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A distributed protocol for finite-time supremum or infimum dynamic consensus: The directed graph case☆

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ABSTRACT

This paper proposes a distributed protocol for tracking the global maximum supremum (or minimum infimum) of a set of exogenous time-varying signals. Specifically, each agent has only access to one of these signals and, by implementing the proposed protocol, it is able to track in finite-time the maximum supremum (or the minimum infimum) of the exogenous time-varying signals in a distributed fashion. No assumption is made on the network size and, remarkably, the communication graph is directed (possibly switching) with the only requirement to be strongly connected at all times. The behavior of the protocol with open networks is discussed and numerical simulations are provided to corroborate the theoretical findings.

1. Introduction

Deploying multiple autonomous agents that cooperate to realize a common goal has proven to be highly beneficial in a variety of applications [1]. In the absence of a centralized control unit, this objective must be reached by having each agent only rely on local information and interactions with neighboring agents. One of the most significant problems in distributed networked multi-agent systems is to achieve consensus [2], which refers to the process of reaching a common agreement among the agents. Different types of consensus problems can be identified, depending on the nature of the quantity of interest and the assumptions about the agents and their communication. Regarding the latter aspect, it should be noted that in the case of directed communication graphs, or digraphs, the exchange of information among the agents is asymmetrical, which significantly complicates the theoretical characterization compared to the case of undirected graphs [3]. A fundamental problem in distributed control is the *static average consensus*, where the agents are required to agree on the average of their initial values. This is among the first consensus problems that have been studied in the literature, with [4] being a seminal paper. Starting from this work, different aspects in static average consensus have been addressed, such as in [5,6]. The *dynamic* version of the average consensus, also referred to as *average consensus tracking*, has also been extensively investigated in the literature. In this case, the agents are required to track the average of time-varying exogenous signals, and relevant works can be found, for instance, in [7–9].

Additionally, the problem of reaching the consensus among the maximum or minimum of a given set of signals, also known as *max/min* consensus problem, has been investigated in the literature both for the static and the dynamic versions. Specifically, in the former case, the agents are required to agree on the maximum or minimum value of the initial states, while in the latter case, they have to track the maximum or minimum value among exogenous time-varying reference signals. Numerous contributions can be found in the literature which focus on the static version. Among these, asymptotic convergence is proved in [10] under the assumption of a weakly connected and weakly-balanced interaction digraph, while finite-time convergence is demonstrated in the case of a strongly connected interaction digraph. Asymptotic convergence with jointly connected communication graphs is reached, instead, in [11] with double-integrator agents, and further results are provided in [12,13]. With regard to the dynamic case, this is investigated in [14] where *bounded* error in case of time-varying signals is guaranteed.

Differently from max/min consensus, the objective in the present work is to achieve consensus on the *maximum supremum (or minimum infimum)* of a given set of signals, which we refer to as supremum (or infimum) consensus problems. Notably, these problems, which were formulated in our previous papers [15,16], can also be divided into static and dynamic according to whether the reference signals are constant or time-varying, respectively. Since the static case can be restated as a static max/min consensus problem, we focus our attention on the

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Table 1
Main notation adopted in this paper.

Variable	Meaning
$\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$	Time-varying graph of the network, with sets of agents $\mathcal{V}$ and time-varying edges $\mathcal{E}(t)$
$\mathcal{N}_i(t)$	Neighborhood of agent $i \in \mathcal{V}$ at time t
$\widetilde{\mathcal{N}}_i(t)$	Extended neighborhood of agent $i \in \mathcal{V}$ at time t, that is $\widetilde{\mathcal{N}}_i(t) = \mathcal{N}_i(t) \cup \{i\}$
$\widetilde{\mathcal{N}}_i^+(t)$	Subset of agents with local maximum state value in $\widetilde{\mathcal{N}}_i(t)$ at time t
i^+	Generic agent in the subset $\widetilde{\mathcal{N}}_i^+(t)$
$\bar{r}_i(t)$ ($\underline{r}_i(t)$)	Supremum (infimum) of the exogenous reference signal $r_i(t)$ local to agent i
$\bar{r}(t)$ ($\underline{r}(t)$)	Piecewise maximum (minimum) of the supremum (infimum) of the exogenous reference signals $\bar{r}_i(t)$
$x_i(t)$	State variable of agent $i \in \mathcal{V}$
$\boldsymbol{x}(t)$	Collective state vector
α	Gain of the proposed update law
$\phi_i(x_i(t), t)$	Selection function of the agent i
$\mathcal{I}^M(\boldsymbol{x}(t))$	Set of agents $M \in \mathcal{V}$ with maximum state in the network at time t
$\mathcal{I}^m(\boldsymbol{x}(t))$	Set of agents $m \in \mathcal{V}$ with minimum state in the network at time t such that $y_m(\mathbf{x}, t) > 0$
$\mathcal{I}^z(\boldsymbol{x}(t))$	Set of agents $z \in \mathcal{V}$ with minimum state in the network at time t such that $y_z(\mathbf{x}, t) = 0$

dynamic formulation. Note that the latter differs from the dynamic max (or min) consensus problem since the agents are not required to "track down" (or "track up") the maximum (or minimum) signal when it decreases (or increases) as required instead for the dynamic max (or max) consensus. A protocol achieving dynamic supremum (or infimum) consensus could be useful in various settings, such as anomaly detection, where the protocol could enable the timely recognition of signals surpassing critical values, or bound identification, where the protocol could determine the bounds for variables of interest, which can be necessary for gain tuning in subsequent protocols. In our previous works [15,16], we resorted to a dynamic supremum consensus protocol for monitoring box filling signals in a precision agriculture harvesting scenario, in order to trigger the prompt intervention for emptying the boxes when necessary. In all the above-mentioned circumstances, finite-time convergence is crucial. In the literature, the dynamic supremum (or infimum) consensus problem has been addressed in our previous works [15,16] and in [17]. Specifically, in [15] we designed a protocol dealing with undirected communication graphs with known bounds of the exogenous signal derivatives and proved the finite-time convergence of the protocol, while in [16] we relaxed the assumption of known bounds and designed an adaptive distributed protocol achieving finite-time convergence with undirected graphs. Authors of [17] consider switching digraphs. However, in [17] only *asymptotic* convergence is guaranteed and the reference signals are assumed to be bounded.

In this paper, we advance the state-of-the-art by designing a distributed protocol solving the dynamic supremum or infimum consensus problem in *finite-time* with *switching directed* graphs. Specifically, we only require the graph to be strongly connected at all times and the derivatives of the exogenous signals to be bounded with known bound, without requiring the exogenous signals to be bounded as in [17]. Moreover, we discuss the behavior of the proposed protocol in practical scenarios where dynamic changes in network topology occur not only in communication links (modeled by switching graphs) but also involve variations in the set of agents. The latter case is modeled by *open* networks, where agents could dynamically enter or exit the network. Simulative results validate the effectiveness of the proposed protocol using sinusoidal reference signals.

2. Preliminaries

This section presents preliminary notions for modeling the network of agents and for nonsmooth analysis. Table 1 shows the main notation adopted in this paper.

2.1. Network modeling

Let us consider a network composed of n agents with a directed and switching topology. We model the underlying topology via a time-varying digraph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of nodes representing the agents and $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ is the set of edges at time t. The existence of an edge $(i, j) \in \mathcal{E}(t)$ implies that agent i can send information to agent j at time t. We denote with $\mathcal{N}_i(t) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}(t)\}$ the neighborhood of agent i at time t and we define the respective extended neighborhood, including the agent i itself, as $\widetilde{\mathcal{N}}_i(t) = \mathcal{N}_i(t) \cup \{i\}$. Additionally, we denote the time instants in which the communication topology switches as $t_s > t_{s-1}$, with $s = 1, 2, ..$ and $t_0 = 0$. Finally, the graph is defined as strongly connected at time t if there exists a directed path connecting each pair of nodes of the graph at the same time t.

2.2. Nonsmooth analysis

The theoretical analysis of this paper relies on some fundamental concepts from nonsmooth analysis, which we briefly recall in the following. Let us consider a possibly discontinuous dynamical system with state vector $\boldsymbol{x} \in \mathbb{R}^n$

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Lebesgue measurable function with respect to $(\boldsymbol{x}(t), t)$ and essentially locally bounded [18]. If the right-hand side of the differential equation (1) is discontinuous, the solution is defined in the Filippov sense [19]. For the sake of notation convenience, we will omit the time dependency of $\boldsymbol{x}(t)$, unless necessary.

Let us now recall the definition of *absolutely continuous* function [18].

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each finite collection $\{(a_1, b_1), \dots, (a_n, b_n)\}$ of disjoint open intervals contained in $[a, b]$ with $\sum_{i=1}^n |b_i - a_i| < \delta$, it holds

$$\sum_{i=1}^n \left| f(b_i) - f(a_i) \right| < \varepsilon.$$

Definition 2 (*Filippov Solution [19]*)**.** A vector function $\boldsymbol{x}(t)$ is a Filippov solution of (1) on a time interval $[t_0, t_1]$ if $\boldsymbol{x}(t)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$ it holds $\dot{\boldsymbol{x}} \in K[f](\boldsymbol{x}, t)$, where $K[f](\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R} \to 2^{\mathbb{R}^n}$, with $2^{\mathbb{R}^n}$ the set of all subsets of $\mathbb{R}^n$, is a set-valued map defined as

$$K[f](\boldsymbol{x}, t) = \bigcap_{\delta > 0} \bigcap_{\mu\{H\}=0} \overline{\mathrm{co}} \left\{ f(B(\boldsymbol{x}, \delta) \setminus H, t) \right\}, \tag{2}$$

where $\mu(\cdot)$ denotes the Lebesgue measure of its set argument and $\bigcap_{\mu\{H\}=0}$ denotes the intersection over all sets H of Lebesgue measure zero, $B(\boldsymbol{x}, \delta)$ is the ball of radius δ centered at $\boldsymbol{x}$, and $\overline{\mathrm{co}}$ represents the convex closure.

Note that a sufficient condition for a function to be Lebesgue measurable is that it is Borel measurable [20].

In order to differentiate Lipschitz regular functions along Filippov solutions, we review the concept of Clarke's generalized gradient and the chain rule.

Definition 3 (*Clarke's Generalized Gradient [19]*). Let $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke's generalized gradient at $(\boldsymbol{x}, t)$ is given by

$$\partial V(\boldsymbol{x}, t) \triangleq \mathrm{co}\left\{ \lim_{k\to\infty} \nabla V(\boldsymbol{x}_k, t_k) : (\boldsymbol{x}_k, t_k) \to (\boldsymbol{x}, t), (\boldsymbol{x}_k, t_k) \notin \Omega_V \right\}, \tag{3}$$

with ∇V the gradient function, $(\boldsymbol{x}_k, t_k) \in \mathbb{R}^n \times \mathbb{R}$ is a pair of an infinite succession converging to $(\boldsymbol{x}, t)$, and Ω_V a set of Lebesgue measure zero containing all points where $\nabla V(\boldsymbol{x}, t)$ is not defined.

Theorem 1 (*Chain Rule [19]*). *Let $\boldsymbol{x}(\cdot)$ be a Filippov solution to (1) on an interval containing t and $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Lipschitz and regular function. Then, $V(\boldsymbol{x}, t)$ is absolutely continuous, $\dot{V}(\boldsymbol{x}, t)$ exists almost everywhere and $\dot{V}(\boldsymbol{x}, t) \in^{a.e.} \dot{\tilde{V}}(\boldsymbol{x}, t)$ with $\dot{\tilde{V}}(\boldsymbol{x}, t)$ defined as*

$$\dot{\tilde{V}}(\boldsymbol{x}, t) = \bigcap_{\xi \in \partial V(\boldsymbol{x}, t)} \xi^T \begin{pmatrix} K[f](\boldsymbol{x}, t) \\ 1 \end{pmatrix}. \tag{4}$$

We now recall the revised version of the generalized Lyapunov theorem for finite-time stability presented in [21].

Theorem 2 (*Finite-Time Stability Theorem*). *Let $\boldsymbol{x}(t)$ be a Filippov solution to (1) and $V(\boldsymbol{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, be a time dependent regular function such that $V(\boldsymbol{x}, t) = 0$, $\forall \boldsymbol{x}(t) \in C(t)$ and $V(\boldsymbol{x}, t) > 0$, $\forall \boldsymbol{x}(t) \notin C(t)$, with $C(t) \subset \mathbb{R}^n$ a compact set. Furthermore, let $\boldsymbol{x}(t)$ and $V(\boldsymbol{x}, t)$ be absolutely continuous in $[0, \infty)$ with $\dot{V}(\boldsymbol{x}, t) \leq -\epsilon < 0$ almost everywhere on $\{t : \boldsymbol{x}(t) \notin C(t)\}$. Then, $V(\boldsymbol{x}, t)$ converges to 0 in finite-time, and $\boldsymbol{x}(t)$ reaches the compact set $C(t)$ in finite-time as well.*

Finally, we introduce the discontinuous sign and sign^+ functions of $z \in \mathbb{R}$ and the respective set-valued functions SIGN and SIGN^+ as follows

$$\mathrm{sign}(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ -1 & \text{if } z < 0, \end{cases} \quad \mathrm{SIGN}(z) = \begin{cases} \{1\} & \text{if } z > 0, \\ [-1, 1] & \text{if } z = 0, \\ \{-1\} & \text{if } z < 0. \end{cases}$$

$$\mathrm{sign}^+(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z \leq 0, \end{cases} \quad \mathrm{SIGN}^+(z) = \begin{cases} \{1\} & \text{if } z > 0, \\ [0, 1] & \text{if } z = 0, \\ \{0\} & \text{if } z < 0, \end{cases}$$

3. Problem setting and proposed solution

3.1. Problem setting

To formally state the dynamic supremum and infimum consensus problems, let us first introduce the concepts of supremum and infimum of a function.

Definition 4. Consider a function $g : [0, \infty) \to \mathbb{R}$. Its supremum (infimum) function is defined as

$$\bar{g}(t) = \sup_{\tau \in [0,t]} \{g(\tau)\}, \left(\underline{g}(t) = \inf_{\tau \in [0,t]} \{g(\tau)\} \right).$$

We consider a networked system consisting of n agents interconnected by a time-varying digraph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$, for which we make the following assumption.

Assumption 1. The time-varying digraph $\mathcal{G}(t)$ that encodes the switching communication topology of the network is strongly connected at every time t. Furthermore, as in [22], any two consecutive switching time instants, denoted as t_{s-1}, t_s, are separated by an arbitrarily small positive dwell-time $t_{DW} > 0$, i.e., $t_s - t_{s-1} \geq t_{DW}$, thus ensuring that the switching digraph is non-chattering.

Each agent $i \in \mathcal{V}$ is endowed with a state $x_i(t) \in \mathbb{R}$ that evolves according to a first-order dynamics

$$\dot{x}_i(t) = u_i(t). \tag{5}$$

Additionally, each agent i has access to a scalar exogenous reference signal $r_i(t)$, for which we make the following assumption.

Assumption 2. The reference signals $r_i(t)$ are absolutely continuous $\forall i \in \mathcal{V}$. Moreover, there exists a positive constant $\psi_r \geq 0$ such that for all $\psi \in K[\dot{r}_i](t)$, $\forall i, t$, it holds $|\psi| \leq \psi_r$.

Let $\bar{r}(t)$ ($\underline{r}(t)$) be the maximum (minimum) of the supremum (infimum) of the reference signals $r_i(t)$, i.e.,

$$\bar{r}(t) = \max_{i \in \mathcal{V}} \{\bar{r}_i(t)\}, \left(\underline{r}(t) = \min_{i \in \mathcal{V}} \{\underline{r}_i(t)\} \right).$$

The distributed finite-time dynamic supremum (infimum) consensus problem is defined as follows.

Problem 1. Consider a multi-agent system with n agents with first-order dynamics (5) and let Assumptions 1–2 hold. The finite-time dynamic supremum (infimum) problem aims to define the control input $u_i(t)$ in (5) that guarantees finite-time tracking of the maximum (minimum) supremum (infimum) of the reference signals, i.e., such that there exists a finite-time $T > 0$ for which it holds

$$|x_i(t) - \bar{r}(t)| = 0, \quad (|x_i(t) - \underline{r}(t)| = 0) \quad \forall t \geq T, i \in \mathcal{V}. \tag{6}$$

3.2. Proposed distributed protocol

We now present the designed update law $u_i(t)$ for (5) solving the finite-time dynamic supremum (infimum) consensus problem described above. With regard to the supremum case, we define the following set

$$\widetilde{\mathcal{N}}_i^+(t) = \left\{ j \in \widetilde{\mathcal{N}}_i(t) : x_j(t) = \max_{\ell \in \widetilde{\mathcal{N}}_i(t)} \{x_\ell(t)\} \right\}, \tag{7}$$

collecting the agents with *local* maximum state in the extended neighborhood of agent i, and indicate with i^+ the index of any agent belonging to $\widetilde{\mathcal{N}}_i^+(t)$, i.e., $i^+ \in \widetilde{\mathcal{N}}_i^+(t)$. Note that all agents in $\widetilde{\mathcal{N}}_i^+(t)$ have same state value by construction, i.e., x_{i^+} is the same for all agents $i^+ \in \widetilde{\mathcal{N}}_i^+(t)$.

Furthermore, by denoting with $\boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ the stacked vector of the agents' states, we introduce the variable $y_i(\boldsymbol{x}, t) \in \mathbb{R}$ as follows

$$y_i(\boldsymbol{x}, t) = \left(x_{i^+}(t) - x_i(t) \right) + \phi_i \left(x_i(t), t \right) \left(\bar{r}_i(t) - x_i(t) \right), \tag{8}$$

which is non-negative by construction and where $\phi_i(\cdot)$ is a selection function equal to

$$\phi_i(x_i(t), t) = \begin{cases} 0, & \text{if } x_i(t) \geq \bar{r}_i(t), \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

Then, for each agent $i \in \mathcal{V}$, we propose the following monotonically non-decreasing control input for the dynamic supremum consensus problem

$$u_i(t) = \alpha \, \mathrm{sign}\left(y_i(\boldsymbol{x}, t) \right) \tag{10}$$

with α a positive constant.

Remark 1. In the case of dynamic infimum consensus, the same form of the control law in (10) is preserved, where y_i is adapted as follows

$$y_i(\boldsymbol{x}, t) = \left(x_{i^-}(t) - x_i(t) \right) + \phi_i \left(x_i(t), t \right) \left(\underline{r}_i(t) - x_i(t) \right), \tag{11}$$

which is non-positive by construction and where i^- represents any agent in the extended neighborhood of i with minimum state and $\phi_i(\cdot)$ is equal to 0 if $x_i(t) \leq \underline{r}_i(t)$ and is 1 otherwise.

4. Theoretical analysis

In this section, we focus on the theoretical analysis of the proposed protocol. Since the formal analyses of the supremum and the infimum consensus cases follow the same reasonings, we only focus on the former problem, thus proving that the protocol (10) solves Problem 1.

Before enunciating the main theorem of the paper (i.e., Theorem 3) some required intermediate results are presented in the following, which are necessary to demonstrate that the proposed discontinuous update law is measurable and locally essentially bounded, as required for nonsmooth analysis recalled in Section 2.2, and to compute the set-valued map $K[\text{sign} \circ y_i](\boldsymbol{x}, t)$, where $\circ$ denotes the function composition.

Proposition 1. *Consider a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $f(\boldsymbol{x}, t)$ is continuous almost everywhere, except at a countable set of points. Then, $f(\boldsymbol{x}, t)$ is Borel measurable.*

Proof. Proof in Appendix A. □

In the following, we report relevant properties related to the function $y_i(\boldsymbol{x}, t)$, for all $i \in \mathcal{V}$.

Property 1. *Let Assumptions 1–2 hold. Then, for all $i \in \mathcal{V}$, (i) the function $y_i(\boldsymbol{x}, t)$ is continuous w.r.t. $\boldsymbol{x}$ and continuous almost everywhere w.r.t. $(\boldsymbol{x}, t)$, except at a countable set of points, and (ii) the function $\text{sign}(y_i(\boldsymbol{x}, t))$ is Lebesgue measurable w.r.t. $(\boldsymbol{x}, t)$ and locally essentially bounded.*

Proof. Proof in Appendix B. □

In the following, by resorting to Definition 2, we characterize the Filippov map $K[\text{sign} \circ y_i](\boldsymbol{x}, t)$.

Proposition 2. *The set-valued map $K[\text{sign} \circ y_i](\boldsymbol{x}, t)$ is given by*

$$K[\text{sign} \circ y_i](\boldsymbol{x}, t) = \text{SIGN}^+(y_i(\boldsymbol{x}, t)).$$

Proof. The proof is provided in Appendix C. □

At this point, let us introduce the set $\mathcal{I}^M(\boldsymbol{x}(t))$ collecting the agents with the maximum state in the network at time t, i.e.,

$$\mathcal{I}^M(\boldsymbol{x}(t)) = \{i \in \mathcal{V} \mid x_i(t) = \max_{\ell \in \mathcal{V}}\{x_\ell(t)\}\}. \tag{12}$$

The following proposition holds true regarding the dynamics of the agents in $\mathcal{I}^M(\boldsymbol{x}(t))$.

Proposition 3. *Assume that each agent runs the control input in (10). Consider any agent M in $\mathcal{I}^M(\boldsymbol{x}(t))$ and assume that $x_M(t) > \bar{r}(t)$. Then,*

$$\dot{x}_M(t) =^{a.e.} 0, \quad M \in \mathcal{I}^M(\boldsymbol{x}(t)). \tag{13}$$

Proof. Proof in Appendix D. □

The following lemma provides an inequality between the agents' states and the maximum supremum signal.

Lemma 1. *Let Assumption 2 hold. Assume that the agents run the control input in (10). Then, if $x_M(0) \le \bar{r}(0)$, $\forall M \in \mathcal{I}^M(\boldsymbol{x}(0))$, it holds $x_i(t) \le \bar{r}(t)$, $\forall i \in \mathcal{V}, t$.*

Proof. Proof in Appendix E. □

We additionally define the set $\mathcal{I}^m(\boldsymbol{x})$ of agents holding the minimum state in the network at time t, i.e.,

$$\mathcal{I}^m(\boldsymbol{x}) = \{i \in \mathcal{V} \mid x_i(t) = \min_{\ell \in \mathcal{V}}\{x_\ell(t)\}\}, \tag{14}$$

and the respective subset $\mathcal{I}^z(\boldsymbol{x}) \subseteq \mathcal{I}^m(\boldsymbol{x})$ of agents having zero variable $y_z(\cdot)$, with $z \in \mathcal{I}^z(\boldsymbol{x})$, i.e.,

$$\mathcal{I}^z(\boldsymbol{x}) = \{z \in \mathcal{I}^m(\boldsymbol{x}) \mid y_z(\boldsymbol{x}, t) = 0\}, \tag{15}$$

Given these sets, we can prove the following propositions.

Proposition 4. *Let $z \in \mathcal{I}^z(\boldsymbol{x})$. Then, it follows that $x_z(t) \ge \bar{r}_z(t)$ and*

$$\mathcal{N}_z(t) \subseteq \mathcal{I}^m(\boldsymbol{x}). \tag{16}$$

Proof. Let us first observe that, being $z \in \mathcal{I}^z(\boldsymbol{x})$, it holds $y_z(\boldsymbol{x}, t) = 0$ by definition as in (15). Hence, since $y_z(\cdot)$ is the sum of two non-negative terms as in the definition (8), it follows that the two terms $x_{z^+}(t) - x_z(t)$ and $\phi_z(\cdot)(\bar{r}_z(t) - x_z(t))$ are zero as well. The equation $x_{z^+}(t) - x_z(t) = 0$ implies that $x_{z^+}(t) \in \mathcal{I}^m(\boldsymbol{x})$, leading to the result in (16). The equation $\phi_z(\cdot)(\bar{r}_z(t) - x_z(t)) = 0$ leads immediately to the inequality $x_z(t) \ge \bar{r}_z(t)$, concluding the proof. □

Proposition 5. *Let Assumption 1 hold. Consider that the agents run the control input in (10). Moreover, suppose that $\mathcal{I}^m(\boldsymbol{x}) \equiv \mathcal{I}^z(\boldsymbol{x})$ holds. Then, all the agents belong to $\mathcal{I}^z(\boldsymbol{x})$, i.e., $\mathcal{V} \equiv \mathcal{I}^z(\boldsymbol{x})$.*

Proof. This proposition easily follows by considering that, since $\mathcal{I}^m(\boldsymbol{x}) \equiv \mathcal{I}^z(\boldsymbol{x})$ and being the digraph strongly connected, then starting from any agent $m \in \mathcal{I}^z(\boldsymbol{x})$ we can recursively apply (16) until all agents are reached, leading to $\mathcal{V} \equiv \mathcal{I}^z(\boldsymbol{x})$. □

Based on the previous results, we can prove the following lemma stating that if the network minimum state, i.e., x_m with $m \in \mathcal{I}^m(\boldsymbol{x})$, is equal to the maximum supremum signal, then all the minimum agents in $\mathcal{I}^m(\boldsymbol{x})$ have zero signal $y_m(\cdot)$ and vice-versa.

Lemma 2. *Let Assumption 1 hold and assume that $x_M(0) \le r_M(0)$, $\forall M \in \mathcal{I}^M(\boldsymbol{x}(0))$. Then, it holds*

$$x_m(t) = \bar{r}(t),\ m \in \mathcal{I}^m(\boldsymbol{x}) \quad \Longleftrightarrow \quad \mathcal{I}^m(\boldsymbol{x}) \equiv \mathcal{I}^z(\boldsymbol{x}). \tag{17}$$

Proof. Part ⇒) Suppose that $x_m(t) = \bar{r}(t)$ for $m \in \mathcal{I}^m(\boldsymbol{x})$. In order to prove that $\mathcal{I}^m(\boldsymbol{x}) \equiv \mathcal{I}^z(\boldsymbol{x})$, we show that the condition $y_m(\boldsymbol{x}, t) = 0$ holds for all $m \in \mathcal{I}^m(\boldsymbol{x})$. To this end, as per (8), we show that for such agents both the terms $x_{m^+}(t) - x_m(t)$ and $\phi_m(\cdot)(\bar{r}_m(t) - x_m(t))$ are zero. Since by assumption it holds $x_m(t) = \bar{r}(t)$ for $m \in \mathcal{I}^m(\boldsymbol{x})$ and since, by definition, $\bar{r}(t) \ge \bar{r}_i(t)$ for all $i \in \mathcal{V}$, it follows that $x_m(t) \ge \bar{r}_m(t)$, by which, from (9), we have $\phi_i(\cdot)(\bar{r}_m(t) - x_m(t)) = 0$. Let us now observe that, in view of Lemma 1, it holds $x_i(t) \le \bar{r}(t)$ for all $i \in \mathcal{V}$. This implies that

$$\bar{r}(t) = x_m(t) \le x_i(t) \le \bar{r}(t) \quad \forall i \in \mathcal{V},$$

by which it follows that $\mathcal{V} \equiv \mathcal{I}^m(\boldsymbol{x})$, hence, $x_{m^+}(t) - x_m(t) = 0$ for all $m \in \mathcal{I}^m(\boldsymbol{x})$.

Part ⇐) Suppose that $\mathcal{I}^m(\boldsymbol{x}) \equiv \mathcal{I}^z(\boldsymbol{x})$. By Proposition 5, it holds $\mathcal{V} \equiv \mathcal{I}^z(\boldsymbol{x})$. Moreover, by Proposition 4, it holds $x_i(t) \ge \bar{r}_i(t)$ for all $i \in \mathcal{V}$. Since $x_m(t) = x_i(t)$ for all i, it follows that

$$x_m(t) \ge \max_{i \in \mathcal{V}}\{\bar{r}_i(t)\} = \bar{r}(t), \quad \forall m \in \mathcal{I}^m(\boldsymbol{x}),$$

which combined with the result of Lemma 1, leads to the condition $x_m(t) = \bar{r}(t)$ for all $m \in \mathcal{I}^m(\boldsymbol{x})$. □

We can now state the main theorem of the manuscript showing that the protocol in (10) solves Problem 1 and providing an upper bound on the convergence time.

Theorem 3. *Consider a multi-agent system of n agents interconnected by a time-varying directed graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ and let Assumptions 1–2 hold. Suppose that the agents run (10) with gain α satisfying*

$$\alpha > \psi_r + \varepsilon, \tag{18}$$

with $\varepsilon > 0$, and assume that $x_M(0) \le r_M(0)$, $\forall M \in \mathcal{I}^M(\boldsymbol{x}(0))$. Then, the agents track the maximum supremum $\bar{r}(t)$ in finite-time and an upper bound to the convergence time is T, defined as

$$T = \frac{1}{\varepsilon}\left(\bar{r}(0) - x_m(0)\right), \quad m \in \mathcal{I}^m(\boldsymbol{x}(0)). \tag{19}$$

Proof. To prove our result, let us define the set $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) = \mathcal{I}^m(\boldsymbol{x}) \setminus \mathcal{I}^z(\boldsymbol{x})$, collecting the agents m in $\mathcal{I}^m(\boldsymbol{x})$ with positive function $y_m(\boldsymbol{x}, t)$, and the function $h(\boldsymbol{x}, t)$ with $m \in \mathcal{I}^{m\setminus z}(\boldsymbol{x})$ as follows

$$h(\boldsymbol{x}, t) = \begin{cases} \bar{r}(t) - x_m(t), & \text{if } \mathcal{I}^{m\setminus z}(\boldsymbol{x}) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

which is non-negative by virtue of Lemma 1. We consider the following Lyapunov candidate

$$V(\boldsymbol{x},t) = |h(\boldsymbol{x},t)|. \tag{20}$$

Note that, being $\bar{r}(t)$ and $x_m(t)$ absolutely continuous by Assumption 2 and Definition 2, respectively, and since, by Lemma 2, it holds $x_m(t) = \bar{r}(t)$ for all $m \in \mathcal{I}^m(\boldsymbol{x})$ if and only if $\mathcal{I}^m(\boldsymbol{x}) \setminus \mathcal{I}^z(\boldsymbol{x}) = \mathcal{I}^{m\setminus z}(\boldsymbol{x}) = \emptyset$, we have that $h(\boldsymbol{x},t)$ is an absolutely continuous function, implying the absolute continuity of $V(\boldsymbol{x},t)$ as required by Theorem 2. Results of Lemma 2 also establish that $V(\boldsymbol{x},t) = 0$ if and only if $x_m(t) = \bar{r}(t)$, which in view of Lemma 1 leads to $V(\boldsymbol{x},t) = 0$ if and only if $x_i(t) = \bar{r}(t)$ for all $i \in \mathcal{V}$.

Let us first analyze the case when $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) \neq \emptyset$. In this case, we can observe that an equivalent form of (20) is

$$V(\boldsymbol{x},t) = \left| \bar{r}(t) - \frac{1}{|\mathcal{I}^{m\setminus z}(\boldsymbol{x})|} \sum_{m\in\mathcal{I}^{m\setminus z}(\boldsymbol{x})} x_m(t) \right|. \tag{21}$$

The Clarke's generalized gradient $\partial V(\boldsymbol{x},t)$, defined in (3), can be expressed as $\partial V(\boldsymbol{x},t) = \left[\partial_x V(\boldsymbol{x},t)^T,\, \partial_t V(\boldsymbol{x},t)\right]^T$, with

$$\begin{aligned} \partial_x V(\boldsymbol{x},t) &= -\,\mathrm{SIGN}(h(\boldsymbol{x},t)) \frac{1}{|\mathcal{I}^{m\setminus z}(\boldsymbol{x})|} \boldsymbol{s}, \\ \partial_t V(\boldsymbol{x},t) &= \mathrm{SIGN}(h(\boldsymbol{x},t)) K[\dot{\bar{r}}], \end{aligned} \tag{22}$$

where $\boldsymbol{s} \in \mathbb{R}^n$ is a selection vector with component i equal to 1 if $i \in \mathcal{I}^{m\setminus z}(\boldsymbol{x})$, 0 otherwise. Moreover, in view of [23], the set-valued map $K[\dot{\boldsymbol{x}}](\boldsymbol{x},t)$ can be computed as follows

$$K[\dot{\boldsymbol{x}}](\boldsymbol{x},t) \subseteq \left[K[u_1](\boldsymbol{x},t),\, K[u_2](\boldsymbol{x},t),\, \dots,\, K[u_n](\boldsymbol{x},t)\right]^T. \tag{23}$$

By combining the above results, from Theorem 1, the generalized derivative $\dot{\tilde{V}}(\boldsymbol{x},t)$ is given by

$$\dot{\tilde{V}}(\boldsymbol{x},t) = \bigcap_{\substack{\eta\,\in\,\mathrm{SIGN}(h(\cdot)) \\ \psi\,\in\,K[\dot{\bar{r}}](t)}} \underbrace{\eta \left(-\frac{1}{|\mathcal{I}^{m\setminus z}(\boldsymbol{x})|} \sum_{m\in\mathcal{I}^{m\setminus z}(\boldsymbol{x})} K[\dot{x}_m](\boldsymbol{x},t) + \psi \right)}_{g(\eta,\psi,t)}, \tag{24}$$

where η and ψ represent scalar values belonging to the respective sets. By virtue of Proposition 2, it holds

$$K[\dot{x}_m](\boldsymbol{x},t) = \alpha\ \mathrm{SIGN}^+\left(y_m(\boldsymbol{x},t)\right), \quad \forall m \in \mathcal{I}^{m\setminus z}(\boldsymbol{x})$$

hence,

$$g(\eta,\psi,t) = \eta \left(-\alpha \frac{1}{|\mathcal{I}^{m\setminus z}(\boldsymbol{x})|} \sum_{m\in\mathcal{I}^{m\setminus z}(\boldsymbol{x})} \mathrm{SIGN}^+(y_m(\boldsymbol{x},t)) + \psi \right).$$

Our objective is to demonstrate that as long as $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) \neq \emptyset$, it holds $\dot{V}(\boldsymbol{x},t) < -\varepsilon < 0$. To this aim, let us study the term $g(\eta,\psi,t)$. We can observe that, $\forall m \in \mathcal{I}^{m\setminus z}(\boldsymbol{x})$ it holds by construction $y_m(\boldsymbol{x},t) > 0$ and

$$\mathrm{SIGN}^+(y_m(\boldsymbol{x},t)) = \{1\}.$$

Moreover, being $h(\boldsymbol{x},t)$ positive by virtue of Lemmas 1 and 2, we obtain

$$\mathrm{SIGN}(h(\boldsymbol{x},t)) = \{1\},$$

leading to

$$-\alpha\,\eta \frac{1}{|\mathcal{I}^{m\setminus z}(\boldsymbol{x})|} \sum_{m\in\mathcal{I}^{m\setminus z}(\boldsymbol{x})} \mathrm{SIGN}^+(y_m(\boldsymbol{x},t)) = \{-\alpha\}.$$

By combining the above results, we obtain

$$\varphi = \psi - \alpha < \psi_r - \alpha < -\varepsilon, \quad \forall \varphi \in g(\eta,\psi,t), \tag{25}$$

where we exploited the fact that, by Assumption 2, $\psi < \psi_r$ for all $\psi \in K[\dot{\bar{r}}](t)$, and, for α satisfying (18), it holds $\psi_r - \alpha < -\varepsilon$. At this point, by plugging (25) in (24) and by recalling that $\dot{V}(\boldsymbol{x},t) \in^{a.e.} \dot{\tilde{V}}(\boldsymbol{x},t)$, we obtain that the following inequality

$$\dot{V}(\boldsymbol{x},t) < -\varepsilon, \tag{26}$$

holds true as long as $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) \neq \emptyset$. It follows that $V(\boldsymbol{x},t)$ is decreasing with a constant rate, leading to $h(\boldsymbol{x},t) = 0$ in finite-time which, as stated above, by Lemma 2 implies that $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) = \emptyset$.

At this point, let us analyze the case when $\mathcal{I}^{m\setminus z}(\boldsymbol{x}) = \emptyset$ and consider an interval of time $(t, t+\delta)$, with $\delta > 0$ sufficiently small. By continuity of $h(\boldsymbol{x},t)$, it can occur either that $h(\boldsymbol{x},\tau) = 0$ or $h(\boldsymbol{x},\tau) > 0$ for all $\tau \in (t, t+\delta)$. In other words, in view of Lemma 2, either the set $\mathcal{I}^{m\setminus z}(\boldsymbol{x}(\tau))$ is empty or non-empty for the entire interval. Based on the above considerations, we can prove that, if $V(\boldsymbol{x},t) = 0$, then $V(\boldsymbol{x},\tau) = 0$ for all $\tau \geq t$. Indeed, let us assume that $h(\boldsymbol{x},t) = 0$ and that, by contradiction, there exists an instant $t_1 \in (t, t+\delta)$ such that $h(\boldsymbol{x},t_1) > 0$. Note that, by construction, it holds $h(\boldsymbol{x},\tau) > 0$ for all $\tau \in (t, t_1)$. It follows that

$$V(\boldsymbol{x}(t_1),t_1) = \underbrace{V(\boldsymbol{x},t)}_{=0} + \int_t^{t_1} \underbrace{\dot{V}(\boldsymbol{x}(\tau),\tau)}_{<-\varepsilon}\, d\tau < 0,$$

which contradicts our assumption that $h(\boldsymbol{x},t_1) > 0$, which is equivalent to $V(\boldsymbol{x},t_1) > 0$. By combining the above result and (26), we achieve that $V(\boldsymbol{x},t)$ vanishes in finite-time and then remains zero. To conclude our proof, we derive the bound in (19). By noting that it holds $\dot{V}(\boldsymbol{x},t) < -\varepsilon$ until $V(\boldsymbol{x},t) \neq 0$, we obtain

$$\begin{aligned} V(\boldsymbol{x},t) &= V(\boldsymbol{x}(0),0) + \int_0^t \underbrace{\dot{V}(\boldsymbol{x}(\tau),\tau)}_{<-\varepsilon}\, d\tau \\ &< V(\boldsymbol{x}(0),0) - \varepsilon t, \end{aligned} \tag{27}$$

from which the bound in (19) is obtained by solving

$$V(\boldsymbol{x}(0),0) - \varepsilon T = 0.$$

This concludes the proof. □

Note that the hypothesis on $x_M(0) \leq r_M(0),\ \forall M \in \mathcal{I}^M(\boldsymbol{x}(0))$ can be easily fulfilled by initializing the state of each agent equal to the initial reference value, i.e., $x_i(0) = r_i(0)$.

5. Discussion on open networks

In this section, we aim to investigate the behavior of the proposed protocol with open networks. The first aspect to address with such networks is to appropriately extend the definition of the maximum supremum and minimum infimum signals provided in Definition 4. In detail, let $\mathcal{V}(t)$ denote the set of agents at time t. The local supremum (infimum) reference signal of the agent i is defined as

$$\bar{r}_i(t) = \sup_{\tau\in[t_{0,i},t]} \{r_i(\tau)\},\ \left(\underline{r}_i(t) = \inf_{\tau\in[t_{0,i},t]} \{r_i(\tau)\}\right),$$

with $t_{0,i}$ denoting the time instant when the agent i joins the network. Based on this, the open network case leads to two possible definitions of the maximum supremum (minimum infimum) signal:

1. *History-based*: taking into account the references of all agents that have been part of the network until time t

$$\bar{r}^H(t) = \max_{i\in\mathcal{V}_c(t)} \{\bar{r}_i(t)\},\ \left(\underline{r}^H(t) = \min_{i\in\mathcal{V}_c(t)} \{\underline{r}_i(t)\}\right), \tag{28}$$

with $\mathcal{V}_c(t) = \bigcup_{\tau\in[0,t]} \mathcal{V}(\tau)$ accumulating all the nodes until time t. Note that this set does not change when an agent exits the network, but expands when new agents join the network.

2. *Instant-based:* taking into account only the references of the agents currently belonging to the network

$$\bar{r}^I(t) = \max_{i\in\mathcal{V}(t)} \{\bar{r}_i(t)\},\ \left(\underline{r}^I(t) = \min_{i\in\mathcal{V}(t)} \{\underline{r}_i(t)\}\right). \tag{29}$$

In the following, we focus on the maximum supremum dynamic consensus problem as similar considerations can be made for the minimum infimum case. Let t_c denote the time instant when the set of nodes changes, i.e., an agent enters/leaves the network, and t_c^- denote the moment immediately preceding this variation. We make the following assumption in the following.

Assumption 3. Each agent persists in the network for a minimum duration of time necessary to achieve tracking of the maximum supremum signal.

Note that such a duration is finite by virtue of Theorem 3. This assumption is motivated by the fact that, if an agent leaves the network before tracking is achieved, the information related to its local supremum might become unaccounted for, since it becomes unavailable to the other agents.

We first analyze the behavior of the proposed protocol with the history-based supremum definition. If a new agent i joins the network, with $x_i(t_c) \leq r_i(t_c)$, the following situations can occur:

- $r_i(t_c) \leq \bar{r}^H(t_c^-)$. In this case, the agent i does not influence the maximum supremum signal $\bar{r}^H(t_c)$ which is being tracked by the other agents within the network and will achieve tracking of the maximum supremum signal in finite-time as per Theorem 3.
- $r_i(t_c) > \bar{r}^H(t_c^-)$. This results in a discontinuity in the maximum supremum signal, which rises to $\bar{r}^H(t_c) = r_i(t_c)$. The proposed protocol guarantees that all the agents will reach and track the maximum supremum signal in finite-time.

If an agent leaves the network, there is no effect on the history-based maximum supremum signal and the tracking of the latter is guaranteed by Theorem 3. In summary, the proposed protocol allows to handle open networks in the case of history-based definition.

When resorting to the instant-based supremum definition, the same considerations made above for a new agent entrance hold. Regarding the scenario of an agent i exiting the network, the following situations can occur:

- $\bar{r}_i(t_c^-) < \bar{r}^I(t_c^-)$. In this case, the departing agent does not influence the instant-based maximum supremum signal and has no impact on the other agents, which keep tracking the maximum supremum signal.
- $\bar{r}_i(t_c^-) = \bar{r}^I(t_c^-)$ and $\bar{r}^I(t_c^-) = \bar{r}_j(t_c^-)$ with $j \in \mathcal{V}(t_c)$. In this case, the departing agent possesses the maximum supremum signal, and this signal is also held by other agents still in the network. Similar observations as in the previous point apply.
- $\bar{r}_i(t_c^-) = \bar{r}^I(t_c^-)$ and $\bar{r}^I(t_c^-) > \bar{r}_j(t_c^-)$, $\forall j \in \mathcal{V}(t_c)$. When the agent i is the only one holding the maximum supremum signal, its departure causes a discontinuity in the instant-based maximum supremum, which drops to $\bar{r}^I(t_c) < \bar{r}^I(t_c^-)$. However, as the proposed update law is monotonically non-decreasing, they would keep tracking the signal $\bar{r}^I(t_c^-)$.

In summary, for the instant-based definition, while new entrances pose no challenges, a re-initialization mechanism is necessary for agents to manage possible drops in the maximum supremum signals due to departures. More in detail, by following the operational conditions in [13], two settings can be considered. In the first setting, agents can announce their departures, allowing the remaining agents to re-initialize the maximum supremum protocol when a departure notification is received. Specifically, we let agents announce the departure only when their local supremum is equal to the maximum supremum signal (estimated through the state variable). The re-initialization is realized by fulfilling the conditions of Theorem 3, which, practically speaking, can be achieved by setting $x_i(t) = \bar{r}_i(t)$, $\forall i \in \mathcal{V}(t)$. In the second setting, agents do not announce the departure. In this case, a heartbeat mechanism can be used (e.g., by following the principles of [24]). Hence, the agents with local supremum equal to the maximum supremum signal have to periodically broadcast a signal testifying that they are still alive. In the meantime, the other agents monitor the receipt of these heartbeat signals, and, in the event of an absence of a heartbeat signal beyond a timeout period, the re-initialization is triggered. Notably, the possible occurrence of delays in recognizing the departure of the agent with maximum supremum might result in the other agents temporarily tracking the signal associated with the departed agent. Nevertheless, the use of the proposed protocol guarantees the absence of any instability issues during this time interval.

6. Simulation results

In this section, we validate the proposed control law and corroborate the bound on the convergence time with simulation results. We consider $n = 10$ agents having sinusoidal exogenous reference signals $r_i(t)$, depicted in Fig. 2 and expressed as

$$r_i(t) = a_i \sin(w_i\, t) + b_i\, t, \tag{30}$$

where the stacked vectors $\boldsymbol{a}, \boldsymbol{w}, \boldsymbol{b}$ of the coefficients a_i, w_i, b_i, respectively, are chosen as

$$\boldsymbol{a} = [10,\ -7.5,\ -7.5,\ 10,\ -10,\ 7.5,\ -7.5,\ 12.5,\ -12.5,\ 10]^T,$$
$$\boldsymbol{w} = [10,\ 5,\ 5,\ 10,\ 10,\ 12.5,\ 12.5,\ 12.5,\ 15,\ 15]^T,$$
$$\boldsymbol{b} = [1.25,\ 3.75,\ -2.5,\ 6.25,\ 6.25,\ 7.5,\ 7.5,\ 5,\ 5,\ 5]^T,$$

leading to the bound ψ_r on $K[\dot{r}_i](t)$ equal to $\psi_r = 162$. The initial values of the exogenous reference signals are set as

$$\boldsymbol{r}(0) = [40,\ 38,\ 36,\ 34,\ 33,\ 27,\ 25,\ 22,\ 18,\ 15]^T.$$

As required by Theorem 3, the agents' states are initialized such that $x_M(0) \leq r_M(0)$, $\forall M \in \mathcal{I}^M(\boldsymbol{x}(0))$, as follows

$$\boldsymbol{x}(0) = [12,\ 15,\ 10,\ 14,\ 13,\ 13,\ 12,\ 11,\ 10,\ 13]^T.$$

In the simulation, we set $\alpha = 200$ and randomly switch the communication digraph $\mathcal{G}(t)$ every 0.5 s, by ensuring that it is always strongly connected. The resulting graphs are shown in Fig. 1. In addition, Fig. 3 depicts the agents' states $x_i(t)$ (solid lines), the supremum of the reference signals $\bar{r}_i(t)$ (fine dotted lines) and the maximum supremum $\bar{r}(t)$ (thick dark blue dotted line). From the figure, it can be noted that the agents reach consensus on $\bar{r}(t)$ at $t \approx 0.2$ s, hence before the theoretical bound on convergence time $T = 1$ s. Moreover, the agents continue tracking $\bar{r}(t)$ from $t \approx 0.2$ s onward and do not lose track of the signal also at time instants when there is a change in the index of the agent with maximum supremum signal. An example of such instants is $t \approx 2$ s, when $\bar{r}_4(t)$ (red) surpasses $\bar{r}_5(t)$ (light blue), becoming the maximum supremum signal $\bar{r}(t)$. Note that the observed convergence time is coherent with the theoretical one defined in (19). In fact, by selecting $\epsilon = 30$ we obtain $T = 1$ s. Finally, Fig. 4 depicts the trend of the Lyapunov function $V(\boldsymbol{x}, t)$. Starting from the initial value $V(\boldsymbol{x}(0), 0) = |\bar{r}(0) - x_m(0)| = 30$, the function slopes down to zero in $t \approx 0.2$ s and remains zero throughout the remainder of the simulation.

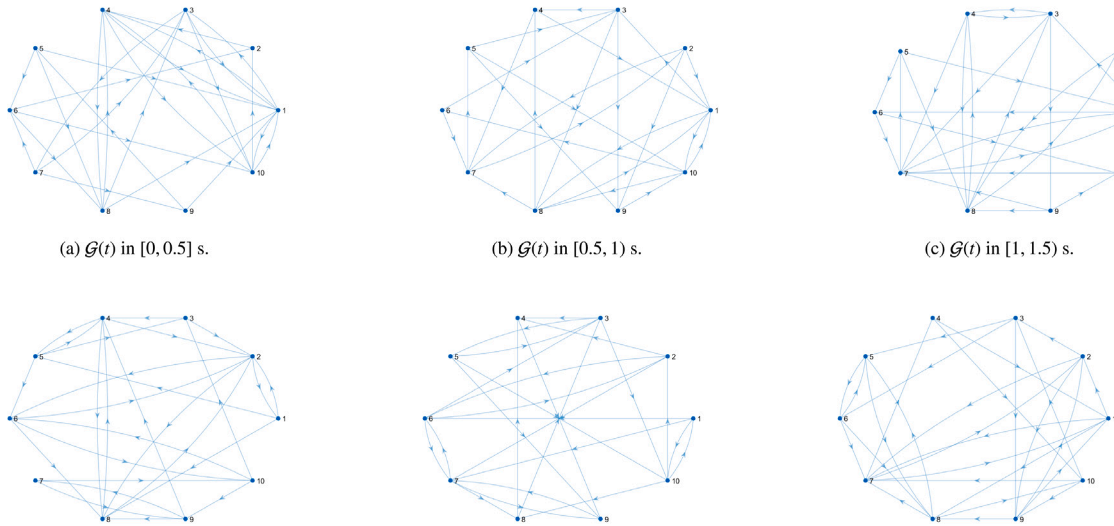
6.1. Results with open networks

In this section, we analyze the outcomes of the proposed protocol with open networks. For the sake of space, we focus on the instant-based supremum formulation, given its greater complexity compared to the history-based counterpart, and consider the setting where agents can announce the departure. In this case study, both the edges and the nodes change simultaneously every second. Focusing on the set of agents $\mathcal{V}(t)$, initially, the network consists of $n = 8$ agents, with $\mathcal{V}(t) = \{1, \ldots, 8\}$ for all $t \in [0, 1)$ s. At time $t = 1$ s, agents 5, 6, 7, 8 leave the network, at $t = 2$ s, agents 9 and 10 join the network, at $t = 3$ s, agent 10 leaves the network, and, finally, at $t = 4$ s, agent 11 joins the network. Concerning the set of edges $\mathcal{E}(t)$, this is randomly generated preserving strong connectivity.

Sinusoidal reference signals with the same form in (30) are considered, where $\boldsymbol{a}, \boldsymbol{w}, \boldsymbol{b}$ are chosen as

$$\boldsymbol{a} = [2.5, -12.5, 5,\ 10,\ 7.5,\ 7.5,\ -12.5, -12.5,\ 2.5 - 2.5,\ 2.5]^T,$$
$$\boldsymbol{w} = [10,\ 15,\ 12.5,\ 5,\ 10,\ 7.5,\ 5,\ 7.5,\ 7.5,\ 7.5,\ 7.5]^T,$$
$$\boldsymbol{b} = [5,\ 2.5,\ 5,\ 5,\ 5,\ 5,\ 2.5,\ 2.5,\ 5,\ 5,\ 2.5]^T.$$

All agents' states are initialized to values that are not greater than the initial values of the respective local reference signals. As per Assumption 2, we consider the bound on the derivative of the reference signals $\psi_r = 185$ and set the gain $\alpha = 187$, with $\epsilon = 1$. Note that, at



(a) $\mathcal{G}(t)$ in $[0, 0.5]$ s.
(b) $\mathcal{G}(t)$ in $[0.5, 1)$ s.
(c) $\mathcal{G}(t)$ in $[1, 1.5)$ s.
(d) $\mathcal{G}(t)$ in $[1.5, 2)$ s.
(e) $\mathcal{G}(t)$ in $[2, 2.5)$ s.
(f) $\mathcal{G}(t)$ in $[2.5, 3)$ s.

Fig. 1. Temporal evolution of the digraph $\mathcal{G}(t)$ considered in the simulation, depicted at intervals of 0.5 s.

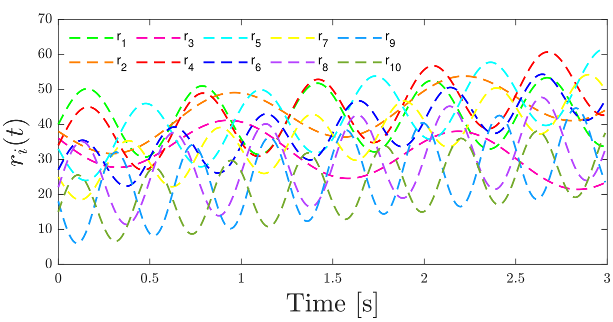


Fig. 2. Temporal evolution of the reference signals $r_i(t)$.

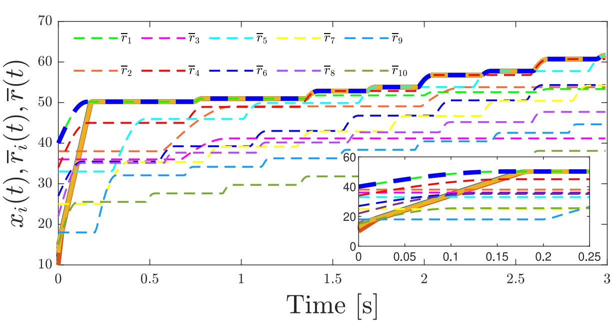


Fig. 3. Temporal evolution of the agents' states $x_i(t)$ (solid lines), the supremum reference signals $\bar{r}_i(t)$ (fine dotted lines) and the maximum supremum $\bar{r}(t)$ (thick dotted line).

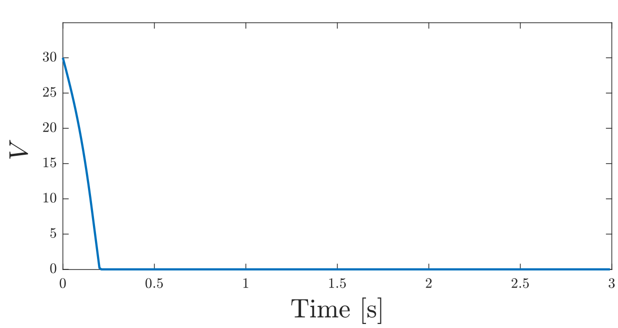


Fig. 4. Temporal evolution of the Lyapunov function $V(\boldsymbol{x}, t)$.

each time interval in which the network topology remains constant, the conditions of Theorem 3 are satisfied.

Fig. 5 depicts the trajectories of the agents' state values $x_i(t)$ (solid lines), the supremum of their local references $\bar{r}_i(t)$ (thin dotted lines) and the instant-based maximum supremum signal $\bar{r}^I(t)$ (thick dotted line). Moreover, vertical dotted lines highlight the instants at which the sets of nodes and edges vary. The figure shows that, in each interval, the agents are able to track the maximum supremum after a finite transient time, upper bounded by (19). This outcome was expected since, as stated above, the conditions of Theorem 3 are met in each interval. By focusing on the individual intervals, we can observe that the first interval $[0, 1)$ s is equivalent to the maximum supremum problem with non-open networks and the agents reach and track the maximum supremum signal in finite-time. Then, the departure of the agent 7 at time $t = 1$ s causes the instant-based maximum supremum to lower to the reference $r_1(1)$, which becomes the new maximum supremum. The departure is announced to all the remaining agents which reset their states to the values of their local supremum and, after a finite transient, resume tracking the maximum supremum. Regarding the third interval $[2, 3)$ s, the entrance of the agent 9 at $t = 2$ s leads the maximum supremum to discontinuously rise to the reference $r_9(2)$, which becomes the new maximum supremum signal. Following this, all the agents resume the tracking of the maximum supremum after a finite-time transient. Regarding the fourth interval $[3, 4)$ s, the departure of the agent 10 at time $t = 3$ s does not influence the instant-based maximum supremum signal and no announcement is made, as the local supremum signal $\bar{r}_{10}(3)$ is lower than the state value $x_{10}(3)$. Hence, the remaining agents continue tracking the maximum supremum signal after $t = 3$ s. Finally, concerning the fifth interval $[4, 5]$ s, the entrance of the agent 11 at $t = 4$ s does not impact the maximum supremum signal, since its reference signal $r_{11}(t)$ is lower than the current value of the maximum supremum. After a finite transient, also the newly joined agent 11 tracks the maximum supremum signal.

7. Conclusions

In this paper, we proposed a distributed protocol for tracking the maximum supremum or the minimum infimum of exogenous time-varying signals in finite time. Specifically, we assumed that each agent has access to a time-varying reference signal, with bounded derivative, and the communication topology is described by directed graphs that can be possibly switching. The finite-time convergence of the protocol was formally proved. The behavior of the protocol in the case of open

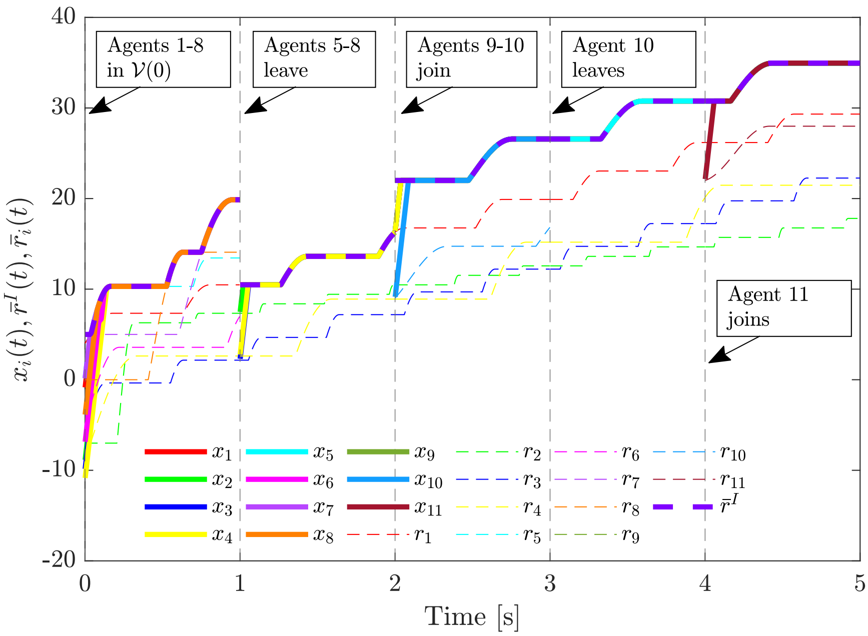


Fig. 5. Results of the simulation for the instant-based supremum problem. The agents' states $x_i(t)$ (solid lines), the supremum of the reference signals $\bar{r}_i$ (thin dotted lines) and the instant-based supremum $\bar{r}^I(t)$ (thick dotted line) are depicted.

networks was also discussed. Finally, simulation results with sinusoidal signals were carried out to validate the proposed protocol. As future work, we aim to relax the assumptions about the knowledge on the derivatives' bound by defining an adaptive distributed protocol and about the graph strong connectivity graph by handling jointly strongly connected graphs, i.e., such that the union of the graphs in an interval of time is strongly connected.

CRediT authorship contribution statement

Antonio Furchì: Formal analysis, Methodology, Validation, Writing – original draft. **Martina Lippi:** Methodology, Writing – review & editing. **Alessandro Marino:** Supervision, Writing – review & editing. **Andrea Gasparri:** Supervision, Writing – review & editing, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Proposition 1

To prove this result, we resort to Theorem 2.1.2 in [25], which, for the case of the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n \times \mathbb{R})$ (see Definition 1.2.10 in [25]), states that $f(\boldsymbol{x}, t)$ is Borel measurable if the following condition is satisfied

$$f^{-1}(S) \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}), \quad \forall S \in \mathcal{B}(\mathbb{R}), \tag{A.1}$$

where $f^{-1}(S)$ is the inverse image under f of the set S, defined as $f^{-1}(S) = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(\boldsymbol{x}, t) \in S\}$, and the sets in $\mathcal{B}(\cdot)$ are called Borel sets. Let $\mathcal{C}, \mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}$ be the set of points at which $f(\cdot)$ is continuous and discontinuous, respectively. We can observe that, since $\mathcal{C} \cup \mathcal{D} = \mathbb{R}^n \times \mathbb{R}$, it holds

$$f^{-1}(S) = \left(f^{-1}(S) \cap \mathcal{C}\right) \cup \left(f^{-1}(S) \cap \mathcal{D}\right).$$

In order to show that $f^{-1}(S) \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R})$, we exploit the closure property of σ-algebras under the union (see Definition 1.2.2 in [25]). Therefore, we prove that, for all $S \in \mathcal{B}(\mathbb{R})$, *(i)* the sets $f^{-1}(S) \cap \mathcal{C}$ and *(ii)* $f^{-1}(S) \cap \mathcal{D}$ belong to $\mathcal{B}(\mathbb{R}^n \times \mathbb{R})$. To prove point *(i)*, we consider the restriction of f to $\mathcal{C}$, denoted by $f|_{\mathcal{C}}(\boldsymbol{x}, t)$, i.e., the function f restricted to the domain $\mathcal{C}$. We can observe that $f^{-1}(S) \cap \mathcal{C}$ is equivalent to the inverse image under $f|_{\mathcal{C}}$ of the set S, i.e., $f|_{\mathcal{C}}^{-1}(S)$. Moreover, since the restriction $f|_{\mathcal{C}}(\boldsymbol{x}, t)$ is, by construction, continuous in $\mathcal{C}$ and continuous functions are Borel measurable [25], we have that $f|_{\mathcal{C}}(\boldsymbol{x}, t)$ satisfies (A.1), implying that

$$f^{-1}(S) \cap \mathcal{C} \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}), \quad \forall S \in \mathcal{B}(\mathbb{R}).$$

To show point *(ii)*, i.e., that $f^{-1}(S) \cap \mathcal{D} \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R})$, we simply need to observe that since, for all $S \in \mathcal{B}(\mathbb{R})$, it holds $f^{-1}(S) \cap \mathcal{D} \subseteq \mathcal{D}$ and since $\mathcal{D}$ is a countable set by assumption, the set $f^{-1}(S) \cap \mathcal{D}$ is also countable. By noticing that countable sets are Borel sets, it follows that

$$f^{-1}(S) \cap \mathcal{D} \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}), \quad \forall S \in \mathcal{B}(\mathbb{R}).$$

Appendix B. Proof of Property 1

Concerning the continuity of $y_i(\boldsymbol{x}, t)$ (i.e., part *(i)*), let us observe that, for all i, it holds

$$x_{i^+}(t) - x_i(t) = \max_{j \in \widetilde{\mathcal{N}}_i(t)} \{x_j(t)\} - x_i(t), \tag{B.1}$$

$$\phi_i(x_i, t)\,(\bar{r}_i(t) - x_i(t)) = \max_{j=1,2}\{\chi_{i,j}(t)\} - x_i(t) \tag{B.2}$$

where $\chi_{i,1} = x_i(t)$ and $\chi_{i,2} = \bar{r}_i(t)$. Then, $y_i(\boldsymbol{x}, t)$ can be equivalently expressed as follows

$$y_i(\boldsymbol{x}, t) = \max_{j \in \widetilde{\mathcal{N}}_i(t)} \{x_j(t)\} - x_i(t) + \max_{j=1,2}\{\chi_{i,j}(t)\} - x_i(t). \tag{B.3}$$

We can easily observe that $y_i(\boldsymbol{x}, t)$ is continuous with respect to the variable $\boldsymbol{x}$ since $\max(\cdot)$ is a continuous function. At this point, we study the continuity of the terms in (B.3) with respect to variable $(\boldsymbol{x}, t)$. Regarding the difference $\max_{j\in\widetilde{\mathcal{N}}_i(t)}\{x_j(t)\} - x_i(t)$, we observe that $\max(\cdot)$ is a continuous function, $\boldsymbol{x}$ is continuous by Definition 2, and there can only exist countable discontinuities in the difference due to switches in the communication graph, i.e., changes in $\widetilde{\mathcal{N}}_i(t)$, implying that the considered term is continuous almost everywhere, except at a countable set of points. Regarding the difference $\max_{j=1,2}\{\chi_{i,j}(t)\} - x_i(t)$, we additionally observe that $\bar{r}_i(t)$ is continuous by construction for all i, since $r_i(t)$ is continuous by Assumption 2 and $\sup(\cdot)$ is a continuous function, leading to a continuous difference. Therefore, the function $y_i(\cdot)$ is continuous almost everywhere with respect to $(\boldsymbol{x}, t)$, except at a countable set of points.

With regard to the part *(ii)* of the proposition, let us observe that since the sign function takes values in the bounded set $\{-1, 0, 1\}$, for all i, the function $\mathrm{sign}(y_i(\boldsymbol{x}, t))$ is bounded, hence also locally essentially bounded [18]. To prove that, for all i, the function $\mathrm{sign}(y_i(\boldsymbol{x}, t))$ is Lebesgue measurable, we resort to the fact that Borel measurable functions are Lebesgue measurable [20]. As proved above, $y_i(\boldsymbol{x}, t)$ is continuous almost everywhere w.r.t. $(\boldsymbol{x}, t)$, except at a countable set of points, implying by Proposition 1 that $y_i(\boldsymbol{x}, t)$ is Borel measurable. Borel measurability is verified for the sign function as well since it is a simple function [25], taking finitely many values (that are -1, 0 and 1) in Borel sets. Finally, since Borel measurability is preserved under composition [25], we obtain that, for all i, $\mathrm{sign}(y_i(\boldsymbol{x}, t))$ is Borel measurable.

Appendix C. Proof of Proposition 2

The proof can be easily deduced by considering the non-negativity of the function $y_i(\boldsymbol{x}, t)$. Hence, the image of $\mathrm{sign}(y_i(\boldsymbol{x}, t))$ is $\{0, 1\}$, which by applying the definition in (2), leads to

$$K[(\mathrm{sign} \circ y_i)](\boldsymbol{x}, t) = \mathrm{SIGN}^{+}(y_i(\boldsymbol{x}, t)),$$

concluding the proof.

Appendix D. Proof of Proposition 3

Let us first observe that, by the definition of Filippov solution in Definition 2, it holds $\dot{x}_M(t) \overset{a.e.}{\in} K[u_M](\boldsymbol{x}, t)$, where $u_M(\boldsymbol{x}, t)$ is the discontinuous control input defined in (10). To prove this proposition, we show that

$$K[u_M](\boldsymbol{x}, t) = \{0\}, \tag{D.1}$$

when $x_M(t) > \bar{r}(t)$ with M in $\mathcal{I}^M(\boldsymbol{x}(t))$. To this end, let us first demonstrate that, for all positive values of δ such that $\delta < x_M(t) - \bar{r}(t)$, it holds

$$u_M(\overline{\boldsymbol{x}}, t) = 0, \quad \forall\, \overline{\boldsymbol{x}} \in B(\boldsymbol{x}, \delta), \tag{D.2}$$

or equivalently that

$$y_M(\overline{\boldsymbol{x}}, t) = 0, \quad \forall\, \overline{\boldsymbol{x}} \in B(\boldsymbol{x}, \delta). \tag{D.3}$$

We can observe that, since $x_M(t) > \bar{r}(t)$ by assumption and $\delta < x_M(t) - \bar{r}(t)$, it holds $\overline{x}_M(t) > \bar{r}(t)$ for all $\overline{\boldsymbol{x}} \in B(\boldsymbol{x}, \delta)$. leading to $\overline{x}_{\overline{M}}(t) > \bar{r}(t)$ for all $\overline{M} \in \mathcal{I}^M(\overline{\boldsymbol{x}})$. In view of (9), we obtain

$$\phi_M(\overline{x}_{\overline{M}}, t) = 0, \quad \forall\, \overline{\boldsymbol{x}} \in B(\boldsymbol{x}, \delta). \tag{D.4}$$

Moreover, since $\overline{M} \in \mathcal{I}^M(\overline{\boldsymbol{x}})$, by construction, it holds

$$\overline{x}_{\overline{M}^+} - \overline{x}_{\overline{M}} = 0, \quad \forall\, \overline{\boldsymbol{x}} \in B(\boldsymbol{x}, \delta). \tag{D.5}$$

By plugging (D.4) and (D.5) in (10), the equalities in (D.3) and (D.2) follow.

At this point, we can notice that (D.2) implies that $\overline{\mathrm{co}}\,\{u_M(B(\boldsymbol{x}, \delta) \setminus \mathcal{H}, t)\} = \{0\}$, for all $\delta < x_M(t) - \bar{r}(t)$ and all sets $\mathcal{H}$ of Lebesgue measure zero. Hence, when considering $\delta > 0$ such that $\delta \geq x_M(t) - \bar{r}(t)$, it holds

$$\{0\} \subseteq \overline{\mathrm{co}}\,\{u_M(B(\boldsymbol{x}, \delta) \setminus \mathcal{H}, t)\},$$

for all $\delta \geq x_M(t) - \bar{r}(t)$ and all sets $\mathcal{H}$ of Lebesgue measure zero, since the neighborhoods $B(\boldsymbol{x}, \delta)$ contain those associated with smaller radii $\delta < x_M(t) - \bar{r}(t)$. Therefore, by applying the definition in (2), the intersection of the convex closures for all $\delta > 0$ and all sets $\mathcal{H}$ of Lebesgue measure zero leads to (D.1).

Appendix E. Proof of Lemma 1

To prove this result, we equivalently show that the maximum network state is lower than or equal to $\bar{r}(t)$, i.e.,

$$x_M(t) \leq \bar{r}(t), \quad \forall\, t,\ M \in \mathcal{I}^M(\boldsymbol{x}(t)). \tag{E.1}$$

To this end, let us assume by contradiction that there exists a time instant $t_2 > 0$ such that $x_M(t_2) - \bar{r}(t_2) > 0$. By the continuity of $x_M(t) - \bar{r}(t)$ and since $x_M(0) - \bar{r}(0) \leq 0$, there exists an instant $t_1 \in [0, t_2)$ such that $x_M(t_1) - \bar{r}(t_1) = 0$ and $x_M(t) - \bar{r}(t) > 0$ for all $t \in (t_1, t_2]$. Then, by virtue of Proposition 3, it holds $\dot{x}_M(t) = 0$ for all $t \in (t_1, t_2)$. Let $\mathcal{T}$ be the set of measure zero composed of the time instants where $\dot{\bar{r}}(t)$ is not defined. By Assumption 2 it holds $\dot{\bar{r}}(t) \leq \psi_r$, $\forall t \in (t_1, t_2) \setminus \mathcal{T}$, leading to

$$\underbrace{\dot{x}_M(t)}_{0} - \underbrace{\dot{\bar{r}}(t)}_{\leq \psi_r} \leq 0 \quad \forall t \in (t_1, t_2) \setminus \mathcal{T},$$

where the last inequality holds since $\psi_r \geq 0$. Then, being $x_M(t) - \bar{r}(t)$ absolutely continuous, it follows that

$$x_M(t_2) - \bar{r}(t_2) = \underbrace{x_M(t_1) - \bar{r}(t_1)}_{=0} + \int_{t_1}^{t_2} \underbrace{\dot{x}_M(t) - \dot{\bar{r}}(t)}_{\leq 0}\, dt \leq 0, \tag{E.2}$$

which yields to $x_M(t_2) - \bar{r}(t_2) \leq 0$, thus contradicting the assumption that $x_M(t_2) - \bar{r}(t_2) > 0$ and proving the result.

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