only the origin is unconstrained), and the fact that for small  $q$  the front velocity along the coordinate axes is much smaller than the minimal velocity in any other direction pointing towards the bulk of  $\Lambda_n$  (cf. part A of Theorem 1). A cutoff result with e.g. a different choice of the geometry of  $\Lambda_n$  or of the boundary conditions (e.g. any vertex on the coordinate axes is unconstrained) would require proving at least the existence of the front velocity.

### 3 Two key tools

In this section we describe the two main tools that we use in order to get upper and lower bounds on  $v_{\max}(\mathbf{x}), v_{\min}(\mathbf{x})$ .

#### 3.1 Lower bounds on a Dirichlet eigenvalue

In the sequel we adopt the following convention for the process on  $\Lambda \subset \mathbb{Z}_+^d$  with boundary condition  $\sigma$ . If either  $\sigma$  is absent because  $\partial_\downarrow \Lambda = \emptyset$  or  $\sigma \equiv 1$ , then the superscript  $\sigma$  is dropped from the notation. Given integers  $(L_1, \dots, L_d)$  the set  $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$  will be called the *box* with side lengths  $(L_1, \dots, L_d)$ . We will write  $x_\Lambda$  for the vertex  $(L_1, \dots, L_d)$ . Notice that  $\partial_\downarrow \Lambda = \emptyset$ . Given a box  $\Lambda$  with side lengths  $(L_1, \dots, L_d)$  the set  $x + \Lambda$  will be called the *box* with side lengths  $L_1, \dots, L_d$  and origin at  $x$ . Unless otherwise specified a box will always have its origin at  $x = 0$ .

Recall now that the origin is always unconstrained. Given a box  $\Lambda$  possibly depending on  $q$ , it is well known (see e.g. [2, Section 6]) that the hitting time  $\tau_{x_\Lambda}$  satisfies

$$\mathbb{P}_\mu(\tau_{x_\Lambda} > t) \leq e^{-\lambda^D(\Lambda)t}, \tag{3.1}$$

where

$$\lambda^D(\Lambda) = \inf\{\mathcal{D}_\Lambda(f)/\mu_\Lambda(f^2) : f : \Omega_\Lambda \mapsto \mathbb{R}, f \upharpoonright_{\{\omega: \omega_{x_\Lambda}=0\}} = 0\} \tag{3.2}$$

is the smallest eigenvalue for the Dirichlet problem

$$-\mathcal{L}_\Lambda f = \lambda f, \quad f \upharpoonright_{\{\omega: \omega_{x_\Lambda}=0\}} = 0.$$

A lower bound on  $\lambda^D(\Lambda)$  is obtained via the spectral gap  $\gamma(\Lambda) > 0$  of the East chain in  $\Lambda$ . Using  $\text{Var}_\Lambda(f) \geq q\mu_\Lambda(f^2)$  for all  $f$  such that  $f \upharpoonright_{\{\omega: \omega_{x_\Lambda}=0\}} = 0$ , we get immediately

$$\lambda^D(\Lambda) \geq q\gamma(\Lambda). \tag{3.3}$$

Using Lemma A.2 it follows that  $\gamma(\Lambda) = \gamma_{d=1}^{(1+o(1))}$  as soon as  $\max_i L_i \geq 2^{\theta_q}$  because of the slow relaxation process mode along the edges of  $\Lambda$  on the coordinate axes.

If  $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) = O(1)$  as  $q \rightarrow 0$ , (3.3) is a very pessimistic bound when  $d \geq 2$  because  $\lambda^D(\Lambda)$  should be mostly influenced by the *d-dimensional bulk dynamics* rather than by the *one dimensional dynamics* along the edges of  $\Lambda$ . In this case it is natural to conjecture that, to the leading order as  $q \rightarrow 0$ ,  $\lambda^D(\Lambda)$  is lower bounded by  $\gamma_d$ . In order to prove the conjecture the following provides a better bound than (3.3).

For any  $V \subset \mathbb{Z}_+^d$  let  $\gamma(V)$  be the spectral gap of the East chain in  $V$  with boundary conditions identically equal to 1 on  $\partial_\downarrow V$ .

**Claim 3.1.**

$$\begin{aligned} \lambda^D(\Lambda) &\geq \max\{\lambda^D(V) : V \subseteq \Lambda, V \supset \{0, x_\Lambda\}\} \\ &\geq q \max\{\gamma(V) : V \subseteq \Lambda, V \supset \{0, x_\Lambda\}\} > 0. \end{aligned} \tag{3.4}$$

*Proof of the claim.* Clearly  $\max\{\gamma(V) : V \subseteq \Lambda, V \supset \{0, x_\Lambda\}\} \geq \gamma(\Lambda) > 0$ . Now fix  $\Lambda \supseteq V \ni \{0, x_\Lambda\}$  together with  $f$  such that  $f \upharpoonright_{\{\omega: \omega_{x_\Lambda}=0\}} = 0$ , and observe that monotonicity in the constraints implies that

$$\mathcal{D}_\Lambda(f) \geq \sum_{\omega \in \Omega_{\Lambda \setminus V}} \mu_{\Lambda \setminus V}(\omega) \mathcal{D}_V(f(\omega \cdot)).$$

Since  $V \ni x_\Lambda$ , for any  $\omega \in \Omega_{\Lambda \setminus V}$  the function  $\Omega_V \ni \omega' \mapsto f(\omega \cdot \omega')$  vanishes if  $\omega'_{x_\Lambda} = 0$ . Therefore, (3.2) implies that for any  $\omega \in \Omega_{\Lambda \setminus V}$

$$\mathcal{D}_V(f(\omega \cdot)) \geq \lambda^D(V) \mu_V(f^2(\omega \cdot)).$$

By averaging over  $\omega$  both sides of the above inequality w.r.t.  $\mu_{\Lambda \setminus V}(\omega)$  we conclude that  $\mathcal{D}_\Lambda(f) \geq \lambda^D(V) \mu_\Lambda(f^2)$  and the first inequality of the claim follows. The second inequality follows from the general inequality (3.3).  $\square$

In order to bound from below the r.h.s. of (3.4) according to whether  $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) = O(1)$  as  $q \rightarrow 0$  or not, it is convenient to introduce the following geometrical definition.

**Definition 3.2.** Fix  $d \geq 2, \beta \geq 0$ , and  $\kappa \geq 1$ . For any given  $q \in (0, 1)$  let  $S(\beta, \kappa; \theta_q)$  be the collection of  $d$ -tuple of integers  $(L_1, \dots, L_d)$  such that  $\max_{i,j}(L_i \vee 1)/(L_j \vee 1) \leq \kappa 2^{\beta \theta_q}$ . We say that a box  $\Lambda$  with side lengths  $(L_1, \dots, L_d)$  is  $(\beta, \kappa; \theta_q)$ -outstretched if  $(L_1, \dots, L_d) \in S(\beta, \kappa; \theta_q)$ , i.e. the maximum aspect ratio between its sides does not exceed  $\kappa 2^{\beta \theta_q}$ . Notice that  $S(\beta, \kappa; \theta_q) \subseteq S(\beta', \kappa; \theta_q)$  if  $\beta \leq \beta'$ .

**Remark 3.3.** Although the class of  $(\beta, \kappa; \theta_q)$ -outstretched boxes contains very regular boxes, e.g. cubes, our focus will be on the most extreme cases where the aspect ratio between the box's sides is close to  $\kappa 2^{\beta \theta_q}$ .

In the sequel, the parameters  $\beta, \kappa$  will always be chosen independent of  $q$ . Moreover, whenever the value of  $q$  is understood we will simply write  $(\beta, \kappa)$ -outstretched instead of  $(\beta, \kappa; \theta_q)$ -outstretched.

**Definition 3.4.** Given  $\beta \geq 0$  we say that  $\lambda > 0$  satisfies condition  $\mathcal{H}(\beta)$  and write  $\lambda \sim \mathcal{H}(\beta)$  if for any  $\kappa \geq 1, \varepsilon > 0$  there exists  $q(\beta, \kappa, \varepsilon) > 0$  such that  $\forall q \leq q(\beta, \kappa, \varepsilon)$  the following occurs:  $\forall (\beta, \kappa; \theta_q)$ -outstretched box  $\Lambda \exists V \subset \Lambda$  with  $V \supset \{0, x_\Lambda\}$  such that  $\gamma(V) \geq 2^{-(1+\varepsilon)\lambda \frac{\theta_q^2}{2}}$ . We then let  $\phi(\beta; d) = \min\{\lambda > 0 : \lambda \sim \mathcal{H}(\beta)\}$ .

**Remark 3.5.** For  $d = 1$  any box  $\Lambda_L = \{0, 1, \dots, L\}$ , is  $(\beta, \kappa)$ -outstretched for all  $\beta \geq 0, \kappa \geq 1$ . Therefore,  $\phi(\beta; 1) = 1$  because  $\inf_L \gamma(\Lambda_L) = 2^{-\frac{\theta_q^2}{2}(1+o(1))}$  [8].

Thus, if  $\lambda \sim \mathcal{H}(\beta)$  then Claim 3.1 implies that for all  $\varepsilon > 0$  the Dirichlet eigenvalue  $\lambda^D(\Lambda)$  is greater than  $2^{-(1+\varepsilon)\lambda \frac{\theta_q^2}{2}}$  for all  $(\beta, \kappa; \theta_q)$ -outstretched box  $\Lambda$  and for all  $q$  small enough depending only on  $\beta, \kappa, \varepsilon$ . In particular,

$$\lambda^D(\Lambda) \geq 2^{-(1+\varepsilon)\phi(\beta; d) \frac{\theta_q^2}{2}}. \tag{3.5}$$

A major problem is then to bound the constant  $\phi(\beta; d)$  for  $d \geq 2$ . Lemma A.2 implies that  $\phi(\beta, d) \leq 1$ . The next result, which in a sense represents the technical core of the paper and whose proof is deferred to Section 5, goes beyond this bound.

**Proposition 3.6.** For  $d \geq 2$  the coefficient  $\phi(\beta; d)$  satisfies:

- (i)  $\phi(0; d) = 1/d;$
- (ii)  $\phi(\beta; d) < 1 \quad \forall \beta \in (0, 1);$
- (iii)  $\phi(\beta; d) = 1 \quad \forall \beta \geq 1.$



In particular, for any  $d \geq 2$  and any  $(\beta, \kappa)$ -outstretched box  $\Lambda$  with  $\beta < 1$  the Dirichlet eigenvalue  $\lambda^D(\Lambda) \gg \gamma_{d=1}$  as  $q \rightarrow 0$ .

A first consequence for the hitting times  $\tau_x$ ,  $x \in \mathbb{Z}_+^d$ , is provided by the next result.

**Lemma 3.7.** Fix  $\varepsilon > 0, \beta \geq 0, \kappa \geq 1$ . Then there exists  $q(\varepsilon, \beta, \kappa)$  such that for any  $q \leq q(\varepsilon, \beta, \kappa)$  and any  $\Lambda = \Lambda_q$  a  $(\beta, \kappa; \theta_q)$ -outstretched box of side lengths  $(L_1, \dots, L_d)$  satisfying  $2^{\theta_q^{3/2}}/2 \leq \min_i L_i \leq 2^{\theta_q^{3/2}}$ , the following holds:

$$\sup_{x \in \mathbb{Z}_+^d} \sup_{\omega \in \{\omega : \omega_x = 0\}} \mathbb{E}_\omega(\tau_{x+x_\Lambda}) \leq 2^{(1+\varepsilon)\phi(\beta;d)\frac{\theta_q^2}{2}}.$$

*Proof.* Fix  $x \in \mathbb{Z}_+^d, \varepsilon > 0$  and let  $T(\varepsilon) = 2^{(1+\varepsilon)\phi(\beta;d)\frac{\theta_q^2}{2}}, T^* = 2^{2\theta_q^2}$ . Then

$$\begin{aligned} \mathbb{E}_\omega(\tau_{x+x_\Lambda}) &= \int_0^{T(\varepsilon)} dt \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) + \int_{T(\varepsilon)}^{T^*} dt \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) + \int_{T^*}^{+\infty} dt \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) \\ &\leq T(\varepsilon) + T^* \mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon)) + \int_{T^*}^{+\infty} dt \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t). \end{aligned} \tag{3.6}$$

We will now prove that the supremum over  $\omega \in \{\omega : \omega_x = 0\}$  of the second and third term in the r.h.s. of (3.6) tend to zero as  $q \rightarrow 0$ . We first need the following general bound whose proof will be provided shortly.

**Lemma 3.8.** There exist positive constants  $c, c'$  independent of  $q$  such that the following holds. Fix  $\ell \in \mathbb{N}$  and for  $x \in \mathbb{Z}_+^d$  write  $V_{x,\ell} = \{x_1 - \ell, \dots, x_1\} \times \dots \times \{x_d - \ell, \dots, x_d\} \cap \mathbb{Z}_+^d$ . Then for any box  $\Lambda$  with side lengths  $(L_1, \dots, L_d)$  and any  $t > 0$  it holds that

$$\sup_{\omega : \omega_x = 0} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) \leq c' t \ell^d e^{-c q \ell} + 2^{\theta_q(\ell + \max_i L_i)^d - t \ell^{-d} \min_{y \in V_{x,\ell}} \lambda^D(\Lambda_y)}, \tag{3.7}$$

where  $\Lambda_y = \{y_1, \dots, y_1 + L_1\} \times \dots \times \{y_d, \dots, y_d + L_d\}$ .

**Remark 3.9.** The length scale  $\ell$  in the lemma is a free parameter that in the applications we will suitably choose depending on  $x, t, \Lambda$ .

Consider now the second term in the r.h.s. of (3.6). In this case we apply Lemma 3.8 with  $t = T(\varepsilon)$  and  $\ell = \lfloor \frac{1}{2} \min_i L_i \rfloor$  to bound from above  $\mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon))$ . The assumption  $\min_i L_i = \Theta(2^{\theta_q^{3/2}})$  and the choice of  $\ell$  imply that the first term in the r.h.s. of (3.7) after multiplication by  $T^*$  is  $o(1)$  as  $q \rightarrow 0$ . Moreover, the fact that  $\Lambda$  is  $(\beta, \kappa)$ -outstretched implies that  $\Lambda + y$  is  $(\beta, \kappa + 1)$ -outstretched for any  $y \in V_{x,\ell}$ . In particular, for all  $q$  small enough depending only on  $\varepsilon, \beta, \kappa$ , and for any  $y \in V_{x,\ell}$

$$\lambda^D(\Lambda_y) \geq 2^{-(1+\varepsilon/2)\phi(\beta;d)\frac{\theta_q^2}{2}}. \tag{3.8}$$

Hence, as  $q \rightarrow 0$

$$T^* \times (\text{the second term in the r.h.s. of (3.7)}) \leq 2^{2\theta_q^2 + \theta_q} 2^{O(\theta_q^{3/2})} e^{-2^\varepsilon \phi(\beta;d)\frac{\theta_q^2}{4}} = o(1).$$

We finally consider the third term in the r.h.s. of (3.6). In this case, for any  $t > T^*$  we apply (3.8) with  $\ell = \ell_t = t^{1/4d}$ . Observe that for some  $y \in V_{x,\ell}$  the box  $\Lambda_y$  could be extremely outstretched in some direction preventing us from using Proposition 3.6. Hence we are forced to use the spectral gap bound (2.2)

$$\min_{y \in V_{x,\ell}} \lambda^D(\Lambda_y) \geq 2^{-(1+\varepsilon)\frac{\theta_q^2}{2}}$$

to get that for any  $t \geq T^*$

$$\begin{aligned} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) &\leq c't\ell_t^d e^{-cqt^{1/4d}} + e^{O(\theta_q)t^{1/4} - t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}} \\ &\leq c't^{5/4} e^{-cqt^{1/4d}} + e^{-t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}/2}. \end{aligned}$$

It now suffices to observe that

$$\int_{T^*}^{+\infty} dt [c't^{5/4} e^{-cqt^{1/4d}} + e^{-t^{3/4}2^{-\frac{\theta_q^2}{2}(1+\varepsilon)}/2}] = o(1) \quad \text{as } q \rightarrow 0. \quad \square$$

*Proof of Lemma 3.8.* Given  $\ell \in \mathbb{N}$  and  $x \in \mathbb{Z}_+^d$  let  $\mathcal{G}(t, \ell), t > 0$ , be the event that there exists  $z \in V_{x, \ell}$  such that

$$\mathcal{T}_t(z) = \int_0^t ds 1_{\{c_z(\omega(s))=1\}} > t/\ell^d.$$

In other words  $z$  is unconstrained for a fraction  $\ell^{-d}$  of the time  $t$ . When such a vertex exists we will write  $\xi \in V_{x, \ell}$  for the smallest one in the lexicographical order. In [11, Corollary 4.2] it has been proved that there exist constants  $c, c' > 0$  such that

$$\sup_{\omega \in \{\omega_x=0\}} \mathbb{P}_\omega(\mathcal{G}(t, \ell)^c) \leq c't\ell^d e^{-cql}. \quad (3.9)$$

**Remark 3.10.** If  $t$  is so large that  $V_{x, \ell_t}$  coincides with the box of side lengths  $(x_1, \dots, x_d)$ , then the event  $\mathcal{G}(t, \ell_t)^c = \emptyset$  because the origin is always unconstrained.

Thus, for any  $\omega$  such that  $\omega_x = 0$ ,

$$\mathbb{P}_\omega(\tau_{x+x_\Lambda} > t) \leq c't\ell^d e^{-cql} + \mathbb{P}_\omega(\tau_{x+x_\Lambda} > T(\varepsilon); \mathcal{G}(x, \ell)).$$

Recall that  $\Lambda_y = \{y_1, \dots, x_1 + L_1\} \times \dots \times \{y_d, \dots, x_d + L_d\}$  and let  $\mathcal{F}_{y, t}$  be the  $\sigma$ -algebra generated by the variables  $\{\omega_z(s) : z \in \partial_\downarrow \Lambda_y, s \leq t\}$ . Notice that  $\{c_y(\omega(s))\}_{s \leq t}$  is measurable w.r.t.  $\mathcal{F}_{y, t}$  so that

$$\begin{aligned} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t; \mathcal{G}(x, \ell)) &= \sum_{y \in V_{x, \ell}} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t; \xi = y) \\ &= \mathbb{E}_\omega(1_{\{\xi=y\}} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t | \mathcal{F}_{y, t})). \end{aligned}$$

The orientation of the East process implies that, conditionally on  $\mathcal{F}_{y, t}$ , the event  $\{\tau_{x+x_\Lambda} > t\}$  coincides with the same event for the *time-inhomogeneous* East chain in  $\Omega_{\Lambda_y}$  with *deterministic, time-dependent* boundary conditions on  $\partial_\downarrow \Lambda_y$ . We denote the law of the latter chain with initial state  $\omega \upharpoonright_{\Lambda_y}$  by  $\hat{\mathbb{P}}_\omega(\cdot)$ . Thus,

$$\begin{aligned} \mathbb{P}_\omega(\tau_{x+x_\Lambda} > t | \mathcal{F}_{y, t}) &= \hat{\mathbb{P}}_\omega(\tau_{x+x_\Lambda} > t) \\ &\leq \mu(\omega \upharpoonright_{\Lambda_y})^{-1} \sum_{\eta \in \Omega_{\Lambda_y}} \mu(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > t) \\ &\leq 2^{\theta_q |\Lambda_y|} \sum_{\eta \in \Omega_{\Lambda_y}} \mu(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > t). \end{aligned} \quad (3.10)$$

Let now  $t_0 \equiv 0 < t_1 < t_2 < \dots < t_n < t_{n+1} \equiv t$  be the times at which the boundary conditions on  $\partial_\downarrow \Lambda_y$  change and let  $\sigma^{(i)}$  denote the boundary condition during the time interval  $(t_{i-1}, t_i)$ . Let also  $\hat{\mathcal{L}}^{(i)}$  be the generator of the East chain on  $\Omega_{\Lambda_y}$  with boundary

conditions  $\sigma^{(i)}$  and let  $\mathcal{A}^{(i)} = 1_{A^c} \hat{\mathcal{L}}^{(i)} 1_{A^c}$  be the generator  $\hat{\mathcal{L}}^{(i)}$  with Dirichlet boundary condition on  $A = \{\eta \in \Omega_{\Lambda_y} : \eta_{x+x_\Lambda} = 0\}$ . Then,

$$\sum_{\eta \in \Omega_{\Lambda_y}} \mu_{\Lambda_y}(\eta) \hat{\mathbb{P}}_\eta(\tau_{x+x_\Lambda} > t) = \langle \mathbf{1}, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2-t_1) \mathcal{A}^{(2)}} \times \dots \times e^{(t_{n+1}-t_n) \mathcal{A}^{(n+1)}} \mathbf{1} \rangle,$$

where  $\mathbf{1}(\eta) = 1 \forall \eta \in \Omega_{\Lambda_y}$  and  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\ell^2(\Omega_{\Lambda_y}, \mu_{\Lambda_y})$ . Let  $\lambda_i \geq 0$  be the smallest eigenvalue of  $-\mathcal{A}^{(i)}$ . Clearly,

$$\langle \mathbf{1}, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2-t_1) \mathcal{A}^{(2)}} \times \dots \times e^{(t_{n+1}-t_n) \mathcal{A}^{(n+1)}} \mathbf{1} \rangle \leq e^{-\sum_{i=1}^{n+1} (t_i - t_{i-1}) \lambda_i}. \tag{3.11}$$

If during the time interval  $(t_i, t_{i+1})$  the constraint  $c_y$  at the vertex  $y$  is zero then we simply use  $\lambda_i \geq 0$ . If instead  $c_y = 1$  we use monotonicity of  $\lambda_i$  in the boundary conditions  $\sigma^{(i)}$  to write  $\lambda_i \geq \lambda^D(\Lambda_y)$ . Thus, recalling that  $\int_0^t ds 1_{\{c_y=1\}} \geq t\ell^{-d}$ , we get

$$\begin{aligned} & \langle \mathbf{1}, e^{t_1 \mathcal{A}^{(1)}} \times e^{(t_2-t_1) \mathcal{A}^{(2)}} \times \dots \times e^{(t_{n+1}-t_n) \mathcal{A}^{(n+1)}} \mathbf{1} \rangle \\ & \leq e^{-\lambda^D(\Lambda_y) \int_0^t ds 1_{\{c_y=1\}}} = e^{-t\ell^{-d} \lambda^D(\Lambda_y)}. \end{aligned}$$

In conclusion,

$$\mathbb{P}_\omega(\tau_{x+x_\Lambda} > t; \mathcal{G}(x, \ell)) \leq 2^{\theta_q |\Lambda_y| - t\ell^{-d} \lambda^D(\Lambda_y)} \leq 2^{\theta_q (\ell + \max_i L_i)^d - t\ell^{-d} \lambda^D(\Lambda_y)}$$

and the statement of the lemma follows. □

### 3.2 A bottleneck on scale $2^{\frac{\theta_q}{d}}$

**Definition 3.11** (Legal updates and legal path). Consider  $\Lambda \subseteq \mathbb{Z}_+^d$  together with a boundary condition  $\sigma$  for  $\Lambda$  if  $\Lambda \neq \mathbb{Z}_+^d$ . Given  $\omega \in \Omega_\Lambda$  and  $x \in \Lambda$  we say that the update  $\omega \rightarrow \omega^x$  is  $\sigma$ -legal iff  $c_x^{\Lambda, \sigma}(\omega) = 1$ . A sequence  $(\omega^{(1)}, \dots, \omega^{(n)})$  of configurations in  $\Omega_\Lambda$  such that  $\omega^{(i+1)}$  is obtained from  $\omega^{(i)}$  by means of a (non-trivial)  $\sigma$ -legal update will be referred to as a  $\sigma$ -legal path in  $\Omega_\Lambda$  joining  $\omega^{(1)}$  to  $\omega^{(n)}$ . When  $\Lambda = \mathbb{Z}_+^d$  and  $\sigma$  is missing we will simply write legal update and legal path.

Before discussing the core of this section, we point out the following *monotonicity property of legal updates*. Take two sets  $\Lambda \subset \Lambda' \subset \mathbb{Z}_+^d$  together with two boundary conditions  $\sigma, \sigma'$  on  $\partial_\downarrow \Lambda$  and  $\partial_\downarrow \Lambda'$  respectively such that  $\sigma_x = 0 \forall x \in \partial_\downarrow \Lambda \cap \Lambda'$  and  $\sigma'_x \leq \sigma_x \forall x \in \partial_\downarrow \Lambda \cap \partial_\downarrow \Lambda'$ . Then any  $\sigma'$ -legal update inside  $\Lambda$  is also a  $\sigma$ -legal update.

**Definition 3.12** (Bottleneck). Let  $\Lambda_L = \{0, \dots, L\}^d$ , and for  $x \in \mathbb{Z}_+^d \setminus \Lambda_L$  let  $V_{x,L} = (\Lambda_L + x - x_{\Lambda_L}) \cap \mathbb{Z}_+^d$ . We say that  $A \subset \Omega_{V_{x,L}}$  is an  $(x, L)$ -bottleneck if any legal path in  $\Omega$  joining  $E_{x,L} \equiv \{\omega \in \Omega : \omega|_{V_{x,L}} = 1\}$  with  $\{\omega : \omega_x = 0\}$  hits  $\{\omega : \omega|_{V_{x,L}} \in A\}$ .

**Proposition 3.13.** In the setting of Definition 3.12 for any  $\varepsilon > 0$  there exists  $q(\varepsilon) > 0$  such that for  $q \leq q(\varepsilon)$  the following holds. For any  $L \leq 2^{\theta_q/d}$  and  $x \in \mathbb{Z}_+^d \setminus \Lambda_L$  there exists a  $(x, L)$ -bottleneck  $A$  with  $\mu(A) \leq 2^{-(n\theta_q - d\binom{n}{2})(1-\varepsilon)}$  where  $n := \lfloor \log_2(L) \rfloor$ .

*Proof.* Fix  $\varepsilon > 0, L \leq 2^{\theta_q/d}$  and  $x \in \mathbb{Z}_+^d \setminus \Lambda_L$ , and w.l.o.g. suppose that  $V_{x,L} \subset \mathbb{Z}_+^d$ . The case when this assumption fails follows immediately from the monotonicity property of legal updates described above. Fix a legal path  $\Gamma = (\omega^{(1)}, \dots, \omega^{(k)})$  in  $\Omega$  such that  $\omega^{(1)} \in E_{x,L}$  and  $\omega_x^{(k)} = 0$ . Finally, write  $\omega_V^{(j)}$  for the restriction to  $V_{x,L}$  of  $\omega^{(j)}$  and let  $1 \leq j_1 < j_2 < \dots < j_m \leq k$  be those indices such that the legal update connecting  $\omega^{(j_i)}$  to  $\omega^{(j_{i+1})}$  occurs inside  $V_{x,L}$ . Let  $\sigma_{\max}$  denotes the *maximal* boundary condition for  $V_{x,L}$ . Using the monotonicity of legal updates, the sequence  $\hat{\Gamma} = (\omega_V^{(j_1)}, \dots, \omega_V^{(j_m)})$  is a  $\sigma_{\max}$ -legal path in  $\Omega_{V_{x,L}}$  connecting the configuration in  $\Omega_{V_{x,L}}$  with no vacancies to  $\{\omega \in \Omega_{V_{x,L}} : \omega_x = 0\}$ . The results of [12, Section 4] imply that  $\hat{\Gamma}$  must hit a fixed subset  $A$  of  $\Omega_{V_{x,L}}$  (called  $\partial A_*$  there) whose equilibrium probability satisfies the required bound. □

**Corollary 3.14.** *In the same setting*

$$\max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_x < t) \leq O(t) \times 2^{-(n\theta_q - d \binom{n}{2})(1-\varepsilon)}.$$

Notice that for  $L = 2^{\theta_q/d}$  the r.h.s. above becomes equal to  $O(t) \times 2^{-\frac{\theta_q^2}{2d}(1-\varepsilon)}$ .

*Proof.* We only give a quick sketch because the proof of similar statements has already appeared elsewhere (see e.g. [10]). Fix  $L \leq 2^{\theta_q/d}$  and  $x \in \mathbb{Z}_+^d \setminus \Lambda_L$ . Using Proposition 3.13 there exists  $A \subset \Omega_{V_{x,L}}$  such that

$$\max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_x < t) \leq \max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_A \leq t).$$

For a given  $\omega \in \Omega_{V_{x,L}}^c$  write  $\delta_\omega \otimes \mu_{V_{x,L}}$  for the product measure on  $\Omega$  whose marginals on  $\Omega_{V_{x,L}}^c \otimes \Omega_{V_{x,L}}$  are the Dirac mass at  $\omega$  and  $\mu_{V_{x,L}}$  respectively. Using  $L \leq 2^{\theta_q/d}$  we get that  $\mu_{V_{x,L}}(\omega \upharpoonright_{V_{x,L}} = 1)^{-1} = O(1)$  as  $q \rightarrow 0$ . Hence,

$$\begin{aligned} \max_{\omega \in E_{x,L}} \mathbb{P}_\omega(\tau_A \leq t) &\leq O(1) \times \max_{\omega \in \Omega_{V_{x,L}}^c} \mathbb{P}_{\delta_\omega \otimes \mu_{V_{x,L}}}(\tau_A \leq t) \\ &\leq O(tL^d) \max_{\omega \in \Omega_{V_{x,L}}^c} \sup_{s \leq t} \mathbb{P}_{\delta_\omega \otimes \mu_{V_{x,L}}}(\omega(s) \upharpoonright_{V_{x,L}} \in A). \end{aligned}$$

It is easy to check (see [11, Section 3]) that  $\mu_{V_{x,L}}$  is stationary for the marginal on  $\Omega_{V_{x,L}}$  of the East process with initial distribution  $\delta_\omega \otimes \mu_{V_{x,L}}$ . Hence, the r.h.s. above is equal to  $O(tL^d)\mu(A) \leq O(t)2^{-(n\theta_q - d \binom{n}{2})(1-2\varepsilon)}$  for  $q$  small enough depending on  $\varepsilon$ .  $\square$

## 4 Proof of Theorems 1, 2, and 3

### 4.1 Proof of Theorem 1: (A)

In the sequel  $\mathbf{x} \in \mathbb{R}_+^d$  will denote a unit vector independent of  $q$  with  $\min_i \mathbf{x}_i > 0$ .

#### 4.1.1 Lower bound on $v_{\min}(\mathbf{x})$

Let  $\ell = \lfloor 2^{\theta_q^{3/2}} \rfloor$  and let  $x^{(n)} = \lfloor n\ell\mathbf{x} \rfloor$ ,  $n \in \mathbb{N}$ . We begin by proving that

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{\omega^*}(\tau_{x^{(n)}})}{n} \leq 2^{\frac{\theta_q^2}{2d}(1+o(1))} \quad \text{as } q \rightarrow 0. \tag{4.1}$$

Clearly

$$\tau_{x^{(n+1)}} \leq \inf\{s \geq \tau_{x^{(n)}} : \omega_{x^{(n+1)}}(s) = 0\},$$

so that, using the strong Markov property,

$$\mathbb{E}_{\omega^*}(\tau_{x^{(n+1)}}) \leq \mathbb{E}_{\omega^*}(\tau_{x^{(n)}}) + \max_{\omega \in \{\omega : \omega_{x^{(n)}} = 0\}} \mathbb{E}_\omega(\tau_{x^{(n+1)}}).$$

Let  $L_i = x_i^{(n+1)} - (x_i^{(n)} + 1)$ ,  $i \in [d]$ . Clearly the box with sides length  $(L_1, \dots, L_d)$  is  $(0, \kappa)$ -outstretched with  $\kappa = \max_{i,j} \mathbf{x}_i/\mathbf{x}_j + 1$  and Lemma 3.7 implies that, uniformly in  $n$ , for any  $\varepsilon > 0$

$$\max_{\omega \in \{\omega : \omega_{x^{(n)}} = 0\}} \mathbb{E}_\omega(\tau_{x^{(n+1)}}) \leq 2^{\frac{\theta_q^2}{2d}(1+\varepsilon)}, \tag{4.2}$$

for any  $q$  sufficiently small depending on  $\varepsilon$ . Equation (4.1) now follows immediately.

In order to complete the proof of (A) we write

$$\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}}) \leq \mathbb{E}_{\omega^*}(\tau_{x^{(\lfloor n/\ell \rfloor)}}) + \max_{\omega \in \{\omega : \omega_{x^{(\lfloor n/\ell \rfloor)}} = 0\}} \mathbb{E}_\omega(\tau_{n\mathbf{x}}).$$

By using the arguments entering into the proof of Lemma 3.7 it is easy to see that  $\sup_n \max_{\omega \in \{\omega: \omega_{\lfloor n/\ell \rfloor} = 0\}} \mathbb{E}_\omega(\tau_{n\mathbf{x}}) < +\infty$ . Therefore

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{\omega^*}(\tau_{n\mathbf{x}})}{n} \leq \ell^{-1} 2^{\frac{\theta_q^2}{2d}(1+o(1))} = 2^{\frac{\theta_q^2}{2d}(1+o(1))},$$

because of the choice of  $\ell$ . In conclusion we have proved that  $v_{\min}(\mathbf{x}) \geq 2^{-\frac{\theta_q^2}{2d}(1+o(1))}$  as  $q \rightarrow 0$ .  $\square$

#### 4.1.2 Upper bound on $v_{\max}(\mathbf{x})$

For any  $y \in \mathbb{Z}_+^d$  and  $n \leq \|y\|_1$  let  $H_{y,n} = \{z : z \prec y, \|y - z\|_1 \leq n\}$ . Fix now  $y \in \mathbb{Z}_+^d$  with  $\|y\|_1 \geq \ell_q = \lfloor 2^{\theta_q/d} \rfloor$  and observe that if the starting configuration of the East process on  $\mathbb{Z}_+^d$  is  $\omega^*$ , then  $\tau_{\partial_\downarrow H_{y,\ell_q}} < \tau_y$  a.s. Hence, for all  $\lambda > 0$  the strong Markov property gives

$$\begin{aligned} \mathbb{E}_{\omega^*}(e^{-\lambda\tau_y}) &= \mathbb{E}_{\omega^*}(e^{-\lambda\tau_{\partial_\downarrow H_{y,\ell_q}}} \mathbb{E}_{\omega_{\tau_{\partial_\downarrow H_{y,\ell_q}}}}(e^{-\lambda\tau_y})) \\ &\leq W(\lambda) \sum_{z \in \partial_\downarrow H_{y,\ell_q}} \mathbb{E}_{\omega^*}(e^{-\lambda\tau_z}), \end{aligned} \tag{4.3}$$

where  $W(\lambda) := \sup_{z: \|z\|_1 \geq \ell} \max_{\omega \in \{\omega: \omega_{\uparrow H_{z,\ell_q}} = 1\}} \mathbb{E}_\omega(e^{-\lambda\tau_z})$ . Using  $|\partial_\downarrow W_{y,\ell_q}| \leq O(\ell^{d-1})$  we can iterate (4.3) to get that

$$\mathbb{E}_{\omega^*}(e^{-\lambda\tau_y}) \leq \left(O(\ell^{d-1})W(\lambda)\right)^{\lfloor \|y\|_1/\ell \rfloor}.$$

**Claim 4.1.** For any  $\varepsilon > 0$  sufficiently small let  $T(\varepsilon) = 2^{\frac{\theta_q^2}{2d}(1-\varepsilon)}$  and choose  $\lambda = \lambda(\varepsilon, q) = \varepsilon\theta_q^2 T(\varepsilon)^{-1}$ . Then  $W(\lambda(\varepsilon, q)) \leq e^{-\Omega(\varepsilon\theta_q^2)}$  as  $q \rightarrow 0$ .

*Proof of the claim.* Using Corollary 3.14, for any  $z$  with  $\|z\|_1 \geq \ell_q$  and any  $q$  small enough depending on  $\varepsilon$ , we get

$$\begin{aligned} \max_{\omega \in \{\omega: \omega_{\uparrow H_{z,\ell_q}} = 1\}} \mathbb{E}_\omega(e^{-\lambda\tau_z}) &\leq e^{-\lambda T(\varepsilon)} + \max_{\omega \in \{\omega: \omega_{\uparrow H_{z,\ell_q}} = 1\}} \mathbb{P}_\omega(\tau_z \leq T(\varepsilon)) \\ &\leq e^{-\varepsilon\theta_q^2} + O(T(\varepsilon))2^{-\frac{\theta_q^2}{2d}(1-\varepsilon/2)} = e^{-\Omega(\varepsilon\theta_q^2)}. \end{aligned} \quad \square$$

Using  $e^{-\lambda\mathbb{E}_{\omega^*}(\tau_y)} \leq \mathbb{E}_{\omega^*}(e^{-\lambda\tau_y})$  and choosing  $\lambda$  as in the claim, we finally obtain

$$\mathbb{E}_{\omega^*}(\tau_y) \geq \Omega(2^{\frac{\theta_q^2}{2d}(1-\varepsilon)}) \lfloor 2^{-\theta_q/d} \|y\|_1 \rfloor. \tag{4.4}$$

In particular, (4.4) implies that  $v_{\max}(\mathbf{x}) \leq 2^{-\frac{\theta_q^2}{2d}(1-o(1))}$  as  $q \rightarrow 0$ .  $\square$

**Remark 4.2.** Exactly the same proof applies to get the following result. For any  $\varepsilon > 0$  there exists  $q(\varepsilon) > 0$  and  $c(\varepsilon) > 0$  such that the following holds for  $q \leq q(\varepsilon)$ . For any  $y \in \mathbb{Z}_+^d$  and  $n \leq \|y\|_1$

$$\max_{\omega: \omega_{\uparrow H(y,n)} = 1} \mathbb{P}_\omega(\tau_y \leq nT(\varepsilon)) \leq e^{-c\varepsilon\theta_q^2 \lfloor n2^{-\frac{\theta_q}{d}} \rfloor}.$$

**4.2 Proof of Theorem 1: (B)**

The proof is identical to that of Section 4.1 with the following modification. The box  $\Lambda$  with side lengths  $L_i = x_i^{(n+1)} - (x_i^{(n)} + 1), i \in [d]$ , is now  $(\beta, \kappa + 1)$ -outstretched because of the assumption on the direction  $x = x(q)$ . Using again Lemma 3.7 we get the analogue of (4.2):

$$\max_{\omega \in \{\omega: \omega_{x^{(n)}} = 0\}} \mathbb{E}_\omega(\tau_{x^{(n+1)}}) \leq 2^{\phi(\beta; d) \frac{\theta^2}{2} (1+\varepsilon)}. \tag{4.5}$$

The rest of the argument remains unchanged and the conclusion is that

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}_{\omega^*}(\tau_{nx})}{n} \leq \ell^{-1} 2^{\phi(\beta; d) \frac{\theta^2}{2} (1+\varepsilon)},$$

i.e.

$$\limsup_{q \rightarrow 0} -\frac{1}{\theta^2 q} \log_2(v_{\min}(x)) \leq \frac{\phi(\beta; d)}{2} < \frac{1}{2}$$

because  $\phi(\beta; d) < 1$  if  $\beta \in [0, 1)$ . □

**4.3 Proof of Theorem 1: (C)**

Fix a  $q$ -dependent unit vector  $\mathbf{x} \in \mathbb{R}_+^2$  such that  $0 < x_2 \leq x_1 2^{-\theta^2 \alpha}$  with  $\alpha > 0$ . In order to track how a vacancy can propagate from the origin to the vertex  $\lfloor n\mathbf{x} \rfloor \in \mathbb{Z}_+^2$  we introduce the following construction.

Let  $0 < \varepsilon \ll 1$  and let  $L = L(\varepsilon, \alpha, q) = \lfloor 2^{\theta^2 \alpha (1-\varepsilon/2)} \rfloor$ . W.l.o.g. we assume that  $q$  is so small that  $L \gg 2^{\theta^2}$ .

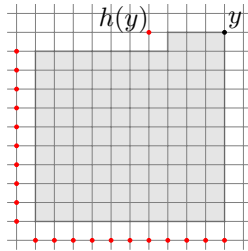


Figure 2: Example for a set  $U_y$  (the gray region). The red vertices denote  $\partial_\downarrow U_y$ .

**Definition 4.3.** For  $y = (y_1, y_2) \in \mathbb{Z}_+^2$  such that  $1 \leq y_2 \leq 2^{-\theta^2 \alpha} y_1$  let  $B_{y,L} \subset \mathbb{Z}^2$  be the box of side lengths  $(L, L)$  and upper-right corner at  $y$  and let (see Figure 2)

$$U_y = \left( B_{y,L} \setminus \cup_{i=\lfloor 1/q \rfloor + 1}^L \{y - i\mathbf{e}^{(1)}\} \right) \cap \mathbb{Z}_+^2.$$

Let also  $h(y) := y - (\lfloor 1/q \rfloor + 1)\mathbf{e}^{(1)}$  and note that  $h(y) \in \partial_\downarrow U_y$ .

If the starting configuration of the East process on  $\mathbb{Z}_+^2$  is  $\omega^*$ , then  $\tau_{\partial_\downarrow U_y} < \tau_{U_y} < \tau_y$ . This observation justifies the following definition. In the sequel  $\{\omega_t\}_{t \geq 0}$  denotes the East process in  $\mathbb{Z}_+^2$  with  $\omega_0 = \omega^*$ .

**Definition 4.4** (Infection sequence for  $y$ ). Let  $\xi^{(0)} = y$  and define recursively  $\xi^{(i)}$  as the unique vertex  $z \in \partial_\downarrow U_{\xi^{(i-1)}}$  such that  $\omega_{\tau_{\partial_\downarrow U_{\xi^{(i-1)}}}}(z) = 0$ . We also let  $\nu := \inf\{i \in \mathbb{N} : 0 \in U_{\xi^{(i)}}\}$  and call the random sequence  $\xi(y) = \{\xi^{(i)}\}_{i \in [\nu]}$  the infection sequence for  $y$ . The collection of all possible infection sequences is denoted by  $\mathcal{S}(y)$ . Given  $\mathbf{v} = \{v^{(i)}\}_i \in \mathcal{S}(y)$  we say that  $v^{(i)}$  is good if  $v^{(i+1)} = h(v^{(i)})$  and bad otherwise.

**Remark 4.5.** By construction any possible infection sequence  $\mathbf{v}$  is such that  $\|v^{(i)} - v^{(i+1)}\|_1 \geq \lfloor 1/q \rfloor$ .

**Lemma 4.6.** For any  $q$  small enough, any infection sequence in  $\mathcal{S}(y)$  contains at most  $y_2$  bad points and at least  $\lfloor y_1 \frac{q}{2} \rfloor$  good points.

*Proof.* Given an infection sequence  $\mathbf{v}$  let  $n_g$  be the number of its good points and observe that if  $v^{(i)}$  is bad then  $v_2^{(i+1)} < v_2^{(i)}$  and  $v_1^{(i)} - v_1^{(i+1)} \leq L$ . Hence,  $(n - n_g) \leq y_2$  and

$$(n - n_g)L + n_g/q \geq y_1 - L,$$

i.e.  $n_g \geq q(y_1 - L(1 + y_2))$ . In particular, if  $1 \leq y_2 \leq 2^{-\theta_q^2} y_1$  then  $n_g \geq \lfloor y_1 q/2 \rfloor$  for  $q$  small enough.  $\square$

For any  $y \in \mathbb{Z}_+^d$  let  $n_y = \lfloor y_1 \frac{q}{2} \rfloor$  and for any given  $\mathbf{v} \in \mathcal{S}(y)$  let  $(w^{(1)}, w^{(2)}, \dots, w^{(n_y)})$  be the collection of the first  $n_y$  good points of  $\mathbf{v}$  ordered from the last one to the first one. By construction, for all  $k$ ,  $w^{(k-1)} \prec h(w^{(k)})$ . Using Definition 4.4, the event  $\{\xi(y) = \mathbf{v}\}$  implies the event

$$G_{\mathbf{v}} := \cap_k \{ \tau_{U_{w^{(k)}}} = \tau_{h(w^{(k)})}; \tau_{h(w^{(k)})} \geq \tau_{w^{(k-1)}} \},$$

and  $\tau_y \geq \sum_k (\tau_{w^{(k)}} - \tau_{h(w^{(k)})})$ . Therefore, for all  $\lambda > 0$  the definition of the event  $G_{\mathbf{v}}$  together with a repeated use of the strong Markov property implies that

$$\begin{aligned} e^{-\lambda \mathbb{E}_{\omega_*}(\tau_y)} &\leq \mathbb{E}_{\omega_*}(e^{-\lambda \tau_y}) \leq \sum_{\mathbf{v} \in \mathcal{S}(y)} \mathbb{E}_{\omega_*}(\mathbf{1}_{G_{\mathbf{v}}} e^{-\lambda \sum_{k=1}^{n_y} (\tau_{w^{(k)}} - \tau_{h(w^{(k)})})}) \\ &\leq |\mathcal{S}(y)| \max_{\mathbf{v}} \mathbb{E}_{\omega_*}(\mathbf{1}_{G_{\mathbf{v}}} \prod_{k=1}^{n_y} e^{-\lambda (\tau_{w^{(k)}} - \tau_{h(w^{(k)})})}) \\ &\leq |\mathcal{S}(y)| F(\lambda)^{n_y}, \end{aligned} \tag{4.6}$$

where  $|\mathcal{S}(y)|$  denotes the cardinality of  $\mathcal{S}(y)$  and

$$F(\lambda) := \max_{z \in \mathbb{Z}_+^d; h(z) \in \mathbb{Z}_+^2} \max_{\omega: \omega(h(z))=0, \omega|_{U_z}=1} \mathbb{E}_{\omega}(e^{-\lambda \tau_z}). \tag{4.7}$$

The next two lemmas provide the necessary bounds on  $|\mathcal{S}(y)|$  and  $F(\lambda)$ .

**Lemma 4.7.** For any  $y \in \mathbb{Z}_+^2$  with  $1 \leq y_2 < y_1 2^{-\alpha \theta_q^2}$  as  $q \rightarrow 0$ , we have

$$|\mathcal{S}(y)| \leq (y_1/y_2)^{O(y_2)}. \tag{4.8}$$

*Proof.* Recall that a good point of an infection sequence specifies uniquely the next point of the sequence. Hence, we can reconstruct the full infection sequence by specifying which points are bad together with their relative position w.r.t. the previous point. Using Remark 4.5 together with  $n_y = \lfloor y_1 \frac{q}{2} \rfloor$ , it also follows that the length  $n$  of any infection sequence satisfies  $n \in [n_y, q(y_1 + y_2)]$ . Thus for  $q$  small enough

$$\begin{aligned} |\mathcal{S}(y)| &\leq \sum_{n=n_y}^{\lceil q(y_1+y_2) \rceil} \sum_{m=0}^{y_2} \binom{n}{m} (2L)^m \leq \sum_{n=n_y}^{\lceil q(y_1+y_2) \rceil} \binom{n}{y_2} (y_2 + 1) (2L)^{y_2} \\ &\leq e^{O(\theta_q^2) y_2} \times O(q) y_1 \times \binom{\lceil q(y_1 + y_2) \rceil}{y_2} \leq (y_1/y_2)^{O(y_2)}. \end{aligned} \tag{4.8} \quad \square$$

**Lemma 4.8.** Fix  $0 < \varepsilon \ll 1$  and let  $T_{\alpha} = T_{\alpha}(\varepsilon, q) = 2^{\frac{\theta_q^2}{4} ((1+4\alpha) \wedge 2)(1-2\varepsilon)}$ . Then for any  $q$  sufficiently small and any  $\lambda > 0$

$$F(\lambda) \leq e^{-\lambda T_{\alpha}} + 2^{-\Omega(\varepsilon) \theta_q^2}.$$

*Proof.* Fix  $z \in \mathbb{Z}_+^2$  such that  $h(z) \in \mathbb{Z}_+^2$  together with  $\omega$  such that  $\omega(h(z)) = 0$  and  $\omega \upharpoonright_{U_z} = 1$ . Let also  $A := \{h(z) + \mathbf{e}^{(1)} - \mathbf{e}^{(2)}, h(z) + 2\mathbf{e}^{(1)} - \mathbf{e}^{(2)}, \dots, z - \mathbf{e}^{(2)}\}$ . Then,

$$\begin{aligned} \mathbb{E}_\omega(e^{-\lambda\tau_z}) &\leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\tau_z < T_\alpha) \\ &\leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) + \mathbb{P}_\omega(\tau_A \leq T_\alpha) \\ &\leq e^{-\lambda T_\alpha} + \mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) + \sum_{a \in A} \mathbb{P}_\omega(\tau_a \leq T_\alpha). \end{aligned}$$

Let  $\mathcal{F}_{T_\alpha}$  be the  $\sigma$ -algebra generated by the variables  $\omega_z(s), s \in [0, T_\alpha]$  where  $z \in \{a \in \mathbb{Z}_+^2 : a \prec h(z)\} \cup \{a \in \mathbb{Z}_+^2 : a \prec b \text{ for some } b \in A\}$ . Clearly  $\{\tau_A > T_\alpha\} \in \mathcal{F}_{T_\alpha}$ . Moreover, conditionally on  $\mathcal{F}_{T_\alpha}$  and on the event  $\{\tau_A > T_\alpha\}$ , the East process on  $A + \mathbf{e}^{(2)}$  coincides up to time  $T_\alpha$  with the one-dimensional East chain on  $A + \mathbf{e}^{(2)}$  with a boundary value at  $\{\omega_{h(w)}(s)\}_{s \leq T}$  which is measurable w.r.t.  $\mathcal{F}_{T_\alpha}$ . We can then apply Corollary 3.14 with  $d = 1$  and  $n = \lfloor \theta_q \rfloor$  to obtain:

$$\begin{aligned} \mathbb{P}_\omega(\{\tau_z < T_\alpha\} \cap \{\tau_A > T_\alpha\}) &\leq O(T_\alpha) 2^{-\frac{\theta_q^2}{4}(1-\varepsilon)} \\ &= O(2^{-\frac{\theta_q^2}{4}((2-(1+4\alpha)\wedge 2)(1-2\varepsilon)+2\varepsilon)}) \leq 2^{-\Omega(\varepsilon)\theta_q^2}. \end{aligned} \tag{4.9}$$

Let  $n_A = \min_{a \in A} \min_{z' \prec a, z' \notin U_z} \|a - z'\|_1$ , and observe that  $\exists \varepsilon(\alpha) > 0$  such that  $\forall \varepsilon \leq \varepsilon(\alpha)$  and all  $q$  small enough depending on  $\varepsilon, T_\alpha \leq n_A 2^{\frac{\theta_q^2}{4}(1-\varepsilon)}$ . We can then use Remark 4.2 to get that

$$\sum_{a \in A} \max_{\omega: \omega \upharpoonright_{U_z} = 1} \mathbb{P}_\omega(\tau_a \leq T_\alpha) \leq e^{-\Omega(\varepsilon\theta_q^2 \lfloor n_A 2^{-\frac{\theta_q^2}{4}} \rfloor)} \leq 2^{-\Omega(\varepsilon)\theta_q^2},$$

because  $n_A \geq L - 2^{\theta_q} \gg 2^{\theta_q/2}$ . □

We can now conclude the proof. By combining the two lemmas above and choosing  $\lambda = \lambda_\alpha(q) = T_\alpha^{-1} \varepsilon \theta_q^2$ , we get from (4.6) that

$$e^{-\lambda \mathbb{E}_{\omega^*}(\tau_y)} \leq |\mathcal{S}(y)| F(\lambda)^{n_y} \leq (y_1/y_2)^{O(y_2)} e^{-\Omega(\varepsilon)\theta_q^2 n_y},$$

where we recall that  $n_y := \lfloor y_1 \frac{q}{2} \rfloor$ . If  $y = \lfloor n\mathbf{x} \rfloor$  with  $\mathbf{x}$  such that  $0 < x_2 \leq x_1 2^{-\theta_q^2 \alpha}$ , the above inequality implies

$$\mathbb{E}_{\omega^*}(\tau_{\lfloor n\mathbf{x} \rfloor}) \geq \Omega(q T_\alpha) \times n \quad \text{as } n \rightarrow \infty.$$

In particular  $v_{\max}(\mathbf{x}) \leq 2^{-\frac{\theta_q^2}{4}((1+4\alpha)\wedge 2)(1-o(1))}$ .

#### 4.4 Proof of Theorem 2

We begin with the case  $\delta = 0$ .

Recall Remark 1.2 and that  $v_{\min}(\mathbf{e}^{(i)}) = v_{\max}(\mathbf{e}^{(i)}) = 2^{-\frac{\theta_q^2}{4}(1+o(1))} \forall i \in [d]$ . Take  $0 < \varepsilon \ll 1$  and let  $x_t = \lfloor 2^{-\frac{\theta_q^2}{4}(1+\varepsilon)} t \rfloor \mathbf{e}^{(1)}, t \gg 0$ . By construction  $x_t \in \Lambda(\delta = 0, \varepsilon, t)$ . Let also

$$A_t = \{\omega : \exists y \in \{x_t - \lfloor 2^{2\theta_q} \rfloor \mathbf{e}^{(1)}, \dots, x_t\} \text{ such that } \omega_y(t) = 0\},$$

and use

$$\|\nu_t^{\delta, \varepsilon} - \mu_{\Lambda(\delta, \varepsilon, t)}\|_{TV} \geq |\mu(A_t) - \nu_t^{\delta, \varepsilon}(A_t)|.$$

For any  $t$  large enough  $\mu(A_t) = 1 - e^{-\Omega(2^{\theta_q})}$ , while Remark 4.2 gives  $\limsup_{t \rightarrow \infty} \nu_t^{\delta, \varepsilon}(A_t) = 0$ . Hence,

$$\liminf_{q \rightarrow 0} \liminf_{t \rightarrow \infty} \|\nu_t^{\delta, \varepsilon} - \mu_{\Lambda(\delta, \varepsilon, t)}\|_{TV} = 1.$$



We now consider the case  $0 < \delta < 1$ .

Fix  $0 < \varepsilon \ll 1$  and observe (see [11, Lemma 5.5]) that equilibrium in the region  $\Lambda(\delta, \varepsilon, t)$  is achieved very rapidly, within a time  $O(\log(|\Lambda(\delta, \varepsilon, t)|)^{4d})$ , if the initial configuration has a vacancy in every interval of  $\Lambda(\delta, \varepsilon, t)$  parallel to a coordinate direction and containing  $O((\log(|\Lambda(\delta, \varepsilon, t)|))^2)$  vertices. Hence, if the above condition is satisfied by the East process at time  $t/2$  then at time  $t$  the measure  $\nu_t^{\delta, \varepsilon}$  will be very close to  $\mu_{\Lambda(\delta, \varepsilon, t)}$  in the total variation distance. The second observation (cf. [11, Lemma 5.3]) is the following. Recall that  $\tau_x$  is the first time a vacancy appears at  $x$ . Then the above requirement for the East process at time  $t/2$  will be fulfilled with w.h.p. if  $\tau_x \leq t/2 - O((\log(|\Lambda(\delta, \varepsilon, t)|))^2) \forall x \in \Lambda(\delta, \varepsilon, t)$ .

A more precise formulation of the above two steps is as follows. For any  $t$  large enough depending on  $q, \delta, \varepsilon$

$$\|\mu_{\Lambda(\delta, \varepsilon, t)} - \nu_t^{\delta, \varepsilon}\|_{TV} \leq \varepsilon + \sum_{x \in \Lambda(\delta, \varepsilon, t)} \mathbb{P}_{\omega^*}(\tau_x > t/3). \tag{4.10}$$

We decided to skip the proof of (4.10) as it follows very closely the proofs of Lemma 5.3. and 5.5 of [11]. The proof of the theorem then boils down to proving that the second term in the r.h.s. of (4.10) vanishes as  $t \rightarrow \infty$ . For future needs we actually prove a slightly stronger result.

**Lemma 4.9.** *For any  $\delta, \varepsilon$  in  $(0, 1)$  there exists  $q(\delta, \varepsilon) > 0$  such that for any  $q \leq q(\delta, \varepsilon)$  and all  $t$  large enough*

$$\sup_{y \in \mathbb{Z}_+^d} \sum_{x \in \Lambda(\delta, \varepsilon, t) + y} \sup_{\omega: c_y(\omega) = 1} \mathbb{P}_{\omega}(\tau_x > t/3) \leq e^{-\Omega\left(2^{-(1+\varepsilon/2)} \frac{\theta^2}{2^d} \log^2(t)\right)}. \tag{4.11}$$

*Proof of the lemma.* Fix  $y \in \mathbb{Z}_+^d$  together with  $\omega$  such that  $c_y(\omega) = 1$ . In the sequel all estimates will be uniform in  $y, \omega$ . Fix  $x \in \Lambda(\delta, \varepsilon, t) + y$  and let  $\mathbf{x} = (x - y)/|x - y|$  be the associated unit vector in  $\mathbb{R}_+^d$ . Clearly the components of  $\mathbf{x}$  satisfy  $\min_{i,j} \mathbf{x}_i/\mathbf{x}_j \geq \delta$ . Let  $\ell_q = 2^{\theta_q^{3/2}}$ , let  $n_x = \lfloor |x - y|/\ell_q \rfloor$ , and define the sequence of vertices  $\{x^{(n)}\}_{n=0}^{n_x+1}$  by  $x^{(n)} = \lfloor n\ell_q \mathbf{x} \rfloor$  if  $0 \leq n \leq n_x$  and  $x^{(n_x+1)} = x$ . By construction  $|x^{(n+1)} - x^{(n)}| \leq \ell_q + 1$ , and  $\exists \kappa(\delta) \geq 1, q(\delta) < 1$  such that  $\forall q \leq q(\delta)$

$$\max_{0 \leq n \leq n_x} \max_{i,j} \frac{(x^{(n+1)} - x^{(n)})_i}{(x^{(n+1)} - x^{(n)})_j} \leq \kappa(\delta).$$

For the East process with initial condition  $\omega$  recursively define

$$\tau^{(0)} = \inf\{s \geq 0, \omega_{x^{(0)}}(s) = 0\}, \quad \tau^{(n)} = \inf\{s \geq \tau^{(n-1)} : \omega_{x^{(n)}}(s) = 0\},$$

and set  $\Delta_n = \tau^{(n)} - \tau^{(n-1)}$ . Finally, let  $M = \log(t)^{5d} \times 2^{\frac{\theta^2}{2^d}(1+\varepsilon/2)}$ . Using  $\tau_x \leq \sum_{n=1}^{n_x+1} \Delta_n$  we write

$$\begin{aligned} & \mathbb{P}_{\omega}(\tau_x \geq t/3) \\ & \leq \mathbb{P}_{\omega}\left(\sum_{n=1}^{n_x+1} \Delta_n \mathbb{1}_{\{\Delta_n \leq M\}} \geq t/3\right) + \sum_{n=1}^{n_x+1} \sup_{\omega: \omega_{x^{(n-1)}} = 0} \mathbb{P}_{\omega}(\Delta_n \geq M). \end{aligned} \tag{4.12}$$

In order to bound from above the second term in (4.12) we apply Lemma 3.8 to  $x = x^{(n-1)}$ ,  $\Lambda$  the box with sides  $L_i = x_i^{(n)} - x_i^{(n-1)}$ ,  $t = M$ , and  $\ell = \ell_t = \log^2(t)$  to get

$$\sup_{\omega: \omega_{x^{(n-1)}} = 0} \mathbb{P}_{\omega}(\Delta_n \geq M) \leq c' M \ell_t^d e^{-cq\ell_t} + 2^{\theta_q(\ell_t + \ell_q + 1)^d} - M \ell_t^{-d} 2^{-\frac{\theta^2}{2^d}(1+\varepsilon)}.$$

Using  $M\ell_t^{-d} = \Omega(\log(t)^{3d})$  as  $t \rightarrow +\infty$ , for any  $t$  large enough depending on  $q$  the second term in the r.h.s. of (4.12) satisfies

$$\sum_{n=1}^{n_x+1} \sup_{\omega: \omega_{x(n-1)}=0} \mathbb{P}_\omega(\Delta_n \geq M) \leq e^{-\Omega(q \log^2(t))}. \tag{4.13}$$

We now tackle the first term in the r.h.s. of (4.12) via the exponential Chebyshev inequality with  $\lambda = 2^{-\frac{\theta^2}{2d}(1+\varepsilon/2)} \log^2(t)/t$ . Using the strong Markov property and  $\lambda M \leq 1$  for any large enough  $t$  we obtain

$$\begin{aligned} \mathbb{P}_\omega\left(\sum_{n=1}^{n_x+1} \Delta_n \mathbb{1}_{\{\Delta_n \leq M\}} \geq t/3\right) &\leq e^{-\lambda t/3} \times \mathbb{E}_\omega\left(\prod_{n=1}^{n_x+1} e^{\lambda \Delta_n \mathbb{1}_{\{\Delta_n \leq M\}}}\right) \\ &\leq e^{-\lambda t/3} \times \left(\sup_n \sup_{\omega: \omega_{x(n-1)}=0} \mathbb{E}_\omega\left(e^{\lambda \Delta_n \mathbb{1}_{\{\Delta_n \leq M\}}}\right)\right)^{n_x+1} \\ &\leq e^{-\lambda t/3} \times \left(1 + e\lambda \sup_n \sup_{\omega: \omega_{x(n-1)}=0} \mathbb{E}_\omega(\Delta_n)\right)^{n_x+1}, \end{aligned}$$

where we used  $e^a \leq 1 + ea, \forall 0 \leq a \leq 1$  in the last inequality. We can finally appeal to Lemma 3.7 to get that for all  $q$  small enough depending on  $\delta, \varepsilon$

$$1 + e\lambda \sup_n \sup_{\omega: \omega_{x(n-1)}=0} \mathbb{E}_\omega(\Delta_n) \leq 1 + e\lambda 2^{(1+\varepsilon/2)\frac{\theta^2}{2d}} \leq e^{e \log^2(t)/t}.$$

In conclusion,

$$\mathbb{P}_\omega\left(\sum_{n=1}^{n_x+1} \Delta_n \mathbb{1}_{\{\Delta_n \leq M\}} \geq t/3\right) \leq e^{-\lambda t/3 + e(n_x+1) \log^2(t)/t} \leq e^{-\lambda t/6}, \tag{4.14}$$

where we used  $(n_x + 1) \leq |x - y| + 1 \leq t 2^{-\frac{\theta^2}{2d}(1+\varepsilon)} + 1$  to obtain the last inequality for  $q$  small enough depending on  $\varepsilon$ . The claim of the lemma now follows from (4.12), (4.13) and (4.14).  $\square$

### 4.5 Proof of Theorem 3

Using Remark 1.2  $d(t) \geq \bar{d}(t)$ , where  $\bar{d}(t)$  is defined as  $d(t)$  but for the *one dimensional* East chain on  $\{0, \dots, n\}$ . Hence (2.5) follows directly from the cutoff result for the latter chain (see [16, Theorem 2]). We now turn to the proof of (2.6).

Let  $w_n = n^{2/3}$  and let  $\hat{T}_n = T_n + w_n/2$ . As in the proof of Theorem 2 (see (4.10) and the explanation immediately before) the following can be proved by following very closely the proof of Lemma 5.3 and Lemma 5.5 of [11].

**Lemma 4.10.** *For any  $q \in (0, 1)$*

$$\limsup_{n \rightarrow \infty} d(T_n + w_n) \leq \limsup_{n \rightarrow \infty} \max_{\omega \in \Omega_{\Lambda_n}} \mathbb{P}_\omega(\exists x \in \Lambda_n : \tau_x \geq \hat{T}_n). \tag{4.15}$$

We will now prove that for  $q$  small enough

$$\limsup_{n \rightarrow \infty} \max_{\omega \in \Omega_{\Lambda_n}} \sum_{x \in \Lambda_n} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) = 0. \tag{4.16}$$

We will give the full details for  $d = 2$  and only sketch the additional steps needed for  $d \geq 3$ . In the sequel  $\varepsilon$  will be a small positive constant,  $q$  will be assumed to be sufficiently small depending on  $\varepsilon$ , and  $c(q)$  will denote a positive constant depending on  $q$  whose value may change from line to line.

The intuition behind (4.16) is as follows. Fix  $x \in \Lambda_n$  and w.l.o.g. suppose that  $x_1 = \max(x_1, x_2)$ . Then the infection time  $\tau_x$  should be dominated by the sum of the infection time of the vertex  $x' = (x_1 - x_2, 0)$  plus the infection time of  $x$  starting from  $\omega_{\tau_{x'}}$ . Using [16, Theorem 2] the first time is, with great accuracy,  $(x_1 - x_2)/v$ , while part (A) of Theorem 1 suggests that w.h.p. the second time is  $O(x_2/v_{\min}(\hat{e}))$  where  $\hat{e} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ . Hence, we expect  $\tau_x$  to satisfy w.h.p.

$$\tau_x \lesssim (x_1 - x_2)/v + x_2/v_{\min}(\hat{e}) \lesssim n/v \quad \forall x \in \Lambda_n,$$

because  $v_{\min}(\hat{e}) \gg v$  for  $q$  small enough. In other words, the time needed to infect all vertices of  $\Lambda_n$  should be dominated by the time needed to infect at least once all vertices of the form  $x = (j, 0)$  or  $x = (0, j), j \in \{0, \dots, n\}$ . In turn, using the one dimensional cutoff result, the latter time is smaller than  $\hat{T}_n$  w.h.p.

We will now detail the intuition above. We cover  $\Lambda_n$  with two regions:

$$\begin{aligned} \Lambda_n^{(1)} &= \{x \in \Lambda_n : \max_i x_i \leq \log(n)^4\}, \\ \Lambda_n^{(2)} &= \{x \in \Lambda_n : \max_i x_i \geq \log(n)^4\}, \end{aligned}$$

and we will prove that

$$\limsup_{n \rightarrow \infty} \max_{\omega \in \Omega_{\Lambda_n}} \sum_{x \in \Lambda_n^{(i)}} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) = 0, \quad \forall i \in [2]. \tag{4.17}$$

( $i = 1$ ) W.l.o.g. fix  $x \in \Lambda_n^{(1)}$  with  $x_2 \leq x_1$  and write  $\hat{x}$  for the vertex  $(x_1, 0)$ . Using the strong Markov property we get

$$\max_{\omega} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) \leq \max_{\omega} \mathbb{P}_\omega(\tau_{\hat{x}} \geq \hat{T}_n - w_n/4) + \max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4).$$

Using once again [16, Theorem 2]

$$\limsup_{n \rightarrow \infty} \sum_{x \in \Lambda_n^{(1)}} \max_{\omega} \mathbb{P}_\omega(\tau_{\hat{x}} \geq \hat{T}_n - w_n/4) = 0.$$

Notice that  $\|x - \hat{x}\|_1 = 2x_2 \leq \log(n)^4 \ll w_n^{3/8}$ . Hence, the term  $\max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4)$  can be bounded from above by applying Lemma 3.8 with  $\Lambda = \{0\} \times \{x_2\}$ , the vertex  $x$  equal to  $\hat{x}$ ,  $t = w_n/4$  and e.g.  $\ell = w_n^{1/4}$ . Using (3.7) for any  $n$  large enough we get

$$\max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4) \leq e^{-c(q)w_n^{1/4}},$$

so that

$$\limsup_{n \rightarrow \infty} \sum_{x \in \Lambda_n^{(1)}} \max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > w_n/4) = 0.$$

( $i = 2$ ) Fix  $x \in \Lambda_n^{(2)}$  with e.g.  $x_2 \leq x_1$  and  $x_1 \geq \log(n)^4$ . We can assume further that  $x_2/x_1 \leq 1/2$  since otherwise  $\max_{\omega} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n)$  could be bounded from above using Lemma 4.9 to get  $\max_{\omega} \mathbb{P}(\tau_x \geq \hat{T}_n) \leq e^{-c(q)\log(n)^2}$ . If  $x_2 = 0$  we can simply apply [16, Theorem 2] to get  $\max_{\omega} \mathbb{P}(\tau_x \geq \hat{T}_n) \leq e^{-c(q)n^{1/3}}$  for some constant  $c(q) > 0$ . Otherwise, let  $\phi(x) = x_1 - x_2$  and set now  $\hat{x} = (\phi(x) - 1, 0)$ . By construction, the direction of the vector  $x - (\hat{x} + e^{(1)})$  is the  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ -direction. Let also  $\varphi_n(x) = \max(\frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4}, w_n/5)$ . As in the previous step we write

$$\begin{aligned} &\max_{\omega} \mathbb{P}_\omega(\tau_x \geq \hat{T}_n) \\ &\leq \max_{\omega} \mathbb{P}_\omega(\tau_{\hat{x}} \geq \varphi_n(x)) + \max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_\omega(\tau_x > \hat{T}_n - \varphi_n(x)). \end{aligned} \tag{4.18}$$

Using [16, Theorem 2]) applied to the interval  $\{0, \dots, \phi(x)\}$  we get that the first term in the r.h.s. of (4.18) is bounded from above by  $e^{-c(q)w_n^{1/3}}$  for large  $n$ , so that

$$\limsup_{n \rightarrow \infty} \sum_{x \in \Lambda_n^{(3)}} \max_{\omega} \mathbb{P}_{\omega}(\tau_{\hat{x}} \geq \varphi_n(x)) = 0.$$

For the second term in the r.h.s. of (4.18) we crucially observe that

$$\hat{T}_n - \varphi_n(x) \geq \begin{cases} \frac{w_n}{4} + \frac{x_2}{v} & \text{if } \frac{\phi(x)}{v} + \frac{\phi(x)^{2/3}}{4} \geq w_n/5, \\ T_n + \frac{3}{10}w_n & \text{otherwise.} \end{cases}$$

In both cases, using  $v \leq 2^{-\frac{\theta^2}{2}(1-\varepsilon)}$ , we get that  $\hat{T}_n - \varphi_n(x) \gg 2^{\frac{\theta^2}{4}(1+\varepsilon)}\|x - \hat{x}\|_1$ . Hence, we can apply Lemma 4.9 with  $y = \hat{x} + e^{(1)}$ ,  $\delta = \frac{1}{3}$ , and  $t = 3(\hat{T}_n - \varphi_n(x))$  to get that

$$\begin{aligned} \max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_{\omega}(\tau_x > \hat{T}_n - \varphi_n(x)) &\leq e^{-\Omega\left(2^{-(1+\varepsilon/2)}\frac{\theta^2}{4} \log^2(\hat{T}_n - \varphi_n(x))\right)} \\ &\leq e^{-c(q) \log(w_n)^2}. \end{aligned}$$

In conclusion

$$\limsup_{n \rightarrow \infty} \sum_{x \in \Lambda_n^{(2)}} \max_{\omega: \omega_{\hat{x}}=0} \mathbb{P}_{\omega}(\tau_x > \hat{T}_n - \varphi_n(x)) = 0.$$

We will now briefly discuss the proof of (4.16) when  $d \geq 3$ . The proof of (4.17) for  $i = 1$  does not change. The proof for  $i = 2$  needs instead a few changes.

Fix  $x \in \Lambda_n^{(2)}$  and w.l.o.g. assume that  $1 \leq x_d \leq x_{d-1} \leq \dots \leq x_1$ . For  $k \in [d - 1]$  define recursively

$$\hat{x}^{(0)} = x, \quad \hat{x}^{(k)} = \hat{x}^{(k-1)} - \hat{x}_{d-k+1}^{(k-1)} \sum_{j=1}^{d-k+1} e^{(j)},$$

so that  $\hat{x}_j^{(k)} = x_j - x_{d-k+1}$  if  $j < d - k + 1$  and  $\hat{x}_j^{(k)} = 0$  otherwise. Notice that the direction vector  $\mathbf{w}^{(k)}$  corresponding to each  $x^{(k-1)} - x^{(k)}$  when the latter is non-zero has the form  $\mathbf{w}^{(k)} = \sum_{j=1}^{d-k+1} e^{(j)}$ . Hence, using Remark 1.2 and part (A) of Theorem 1, the corresponding minimal velocity  $v_{\min}(\mathbf{w}^{(k)})$  satisfies  $v_{\min}(\mathbf{w}^{(k)}) \geq 2^{-\frac{\theta^2}{4}(1+o(1))} \gg v$ . Let also

$$\varphi_n^{(k-1)}(x) = \begin{cases} \max\left(2^{\frac{\theta^2}{4}(1+\varepsilon)}\|x^{(k-1)} - x^{(k)}\|_1, \frac{w_n}{5d}\right) & \text{if } k \leq d - 1, \\ \max\left(\frac{x_1^{(d-1)}}{v} + \frac{(x_1^{(d-1)})^{2/3}}{4}, \frac{w_n}{5d}\right) & \text{if } k = d. \end{cases}$$

Using  $v \ll 2^{-\frac{\theta^2}{4}(1+o(1))}$  for  $q$  small enough, it is easy to check that

$$\sum_{k=1}^d \varphi_n^{(k-1)}(x) \leq \frac{9}{20}w_n + \frac{x_1}{v} \leq \hat{T}_n.$$

Hence, by setting recursively  $\sigma_d = 0$  and  $\sigma_{k-1} = \inf\{t > \sigma_k, \omega_{x^{(k-1)}}(t) = 0\}$ , we get

$$\begin{aligned} \max_{\omega} \mathbb{P}_{\omega}(\tau_x \geq \hat{T}_n) &\leq \max_{\omega} \mathbb{P}_{\omega}(\exists k : \sigma_{k-1} - \sigma_k \geq \varphi_n^{(k-1)}(x)) \\ &\leq \sum_k \max_{\omega: \omega_{x^{(k)}}=0} \mathbb{P}_{\omega}(\sigma_{k-1} - \sigma_k \geq \varphi_n^{(k-1)}(x)). \end{aligned}$$

As in the  $d = 2$  case, we apply Lemma 4.9 to each term in the above sum with  $k < d - 1$  and [16, Theorem 2]) to the term  $k = d - 1$  to conclude that the r.h.s. above is smaller than  $e^{-c(q) \log(n)^2}$ .  $\square$

## 5 Proof of Proposition 3.6

### 5.1 Proof of (i)

We proceed in two steps: we first prove that  $\phi(0; d) \geq 1/d$  using a bottleneck argument and then, inspired by [12], that  $\phi(0; d) \leq 1/d$ .

**The lower bound** Let  $\Lambda$  be the equilateral box of side length  $\lfloor 2^{\theta_q/d} \rfloor$  and let  $\Lambda \supset V \supset \{0, x_\Lambda\}$  be such that  $\gamma(V) > 0$ .

**Claim 5.1.** For any  $\varepsilon > 0$  there exists  $q(\varepsilon) > 0$  such that for any  $q \leq q(\varepsilon)$

$$\gamma(V) \leq 2^{-(1-\varepsilon)\frac{\theta_q^2}{2d}}.$$

*Proof.* Let  $A_* \subset \Omega_\Lambda$  be the event defined in [12, Definition 4.3] and let  $A_V = \{\omega \in \Omega_V : 1_{V^c} \cdot \omega \in A_*\}$ , where  $1_{V^c}$  denotes the configuration in  $\Omega_{V^c}$  identically equal to one. As observed in [12, Remark 4.4]  $1_V \notin A_V$  while the configuration with exactly one vacancy at  $x_\Lambda$  belongs to  $A_V$ . Therefore,  $\text{Var}(\mathbb{1}_{A_V}) \geq (1-q)^{2|V|-1}q = \Theta(q)$  because  $|V| \leq 1/q$ . Next we bound the Dirichlet form of  $\mathbb{1}_{A_V}$ . Let  $\partial A_V$  consists of those elements of  $A_V$  which are connected to  $A_V^c$  via a legal update for the East chain on  $V$ . Then

$$\mathcal{D}_V(\mathbb{1}_{A_V}) \leq |\Lambda|\mu_V(\partial A_V) \leq |\Lambda|\mu_{V^c}(1_{V^c})^{-1}\mu_\Lambda(\partial A_*) \leq 2^{-(1-o(1))\frac{\theta_q^2}{2d}},$$

where we used [12, Section 4.3]. The claim now follows from the variational characterization of the spectral gap  $\gamma(V)$ .  $\square$

Since the box  $\Lambda$  is  $(0, 1; \theta_q)$ -outstretched, the claim implies that if  $\lambda \sim \mathcal{H}(0)$  then  $\lambda \geq 1/d$ . Hence  $\phi(0; d) \geq 1/d$ .

**The upper bound** The proof that  $\phi(0; d) \leq 1/d$  requires a bootstrap procedure like the one introduced in [12]. The *base case* is Lemma A.2 which gives that  $\lambda = 1 \sim \mathcal{H}(0)$ . We then prove the *recursive step*, namely that  $\lambda \sim \mathcal{H}(0)$  implies  $F(\lambda) \sim \mathcal{H}(0)$ , where

$$F(\lambda) = ((2d-1)\lambda - 1)/(d^2\lambda - 1) < \lambda \quad \forall \lambda \in [1/d, 1]. \tag{5.1}$$

Since the mapping  $F$  has an attractive fixed point in  $1/d$ , the sought claim follows by iteration.

**Proof of the recursive step** We find it easier to work with *equilateral* boxes, i.e.  $(0, 1; \theta_q)$ -outstretched boxes. For this purpose we first introduce a new condition, equivalent to  $\mathcal{H}(0)$ , which only requires a check on the spectral gap of suitable subsets of equilateral boxes.

**Definition 5.2.** We say that  $\lambda \sim \mathcal{H}'(0)$  if  $\forall \varepsilon > 0$  there exists  $q(\varepsilon) > 0$  such that  $\forall q \leq q(\varepsilon)$  and for any equilateral box  $\Lambda$  there exists  $\Lambda \supset V \supset \{0, x_\Lambda\}$  such that  $\gamma(V) \geq 2^{-\lambda(1+\varepsilon)\frac{\theta_q^2}{2}}$ .

**Lemma 5.3.**  $\lambda \sim \mathcal{H}'(0)$  iff  $\lambda \sim \mathcal{H}(0)$ .

The proof of the lemma is postponed to the appendix. Next, motivated by [12, Definition 5.2], we construct three useful auxiliary Markov chains. The first one, dubbed the *\*East chain*, is a natural generalisation of the East chain when the single site state space is a general finite set and not just the set  $\{0, 1\}$ . The other two chains, dubbed the *Knight Chain* and *\*Knight Chain* respectively, require a somewhat more involved geometric setting.

**Definition 5.4** (The \*East chain). Let  $q^* \in (0, 1)$  and let  $\{\Omega_x^*, \mu_x^*\}_{x \in \mathbb{Z}_+^d}$  be a family of finite probability spaces. For each  $x \in \mathbb{Z}_+^d$  let  $G_x^* \subset \Omega_x^*$  be an event such that  $\mu_x^*(G_x^*) = q^*$ . In the sequel we will refer to  $G_x^*$  as the facilitating event at  $x$ . Let  $V \subset \mathbb{Z}_+^d$  be a finite subset that contains the origin. Then the \*East chain on  $\Omega_V^* := \otimes_{x \in V} \Omega_x^*$  is the continuous time Markov chain, reversible w.r.t.  $\mu_V^* = \otimes_{x \in V} \mu_x^*$ , evolving as follows. With rate one and independently across  $V$  the chain attempts to update its current state  $\omega_x$  at any given vertex  $x \in V$  by proposing a new state  $\omega_x^{\text{new}}$  sampled from  $\mu_x^*$ . The attempt is successful, i.e. the proposal is accepted iff the constraint  $c_x^*(\omega) = 1$  where

$$c_x^*(\omega) = \begin{cases} 1 & \text{if } x = 0 \text{ or } \exists e \in \mathcal{B} \text{ such that } x - e \in V \text{ and } \omega_{x-e} \in G_{x-e}^*, \\ 0 & \text{else.} \end{cases}$$

**Remark 5.5.** If for all  $x$  the probability space  $\{\Omega_x^*, \mu_x^*\}$  and the facilitating event  $G_x^*$  coincide with the two points space  $\{\{0, 1\}, \text{Bernoulli}(p)\}$  and with the event  $\omega_x = 0$  respectively, then the \*East chain coincides with the standard East chain discussed so far. However, as we will see in the proof of Proposition 5.10, in a natural renormalisation procedure in which  $\mathbb{Z}_+^d$  is partitioned into equal disjoint blocks indexed by  $x \in \mathbb{Z}_+^d$  and the 0/1 variables associated to the vertices of each “block” are treated together as a single block-variable, the natural choice for the pair  $(\Omega_x^*, \mu_x^*)$  is the probability state space  $(\{0, 1\}^{B_x}, \otimes_{i \in B_x} \mu_i)$ . In this case the natural candidate for the facilitating event  $G_x^*$  is the event that inside the block  $B_x$  there is at least one vacancy.

As in [12, Proposition 3.4] it is possible to prove that the spectral gap  $\gamma^*(V)$  of the \*East chain in  $V$  coincides with the spectral gap  $\gamma(V; q^*)$  of the standard East chain with vacancy density  $q^*$ .

The construction of the Knight chain and \*Knight chain requires first the construction of the Knight graph (see Fig. 3).

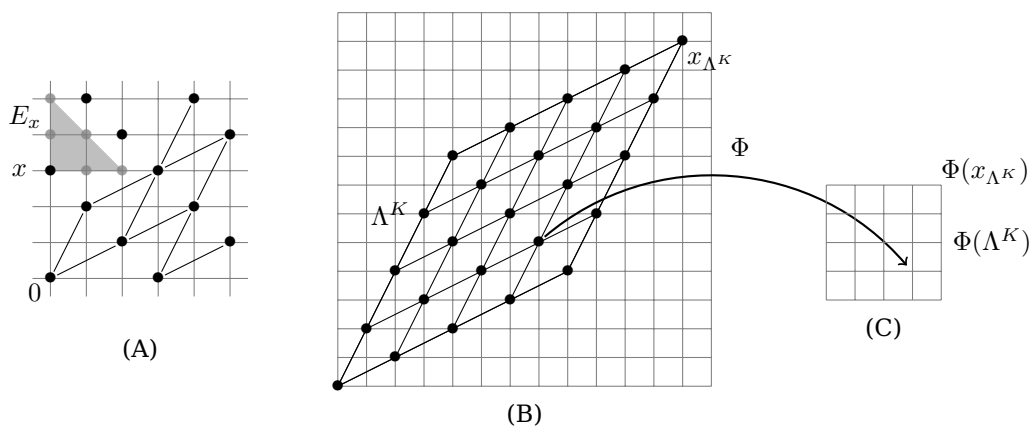


Figure 3: (A) A piece of the Knight graph (the black dots and the Knight edges) for  $d = 2$ . The gray triangle corresponds to the enlargement  $E_x$  of  $x$ . (B) The graph of the largest Knight equilateral box  $\Lambda^K$  of side length 4 inside an equilateral box of side length 13. (C) Under the natural isomorphism  $\Phi$  the graph  $\Lambda^K$  becomes an equilateral box.

**Definition 5.6** (The Knight graph). Given two vertices  $x, y \in \mathbb{Z}^d$  we say that they form a Knight edge if there exists  $j \in [d]$  such that  $y_i = x_i - 1$  for all  $i \neq j$  and  $y_j = x_j - 2$  or vice versa. We then consider the unique graph  $G = (W, E)$ ,  $W \subset \mathbb{Z}^d$ , constructed as follows. The vertex set  $W$  contains the origin and those  $x \in \mathbb{Z}^d$  which are connected to the origin via a path of Knight edges. The edge set  $E$  consists of all the Knight edges of

$W \times W$ . It is easy to see that  $G$  is isomorphic to  $\mathbb{Z}^d$  via the natural isomorphism  $\Phi$  which is unique if we set  $\Phi(0) = 0$ .

The graph  $G$  will inherit the notation used so far for  $\mathbb{Z}^d$  via the isomorphism  $\Phi$ . We write  $W_+ = \Phi^{-1}(\mathbb{Z}_+^d)$  and we say that  $\Lambda^K \subset W_+$  is a Knight equilateral box containing the origin if  $\Phi(\Lambda^K)$  is an equilateral box in  $\mathbb{Z}_+^d$  containing the origin. In the latter case we write  $x_{\Lambda^K} \in \Lambda^K$  for the vertex  $\Phi^{-1}(x_{\Phi(\Lambda^K)})$ . Notice that  $\exists c > 0$  such that for any equilateral box  $\Lambda \subset \mathbb{Z}_+^d$  containing the origin there exists a Knight equilateral box  $\Lambda \supset \Lambda^K \ni 0$  such that  $\|x_\Lambda - x_{\Lambda^K}\|_1 \leq c$ .

Recall that  $\|z - z'\|_1 = d + 1 \forall z, z' \in W$  connected by a Knight edge and  $\forall x \in W$  let  $E_x = \{y \in W^c : y \succ x, \|x - y\|_1 \leq d\}$  be the enlargement of  $x$  (see Figure 3). The enlargement of a subset  $V^K$  of the Knight graph  $W$  is the set  $EV^K = \cup_{x \in V^K} E_x$ .

We are now ready to define the Knight and \*Knight chains. As in Definition 5.4 we assume that we are given  $q^* \in (0, 1)$ , a family  $\{\Omega_x^*, \mu_x^*\}_{x \in \mathbb{Z}_+^d}$  of finite probability spaces and a facilitating event  $G_x^* \subset \Omega_x^*$  for each  $x \in \mathbb{Z}_+^d$ .

**Definition 5.7** (The Knight chain). *Given an equilateral box  $\Lambda \subset \mathbb{Z}_+^d$  with origin at 0 and  $V \subset \Lambda$  containing the origin, let  $V^K := \Phi^{-1}(V)$ . Then the Knight chain on  $\Omega_{V^K}^*$  is the image under  $\Phi^{-1}$  of the \*East chain on  $\Omega_V^*$ .*

**Definition 5.8** (The \*Knight chain). *Given an equilateral box  $\Lambda \subset \mathbb{Z}_+^d$  with origin at 0 and  $V \subset \Lambda$  containing the origin the \*Knight chain on  $\Omega_{EV^K \cap \Lambda}^*$  is the continuous time Markov chain evolving as follows. At any legal update at  $z \in V^K$  of the Knight chain on  $\Omega_{V^K}^*$  the whole configuration in  $E_z \cap \Lambda$  is resampled from  $\mu_{E_z \cap \Lambda}^*$ .*

It is immediate to verify that the \*Knight chain is reversible w.r.t.  $\mu_{EV^K \cap \Lambda}^*$  with a positive spectral gap  $\gamma^{*K}(EV^K \cap \Lambda)$ . In the appendix will prove the following result:

**Lemma 5.9.**  $\gamma^{*K}(EV^K \cap \Lambda) = \gamma(V; q^*)$ .

We can finally state the main result of this section.

**Proposition 5.10.** *Fix  $\lambda \in (1/d, 1]$  and let  $F(\cdot)$  be the mapping in (5.1). Then  $\lambda \sim \mathcal{H}'(0)$  implies that  $F(\lambda) \sim \mathcal{H}'(0)$ .*

*Proof.* Let  $\lambda \in (1/d, 1]$  with  $\lambda \sim \mathcal{H}'(0)$  and let  $\Lambda \subset \mathbb{Z}_+^d$  be an equilateral box with side length  $L$ . Using a suitable  $\lambda$ -dependent \*Knight chain, we will now construct a set  $V \subset \Lambda$  such that  $\gamma(V) \geq 2^{-F(\lambda) \frac{\theta_q^{*2}}{2} (1+\varepsilon)}$ .

Let  $\ell = \lfloor 2^{m\theta_q} \rfloor$ , where  $m = (d\lambda - 1)/(d^2\lambda - 1)$  and observe that  $\ell \leq 2^{\theta_q/d}$ . If  $L \leq \ell$  we can use Lemma A.2 to get that

$$\gamma(\Lambda) \geq 2^{-(m-m^2/2)\theta_q^2(1+o(1))} \geq 2^{-F(\lambda) \frac{\theta_q^{*2}}{2} (1+o(1))}.$$

In this case we simply choose  $V = \Lambda$ . If instead  $L > \ell$  we proceed as follows.

Let  $B_0$  be the equilateral box with side length  $\ell$ , let  $\Lambda_B := \{0, \dots, \lfloor L/\ell \rfloor\}^d$  and for  $\mathbf{j} \in \mathbb{Z}_+^d$  let  $B_{\mathbf{j}} = B_0 + \mathbf{j}\ell$ . Thus  $\cup_{\mathbf{j} \in \Lambda_B} B_{\mathbf{j}} \subset \Lambda$  and  $\min_{x \in B_{\mathbf{j}\Lambda_B}} \|x - x_\Lambda\|_1 \leq O(\ell)$ . We say that  $B_{\mathbf{j}}$  is *good* if it contains at least one vacancy and observe that the density  $q^* = 1 - (1 - q)^{\ell^d}$  of good boxes satisfies (we use the Bonferroni inequality for the lower bound)

$$q\ell^d/2 \leq q^* \leq q\ell^d \leq 1 \quad \Rightarrow \quad \theta_{q^*} \in [\theta_q(1 - dm), \theta_q(1 - dm) + 1].$$

In the sequel we will use the Knight chain and the \*Knight chain with  $\Omega_{\mathbf{j}}^* = \{0, 1\}^{B_{\mathbf{j}}}$ ,  $\mu_{\mathbf{j}}^* = \otimes_{x \in B_{\mathbf{j}}} \mu_x$ , and facilitating events  $G_{\mathbf{j}}^* = \{B_{\mathbf{j}} \text{ is good}\}$ .

Let  $\Lambda_B^K \subset \Lambda_B$  be the largest Knight equilateral box containing the origin and for  $V^K \subset \Lambda_B^K$  consider the \*Knight chain on  $\Omega_{EV^K \cap \Lambda_B}$ . Using  $\lambda \sim \mathcal{H}'(0)$  we can choose

East model front evolution

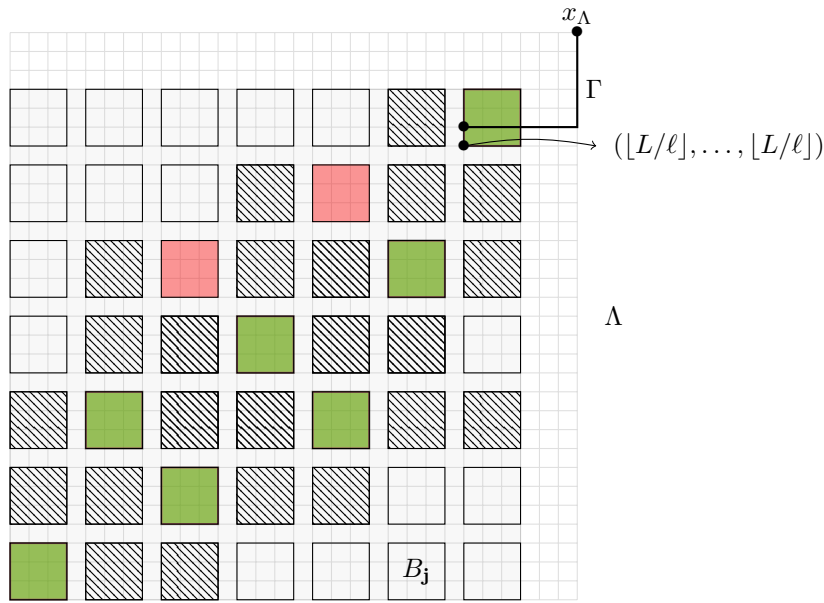


Figure 4: The setting in the proof of Proposition 5.10 with  $\ell = 3$  and  $L = 30$ . The  $3 \times 3$  boxes  $B_j$  are those with  $j \in \Lambda_B$ , the coloured (red/green) ones are those with  $j \in \Lambda_B^K$ , the green ones are those with  $j \in V^K$ , and the dashed ones are those with  $j \in (EV^K \cap \Lambda_B) \setminus V^K$ . The set  $V$  with  $\gamma(V) \geq 2^{-F(\lambda)\frac{\theta_q^2}{2}(1+o(1))}$  is the union of the green and dashed boxes together with the path  $\Gamma$ .

$V^K \subset \Lambda_B^K$  such that  $V^K \supset \{0, \mathbf{j}_{\Lambda_B^K}\}$  and  $\forall \varepsilon > 0$  and  $q$  small enough depending on  $\varepsilon$

$$\gamma^{*K}(EV^K \cap \Lambda_B) = \gamma(\Phi(V^K); q^*) \geq 2^{-\lambda\frac{\theta_q^2}{2}(1+\varepsilon/2)}, \quad (5.2)$$

where in the equality we used Lemma 5.9. We then take  $V = V_1 \cup \Gamma \subset \Lambda$ , where  $V_1 = \cup_{j \in EV^K \cap \Lambda_B} B_j$  and  $\Gamma = (x^{(0)}, x^{(1)}, \dots, x^{(N)})$  is any path in  $\Lambda$  satisfying: (i)  $x^{(0)} \in V_1$ ,  $x^{(N)} = x_\Lambda$ , (ii)  $x^{(i-1)} \prec x^{(i)} \forall i \in [N]$ , and (iii)  $N = O(\ell)$ . By construction such a path always exists.

**Claim 5.11.** For any  $\varepsilon > 0$  there exists  $q(\varepsilon) > 0$  such that for all  $q \leq q(\varepsilon)$

$$\gamma(V) \geq 2^{-\frac{1}{2}(\lambda\theta_{q^*}^2 + (2m-m^2)\theta_q^2)(1+\varepsilon)} = 2^{-F(\lambda)\frac{\theta_q^2}{2}(1+\varepsilon)}.$$

Clearly the claim proves the proposition. □

*Proof of the claim.* Fix  $\varepsilon > 0$  and choose  $q$  small enough depending on  $\varepsilon$ . Let  $V_2 = \Gamma \setminus x^{(0)}$  and use Lemma A.1 to get that  $\gamma(V) \geq 2^{-(\theta_q+2)} \min(\gamma(V_1), \gamma^\sigma(V_2))$  where  $\sigma \in \Omega_{\partial_\downarrow V_2}$  consists of a unique vacancy at  $x^{(0)}$ . Lemma A.3 together with (5.2) and the fact that  $\gamma(B_0) \geq 2^{-(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$  give that  $\gamma(V_1) \geq 2^{-\frac{1}{2}(\lambda\theta_{q^*}^2 + (2m-m^2)\theta_q^2)(1+\varepsilon)}$ . Moreover, using Lemma A.2 we have that  $\gamma^\sigma(V_2) \geq 2^{-(2m-m^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$ . The claim then follows from the observation that

$$(\lambda\theta_{q^*}^2 + (2m-m^2)\theta_q^2) = F(\lambda)\theta_q^2(1+o(1)) \quad \text{as } q \rightarrow 0. \quad \square$$

The recursive step  $\lambda \sim \mathcal{H}(0) \Rightarrow F(\lambda) \sim \mathcal{H}(0)$  now follows immediately from Lemma 5.10 and Lemma 5.3.



**5.2 Proof of (ii)**

The proof consists of two different steps. We first prove that  $\phi(\beta; 2) < 1$  for all  $\beta < 1$  implies that the same holds for any  $d \geq 3$  and then we deal with the two dimensional case.

**5.2.1 The induction step**

Fix  $d \geq 3$  and  $\beta < 1$  and assume  $\phi(\beta; d') < 1$  for any  $2 \leq d' \leq d - 1$ . We are going to prove that  $\phi(\beta; d) < 1$  as well. Fix  $\kappa \geq 1$  together with a  $(\beta, \kappa)$ -outstretched box  $\Lambda$  with side lengths  $(L_1, \dots, L_d)$  and set (see Fig. 5)

$$\begin{aligned} \Lambda_1 &= \{x \in \Lambda : x_1 \leq \lfloor L_1/2 \rfloor, x_d = 0\}, \\ \Lambda_2 &= \{x \in \Lambda : x_1 > \lfloor L_1/2 \rfloor, x_i = L_i, 2 \leq i \leq d - 1\}. \end{aligned}$$

By construction, the origin of the box  $\Lambda_2$  is at  $x_{\Lambda_1} + e_1$  and  $x_{\Lambda_2} = x_\Lambda$ . Moreover, both

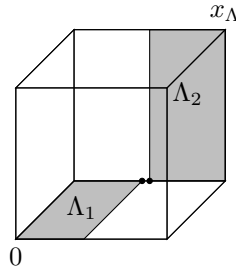


Figure 5: The boxes  $\Lambda_1, \Lambda_2$ . The two black dots denote  $x_{\Lambda_1}$  and the origin of  $\Lambda_2$  at  $x_{\Lambda_1} + e_1$  respectively.

$\Lambda_1$  and  $\Lambda_2$  are  $(\beta, \kappa)$ -outstretched boxes in  $\mathbb{Z}_+^{d-1}$  and  $\mathbb{Z}_+^2$  respectively. The induction hypothesis implies that for all  $\varepsilon > 0$  and all  $q$  small enough depending on  $\varepsilon, \beta, \kappa$  there exist  $V_i \subset \Lambda_i, i = 1, 2$ , such that

- $V_1 \supset \{0, x_{\Lambda_1}\}$  and  $V_2 \supset \{x_{\Lambda_1} + e_1, x_\Lambda\}$ ;
- $\gamma(V_1) \geq 2^{-(1+\varepsilon)\phi(\beta; d-1)\frac{\theta_q^2}{2}}$  and  $\gamma^\sigma(V_2) \geq 2^{-(1+\varepsilon)\phi(\beta; 2)\frac{\theta_q^2}{2}}$ , where  $\sigma \in \Omega_{\partial_\downarrow V_2}$  has a unique vacancy at  $x_{\Lambda_1}$ .

Lemma A.1 then implies that  $\gamma(V) \geq 2^{-(1+2\varepsilon)(\phi(\beta; d-1) \vee \phi(\beta; 2))\frac{\theta_q^2}{2}}$ , i.e.  $\phi(\beta; d) \leq \phi(\beta; d - 1) \vee \phi(\beta; 2) < 1$ .

**5.2.2 The base case  $d = 2$**

We will prove that  $\forall \beta \in (0, 1)$

$$\phi(\beta; 2) \leq \frac{1}{2}(1 - \beta)^2 + 2\beta - \beta^2, \tag{5.3}$$

which, in particular, implies that  $\phi(\beta; 2) < 1 \forall \beta < 1$ . The main idea here is to partition a  $(\beta, \kappa)$ -outstretched box  $\Lambda$  into suitably chosen mesoscopic boxes in such a way that the coarse-grained version of  $\Lambda$  becomes a  $(0, 2)$ -outstretched box on which the control of the Dirichlet eigenvalue gap is assured by part (i) of the proposition.

Fix  $0 < \beta < 1, \kappa \geq 1$  together with a  $(\beta, \kappa)$ -outstretched box  $\Lambda$  with side lengths  $(L_1, L_2)$ , and assume w.l.o.g. that  $L_1 = \min_i L_i$ . We set  $\ell = \lceil (L_2 + 1)/2(L_1 + 1) \rceil \leq$

$(\kappa/2)2^{\beta\theta_a}$ , and w.l.o.g. we assume that  $(L_2 + 1)/\ell \in \mathbb{N}$ . We then partition  $\Lambda$  into vertical one dimensional boxes  $B_{\mathbf{j}} = B + x_{\mathbf{j}}$ ,  $B = \{0\} \times \{0, \dots, \ell - 1\}$ ,  $x_{\mathbf{j}} = (j_1, j_2\ell)$  where  $\mathbf{j} \in Q = \{0, \dots, L_1 - 1\} \times \{0, \dots, (L_2 + 1)/\ell - 1\}$ . We also write  $\Omega_{\mathbf{j}}^*, \mu_{\mathbf{j}}^*$  for  $\Omega_{B_{\mathbf{j}}}$  and  $\mu_{B_{\mathbf{j}}}$  respectively.

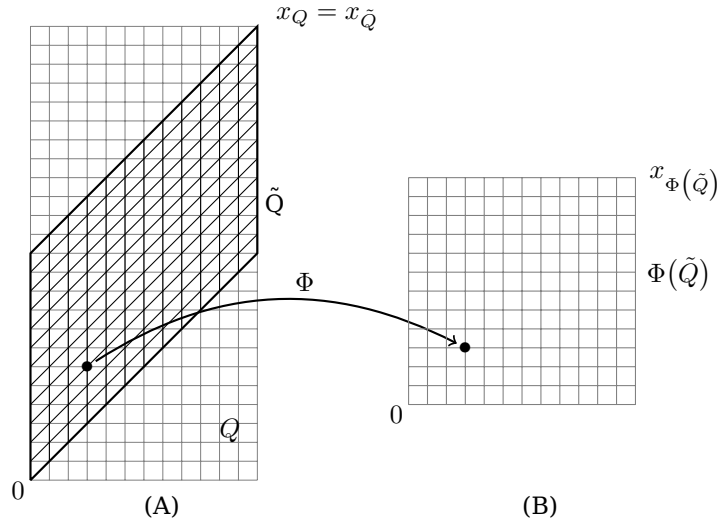


Figure 6: (A). The box  $Q$  and the region  $\tilde{Q}$  with its graph structure. Each vertex  $\mathbf{j} \in Q$  represents the box  $B_{\mathbf{j}}$ . (B). Under the natural isomorphism  $\Phi$  the graph  $\tilde{Q}$  becomes the standard square graph of  $\mathbb{Z}^2$ .

Let  $\tilde{Q}$  be the subset of  $Q$  lying between the two  $45^\circ$ -lines, one through the origin and the other through the point  $x_Q$  and declare that  $\mathbf{j}, \mathbf{j}' \in \tilde{Q}$  form an edge if either  $j_2 = j'_2 + 1$  and  $j_1 \in \{j'_1, j'_1 + 1\}$  or vice versa (see Figure 6). The corresponding graph over the vertex set  $Q$  is isomorphic via the natural graph isomorphism  $\Phi$  to the box  $\Phi(\tilde{Q}) \subset \mathbb{Z}_+^2$  with origin at  $x = 0$  and side lengths  $L_1 - 1, (L_2 + 1)/\ell - L_1$ . In particular, we write  $\mathbf{j}' \prec \mathbf{j}$  iff  $\Phi(\mathbf{j}') \prec \Phi(\mathbf{j})$ .

On any subset  $V$  of  $\tilde{Q}$  we consider the image of the \*East chain on  $\Phi(V)$  (or rather a slightly altered version of it as we see below) with parameters  $\Omega_{\mathbf{j}}^*, \mu_{\mathbf{j}}^*$  and facilitating event  $G_{\mathbf{j}} = \{\omega_{B_{\mathbf{j}}} \neq 1\}$ . Thus  $q^* = 1 - (1 - q)^\ell$  and  $\theta_{q^*} = (1 - \beta)\theta_q + \Theta(1)$ . As the box  $\Phi(\tilde{Q})$  is  $(0, 2)$ -outstretched, part (i) of Proposition 3.6 implies the existence of  $W \subset \Phi(\tilde{Q})$ , containing the origin and  $x_{\Phi(\tilde{Q})}$  such that, for any  $\varepsilon > 0$  and any  $q$  sufficiently small depending on  $\varepsilon$ ,

$$\gamma(W; q^*) \geq 2^{-(1+\varepsilon/2)\frac{\theta_{q^*}^2}{4}}. \tag{5.4}$$

Recall the definition of enlargements  $E_x$  from above Definition 5.8. We define  $E\Phi^{-1}(W) := \cup_{\mathbf{j} \in \Phi^{-1}(W)} E_{\mathbf{j}} \cap Q$  and  $V = \cup_{\mathbf{j} \in E\Phi^{-1}(W)} B_{\mathbf{j}} \subset \Lambda$  and observe that  $V$  contains the origin and the vertex  $x_\Lambda$ .

**Claim 5.12.** For any  $\varepsilon > 0$  and any  $q$  sufficiently small depending on  $\varepsilon$

$$\gamma(V) \geq \gamma(W; q^*) \times 2^{-(\beta-\beta^2/2)\theta_q^2(1+\varepsilon)}.$$

*Proof of the claim.* On  $V$  we define an auxiliary dynamics to the \*East chain. Consider for that a partition of  $E\Phi^{-1}(W)$  into disjoint connected subsets  $U_{\mathbf{j}}$  for  $\mathbf{j} \in \Phi^{-1}(W)$  such that  $\mathbf{j} \in U_{\mathbf{j}} \subset E_{\mathbf{j}}$  and  $\cup_{\mathbf{j} \in \Phi^{-1}(W)} U_{\mathbf{j}} = E\Phi^{-1}(W)$ . In the sequel we write  $B_{U_{\mathbf{j}}} := \cup_{\mathbf{j}' \in U_{\mathbf{j}}} B_{\mathbf{j}'}$  and analogously for  $B_{E_{\mathbf{j}}}$ . Let  $c_{\mathbf{j}}^*(\omega) = 1$  iff either  $\mathbf{j} = 0$  or there exists a neighbor  $\mathbf{j}' \prec \mathbf{j}$  such that there exists at least a vacancy in  $B_{\mathbf{j}'}$ . For such constraints we define the

auxiliary dynamics that updates  $B_{U_j}$  with a configuration sampled from  $\mu_{B_{U_j}}$  if  $c_j^*(\omega) = 1$  and otherwise do nothing. The spectral gap of this chain is, as the one for the enlarged East chain, given by  $\gamma(W, q^*)$ , since the  $j$  that participate in the dynamics are only the ones in  $\Phi^{-1}(W)$  (see the appendix for the proof in the case of enlarged-Knight chains). The Poincaré inequality reads

$$\text{Var}_V(f) \leq \gamma(W; q^*)^{-1} \sum_{j \in \Phi^{-1}(W)} \mu_V(c_j^* \text{Var}_{B_{U_j}}(f)), \quad \forall f, \tag{5.5}$$

We now bound a generic term  $\mu_V(c_j^*(\omega) \text{Var}_{B_{U_j}}(f))$ . Using Lemma A.3, Lemma A.2, and  $\ell \leq O(\kappa)2^{\beta\theta_q}$ , for any  $\varepsilon > 0$  and any  $q$  small enough depending on  $\varepsilon$  we get

$$\mu_V(c_j^*(\omega) \text{Var}_{B_{U_j}}(f)) \leq 2^{(\beta-\beta^2/2)\theta_q^2(1+\varepsilon/2)} \sum_{\substack{z \in B_{E_{j'}} \\ j'=j \text{ or } j' \prec j, j' \text{ neighbor of } j}} \mu_V(c_z^V \text{Var}_z(f)). \tag{5.6}$$

By combining (5.5) and (5.6) and using that  $|E_j| = O(\ell)$  we conclude for  $q$  small enough that

$$\text{Var}_V(f) \leq \gamma(W; q^*)^{-1} \times 2^{(2\beta-\beta^2)\frac{\theta_q^2}{2}(1+\varepsilon)} \mathcal{D}_V(f) \quad \forall f,$$

and the claim follows from the variational characterization of  $\gamma(V)$ . □

The claim together with (5.4) finally implies that  $\gamma(V) \geq 2^{-((1-\beta)^2/2+2\beta-\beta^2)\frac{\theta_q^2}{2}(1+\varepsilon)}$ ,  $\forall q \leq q(\varepsilon)$ , i.e. (5.3).

### 5.3 Proof of (iii)

We already know (cf. Lemma A.2) that  $\phi(\beta; d) \leq 1 \forall \beta$ . Fix now  $\beta \geq 1$  and consider the  $(\beta, 1)$ -outstretched one dimensional box  $\Lambda = \cup_{k=0}^{\lfloor 2^{\beta\theta_q} \rfloor} \{k e_1\}$ . The only subset  $V \subset \Lambda$  containing the origin and  $x_\Lambda$  and such that  $\gamma(V) > 0$  is  $V = \Lambda$ . But  $\gamma(\Lambda) = 2^{-\frac{\theta_q^2}{2}(1+o(1))}$  (see again Lemma A.2) so that  $\phi(\beta; d) \geq 1$ .

## A Appendix

We first state three results which have been used quite often in the previous sections and then we prove Lemmas 5.3 and 5.9.

**Lemma A.1.** *Consider two finite sets  $V_1, V_2 \subset \mathbb{Z}_+^d$  such that  $V_1 \ni 0$  and  $\exists z \in V_1$  such that  $z + e \in V_2$  for some  $e \in \mathcal{B}$  and the East chain on  $V_2$  with boundary condition  $\sigma$  having a unique vacancy at  $z$  is ergodic. Then  $\gamma(V_1 \cup V_2) \geq \frac{q}{4} \min(\gamma(V_1), \gamma^\sigma(V_2))$ .*

*Proof.* Let  $V = V_1 \cup V_2$  and consider the 2-block chain on  $\Omega_V$ , reversible w.r.t.  $\mu_V$ ,

- (i) with rate one  $\omega \upharpoonright_{V_1}$  is resampled from  $\mu_{V_1}$ ;
- (ii) with rate one  $\omega \upharpoonright_{V_2}$  is resampled from  $\mu_{V_2}$  iff  $\omega_z = 0$ .

The block chain has Dirichlet form

$$\mathcal{D}_V^{\text{block}}(f) = \mu_V(\text{Var}_{V_1}(f) + \mathbb{1}_{\{\omega_z=0\}} \text{Var}_{V_2}(f))$$

and spectral gap  $\gamma_V^{\text{block}}(f) = 1 - \sqrt{1-q} \geq q/2$  (see [8, Proposition 4.4]). Therefore, the Poincaré inequality for the block chain reads

$$\text{Var}_V(f) \leq 2/q \mu_V(\text{Var}_{V_1}(f) + \mathbb{1}_{\{\omega_z=0\}} \text{Var}_{V_2}(f)) \quad \forall f. \tag{1.1}$$

The definition of  $\gamma(V_1)$  and  $\gamma^\sigma(V_2)$  implies that

$$\text{Var}_{V_1}(f) \leq \gamma(V_1)^{-1} \sum_{x \in V_1} \mu_{V_1}(c_x^{V_1,1} \text{Var}_x(f)), \tag{1.2}$$

$$\mathbb{1}_{\{\omega_z=0\}} \text{Var}_{V_2}(f) \leq \gamma^\sigma(V_2)^{-1} \sum_{x \in V_2} \mathbb{1}_{\{\omega_z=0\}} \mu_{V_2}(c_x^{V_2, \omega^\uparrow \partial_\downarrow V_2} \text{Var}_x(f)). \tag{1.3}$$

It is now sufficient to insert the r.h.s. of (1.2), (1.3) into the r.h.s. of (1.1) and use the fact that both  $c_x^{V_1,1}, x \in V_1$ , and  $\mathbb{1}_{\{\omega_z=0\}} c_x^{V_2, \omega^\uparrow \partial_\downarrow V_2}, x \in V_2$ , are dominated by the constraint  $c_x^{V_1,1}$  to conclude that

$$\text{Var}_V(f) \leq 4/q \times \max(\gamma(V_1)^{-1}, \gamma^\sigma(V_2)^{-1}) \mathcal{D}_V^1(f) \quad \forall f,$$

where the additional factor of 2 appears if  $V_1 \cap V_2 \neq \emptyset$ . □

**Lemma A.2** ([12, Lemma 3.1 and eq. (2.9)]). *Consider the box  $\Lambda$  with side lengths  $(L_1, \dots, L_d)$  and let  $n \in \mathbb{N}$  be such that  $\max_i L_i \in (2^{n-1}, 2^n]$ . Then, as  $q \rightarrow 0$*

$$\gamma(\Lambda) = \begin{cases} 2^{-(n\theta_q - \binom{n}{2})(1+o(1))} & \text{if } n \leq \theta_q \\ 2^{-\frac{\theta_q^2}{2}(1+o(1))} & \text{otherwise.} \end{cases}$$

**Lemma A.3** ([12, Lemma 3.6]). *Let  $\Lambda_x = \Lambda + x$  where  $\Lambda$  is a box of  $\mathbb{Z}_+^d$  and  $x$  an arbitrary vertex. Let  $V \subset \Lambda_x$  be such that  $x \prec V$  and let  $A = \{\exists z \in \Lambda_x, z \prec V : \omega_z = 0\}$ . Then,*

$$\mu_{\Lambda_x}(\mathbb{1}_A \text{Var}_V(f)) \leq \gamma(\Lambda)^{-1} \mathcal{D}_{\Lambda_x}(f).$$

**Proof of Lemma 5.3** Clearly,  $\lambda \sim \mathcal{H}(0) \Rightarrow \lambda \sim \mathcal{H}'(0)$ . Suppose now that  $\lambda \sim \mathcal{H}'(0)$ , fix  $\kappa \geq 1, \varepsilon > 0$  and let  $\Lambda$  be a  $(0, \kappa, \theta_q)$ -outstretched box with side lengths  $(L_1, \dots, L_d)$ . Let  $N = \min_j L_j$  and for any  $i \in [d]$  choose a partition of the discrete interval  $\{0, 1, \dots, L_i\}$  into  $N+1$  discrete intervals,  $B_0^{(i)}, \dots, B_N^{(i)}$ , ordered from left to right, each one containing at least one vertex and at most  $\kappa + 1$  vertices. For  $\mathbf{j} \in \Lambda_B := \{0, \dots, N\}^d$  write  $B_{\mathbf{j}} = \prod_{i=1}^d B_{j_i}^{(i)}$  so that  $\cup_{\mathbf{j} \in \Lambda_B} B_{\mathbf{j}} = \Lambda$ . Furthermore, let  $\Omega_{\mathbf{j}}^* := \Omega_{B_{\mathbf{j}}}$ ,  $\mu_{\mathbf{j}}^* := \mu_{B_{\mathbf{j}}}$  and choose as facilitating event  $G_{\mathbf{j}}$  the event that the smallest vertex in  $B_{\mathbf{j}}$  in the  $\prec$ -ordering (for example the lowest-left corner if  $d = 2$ ) has a vacancy. Clearly  $\mu_{\mathbf{j}}^*(G_{\mathbf{j}}) = q \forall \mathbf{j} \in \Lambda_B$ , i.e.  $q^* = q$ . Recall now Definition 5.4. Using  $\lambda \sim \mathcal{H}'(0)$  there exists  $V^* \subset \Lambda_B$  containing the origin and  $x_{\Lambda_B}$  such that

$$\gamma^*(V^*) = \gamma(V^*; q^*) = \gamma(V^*) \geq 2^{-\lambda(1+\varepsilon/2) \frac{\theta_q^2}{2}}.$$

Hence, if we set  $V = \cup_{\mathbf{j} \in V^*} B_{\mathbf{j}}$  and write  $\text{Var}^*$  for the variance w.r.t.  $\mu^*$  we get

$$\text{Var}_V(f) = \text{Var}_{V^*}^*(f) \leq 2^{\lambda(1+\varepsilon/2) \frac{\theta_q^2}{2}} \sum_{\mathbf{j} \in V^*} \mu_V(c_{\mathbf{j}}^* \text{Var}_{B_{\mathbf{j}}}(f)).$$

Using Lemma A.3, A.2 and the fact that each box  $B_{\mathbf{j}}$  contains at most  $\kappa^d$  vertices, we get that the r.h.s. above is not larger than  $2^{\lambda(1+\varepsilon/2) \frac{\theta_q^2}{2}} 2^{O(\kappa^d)\theta_q} \mathcal{D}_\Lambda(f)$  so that

$$\text{Var}_V(f) \leq 2^{\lambda(1+\varepsilon/2) \frac{\theta_q^2}{2}} 2^{O(\kappa^d)\theta_q} \mathcal{D}_V(f) \leq 2^{\lambda(1+\varepsilon) \frac{\theta_q^2}{2}} \mathcal{D}_V(f).$$

Hence, for any  $q$  small enough depending on  $(\varepsilon, \kappa)$ ,  $\gamma(V) \geq 2^{-\lambda(1+\varepsilon) \frac{\theta_q^2}{2}}$  implying that  $\lambda \sim \mathcal{H}(0)$ . □

**Proof of Lemma 5.9** Recall Definition 5.8 and consider a partition  $\{Q_x\}_{x \in V^K}$  of  $(EV^K \setminus V^K) \cap \Lambda$  such that  $Q_x \subset E_x \forall x$ . The important point here is that the sets  $\{Q_x\}_{x \in V^K}$  are mutually disjoint, a feature not necessarily shared by the sets  $\{E_x \setminus \{x\}\}_{x \in V^K}$  (see Fig. 4). Instead of the \*Knight chain on  $\Omega_{EV^K \cap \Lambda}^*$  consider now the (very closely related) chain which at any legal update of the Knight chain at  $x \in V^K$  resamples the whole configuration in  $x \cup Q_x$ . This chain can be viewed as a new Knight chain on  $\Omega_{V^K}^*$  with new parameters  $\tilde{\Omega}_x^* = \otimes_{z \in x \cup Q_x} \Omega_z^*, \tilde{\mu}_x^* = \otimes_{z \in x \cup Q_x} \mu_z^*, x \in V^K$ , and the same facilitating events as the original Knight chain. Of course  $\otimes_{x \in V^K} (\tilde{\Omega}_x^*, \tilde{\mu}_x^*) = (\Omega_{V^K}^*, \mu_{V^K}^*)$ . Hence, the spectral gap of the new chain, as discussed after Definition 5.4, coincides with  $\gamma(V; q^*)$  and  $\forall f$

$$\begin{aligned} \text{Var}_{V^K}^*(f) &\leq \gamma(V; q^*)^{-1} \sum_{x \in V^K} \mu_{V^K}^*(K_x \text{Var}_{x \cup Q_x}^*(f)) \\ &\leq \gamma(V; q^*)^{-1} \sum_{x \in V^K} \mu_{V^K}^*(K_x \text{Var}_{E_x}^*(f)). \end{aligned}$$

where  $K_x$  is the Knight constraint at  $x$ . Above we used the fact that  $K_x$  does not depend on  $\{\omega_z\}_{z \in x \cup Q_x}$  and that  $\mu_{E_x}^*(\text{Var}_{x \cup Q_x}^*(f)) \leq \text{Var}_{E_x}^*(f)$ . The sum in the r.h.s. above is the Dirichlet form of the \*Knight chain and we conclude that its spectral gap is at least  $\gamma(V; q^*)$ . The reverse inequality follows immediately by projection onto the variables  $\eta_x = 1 - \mathbb{1}_{\{\omega_x \in G_x^*\}}, x \in V^K$ , where  $G_x^*$  is the facilitating event.  $\square$

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