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Dynamics of some Kirchhoff-type PDEs

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Alla mia famiglia e a Marina.

Abstract

In this thesis we study the long-time dynamics of a class of Kirchhoff-type partial differential equations, which provide fundamental examples of quasilinear Hamiltonian PDEs with nonlocal nonlinearities.

In the first part of the thesis we focus on a special Kirchhoff equation introduced by Pohozaev, for which global-in-time existence is known to hold for sufficiently regular initial data. Our goal is to understand this phenomenon from a dynamical systems perspective. To this end, we perform a quasilinear normal form reduction of the associated vector field and analyze the structure of the resonant terms at low orders. We show that all resonant interactions up to a relatively high degree are ineffective in producing growth of Sobolev norms. As a consequence, we obtain long-time stability results for small-amplitude solutions and improved lower bounds on their lifespan.

In order to further investigate the dynamical properties of the equation, we study the formal Birkhoff normal form of the corresponding Hamiltonian. By means of an algebraic analysis based on formal power series, we prove that, despite the absence of effective resonances at low orders, the system fails to be integrable at higher orders due to the presence of nontrivial resonant terms.

In the second part of the thesis we consider a semilinear Kirchhoff-type equation and address the existence of small-amplitude almost-periodic solutions. Adopting a dynamical systems viewpoint, the equation can be interpreted as an infinite chain of coupled harmonic oscillators. We construct invariant tori supporting almost-periodic solutions by means of a KAM scheme. Although the linear dispersion relation leads to a delicate small divisor problem, the special structure of the Kirchhoff nonlinearity allows us to overcome these difficulties. We prove that, for almost every choice of the external potential, the equation admits infinitely many weak almost-periodic solutions, among which infinitely many are classical and infinitely many are non-classical.

Contents

Chapter 1

Introduction

In the last years, a growing interest has been devoted to the study of the long-time dynamics of nonlinear Hamiltonian partial differential equations. These equations naturally arise as perturbations of linear, completely integrable models and describe a wide range of physical phenomena.
A central question in this context is to understand whether the integrable structure of the linear equation persists, at least in an approximate sense, when nonlinear effects are taken into account, and how this reflects on the long-time behavior of its solutions.

Among the various techniques developed to face this problem, normal form methods play a fundamental role.
By means of suitable canonical transformations, these techniques allow one to simplify the structure of the Hamiltonian near an equilibrium, isolating resonant interactions and eliminating non-resonant ones up to a prescribed order.
This strategy has proved extremely effective in finite-dimensional Hamiltonian systems, where it leads to precise stability results near elliptic equilibria, and it has been progressively extended to infinite-dimensional settings, in particular to Hamiltonian PDEs.

A particularly complex class of problems is given by quasi-linear Hamiltonian equations, namely equations whose highest-order derivatives appear linearly, but with coefficients that depend on the unknown function or its lower-order derivatives.
In this case, the construction of normal form transformations becomes significantly more delicate, due to the appearance of unbounded operators and the possible loss of regularity. As a consequence, only a few equations are known for which a detailed normal form analysis can be successfully carried out.

Among these few examples, one finds Kirchhoff-type equations.
In their general form, they can be written as

$$\partial_{tt} u - \mathbf{f}\left(\int_{\mathbb{T}^n} |\nabla u|^2 \, dx \right) \Delta u = 0, \tag{1.1}$$

or, more generally, as second-order evolution equations of the form

$$u''(t) + \mathbf{f}\big(\langle \mathfrak{B} u(t), u(t) \rangle\big)\, \mathfrak{B} u(t) = 0,$$

where $\mathfrak{B}$ is a positive self-adjoint operator and $\mathbf{f}$ is strictly positive.
From a pure mathematical point of view, (1.1) is arguably the simplest example of quasilinear hyperbolic equation, while from the mechanical point of view, these models naturally arise in

the description of small transversal vibrations of an elastic string ($n = 1$) or membrane ($n = 2$). However, despite their relatively simple appearance, Kirchhoff equations exhibit a highly non-trivial dynamical behavior, and only in very special cases their long-time dynamics can be understood in detail.
In particular, this thesis will focus on two different Kirchhoff-type problems:

1. In the first part (Chapters 2 and 3), we will consider the nonlinearity $\mathbf{f}(y) = \dfrac{1}{1+cy}$, $c \in \mathbb{R}$ and the corresponding Kirchhoff equation

$$\partial_{tt} u - \Big(\frac{1}{1 + c\int_{\mathbb{T}^n} |\nabla u|^2 \, dx}\Big)^2 \Delta u = 0. \tag{1.2}$$

   Equation (1.2) has been introduced in 1985 by Pohozaev [81] and, to the best of our knowledge, it represents the only example of a Kirchhoff-type equation for which global-in-time existence holds for all solutions in sufficiently regular Sobolev spaces (H^s, $s \geq 2$). This property contrasts with the behavior one expects from a quasilinear equation.
   Even in the context of generic Kirchhoff equations, for which only local or almost-global existence results are available, it represents a unique case.
   Our aim is to understand the structural reasons behind the global well-posedness property of (1.2), by investigating its dynamics, from the perspective of normal form theory. In particular, we aim to clarify how the special nonlinear structure of the equation that simplifies the possible resonances among Fourier modes influences the long-time behavior of solutions. The analysis is carried out by combining rigorous normal form transformations at the level of the vector field with a complementary algebraic approach based on the Birkhoff normal form of the associated Hamiltonian.
   A key aspect of our study is the identification of resonant and non-resonant terms at low orders and the analysis of their contribution to the evolution of Sobolev norms. Surprisingly, we find that all resonant terms up to a relatively high order are ineffective in producing growth of Sobolev norms. This leads to long-time stability results for small-amplitude solutions and provides a normal-form-based explanation for the global existence phenomenon observed for the Kirchhoff equation.
   To this aim, the invariance of the Fourier support of solutions, which is a key feature of the Kirchhoff equations, plays a crucial role both in the rigorous normal form analysis and in the formal Birkhoff normal form computations, and it enables us to bypass several technical difficulties typically encountered in quasi-linear problems.

2. The second part of the thesis (Chapter 4) is devoted to the construction of small-amplitude almost periodic solutions for the semilinear Kirchhoff-type equation

$$\partial_{tt} u - \Lambda u + W * u(t) + \mathbf{f}\big(\langle \mathfrak{B} u(t), u(t)\rangle\big)\, \mathfrak{B} u(t) = 0 \tag{1.3}$$

   where $\mathbf{f}$ is real analytic near the origin, W is a convolutional potential, $\Lambda = \sqrt{-\Delta}$ and $\mathfrak{B}$ is a bounded Fourier multiplier.
   Adopting a dynamical systems point of view, equation (1.3) can be interpreted as an infinite chain of harmonic oscillators coupled through the nonlinearity $\mathbf{f}$.
   Following the pioneering approach introduced by Bourgain [33] for the quintic nonlinear Schrödinger equation, and later refined and extended by Biasco, Massetti, and Procesi [24], [23], [25] to nonlinear Schrödinger equations with generic, non-translation-invariant nonlinearities, the goal is to construct invariant tori supporting almost-periodic solutions by means of a KAM scheme.
   However, this approach strongly relies on the superlinear dispersive relation typical of

Schrödinger-type equations, and it is not a priori clear whether it can be implemented for wave-type equations such as (1.3).
The key point is that the structural properties inherited from the Kirchhoff nonlinearity make it possible to overcome these difficulties. In particular, it will be shown that, for almost every choice of the external potential, equation (1.3) admits infinitely many small-amplitude weak almost-periodic solutions, among which infinitely many are classical and infinitely many are non-classical (Sobolev regularity).

1.1 Historical Preface

Equation (1.1) was introduced in 1876 by Kirchhoff [67] in one dimension with Dirichlet boundary conditions, namely

$$u_{tt} - \mathbf{f}\left(\int_0^\pi u_x^2\,dx\right)u_{xx} = 0, \quad u(0,t) = u(\pi,0)$$

as effective models for the vibrations of elastic structures when one takes into account the dependence of the tension on the deformation.
Independently, the same equation was rediscovered in 1945 by Carrier [34] and in 1968 by Narasimha [78], as a nonlinear approximation of the exact model for the stretched string.
During the years, many authors studied the Kirchhoff equation from the point of view of the Cauchy problem

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x).$$

The earliest contribution in this direction was the seminal work of Bernstein [18] in 1940 in which he proved the local well-posedness for Sobolev initial data $(u_0, v_0) \in H^2 \times H^1$ and the global well-posedness for (u_0, v_0) analytic.
Following this pioneering work, the study of Kirchhoff-type equations has developed along several directions, the principal ones concern:

1. Local well-posedness;
2. Global well-posedness;
3. Long-time existence and qualitative dynamical properties of solutions.

While the questions related to the first direction are by now largely understood, with only a few classical issues still open (we refer to [68] for a comprehensive survey), the second one still presents several challenging open problems, which, to the best of our knowledge, remain far from being completely resolved. The third direction, concerning long-time existence and the qualitative dynamics of solutions, represents instead one of the most active and lively research areas. It is precisely in this setting that techniques from dynamical systems theory play a fundamental role, and it is within this framework that the present thesis is placed.
We collect the most important contributions:

1. **Local well-posedness**
   Local existence has been proved in the last century with two opposite sets of assumptions.
   Sobolev Regularity:
   As we mentioned before, under these assumptions the first local existence result was proved by S. Bernstein [18] and then extended by numerous authors, such as Dickey [48], Medeiros and Milla Miranda [74] and Arosio and Panizzi [1].

All these results, although in different settings, conclude that the Cauchy problem associated with (1.1) is well posed in the phase space $(u_0, v_0) \in H^{\frac{3}{2}} \times H^{\frac{1}{2}}$ with time of existence of the order $(\|u_0\|_{\frac{3}{2}} + \|v_0\|_{\frac{1}{2}})^{-2}$.

Analytic Regularity:
Local existence under these hypotheses has been proved through several stages of generalization by Bernstein [18], Pohozaev [80], Arosio and Spagnolo [2], D'Ancona and Spagnolo [40], and Hirosawa [63]. In these settings, the minimal assumption on the nonlinearity f is continuity.

2. **Global existence**:
Global existence for Kirchhoff equations is more challenging. Despite extensive research efforts, the problem of global-in-time existence for Kirchhoff-type equations remains open and is still far from being understood in a general setting.
Even on compact domains, where dispersion, scattering and time-decay mechanisms are not available, there are no results of global existence, nor of finite time blowup, for initial data with Sobolev, or C^∞, or Gevrey regularity There are no examples of local-in-time solutions that blow-up in some sense in finite time. This remains one of the oldest and most important open questions for Kirchhoff-type equations.
Throughout all these years, a few global existence results have been proved for the Kirchhoff, in very special cases. We collect some of them:
Analytic Data:
As mentioned before, problem (1.1) admits a global solution if both the initial data are analytic with respect to the space variables (this is true also with external forcing terms). Given the restrictive hypotheses on the initial data, this case requires rather minimal assumption on **f**, which is only required to be continuous and nonnegative.
This result was first proved by Bernstein [18] in the one-dimensional case and then extended by Pohozaev [80] to problems like (1.3) in several space dimensions.
Quasi-analytic data:
Nishihara in 1984 [79] proved global existence for a class of initial data which strictly contains analytic functions and is strictly contained in the Sobolev class.
The more general assumptions on the initial data, compared to the analytic case, require in an essential way the Lipschitz continuity and the strict hyperbolicity of the nonlinearity.
In this framework, we also mention the work of Ghisi and Gobbino in 2011 [57] that extended Nishihara's result.
Special Nonlinearities:
In a completely different direction, S. I. Pohozaev [81] considered the special case where the nonlinearity is given by

$$\mathbf{f}(y) = (1 + cy)^{-2}, \quad c \in \mathbb{R} \tag{1.4}$$

and proved global existence for all initial data in $H^2 \times H^1$.
The central point is that, in the case of nonlinearity (1.4) (and in a certain sense only in this case), equation (1.1) admits a second order nonnegative invariant which is constant (see section A.1 for more details). From this, it is not difficult to obtain a uniform bound in time on $\|u(t)\|_2, \|u_t(t)\|_1$ and the global existence follows in a standard way.

For recent papers explaining this second-order conservation, we refer to the work of Boiti and Manfrin [29]. Within the same framework, in [30] the authors also identified

a third-order conservation law for equation (1.2), provided additional regularity is assumed. Furthermore, under similar conditions, they demonstrated in [31] that equation (1.2) actually admits conservation laws of all orders..
Dispersive equations
Global existence results have been obtained in the whole space $\mathbb{R}^d$ or in an external domain $\mathbb{R}\backslash K$ for K compact.
In this setting, initial data have Sobolev regularity in the space variables, and satisfy suitable smallness assumptions and decay conditions at infinity. We refer to [41] [60], [73],[86] for precise statements.
Spectral-gap data and operators.
Global existence results have been established for 'spectral gap' initial data, characterized by a spectrum containing a sequence of large "holes". Equivalently, such results hold whenever the eigenvalues of the operator $\mathfrak{B}$ in (1.3) form a sufficiently fast-growing sequence.
We refer to [56],[58], [70] for exhaustive discussions.

3. **Long-time existence and qualitative dynamical properties of solutions**
   From a dynamical systems approach, several authors have improved lower bounds for the time of existence of solutions to the Kirchhoff equation and have shown the presence of recurrent or chaotic behaviors.

   In [3], the existence of periodic solutions for the one-dimensional forced Kirchhoff equation is proved via the Nash–Moser method, both for periodic and Dirichlet boundary conditions, exploiting the special structure of the nonlinearity.
   Quasi-periodic solutions with small amplitude and Sobolev regularity are constructed by Montalto in [76] for the one-dimensional forced Kirchhoff equation with periodic boundary conditions. Related results in higher dimensions are obtained by Montalto and Corsi in [38], where stability is not addressed. A reducibility result for the forced Kirchhoff equation on $\mathbb{T}^n$ is proved in [77].
   In [69], Liu and Xiang establish almost global existence and stability in Sobolev and Gevrey regularity for the Kirchhoff equation on $\mathbb{T}^1$ with nonlinearity $\mathbf{f}(y) = 1 + y$. This result was achieved using the rational normal form techniques developed in [14].
   Lower bounds on the lifespan of small-amplitude solutions on $\mathbb{T}^n$ are obtained by Baldi and Haus in [6], improving the bound given by standard local theory. The optimality of such bounds for general initial data is discussed in [7]. Under additional non-resonance assumptions on the initial data, longer lifespans are proved in [5].
   Finally, in [4], Baldi, Giuliani, Guardia, and Haus construct solutions whose Sobolev norms exhibit chaotic oscillations over long time scales, a phenomenon referred to as effective chaos.

1.1.1 Literature on Normal Form Methods For PDEs

Since the analysis carried out in this thesis relies on normal form arguments, we want to recall some related results in the literature concerning long-time existence for Hamiltonian (semi-linear and quasilinear) partial differential equations:

1. **Semi-Linear Case**
   The literature on the long-time behavior of semi-linear PDEs has grown significantly, starting from the seminal developments in Birkhoff normal form theory by Bambusi [8], Bambusi and Grebért [13], Delort and Szeftel [46, 47].
   These early frameworks laid the groundwork for extending such techniques to reversible

systems [52] and for analyzing the dynamics on Zoll manifolds, where many foundational insights were first consolidated [10].
A substantial part of the research has focused on the specific challenges posed by the Schrödinger equation on the torus $\mathbb{T}^d$. This includes the study of normal forms in fully resonant regimes [84], as well as stability analyses for specific solutions, such as plane waves [50] and small finite-gap solutions on $\mathbb{T}^2$ [72].
The regularity of the initial data also plays a crucial role. While Faou and Grébert [51] initially obtained sub-exponential stability times for analytic data on $\mathbb{T}^d$, these results were later optimized in the 1D case by Biasco, Massetti, and Procesi [23] via refined Diophantine conditions. Parallel to this, recent efforts by Bernier and Grébert [16, 17] have pushed the boundaries of normal form theory into the regime of low-regularity solutions.
A common thread among many of these results is the reliance on robust lower bounds for the small divisors. However, the scenario changes when only weak non-resonance conditions can be satisfied.
In this more restrictive setting, we refer to [53, 15, 65, 43], as well as to the recent work of Bambusi, Feola, and Montalto [12]. Their analysis establishes almost global existence for a wide class of Schrödinger-type equations, even on irrational tori where the small divisor problem is particularly delicate.
We also mention the work [14], of Bernier and Grébert in which they develop and apply the rational normal form to general classes of nonlinear Schrödinger equations on the circle with a nontrivial cubic part and without external parameters.
For results in a more general setting, we refer to the work of Bambusi, Feola, Langella, and Monzani [11], where an abstract almost global existence result is established for small, smooth solutions of certain semilinear PDEs on Riemannian manifolds with globally integrable geodesic flow. More recently, in [9], Bambusi, Grébert, Bernier and Imekraz proved a global existence result for semilinear Hamiltonian PDEs on generic compact boundaryless manifolds.

2. **Quasi-Linear Case**

The transition from semi-linear to quasi-linear PDEs introduces a fundamental difficulty:
indeed, when the nonlinearity involves derivatives of the unknown u, the standard change of coordinates used in Birkhoff normal form theory becomes unbounded.
An early attempt to address this problem for pure-gravity water waves is due to Craig and Worfolk [39]. The first fully rigorous results for quasi-linear equations were later obtained by Delort.
In his studies of the Klein–Gordon equation, both on the circle [42] and on higher-dimensional spheres [44], he introduced a paradifferential calculus based on multilinear maps. These results strongly rely on the special structure of the Klein–Gordon equation, namely a linear dispersion law (as in (1.1)) and a first-order nonlinearity.
A different approach was subsequently developed for equations with super-linear dispersion. This method was first applied to irrotational water waves in [19] and later to the quasi-linear Schrödinger equation in [54].
The analysis in [19] makes essential use of time reversibility, which limits the long-time existence result to the invariant subspace of standing waves. More recently, the paradifferential Hamiltonian Birkhoff normal form has become a central tool in the study of water wave equations. It has been used, for instance, to treat constant vorticity flows [21] and to prove the Zakharov–Dyachenko conjecture [20].
Building on these ideas, Feola, Montalto, and Terracina [55] have recently proved the

existence of time quasi-periodic traveling waves for three-dimensional gravity water waves in finite depth on flat tori.

1.1.2 Hamiltonian Formalism And Fourier Spaces

We now introduce the Hamiltonian formulation of the Kirchhoff equation.
Although Kirchhoff-type equations are often studied from a purely PDE perspective, their dynamics can be naturally embedded into an infinite-dimensional Hamiltonian framework. This viewpoint is fundamental to investigate important structural properties of the equation, such as the presence of conserved quantities and an underlying symplectic structure, and it provides the natural setting for the application of normal form and dynamical systems techniques.

Let us consider the space of Sobolev regular functions

$$H^s(\mathbb{T}^n,\mathbb{C}) := \{u : \mathbb{T}^n \to \mathbb{C} :\ \|u\|_s^2 := \sum_{j\in\mathbb{Z}^n} \langle j\rangle^{2s}|u_j|^2 < \infty\},$$

where $\langle j\rangle = \max\{1, |j|\}$ and $\{u_j\}_{j\in\mathbb{Z}^n}$ is the sequence of Fourier coefficients associated to u:

$$u_j = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} u(x)e^{-ij\cdot x}dx.$$

In particular, we will identify a function $u(x) = \sum_{k\in\mathbb{Z}^n} u_k e^{ik\cdot x}$, with the sequence of its coefficients.

By adopting the set of variables $(u,\, v) = (u,\, u_t)$, equation (1.3) can be decoupled in a system of two partial differential equations of order one with respect to time, specifically:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \mathbf{f}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)\Delta u \end{cases} . \tag{1.5}$$

By considering the Hamiltonian $H :\ H^s(\mathbb{T}^n,\mathbb{R}) \times H^{s-1}(\mathbb{T}^n,\mathbb{R}) \to \mathbb{R}$,

$$H(u,v) = \frac{1}{2}\int_{\mathbb{T}^n} v^2 dx + \frac{1}{2}\mathbf{F}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right), \quad \mathbf{F}(y) = \int_0^y \mathbf{f}(s)ds, \tag{1.6}$$

we also have that system (1.5) can be written in the form

$$\begin{cases} \partial_t u = \nabla_v H(u,v) \\ \partial_t v = -\nabla_u H(u,v) \end{cases} ,$$

where $\nabla_u H$ and $\nabla_v H$ are the gradients of H with respect to the real scalar product on L^2:

$$\langle f, g\rangle := \int_{\mathbb{T}^n} f(x)g(x)dx, \quad f,\, g \in L^2(\mathbb{T}^n,\mathbb{R}).$$

In terms of the Fourier expansion of its solution $u(t,\cdot) : \mathbb{T}^n \to \mathbb{R}$, system (1.5) turns into the (infinite-dimensional) system of ordinary differential equations

$$\ddot{u}_k(t) + \mathbf{f}\left(\sum_{j\in\mathbb{Z}^n} |j|^2|u_j(t)|^2\right)|k|^2 u_k(t) = 0, \quad k \in \mathbb{Z}^n, \tag{1.7}$$

parametrized by $k \in \mathbb{Z}^n$.
The equation corresponding to the index $k = 0$ has the simpler form

$$\ddot{u}_0(t) = 0.$$

It is clear that one can decouple the equation for the coefficient u_0, which corresponds to the spatial mean of u

$$\int_{\mathbb{T}^n} u(t,x)dx,$$

from the others.
For this reason, without loss of generality, we can consider equation (1.1) restricted to the space of zero-mean functions. In particular, we will set our problem in the space of complex zero-mean Sobolev functions in the x-variable:

$$H_0^s(\mathbb{T}^n,\mathbb{C}) := \left\{ u(x) = \frac{1}{(2\pi)^{n/2}} \sum_{k\in\mathbb{Z}^n\setminus\{0\}} u_k e^{ik\cdot x} \ : \ u_k \in \mathbb{C} \text{ and } \|u\|_s < \infty \right\}$$

and its subspace of real Sobolev functions

$$H_0^s(\mathbb{T}^n,\mathbb{R}) := \{u \in H_0^s(\mathbb{T}^n,\mathbb{C}) \ : \ u \text{ is real}\},$$

where

$$\|u\|_s^2 := \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{2s}|u_k|^2.$$

Finally, throughout the rest of this thesis, we will denote the open ball of radius $r > 0$ cenetrd at the origin of $H^s(\mathbb{T}^n,\mathbb{C})$, for $s \in \mathbb{R}$, with

$$B_r\left(H^s(\mathbb{T}^n,\mathbb{C})\right) := \{u \in H^s(\mathbb{T}^n,\mathbb{C}) \ : \ \|u\|_s < r\}.$$

A similar notation will be adopted also for $H^s(\mathbb{T}^n,\mathbb{R})$, $H_0^s(\mathbb{T}^n,\mathbb{C})$ and $H_0^s(\mathbb{T}^n,\mathbb{R})$.

1.1.3 Symmetries Of The Kirchhoff Equation

Following the results in [6], we summarize several fundamental structural properties of the Kirchhoff equation. While these features are not central to the aim of this chapter, they will be fundamental for the analysis performed in the next one.
These properties, which are ultimately linked to the special nonlinearity of the Kirchhoff-type equations i.e. the integral terms, turn out to be very useful when one tries to perform a Birkhoff normal form algorithm.

(1) **First Integrals for the Kirchhoff:**
If we consider the Hamiltonian momentum

$$M := \int_{\mathbb{T}} (\partial_t u)\nabla u dx, \tag{1.8}$$

a simple calculation shows that it is conserved by the dynamics generated by the Hamiltonian H.
In Fourier coordinates we have that

$$M = i\sum_{j\in\mathbb{Z}} j u_j(\partial_t u_{-j}) = \frac{i}{2}\sum_{j\in\mathbb{Z}} j\left[u_j(\partial_t u_{-j}) - u_{-j}(\partial_t u_j)\right].$$

The momentum is hence the sum $M = \sum_{j\in\mathbb{Z}} M_j$, of infinitely many quantities

$$M_j := \frac{1}{2} ij \left[u_j(\partial_t u_{-j} - u_{-j}(\partial_t u_j)\right],$$

moreover, thanks to the special structure of the Kirchhoff equation, all of the M_j are prime integrals. In fact, one has

$$\begin{aligned} \partial_t M_j &:= \frac{1}{2} ij \left[u_j(\partial_{tt} u_{-j}) - u_{-j}(\partial_{tt} u_j)\right] \\ &\overset{(1.7)}{=} 0 \end{aligned}$$

(2) **Invariance of the Fourier Support:**
If (u, u_t) is a solution of (1.1) (and of (1.3)), the Fourier support at a given time t is defined as the set of indices

$$\mathcal{S}(t) = \{j \in \mathbb{Z}^n : (u_j(t), \partial_t u_j(t)) \neq (0,0)\}. \tag{1.9}$$

Lemma 1.1.1 *Let (u, u_t) be a solution of* (1.1) *with initial data (u_0, v_0), defined over an interval of time $[0, T]$, then*

$$\mathcal{S}(t) = \mathcal{S}(0) \quad \forall t \in [0, T].$$

Proof: By looking at (1.7) we have that if $u_{0,j} = v_{0,j} = 0$ for some j, then it must holds $u_j(t) = 0$ for all t. □

This in particular implies that initial data supported on a finite subset of modes, gives rise to global-in-time solutions, since in this case, equation (1.1) reduces to a finite dimensional Hamiltonian system with analytic Hamiltonian.

1.2 Main Results

We now collect the main results of this thesis:

1.2.1 Quasilinear Normal Form

In Chapter 2 we consider the Cauchy problem associated with the Kirchhoff equation (1.2) on the n-dimensional torus $\mathbb{T}^n$ and with initial data $u_0 := u(0, \cdot)$, $v_0 := u_t(0, \cdot)$ in Sobolev class $H^s(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}(\mathbb{T}^n, \mathbb{R})$ and s greater than the regularity threshold given in Theorem (1.2.1).

Equation (1.2) was first introduced in the work of Pohožaev [81] and represents, as far as we know, one of the few examples of Kirchhoff equation for which global-in-time existence holds for all solutions (see section A.1 for a detailed proof). The aim of this chapter is to investigate this property by drawing a parallel with the known results of long-time existence obtained via normal form theory.
Since this problem is set on the $n-$dimensional torus, the key step in our study is the analysis of the resonances:
after reducing equation (1.2) to a first order system in complex variables

$$z_t = \mathcal{L}[z] + \sum_{k\in\mathbb{N}} \mathcal{N}_k(z) \tag{1.10}$$

with $\mathcal{L}$ linear part and $\mathcal{N}_k$ nonlinear terms of degree k, we seek a transformation, Φ, that conjugates system (1.10) to one in Normal Form up to order K, i.e. a system

$$w_t = \mathcal{L}[w] + \sum_{k=2}^{K} \mathcal{Z}_k(z) + \mathcal{R}_{\geq 7}(z) \tag{1.11}$$

for which the vector fields $\mathcal{Z}_k$ commute with the linear part.
The main issue in our in our setting is to propagate energy estimates for the remainder terms of system (1.11), verifying in particular that $\mathcal{R}_{\geq 7}$ is a bounded vector field in H^s.
An additional difficulty arises from the quasi-linear structure of (1.2), which makes the construction of the normal form map Φ more delicate:
ensuring that Φ is bounded from a ball of $H^s(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}(\mathbb{T}^n, \mathbb{R})$ around the origin and close to the identity presents technical challenges that are directly addressed to the presence of derivatives in the nonlinearity.

In order to overcome this structural difficulties one is forced to perform a preliminary nonlinear transformation to diagonalize the operator at its highest orders. Such a transformation is achieved by making extensive use of the special nonlinear structure of the Kirchhoff equation, that on one hand restricts the possible interactions and resonances between the different modes and, on the other hand, allows one to directly apply tools from the para-differential calculus to (1.2), simplifying the energy estimates.

The methodologies used to tackle our problem have been developed by Baldi and Haus in [6], where they studied an analogue problem but with a different aim:
in their work, in fact, they considered the Cauchy problem for the Kirchhoff equation

$$u_{tt} - \left(1 + \int_{\mathbb{T}^n} |\nabla u|^2 dx\right) \Delta u = 0,$$

with initial data of small amplitude ε in Sobolev class with the purpose of improving the lower bound for the existence time of the solutions, given by the standard local theory.

Their method is deeply inspired by the works of Delort in [42], [45] in which he constructs a normal form for quasilinear Klein-Gordon equations on the circle. A pivotal role in our analysis is played by Bony's paradifferential calculus (see [32], [64] and [75]), a framework based on a multi-scale decomposition to handle nonlinearities.
A key idea from J-M. Bony is that one can replace (para-linearize) nonlinear expressions, hence nonlinear equations, by para-differential expressions, up to smoothing remainders (that are presumed to be harmless in the derivation of estimates).
This decomposition is crucial for treating quasi-linear equations, where the main challenge is the presence of derivatives in the nonlinear terms. As shown by Delort, by replacing standard nonlinearities with paradifferential ones, one can perform a paradifferential normal form reduction.
Ultimately, this procedure removes problematic non-resonant terms without increasing the order of the operators, thus avoiding the loss of regularity that typically arises in the quasilinear setting. In this respect, the Kirchhoff equation is particularly well suited to this approach, since it is already written in a paralinearized form.

Let us consider the Hamiltonian formulation of equation (1.2), namely

$$\begin{cases} \partial_t u = \nabla_v H(u,v) \\ \partial_t v = -\nabla_v H(u,v) \end{cases} . \tag{1.12}$$

The main result of this chapter is the following:

Theorem 1.2.1 *[61] For $n \in \mathbb{N}$, let*

$$m_0 = \begin{cases} 1 & \text{if } n = 1 \\ \frac{3}{2} & \text{if } n \geq 2 \end{cases} \tag{1.13}$$

For every $s \geq m_0$ there exists a $r \geq 0$ and a bounded, injective transformation

$$\Phi : B_r\left(H^s(\mathbb{T}^n, \mathbb{R})\right) \times B_r\left(H^s(\mathbb{T}^n, \mathbb{R})\right) \to H^{s+\frac{1}{2}}(\mathbb{T}^n, \mathbb{C}) \times H^{s-\frac{1}{2}}(\mathbb{T}^n, \mathbb{C})$$

that conjugates system (1.12) *to a system of the form*

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \mathcal{X}(z, \bar{z}) = \mathcal{D}_1(z, \bar{z}) + \mathcal{Z}_3(z, \bar{z}) + \mathcal{Z}_5(z, \bar{z}) + \mathcal{R}_{\geq 7}(z, \bar{z})$$

The vector field $\mathcal{D}_1$ is linear. The vector fields $\mathcal{Z}_3$, $\mathcal{Z}_5$ commute with $\mathcal{D}_1$ and contain only terms of homogeneity 3 and 5, respectively.
Moreover $\mathcal{D}_1$, $\mathcal{Z}_3$ and $\mathcal{Z}_5$ do not contribute to the energy estimates, namely the Sobolev norms of the solutions of the truncated system

$$\partial_t(z, \bar{z}) = \mathcal{D}_1(z, \bar{z}) + \mathcal{Z}_{\leq 6}(z, \bar{z})$$

are constant. Finally, the vector field $\mathcal{R}_{\geq 7}$ contains only terms of homogeneity ≥ 7 and is bounded.

Remark 1.2.1 *We emphasize that the fact that the remainder $\mathcal{R}_{\geq 7}$ maps H^s into itself it is crucial for performing the energy estimates on the solutions.*

Remark 1.2.2 *The variation of the regularity threshold m_0 in* (1.13) *with respect to the dimension n, is linked to the fact that the coefficients of the map Φ have denominators of the form $|j| - |k|$, for j, $k \in \mathbb{Z}^n \backslash \{0\}$ (see Remark 2.4.1). While in the one-dimensional case this difference is always greater than 1, for $n \geq 2$ it can accumulate near zero with a lower bound of the form $||j| - |k|| \geq 1/(|j| + |k|)$. This causes the additional loss of half a derivative that occurs in the multidimensional case.*

Remark 1.2.3 *As anticipated before, the proof of Theorem 1.2.1 relies on the techniques developed in [6], [7] for a related problem. In these works, however, the construction of such a transformation shows an analogue integrable behaviour only for the resonant cubic terms, while it fails for the quintic ones. This is, as will be shown in Section 2.5, ultimately a consequence of the particular nonlinearity of equation* (1.2).

Theorem 1.2.1 yields two dynamical consequences.

1. The works of Dickey [48], Arosio-Panizzi [1] and Pohožaev [80], guarantee the existence of the solutions of (1.2) for initial data in $H^s \times H^{s-1}$ with $s \geq \frac{3}{2}$ and provide an upper bound on the time of existence of order ε^{-2}, where ε denotes the size of the initial data.
   On the other hand, the theorem of Pohožaev [81] ensures existence for all times, only above the regularity threshold of $s = 2$.
   In this regard, Theorem 1.2.1 provides us an improved lower bound for the case $s \in [3/2, 2)$:

Theorem 1.2.2 *There exist universal constants ε_0, C such that the following holds: if $(u_0, v_0) \in H^s(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}(\mathbb{T}^n, \mathbb{R})$ with $s \in \left[\frac{3}{2}, 2\right)$ and $\|u_0\|_s + \|v_0\|_{s-1} \leq \varepsilon_0$, the equation* (1.2) *admits a unique solution $u \in C^0([0,T], H^s(\mathbb{T}^n,\mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{T}^n,\mathbb{R}))$ with*

$$T \sim (\|u_0\|_s + \|v_0\|_{s-1})^{-6}$$

and

$$\max_{t\in[0,T]} \|u(t)\|_s + \|\partial_t u(t)\|_{s-1} \leq C(\|u_0\|_s + \|v_0\|_{s-1}). \tag{1.14}$$

2. Theorem 1.2.1 shows that equation (1.2) can be transformed into an equation that is in normal form up to order seven. This means that nonlinear terms up to this order only produce effective interactions among the modes corresponding to the same Sobolev weight, namely terms of the form

$$\sum_{j\in\mathbb{Z}^n:|j|=k} |u_j|^2, \quad k \in \mathbb{N}.$$

These quantities, called superactions, are then constants of motion for the truncated system.
As a consequence, the dynamics of the transformed equation exhibits long-time stability and no effective energy transfer occurs at these orders.
As for the original equation, this implies that equation (1.2) exhibits long-time stability for small initial data, and that any possible energy transfer or instability mechanism can only arise from nonlinear effects of order eight or higher:

Corollary 1.2.1 *For sufficiently small initial data of size ε the dynamics of the superactions is approximately constant for times of order ε^{-6}*

Remark 1.2.4 *In the one-dimensional setting, both Theorem 1.2.1 and Corollary 1.2.1 lead to stronger consequences: the superactions reduce, in this case, to the usual actions $\{|u_j|^2\}_{j\in\mathbb{Z}}$, and our results translate, respectively, into the integrability of the transformed system up to order six and the stability of the actions over time scales of order ε^{-6}.*

Given the unexpected properties of the resonant terms highlighted by Theorem 1.2.1, one may conjecture that the resonant terms exhibit the same behaviour at all orders of the normal form. This would provide strong evidence for the integrability of equation (1.2) in the one-dimensional case, and would also imply the conservation in time of the Sobolev norms of its solutions in all dimensions. In the next chapter, we will continue our calculations at a formal level, up to the eighth-order Birkhoff normal form, showing that the expectation of integrability in the above sense fails.

1.2.2 Sketch of the Proof

As discussed previously, (1.2) is a Hamiltonian PDE, in the sense that it can be written in the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J \begin{pmatrix} \nabla_u H \\ \nabla_v H \end{pmatrix}, \tag{1.15}$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{1.16}$$

and

$$H(u,v) = \frac{1}{2}\int_{\mathbb{T}^n} v^2 dx - \frac{1}{2c}\left(\frac{1}{1+c\int_{\mathbb{T}^n}|\nabla u|^2\, dx}\right).$$

The main tool in our study is the analysis of the resonant terms of order up to 5 in (1.15). The strategy consists in performing preliminary changes of variables in order to put (1.15) in a more suitable form and then apply a nonlinear normal form transformation.

Diagonalization and Symmetrization:
We start by introducing a set of symmetrized complex coordinates

$$(z,\bar{z}) = \frac{1}{\sqrt{2}}(\Lambda^{\frac{1}{2}}u + \Lambda^{-\frac{1}{2}}v, \Lambda^{\frac{1}{2}}u - \Lambda^{-\frac{1}{2}}v),$$

where $\Lambda = \sqrt{-\Delta}$.
With this change, the linear wave operator becomes diagonal, and system (1.15) becomes

$$\begin{cases} \partial_t z = -i\Lambda z - \frac{i}{2}\mu\left(Q(z,\bar{z})\right)(\Lambda z + \Lambda \bar{z}), \\ \partial_t \bar{z} = i\Lambda \bar{z} + \frac{i}{2}\mu\left(Q(z,\bar{z})\right)(\Lambda z + \Lambda \bar{z}), \end{cases} \tag{1.17}$$

where

$$\mu(x) = \frac{1}{(1+cx)^2} - 1, \quad Q(z,\bar{z}) = \frac{1}{2}\left\langle \Lambda(z+\bar{z}), z+\bar{z}\right\rangle, \tag{1.18}$$

and $\langle\cdot,\cdot\rangle$ is the standard scalar product on $L^2(\mathbb{T}^n,\mathbb{R})$.
Let us note that (1.17) is again a Hamiltonian system with Hamiltonian

$$\begin{aligned} H: \qquad & H^{\frac{1}{2}}(\mathbb{T}^n,\mathbb{C}) \times H^{\frac{1}{2}}(\mathbb{T}^n,\mathbb{C}) \longrightarrow \mathbb{R} \\ H(z,\bar{z}) = \quad & \langle \Lambda z, \bar{z}\rangle - \frac{1}{2c}\left[\frac{\left(\frac{c}{2}\langle\Lambda(z+\bar{z}), z+\bar{z}\rangle\right)^2}{1+\frac{c}{2}\langle\Lambda(z+\bar{z}), z+\bar{z}\rangle}\right]. \end{aligned} \tag{1.19}$$

This set of coordinates turns out to be particularly useful for studying the structure of the resonant terms. In fact, since system (1.17) has the form

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \mathcal{F}(z,\bar{z}) = \begin{pmatrix} \mathcal{F}_1(z,\bar{z}) \\ \mathcal{F}_2(z,\bar{z}) \end{pmatrix}, \tag{1.20}$$

where the vector field $\mathcal{F}$ has the real structure $\mathcal{F}_2(z,\bar{z}) = \overline{\mathcal{F}_1(z,\bar{z})}$, one has that the energy estimates for its solutions yields:

$$\partial_t \|z\|_s^2 = \partial_t \langle \Lambda^s z, \Lambda^s \bar{z}\rangle = 2\mathrm{Re}\Big(\langle \Lambda^{2s} z, \overline{\mathcal{F}_1(z,\bar{z})}\rangle\Big). \tag{1.21}$$

Block Diagonalization:
System (1.17) can be interpreted as a (para-)linear system of the form

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = i\mathcal{L}(z,\bar{z}) \begin{pmatrix} \Lambda z \\ \Lambda \bar{z} \end{pmatrix}, \tag{1.22}$$

where $\mathcal{L}(z,\bar{z})$ is a matrix of the form

$$\begin{pmatrix} \mathcal{L}_{1,1}(z,\bar{z}) & \mathcal{L}_{1,2}(z,\bar{z}) \\ \mathcal{L}_{2,1}(z,\bar{z}) & \mathcal{L}_{2,2}(z,\bar{z}) \end{pmatrix},$$

whose entries are multiplicative operators of the form

$$\mathcal{L}_{1,1}(z_1, z_2) = -\mathcal{L}_{2,2}(z_1, z_2) = \Big(-1 - \frac{1}{2}\mu\left(Q(z_1, z_2)\right)\Big)[\cdot]$$
$$\mathcal{L}_{1,2}(z_1, z_2) = -\mathcal{L}_{2,1}(z_1, z_2) = -\frac{1}{2}\mu\left(Q(z_1, z_2)\right)[\cdot]. \tag{1.23}$$

When one tries to perform energy estimates on the Sobolev norm $\|z\|_s$, the off-diagonal unbounded terms cause a loss of half a derivative, in particular

$$\partial_t \|z\|_s^2 \leq c\|z\|_{\frac{1}{2}}^2 \|z\|_{s+\frac{1}{2}}^2$$

and this is closely related to the quasilinear structure of the problem.

In order to overcome this difficulty, one must perform a nonlinear transformation and diagonalize, up to bounded remainders, the operator valued matrix $\mathcal{L}$.
This transformation, namely $\Phi^{(2)}$, is bounded from $H^s(\mathbb{T}^n, \mathbb{C})$ into itself and has the form (2.24).
The variables $(z, \bar{z})$ are conjugated to a new pair of complex-conjugate variables, namely $(\eta, \bar{\eta})$. In the new coordinates system (1.22) is:

$$\begin{cases} \partial_t \eta = -i\left(1 - cQ(\eta, \bar{\eta})\right)\Lambda\eta - \dfrac{ic\left(1 - cQ(\eta, \bar{\eta})\right)}{2}\left(\langle \Lambda\bar{\eta}, \Lambda\bar{\eta}\rangle - \langle \Lambda\eta, \Lambda\eta\rangle\right)\bar{\eta}, \\ \\ \partial_t \bar{\eta} = i\left(1 - cQ(\eta, \bar{\eta})\right)\Lambda\bar{\eta} - \dfrac{ic\left(1 - cQ(\eta, \bar{\eta})\right)}{2}\left(\langle \Lambda\bar{\eta}, \Lambda\bar{\eta}\rangle - \langle \Lambda\eta, \Lambda\eta\rangle\right)\eta. \end{cases} \tag{1.24}$$

The advantage of this form is that non-diagonal terms of (1.24) are operators of order zero, while the diagonal ones, namely those of the form

$$\mathcal{D}(\eta, \bar{\eta}) = -i\left(1 - cQ(\eta, \bar{\eta})\right)\begin{pmatrix} \Lambda\eta \\ \Lambda\bar{\eta} \end{pmatrix},$$

are harmless. Indeed, let us write system (1.24) in the form

$$\partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = \mathcal{D}(\eta, \bar{\eta}) + \mathcal{B}(\eta, \bar{\eta})$$

where

$$\mathcal{B}(\eta, \bar{\eta}) = -\frac{ic(1 - Q(\eta, \bar{\eta}))}{2}\left(\langle \Lambda\bar{\eta}, \Lambda\bar{\eta}\rangle - \langle \Lambda\eta, \Lambda\eta\rangle\right)\begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix}$$

is the non-diagonal part of (1.24). By computing the time derivative of the Sobolev norm of a solution η of (1.24), we have

$$\partial_t \|\eta\|_s^2 = 2\mathrm{Re}\Big(\langle \Lambda^{2s}\eta, \overline{\mathcal{B}(\eta, \bar{\eta})}\rangle\Big).$$

The growth of η in norm, therefore, is entirely determined by the non-diagonal terms.

First Normal Form Step:
In order to understand the dynamics of the solutions, we have to quantify more precisely the contribution of X_3 and hence to analyze its resonant terms:
we then perform a normal form step by constructing a transformation $\Phi^{(3)}$ that conjugates the system (1.24) into one without non-resonant, third-order terms.
The main issue is to ensure that $\Phi^{(3)}$ is well-defined and invertible in a ball around the origin of some Sobolev space H^s. This is closely related to the presence of small divisors of the form

$$\frac{1}{\big||j|-|k|\big|}, \quad j,\, k \in \mathbb{Z}^n.$$

Quantities of this form are uniformly bounded by a constant if $n = 1$ and by $2|j|$, $2|k|$ for $n \geq 2$, and this causes the additive loss of half a derivative in the definition of the regularity threshold m_0 in (1.13).
After the normal form step procedure, we obtain a system of the form

$$X^{(1)}(w,\bar{w}) = (I + \mathcal{P}(w,\bar{w}))\,\mathcal{D}_1 + X_{\mathbf{Res},3}(w,\bar{w}) + X^{(1)}_{\geq 5}(w,\bar{w}), \tag{1.25}$$

where $\mathcal{P}$ is a real, negative function and $X_{\mathbf{res},3}$ is the resonant cubic part.
Unexpectedly, the surviving cubic term does not contribute to the Sobolev norm growth of the solutions, indeed

$$\mathrm{Re}\Big(\langle \Lambda^{2s} w \rangle X_{\mathbf{Res},3}(w,\bar{w})\Big) = 0, \quad s \geq m_0$$

and hence, by (1.21)

$$\partial_t \|w\|_s^2 = \mathrm{Re}\Big(\langle \Lambda^{2s} w,\, \overline{X^{(1)}_{\geq 5}(w,\bar{w})} \rangle\Big).$$

It follows that the norm evolution of the solution of (1.25) is truly determined by the terms of order greater than 5.

Second Normal Form Step:
In Section 2.5, we perform a similar analysis on the quintic resonant terms:
via the normal form transformation $\Phi^{(4)}$, we conjugate system (1.25) to another one of the form

$$X^{(2)}(q,\bar{q}) = \Bigg(1 + \tilde{\mathcal{P}}(q,\bar{q})\Bigg) \cdot (\mathcal{D}_1(q,\bar{q}) + X_{\mathbf{Res},3}(q,\bar{q})) + X_{\mathbf{Res},5}(q,\bar{q}) + X^{(2)}_{\geq 7}(q,\bar{q}),$$

The construction of $\Phi^{(4)}$ is carried out with the same techniques as in the previous step, the only difference lies in the higher complexity given by the quintic order.
Surprisingly, the structure of the small divisors (and hence of the resonant terms) is unchanged with respect to the first step.
This is ultimately related to the particular nonlinearity of (1.2) that prevents the presence of more complex small divisors of the form

$$\frac{1}{\Big||j_1| \pm |j_2| \mp |j_3|\Big|}, \quad j_1,\, j_2,\, j_3 \in \mathbb{Z}^n. \tag{1.26}$$

In fact, the resonant terms in $X_{\mathbf{Res},5}$ of the form

$$q_{j_1} q_{-j_1} (q_{j_2} q_{-j_2})^{\pm} q^{\mp}_{j_3}, \quad q^+ = q,\, q^- = \bar{q},$$

that give rise to small divisors of the form (1.26), have vanishing coefficients.
Thanks to this special property, also in this case, the quintic resonant terms do not contribute to the growth of solutions, in the sense that

$$\mathrm{Re}\Big(\langle \Lambda^{2s} q,\, \overline{X_{\mathbf{Res},5}(q,\bar q)}\rangle\Big) = 0, \quad s \geq m_0$$

1.2.3 Formal Birkhoff Normal Form

The aim of Chapter 3 is to continue the analysis of the previous chapter. It will be shown that, if we consider the Hamiltonian (1.19) associated with the equation (1.2) in complex coordinates, then it is impossible to find a (symplectic, close to identity) map, such that the resonant degree-8 terms of the resulting Hamiltonian give a non-zero contribution to the growth in Sobolev norm of its solution. This is implied by Theorem 1.2.3.
For this purpose, we adopt a different and more algebraic approach based on the use of the Birkhoff normal form within the framework of formal power series. This is outlined in Section 3.1.1. The main advantage of this method is that all computations are performed only at a formal level (although one can show that the corresponding maps are indeed well defined), which enables us to write everything in a cleaner way and to bypass the difficulties arising from the quasi-linear nature of the Kirchhoff equation. On the other hand, this approach makes it possible to better exploit the symmetries inherent in the nonlinear structure of the Kirchhoff equation. The main tools used in this section are the following:

(1) **Formal BNF:**
We translate our problem into the more abstract language of formal power series defined on complex sequences, which arise naturally from the Fourier expansions of Hamiltonian partial differential equations.
Starting from a Hamiltonian of the form

$$H = H_2 + \sum_{j>2} H_j$$

where H_2 represents the diagonal quadratic part and each H_j is a formal homogeneous polynomial of degree j, we construct a formal canonical transformation Φ in order to remove all non-resonant monomials up to an arbitrary finite order k.
The resulting transformed Hamiltonian is in Birkhoff normal form, taking the form:

$$H \circ \Phi = H_2 + \sum_{2<j\leq k} Z_j + R_{k+1}$$

where the Z_j are resonant homogeneous polynomials of degree j, in the sense that they Poisson commute with H_2.
Consequently, the Hamiltonian is reduced, at any prescribed order, to a normal form whose dynamics is entirely determined by its resonant terms, while the non-resonant dynamics are relegated to higher-order remainders. Finally, this analysis is conducted in a purely formal setting, focusing on the algebraic structure of the transformation without addressing convergence or analytic estimates.
Within this framework, we can reproduce more efficiently the first two normal-form steps carried out in the previous chapter, obtaining the same results concerning the resonant terms of degree 4 and 6 of the Hamiltonian (that correspond to degree 3 and 5 of the vector field), and then focus our attention on the analysis of the degree-8 terms.

(2) **Formal Variables:**
The key step consists of rewriting the Hamiltonian (1.19) associated with the equation in a new set of (formal) variables, denoted by $I = (|z_j|^2)_{j\in\mathbb{Z}}$ and $W = (z_j z_{-j})$, which highlights the special structure of the Kirchhoff equation and particularly those terms whose dynamics preserve the linear (super)actions.

Lemma 1.2.1 *Let us define the quantities* $H_2 := \sum_{k\in\mathbb{N}} k(I_k+I_{-k})$ *and* $P := \sum_{k\in\mathbb{N}} k(W_k+\bar{W}_k)$.
The Hamiltonian (1.19) *associated to system* (1.17) *can be formally written as*

$$H = H_2 + \sum_{n\geq 1} H_{2n}$$

where

$$H_{2n} = \frac{(-1)^{n+1}}{2} c^n \left(H_2 + P\right)^n .$$

Note that these variables are, however, not a symplectically conjugated pair, they merely provide a more convenient representation of the Hamiltonian that allows one to distinguishing the terms that are harmless, i.e. the formally action preserving terms that can be written only in terms of I_j, from other terms.
This particular writing is possible only for an equation with integral nonlinearity like the Kirchhoff. In fact the structure of the Kirchhoff-correction term drastically reduces the possible interactions between different modes.

(3) **Invariance of the Fourier Support:**
As we pointed out in Remark 2.3.3, the Fourier support is an invariant for system (1.17). This can alternatively be rephrased

Lemma 1.2.2 *Fixed any set of indices* $\mathcal{S} \subset \mathbb{N}$ *then the subspace*

$$\Omega_{\mathcal{S}} := \{z \in H^s(\mathbb{T}) : \, z_k = 0 \text{ if } |k| \notin \mathcal{S}\}$$

is invariant for the flow of (1.17)

This is fundamental for two reasons: We can reduce our problem to the study of Hamiltonians that are formal power series on spaces of sequences, bypassing all regularity issues. The arguments are then made rigorous by considering solutions supported on finite sets of modes.
Second, while our analysis is restricted to the one-dimensional case, this does not entail a loss of generality. Due to the invariance of the Fourier support (Lemma 1.2.2), one can always reduce the problem to one-dimensional invariant subsets to prove the non-integrability of the resonant degree-8 terms.

The main result of this chapter is the following

Theorem 1.2.3 *[71] There exists a (formally) symplectic transformation* Φ *such that*

$$H_{\mathbf{BNF}} := H \circ \Phi = H_2 + Z_4 + Z_6 + \tilde{Z}_8 + R_{\geq 10},$$

where H_2, Z_4, Z_6 *are resonant, homogeneous terms of degree respectively two, four and six, which are also formally action preserving. The term* $\tilde{Z}_8$ *collects the resonant degree-8 monomials, of which infinitely many are not action preserving. Finally, the term* $R_{\geq 10}$ *collects all the remaining monomials with degree greater than or equal to 10.*

Even if our result is only on a formal level, thanks to Lemma 1.2.2 it can be made rigorous by restricting ourselves to finite dimensional subspaces of sequences.
From theorem 1.2.3 it follows that

Corollary 1.2.2 *The Hamiltonian* (1.19) *is not formally action preserving, in sense of Definition 3.1.6.*

It is important to stress that, the above theorem holds also in a non-formal way thanks to the subspace invariance of the Kirchhoff.
One may ask if there is a particular choice of the nonlinearity $\mathbf{F}$ in (1.6) for which the Hamiltonian is formally action preserving.
With the same techniques used for Theorem 1.2.2 we can prove the following result:

Theorem 1.2.4 *Every Hamiltonian H of the form* (1.6)*, is not formally action preserving. In particular, if we consider the 8th-order formal Birkhoff normal form of H, it is always possible to exhibit infinitely many degree-8 resonant terms which are not action preserving.*

1.2.4 Formal BNF vs Normal Form

In Chapter 2, we adopted a non-formal normal form procedure that consists of a preliminary block-diagonalization procedure performed in section 2.3.2 together with two steps of Poincaré normal form on the resulting vector field. The main advantage of this technique is that all the maps involved are bounded transformations of the phase space H^s, which allows one to perform energy estimates on the solutions.
The drawback, however, is the loss of the original symplectic structure of the Kirchhoff equation.The method of Chapter 3, on the other hand, is based on an abstract version of the usual Birkhoff normal form (BNF) algorithm, adapted for the framework of formal power series on sequence spaces. Even if the formal BNF method does not require extension to Hamiltonians defined on spaces of Sobolev class functions, it can be carried out with cleaner calculations. Moreover, the maps involved are all formally symplectic. We also stress that the formal BNF procedure can be made rigorous by restricting ourselves to the space of finite support sequences (initial data with finite Fourier support).
Despite their structural differences, both methods rely on the cancellation of non-resonant terms and the study of the remaining resonant ones. The former is performed directly on the vector field, while the latter is applied to the Hamiltonian. At this point, one may ask if the resonant terms produced by the two approaches coincide.
A similar identification argument was used by Berti, Feola, and Pusateri in [20] in the context of the water wave equation. Their goal was to identify the degree-4 resonant monomials obtained through a formal BNF procedure with the ones obtained by means of a rigorous (yet non-symplectic) normal form transformation. The core of their argument relies on the uniqueness of the solutions of the quadratic homological equation, which stems from the absence of 3-wave resonances. This result, however, holds only for resonances of degree-4 and, to our knowledge, cannot be extended to higher orders.
A related result of this nature was provided by Kappeler and Pöschel [66] (Theorem G.1).
Their result ensures that the normal form is unique at each step, provided that the eigenvalues of the linearized operator are non-resonant. Both results stated above rely on a strong non-resonance condition on the linear part. Due to the strongly resonant structure of the Kirchhoff equation, whether such an identification argument could also work in our case deserves further investigation.

1.2.5 Almost Periodic Solutions

Chapter 4 is devoted to proving the existence of almost-periodic solutions for equation (4.2) that is a semilinear version of the Kirchhoff equation.
In complex variables we consider a family of PDEs on the one-dimensional torus, with an external parameter (potential):

$$iu_t = -\Lambda u + V * u + \mathbf{f}\left(\int_{\mathbb{T}} (u+\bar{u})\mathfrak{B}[u+\bar{u}]\right)\mathfrak{B}[u+\bar{u}], \tag{1.27}$$

where $\mathbf{f}$ is a real analytic function in a neighborhood of $y = 0$ with a zero of order at least 2 in $y = 0$ and Λ, V, $\mathfrak{B}$ are Fourier multipliers such that

1. $V * e^{ikx} = |k|\tilde{V}_k e^{ikx}$ for $\{\tilde{V}_k\}_{k\in\mathbb{Z}} \in \ell^\infty$ symmetric.
2. $\Lambda = \sqrt{-\Delta}$ and $\Lambda e^{ikx} = |k|e^{ikx}$
3. $\mathfrak{B}[e^{ik\cdot x}] = \mathfrak{b}_k e^{ik\cdot x}$ for $\{\mathfrak{b}_j\}_{j\in\mathbb{Z}} \in \ell^\infty$

Much of the literature concerning PDEs of this kind focuses on the construction of so-called quasi-periodic solutions in time, that is, functions of the form $u(t,x) = U(t\omega_1,\dots,t\omega_n,x)$, with $x \in \mathbb{T}$, $\omega = (\omega_1,\dots,\omega_n)$ a vector of rationally independent entries, and $U : \mathbb{R}^{n+1} \to \mathbb{R}$ periodic.
The principal techniques employed to face this problem, come from the theory of dynamical systems and have been extended to the infinite dimensional setting. One of the most important is to look for invariant manifolds on which the dynamics is linear and is given by the map $\varphi \to \varphi + t\omega$.
This idea is implemented by means of the so-called KAM procedure:
it consists in an iterative super-quadratic scheme, which, by performing infinitely many canonical transformations, brings the Hamiltonian associated to the PDE into another one which has a finite dimensional invariant torus at the origin.
The principal problem arising in the search of such quasi-periodic solutions is the presence of the so-called "small divisors", which are arbitrary small quantities appearing in the denominators when one computes explicitly the approximate perturbative series. In particular they arise at each step of the iteration in solving the so-called homological equation.
In the usual KAM framework, such equations are constant coefficients linear PDE, which can be solved by imposing the strong non resonance conditions on the frequency ω.
Unfortunately, being supported on finite dimensional tori, such solutions have no chance to be typical with respect to any reasonable measure one can define on the phase space: even for integrable models typical solutions are supported on infinite dimensional tori.
It is reasonable to suppose that such behavior is preserved when it comes to close to integrable PDEs.
With these premises, a natural direction is the study of the almost-periodic solutions, i.e. uniform limits of quasi-periodic ones see [27], [28], [26] and the references therein).
A first main difficulty is that the structure of the small divisors involved:
in fact, it is well known that in constructing finite-dimensional invariant tori the non-resonance conditions become weaker as the dimension increases.
To overcome this difficulty, we need to impose some strong (Diophantine) condition on the eigenvalues of the linear part. For this purpose, it is convenient to consider families of PDEs depending on as many parameters as the dimension of the invariant tori one aims to construct. The simplest way of introducing such parameters is through an external potential.
This is the core of the method developed by Bourgain for the quintic NLS, in his seminal

work [33] and then refined and generalized by Biasco-Massetti-Procesi [24], [23], [25] to the NLS with generic, non-translation invariant, non-linearity.
Following this approach, our goal is to prove the existence of almost-periodic solutions for (1.27), both with high regularity (Gevrey, Analytical) and low regularity (Sobolev).
The main difference with the above mentioned problems resides in dispersion law (quadratic for the NLS, linear for (1.27)), that a priori results in more restrictive Diophantine conditions. This feature is crucially exploited in the strategy developed by Cong and Yuan in [36] for the NLW equation.
In our case, in order to overcome this difficulty while still taking advantage of the non-resonance conditions introduced in [33], we rely on the special structure of the nonlinearity: in particular, as is shown in section 4.1.1, equation (1.27) is closely related to the Kirchhoff equation with which it shares the same structural symmetries. This allows to carry on the KAM scheme imposing similar non-resonance conditions with the one used for the NLS.

Hamiltonian Structure and Almost-Periodic Solutions
Equation (1.27) is a Hamiltonian system with Hamiltonian

$$\begin{aligned} H(u,\bar{u}) &= \int_{\mathbb{T}} u \cdot \Lambda \bar{u} + \frac{1}{2}\mathbf{F}\left(\frac{1}{2}\int_{\mathbb{T}}(u+\bar{u})\mathfrak{B}[u+\bar{u}]\right) + \frac{1}{2}\int_{\mathbb{T}} V * |u|^2, \\ \mathbf{F}(y) &= \int_0^y \mathbf{f}(s), ds \end{aligned} \tag{1.28}$$

with respect to the classical symplectic form $\omega(u,v) = \mathrm{Im}\left(\int u\bar{v}\right)$.

In the following, we will consider analytic non-linearity $\mathbf{f}$ such that

$$|\mathbf{f}|_R := \sum_{d=1}^{\infty} |\mathbf{f}^{(d)}| R^d < \infty \tag{1.29}$$

for some $R > 0$, where $\mathbf{f}(y) = \sum_{d=1}^{\infty} \mathbf{f}^{(d)} y^d$ is the Taylor expansion of $\mathbf{f}$ at the origin.
We are particularly interested in weak solutions of (1.27), according to the following

Definition 1.2.5 (Global in Time Weak Solution)
A function $u : \mathbb{R}^2 \to \mathbb{C}$, 2π-periodic in the second variable x,is a weak solution of (1.27), *if*

1. *The map $t \to u(t,\cdot) \in L^2(\mathbb{T})$ is continuous;*
2. *For every compactly supported, smooth function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ one has*

$$\int_{\mathbb{R}^2}\left[(-i\varphi_t + \varphi_{xx})\,u - \left(V * u - \mathbf{f}\left(\int_{\mathbb{T}}(u+\bar{u})\mathfrak{B}[u+\bar{u}]\right)\mathfrak{B}[u+\bar{u}]\right)\varphi\right] dtdx = 0.$$

The main results of this Chapter are summarized in the following statement:

Theorem 1.2.6 *For almost every Fourier multiplier V there exist infinitely many (small-amplitude) weak almost-periodic solutions of* (1.27).
Infinitely many of them are classical while infinitely many are non-classical.

From now on, we will operate in the Fourier setting and identify each function u with the sequence of its Fourier coefficients $\{u_j\}_{j\in\mathbb{Z}}$.

We consider the space $\ell^2(\mathbb{C})$ endowed with the standard symplectic form $idu_j \wedge d\bar{u}_j$.
Within this framework, equation (1.27) is the Hamiltonian system

$$\dot{u}_j = (|j| + V_j)u_j + \mathbf{f}\left(\frac{1}{2}\sum_{j\in\mathbb{Z}} \mathfrak{b}_j(u_j + \bar{u}_{-j})^2\right)\mathfrak{b}_j(u_j + \bar{u}_{-j}), \quad j \in \mathbb{Z},$$

namely, an infinite lattice of harmonic oscillators, with frequencies $|j| + V_j$, coupled by the non-linearity $\mathbf{f}$.
The associated Hamiltonian (1.28) on the other hand becomes

$$H(u) = \sum_{j\in\mathbb{Z}}(|j| + V_j)|u_j|^2 + \mathbf{F}\left(\frac{1}{2}\sum_{j\in\mathbb{Z}} \mathfrak{b}_j(u_j + u_{-j})^2\right).$$

If we restrict ourselves to the linear case, i.e. when $\mathbf{f} = 0$, the Hamiltonian H_{L} consists only of its quadratic part and the linear actions $|u_j|^2$ are all conserved, moreover, for $u(0) = \{u_j(0)\}_{j\in\mathbb{Z}}$, the corresponding solution is given by

$$u_{\mathrm{L}}(t,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{i(k\cdot x + \lambda_k t)}, \quad \lambda_k = |k| + V_k.$$

Let us note that, if only a finite number of initial modes $u_j(0)$ are non-zero, the associated solution u is quasi-periodic and inherits the regularity from the initial data.
On the other hand if the set of initial excited modes is infinite, then u is time almost periodic, being the uniform limit of the truncations $\sum_{|j|\le N} u_j e^{i(j\cdot x+\lambda_j t)}$ but its regularity is more complex to determine and requires further discussions.
In this case, being the linear actions constants of motion, we have that every solution is supported on an invariant torus in the sense that its dynamics is confined to the set

$$\mathcal{T}_{\mathbf{I}} := \{u : |u_j|^2 = \mathbf{I}_j\}, \tag{1.30}$$

for a given sequence $\mathbf{I}_j = |u_j(0)|^2$.
By construction, moreover it follows that the dynamics of H_{L} restricted to $\mathcal{T}_{\mathbf{I}}$ is linear with frequencies λ_j.

A natural question is whether this kind of solutions continue to exist once we take into account the non-linearity.
In this direction, a natural attempt is to try to construct a symplectic change of variables Φ defined in some open ball centered at the origin, such that the resulting Hamiltonian $H \circ \Phi$ still has the invariant tori with given linear frequency ω. To this aim, we consider the following definition:

Definition 1.2.7 (KAM Torus)
Let us consider a sequence $\mathbf{I}_j$ such that $\sqrt{\mathbf{I}} := \{\sqrt{\mathbf{I}_j}\}_{j\in\mathbb{Z}}$ and the associated flat torus $\mathcal{T}_{\mathbf{I}}$ defined in (1.30).
We say that $\mathcal{T}_{\mathbf{I}}$ is a KAM torus of frequency ω for the Hamiltonian

$$N(u) = \sum_{j\in\mathbb{Z}} \omega_j|u_j|^2 + P(u),$$

if the Hamiltonian vector field $X_P := i\frac{\partial P}{\partial \bar{u}}$ vanishes on $\mathcal{T}_{\mathbf{I}}$.

It is clear that every KAM torus is invariant under th flow of N, moreover, the restricted dynamics is linear with frequency ω, namely

$$u_j(t) = u_j(0)e^{i\omega_j t}, \quad |u_j(0)|^2 = \mathbf{I}_j.$$

The construction of such tori relies crucially on solving the *Homological Equation* (equation (4.42)) in a suitable Hamiltonian functional space.
It is well known that this cannot be achieved for all the values of ω, in fact we have to restrict ourselves to a set of "good frequencies" satisfying suitable non-resonance conditions: following the theoretical framework delineated in [33] we fix the hypercube

$$\mathcal{Q} = \left\{ \omega = (\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \;:\quad \sup_j |\omega_j - |j|| \le \frac{1}{2} \right\}.$$

and define the set of the Diophantine frequencies as

Definition 1.2.8 (Diophantine Frequencies)

$$\mathtt{D}_\gamma := \left\{ \omega \in \mathcal{Q} \,:\, |\omega \cdot \ell| > \gamma \prod_{j\in\mathbb{Z}} \frac{1}{(1 + |\ell_j|^2 \langle j \rangle^2)}, \, \forall \ell \in \mathbb{Z}^{\mathbb{Z}}, \; with \; |\ell| := \sum_{j\in\mathbb{Z}} |\ell_j| < \infty \right\}.$$

The goal of the above definition is that we are able to provide quite stringent non-resonant conditions on a set of frequency of full measure (see [23] for rigorous estimates).

Weighted Spaces and Regular Hamiltonians:
We consider Hamiltonians that are absolutely convergent power series defined on the sequence space

$$\mathfrak{g}_{\boldsymbol{h},s} := \left\{ u := (u_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) : \|u\|_{\boldsymbol{h},s} = \sup_{j\in\mathbb{Z}} |u_j| e^{s\cdot \boldsymbol{h}_j} \right\} \tag{1.31}$$

where $\{\boldsymbol{h}_j\}_{j\in\mathbb{Z}}$ is a sequence satisfying:

$$\textbf{(i)} \qquad \boldsymbol{h}_j = \boldsymbol{h}_{-j} \quad j \in \mathbb{Z} \tag{1.32}$$

$$\textbf{(ii)} \qquad \boldsymbol{h}_{j_1+j_2} \le \boldsymbol{h}_{j_1} + \boldsymbol{h}_{j_2} \quad j_1, j_2 \in \mathbb{N} \tag{1.33}$$

$$\textbf{(iii)} \qquad \lim_{i\in\mathbb{N}} \frac{\ln\langle i \rangle}{\boldsymbol{h}_i} = 0 \quad . \tag{1.34}$$

In particular, we will operate within the class of **regular Hamiltonians** $\mathcal{H}_{r,\boldsymbol{h},s}$ that, roughly speaking, are analytic functions whose Cauchy majorant (defined in (4.11)) still is analytic from some ball $B_r(\mathfrak{g}_{\boldsymbol{h},s})$ to $\mathbb{R}$. Such class, moreover, is a Banach scale, once endowed with the norm

$$|H|_{r,\boldsymbol{h},s} := \frac{1}{r} \left(\sup_{\|u\|_{\mathfrak{g}_s} \le r} \|X_{\underline{H}}\|_{\mathfrak{g}_{\boldsymbol{h},s}} \right),$$

where $X_{\underline{H}}$ is the Hamiltonian vector field associated with the Cauchy majorant.
The choice of $\boldsymbol{h}$ is fundamental since it is closely related to the norm $|H|_{r,\boldsymbol{h},s}$. In the specific, conditions (1.32) and (1.33) ensures the monotonicity of the latter with respect to the parameters s and r while condition (1.34) allows us to obtain the bound

$$|L_\omega^{-1}(F)|_{r,\boldsymbol{h},s+\sigma} \le \gamma^{-1} e^{\mathtt{C}(\sigma)} |F|_{r,\boldsymbol{h},s} \tag{1.35}$$

on the solutions of the Homological equation, with $\omega \in \mathtt{D}_\gamma$.
This only holds for equations with this type of nonlinearity, it does not happen for the NLS, for instance.
However, in order to perform the KAM scheme, we must be able to quantify precisely the constant $\mathtt{C}(\sigma)$ (1.35). Thus, in the last part of Chapter 4 we will adopt a weight of the form $\boldsymbol{h}_j = \ln(1+|j|)^\tau$, $\tau > 2$.

As it is easy to imagine, the structure of a solution which is supported on an invariant flat torus $\mathcal{T}_{\mathbf{I}}$, varies depending on whether such torus is maximal or not (i.e. whether all the actions $I_j \neq 0$ or not).
In the case of maximal tori, we construct solutions that are very regular (at least C^∞). Indeed, to face the small divisors we must impose fast growth conditions on the sequence $\boldsymbol{h}_j$, in (1.31). This heuristically comes from the fact that finding very regular almost-periodic solutions is simpler since they are very close to finite dimensional tori.
On the other hand, when we work on non-maximal tori, we have a control on which action is turned on, therefore one can look for special tori which are supported, in Fourier space, on a sparse subset $\mathcal{S}$ of $\mathbb{N}$ called "tangential sites".
This idea was successfully used in [23] to construct lower-regularity (at least Sobolev) solutions for the NLS.
The central fact is that the choice of such sites provides an extra set of parameters which can be used in order to avoid resonances or simplify small divisor estimates. However dealing with the interaction between tangential and normal sites (i.e. $\mathcal{S}^c$) requires a careful case analysis.
In the case of (1.27), this method can be performed without the complications related to the interactions between tangential and normal modes:
in fact, the special nonlinearity of (1.27) guarantees the invariance of the Fourier support of its solutions, preventing these interactions.

With these premises, we will divide Theorem (1.2.6) in two different results:

(i) Maximal Tori
In this setting we produce almost-periodic solutions of high regularity (at least C^∞) for equation (1.27), which are supported on a maximal invariant tori. The main result is the following:

Theorem 1.2.9 *[83] For any $s > 0$, $\gamma > 1$ there exists $\varepsilon_* = \varepsilon_*(s,\gamma) > 0$ such that, for all $r > 0$ that satisfy*

$$\frac{|\mathbf{f}|_R}{\gamma R} r^2 \leq \varepsilon_* \tag{1.36}$$

the following holds:
For every $\omega \in \mathtt{D}_\gamma$ and every $\sqrt{\mathbf{I}} \in \bar{B}_r(\mathfrak{g}_s)$ there exists a potential V and a symplectic analytic change of variables $\Phi : \bar{B}_{2r}(\mathfrak{g}_s) \to \bar{B}_{4r}(\mathfrak{g}_s)$such that $\mathcal{T}_{\mathbf{I}}$ is a KAM torus of frequency ω for $H \circ \Phi$

(ii) Non-Maximal Tori
In the setting delineated above, we use the invariance of (1.27) a to construct almost periodic solutions in spaces of lower regularity, namely

$$\mathtt{w}_p := \big\{u = (u_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) : |u|_p := \sum_{j\in\mathbb{Z}} |u_j| \langle j \rangle^p \leq \infty \big\},$$

that are supported in non-maximal invariant tori.
To this end, we impose stronger non-resonance conditions on the frequency vector ω. More precisely, let $\mathcal{S} \subset \mathbb{N}$ be the subset defined in (4.141), depending only on p, and consider the map $s : \mathbb{N} \to \mathcal{S}$ defined in (4.140) and it's inverse $i : \mathcal{S} \to \mathbb{N}$.
Let us write moreover the hypercube $\mathcal{Q}^{\mathbf{Sym}} = \mathcal{Q}^{\mathbf{Sym}}_{\mathcal{S}} \times \mathcal{Q}^{\mathbf{Sym}}_{\mathcal{S}^c}$, where

$$\mathcal{Q}^{\mathbf{Sym}}_{\mathcal{S}} = \{\omega \in \mathcal{Q}^{\mathbf{Sym}} : \ \omega_j = 0 \text{ if } |j| \notin \mathcal{S}\}$$

and define the Diophantine conditions

$$\mathtt{D}_{\gamma,\mathcal{S}} := \Big\{\omega \in \mathcal{Q}_{\mathcal{S}} \ : \ |\omega \cdot \ell| > \gamma \prod_{i \in \mathbb{N}} \frac{1}{\big(1 + |\ell_i|^2 |i|^2\big)^2}, \ \forall \, \ell \in \mathbb{Z}^{\mathbb{N}}_f \Big\}.$$

Finally, let us consider

$$\mathtt{w}^{\mathcal{S}}_p := \big\{u = (u_i)_{i \in \mathbb{Z}} \in \mathtt{w}_p \ : \ u_i = 0 \text{ for all } |i| \notin \mathcal{S}\big\}\,.$$

Moreover, we decompose $V = (V_{\mathcal{S}}, V_{\mathcal{S}^c})$ with

$$(V_{\mathcal{S}})_j = \begin{cases} 0 & \text{if } |j| \neq \mathcal{S} \\ V_j & \text{otherwise} \end{cases}$$

and similarly for $\mathbf{I} = (\mathbf{I}_{\mathcal{S}}, \mathbf{I}_{\mathcal{S}^c})$. The main result is the following:

Theorem 1.2.10 (Lower-Regularity) *Under the same hypotheses of Theorem 1.2.9, there exists $r > 0$ such that for every $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$ and every $\sqrt{\mathbf{I}_{\mathcal{S}}} \in \bar{B}_r(\mathtt{w}^{\mathcal{S}}_p)$, there exists a potential V and a symplectic analytic change of variables $\Phi : \bar{B}_{2r}(\mathtt{w}_p) \to \bar{B}_{4r}(\mathtt{w}_p)$ such that $\mathcal{T}_{\mathbf{I}_{\mathcal{S}}}$ is a KAM torus of frequency ω for $H \circ \Phi$.*

We note that, depending on the value of p, the almost-periodic solutions obtained via Theorem 1.2.10 are either classical or weak in the sense of Definition 1.2.5. Moreover, these solutions are supported on a non-maximal torus, since their dynamics is confined to

$$\mathbb{T}^{\mathcal{S}}_{\mathcal{I}} := \big\{u = (u_j)_{j \in \mathbb{Z}} \ : \ |u_j|^2 = \mathcal{I}_j \text{ for } |j| \in \mathcal{S}, \quad u_j = 0 \text{ for } |j| \notin \mathcal{S}\big\}\,.$$

1.2.6 Sketch of the Proof

- **Projectors and decomposition of the Hamiltonian**

  The space of regular Hamiltonians is decomposed into subspaces according to their order of vanishing at a prescribed invariant torus $\mathcal{T}_{\mathbf{I}}$. This leads to a direct sum decomposition of the form

  $$\mathcal{H} = \mathcal{H}^{(-2)} \oplus \mathcal{H}^{(0)} \oplus \mathcal{H}^{(\geq 2)},$$

  where:

  $\mathcal{H}^{(-2)}$ contains the terms that do not vanish on $\mathcal{T}_{\mathbf{I}}$, $\mathcal{H}^{(0)}$ contains terms whose Hamiltonian vector field is tangent but not zero on $\mathcal{T}_{\mathbf{I}}$, $\mathcal{H}^{(\geq 2)}$ consists of terms whose vector field vanishes on $\mathcal{T}_{\mathbf{I}}$. Explicit projection operators onto these subspaces are constructed via an auxiliary Hamiltonian depending on artificial action variables. This decomposition is stable under Poisson brackets and does not introduce singularities at the origin. The projectors play a fundamental role in identifying the obstructions to the persistence of invariant tori.

- **KAM algorithm**

  The KAM iteration is formulated as a sequence of near-identity symplectic transformations acting on the Hamiltonian. At each step, the terms belonging to $\mathcal{H}^{(-2)}$ and $\mathcal{H}^{(0)}$ are eliminated by solving suitable homological equations. The remaining Hamiltonian is progressively conjugated to a normal form which belongs to $\mathcal{H}^{(\geq 2)}$. The iteration produces a *twisted conjugacy*, where counter-terms appear as corrections to the frequencies. These counter-terms depend Lipschitz-continuously on the external parameters and can be removed by a suitable choice of such parameters. The scheme converges uniformly with respect to the dimension of the invariant torus, allowing both finite-dimensional and infinite-dimensional tori to be treated in the same framework. The limiting Hamiltonian is in normal form at the torus $\mathcal{T}_{\mathbf{I}}$, which is therefore invariant. The induced dynamics on $\mathcal{T}_{\mathbf{I}}$ is linear with Diophantine frequency ω.
  The resulting invariant torus supports quasi-periodic or almost-periodic solutions depending on the cardinality of the excited modes.

- **Homological equation**

  The KAM scheme relies on solving homological equations of the form

  $$\mathcal{L}_\omega F = G,$$

  where $\mathcal{L}_\omega := \{\sum_{j\in\mathbb{Z}} \omega_j |u_j|^2, \cdot\}$ is the linearized operator associated with the integrable part of the Hamiltonian. The right-hand side G belongs to specific sub-spaces determined by the order of vanishing at the reference torus. Strong infinite-dimensional Diophantine conditions on the frequency vector ω are imposed to control small divisors uniformly. A crucial feature is that G is analytic in a ball centered at the origin, which allows one to solve the homological equation with no loss of regularity in the chosen Banach spaces. Sharp bounds are obtained for the inverse operator $\mathcal{L}_\omega^{-1}$, ensuring tame estimates compatible with the iteration.

1.3 Organization of the Thesis

The thesis is organized as follows:

- Chapter 2 is devoted to the study of (1.2) on $\mathbb{T}^n$ and to the analysis of its resonances: After rewriting the equation as an infinite-dimensional Hamiltonian system, in Sections 2.3 and 2.3.2 a sequence of preliminary transformations is performed in order to diagonalize the linear part and block-diagonalize the highest-order terms.
  In Sections 2.4 and 2.5 we perform two steps of normal form and we show that the cubic and quintic resonant terms obtained do not contribute to the growth of Sobolev norms. As a consequence, we can prove Theorems 1.2.1 and 1.2.2.

- Chapter 3 develops a formal Birkhoff normal form approach for Kirchhoff-type Hamiltonians.
  Working in the framework of formal power series introduced in Section 3.1.1, the chapter reproduces the lower-order cancellations observed in Chapter 2 in the setting of formal Birkhoff normal forms and allows one to extend the analysis to higher order resonances of the Hamiltonian. In particular, using Wolfram Mathematica the degree-eight resonant terms are explicitly computed and shown to contain monomials that are not action-preserving, thus providing evidence against integrability beyond the lower orders.

In Subsections 3.2.4 and 3.4.2 we explain the codes used to prove Theorems 1.2.3 and 1.2.4.

- Chapter 4 is devoted to proving the existence of almost-periodic small-amplitude solutions for equation (1.27) After introducing a suitable functional setting in Section 4.2, we solve the homological equation (Section 4.3). In Section 4.4 we introduce the projector operators (4.71). In Section 4.5 we prove the KAM Iterative Lemma 4.5.1 that implies Theorem 1.2.9.
  Finally in Section 4.6 we prove Theorem 1.2.10.

Chapter 2

Quasi-Linear Normal Form Approach

2.1 Introduction

In this Chapter we consider the Cauchy problem associated with the Kirchhoff equation on the n-dimensional torus $\mathbb{T}^n$

$$\partial_{tt}u - \left(\frac{1}{1 + c\int_{\mathbb{T}^n} |\nabla u|^2\, dx}\right)^2 \Delta u = 0, \quad c \in \mathbb{R} \tag{2.1}$$

with initial data $u_0 := u(0, \cdot)$, $v_0 := u_t(0, \cdot)$ in Sobolev class.
The main results of this chapter are stated in Theorems 1.2.1 and 1.2.2. We briefly recall the fundamental steps:
The proof is based on a normal form analysis of the Kirchhoff equation within its Hamiltonian formulation. In Section 2.3, we rewrite the equation (2.1) as the first-order system (2.2) by employing the complex variables (2.6) that diagonalize the linear part. This allows us to express the Kirchhoff in the form (2.7) that is convenient for energy estimates and for the identification of resonant interactions.
In the quasilinear setting, the use of normal form techniques is obstructed by the difficulty of closing the energy estimates after the associated change of variables.
To overcome this difficulty, in Section 2.3.2 we block-diagonalize the highest-order terms of (2.7), using ideas from paradifferential calculus. This transformation reduces the system to one of the form (2.12), for which the remaining non-diagonal terms are bounded operators.
Once this preliminary reduction is achieved, in Section 2.4 and 2.5 we construct two nonlinear normal form transformations ((2.36) and (2.71)) aimed at eliminating non-resonant terms with degree respectively 3 and 5. A key point of the analysis is the study of the resonant terms that survive these transformations. We show that,the remaining resonant terms do not contribute to the growth of Sobolev norms.
As a consequence, the evolution of the solutions is effectively governed by higher-order remainders, which yields long-time stability results for small-amplitude initial data.

2.2 Hamiltonian Structure and Functional Setting

Recalling Subsection 1.1.2 we consider the set of variables $(u, v) = (u, u_t)$, that decouples equation (2.1) in a system of two partial differential equations of order one with respect to time, specifically:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \left(\dfrac{1}{1 + c \int_{\mathbb{T}^n} |\nabla u|^2 \, dx} \right)^2 \Delta u \end{cases} \tag{2.2}$$

By considering the Hamiltonian $H : H^1_0(\mathbb{T}^n, \mathbb{R}) \times L^2(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$,

$$H(u, v) = \frac{1}{2} \int_{\mathbb{T}^n} v^2 dx - \frac{1}{2} \left(\frac{c}{1 + c \int_{\mathbb{T}^n} |\nabla u| \, dx} \right), \tag{2.3}$$

we also have that system (2.2) can be written in the form

$$\begin{cases} \partial_t u = \nabla_v H(u, v) \\ \partial_t v = -\nabla_u H(u, v) \end{cases},$$

where $\nabla_u H$ and $\nabla_v H$ are the gradients of H with respect to the real scalar product on L^2:

$$\langle f, g \rangle := \int_{\mathbb{T}^n} f(x) g(x) dx, \quad f, \, g \in L^2(\mathbb{T}^n, \mathbb{R}).$$

In terms of the Fourier expansion of its solution $u(t, \cdot) : \mathbb{T}^n \to \mathbb{R}$

$$u(t, x) = \frac{1}{(2\pi)^{n/2}} \sum_{k \in \mathbb{Z}^n} u_k(t) e^{ik \cdot x}, \quad u_k(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} u(t, x) e^{-ik \cdot x} dx,$$

system (2.2) turns into the (infinite-dimensional) system of ordinary differential equations

$$\ddot{u}_k(t) + \left(\frac{1}{1 + c \sum\limits_{j \in \mathbb{Z}^n} |j|^2 |u_j(t)|^2} \right) |k|^2 u_k(t) = 0, \quad k \in \mathbb{Z}^n,$$

parametrized by $k \in \mathbb{Z}^n$. By adopting a dynamical system point of view, this implies that equation (2.1) can be seen as an infinite-dimensional Hamiltonian system in the phase space $H^s_0(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}_0(\mathbb{T}^n, \mathbb{R})$, $s \geq 1$.
Let us note, moreover, that the nonlinearity in Hamiltonian (2.3) and the corresponding one in equation (2.1), are well-defined only when the denominator

$$1 + c \int_{\mathbb{T}^n} |\nabla u| \, dx \neq 0.$$

Since $\int_{\mathbb{T}^n} |\nabla u| = \|u\|_1^2$, it is sufficient to consider solutions supported in the ball

$$u \in B_{\delta_0} \left(H^1_0(\mathbb{T}^n, \mathbb{R}) \right),$$

where

$$\delta_0 := \frac{1}{\sqrt{|c|}} - \varepsilon, \quad \varepsilon > 0. \tag{2.4}$$

In this way both the nonlinearities involved in Hamiltonian (2.3) and in equation (2.1) are well-defined.

2.3 Preliminary Transformations

The purpose of this section is to perform two preliminary changes of variables, one linear and one nonlinear, in order to conjugate system (2.2) to one in a more suitable form.
The first one diagonalizes the linear part of the system, while the second one has the purpose of removing, up to a bounded remainder, the off-diagonal unbounded part.
This strategy of eliminating the off-diagonal unbounded terms before the normal form construction is necessary due to the quasi-linear structure of the Kirchhoff equation.

2.3.1 Diagonalization of the highest order

We want to diagonalize the linear part of system (2.2), namely

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u \end{cases} . \tag{2.5}$$

Let us consider the complex linear isomorphism:

$$\Phi^{(1)} : H_0^s(\mathbb{T}^n, \mathbb{R}) \times H_0^{s-1}(\mathbb{T}^n, \mathbb{R}) \longrightarrow H_0^{s-\frac{1}{2}}(\mathbb{T}^n, \mathbb{C}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^n, \mathbb{C})$$

$$\begin{cases} z = \dfrac{\left(\Lambda^{\frac{1}{2}} u + i\Lambda^{-\frac{1}{2}} v\right)}{\sqrt{2}} \\[2ex] \bar{z} = \dfrac{\left(\Lambda^{\frac{1}{2}} u - i\Lambda^{-\frac{1}{2}} v\right)}{\sqrt{2}} \end{cases} \overset{\Phi^{(1)}}{\longleftrightarrow} \begin{cases} u = \dfrac{\Lambda^{-\frac{1}{2}} (z + \bar{z})}{\sqrt{2}} \\[2ex] v = \dfrac{\Lambda^{\frac{1}{2}} (z - \bar{z})}{i\sqrt{2}} \end{cases} \tag{2.6}$$

where $\Lambda = \sqrt{-\Delta}$ is the Fourier multiplier defined by

$$|\Lambda|\, e^{ik\cdot x} = |k| e^{ik\cdot x}, \qquad \text{for } k \in \mathbb{Z}^n.$$

When, as in our case, $\Phi^{(1)}$ is restricted to a pair of real functions, it becomes a real isomorphism from $H_0^s(\mathbb{T}^n, \mathbb{R}) \times H_0^s(\mathbb{T}^n, \mathbb{R})$ into the space of complex-conjugate, zero-mean functions

$$H_0^s(\mathbb{T}^n, c.c.) = \{(f,\, g) \in H_0^s(\mathbb{T}^n, \mathbb{C}) \times H_0^s(\mathbb{T}^n, \mathbb{C}) \,:\, g = \bar{f}\}.$$

In these new variables, the linear part (2.5) takes the diagonal form

$$\begin{cases} \partial_t z = -i\Lambda z \\ \partial_t \bar{z} = i\Lambda \bar{z} \end{cases}$$

while the whole system (2.2) transforms into:

$$\begin{cases} \partial_t z = -i\Lambda z - \dfrac{i}{2}\left(\dfrac{1}{\left(1 + \frac{c}{2}\langle \Lambda(z+\bar{z}), z+\bar{z}\rangle\right)^2} - 1\right)(\Lambda z + \Lambda \bar{z}) \\[2ex] \partial_t \bar{z} = i\Lambda \bar{z} + \dfrac{i}{2}\left(\dfrac{1}{\left(1 + \frac{c}{2}\langle \Lambda(z+\bar{z}), z+\bar{z}\rangle\right)^2} - 1\right)(\Lambda z + \Lambda \bar{z}) \end{cases}, \tag{2.7}$$

where

$$\langle \Lambda(z+\bar{z}), z+\bar{z}\rangle = \int_{\mathbb{T}^n} \left[(z(x) + \bar{z}(x))\,(\Lambda z(x) + \Lambda \bar{z}(x))\right] dx.$$

Finally, by recalling the definition of μ and Q in (1.18), system (2.7) can be rewritten as

$$\begin{cases} \partial_t z = -i\Lambda z - \frac{i}{2}\mu\left(Q(z,\bar{z})\right)(\Lambda z + \Lambda\bar{z}) \\ \partial_t \bar{z} = i\Lambda\bar{z} + \frac{i}{2}\mu\left(Q(z,\bar{z})\right)(\Lambda z + \Lambda\bar{z}) \end{cases} . \tag{2.8}$$

Note that the second equation of (2.8) is redundant, being the complex conjugate of the first one.

Remark 2.3.1 *The quantity $Q(z,\bar{z})$ is real and positive, since it corresponds, in the new variables, to the term $\int_{\mathbb{T}^n} |\nabla u|^2$.*
Therefore, system (2.8) *is well-defined only when it holds*

$$Q(z,\bar{z}) < 1/|c|. \tag{2.9}$$

Since one has

$$Q(z,\bar{z}) \le \|z\|^2_{\frac{1}{2}}, \quad (z,\bar{z}) \in H_0^{\frac{1}{2}}(\mathbb{T}^n, c.c.) \tag{2.10}$$

condition (2.9) *is verified for $(z,\bar{z})$ in the ball*

$$(z,\bar{z}) \in B_{\delta_0}\left(H_0^{\frac{1}{2}}(\mathbb{T}^n, c.c)\right),$$

where

$$B_r\left(H_0^{\frac{1}{2}}(\mathbb{T}^n, c.c)\right) := \{(z,\bar{z}) \,:\, \|z\|_{\frac{1}{2}} \le r\}$$

and δ_0 is the same as in (2.4).

System (2.8) is again Hamiltonian of the form

$$\begin{cases} \partial_t z = -i\nabla_{\bar{z}} H(z,\bar{z}) \\ \\ \partial_t \bar{z} = i\nabla_z H(z,\bar{z}) \end{cases} \tag{2.11}$$

with

$$H: \qquad H^{\frac{1}{2}}(\mathbb{T}^n,\mathbb{C}) \times H^{\frac{1}{2}}(\mathbb{T}^n,\mathbb{C}) \longrightarrow \mathbb{R}$$

$$H(z,\bar{z}) = \quad \langle \Lambda z, \bar{z}\rangle - \frac{1}{2c}\left[\frac{\left(\frac{c}{2}\langle \Lambda(z+\bar{z}), z+\bar{z}\rangle\right)^2}{1+\frac{c}{2}\langle \Lambda(z+\bar{z}), z+\bar{z}\rangle}\right].$$

Here, ∇_z is the gradient with respect to the complex inner product

$$\langle f, g\rangle_{\mathbb{C}} = \langle f, \bar{g}\rangle, \quad f, g \in L^2(\mathbb{T}^n,\mathbb{C}).$$

The map $\Phi^{(1)}$ is also a symplectomorphism, in the sense that the conjugated Hamiltonian system (2.11) conserves the same structure as the previous one, in fact it can be written in the form

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = iJ \begin{pmatrix} \nabla_z H \\ \nabla_{\bar{z}} H \end{pmatrix}$$

where J is defined in (1.16).

Remark 2.3.2 (Conserved Quantities in Complex Variables)
From Subsection 1.1.3 we know that the quantities

$$M_j := \frac{1}{2} ij\left[u_j(\partial_t u_{-j}) - u_{-j}(\partial_t u_j)\right], \quad j \in \mathbb{Z}$$

are conserved for the flow of (2.1). *Where* $\{u_j\}_{j\in\mathbb{Z}}$ *are the Fourier coefficients for the solution of* (2.1) *in the original variables.*
In the complex coordinates given by (2.6)*, the quantities* M_j *take the form*

$$\begin{aligned} M_j &= \frac{1}{4} j\left[(z_j + \bar{z}_{-j})(z_{-j} - \bar{z}_j) - (z_{-j} + \bar{z}_j)(z_j - \bar{z}_{-j})\right] \\ &= \frac{1}{2} j[z_{-j}\bar{z}_{-j} - z_j\bar{z}_j] \\ &= \frac{1}{2} j[|z_{-j}|^2 - |z_j|^2]. \end{aligned}$$

We then have that the quantities $|z_j|^2 - |z_{-j}|^2$*,* $j \in \mathbb{N}$ *are conserved by the flow of* (2.1).

Remark 2.3.3 (Invariance of the Fourier Support in Complex Variables)
The Fourier support for the original variables, defined in (1.9)*, translates, in the new complex variables, in the set*

$$\mathcal{S}_{\mathbb{C}}(t) := \{j \in \mathbb{Z}^n : z_j(t) - \bar{z}_{-j}(t) \neq (0,0)\}.$$

This, together with Lemma 1.1.1, implies that if the initial data z_0 *for system* (2.7) *is supported on a symmetric set of frequencies, then this support is invariant for the corresponding solution.*

Even if, in the new variables the linear operator is diagonal, when one tries to perform an energy estimate on the Sobolev norm H^s, $s \geq \frac{1}{2}$ of a solution z of (2.8), one obtains

$$\begin{aligned} \partial_t \|z\|_s^2 &\overset{(1.21)}{=} 2\mathrm{Re}\left(i\langle \Lambda z,\, \Lambda\bar{z} + \frac{1}{2}\mu\left(Q(z,\bar{z})\right)(\Lambda z + \Lambda\bar{z})\rangle \right) \\ &= |\mu\left(Q(z,\bar{z})\right)|\,\mathrm{Re}\left(i\langle \Lambda z,\, \Lambda z\rangle\right). \end{aligned}$$

Since

$$|\mu\left(Q(z_1,\bar{z}_2)\right)| \leq \mathfrak{c}_0(\varepsilon,c)\|z_1\|_{\frac{1}{2}}\|z_2\|_{\frac{1}{2}}$$

for $z_1,\, z_2 \in B_{\delta_0}\left(H^{\frac{1}{2}}(\mathbb{T}^n, c.c.)\right)$ and $\mathfrak{c} = \mathfrak{c}(\varepsilon, c)$, one has an additional loss of half derivative

$$\partial_t \|z\|_s^2 \leq \mathfrak{c}(\varepsilon,c)\|z\|_{\frac{1}{2}}^2\|z\|_{s+\frac{1}{2}}^2$$

that prevents us from performing the energy estimate.
The responsible for this loss are the off-diagonal terms of system (2.8), i.e. the ones of the form

$$\begin{cases} \partial_t z = -\frac{i}{2}\mu\left(Q(z,\bar{z})\right)\Lambda\bar{z} \\ \partial_t \bar{z} = \frac{i}{2}\mu\left(Q(z,\bar{z})\right)\Lambda z \end{cases}.$$

We then have to perform a preliminary step in order to remove those terms.

2.3.2 Block Diagonalization

The aim of this section is to transform system (2.8) into the system (2.12) below, which is block-diagonal modulo bounded remainders.
The main result we will prove is:

Theorem 2.3.1 *There exists a* $\delta_0 \geq 0$ *and an invertible, close-to-identity map*

$$\Phi^{(2)} : B_{\delta_0}\left(H_0^s(\mathbb{T}^n, c.c.)\right) \to H_0^s(\mathbb{T}^n, c.c)$$

that conjugates system (2.8) *to one of the form*

$$\begin{cases} \partial_t \eta = -i\left(1 - cQ(\eta,\bar{\eta})\right)\Lambda\eta - \dfrac{ic\left(1 - cQ(\eta,\bar{\eta})\right)}{2}\left(\langle\Lambda\bar{\eta},\Lambda\bar{\eta}\rangle - \langle\Lambda\eta,\Lambda\eta\rangle\right)\bar{\eta} \\ \\ \partial_t \bar{\eta} = i\left(1 - cQ(\eta,\bar{\eta})\right)\Lambda\bar{\eta} - \dfrac{ic\left(1 - cQ(\eta,\bar{\eta})\right)}{2}\left(\langle\Lambda\bar{\eta},\Lambda\bar{\eta}\rangle - \langle\Lambda\eta,\Lambda\eta\rangle\right)\eta \end{cases} . \tag{2.12}$$

Recalling what we said in subsection (1.2.2), we have that system (2.8) is of the form

$$\partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = i\mathcal{L}(z,\bar{z})\begin{pmatrix} \Lambda z \\ \Lambda \bar{z} \end{pmatrix}, \tag{2.13}$$

where the coefficients of the matrix

$$\mathcal{L}(z,\bar{z}) = \begin{pmatrix} \mathcal{L}_{1,1}(z,\bar{z}) & \mathcal{L}_{1,2}(z,\bar{z}) \\ \mathcal{L}_{2,1}(z,\bar{z}) & \mathcal{L}_{2,2}(z,\bar{z}) \end{pmatrix}$$

are defined in (1.23).
For the sake of simplicity, let us rewrite the diagonal coefficients as

$$\mathcal{L}_{1,1}(z_1,z_2) = -\mathcal{L}_{2,2}(z_1,z_2) = -1 - \frac{1}{2}\mathcal{Q}(z_1,z_2)$$

and the non-diagonal ones as

$$\mathcal{L}_{1,2}(z_1,z_2) = -\mathcal{L}_{2,1}(z_1,z_2) = -\frac{1}{2}\mathcal{Q}(z_1,z_2)$$

where

$$\mathcal{Q}(z_1,z_2) := \mu\left(Q(z_1,z_2)\right) \tag{2.14}$$

and μ, Q are defined in (1.18).
Since the off-diagonal terms of $\mathcal{L}$ are responsible for the loss of half-derivative when one tries to perform an energy estimate on the H_0^s norm of the solution of (2.13), our aim is to construct a transformation and remove such terms.
By adopting a para-differential point of view, one can consider system (2.13) as a linear one for which $\mathcal{Q}(z,\bar{z})$ plays the role of a coefficient. Our strategy therefore is to diagonalize the associated matrix

$$\mathtt{M} = \begin{pmatrix} -1-\mathtt{y} & -\mathtt{y} \\ \mathtt{y} & 1+\mathtt{y} \end{pmatrix}$$

where $\mathtt{y} = \frac{1}{2}\mathcal{Q}(z,\bar{z})$.
We have $\mathtt{M}$ has eigenvalues of the form

$$\mathtt{m}_1 = -\sqrt{1+2\mathtt{y}} \quad \text{and} \quad \mathtt{m}_2 = \sqrt{1+2\mathtt{y}},$$

with associated eigenvectors $\begin{pmatrix} 1 \\ \rho(\mathrm{y}) \end{pmatrix}, \begin{pmatrix} \rho(\mathrm{y}) \\ 1 \end{pmatrix}$ where

$$\rho(\mathrm{y}) = \frac{-\mathrm{y}}{1+\mathrm{y}+\sqrt{1+2\mathrm{y}}}. \tag{2.15}$$

Therefore we have

$$\begin{pmatrix} 1 & \rho(\mathrm{y}) \\ \rho(\mathrm{y}) & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1-\mathrm{y} & -\mathrm{y} \\ \mathrm{y} & 1+\mathrm{y} \end{pmatrix} \begin{pmatrix} 1 & \rho(\mathrm{y}) \\ \rho(\mathrm{y}) & 1 \end{pmatrix} = \begin{pmatrix} -\sqrt{1+2\mathrm{y}} & 0 \\ 0 & \sqrt{1+2\mathrm{y}} \end{pmatrix}. \tag{2.16}$$

It is natural to define a new set of variables of the form

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \mathcal{M}(z_1, z_2) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \\ \mathcal{M}(z_1, z_2) &:= \frac{1}{\sqrt{1-\rho^2\left(\mathcal{Q}(z_1, z_2)\right)}} \begin{pmatrix} 1 & \rho\left(\mathcal{Q}(z_1, z_2)\right) \\ \rho\left(\mathcal{Q}(z_1, z_2)\right) & 1 \end{pmatrix}. \end{aligned} \tag{2.17}$$

Remark 2.3.4 *As pointed out in [6], the corrective term*

$$\frac{1}{\sqrt{1-\rho^2\left(\mathcal{Q}(z_1, z_2)\right)}}$$

used in formula (2.17) *is essential. Indeed, using any other similar change will generate a diagonal term of order zero in the new system which gives a non-trivial contribution to the energy estimate.*
This corrective term is the only way, up to constant factors, to eliminate those diagonal terms and is related to the symplectic structure of (2.12).

Remark 2.3.5 *We can invert expression* (2.17) *with respect to the variables* z_1, z_2 *and get*

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{M}^{-1}(z, \bar{z}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.18}$$

where

$$\mathcal{M}^{-1}(z_1, z_2) = \frac{1}{\sqrt{1-\rho^2(\mathcal{Q}(z_1, z_2))}} \begin{pmatrix} 1 & -\rho(\mathcal{Q}(z_1, z_2)) \\ -\rho(\mathcal{Q}(z_1, z_2)) & 1 \end{pmatrix}. \tag{2.19}$$

From (2.18) *it follows that complex conjugated pairs of functions* $(z, \bar{z})$ *are mapped into another pair of complex conjugated functions. Therefore,* (2.17) *is well-posed from the space of complex conjugate functions onto itself. We will then consider* $\eta_2 = \bar{\eta}_1$.

Remark 2.3.6 *From the expression of* ρ *in* (2.15) *and* $\mathcal{Q}$ *in* (2.14) *one has that*

$$\rho(\mathcal{Q}(z, \bar{z})) = \frac{cQ(z, \bar{z})}{2+cQ(z, \bar{z})}. \tag{2.20}$$

It follows that the above quantity is well-defined, since, from Remark 2.3.1, $(z, \bar{z}) \in B_{\delta_0}\left(H_0(\mathbb{T}^n, c.c.)\right)$.
Let us note moreover that, under this hypothesis, also the term $\frac{1}{\sqrt{1-\rho^2(\mathcal{Q}(z,\bar{z}))}}$ *is well-defined.*

Since the change defined in (2.17) is implicit in the variable $(z, \bar{z})$ we have to express $\mathcal{Q}(z, \bar{z})$ in terms of η and $\bar{\eta}$:
since $\mathcal{Q}(z, \bar{z}) = \frac{1}{2}\mu(Q(z, \bar{z}))$, se start by make explicit the dependence of $Q(z, \bar{z}) = \frac{1}{2}\langle\Lambda(z + \bar{z}),\, z + \bar{z}\rangle$ from the variables $(\eta, \bar{\eta})$.
Recalling (2.17), we have

$$
\begin{aligned}
\langle\Lambda(z+\bar{z}), z+\bar{z}\rangle &= \frac{1}{(1-\rho^2(\mathcal{Q}(z,\bar{z})))}\langle(1+\rho(\mathcal{Q}(z,\bar{z})))\Lambda(\eta+\bar{\eta}), (1+\rho(\mathcal{Q}(z,\bar{z})))(\eta+\bar{\eta})\rangle \\
&= \frac{1+\rho(\mathcal{Q}(z,\bar{z}))}{1-\rho(\mathcal{Q}(z,\bar{z}))}\langle\Lambda(\eta+\bar{\eta}), \eta+\bar{\eta}\rangle\,,
\end{aligned} \tag{2.21}
$$

moreover, from the definition of ρ in (2.15), we have that

$$
\frac{1-\rho(\mathrm{y})}{1+\rho(\mathrm{y})} = \sqrt{1+2\mathrm{y}},
$$

hence

$$
\begin{aligned}
Q(\eta,\bar{\eta}) &\overset{(2.21)}{=} \sqrt{1+\mathcal{Q}(z,\bar{z})}Q(z,\bar{z}) \\
&\overset{(1.18)}{=} \frac{Q(z,\bar{z})}{1+cQ(z,,\bar{z})}.
\end{aligned} \tag{2.22}
$$

By inverting expression (2.22) with respect to $Q(z, \bar{z})$ we get

$$
Q(z,\bar{z}) = \frac{Q(\eta,\bar{\eta})}{1-cQ(\eta,\bar{\eta})} := \varphi\left(Q(\eta,\bar{\eta})\right). \tag{2.23}
$$

Let us note that (2.23) is well defined only when $Q(\eta, \bar{\eta}) < 1/|c|$. As in Remark 2.3.1 it is sufficient to impose

$$
(\eta,\bar{\eta}) \in B_{\delta_0}\left(H_0^{\frac{1}{2}}(\mathbb{T}^n, c.c)\right).
$$

We can now make expression (2.17) explicit with respect to the variables $(\eta,\, \bar{\eta})$ and define the map

$$
(f,g) = \Phi^{(2)}(\eta,\bar{\eta}) := \mathcal{M}(\eta,\bar{\eta})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}, \tag{2.24}
$$

where

$$
\mathcal{M}(\eta,\bar{\eta}) = \frac{1}{\sqrt{1-\varrho^2(\eta,\bar{\eta})}}\begin{pmatrix}1 & \varrho(\eta,\bar{\eta})\\ \varrho(\eta,\bar{\eta}) & 1\end{pmatrix}
$$

and

$$
\begin{aligned}
\varrho(\eta,\bar{\eta}) &:= \rho\left(\frac{\mu}{2}\left(\varphi\left(Q(\eta,\bar{\eta})\right)\right)\right) \\
&\overset{(1.18),(2.23)}{=} \frac{cQ(\eta,\bar{\eta})}{2-cQ(\eta,\bar{\eta})}.
\end{aligned} \tag{2.25}
$$

Reasoning as in [6], Lemma 3.1, we have:

Lemma 2.3.1 *For every $s \geq \frac{1}{2}$ the nonlinear map $\Phi^{(2)} : B_{\delta_0}(H_0^s(\mathbb{T}^n, c.c.)) \to H_0^s(\mathbb{T}^n, c.c.)$ is invertible, continuous, with continuous inverse*

$$\left(\Phi^{(2)}\right)^{-1}(z,\bar{z}) = \frac{1}{\sqrt{1-\rho^2\left(\mathcal{Q}(z,\bar{z})\right)}} \begin{pmatrix} 1 & -\rho\left(\mathcal{Q}(z,\bar{z})\right) \\ -\rho\left(\mathcal{Q}(z,\bar{z})\right) & 1 \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}$$

defined for every $(z,\bar{z}) \in B_{\delta_0}(H_0^s(\mathbb{T}^n, c.c.))$ with δ_0 defined in (2.4).
Moreover, for all $s \geq \frac{1}{2}$, all $(\eta,\bar{\eta}) \in H_0^s(\mathbb{T}^n, c.c.)$ satisfying $\|\bar{\eta}\|_{\frac{1}{2}} \leq \delta_0$ one has

$$\|\Phi^{(2)}(\eta,\bar{\eta})\|_s \leq \mathfrak{C}(\|\eta,\bar{\eta}\|_{\frac{1}{2}})\|\eta,\bar{\eta}\|_s$$

for some increasing function $\mathfrak{C}$. The same estimate is satisfied by $\left(\Phi^{(2)}\right)^{-1}$.

Proof: The bounds $\|\eta\|_{\frac{1}{2}}, \|\eta\|_{\frac{1}{2}} \leq \delta_0$ guarantees that the denominators $\sqrt{1-\rho^2\left(\mathcal{Q}(z,\bar{z})\right)}$, $\sqrt{1-\rho^2\left(\mathcal{Q}(\eta,\bar{\eta})\right)}$ are well defined, while the estimates on $\Phi^{(2)}$ and $\left(\Phi^{(2)}\right)^{-1}$ follow from (2.15), (2.20), (2.24) and estimate (2.10). □

Let us see how system (2.13) behaves under the change (2.17):
By (2.24) one has

$$\begin{aligned} \partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} &= \partial_t \Phi^{(2)}(\eta,\bar{\eta}) = \partial_t \left(\mathcal{M}(\eta,\bar{\eta}) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} \right) \\ &= \mathcal{M}(\eta,\bar{\eta}) \partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} + \partial_t \Big(\mathcal{M}(\eta,\bar{\eta}) \Big) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}. \end{aligned} \tag{2.26}$$

On the other hand, by using (1.22)

$$\begin{aligned} \partial_t \begin{pmatrix} z \\ \bar{z} \end{pmatrix} &= i\mathcal{L}(z,\bar{z}) \begin{pmatrix} \Lambda z \\ \Lambda \bar{z} \end{pmatrix} \\ &= i \begin{pmatrix} -1-\mathcal{Q}(z,\bar{z}) & -\mathcal{Q}(z,\bar{z}) \\ \mathcal{Q}(z,\bar{z}) & 1+\mathcal{Q}(z,\bar{z}) \end{pmatrix} \begin{pmatrix} \Lambda z \\ \Lambda \bar{z} \end{pmatrix} \\ &\overset{(2.17)}{=} i \begin{pmatrix} -1-\mathcal{Q}(z,\bar{z}) & -\mathcal{Q}(z,\bar{z}) \\ \mathcal{Q}(z,\bar{z}) & 1+\mathcal{Q}(z,\bar{z}) \end{pmatrix} \mathcal{M}(z,\bar{z}) \begin{pmatrix} \Lambda \eta \\ \Lambda \bar{\eta} \end{pmatrix} \\ &\overset{(2.16)}{=} i\mathcal{M}(z,\bar{z}) \begin{pmatrix} -\sqrt{1+2\mathcal{Q}(z,\bar{z})} & 0 \\ 0 & \sqrt{1+2\mathcal{Q}(z,\bar{z})} \end{pmatrix} \begin{pmatrix} \Lambda \eta \\ \Lambda \bar{\eta} \end{pmatrix} \\ &\overset{(2.14),(2.23)}{=} i\mathcal{M}(\eta,\bar{\eta}) \begin{pmatrix} -\sqrt{1+\mu\left(\varphi\left(Q(\eta,\bar{\eta})\right)\right)} & 0 \\ 0 & \sqrt{1+2\mu\left(\varphi\left(Q(\eta,\bar{\eta})\right)\right)} \end{pmatrix} \begin{pmatrix} \Lambda \eta \\ \Lambda \bar{\eta} \end{pmatrix} \\ &\overset{(1.18),(2.23)}{=} i\mathcal{M}(\eta,\bar{\eta}) \begin{pmatrix} -(1-cQ(\eta,\bar{\eta})) & 0 \\ 0 & 1-cQ(\eta,\bar{\eta}) \end{pmatrix} \begin{pmatrix} \Lambda \eta \\ \Lambda \bar{\eta} \end{pmatrix}. \end{aligned} \tag{2.27}$$

Finally, by matching (2.26) and (2.27) one gets

$$\partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} + \mathcal{M}(\eta,\bar{\eta})^{-1} \partial_t \Big(\mathcal{M}(\eta,\bar{\eta}) \Big) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = i(1-cQ(\eta,\bar{\eta})) \begin{pmatrix} -\Lambda \eta \\ \Lambda \bar{\eta} \end{pmatrix}. \tag{2.28}$$

We have to study the left-hand side of (2.28):
by (2.19) and (2.15)

$$\mathcal{M}(\eta,\bar{\eta})^{-1} = \frac{1}{\sqrt{1-\varrho^2(\eta,\bar{\eta})}} \begin{pmatrix} 1 & -\varrho\left(\eta,\bar{\eta}\right)) \\ -\varrho\left(\eta,\bar{\eta}\right) & 1 \end{pmatrix},$$

moreover

$$\partial_t\Big(\mathcal{M}(\eta,\bar\eta)\Big) = \frac{1}{(1-\varrho^2(\eta,\bar\eta))^{\frac{3}{2}}}\begin{pmatrix}\varrho\,(\eta,\bar\eta)) & 1\\ 1 & \varrho\,(\eta,\bar\eta)\end{pmatrix}\partial_t\varrho(\eta,\bar\eta).$$

We then have

$$\mathcal{M}(\eta,\bar\eta)^{-1}\partial_t\Big(\mathcal{M}(\eta,\bar\eta)\Big) = \frac{1}{1-\varrho^2(\eta,\bar\eta)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\partial_t\varrho(\eta,\bar\eta). \tag{2.29}$$

Now, keeping in mind the definition of ϱ in (2.25), we have that

$$\partial_t\varrho(\eta,\bar\eta) = \frac{2c}{(2-cQ(\eta,\bar\eta))^2}\partial_t Q(\eta,\bar\eta) = \frac{2c}{(2-cQ(\eta,\bar\eta))^2}\left\langle\Lambda(\eta+\bar\eta),\partial_t\eta+\partial_t\bar\eta\right\rangle$$

and

$$1-\rho^2(\eta,\bar\eta) = \frac{4-4cQ(\eta,\bar\eta)}{(2-cQ(\eta,\bar\eta))^2}.$$

Putting these together with (2.29), we get

$$\mathcal{M}(\eta,\bar\eta)^{-1}\partial_t\left\{\mathcal{M}(\eta,\bar\eta)\right\} = \frac{c}{2\,(1-cQ(\eta,\bar\eta))}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left\langle\Lambda(\eta+\bar\eta),\partial_t\eta+\partial_t\bar\eta\right\rangle.$$

Let us consider the operator

$$K_1(\alpha_1,\alpha_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} := \frac{c}{2\,(1-cQ(\alpha_1,\alpha_2))}\left\langle\Lambda(\alpha_1+\alpha_2),\beta_1+\beta_2\right\rangle\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}.$$

We have that system (2.28) can be rewritten as

$$\big(Id+K_1(\eta,\bar\eta)\big)\begin{pmatrix}\partial_t\eta\\ \partial_t\bar\eta\end{pmatrix} = i(1-cQ(\eta,\bar\eta))\begin{pmatrix}-\Lambda\eta\\ \Lambda\bar\eta\end{pmatrix}. \tag{2.30}$$

Using the Neumann series, we have that a formal inverse for $I+K$ is given by

$$(\mathrm{Id}+K_1(\eta,\bar\eta))^{-1} = \mathrm{Id}+\sum_{n=1}^{\infty}(-1)^n K_1(\eta,\bar\eta)^n$$

provided the right-hand side series converges. By defining

$$F(\eta,\bar\eta) := \frac{c}{2\,(1-cQ(\eta,\bar\eta))}$$

we have that

$$K_1^n(\eta,\bar\eta)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \begin{pmatrix}\bar\eta\\ \eta\end{pmatrix}\left\langle\Lambda(\eta+\bar\eta),\alpha_1+\alpha_2\right\rangle F^n(\eta,\bar\eta)\left\langle\Lambda(\eta+\bar\eta),\eta+\bar\eta\right\rangle^{n-1},$$

so the Neumann series converges if $|F(\eta,\bar\eta)\left\langle\Lambda(\eta+\bar\eta),\eta+\bar\eta\right\rangle| < 1$. Since we have

$$F(\eta,\bar\eta)\left\langle\Lambda(\eta+\bar\eta),\eta+\bar\eta\right\rangle = \frac{cQ(\eta,\bar\eta)}{1-cQ(\eta,\bar\eta)},$$

for the series to converge, it is sufficient that $|Q(\eta,\bar\eta)| < 1/2|c|$ and hence

$$(\eta,\bar\eta)\in B_{\frac{\delta_0}{2}}\left(H_0^{\frac{1}{2}}(\mathbb{T}^n,c.c)\right).$$

Therefore we have

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n K_1(\eta,\bar\eta)^n \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} &= \begin{pmatrix}\bar\eta\\ \eta\end{pmatrix}\left(-F(\eta,\bar\eta)\right)\langle\Lambda(\eta+\bar\eta),\alpha+\beta\rangle\sum_{n=0}^{\infty}\left(-F(\eta,\bar\eta)\langle\Lambda(\eta+\bar\eta),\eta+\bar\eta\rangle\right)^n\\
&= \begin{pmatrix}\bar\eta\\ \eta\end{pmatrix}\frac{-F(\eta,\bar\eta)}{1+F(\eta,\bar\eta)\langle\Lambda(\eta+\bar\eta),\eta+\bar\eta\rangle}\langle\Lambda(\eta+\bar\eta),\alpha_1+\alpha_2\rangle\\
&= \begin{pmatrix}\bar\eta\\ \eta\end{pmatrix}\left[-\frac{c}{2}\langle\Lambda(\eta+\bar\eta),\alpha_1+\alpha_2\rangle\right].
\end{aligned}$$

Thanks to this calculation we can rewrite system (2.30) in the following way:

$$\begin{aligned}
\partial_t\begin{pmatrix}\eta\\ \bar\eta\end{pmatrix} &= i\left(1-cQ(\eta,\bar\eta)\right)\left(I+K(\eta,\bar\eta)\right)^{-1}\begin{pmatrix}-\Lambda\eta\\ \Lambda\bar\eta\end{pmatrix}\\
&= i\left(1-cQ(\eta,\bar\eta)\right)\begin{pmatrix}-\Lambda\eta\\ \Lambda\bar\eta\end{pmatrix}-\frac{ic\left(1-cQ(\eta,\bar\eta)\right)}{2}\left(\langle\Lambda\bar\eta,\Lambda\bar\eta\rangle-\langle\Lambda\eta,\Lambda\eta\rangle\right)\begin{pmatrix}\bar\eta\\ \eta\end{pmatrix},
\end{aligned}$$

that is system (2.12).

Let $(\eta,\bar\eta)$ be a solution of (2.12), then we have an a-priori energy estimate of the form:

$$\begin{aligned}
\partial_t\|\eta\|_s^2 &= 2\mathrm{Re}\left(i\left(1-cQ(\eta,\bar\eta)\right)\langle\Lambda^{2s}\eta,\,\Lambda\bar\eta-\frac{c}{2}\left(\langle\Lambda\bar\eta,\Lambda\bar\eta\rangle-\langle\Lambda\eta,\Lambda\eta\rangle\right)\eta\rangle\right)\\
&= 2\mathrm{Re}\left(\frac{ic}{2}\left(1-cQ(\eta,\bar\eta)\right)\left(\langle\Lambda\bar\eta,\Lambda\bar\eta\rangle-\langle\Lambda\eta,\Lambda\eta\rangle\right)\langle\Lambda^{2s}\eta,\,\eta\rangle\right) \qquad (2.31)\\
&\overset{(2.10)}{\leq} |c|\|\eta\|_1^2\|\eta\|_s^2+c^2\|\eta\|_1^4\|\eta\|_s^2.
\end{aligned}$$

This gives the local existence for solutions of (2.1) with initial data $(\eta_0,\bar\eta_0)\in H_0^1(\mathbb{T}^n,c.c.)$ in a time interval $T\sim\|\eta_0\|_1^{-2}$. Thanks to a bootstrap argument, the same result can be generalized also for the case of H_0^s with $s>1$. This lower bound for the time of existence of solutions is consistent with the one obtained by Dickey in [48].

Let us note moreover that the diagonal part of (2.12), namely $\frac{ic\left(1-cQ(\eta,\bar\eta)\right)}{2}\begin{pmatrix}-\Lambda\eta\\ \Lambda\bar\eta\end{pmatrix}$, does not contribute to the energy estimate (2.31).

Indeed, the first term that gives a non-trivial contribution is the off-diagonal cubic one, i.e.

$$\mathcal{B}_3(\eta,\bar\eta):=-\frac{ic}{2}\left(\langle\Lambda\bar\eta,\Lambda\bar\eta\rangle-\langle\Lambda\eta,\Lambda\eta\rangle\right)\begin{pmatrix}\bar\eta\\ \eta\end{pmatrix}, \qquad (2.32)$$

while the remaining terms give a contribution of higher order. The next step then is the cancellation of $\mathcal{B}_3$.

2.4 Normal Form: First Step

The aim of this section is to remove the off-diagonal cubic term $\mathcal{B}_3$. This will be done by constructing a (normal form) transformation, $\Phi^{(3)}$ with the aim of removing the resonant cubic terms. The main result we will prove is the following

Lemma 2.4.1 (Normal Form First Step)
There exists $\delta_2 > 0$ *(defined in* (2.52) *) and a map*

$$\Phi^{(3)} : B_{2\delta_2}(H_0^{m_0}(\mathbb{T}^n, c.c.)) \to B_{\delta_2}(H_0^{m_0}(\mathbb{T}^n, c.c.)) \tag{2.33}$$

that conjugates system (2.12) *to one of the form*

$$\partial_t(w, \bar{w}) = X^{(1)}(w, \bar{w}) = (I + \mathcal{P}(w, \bar{w}))\,\mathcal{D}_1 + X_{\mathbf{Res},3}(w, \bar{w}) + X^{(1)}_{\geq 5}(w, \bar{w}),$$

where $\mathcal{P}$ *is a function of time only, defined in* (2.39), $\mathcal{D}_1$ *is diagonal,* $X_{\mathbf{Res},3}(w,\bar{w})$ *is made of homogeneous cubic resonant terms and is defined in* (2.43), (2.44) *and* $X^{(1)}_{\geq 5}(w, \bar{w})$ *is bounded and contains the remaining terms of homogeneity greater than 5.*
Finally, the terms $(I + \mathcal{P}(w, \bar{w}))\,\mathcal{D}_1$ *and* $X_{\mathbf{Res},3}(w, \bar{w})$ *give no contribution to the energy estimates, namely the Sobolev norms of the solutions of the system* $\partial_t(w, \bar{w}) = (I + \mathcal{P}(w, \bar{w}))\,\mathcal{D}_1 + X_{\mathbf{Res},3}(w, \bar{w})$ *are constants.*

We start by grouping together the terms of system (2.12) with the same homogeneity :

$$\partial_t(\eta, \bar{\eta}) = X(\eta, \bar{\eta}) := \mathcal{D}_1(\eta, \bar{\eta}) + \mathcal{D}_3(\eta, \bar{\eta}) + \mathcal{B}_3(\eta, \bar{\eta}) + \mathcal{B}_5(\eta, \bar{\eta}), \tag{2.34}$$

where

$$\mathcal{D}_1(\eta, \bar{\eta}) = \begin{pmatrix} -i\Lambda\eta \\ i\Lambda\bar{\eta} \end{pmatrix}$$

is the linear diagonal part of (2.12),

$$\mathcal{D}_3(\eta, \bar{\eta}) = -cQ(\eta, \bar{\eta})\mathcal{D}_1(\eta, \bar{\eta})$$

is its diagonal cubic part, $\mathcal{B}_3$ is the cubic off-diagonal part defined in (2.32) and,

$$\mathcal{B}_5(\eta, \bar{\eta}) = -cQ(\eta, \bar{\eta})\mathcal{B}_3(\eta, \bar{\eta}) \tag{2.35}$$

is the remaining quintic part.
Let us formally define the map

$$\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = \Phi^{(3)}(w, \bar{w}) := (\mathrm{Id} + M(w, \bar{w})) \begin{pmatrix} w \\ \bar{w} \end{pmatrix}, \tag{2.36}$$

where M is a bi-linear map with values in the space of 2×2 matrices, more precisely

$$M(w_1, w_2) = \begin{pmatrix} M_{11}(w_1, w_2) & M_{12}(w_1, w_2) \\ M_{21}(w_1, w_2) & M_{22}(w_1, w_2) \end{pmatrix}$$

with $M_{ij}(w_1, w_2) = A_{ij}[w_1, w_1] + B_{ij}[w_1, w_2] + C_{ij}[w_2, w_2]$. We moreover denote

$$A[w, w] := \begin{pmatrix} A_{11}\left[w_1^{(1)}, w_1^{(2)}\right] & A_{12}\left[w_1^{(1)}, w_1^{(2)}\right] \\ A_{21}\left[w_1^{(1)}, w_1^{(2)}\right] & A_{22}\left[w_1^{(1)}, w_1^{(2)}\right] \end{pmatrix},$$

where each component A_{ℓ_1,ℓ_2} is an operator that acts on the Fourier series of $w = \sum_{k\in\mathbb{Z}^n} w_j e^{ik\cdot x}$ in the following way:

$$A_{\ell_1,\ell_2}[w, w]h := \sum_{j,k\in\mathbb{Z}^n\setminus\{0\}} a_{\ell_1,\ell_2}(j, k) w_j w_{-j} h_k e^{ik\cdot x}. \tag{2.37}$$

Similarly for B and C.
We assume A and C to be symmetric, i.e.

$$A[w_1^{(1)}, w_1^{(2)}] = A[w_1^{(2)}, w_1^{(2)}], \quad C[w_2^{(1)}, w_2^{(2)}] = C[w_2^{(2)}, w_2^{(1)}] \quad \text{for all } w_1^{(1)},\ w_1^{(2)},\ w_2^{(1)},\ w_2^{(2)}.$$

Moreover, in order for $\Phi^{(3)}$ to be a well-posed map from complex-conjugated functions, we impose

$$\overline{M_{11}(w_1, w_2)} = M_{22}(w_1, w_2), \quad \overline{M_{12}(w_1, w_2)} = M_{21}(w_1, w_2).$$

Let us evaluate how $\Phi^{(3)}$ transforms System (2.34): we have

$$\begin{aligned} \partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} &= \partial_t \Phi^{(3)}(w, \bar{w}) = \partial_t \left[(\mathrm{Id} + M(w, \bar{w})) \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \right] \\ &= (\mathrm{Id} + M(w, \bar{w}))\, \partial_t \begin{pmatrix} w \\ \bar{w} \end{pmatrix} + \partial_t \left\{ M(w, \bar{w}) \right\} \begin{pmatrix} w \\ \bar{w} \end{pmatrix}. \end{aligned} \tag{2.38}$$

Since A and C are symmetric, one has

$$\partial_t \left\{ M(w, \bar{w}) \right\} \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = [2A[w, w_t] + B[w_t, \bar{w}] + B[w, \bar{w}_t] + 2C[\bar{w}, w_t]] \begin{pmatrix} w \\ \bar{w} \end{pmatrix},$$

then, by defining

$$K_2(w, \bar{w}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = M(w, \bar{w}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + M_t(w, \bar{w}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with

$$M_t(w, \bar{w}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} := [2A[w, \alpha] + B[\alpha, \bar{w}] + B[w, \beta] + 2C[\bar{w}, \beta]] \begin{pmatrix} w \\ \bar{w} \end{pmatrix},$$

we have that (2.38) can be written as

$$\partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = (\mathrm{Id} + K_2(w, \bar{w})) \begin{pmatrix} \partial_t w \\ \partial_t \bar{w} \end{pmatrix}.$$

On the other hand, by (2.36)

$$\begin{aligned} \partial_t \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} &= X\left(\Phi^{(3)}(w, \bar{w})\right) \\ &= \mathcal{D}_1\left(\Phi^{(3)}(w, z)\right) + \mathcal{D}_3\left(\Phi^{(3)}(w, z)\right) + \mathcal{B}_3\left(\Phi^{(3)}(w, z)\right) + \mathcal{B}_5\left(\Phi^{(3)}(w, z)\right). \end{aligned}$$

We arrive to

$$\partial_t \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = X^{(1)}(w, \bar{w}) := (\mathrm{Id} + K_2(w, \bar{w}))^{-1} \left(X\left(\Phi^{(3)}(w, \bar{w})\right)\right).$$

Therefore, in order to derive the equation for the variables $(w, \bar{w})$ one must be able to invert the operator $\mathrm{Id} + K_2$:
using the Neumann series we (formally) have

$$(\mathrm{Id} + K_2(w, \bar{w}))^{-1} = I - K_2(w, \bar{w}) + \tilde{K}_2(w, \bar{w}),$$

where

$$\tilde{K}_2(w,\bar{w}) = \sum_{n=2}^{\infty}(-1)^n K_2(w,\bar{w})^n.$$

We then have

$$X^{(1)}(w,\bar{w}) = \mathcal{D}_1(w,\bar{w}) + \tilde{D}_{\geq 3}(w,\bar{w}) + X_3^{(1)}(w,\bar{w}) + X_{\geq 5}^{(1)}(w,\bar{w}),$$

where

(i) The diagonal part of degree 3 is

$$\tilde{\mathcal{D}}_{\geq 3}(w,\bar{w}) := \mathcal{P}(w,\bar{w})\mathcal{D}_1(w,\bar{w})$$

with

$$\mathcal{P}(w,\bar{w}) := -cQ\left(\Phi^{(3)}(w,\bar{w})\right). \tag{2.39}$$

(ii) The non-diagonal cubic terms of $X^{(1)}$ are

$$X_3^{(1)}(w,\bar{w}) := \mathcal{D}_1\left(M(w,\bar{w})\begin{pmatrix} w \\ \bar{w}\end{pmatrix}\right) - K_2(w,\bar{w})\mathcal{D}_1(w,\bar{w}) + \mathcal{B}_3(w,\bar{w}), \tag{2.40}$$

(iii) Finally, the remaining terms with degree greater than 5 are

$$\begin{aligned} X_{\geq 5}^{(1)}(w,\bar{w}) \quad = \quad & -K_2(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\begin{pmatrix} w \\ \bar{w}\end{pmatrix}\right) + \tilde{K}_2(w,\bar{w})\mathcal{D}_1\left(\Phi^{(3)}(w,\bar{w})\right) \\ & +\mathcal{P}(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\begin{pmatrix} w \\ \bar{w}\end{pmatrix}\right) + \left(K_2(w,\bar{w}) + \tilde{K}_2(w,\bar{w})\right)\mathcal{D}_3\left(\Phi^{(3)}(w,\bar{w})\right) \\ & + \left(I + K_2(w,\bar{w})\right)^{-1}\mathcal{B}_5\left(\Phi^{(3)}(w,\bar{w})\right) + \left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar{w})\right) - \mathcal{B}_3(w,\bar{w})\right] \\ & + \left(-K_2(w,\bar{w}) + \tilde{K}_2(w,\bar{w})\right)\mathcal{B}_3\left(\Phi^{(3)}(w,\bar{w})\right). \end{aligned} \tag{2.41}$$

Let us analyze the first component of the cubic vector field $X_3^{(1)}(w,\bar{w})$:

$$\begin{aligned} \left(X_3^{(1)}(w,\bar{w})\right)_1 \quad = \quad & -i\Lambda M_{11}(w,\bar{w})w - i\Lambda M_{12}(w,\bar{w})z + iM_{11}(w,\bar{w})\Lambda w - iM_{12}(w,\bar{w})\Lambda\bar{w} \\ & -i\left\{-2A_{11}(w,\Lambda w) - B_{11}(\Lambda w,\bar{w}) + B_{11}(w,\Lambda\bar{w}) + 2C_{11}(\bar{w},\Lambda\bar{w})\right\}w \\ & -i\left\{-2A_{11}(w,\Lambda w) - B_{11}(\Lambda w,\bar{w}) + B_{11}(w,\Lambda\bar{w}) + 2C_{11}(\bar{w},\Lambda\bar{w})\right\}\bar{w} \\ & -\frac{ic}{2}\left(\langle\Lambda\bar{w},\Lambda\bar{w}\rangle - \langle\Lambda w,\Lambda w\rangle\right)\bar{w}. \end{aligned}$$

Since our aim is to choose the coefficients $M_{i,j}$ in order to eliminate as many terms as possible from $X_3^{(1)}$, we set

$$M_{11}, B_{12} = 0.$$

Under these assumptions it remains

$$\begin{aligned} \left(X_3^{(1)}(w,\bar{w})\right)_1 \quad = \quad & -i\Lambda A_{12}[w,w]\bar{w} - i\Lambda C_{12}[\bar{w},\bar{w}]\bar{w} - iA_{12}[w,w]\Lambda\bar{w} - iC_{12}[\bar{w},\bar{w}]\Lambda\bar{w} \\ & +2iA_{12}[w,\Lambda w]\bar{w} - 2iC_{12}[\bar{w},\Lambda\bar{w}]\bar{w} - \frac{ic}{2}\left(\langle\Lambda\bar{w},\Lambda\bar{w}\rangle - \langle\Lambda w,\Lambda w\rangle\right)\bar{w}. \end{aligned}$$

By using (2.37), we write all the quantities involved in $\left(X_3^{(1)}(w,z)\right)_1$ in Fourier series and get

$$\begin{aligned}\left(X_3^{(1)}(w,\bar{w})\right)_1 &= \sum_{j,k\neq 0} w_j w_{-j}\bar{w}_{-k}e^{ik\cdot x}\left[2i(|j|-|k|)a_{12}(j,k)+\frac{ic}{2}|j|^2\right]\\ &\quad+\sum_{j,k\neq 0}\bar{w}_j\bar{w}_{-j}\bar{w}_{-k}e^{ik\cdot x}\left[-2i(|j|+|k|)c_{12}(j,k)-\frac{ic}{2}|j|^2\right].\end{aligned}$$

In order to cancel out as many terms as possible we set

$$a_{12}(j,k)=\begin{cases}\dfrac{c|j|^2}{4(|k|-|j|)} & \text{if } j\neq k\\ 0 & \text{if } j=k\end{cases},\qquad c_{12}(j,k)=-\frac{c|j|^2}{4(|j|+|k|)}. \tag{2.42}$$

By defining $X_{\mathbf{Res},3}:=X_3^{(1)}$, from (2.42) we have that

$$\left(X_{\mathbf{Res},3}(w,\bar{w})\right)_1=\frac{ic}{2}\sum_{j\,k\neq 0\,|j|=|k|}|j|^2 w_j w_{-j}\bar{w}_{-k}e^{ik\cdot x}. \tag{2.43}$$

Similar calculations for $\left(X^{(1)}(w,z)\right)_2$ yield to

$$M_{22}=0,\quad B_{21}=0,\quad A_{21}=C_{12},\quad C_{21}=A_{12}$$

and

$$\left(X_{\mathbf{Res},3}(w,\bar{w})\right)_2=-\frac{ic}{2}\sum_{j\,k\neq 0\,|j|=|k|}|j|^2 \bar{w}_j \bar{w}_{-j} w_k e^{ik\cdot x}. \tag{2.44}$$

Remark 2.4.1 *The denominator* $||k|-|j||$ *of* (2.42) *is the one responsible for the different regularity thresholds* m_0 *in* (1.13). *In fact, in dimension* $d=1$, *one has* $||j|-|k||\geq 1$ *for all* $|j|\neq|k|$ *and therefore*

$$\begin{aligned}\|A_{1,2}[w,w]\bar{w}\|_s^2 &= \sum_{k\neq 0}\left|\sum_{j\neq 0,|j|\neq|k|} w_j w_{-j}\frac{c|j|^2}{4(|k|-|j|)}\bar{w}_k\right|^2|k|^{2s}\\ &\leq \frac{|c|}{16}\sum_{k\neq 0}\left(\sum_{j\neq 0,|j|\neq|k|}|j|^2|w_j|^2\right)^2|w_k|^2|k|^{2s}\leq\frac{1}{16}\|w\|_1^4\|w\|_s^2.\end{aligned} \tag{2.45}$$

In dimension $d\geq 2$, *on the other hand, the best possible bound is*

$$\frac{1}{||k|-|j||}\leq 3|j|\quad \forall j,k\in\mathbb{Z}^n\backslash\{0\},\ |j|\neq|k|. \tag{2.46}$$

Indeed, if $||j|-|k||>1$, (2.46) *follows trivially, while if* $||j-|k||\leq 1$ *one has* $|j|\leq|k|+1$ *or* $|k|\leq|j|+1$ *and*

$$\frac{1}{||j|-|k||}\leq\frac{|j|+|k|}{||j|^2-|k|^2|}\leq\max\{3|j|,3|k|\}.$$

Thanks to (2.46)*, a computation similar to the one performed in* (2.45) *yields to*

$$\|A_{1,2}[w,w]\bar{w}\|_s^2 \leq \frac{9|c|}{16}\|w\|_{\frac{3}{2}}^4\|w\|_s^2 \tag{2.47}$$

for the case $d \geq 2$*.*

We can therefore explicitly write the operator matrices

$$M(w,z)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}(A_{12}[w,w] + C_{12}[\bar{w},\bar{w}])\,\beta\\ (A_{21}[w,w] + C_{21}[\bar{w},\bar{w}])\,\alpha\end{pmatrix}$$

and

$$K_2(w,\bar{w})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}A_{12}[w,w]\beta + C_{12}[\bar{w},\bar{w}]\beta + 2A_{12}[w,\alpha]\bar{w} + 2C_{12}[\bar{w},\beta]\bar{w}\\ A_{21}[w,w]\alpha + C_{21}[\bar{w},\bar{w}]\alpha + 2A_{21}[w,\alpha]w + 2C_{21}[\bar{w},\beta]w\end{pmatrix}. \tag{2.48}$$

It holds the following quantitative result:

Lemma 2.4.2 *Let* m_0 *be as in* (1.13)*, for every triple of complex functions* $u,\ v,\ h$ *and all real* $s \geq 0$ *it holds*

$$\|A_{12}[u,v]h\|_s \leq \frac{3|c|}{4}\|u\|_{m_0}\|v\|_{m_0}\|h\|_s, \qquad \|C_{12}[u,v]h\|_s \leq \frac{|c|}{8}\|u\|_1\|v\|_1\|h\|_s.$$

Proof: This follows directly by performing the same computations of Remark 2.4.1. □

Lemma 2.4.3 *For all* $s \geq 0$*, all* $(w,\bar{w}) \in H_0^{m_0}(\mathbb{T}^n,\ c.c.)$*,* $(\alpha,\bar{\alpha}) \in H_0^s(\mathbb{T}^n,\ c.c.)$ *one has*

$$\left\|M(w,\bar{w})\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s \leq \frac{7|c|}{8}\|w\|_{m_0}^2\|\alpha\|_s; \tag{2.49}$$

$$\left\|K(w,\bar{w})\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s \leq \frac{7|c|}{8}\|w\|_{m_0}^2\|\alpha\|_s + \frac{7|c|}{4}\|w\|_{m_0}\|w\|_s\|\alpha\|_{m_0}. \tag{2.50}$$

Finally, there exists $\delta_1 > 0$*, such that, if* $\|w\|_{m_0} < \delta_1$ *the operator*

$$I + K(w,\bar{w}) : H_0^{m_0}(\mathbb{T}^n, c.c.) \longrightarrow H_0^{m_0}(\mathbb{T}^n, c.c.)$$

is invertible with inverse satisfying

$$\left\|(I + K(w,\bar{w}))^{-1}\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s \leq C\left(\|\alpha\|_s + \|w\|_{m_0}\|w\|_s\|\alpha\|_{m_0}\right) \quad \text{for every } s \geq m_0, \tag{2.51}$$

where C *depends on* c*.*

Proof: From (2.50) it is clear that, for a fixed $(w,\bar{w}) \in H_0^{m_0}(\mathbb{T}^n, c.c.)$, $I+K(w,\bar{w})$ is a bounded operator from $H_0^{m_0}(\mathbb{T}^n, c.c.)$ into itself. Having that

$$\left\|K(w,\bar{w})\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_{m_0} \leq \frac{21|c|}{8}\|w\|_{m_0}^2\|\alpha\|_{m_0},$$

Now, using the Neumann series we can compute explicitly that

$$\left\|(I+K(w,\bar{w}))^{-1}\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s=\left\|\left(I+\sum_{n=1}^{\infty}(-1)^nK^n(w,\bar{w})\right)\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s\le\|\alpha\|_s+\sum_{n=1}^{\infty}\left\|K^n(w,\bar{w})\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s.$$

Finally, by induction one can verify that

$$\left\|K^n(w,z)\begin{pmatrix}\alpha\\ \bar{\alpha}\end{pmatrix}\right\|_s\le\left(\frac{7|c|}{8}\|w\|_{m_0}^2\right)^n\|\alpha\|_s+\left(\frac{21|c|}{8}\|w\|_{m_0}^2\right)^{n-1}\frac{7|c|}{4}\|w\|_{m_0}\|w\|_s\|\alpha\|_{m_0}$$

$$+\left[\sum_{j=1}^{n-1}\left(\frac{7|c|}{8}\|w\|_{m_0}^2\right)^j\left(\frac{21|c|}{8}\|w\|_{m_0}^2\right)^{n-1-j}\right]\frac{7|c|}{4}\|w\|_{m_0}\|w\|_s\|\alpha\|_{m_0},$$

so, imposing for instance

$$\|w\|_{m_0}<\delta_1:=\sqrt{4/(21|c|)}\tag{2.52}$$

we can derive (2.51) □

By using the same contraction argument as in [6] Lemma 4.3, we can also prove that the map $\Phi^{(3)}$ is invertible near the origin:

Lemma 2.4.4 *There exists $\delta_2>0$, such that for all $(\eta,\bar{\eta})\in H_0^{m_0}(\mathbb{T}^n,c.c.)$ in the ball*

$$\|\eta\|_{m_0}\le\delta_2,$$

there exists a unique $(w,\bar{w})\in H_0^{m_0}(\mathbb{T}^n,c.c.)$ such that $\Phi^{(3)}(w,\bar{w})=(\eta,\bar{\eta})$, with $\|w\|_{m_0}\le 2\|\eta\|_{m_0}$.
If, in addition, $\eta\in H_0^s(\mathbb{T}^n,c.c.)$ for some $s>m_0$, then w also belongs to H_0^s, and $\|w\|_s\le 2\|\eta\|_s$.
.

Proof: Following the idea of the original proof, for a chosen element $(\eta,\bar{\eta})\in H_0^{m_0}(\mathbb{T}^n,c.c.)$ we want to find a fixed point for the map

$$\Psi(w,\bar{w}):=\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}-M(w,\bar{w})\begin{pmatrix}w\\ \bar{w}\end{pmatrix}.$$

Let us consider the ball $B_R:=B_R\left(H_0^{m_0}(\mathbb{T}^n,c.c.)\right)$, a direct application of (2.49) shows that Ψ maps B_R into itself if

$$\|\eta\|_{m_0}+\frac{7|c|}{8}R^3\le R.\tag{2.53}$$

It remains to check if Ψ is a contraction:

$$\|\Psi(w_1,\bar{w}_1)-\Psi(w_2,\bar{w}_2)\|_{m_0}=\left\|M(w_1,\bar{w}_1)\begin{pmatrix}w_1\\ \bar{w}_1\end{pmatrix}-M(w_2,\bar{w}_2)\begin{pmatrix}w_2\\ \bar{w}_2\end{pmatrix}\right\|_{m_0}$$

$$\le\int_0^1\left\|K\big(w_2+\theta(w_1-w_2),\bar{w}_2+\theta(\bar{w}_1-\bar{w}_2)\big)\begin{pmatrix}w_1-w_2\\ \bar{w}_1-\bar{w}_2\end{pmatrix}\right\|_{m_0}d\theta\le\frac{21|c|}{8}R^2\|w_1-w_2\|_{m_0}.$$

Hence Ψ is a contraction if

$$\frac{21|c|}{8}R^2<1.\tag{2.54}$$

By choosing $R = 2\|\eta\|_{m_0}$ and using (2.53) together with (2.54) we have that Ψ is a contraction in the ball B_R if $\|\eta\|_{m_0} \le \delta_2$

$$\delta_2 = (21|c|)^{-\frac{1}{2}}$$

□

Lemma 2.4.5 *For all complex functions $u,\ v,\ h,\ y$ one has*

$$\begin{aligned}
\langle A_{12}[u,v]y,h\rangle &= \langle y, A_{12}[u,v]h\rangle, & \langle C_{12}[u,v]y,h\rangle &= \langle y, C_{12}[u,v]h\rangle,\\
\overline{A_{12}[u,v]y} &= A_{12}[\bar u,\bar v]\bar y, & \overline{C_{12}[u,v]y} &= C_{12}[\bar u,\bar v]\bar y,\\
A_{12}[u,v]\Lambda^s y &= \Lambda^s A_{12}[u,v]y, & C_{12}[u,v]\Lambda^s y &= \Lambda^s C_{12}[u,v]y
\end{aligned}$$

From these relations it follows that

$$\begin{aligned}
\langle M_{12}(u,v)y,h\rangle &= \langle y, M_{12}(u,v)h\rangle, & \langle M_{21}(u,v)y,h\rangle &= \langle y, M_{21}(u,v)h\rangle,\\
\overline{M_{12}(u,v)h} &= M_{12}(\bar u,\bar v)\bar h, & \overline{M_{21}(u,v)h} &= M_{21}(\bar u,\bar v)\bar h,\\
[M_{12}(u,v),\Lambda^s] &= 0, & [M_{21}(u,v),\Lambda^s] &= 0.
\end{aligned}$$

$$M_{12}(u,v)h = M_{21}(v,u)h$$

and

$$M(u,v)\mathcal{D}_1 + \mathcal{D}_1 M(u,v) = 0. \tag{2.55}$$

Proof: It follows directly from the definition of A and C in (2.42) □

Lemma 2.4.5 implies the reality structure of the vector field $X^{(1)}$:

Lemma 2.4.6 *The vector field $X^{(1)}$ preserves the real structure* (1.20)*, i.e.* $\left(X^{(1)}\right)_1 = \overline{\left(X^{(1)}\right)_2}$.

Let us now analyze $X^{(1)}_{\ge 5}$, defined in (2.41). We have:

$$\begin{aligned}
X^{(1)}_{\ge 5}(w,\bar w) \;=\;& -K_2(w,\bar w)\mathcal{D}_1\left(M(w,\bar w)\begin{pmatrix} w\\ \bar w\end{pmatrix}\right) + \tilde K_2(w,\bar w)\mathcal{D}_1\left(\Phi^{(3)}(w,\bar w)\right)\\
&+\mathcal{P}(w,\bar w)\mathcal{D}_1\left(M(w,\bar w)\begin{pmatrix} w\\ \bar w\end{pmatrix}\right) + \left(K_2(w,\bar w)+\tilde K_2(w,\bar w)\right)\mathcal{D}_3\left(\Phi^{(3)}(w,\bar w)\right)\\
&+\tilde X^{(1)}_{\ge 5}(w,\bar w),
\end{aligned}$$

where

$$\begin{aligned}
\tilde X^{(1)}_{\ge 5}(w,\bar w) \;:=\;& \left(I + K_2(w,\bar w)\right)^{-1}\mathcal{B}_5\left(\Phi^{(3)}(w,\bar w)\right)\\
&+\left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w)\right) - \mathcal{B}_3(w,\bar w)\right] + \left(-K_2(w,\bar w)+\tilde K_2(w,\bar w)\right)\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w)\right).
\end{aligned}$$

Now, we have from (2.55) and (2.40) that

$$\left(M(w,\bar w) + K_2(w,\bar w)\right)\mathcal{D}_1\begin{pmatrix} w\\ \bar w\end{pmatrix} = \mathcal{B}_3(w,\bar w) - X_{\mathbf{Res},3}(w,\bar w), \tag{2.56}$$

hence it follows that

$$\begin{aligned}
& -K_2(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\right)+\tilde{K}_2(w,\bar{w})\mathcal{D}_1\left(I+M(w,\bar{w})\right)\\
= & -K_2(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\right)+\sum_{n=2}^{\infty}(-1)^n K_2^n(w,\bar{w})\mathcal{D}_1\left(I+M(w,\bar{w})\right)\\
= & \; K_2(w,\bar{w})M(w,\bar{w})\mathcal{D}_1+\sum_{n=2}^{\infty}(-1)^n K_2^n(w,\bar{w})\mathcal{D}_1-\sum_{n=2}^{\infty}(-1)^n K_"^n(w,\bar{w})M(w,\bar{w})\mathcal{D}_1\\
= & -\sum_{n=1}^{\infty}(-1)^n K_2^n(w,\bar{w})K(w,\bar{w})\mathcal{D}_1-\sum_{n=1}^{\infty}(-1)^n K_2^n(w,\bar{w})M(w,\bar{w})\mathcal{D}_1\\
= & -\sum_{n=1}^{\infty}(-1)^n K_2^n(w,\bar{w})\left(K(w,\bar{w})+M(w,\bar{w})\right)\mathcal{D}_1\\
= & \; K_2(w,\bar{w})\left(I+K_2(w,\bar{w})\right)^{-1}\left(\mathcal{B}_3(w,\bar{w})-X_{\mathbf{Res},3}(w,\bar{w})\right). \qquad (2.57)
\end{aligned}$$

On the other hand by (2.55) and (2.56) we have

$$\begin{aligned}
& \mathcal{P}(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\begin{pmatrix} w\\ \bar{w}\end{pmatrix}\right)+\left(K_2(w,\bar{w})+\tilde{K}_2(w,\bar{w})\right)\mathcal{D}_3\left(\Phi^{(3)}(w,\bar{w})\right)\\
= & \; \mathcal{P}(w,\bar{w})\sum_{n=1}^{\infty}(-1)^n K^n(w,\bar{w})\mathcal{D}_1+\mathcal{P}(w,\bar{w})\sum_{n=0}^{\infty}(-1)^n K^n(w,\bar{w})\mathcal{D}_1\left(M(w,\bar{w})\right)\\
= & -\sum_{n=0}^{\infty}(-1)^n K^n(w,\bar{w})K(w,\bar{w})\mathcal{D}_1-\mathcal{P}(w,\bar{w})\sum_{n=0}^{\infty}(-1)^n K^n(w,\bar{w})M(w,\bar{w})\mathcal{D}_1\\
= & -\mathcal{P}(w,\bar{w})\left(I+K(w,\bar{w})\right)^{-1}\left(\mathcal{B}_3(w,\bar{w})-X_{\mathbf{Res},3}(w,\bar{w})\right). \qquad (2.58)
\end{aligned}$$

Now, putting together (2.57) and (2.58) we get

$$X^{(1)}(w,\bar{w})=\left(I+\mathcal{P}(w,\bar{w})\right)\mathcal{D}_1+X_{\mathbf{Res},3}(w,\bar{w})+X^{(1)}_{\geq 5}(w,\bar{w}), \qquad (2.59)$$

with

$$X^{(1)}_{\geq 5}(w,\bar{w})=\left(K_2(w,\bar{w})-\mathcal{P}(w,\bar{w})\right)\left(I+K_2(w,\bar{w})\right)^{-1}\left(\mathcal{B}_3(w,\bar{w})-X_{\mathbf{Res},3}(w,\bar{w})\right)+\tilde{X}^{(1)}_{\geq 5}(w,\bar{w}). \qquad (2.60)$$

Lemma 2.4.7 *For all $s\geq 0$ and for any pair of complex conjugate functions $(w,\bar{w})$ we have:*

$$\left\|\mathcal{B}_3(w,\bar{w})\right\|_s\leq |c|\|w\|_1^2\|w\|_s, \quad \left\|X_{\mathbf{Res},3}(w,\bar{w})\right\|_s\leq \frac{|c|}{2}\|w\|_1^2\|w\|_s, \qquad (2.61)$$

moreover, if $\|w\|_{m_0}\leq\delta_0$ and for all complex functions h, we have

$$\begin{aligned}
\left\|\mathcal{P}(w,\bar{w})h\right\|_s &= \left|\mathcal{P}(w,\bar{w})\right|\|h\|_s, \quad 0\leq\left|\mathcal{P}(w,\bar{w})\right|\leq 3|c|\|w\|_{\frac{1}{2}}^2 \qquad (2.62)\\
\left\|\mathcal{B}_5(w,\bar{w})\right\|_s &\leq 2c^2\|w\|_{\frac{1}{2}}^2\|w\|_1^2\|w\|_s \qquad (2.63)
\end{aligned}$$

Proof: Estimate (2.61) follows from the definition of $\mathcal{B}_s$ and $X_{\mathbf{Res},3}$ in (2.32), (2.43), (2.44). Estimate (2.62) follows from (2.39), while estimate can be derived from (2.62) together with (2.35). □

Lemma 2.4.8 *For all $s \geq 0$, all $(w, \bar{w}) \in H_0^s(\mathbb{T}^n, c.c.) \cap H_0^{m_0}(\mathbb{T}^n, c.c.)$ with $\|w\|_{m_0} \leq \delta_0$ we have*

$$\left\| X_{\geq 5}^{(1)}(w, \bar{w}) \right\|_s \leq |c|^2 \|w\|_{m_0}^2 \|w\|_1^2 \|w\|_s. \tag{2.64}$$

Proof: It follows from the expression of $X_{\geq 5}^{(1)}(w, \bar{w})$ in (2.60), together with the estimates contained in Lemma 2.4.7 □

Let us now compute the contribution of the term $X_{\mathbf{Res},3}$ in the energy estimate:

$$\begin{aligned} & 2\mathrm{Re}\left(\langle \Lambda^s w,\, \Lambda^2 X_{\mathbf{Res},3}\rangle\right) \\ = \;& \langle \Lambda^s w\rangle (\Lambda^s X_{\mathbf{Res},3})_2 + \langle (\Lambda^s X_{\mathbf{Res},3})_1\,,\, \Lambda^s \bar{w}\rangle \\ = \;& \frac{ic}{2} \sum_{j\, k\neq 0\, |j|=|k|} |j|^2 w_j w_{-j} \bar{w}_k \bar{w}_{-k} - \frac{ic}{2} \sum_{j\, k\neq 0\, |j|=|k|} |j|^2 \bar{w}_j \bar{w}_{-j} w_k w_{-k} = 0 \end{aligned} \tag{2.65}$$

and then

$$\begin{aligned} \partial_t \|w\|_s^2 & \overset{(2.59)}{=} 2\mathrm{Re}\left(\langle \Lambda^s w,\, \Lambda^s \left((I + \mathcal{P}(w, \bar{w}))\, \mathcal{D}_1 + X_{\mathbf{Res},3}(w, \bar{w}) + X_{\geq 5}^{(1)}(w, \bar{w}) \right) \rangle \right) \\ & \overset{(2.65)}{=} 2\mathrm{Re}\left(\langle \Lambda^s w,\, \Lambda^s X_{\geq 5}^{(1)}(w, \bar{w})\rangle\right) \\ & \overset{(2.64)}{\leq} 2C|c|^2 \|w\|_{m_0}^2 \|w\|_1^2 \|w\|_s^2 \end{aligned}$$

2.5 Normal form: second step

As in the previous section, the goal of this section is to construct a normal form transformation in order to eliminate the resonant degree-5 terms. The main result is the following:

Lemma 2.5.1 (Normal Form Second Step)
There exists $\delta_4 > 0$ (defined in (2.85)) and a map

$$\Phi^{(4)} : B_{2\delta_4}(H_0^{m_0}(\mathbb{T}^n, c.c.)) \to B_{\delta_4}(H_0^{m_0}(\mathbb{T}^n, c.c.))$$

that conjugates system (2.33) into a system of the form

$$\partial_t(q, \bar{q}) = \left(1 + \mathcal{P}\left(\Phi^{(4)}(q, \bar{q})\right) \right) (\mathcal{D}_1(q, \bar{q}) + X_{\mathbf{Res},3}(q, \bar{q})) + X_{\mathbf{Res},5}(q, \bar{q}) + X_{\geq 7}^{(2)}(q, \bar{q}),$$

where $X_{\mathbf{Res},5}(w, \bar{w})$ consists of homogeneity-5 resonant terms and is defined in (2.79), (2.80) and $X_{\geq 7}^{(2)}(w, \bar{w})$ is bounded and contains the remaining terms of homogeneity greater than 7.
Finally, the terms $\left(1 + \mathcal{P}\left(\Phi^{(4)}(q, \bar{q})\right) \right) (\mathcal{D}_1(q, \bar{q}) + X_{\mathbf{Res},3}(q, \bar{q}))$ and $X_{\mathbf{Res},5}(q, \bar{q})$ do not contribute to the energy estimates, namely the Sobolev norms of the solutions of the system

$$\partial_t(q, \bar{q}) = \left(1 + \mathcal{P}\left(\Phi^{(4)}(q, \bar{q})\right) \right) (\mathcal{D}_1(q, \bar{q}) + X_{\mathbf{Res},3}(q, \bar{q})) + X_{\mathbf{Res},5}(q, \bar{q})$$

are constant.

We start by splitting

$$X^{(1)}_{\geq 5} = \mathcal{P} X_{\mathbf{Res},3} + X^{(1)}_5 + X^{(1)}_{\geq 7}$$

where $\mathcal{P} X_{\mathbf{Res},3}$ does not contribute to the energy estimates, $X^{(1)}_5$ only contains terms with degree 5 and $X^{(1)}_{\geq 7}$ contains the remaining terms with degree ≥ 7. In this way, the vector field $X^{(1)}$ in (2.59) can be written as

$$X^{(1)}(w,\bar{w}) = (I + \mathcal{P}(w,\bar{w}))\,(\mathcal{D}_1 + X_{\mathbf{Res},3}(w,\bar{w})) + X^{(1)}_5 + X^{(1)}_{\geq 7}.$$

1. Since the operator K_2 is quadratic, the degree-5 five part of

$$K_2(w,\bar{w})\,(I + K_2(w,\bar{w}))^{-1}\,(\mathcal{B}_3(w,\bar{w}) - X_{\mathbf{Res},3}(w,\bar{w}))$$

is

$$K(w,\bar{w})\,(\mathcal{B}_3(w,\bar{w}) - X_{\mathbf{Res},3}(w,\bar{w}))\,. \tag{2.66}$$

2. By recalling the definition of $\mathcal{P}$ in (2.39), we have that the degree-5 part of

$$-\mathcal{P}(w,\bar{w})\,(I + K_2(w,\bar{w}))^{-1}\,(\mathcal{B}_3(w,\bar{w}) - X_{\mathbf{Res},3}(w,\bar{w}))$$

is

$$cQ(w,\bar{w})\,(\mathcal{B}_3(w,\bar{w}) - X_{\mathbf{Res},3}(w,\bar{w}))\,. \tag{2.67}$$

3. By recalling the definition of $\mathcal{B}_5$ in (2.35) we have that the degree-5 part of

$$(I + K_2(w,\bar{w}))^{-1}\,\mathcal{B}_5\left(\Phi^{(3)}(w,\bar{w})\right)$$

is

$$-cQ(w,\bar{w})\mathcal{B}_3(w,\bar{w}). \tag{2.68}$$

4. Finally, concerning the term

$$\left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar{w})\right) - \mathcal{B}_3(w,\bar{w})\right],$$

by recalling the definition of $\Phi^{(3)} = \mathrm{Id} + M$ and Taylor expanding in $(w,\bar{w})$ we have

$$\mathcal{B}_3'(w,\bar{w})M(w,\bar{w})\begin{pmatrix} w \\ \bar{w} \end{pmatrix}, \tag{2.69}$$

where $\mathcal{B}_3'(u,v)$ is the Gateaux derivative of $\mathcal{B}_3$ in the point (u,v) and acts on pair of functions (α,β) in the following way:

$$\begin{aligned} \mathcal{B}_3'(u,v)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= -ic\big(\,\langle \Lambda v, \Lambda\beta\rangle - \langle \Lambda u, \Lambda\alpha\rangle\,\big)\begin{pmatrix} v \\ u \end{pmatrix} \\ &\quad -\frac{ic}{2}\big(\,\langle \Lambda v, \Lambda v\rangle - \langle \Lambda u, \Lambda u\rangle\,\big)\begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \end{aligned}$$

By collecting (2.66), (2.67), (2.68) and (2.69) we have

$$X^{(1)}_5(w,\bar{w}) = -K_2(w,\bar{w})X_{\mathbf{Res},3}(w,\bar{w}) + \mathcal{B}_3'(w,\bar{w})M(w,\bar{w})\begin{pmatrix} w \\ \bar{w} \end{pmatrix} \tag{2.70}$$

and, consequently

$$
\begin{aligned}
X^{(1)}_{\geq 7}(w,\bar w) &= K_2(w,\bar w)\left(-K_2(w,\bar w)+\tilde K_2(w,\bar w)\right)(\mathcal{B}_3(w,\bar w)-X_{\mathbf{Res},3}(w,\bar w))\\
&\quad -\mathcal{P}(w,\bar w)\left(-K_2(w,\bar w)+\tilde K_2(w,\bar w)\right)(\mathcal{B}_3(w,\bar w)-X_{\mathbf{Res},3}(w,\bar w))\\
&\quad +\mathcal{P}(w,\bar w)\left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w)\right)-\mathcal{B}_3(w,\bar w)\right]\\
&\quad +\left(-K_2(w,\bar w)+\tilde K_2(w,\bar w)\right)\mathcal{B}_5\left(\Phi^{(3)}(w,\bar w)\right)\\
&\quad +\left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w)\right)-\mathcal{B}_3(w,\bar w)-\mathcal{B}_3'(w,\bar w)M(w,\bar w)\begin{pmatrix}w\\ \bar w\end{pmatrix}\right]\\
&\quad -K_2(w,\bar w)\left[\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w\right)-\mathcal{B}_3(w,\bar w)\right]+\tilde K_2(w,\bar w)\mathcal{B}_3\left(\Phi^{(3)}(w,\bar w)\right).
\end{aligned}
$$

From a direct calculation we have:

Lemma 2.5.2 *For all $s\geq 0$, all $(w,z)\in H^s_0(\mathbb{T}^n,c.c.)\cap H^{m_0}_0(\mathbb{T}^n,c.c.)$ with $\|w\|_{m_0}\leq\delta_0$ we have*

$$
\left\|X^{(1)}_5(w,z)\right\|_s\leq\frac{63}{16}|c|^2\|w\|^2_{m_0}\|w\|^2_1\|w\|_s,\quad \left\|X^{(1)}_{\geq 7}(w,z)\right\|_s\leq C|c|^3.\|w\|^4_{m_0}\|w\|^2_1\|w\|_s
$$

Let us consider the following change of variables:

$$
\begin{pmatrix}w\\ \bar w\end{pmatrix}=\Phi^{(5)}(q,\bar q):=(I+\mathcal{M}(q,\bar q))\begin{pmatrix}q\\ \bar q\end{pmatrix}, \tag{2.71}
$$

where

$$
\mathcal{M}(q,\bar q)=\mathcal{A}[q,q,q,q]+\mathcal{B}[q,q,q,\bar q]+\mathcal{C}[q,q,\bar q,\bar q]+\mathcal{D}[q,\bar q,\bar q,\bar q]+\mathcal{F}[\bar q,\bar q,\bar q] \tag{2.72}
$$

is a generic map in the space of quadrilinear operators. We write

$$
\mathcal{A}[u,u,u,u]=\begin{pmatrix}\mathcal{A}_{11}[q,q,q,q] & \mathcal{A}_{12}[q,q,q,q]\\ \mathcal{A}_{21}[q,q,q,q] & \mathcal{A}_{22}[q,q,q,q]\end{pmatrix},
$$

with

$$
\mathcal{A}_{11}[q^{(1)},q^{(2)},q^{(3)},q^{(4)}]h=\sum_{j,l,k}q^{(1)}_j q^{(2)}_{-j}q^{(3)}_l q^{(4)}_{-l}\,h_k\,a_{11}(j,l,k)e^{ikx}
$$

and similarly for the other terms.
We assume moreover the following symmetry relations:

$$
\begin{aligned}
\mathcal{A}[q^{(1)},q^{(2)},q^{(3)},q^{(4)}]&=\mathcal{A}[q^{(2)},q^{(1)},q^{(3)},q^{(4)}]=\mathcal{A}[q^{(1)},q^{(2)},q^{(4)},q^{(3)}]\\
\mathcal{B}[q^{(1)},q^{(2)},q^{(3)},\bar q]&=\mathcal{B}[q^{(2)},q^{(1)},q^{(3)},\bar q]\\
\mathcal{C}[q^{(1)},q^{(2)},\bar q^{(1)},\bar q^{(2)}]&=\mathcal{C}[q^{(2)},q^{(1)},\bar q^{(1)},\bar q^{(2)}]=\mathcal{C}[q^{(1)},q^{(2)},\bar q^{(2)},\bar q^{(1)}]\\
\mathcal{D}[q,\bar q^{(1)},\bar q^{(2)},\bar q^{(3)}]&=\mathcal{D}[q,\bar q^{(1)},\bar q^{(3)},\bar q^{(2)}]\\
\mathcal{F}[\bar q^{(1)},\bar q^{(2)},\bar q^{(3)},\bar q^{(4)}]&=\mathcal{F}[\bar q^{(2)},\bar q^{(1)},\bar q^{(3)},\bar q^{(4)}]=\mathcal{F}[\bar q^{(1)},\bar q^{(2)},\bar q^{(4)},\bar q^{(3)}]
\end{aligned}
$$

Finally, in order for $\Phi^{(4)}$ to map complex conjugated pair of functions into pairs of complex conjugated function, we assume

$$
\mathcal{M}_{11}(q,\bar q)=\overline{\mathcal{M}_{22}(q,\bar q)},\quad \mathcal{M}_{12}(q,\bar q)=\overline{\mathcal{M}_{21}(q,\bar q)}.
$$

Similarly to the first normal form transformation, we have in the new set of variables:

$$(I+\mathcal{K}(q,\bar{q}))\,\partial_t\begin{pmatrix}q\\ \bar{q}\end{pmatrix}=X^{+}\left(\Phi^{(4)}(q,\bar{q})\right)$$

where

$$\mathcal{K}(q,\bar{q})\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=(\mathcal{M}(q,\bar{q})+\mathcal{E}(q,\bar{q}))\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

and

$$\begin{aligned}\mathcal{E}(q,\bar{q})\begin{pmatrix}\alpha\\ \beta\end{pmatrix} &= \Big(2\mathcal{A}[q,\alpha,q,q]+2\mathcal{A}[q,q,q,\alpha]+2\mathcal{B}[q,\alpha,q,\bar{q}]+\mathcal{B}[q,q,\alpha,\bar{q}]\\ &\quad+\mathcal{B}[q,q,q,\beta]+2\mathcal{C}[q,\alpha,\bar{q},\bar{q}]+2\mathcal{C}[q,q,\bar{q},\beta]+\mathcal{D}[\alpha,\bar{q},\bar{q},\bar{q}]\\ &\quad+\mathcal{D}[q,\beta,\bar{q},\bar{q}]+2\mathcal{D}[q,\bar{q},\bar{q},\beta]+2\mathcal{F}[\bar{q},\beta,\bar{q},\bar{q}]+2\mathcal{F}[\bar{q},\bar{q},\bar{q},\beta]\Big)\begin{pmatrix}q\\ \bar{q}\end{pmatrix}.\end{aligned}$$

Hence, we can write

$$\begin{aligned}\partial_t\begin{pmatrix}q\\ \bar{q}\end{pmatrix} &= X^{(2)}(q,\bar{q}):=(I+\mathcal{K}(q,\bar{q}))^{-1}X^{(1)}\left(\Phi^{(4)}(q,\bar{q})\right)\\ &= \left(1+\mathcal{P}\left(\Phi^{(4)}(q,\bar{q})\right)\right)(\mathcal{D}_1(q,\bar{q})+X_{\mathbf{Res},3}(q,\bar{q}))+X_5^{(2)}(q,\bar{q})+X_{\geq 7}^{(2)}(q,\bar{q}),\end{aligned}$$

where

$$X_5^{(2)}(q,\bar{q})=X_5^{(1)}(q,\bar{q})+\mathcal{D}_1\left(\mathcal{M}(q,\bar{q})\begin{pmatrix}q\\ \bar{q}\end{pmatrix}\right)-\mathcal{K}(q,\bar{q})\mathcal{D}_1(q,\bar{q}).$$

is the part with degree-5 and

$$\begin{aligned}X_{\geq 7}^{(2)}(q,\bar{q}) &= \left[1+\mathcal{P}\left(\Phi^{(4)}(q,\bar{q})\right)\right]\left(-\mathcal{K}(q,\bar{q})+\tilde{\mathcal{K}}(q,\bar{q})\right)\left(X_5^{(2)}(q,\bar{q})-X_5^{(1)}(q,\bar{q})\right)\\ &\quad+\mathcal{P}\left(\Phi^{(4)}(q,\bar{q})\right)\left(X_5^{(2)}(q,\bar{q})-X_5^{(1)}(q,\bar{q})\right)\\ &\quad+\left[1+\mathcal{P}\left(\Phi^{(4)}(q,\bar{q})\right)\right]\left(-\mathcal{K}(q,\bar{q})+\tilde{\mathcal{K}}(q,\bar{q})\right)X_{\mathbf{Res},3}(q,\bar{q})\\ &\quad+\left(-\mathcal{K}(q,\bar{q})+\tilde{\mathcal{K}}(q,\bar{q})\right)X_5^{(1)}(q,\bar{q}) \qquad (2.73)\\ &\quad+\left[1+\mathcal{P}\left(\Phi^{(4)}(q,\bar{q})\right)\right](I+\mathcal{K}(q,\bar{q}))^{-1}\left[X_{\mathbf{Res},3}\left(\Phi^{(4)}(q,\bar{q})\right)-X_{\mathbf{res},3}(q,\bar{q})\right]\\ &\quad+(I+\mathcal{K}(q,\bar{q}))^{-1}\left[X_5^{(1)}\left(\Phi^{(4)}(q,\bar{q})\right)-X_5^{(1)}(q,\bar{q})\right]\\ &\quad+(I+\mathcal{K}(q,\bar{q}))^{-1}X_{\geq 7}^{(1)}\left(\Phi^{(4)}(q,\bar{q})\right)\end{aligned}$$

is the remaining part with degree ≥ 7.
We start by analyzing the first component of $X_5^{(2)}$:

$$\left(X_5^{(2)}(q,\bar{q})\right)_1=\left(X_5^{(1)}(q,\bar{q})\right)_1-2i\mathcal{M}_{1,2}(q,\bar{q})\Lambda\bar{q}-\left(\mathcal{E}(q,\bar{q})\begin{pmatrix}-i\Lambda q\\ i\Lambda\bar{q}\end{pmatrix}\right)_1.$$

By recalling the expressions of $X_5^{(1)}$ in (2.70)and of K_2 in (2.48) and recalling that

$$Q(q,\bar{q})=\frac{c}{2}\langle\Lambda q,q\rangle+c\langle\Lambda q,\bar{q}\rangle+\frac{c}{2}\langle\Lambda\bar{q},\bar{q}\rangle$$

and that

$$\mathcal{B}_3(q,\bar{q}) = \left(-\frac{ic}{2}\left\langle \Lambda\bar{q}, \Lambda\bar{q}\right\rangle + \frac{ic}{2}\left\langle \Lambda q, \Lambda q\right\rangle\right)\begin{pmatrix}\bar{q}\\ q\end{pmatrix},$$

we have

$$\begin{aligned}\big(-K_2(q,\bar{q})X_{\mathbf{Res},3}(q,\bar{q})\big)_1 &= -2A_{1,2}\left[q,(X_{\mathbf{Res},3}(q,\bar{q}))_1\right]\bar{q} - 2C_{1,2}\left[\bar{q},(X_{\mathbf{Res},3}(q,\bar{q}))_2\right]\bar{q}\\ &\quad -A_{1,2}[q,q]\,(X_{\mathbf{Res},3}(q,\bar{q}))_2 - C_{1,2}[\bar{q},\bar{q}]\,(X_{\mathbf{Res},3}(q,\bar{q}))_2\,;\end{aligned}$$

$$\begin{aligned}&\left(\mathcal{B}_3'(q,\bar{q})M(q,\bar{q})\begin{pmatrix}q\\ \bar{q}\end{pmatrix}\right)_1\\ &= -ic\Bigg(\left\langle \Lambda\bar{q}, \Lambda A_{1,2}[\bar{q},\bar{q}]q + \Lambda C_{1,2}[q,q]q\right\rangle - \left\langle \Lambda q, \Lambda A_{1,2}[q,q]\bar{q} + \Lambda C_{1,2}[\bar{q},\bar{q}]\bar{q}\right\rangle\Bigg)\bar{q}\\ &\quad -\frac{ic}{2}(\langle \Lambda\bar{q},\Lambda\bar{q}\rangle - \langle\Lambda q,\Lambda q\rangle)\,(A_{1,2}[\bar{q},\bar{q}]q + C_{1,2}[q,q]q)\\ &= -ic\left\langle \Lambda\bar{q}, \Lambda A_{1,2}[\bar{q},\bar{q}]q\right\rangle\bar{q} - ic\left\langle \Lambda\bar{q}, \Lambda C_{1,2}[q,q]q\right\rangle\bar{q} + ic\left\langle \Lambda q, \Lambda A_{1,2}[q,q]\bar{q}\right\rangle\bar{q}\\ &\quad +ic\left\langle \Lambda q, \Lambda C_{1,2}[\bar{q},\bar{q}]\bar{q}\right\rangle\bar{q} - \frac{ic}{2}\left\langle \Lambda\bar{q},\Lambda\bar{q}\right\rangle A_{1,2}[\bar{q},\bar{q}]q - \frac{ic}{2}\left\langle \Lambda\bar{q},\Lambda\bar{q}\right\rangle C_{1,2}[q,q]q\\ &\quad +\frac{ic}{2}\left\langle \Lambda q,\Lambda q\right\rangle A_{1,2}[\bar{q},\bar{q}]q + \frac{ic}{2}\left\langle \Lambda q,\Lambda q\right\rangle C_{1,2}[q,q]q;\end{aligned}$$

$$\begin{aligned}-2i\mathcal{M}_{1,2}(q,\bar{q})\Lambda\bar{q} &= -2i\mathcal{A}_{1,2}[q,q,q,q]\Lambda\bar{q} - 2i\mathcal{B}_{1,2}[q,q,q,\bar{q}]\Lambda\bar{q} - 2i\mathcal{C}_{1,2}[q,q,\bar{q},\bar{q}]\Lambda\bar{q}\\ &\quad -2i\mathcal{D}_{1,2}[q,\bar{q},\bar{q},\bar{q}]\Lambda\bar{q} - 2i\mathcal{F}_{1,2}[\bar{q},\bar{q},\bar{q},\bar{q}]\Lambda\bar{q};\end{aligned}$$

$$\begin{aligned}&-\left(\mathcal{E}(q,\bar{q})\begin{pmatrix}-i\Lambda q\\ i\Lambda\bar{q}\end{pmatrix}\right)_1\\ &= 2i\mathcal{A}_{1,1}[q,\Lambda q,q,q]q + 2i\mathcal{A}_{1,1}[q,q,q,\Lambda q]q + 2i\mathcal{B}_{1,1}[q,\Lambda q,q,\bar{q}]q + i\mathcal{B}_{1,1}[q,q,\Lambda q,\bar{q}]q\\ &\quad -i\mathcal{B}_{1,1}[q,q,q,\Lambda\bar{q}]q + 2i\mathcal{C}_{1,1}[q,\Lambda q,\bar{q},\bar{q}]q - 2i\mathcal{C}_{1,1}[q,q,\bar{q},\Lambda\bar{q}]q + i\mathcal{D}_{1,1}[\Lambda q,\bar{q},\bar{q},\bar{q}]q\\ &\quad -i\mathcal{D}_{1,1}[q,\Lambda\bar{q},\bar{q},\bar{q}]q - 2i\mathcal{D}_{1,1}[q,\bar{q},\bar{q},\Lambda\bar{q}]q - 2i\mathcal{F}_{1,1}[\bar{q},\Lambda\bar{q},\bar{q},\bar{q}]q - 2i\mathcal{F}_{1,1}[\bar{q},\bar{q},\bar{q},\Lambda\bar{q}]q\\ &\quad +2i\mathcal{A}_{1,2}[q,\Lambda q,q,q]\bar{q} + 2i\mathcal{A}_{1,2}[q,q,q,\Lambda q]\bar{q} + 2i\mathcal{B}_{1,2}[q,\Lambda q,q,\bar{q}]\bar{q} + i\mathcal{B}_{1,2}[q,q,\Lambda q,\bar{q}]\bar{q}\\ &\quad -i\mathcal{B}_{1,2}[q,q,q,\Lambda\bar{q}]\bar{q} + 2i\mathcal{C}_{1,2}[q,\Lambda q,\bar{q},\bar{q}]\bar{q} - 2i\mathcal{C}_{1,2}[q,q,\bar{q},\Lambda\bar{q}]\bar{q} + i\mathcal{D}_{1,2}[\Lambda q,\bar{q},\bar{q},\bar{q}]\bar{q}\\ &\quad -i\mathcal{D}_{1,2}[q,\Lambda\bar{q},\bar{q},\bar{q}]\bar{q} - 2i\mathcal{D}_{1,2}[q,\bar{q},\bar{q},\Lambda\bar{q}]\bar{q} - 2i\mathcal{F}_{1,2}[\bar{q},\Lambda\bar{q},\bar{q},\bar{q}]\bar{q} - 2i\mathcal{F}_{1,2}[\bar{q},\bar{q},\bar{q},\Lambda\bar{q}]\bar{q};\end{aligned}$$

Collecting all these terms, we obtain

$$
\begin{aligned}
\left(X_5^{(2)}(q,\bar q)\right)_1 \;=\;& -2A_{1,2}\left[q,\left(X_3^+(q,\bar q)\right)_1\right]\bar q-2C_{1,2}\left[\bar q,\left(X_3^+(q,\bar q)\right)_2\right]\bar q\\
&-A_{1,2}[q,q]\left(X_3^+(q,\bar q)\right)_2-C_{1,2}[\bar q,\bar q]\left(X_3^+(q,\bar q)\right)_2\\
&-ic\left\langle\Lambda\bar q,\Lambda A_{1,2}[\bar q,\bar q]q\right\rangle\bar q-ic\left\langle\Lambda\bar q,\Lambda C_{1,2}[q,q]q\right\rangle\bar q+ic\left\langle\Lambda q,\Lambda A_{1,2}[q,q]\bar q\right\rangle\bar q\\
&+ic\left\langle\Lambda q,\Lambda C_{1,2}[\bar q,\bar q]\bar q\right\rangle\bar q-\frac{ic}{2}\left\langle\Lambda\bar q,\Lambda\bar q\right\rangle A_{1,2}[\bar q,\bar q]q-\frac{ic}{2}\left\langle\Lambda\bar q,\Lambda\bar q\right\rangle C_{1,2}[q,q]q\\
&+\frac{ic}{2}\left\langle\Lambda q,\Lambda q\right\rangle A_{1,2}[\bar q,\bar q]q+\frac{ic}{2}\left\langle\Lambda q,\Lambda q\right\rangle C_{1,2}[q,q]q\\
&-2i\mathcal{A}_{1,2}[q,q,q,q]\Lambda\bar q-2i\mathcal{B}_{1,2}[q,q,q,\bar q]\Lambda\bar q-2i\mathcal{C}_{1,2}[q,q,\bar q,\bar q]\Lambda\bar q\\
&-2i\mathcal{D}_{1,2}[q,\bar q,\bar q,\bar q]\Lambda\bar q-2i\mathcal{F}_{1,2}[\bar q,\bar q,\bar q,\bar q]\Lambda\bar q\\
&+2i\mathcal{A}_{1,1}[q,\Lambda q,q,q]q+2i\mathcal{A}_{1,1}[q,q,q,\Lambda q]q+2i\mathcal{B}_{1,1}[q,\Lambda q,q,\bar q]q+i\mathcal{B}_{1,1}[q,q,\Lambda q,\bar q]q\\
&-i\mathcal{B}_{1,1}[q,q,q,\Lambda\bar q]q+2i\mathcal{C}_{1,1}[q,\Lambda q,\bar q,\bar q]q-2i\mathcal{C}_{1,1}[q,q,\bar q,\Lambda\bar q]q+i\mathcal{D}_{1,1}[\Lambda q,\bar q,\bar q,\bar q]q\\
&-i\mathcal{D}_{1,1}[q,\Lambda\bar q,\bar q,\bar q]q-2i\mathcal{D}_{1,1}[q,\bar q,\bar q,\Lambda\bar q]q-2i\mathcal{F}_{1,1}[\bar q,\Lambda\bar q,\bar q,\bar q]q-2i\mathcal{F}_{1,1}[\bar q,\bar q,\bar q,\Lambda\bar q]q\\
&+2i\mathcal{A}_{1,2}[q,\Lambda q,q,q]\bar q+2i\mathcal{A}_{1,2}[q,q,q,\Lambda q]\bar q+2i\mathcal{B}_{1,2}[q,\Lambda q,q,\bar q]\bar q+i\mathcal{B}_{1,2}[q,q,\Lambda q,\bar q]\bar q\\
&-i\mathcal{B}_{1,2}[q,q,q,\Lambda\bar q]\bar q+2i\mathcal{C}_{1,2}[q,\Lambda q,\bar q,\bar q]\bar q-2i\mathcal{C}_{1,2}[q,q,\bar q,\Lambda\bar q]\bar q+i\mathcal{D}_{1,2}[\Lambda q,\bar q,\bar q,\bar q]\bar q\\
&-i\mathcal{D}_{1,2}[q,\Lambda\bar q,\bar q,\bar q]\bar q-2i\mathcal{D}_{1,2}[q,\bar q,\bar q,\Lambda\bar q]\bar q-2i\mathcal{F}_{1,2}[\bar q,\Lambda\bar q,\bar q,\bar q]\bar q-2i\mathcal{F}_{1,2}[\bar q,\bar q,\bar q,\Lambda\bar q]\bar q.
\end{aligned}
$$

We now select the operators $\mathcal{A}$, $\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{F}$ in (2.72) and the corresponding Fourier symbols a, b, c, d, f in order to cancel out as many terms as possible from $X_5^{(2)}$.
Since, in Fourier variables, the terms involved in the expression of $X_5^{(2)}$ are monomials in the variables q_j, $\bar q_j$ we start by separating the terms with the same homogeneity in the variables q and $\bar q$:

(1) Terms containing the monomial $\mathbf{q_l q_{-l} q_j q_{-j} q_k}$:

$$\begin{aligned}
&\frac{ic}{2}\left\langle \Lambda q, \Lambda q\right\rangle C_{1,2}[q,q]q + 2i\mathcal{A}_{1,1}[q,\Lambda q,q,q]q + 2i\mathcal{A}_{1,1}[q,q,q,\Lambda q]q \\
= \quad & -\frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|l|^2|j|^2}{|l|+|k|}q_jq_{-j}q_lq_{-l}q_ke^{ik\cdot x} + 2i\sum_{j,l,k\neq 0}|j|a_{11}(j,l,k)q_jq_{-j}q_lq_{-l}q_ke^{ik\cdot x} \\
& + \sum |l|a_{11}(j,l,k)q_jq_{-j}q_lq_{-l}q_ke^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\left(2ia_{11}(j,l,k)|j| + 2ia_{11}(j,l,k)|l| - \frac{ic^2}{8}\frac{|l|^2|j|^2}{|l|+|k|}\right)q_lq_{-l}q_jq_{-j}q_ke^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\left(2ia_{11}(j,l,k)(|j|+|l|) - \frac{ic^2}{16}\frac{|l|^2|j|^2}{|j|+|k|} - \frac{ic^2}{16}\frac{|l|^2|j|^2}{|l|+|k|}\right)q_lq_{-l}q_jq_{-j}q_ke^{ik\cdot x},
\end{aligned}$$

then we can choose a_{11} to be

$$a_{11}(j,l,k) = \frac{c^2}{32}\frac{|l|^2|j|^2}{|j|+|l|}\left(\frac{1}{|j|+|k|} + \frac{1}{|l|+|k|}\right). \tag{2.74}$$

There are no resonant terms of this form.

(2) Terms containing the monomial $\mathbf{q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}}$:

$$\begin{aligned}
&-2A_{1,2}\left[q,\left(X_3^+(q,\bar{q})\right)_1\right]\bar{q} + ic\left\langle \Lambda q, \Lambda A_{1,2}[q,q]\bar{q}\right\rangle\bar{q} - ic\left\langle \Lambda\bar{q}, \Lambda C_{1,2}[q,q]q\right\rangle\bar{q} \\
&-2i\mathcal{B}_{1,2}[q,q,q,\bar{q}]\Lambda\bar{q} + 2i\mathcal{B}_{1,2}[q,\Lambda q,q,\bar{q}]\bar{q} + i\mathcal{B}_{1,2}[q,q,\Lambda q,\bar{q}]\bar{q} - i\mathcal{B}_{1,2}[q,q,q,\Lambda\bar{q}]\bar{q}
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{ic^2}{4}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})\delta_{|l|}^{|j|}}{|l|-|k|}q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} - \frac{ic^2}{4}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|j|})}{|j|-|l|}q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
& + \frac{ic^2}{4}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2}{|j|+|l|}q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} - 2i\sum_{j,l,k\neq 0}|k|b_{12}(j,l,k)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
& + 2i\sum_{j,l,k\neq 0}|j|b_{12}(j,l,k)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} + i\sum_{j,l,k\neq 0}|l|b_{12}(j,l,k)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
& - i\sum_{j,l,k\neq 0}|l|b_{12}(j,l,k)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\Bigg(\frac{ic^2}{4}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})\delta_{|l|}^{|j|}}{|l|-|k|} - \frac{ic^2}{4}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|j|})}{|j|-|l|} + \frac{ic^2}{4}\frac{|j|^2|l|^2}{|j|+|l|} \\
& + 2ib_{12}(j,l,k)(|j|-|k|)\Bigg)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x},
\end{aligned}$$

hence

$$b_{1,2}(j,l,k) = \frac{c^2|j|^2|l|^2}{8}\left(\frac{(1-\delta_{|l|}^{|k|})\delta_{|l|}^{|j|}}{|l|-|k|} - \frac{(1-\delta_{|l|}^{|j|})}{|j|-|l|} + \frac{1}{|j|+|l|}\right)\frac{1-\delta_{|j|}^{|k|}}{|k|-|j|}. \tag{2.75}$$

The remaining resonant terms are given by

$$\sum_{|j|=|k|}\frac{ic^2|j|^2|l|^2}{8}\left(\frac{(1-\delta_{|l|}^{|k|})\delta_{|l|}^{|j|}}{|l|-|k|} - \frac{1-\delta_{|l|}^{|j|}}{|j|-|l|} + \frac{1}{|j|+|l|})\right)q_jq_{-j}q_l\bar{q}_l\bar{q}_{-k}e^{ik\cdot x}$$

$$= \sum_{|j|=|k|} \frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|l|} - \frac{1-\delta_{|l|}^{|j|}}{|j|-|l|}\right) q_j q_{-j} q_l \bar{q}_l \bar{q}_{-k} e^{ik\cdot x}. \qquad \text{(R1)}$$

(3) Terms containing the monomial $\mathbf{q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}}$:

$$-2C_{1,2}\left[\bar{q}, \left(X_3^+(q,\bar{q})\right)_2\right]\bar{q} - ic\left\langle \Lambda\bar{q}, \Lambda A_{1,2}[\bar{q},\bar{q}]q\right\rangle \bar{q} + ic\left\langle \Lambda q, \Lambda C_{1,2}[\bar{q},\bar{q}]\bar{q}\right\rangle \bar{q}$$
$$-2i\mathcal{D}_{1,2}[q,\bar{q},\bar{q},\bar{q}]\Lambda\bar{q} + i\mathcal{D}_{1,2}[\Lambda q,\bar{q},\bar{q},\bar{q}]\bar{q} - i\mathcal{D}_{1,2}[q,\Lambda\bar{q},\bar{q},\bar{q}]\bar{q} - 2i\mathcal{D}_{1,2}[q,\bar{q},\bar{q},\Lambda\bar{q}]\bar{q}$$

$$\begin{aligned}
= \quad & -\frac{ic^2}{4}\sum_{j,l,k\neq 0} \frac{|j|^2|l|^2\delta_{|l|}^{|j|}}{|j|+|k|} q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} + \frac{ic^2}{4}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|j|})}{|l|-|j|} q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
& -\frac{ic^2}{4}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2}{|l|+|j|} q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} - 2i\sum_{j,l,k\neq 0} d_{12}(j,l,k)(|j|+|k|) q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}e^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\Bigg(-\frac{ic^2}{4}\frac{|j|^2|l|^2\delta_{|l|}^{|j|}}{|j|+|k|} + \frac{ic^2}{4}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|j|})}{|l|-|j|} - \frac{ic^2}{4}\frac{|j|^2|l|^2}{|l|+|j|} \\
& -2id_{12}(j,l,k)(|j|+|k|)\Bigg) q_j\bar{q}_j\bar{q}_{-l}\bar{q}_l\bar{q}_{-k}e^{ik\cdot x}
\end{aligned}$$

hence

$$d_{12}(j,l,k) = \frac{ic^2|j||l|}{8}\left(-\frac{\delta_{|l|}^{|j|}}{|j|+|k|} + \frac{1-\delta_{|l|}^{|j|}}{|l|-|j|} - \frac{1}{|l|+|j|}\right)\frac{1}{|j|+|k|}. \qquad (2.76)$$

There are no resonant terms of this form.

(4) Terms containing the monomial $\mathbf{\bar{q}_{-j}\bar{q}_j\bar{q}_{-l}\bar{q}_l q_k}$:

$$\begin{aligned}
& -C_{1,2}[\bar{q},\bar{q}]\left(X_3^+(q,\bar{q})\right)_2 - \frac{ic}{2}\left\langle \Lambda\bar{q},\Lambda\bar{q}\right\rangle A_{1,2}[\bar{q},\bar{q}]q - 2i\mathcal{F}_{1,1}[\bar{q},\Lambda\bar{q},\bar{q},\bar{q}]q - 2i\mathcal{F}_{1,1}[\bar{q},\bar{q},\bar{q},\Lambda\bar{q}]q \\
= \quad & -\frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2\delta_{|l|}^{|k|}}{|j|+|k|}\bar{q}_{-j}\bar{q}_j\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} + \frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})}{|j|-|k|}\bar{q}_{-j}\bar{q}_j\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} \\
& -2i\sum_{j,l,k\neq 0} f_{11}(j,l,k)(|j|+|l|)\bar{q}_{-j}\bar{q}_j\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\left(-\frac{ic^2}{8}\frac{|j|^2|l|^2\delta_{|l|}^{|k|}}{|j|+|k|} + \frac{ic^2}{8}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})}{|j|-|k|} - 2if_{11}(j,l,k)(|j|+|l|)\right)\bar{q}_{-j}\bar{q}_j\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x}
\end{aligned}$$

hence

$$f_{11}(j,l,k) = \frac{c^2|j|^2|l|^2}{16}\left(-\frac{\delta_{|l|}^{|k|}}{|j|+|k|} + \frac{(1-\delta_{|l|}^{|k|})}{|j|-|k|}\right)\frac{1}{|j|+|l|} \qquad (2.77)$$

There are no remaining resonant terms of this form.

(5) Terms containing the monomial $\mathbf{q_j q_{-j} \bar{q}_{-l} \bar{q}_l q_k}$:

$$
\begin{aligned}
&-A_{1,2}[q,q]\left(X_3^+(q,\bar{q})\right)_2 - \frac{ic}{2}\langle \Lambda\bar{q},\Lambda\bar{q}\rangle C_{1,2}[q,q]q + \frac{ic}{2}\langle \Lambda q,\Lambda q\rangle A_{1,2}[\bar{q},\bar{q}]q \\
&+2i\mathcal{C}_{1,1}[q,\Lambda q,\bar{q},\bar{q}]q - 2i\mathcal{C}_{1,1}[q,q,\bar{q},\Lambda\bar{q}]q
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|j|}^{|k|})\delta_{|l|}^{|k|}}{|j|-|k|} q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} + \frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2}{|j|+|k|} q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} \\
& -\frac{ic^2}{8}\sum_{j,l,k\neq 0}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})}{|l|-|k|} q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} - 2i\sum_{j,l,k\neq 0} c_{11}(j,l,k)(|l|-|j|) q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x} \\
= \quad & \sum_{j,l,k\neq 0}\Big(-\frac{ic^2}{8}\frac{|j|^2|l|^2(1-\delta_{|j|}^{|k|})\delta_{|l|}^{|k|}}{|j|-|k|} + \frac{ic^2}{8}\frac{|j|^2|l|^2}{|j|+|k|} - \frac{ic^2}{8}\frac{|j|^2|l|^2(1-\delta_{|l|}^{|k|})}{|l|-|k|} \\
& -2ic_{11}(j,l,k)(|l|-|j|)\Big) q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x}
\end{aligned}
$$

hence

$$
c_{11}(j,l,k) = \frac{c^2|j|^2|l|^2}{16}\left(-\frac{(1-\delta_{|j|}^{|k|})\delta_{|l|}^{|k|}}{|j|-|k|} + \frac{1}{|j|+|k|} - \frac{(1-\delta_{|l|}^{|k|})}{|l|-|k|}\right)\frac{1-\delta_{|j|}^{|l|}}{|l|-|j|}. \tag{2.78}
$$

The remaining resonant terms are

$$
\sum_{|j|=|l|}\frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|k|} - \frac{1-\delta_{|l|}^{|k|}}{|l|-|k|}\right) q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x}. \tag{R2}
$$

As they are not needed in the calculation, we set

$$
\mathcal{A}_{12},\ \mathcal{B}_{11},\ \mathcal{C}_{12},\ \mathcal{D}_{11},\ \mathcal{F}_{12} = 0.
$$

By recalling (R1) and (R2), we have that the resonant vector field of degree 5 has first component

$$
\begin{aligned}
(X_{\mathbf{Res},5}(q,\bar{q}))_1 \quad = \quad & \sum_{|j|=|k|}\frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|l|} - \frac{1-\delta_{|l|}^{|j|}}{|j|-|l|}\right) q_j q_{-j} q_l \bar{q}_l \bar{q}_{-k} e^{ik\cdot x} \\
& + \sum_{|j|=|l|}\frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|k|} - \frac{1-\delta_{|l|}^{|k|}}{|l|-|k|}\right) q_j q_{-j}\bar{q}_{-l}\bar{q}_l q_k e^{ik\cdot x}.
\end{aligned} \tag{2.79}
$$

Analogous calculations lead to

$$
\begin{aligned}
&\mathcal{A}_{21},\ \mathcal{B}_{22},\ \mathcal{C}_{21},\ \mathcal{D}_{22},\ \mathcal{F}_{21} = 0, \\
&\mathcal{A}_{22} = \mathcal{F}_{11},\ \mathcal{B}_{21} = \mathcal{C}_{12},\ \mathcal{C}_{22} = \mathcal{B}_{11},\ \mathcal{F}_{22} = \mathcal{A}_{11}
\end{aligned}
$$

and

$$(X_{\mathbf{Res},5}(q,\bar q))_2 = \qquad -\sum_{|j|=|k|}\frac{c^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|l|}-\frac{1-\delta_{|l|}^{|j|}}{|j|-|l|}\right)\bar q_{-j}\bar q_j\bar q_{-l}q_{-l}q_k e^{ik\cdot x}$$

$$-\sum_{|j|=|l|}\frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|k|}-\frac{1-\delta_{|l|}^{|k|}}{|l|-|k|}\right)\bar q_{-j}\bar q_j q_l q_{-l}\bar q_{-k}e^{ik\cdot x}. \tag{2.80}$$

Lemma 2.5.3 *The vector field $X^{(2)}$ preserves the real structure, namely $\overline{\left(X^{(2)}\right)_1}=\left(X^{(2)}\right)_2$*

Proof: It follows from the definitions od a_{11}, b_{12}, c_{11}, d_{12}, f_{11} in (2.74), (2.75), (2.74), (2.78), (2.76), (2.77). □

We now collect some basic estimates on the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{F}$:

Lemma 2.5.4 *For any complex functions u, v, w, z, h and for all real $s\geq 0$, it holds:*

$$\begin{aligned}
\|\mathcal{A}_{11}[u,v,w,z]h\|_s &\leq c^2\|u\|_{\frac12}\|v\|_{\frac12}\|w\|_{\frac12}\|z\|_{\frac12}\|h\|_s\\
\|B_{12}[u,v,w,z]h\|_s &\leq 3c^2\|u\|_{m_0}\|v\|_{m_0}\|w\|_{m_0}\|z\|_{m_0}\|h\|_s\\
\|\mathcal{C}_{11}[u,v,w,z]h\|_s &\leq 2c^2\|u\|_{m_0}\|v\|_{m_0}\|w\|_{m_0}\|z\|_{m_0}\|h\|_s\\
\|\mathcal{D}_{12}[u,v,w,z]h\|_s &\leq c^2\|u\|_1\|v\|_1\|w\|_1\|z\|_1\|h\|_s\\
\|\mathcal{F}_{11}[u,v,w,z]h\|_s &\leq c^2\|u\|_1\|v\|_1\|w\|_1\|z\|_1\|h\|_s
\end{aligned} \tag{2.81}$$

Proof: It follows directly from (2.74), (2.75) , (2.76), (2.77), (2.78), together with a repeated use of (2.46). □

Lemma 2.5.5 *For all $s\geq 0$, all $(u,v)\in H_0^{m_0}(\mathbb{T}^n, c.c.)$, $(\alpha,\beta)\in H_0^s(\mathbb{T}^n,c.c.)$ one has*

$$\left\|\mathcal{M}(u,v)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|_s \leq 12c^2\|u\|_{m_0}^4\|\alpha\|_s \tag{2.82}$$

$$\left\|\mathcal{K}(u,v)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|_s \leq 12c^2\|u\|_{m_0}^3(\|u\|_{m_0}\|\alpha\|_s+4\|u\|_s\|\alpha\|_{m_0}). \tag{2.83}$$

Moreover, there exists δ_3 such that, if $\|u\|_{m_0}<\delta_3$, the operator

$$(I+\mathcal{K}(u,v)) : H_0^{m_0}(\mathbb{T}^n,c.c.)\longrightarrow H_0^{m_0}(\mathbb{T}^n,c.c.)$$

is invertible with inverse satisfying

$$\left\|(I+\mathcal{K}(u,v))^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|_s\leq C\left(\|\alpha\|_s+\|u\|_{m_0}^3\|u\|_s\|\alpha\|_{m_0}\right). \tag{2.84}$$

In our case $C=C(c^2)$ and $\delta_3=\left(60c^2\right)^{-\frac14}$.

Proof: Estimates (2.82) and (2.83) are a direct consequence of (2.81), while (2.84) follows from an application of the Neumann series together with (2.83). □

Lemma 2.5.6 *There exists a universal constant $\delta_4 > 0$ such that, for all $(w, \bar{w}) \in H_0^{m_0}(\mathbb{T}^n, c.c.)$ in the ball $\|w\|_{m_0} \leq \delta$ there exists a unique $(q, \bar{q}) \in H_0^{m_0}(\mathbb{T}^n, c.c.)$ such that $\Phi^{(5)}(u, v) = (w, \bar{w})$, with $\|q\|_{m_0} \leq 2\|w\|_{m_0}$.*
If, in addition, $w \in H_0^s$ for some $s \geq m_0$, then q also belongs to H_0^s and $\|u\|_2 \leq 2\|w\|_s$.
(in our case

$$\delta_4 = \left(385c^2\right)^{-\frac{1}{4}} \tag{2.85}$$

Proof: The proof is the same as in Lemma 2.4.4 with the use of (2.82) and (2.83). □

If we perform an energy estimate on a solution of

$$\partial_t \begin{pmatrix} q \\ \bar{q} \end{pmatrix} = X^{(2)}(q, \bar{q}) = \left(1+\mathcal{P}\left(\Phi^{(4)}(q, \bar{q})\right)\right)(\mathcal{D}_1(q, \bar{q}) + X_{\mathbf{Res},3}(q, \bar{q}))+X_{\mathbf{Res},5}(q, \bar{q})+X_{\geq 7}^{(2)}(q, \bar{q})$$

since the terms $\left(1 + \mathcal{P}\left(\Phi^{(4)}(q, \bar{q})\right)\right)(\mathcal{D}_1(q, \bar{q}) + X_{\mathbf{Res},3}(q, \bar{q}))$ give zero contribution, we obtain

$$\begin{aligned} \partial_t\left(\|q\|_s^2\right) &= \left\langle \left(X^{(2)}(q, \bar{q})\right)_1, \Lambda^{2s}\bar{q}\right\rangle + \left\langle \Lambda^{2s}q, \left(X^{(2)}(q, \bar{q})\right)_2\right\rangle \\ &= \left\langle (X_{\mathbf{Res},5}(q, \bar{q}))_1 + \left(X_{\geq 7}^{(2)}(q, \bar{q})\right)_1, \Lambda^{2s}\bar{q}\right\rangle + \left\langle \Lambda^{2s}q, (X_{\mathbf{Res},5}(q, \bar{q}))_2 + \left(X_{\geq 7}^{(2)}(q, \bar{q})\right)_2\right\rangle. \end{aligned} \tag{2.86}$$

Then the first non-trivial contribution is given by $X_{\mathbf{Res},5}$.
By recalling (2.79) and (2.80) we have

$$\begin{aligned} &\left\langle (X_{\mathbf{Res},5}(q, \bar{q}))_1, \Lambda^{2s}\bar{q}\right\rangle + \left\langle \Lambda^{2s}q, (X_{\mathbf{Res},5}(q, \bar{q}))_2\right\rangle \\ = &\sum_{|j|=|k|} \frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|l|} - \frac{1-\delta_{|l|}^{|j|}}{|j|-|l|}\right) q_j q_{-j} q_l \bar{q}_l \bar{q}_{-k} \bar{q}_k && (2.87) \\ &+ \sum_{|j|=|l|} \frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|k|} - \frac{1-\delta_{|l|}^{|k|}}{|l|-|k|})\right) q_j q_{-j} \bar{q}_{-l} \bar{q}_l q_k \bar{q}_k && (2.88) \\ &- \sum_{|j|=|k|} \frac{c^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|l|} - \frac{1-\delta_{|l|}^{|j|}}{|j|-|l|}\right) \bar{q}_{-j} \bar{q}_j \bar{q}_{-l} q_{-l} q_k q_{-k} \\ &+ \sum_{|j|=|l|} \frac{ic^2|j|^2|l|^2}{8}\left(\frac{1}{|j|+|k|} - \frac{1-\delta_{|l|}^{|k|}}{|l|-|k|})\right) \bar{q}_{-j} \bar{q}_j q_l q_{-l} \bar{q}_{-k} q_{-k}. \end{aligned}$$

By renaming $j \leftrightarrow k$ in equation (2.87) and $j \leftrightarrow l$ in (2.88) we have that

$$\left\langle (X_{\mathbf{Res},5}(q, \bar{q}))_1, \Lambda^{2s}\bar{q}\right\rangle + \left\langle \Lambda^{2s}q, (X_{\mathbf{Res},5}(q, \bar{q}))_2\right\rangle = 0. \tag{2.89}$$

Equation (2.89) implies that also $X_{\mathbf{Res},5}$ does not contribute to the energy estimate, therefore we need to quantify the contribution of the higher order term $X_{\geq 7}^{(2)}$, defined in (2.73). Let us quantify it:

Lemma 2.5.7 *For all $s \geq 0$ all $(q, \bar{q}) \in H_0^s(\mathbb{T}^n, c.c.) \cap B_{\delta_4}(H_0^{m_0}(\mathbb{T}^n, c.c.))$ one has*

$$\left\| X_{\geq 7}^{(2)}(q, \bar{q}) \right\|_s \leq \mathfrak{c}_1 \|q\|_{m_0}^4 \|q\|_1^2 \|q\|_s \tag{2.90}$$

where $\mathfrak{c}_1$ is a constant depending on c.

Proof: It follows from the definition of $X_{\geq 7}^{(2)}$ in (2.73) together with estimates (2.83) and (2.49). □

We can therefore improve the energy estimate on the solutions of the new system:

Lemma 2.5.8 *Let $T > 0$, $s \geq m_0$ and consider a solution $(q, \bar{q}) \in H_0^s(\mathbb{T}^n, c.c.) \cap B_{\delta_4}(H_0^{m_0}(\mathbb{T}^n, c.c.))$ of*

$$\partial_t(q, \bar{q}) = X^{(2)}(q, \bar{q}),$$

then it holds

$$\partial_t(\|q\|_s^2) \leq \mathfrak{c}_2 \|q\|_{m_0}^4 \|q\|_1^2 \|q\|_s^2 \tag{2.91}$$

where $\mathfrak{c}_2$ is a constant depending on c.

Proof: Follows from (2.86), together with (2.89) and (2.90). □

2.5.1 Proof of Theorem 1.2.2

We now consider the maps $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$, $\Phi^{(4)}$ defined in (2.6), (2.24), (2.36), (2.71). Thanks to lemmas 3.5.1, 2.4.4 2.5.6, we have that there exists a constant δ depending on c only, such that the transformation $\Phi = \Phi^{(1)} \circ \Phi^{(2)} \circ \Phi^{(3)} \circ \Phi^{(4)}$ is well defined

$$\Phi : B_\delta(H_0^{m_0}(\mathbb{T}^n, c.c.)) \cap H_0^s(\mathbb{T}^n, c.c.) \to H_0^s(\mathbb{T}^n, \mathbb{R}) \times H_0^{s-1}(\mathbb{T}^n, \mathbb{R})$$

and invertible for all $s \geq m_0$. In particular we have:

1 For all pairs of zero mean real functions $(u, v) \in H_0^{s+\frac{1}{2}}(\mathbb{T}^n, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$ satisfying

$$\|u\|_{m_0+\frac{1}{2}} + \|v\|_{m_0-\frac{1}{2}} \leq \delta, \tag{2.92}$$

there exists a unique pair $(q, \bar{q}) = (q, \bar{q}) \in H_0^s(\mathbb{T}^d, \text{c.c.})$ such that $(u, v) = \Phi(q, \bar{q})$. Moreover, $(q, \bar{q}) = \Phi^{-1}(u, v)$ satisfies the estimate

$$\|w\|_s \leq \mathfrak{C}_0(\|u\|_{s+\frac{1}{2}} + \|v\|_{s-\frac{1}{2}}). \tag{2.93}$$

On the other hand, if $w \in H_0^s(\mathbb{T}^d, \mathbb{C})$ satisfies

$$\|w\|_{m_0} \leq \delta_0,$$

then $(u, v) = \Phi(w, \overline{w}) \in H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})$ is a pair of zero mean real functions satisfying

$$\|u\|_{s+\frac{1}{2}} + \|v\|_{s-\frac{1}{2}} \leq \mathfrak{C}_0 \|w\|_s.$$

2 For every $s \geq m_0$, and every solution of (2.1)

$$u \in C^0([0, T], H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})) \cap C^1([0, T], H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}))$$

on a given time interval $[0, T]$ satisfying

$$\max_{t\in[0,T]} \left(\|u(t)\|_{m_0+\frac{1}{2}} + \|\partial_t u(t)\|_{m_0-\frac{1}{2}} \right) \le \delta_0,$$

the corresponding pair

$$(q, \bar{q}) := \Phi^{-1}(u, u_t) \in C^0([0, T], H_0^s(\mathbb{T}^d, \text{c.c.}))$$

is a solution of system

$$\partial_t(q, \bar{q}) = X^{(2)}(q, \bar{q}). \tag{2.94}$$

Moreover it satisfies the bound

$$\max_{t\in[0,T]} \|q(t)\|_s \le \mathfrak{C}_0 \max_{t\in[0,T]} \left(\|u(t)\|_{s+\frac{1}{2}} + \|\partial_t u(t)\|_{s-\frac{1}{2}} \right).$$

Similarly, if $(w, \overline{w}) \in C^0([0, T], H_0^s(\mathbb{T}^d, \text{c.c.}))$ is a solution of system (2.94) and satisfies

$$\max_{t\in[0,T]} \|q(t)\|_{m_0} \le \delta,$$

then the corresponding pair

$$(u, v) := \Phi(q, \bar{q}) \in C^0([0, T], H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R})) \times C^0([0, T], H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}))$$

are of the form

$$v = \partial_t u, \quad u \in C^0([0, T], H_0^{s+\frac{1}{2}}(\mathbb{T}^d)) \cap C^1([0, T], H_0^{s-\frac{1}{2}}(\mathbb{T}^d)).$$

Moreover u is a solution of equation (2.1) and

$$\max_{t\in[0,T]} \left(\|u(t)\|_{s+\frac{1}{2}} + \|\partial_t u(t)\|_{s-\frac{1}{2}} \right) \le \mathfrak{C}_0 \max_{t\in[0,T]} \|q(t)\|_s.$$

*P*roof of Theorem 1.2.2
We proceed via bootstrap argument: thanks to the classical results of local well-posedness for the Kirchhoff Equation (see [48] or [1]) there exists a solution u of equation (2.1) with initial data $(u_0,\, v_0)$ satisfying (2.92).
This implies the local existence and uniqueness for system (2.94) with initial data $(q_0, \bar{q}_0) := \Phi^{-1}(u_0, v_0)$.

Let us fix a pair of initial data (u_0^*, v_0^*) for equation (2.1) such that

$$\varepsilon := \|u_0^*\|_{m_0+\frac{1}{2}} + \|v_0^*\|_{m_0-\frac{1}{2}} \le \varepsilon_0 := \frac{\delta}{2\mathfrak{C}},$$

then from (2.93) we get the following bound for the new variables $(q_0^*, \bar{q}_0^*)$:

$$\|q_0^*\|_{m_0} \le \frac{\delta}{2}.$$

It follows that the solution $(q(t), \bar{q}(t))$ of the Cauchy problem associated with system (2.94) with initial data $(q_0^*, \bar{q}_0^*)$ is well-defined and unique as long as $(q(t), \bar{q}(t)) \in B_\delta(H_0^{m_0}(\mathbb{T}^n, c.c.))$.
By using the energy estimate (2.91) we have

$$\partial_t \|q\|_{m_0}^2 \le \mathfrak{c}_2 \|q\|_{m_0}^2,$$

and then

$$\|q(t)\|_{m_0} \leq \frac{\|q_0^*\|_{m_0}}{(1 - 3\mathfrak{c}_2\|q_0^*\|_{m_0}^6)^{\frac{1}{6}}},$$

that implies the existence for times of order $T \sim \varepsilon^{-6}$.
The bound (1.14) for $u \in H^{m_0}$ is then implied by (2.92).
If, in addition $(u_0, v_0) \in H^{s+\frac{1}{2}} \times H^{s-\frac{1}{2}}$ for some $s \geq m_0$ then the energy estimate (2.91) together with a Gronwall argument provide us the corresponding bound (1.14). □

Chapter 3

Formal Birkhoff Normal Form

This chapter continues the analysis of the resonances carried out in Chapter 2, using a complementary and more algebraic approach based on the formal Birkhoff normal form outlined in Section (3.1.1). The advantage of this method is that working at a formal level simplifies the computations and avoids several technical issues related to the quasilinear character of the Kirchhoff equation. At the same time, it provides a convenient framework to exploit the symmetries of its nonlinearity.
The main results of this chapter are Theorems 1.2.3 and 1.2.4.
In Section 3.1 we analyze the algebraic structure of Hamiltonian (3.2). In Section 3.1.1 we define the space of formal polynomials $\mathcal{F}$ and recall the formal Birkhoff normal form Theorem 3.1.7 together with Definition 3.1.6, of **formal action preserving** terms .
In Section 3.2 we perform two step of formal BNF putting Hamiltonian (3.2) in normal form up to order eight. We then observe that the resonant terms of degree 4 and 6 are formally action preserving (Lemmas 3.2.1 and 3.2.4).
Finally, in Subsection 3.2.4, we explain the code we used to implement in Wolfram Alpha the proof of Theorem 1.2.3. In Section 3.3 we apply the same method to the generic Kirchhoff Hamiltonian (3.45) and prove Theorem 1.2.4. Finally in Section 3.5 we study the qualitative dynamics of Hamiltonian (3.2) restricted to a finite number of sites (two).

3.1 Algebraic Structure

In the previous chapter, we have seen that, in complex coordinates, the Kirchhoff equation takes the formulation

$$\begin{cases} \partial_t z = -i\Lambda z - \frac{i}{2}\left(\frac{1}{\left(1+\frac{c}{2}\langle\Lambda(z+\bar z), z+\bar z\rangle\right)^2} - 1\right)(\Lambda z + \Lambda \bar z) \\ \partial_t \bar z = i\Lambda \bar z + \frac{i}{2}\left(\frac{1}{\left(1+\frac{c}{2}\langle\Lambda(z+\bar z), z+\bar z\rangle\right)^2} - 1\right)(\Lambda z + \Lambda \bar z)\,. \end{cases} \tag{3.1}$$

Where $\Lambda = |D_x|$ is the Fourier multiplier for which

$$\Lambda e^{ik\cdot x} = |k| e^{ik\cdot x}, \quad \text{for } k \in \mathbb{Z}^d.$$

This is moreover an Hamiltonian partial differential equation, in the sense that, once considered the Hamiltonian function

$$\begin{aligned} H \quad &: \quad H^{\frac{1}{2}}(\mathbb{T}^d) \longrightarrow \mathbb{R} \\ H(z,\bar z) \quad &:= \quad \langle \Lambda z, \bar z\rangle - \frac{1}{2c}\left[\frac{\left(\frac{c}{2}\langle \Lambda(z+\bar z), z+\bar z\rangle\right)^2}{1+\frac{c}{2}\langle \Lambda(z+\bar z), z+\bar z\rangle}\right], \end{aligned} \tag{3.2}$$

system (3.1) can be written in the following form;

$$\begin{cases} \partial_t z = -i\nabla_{\bar z} H(z,\bar z) \\ \partial_t \bar z = i\nabla_z H(z,\bar z) \end{cases}.$$

Remark 3.1.1 *Since the aim of this chapter is to show that the 8-order part of the Hamiltonian $\mathcal{H}$ cannot be Birkhoff integrable, it is sufficient to restrict ourselves to the one-dimensional case $\mathbb{T}$: in fact, thanks to the Fourier-support invariance of the Kirchhoff Type equation enlightened in the previous chapter, any solution of* (3.1) *whose initial data depend only on a single spacial variable will, as time evolves, remain confined to the space of functions of that same single variable.*

Let us take a closer look to the quantities involved in H: we have that

$$\langle \Lambda z, \bar z\rangle = \sum_{j\in\mathbb{Z}} |j||z_j|^2, \qquad \langle \Lambda(z+\bar z), z+\bar z\rangle = \sum_{j\in\mathbb{Z}} |j|(|z_j|^2 + z_j z_{-j} + \bar z_j \bar z_{-j} + |z_{-j}|^2).$$

It appears a natural choice to set

$$I_j := |z_j|^2 \quad \text{and} \quad W_j := z_j z_{-j}. \tag{3.3}$$

We then have

$$\langle \Lambda z, \bar z\rangle = \sum_{j\in\mathbb{Z}} |j| I_j \quad \text{and} \quad \langle \Lambda(z+\bar z), z+\bar z\rangle = \sum_{j\in\mathbb{Z}} |j|(I_j + I_{-j} + W_j + \bar W_j).$$

Now, noting that $W_j = W_{-j}$, we can write

$$\begin{aligned} \langle \Lambda z, \bar z\rangle \quad &= \quad \sum_{j\in\mathbb{Z}} |j| I_j = \sum_{j\in\mathbb{N}} j(I_j + I_{-j}) \\ \langle \Lambda(z+\bar z), z+\bar z\rangle \quad &= \quad \sum_{j\in\mathbb{Z}} |j|(2I_j + W_j + \bar W_j) = 2\sum_{j\in\mathbb{N}} j(I_j + I_{-j} + W_j + \bar W_j). \end{aligned}$$

Finally, by defining

$$H_2 := \sum_{j\in\mathbb{N}} |j|(I_j + I_{-j}) \quad \text{and} \quad P := \sum_{j\in\mathbb{N}} |j|(W_j + \bar W_j), \tag{3.4}$$

the Hamiltonian (3.2) become

$$H = H_2 - \frac{1}{2c}\left[\frac{c^2\cdot(H_2+P)^2}{1+c\cdot(H_2+P)}\right]. \tag{3.5}$$

Remark 3.1.2 *Let us note that the quantities I and W are not actual variables, they are only a formal and more convenient way to group the terms involved in the Kirchhoff non-linearity.*

The Hamiltonian (3.5) can be written in the more compact way

$$H(z,\bar{z}) = H_2 + \mathcal{P}\left(c \cdot (H_2 + P)\right). \tag{3.6}$$

with $\mathcal{P}(y) = -\frac{1}{2c}\frac{y^2}{1+y}$.
We now analyze the nonlinear part $\mathcal{P}\left(c \cdot (H_2+P)\right)$:
in order to separate the different order of homogeneity we perform a Taylor series expansion on $\mathcal{F}$, obtaining

$$\begin{aligned} H &= H_2 + H_4 + H_6 + \ldots + H_{2n} + h.o.t., \\ H_{2n} &= \frac{(-1)^{n+1}}{2} c^n \left(H_2 + P\right)^n, \quad n \geq 2. \end{aligned} \tag{3.7}$$

By (3.7) we have that the Hamiltonian (3.5) can be entirely written in terms of the variables $I = \{|z_j|^2\}_{j\in\mathbb{Z}}$ and $W = \{z_j z_{-j}\}_{j\in\mathbb{Z}}$.
In particular, one has that

$$(H_2+P)^n = \sum_{\substack{a,b\in\mathbb{N}^{\mathbb{N}}_f, m\in\mathbb{N}^{\mathbb{Z}}_f \\ |a|+|b|+|m|=n \\ \mathrm{supp}(a)\cap\mathrm{supp}(b)=\emptyset}} \binom{n}{a,\, b,\, m} \prod_{i\in\mathrm{supp}(a)} |j_{a_i}| W^{a_i}_{j_{a_i}} \prod_{i\in\mathrm{supp}(b)} |j_{b_i}| \bar{W}^{b_i}_{j_{b_i}} \prod_{i\in\mathrm{supp}(m)} |j_{m_i}| I^{m_i}_{j_{m_i}},$$

where $\mathbb{N}^{\mathbb{Z}}_f$ is the set of finitely supported sequence on non negative integers whose index is an element of $\mathbb{Z}$ and similarly for $\mathbb{N}^{\mathbb{N}}_f$.
We write moreover $m = (m^-,\, m^+)$ with $m^\pm \in \mathbb{N}^{\mathbb{N}}_f$, in this way we have

$$I^m = \prod_{i\in\mathbb{N}} I_i^{m_i^+} I_{-i}^{m^-_{-i}}.$$

Hence in the I, $W, \bar{W}$ variables we have that, to a formal level, Hamiltonian (3.5) can be written as

$$H(I,W) = \sum_{j\in\mathbb{N}} j(I_j + I_{-j}) + \mathcal{P}(I,W),$$

with

$$\mathcal{P}(I,W) = \sum_{n\geq 2} \frac{(-1)^{n+1}}{2} c^2 \sum_{\substack{a,b\in\mathbb{N}^{\mathbb{N}}_f, m\in\mathbb{N}^{\mathbb{Z}}_f \\ |a|+|b|+|m|=n \\ \mathrm{supp}(a)\cap\mathrm{supp}(b)=\emptyset}} \binom{n}{a,\, b,\, m} \prod_{i\in\mathrm{supp}(a)} |j_{a_i}| W^{a_i}_{j_{a_i}} \prod_{i\in\mathrm{supp}(b)} |j_{b_i}| \bar{W}^{b_i}_{j_{b_i}} \prod_{i\in\mathrm{supp}(m)} |j_{m_i}| I^{m_i}_{j_{m_i}}.$$

Th Hamiltonian H is then a formal sum of monomials with the form $I^m W^a \bar{W}^b$ for $a, b \in \mathbb{N}^{\mathbb{N}}_f, m \in \mathbb{N}^{\mathbb{Z}}_f$, $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$.
By keeping in mind the definitions of the variables I and W in (3.3), we have

$$I^m W^a \bar{W}^b = z^{\alpha(m,a,b)} \bar{z}^{\beta(m,a,b)},$$

where $\alpha(m,a,b)$, $\beta(m,a,b) \in \mathbb{N}^{\mathbb{Z}}_f$ are the solutions of

$$\begin{aligned} &\alpha_k = m^+_k + a_k, \quad \alpha_{-k} = m^-_k + a_k, \quad \beta_k = m^+_k + b_k, \quad \beta_{-k} = m^-_k + b_k,\ k \neq 0 \\ &\alpha_0 = \beta_0 = 0. \end{aligned} \tag{3.8}$$

We can hence write H as a sum of monomials of the form $z^\alpha \bar{z}^\beta$, namely

$$H = \sum_{\alpha,\beta \in \mathcal{M}_{\mathbf{kirk}}} H_{\alpha,\beta} z^\alpha \bar{z}^\beta, \tag{3.9}$$

where

$$\mathcal{M}_{\mathbf{kirk}} := \big\{\alpha(m,a,b),\ \beta(m,a,b) \in (3.8) :\ a, b \in \mathbb{N}_f^{\mathbb{N}}, m \in \mathbb{N}_f^{\mathbb{Z}},\ \mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset\big\}. \tag{3.10}$$

Remark 3.1.3 (Symmetries of H**)** *Let us note that Hamiltonian* (3.9) *has some important symmetries:*

(1) $H_{\alpha,\beta} = \bar{H}_{\beta,\alpha}$;

(2) $H_{\alpha,\beta} = 0$, *if* $\sum_{i\in\mathbb{Z}} i(\alpha_i - \beta_i) = 0$;

(3) $\alpha_i - \beta_i = \alpha_{-i} - \beta_{-i}$;

(4) H *is invariant under the change* $|u_k|^2 \longleftrightarrow |u_{-k}|^2$.

Remark 3.1.4 *Since Hamiltonian* (3.5) *is invariant under the change* $|u_k|^2 \longleftrightarrow |u_{-k}|^2$ *or, equivalently, in the* I, W *variables,* $I_k longleftrightarrow I_{-k}$, *if we write the index* $m \in \mathbb{N}_f^{\mathbb{Z}}$ *associated to the variable* I *as* $m = (m^+, m^-)$, *with* $m^\pm \in \mathbb{N}_f^{\mathbb{N}}$, *we have the identifications between the coefficients*

$$H_{m^+,m^-,a,b} = H_{m^-,m^+,a,b}. \tag{3.11}$$

3.1.1 Formal Birkhoff Normal Form on Sequence Spaces

This section is devoted to give all the basic definitions and properties needed to perform the Birkhoff normal form.
We will set our problem on a space of complex sequences based on $H_0^s(\mathbb{T}, \mathbb{C})$, namely we consider the weighted space

$$\mathtt{h}_s = \big\{z \in \ell^2(\mathbb{Z}, \mathbb{C}) :\ |z|_s^2 := \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2s} |z_j|^2 < \infty\big\}.$$

Within this theoretical framework, we need a class of objects that model the Hamiltonian (3.2), together with the properties enlightened in Remark 3.1.3. This is the sense of the following definition:

Definition 3.1.1 ***(Formal Power Series)***
We consider the space $\mathcal{F}$ *of Formal Power Series expansions in the variable* $u \in \mathbb{C}^{\mathbb{Z}}$:

$$H(z) = \sum_{\alpha,\, \beta \in \mathbb{N}_f^{\mathbb{Z}}} H_{\alpha,\beta} z^\alpha \bar{z}^\beta, \quad z^\alpha := \prod_{j\in\mathbb{Z}} z_j^{\alpha_j},$$

such that:

1. $H_{0,0} = H_{\mathbf{e}_0,0} = H_{0,\mathbf{e}_0} = 0$;

2. **Reality Condition**

$$H_{\alpha,\beta} = \bar{H}_{\beta,\alpha}; \tag{3.12}$$

3. **Momentum Conservation**

$$H_{\alpha,\beta} = 0 \quad \textit{if} \quad \pi(\alpha,\beta) := \sum_{j\in\mathbb{Z}} j(\alpha_j - \beta_j) \neq 0.$$

If we consider the subspace $\mathcal{F}^d \subset \mathcal{F}$ of homogeneous polynomials with degree d, we have the direct sum decomposition

$$\mathcal{F} = \oplus_{d\in\mathbb{N}} \mathcal{F}^d.$$

This for every element $H \in \mathcal{F}$, induces the decomposition on $H = \sum_{d\in\mathbb{N}} H^{(d)}$, where

$$\mathcal{F}^d \ni H^{(d)} := \sum_{|\alpha|+|\beta|=d} H_{\alpha,\beta} u^\alpha \bar{u}^\beta$$

is the homogeneous component of H with degree d.
Finally we define the spaces $\mathcal{F}^{\le n} = \oplus_{d\le n} \mathcal{F}^d$ and $\mathcal{F}^{>n}$whose elements are the polynomials of degree respectively $\le n$, $> n$ and have the form

$$H = \sum_{d\le n} H^d = \sum_{|\alpha|+|\beta|<n} H_{\alpha,\beta} u^\alpha \bar{u}^\beta \quad \text{and} \quad H = \sum_{d>n} H^d = \sum_{|\alpha|+|\beta|\ge n} H_{\alpha,\beta} u^\alpha \bar{u}^\beta.$$

Since no convergence hypotheses are required in definition 3.1.1, the elements of $\mathcal{F}^d$ are only formal homogeneous polynomials.
However if we restrict to the space of truncated monomials, $z^\alpha \bar{z}^\beta$ such that $|\alpha_j| + |\beta_j| = 0$ for $j > N$, then we have the classical space of polynomials, that can be endowed with the standard symplectic structure $i \sum_{j\le N} dz_j \wedge d\bar{z}_j$.
This structure induces the usual Poisson bracket

$$\begin{aligned}\{F, G\} &= i \sum_{j\le N} \left(\frac{\partial F}{\partial \bar{z}_j} \frac{\partial G}{\partial z_j} - \frac{\partial F}{\partial z_j} \frac{\partial F}{\partial \bar{z}_j} \right) \\ &= i \sum_{\alpha^{(i)},\beta^{(i)} \in \mathbb{N}^{\mathbb{Z}}_f} F_{\alpha^{(1)},\beta^{(1)}} G_{\alpha^{(2)},\beta^{(2)}} \sum_{j\le N} \big(\beta_j^{(1)} \alpha_j^{(2)} - \alpha_j^{(1)} \beta_j^{(2)}\big) u^{\alpha^{(1)}+\alpha^{(2)}-\mathbf{e}_j} \bar{z}^{\beta^{(1)}+\beta^{(2)}-\mathbf{e}_j},\end{aligned}$$

where $\mathbf{e}_j$, $j \in \mathbb{Z}$ is the vector whose entries are zero except for the $j - th$ that is 1.
The following result shows that this structure extends trivially to the space $\mathcal{F}$.

Theorem 3.1.2 ([85], Proposition 2.5)
Given F, $G \in \mathcal{F}$ the Poisson Bracket

$$\{F, G\} := i \sum_{\alpha^{(i)},\beta^{(i)} \in \mathbb{N}^{\mathbb{Z}}_f} F_{\alpha^{(1)},\beta^{(1)}} G_{\alpha^{(2)},\beta^{(2)}} \sum_{j\in\mathbb{Z}} \big(\beta_j^{(1)} \alpha_j^{(2)} - \alpha_j^{(1)} \beta_j^{(2)}\big) u^{\alpha^{(1)}+\alpha^{(2)}-\mathbf{e}_j} u^{\beta^{(1)}+\beta^{(2)}-\mathbf{e}_j}, \tag{3.13}$$

is well defined and endows $\mathcal{F}$ with a Poisson algebra structure which is a filtered Lie algebra w.r.t. to the degree.

Since the Hamiltonian (3.5) is of the form (3.9), it is not restrictive to set our problem in the following sub-space of the formal power series:

Definition 3.1.3 (Kirchhoff-Type Power Series)
An formal power series H is Kirchhoff-type if (3.11) *holds and*

$$H_{\alpha,\beta} = 0 \textit{ if } \alpha,\, \beta \notin \mathcal{M}_{\mathbf{kirk}},$$

where $\mathcal{M}_{\mathbf{kirk}}$ is defined in (3.10).
We denote by $\mathcal{F}_{\mathbf{kir}}$ sub-space of formal power series of Kirchhoff-type.

As we have seen in the previous section, the elements of $\mathcal{F}_{\mathbf{kir}}$ have an equivalent formal writing in terms of monomials of the form $I^m W^a \bar{W}^b$, for $m \in \mathbb{N}_f^{\mathbb{Z}}$, $a, b \in \mathbb{N}_f^{\mathbb{N}}$ such that $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$.

We now want to focus on how the Poisson bracket acts on the elements of $\mathcal{F}_{\mathbf{kir}}$ and on the principal quantities involved in their formal writing, namely I_j, W_j, H_2 and P.

Keeping in mind that

$$\begin{cases} \bar{I}_j = I_j \\ W_{-j} = W_j \end{cases},$$

we have the following commutation rules:

$$\begin{aligned}
&\mathbf{i)} \quad \{I_j, I_k\} = \{W_j, W_k\} = 0 \quad \forall j,\, k \in \mathbb{Z};\\
&\mathbf{ii)} \quad \{W_j, \bar{W}_k\} = -i\delta(|j|, |k|)(I_j + I_{-j}) \quad \forall j,\, k \in \mathbb{Z};\\
&\mathbf{iii)} \quad \{I_j, W_k\} = i\delta(|j|, |k|)W_j \quad \forall j,\, k \in \mathbb{Z};\\
&\mathbf{iv)} \quad \{I_j, \bar{W}_k\} = -i\delta(|j|, |k|)\bar{W}_j \quad \forall j,\, k \in \mathbb{Z};\\
&\mathbf{v)} \quad \{H_2, I_j\} = 0 \quad \forall j \in \mathbb{Z};\\
&\mathbf{vi)} \quad \{H_2, W_j\} = 2i|j|W_j \quad \forall j \in \mathbb{Z};\\
&\mathbf{vii)} \quad \{H_2, \bar{W}_j\} = -2i|j|\bar{W}_j \quad \forall j \in \mathbb{Z};\\
&\mathbf{viii)} \quad \{H_2, P\} = 2i \sum_{j \in \mathbb{N}} j^2 (W_j - \bar{W}_j).
\end{aligned} \tag{3.14}$$

From the commutation rules in (3.14), it follows that the Poisson bracket between monomials in the variables I, W, $\bar{W}$ is again the sum of monomials in the same variables.
We then have that the space $\mathcal{F}_{\mathbf{kir}}$ is closed under the action of the Poisson bracket.

Lemma 3.1.1 *The sub-space $\mathcal{F}_{\mathbf{kir}}$ equipped with the Poisson bracket defined in* (3.13) *is a Poisson sub-algebra of the space of formal power series $\mathcal{F}$.*

Similarly to the Poisson bracket, the space of formal power series $\mathcal{F}$ inherits from the spaces truncated monomials

$$\Big\{ H \in \mathcal{F} : \, H = \sum_{\substack{\alpha,\beta \in \mathbb{N}_f^{\mathbb{Z}} \\ \alpha_j, \beta_j = 0, \text{ if } j \geq N}} H_{\alpha,\beta} z^\alpha \bar{z}^\beta \Big\},$$

the Lie exponential operator (3.15).
Unlike the analytic case, where one must satisfy strict convergence criteria for the flow of a vector field, the formal context allows for a purely algebraic treatment.

Lemma 3.1.2 ([85], Corollary 2.9) *Given $G \in \mathcal{F}^{\geq n}$ with $n \geq 3$, we defined*

$$\mathrm{ad}_G := \{\cdot, G\}, \quad e^{\{\cdot, G\}} := \sum_{k \geq 0} \frac{\mathrm{ad}_G^k}{k!}. \tag{3.15}$$

We have that $e^{\{\cdot,G\}}$ is a linear operator $\mathcal{F}^n \to \mathcal{F}^n$, moreover we have that ad_G and $e^{\{\cdot,G\}} - \mathrm{Id}$ are operators of order $d-2$ in the sense that

$$\mathrm{ad}_G,\ e^{\{\cdot,G\}} - \mathrm{Id} : \mathcal{F}^{\geq h} \to \mathcal{F}^{\geq h+d-2}.$$

Similarly for any sequence $\{b_k\}_{k\geq 0}$ one has that

$$\sum_{k \geq n} b_k \mathrm{ad}_G^k : \mathcal{F}^{\geq h} \to \mathcal{F}^{\geq h + d(n-2)}$$

Definition 3.1.4 *(***Formal Symplectic Change)**
For $G \in \mathcal{F}^{\geq 3}$, the operator $e^{\{\cdot,G\}}$ defined in (3.15) *generates an action on the space $\mathcal{F}$, which is called formal symplectic change of variables on the space $\mathcal{F}$.*

Definition 3.1.5 (Formal Integral of Motion)
An element $F \in \mathcal{F}$ is a formal integral of motion for $H \in \mathcal{F}^{\geq 2}$, if

$$\{H, F\} = 0.$$

Definition 3.1.6 (Formally Action Preserving Terms)
An element $F \in \mathcal{F}$ is said to be Formally Action Preserving if

$$\{I_j, F\} = 0, \quad \forall j \in \mathbb{Z},$$

where $I_j = |z_j|^2$.

The writing of the elements of $\mathcal{F}_{\mathbf{kir}}$ in terms of the formal variables I, W defined in (3.3), is then an useful tool to separate the formal action preserving terms from the others.
Thanks to Remark 2.3.2 we know that the quantities $|z_j^2| - |z_{-j}|^2$, $j \in \mathbb{N}$ are conserved for the actual dynamics of the Kirchhoff equation, the same result holds also for $I_j - I_{-j}$ in terms of formal dynamic:

Lemma 3.1.3 *The quantities $\{I_j - I_{-j}\}_{j\in\mathbb{Z}}$ are formal integral of motions for H in* (3.2).

Proof: We have

$$\begin{aligned}
\{M_j, H\} &\overset{(3.6)}{:=} \{M_j, H_2 + \mathcal{P}(c \cdot (H_2 + P))\} \\
&= \{M_j, H_2\} + \{M_j, \mathcal{P}(c \cdot (H_2 + P))\} \\
&= \{M_j, \mathcal{P}(c \cdot (H_2 + P))\} \\
&= c\mathcal{P}'(c \cdot (H_2 + P))\{M_j, H_2 + P\} = 0
\end{aligned}$$

since

$$\{M_j, H_2 + P\} = 0.$$

□

In order to perform the Birkhoff normal form algorithm, one must be able to invert the operator ad_{H_2} on some subspace of $\mathcal{F}$. To this purpose, we define

$$\mathcal{K} := \ker(\mathrm{ad}_{H_2}), \quad \mathcal{R} := \mathrm{Rg}(\mathrm{ad}_{H_2}).$$

Since $H_2 = \sum_{j\in\mathbb{Z}} |j||z_j|^2$, we have that ad_{H_2} acts on the monomials of the form $z^\alpha \bar{z}^\beta$ $\alpha,\ \beta \in \mathbb{N}^{\mathbb{Z}}_f$ as a diagonal operator of the form

$$\mathrm{ad}_{H_2}(z^\alpha \bar{z}^\beta) = i\omega\cdot(\alpha-\beta) z^\alpha \bar{z}^\beta,$$

with $\omega \in \mathbb{N}^{\mathbb{Z}}$, vector of components $\omega_j = |j|$.
In particular,

$$\begin{aligned}
\mathcal{R} &= \left\{ R\in\mathcal{F}:\ R(z) = \sum_{\substack{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f:\\ \omega\cdot(\alpha-\beta)\neq 0}} R_{\alpha,\beta} z^\alpha \bar{z}^\beta \right\};\\
\mathcal{K} &= \left\{ Z\in\mathcal{F}:\ Z(z) = \sum_{\substack{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f:\\ \omega\cdot(\alpha-\beta)= 0}} R_{\alpha,\beta} z^\alpha \bar{z}^\beta \right\}.
\end{aligned}$$

It is clear that we can decompose $\mathcal{F} = \mathcal{K}\oplus\mathcal{R}$.
We define moreover

$$\mathcal{K}^d := \Pi_{\mathcal{F}^d}(\mathcal{K}),\quad \mathcal{K}^{<d} := \Pi_{\mathcal{F}^{<d}}(\mathcal{K}),\quad \mathcal{K}^{\geq d} := \Pi_{\mathcal{F}^{\geq d}}(\mathcal{K})$$

and similarly for $\mathcal{R}$.
Finally we define the projectors on $\mathcal{K}$ and $\mathcal{R}$ as

$$\Pi^{\mathcal{K}} H := \sum_{\substack{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f\\ \omega(\alpha-\beta)=0}} H_{\alpha,\beta} u^\alpha \bar{u}^\beta,\quad \Pi^{\mathcal{R}} H := H - \Pi^{\mathcal{K}} H.$$

Lemma 3.1.4 ([85], Lemma 2.13)
The operator ad_{H_2} *is invertible in* $\mathcal{R}^d$ *for all d.*

On the subspace $\mathcal{F}_{\mathbf{kir}}$ the Ker and Range of ad_{H_2} have a similar writing in terms of the variables I, W. Indeed, by setting $\tilde{\omega}$ as the vector of $\mathbb{R}^{\mathbb{N}}$ with component $\tilde{\omega}_j = |j|$ and using the commutation rules (3.14), we have that ad_{H_2} acts on the monomials $I^m W^a \bar{W}^b$ as

$$\mathrm{ad}_{H_2}(I^m W^a \bar{W}^b) := 2i\tilde{\omega}\cdot(a-b) I^m W^a \bar{W}^b.$$

We then can write

$$\begin{aligned}
\mathcal{R}_{\mathbf{kir}} &:= \Pi_{\mathcal{F}_{\mathbf{kir}}}\Big(\mathrm{Rg}(\mathrm{ad}_{H_2})\Big) = \left\{ R\in\mathcal{F}:\ R(I,W) = \sum_{\substack{a,b\in\mathbb{N}^{\mathbb{N}}_f, m\in\mathbb{N}^{\mathbb{Z}}_f:\\ \tilde{\omega}\cdot(a-b)\neq 0}} Z_{m,a,b} I^m W^a \bar{W}^b \right\};\\
\mathcal{K}_{\mathbf{kir}} &:= \Pi_{\mathcal{F}_{\mathbf{kir}}}\Big(\ker(\mathrm{ad}_{H_2})\Big) = \left\{ Z\in\mathcal{F}:\ Z(I,W) = \sum_{\substack{a,b\in\mathbb{N}^{\mathbb{N}}_f, m\in\mathbb{N}^{\mathbb{Z}}_f:\\ \tilde{\omega}\cdot(a-b)= 0}} Z_{m,a,b} I^m W^a \bar{W}^b \right\}.
\end{aligned} \tag{3.16}$$

The corresponding projectors are denoted by $\Pi_{\mathcal{K}_{\mathbf{kir}}}$ and $\Pi_{\mathcal{R}_{\mathbf{kir}}}$, moreover we set

$$\mathcal{K}^d_{\mathbf{kir}} := \Pi_{\mathcal{F}^d_{\mathbf{kir}}}(\mathcal{K}_{\mathbf{kir}}), \quad \mathcal{K}^{<d}_{\mathbf{kir}} := \Pi_{\mathcal{F}^{<d}_{\mathbf{kir}}}(\mathcal{K}_{\mathbf{kir}}), \quad \mathcal{K}^{\geq d}_{\mathbf{kir}} := \Pi_{\mathcal{F}^{\geq d}_{\mathbf{kir}}}(\mathcal{K}_{\mathbf{kir}})$$

and similarly for $\mathcal{R}_{\mathbf{kir}}$.
Let us note that the same result of Lemma 3.1.4 holds also for the subspace $\mathcal{F}_{\mathbf{kir}}$.
We can now formulate the main result concerning the Birkhoff normal form on formal polynomials, that we will use throughout all the chapter:

Theorem 3.1.7 ([85], Proposition 2.18)
Let us consider a formal Hamiltonian of the form

$$H = D_\omega + \mathcal{N},$$

where D_ω is a diagonal operator of eigenvalues ω, and $\mathcal{N} \in \mathcal{F}^{\geq d}$ with $d \geq 3$. For every $k \geq d$, there exists $S \in \mathcal{F}^{\geq d}$ such that

$$e^{\{\cdot, S\}}(H) = D_\omega + \sum_{j=d}^{k} Z_j + R_{k+1},$$

with $Z_j \in \mathcal{K}^j$ and $R_{k+1} \in \mathcal{F}^{\geq k+1}$.

Remark 3.1.5 *Keeping in mind Lemma 3.1.1, it is clear that the result of Theorem 3.1.1 transfers automatically to the spaces $\mathcal{F}_{\mathbf{kir}}$ by substituting $\mathcal{R}_{\mathbf{kir}}$ and $\mathcal{K}_{\mathbf{kir}}$ in place of $\mathcal{R}$ and $\mathcal{K}$.*

In our analysis, we perform three steps of formal Birkhoff normal form proving the existence of terms of degree 8 that are not formally action preserving.
Moreover, since each finite subspace is invariant, when restricted to one of these sub-spaces, these terms became non action preserving, not only to a formal level.

3.2 Formal BNF for the Kirchhoff

In this section we apply the Formal Birkhoff normal form Theorem 3.1.1 to Hamiltonian $H \in \mathcal{F}_{\mathbf{kir}}$ defined in (3.2).
All the computations will be carried out in the space of the formal polynomial of Kirchhoff type. For this reason, all the quantity involved will be written in the formal variables I, W defined in (3.3).

3.2.1 First Step of BNF

We want to apply Theorem 3.1.1 in order to put Hamiltonian H, defined in (3.5), in formal Birkhoff normal Form up to homogeneity 4.

Notation.
In what follows, the coordinates given by the Normal form we be designated with the symbol w, however, we adopt the convention of denoting the quantities I, W, P, H_2 defined in (3.3) and (3.4), with the same symbols in both the original and the Birkhoff normal form coordinates. This slight abuse of notation will be maintained throughout, as long as no ambiguity arises.

The main result we are going to prove is:

Lemma 3.2.1 *The formal Birkhoff normal form, up to homogeneity 4, of H is*

$$H^{(4)} := H_2 + Z_4 + R_{\geq 6}, \tag{3.17}$$

where $Z_4 \in \mathcal{K}^4_{\mathbf{kir}}$ has expression (3.20) *and is formally action preserving and $R_{\geq 6} \in \mathcal{F}^{\geq 6}_{\mathbf{kir}}$ has expression* (3.18).

We then need to construct a formal symplectic change of variable $e^{\{\cdot, F_4\}}$, generated by $F_4 \in \mathcal{F}^4_{\mathbf{kir}}$, such that $H^{(4)} := e^{\{\cdot, F_4\}}(H)$.
By Lemma 3.1.2 the full expression of $H^{(4)}$ is

$$\begin{aligned} H^{(4)} &= H_2 + H_4 + \{H_2, F_4\} + \{H_4, F_4\} + \frac{1}{2}\{\{H_2, F_4\}, F_4\} + H_6 \\ &\quad + \sum_{k \geq 3} \frac{\mathrm{ad}_{F_4}}{k!}(H_2) + \sum_{k \geq 2} \frac{\mathrm{ad}_{F_4}}{k!}(H_4) + \sum_{k \geq 1} \frac{\mathrm{ad}_{F_4}}{k!}(H_6) + e^{\{\cdot, F_4\}}\left(\sum_{j \geq 4} H_{2n}\right) \\ &= H_2 + \tilde{H}_4 + R_{\geq 6}, \end{aligned}$$

where

$$H_4 + \{H_2, F_4\} \in \mathcal{F}^4_{\mathbf{kir}},$$

while the remaining order 6 part is

$$R_{\geq 6} := H^{(4)} - H_2 - H_4 - \{H_2, F_4\} \in \mathcal{F}^{\geq 6}_{\mathbf{kir}}. \tag{3.18}$$

In order for $H^{(4)}$ to be in normal form, one should have $\{H_2, F_4\} + H_4 \in \mathcal{K}_{\mathbf{kir}}$ and hence, the generating function F_4 must solve the (homological) equation

$$\{H_2, F_4\} + H_4 = Z_4, \tag{3.19}$$

where $Z_4 := \Pi_{\mathcal{K}_{\mathbf{kir}}}(H_4)$.

Lemma 3.2.2 *The resonant degree-4 part, has the expression*

$$Z_4 = -\frac{c}{2} H_2^2 - c \sum_{j \in \mathbb{N}} j^2 I_j I_{-j}, \tag{3.20}$$

hence it is formally action preserving.

Proof: Le us take a closer look to H_4 in order to analyze the structure of $\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_4)$:
we have

$$H_4 = -\frac{c}{2} H_2^2 - c H_2 P - \frac{c}{2} P^2.$$

Obviously, since $\{H_2, H_2^2\} = 0$, one has $H_2^2 \in \mathcal{K}_{\mathbf{kir}}$, hence

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_2^2) = H_2^2. \tag{3.21}$$

For what concerns the term $H_2 P$ we have that

$$\{H_2, H_2 P\} = H_2 \{H_2, P\} = 2 H_2 \sum_{j \in \mathbb{N}} j^2 (W_j - \bar{W}_j),$$

then $H_2 P \in \mathcal{R}_{\mathbf{kir}}$ and so

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_2 P) = 0. \tag{3.22}$$

Finally

$$P^2 = \sum_{j,k \in \mathbb{N}} jk \left(W_j W_k + \bar{W}_j \bar{W}_k + 2\bar{W}_j W_k \right),$$

moreover, from (3.14) it follows that

$$\{H_2, W_j W_k\} = 2i(j+k) W_j W_k, \quad \{H_2, \bar{W}_j W_k\} = 2i(-j+k)\bar{W}_j W_k,$$

$$\{H_2, \bar{W}_j \bar{W}_k\} = -2i(j+k) W_j W_k, \quad \{H_2, W_j \bar{W}_k\} = 2i(j-k)\bar{W}_j W_k,$$

hence

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) = 2\sum_{j \in \mathbb{N}} j^2 W_j \bar{W}_j = 2\sum_{j \in \mathbb{N}} j^2 I_j I_{-j}. \tag{3.23}$$

We can conclude that the order-4 resonant terms are

$$Z_4 = -\frac{c}{2} H_2^2 - \frac{c}{2} \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2).$$

By putting together (3.21), (3.22) and (3.23) we obtain the expression (3.20). □

Since we will need the explicit expression of the generating function F_4 in the following steps of BNF, we have to solve homological equation (3.19).
By (3.20) we have that the non-resonant terms of H_4 are:

$$\begin{aligned}
\Pi_{\mathcal{R}_{\mathbf{kir}}}(H_4) &= H_4 - Z_4 \\
&= -cH_2 P - \frac{c}{2} P^2 + \frac{c}{2} \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) \\
&= -cH_2 P - \frac{c}{2} \sum_{j,k \in \mathbb{N}} jk \left(W_j W_k + \bar{W}_j \bar{W}_k \right) - c \sum_{j \neq k} \bar{W}_j W_k.
\end{aligned}$$

We have the following

Lemma 3.2.3 *Then order-4 homological equation* (3.19) *is solved by*

$$F_4 := \frac{icH_2}{2} \sum_{j \in \mathbb{N}} (\bar{W}_j - W_j) + \frac{ic}{4} \sum_{j,k \in \mathbb{N}} \frac{jk}{j+k} (\bar{W}_j \bar{W}_k - W_j W_k) + \frac{ic}{2} \sum_{j \neq k \in \mathbb{N}} \frac{jk}{j-k} \bar{W}_j W_k. \tag{3.24}$$

Proof: We look for a solution of (3.19), namely F_4 of the form $F_4^{(1)} + F_4^{(2)}$ with $F_4^{(1)}$ and $F_4^{(2)}$ such that

$$\begin{aligned}
(i) \quad \{H_2, F_4^{(1)}\} &= cH_2 P, \\
(ii) \quad \{H_2, F_4^{(2)}\} &= \frac{c}{2} P^2 - \frac{c}{2} \Pi_{H_2}(P^2) \\
&= \frac{c}{2} \sum_{j,k \in \mathbb{N}} jk (W_j W_k + \bar{W}_j \bar{W}_k) + c \sum_{j \neq k \in \mathbb{N}} jk \bar{W}_j W_k.
\end{aligned} \tag{3.25}$$

We moreover impose $F_4^{(1)}$ to be of the form cH_2G, with G such that

$$\{H_2, G\} = P. \tag{3.26}$$

Let us note that P is made by terms of the form W, $\bar{W}$, hence, by keeping in mind the commutation rules (3.14), one have that G must be of the form

$$G = \sum_{j\in\mathbb{N}}(\alpha_j W_j + \beta_j \bar{W}_j),$$

with α, j, β_j to be determined.

$$\begin{aligned}\{H_2, G\} &= \left\{H_2, \sum_{j\in\mathbb{N}}(\alpha_j W_j + \beta_j \bar{W}_j)\right\} \\ &= \sum_{j\in\mathbb{N}}\left(\alpha_j\{H_2, W_j\} + \beta_j\{H_2, \bar{W}_j\}\right) \\ &= \sum_{j\in\mathbb{N}} 2i|j|(\alpha_j W_j - \beta_j \bar{W}_j)\end{aligned}$$

It follows that, in order for (3.26) to hold, one must have $\alpha_j = -\dfrac{i}{2}$ and $\beta_j = \dfrac{i}{2}$. We then arrive to

$$G = \frac{i}{2}\sum_{j\in\mathbb{N}}(\bar{W}_j - W_j), \tag{3.27}$$

and hence

$$F_4^{(1)} = \frac{icH_2}{2}\sum_{j\in\mathbb{N}}(\bar{W}_j - W_j). \tag{3.28}$$

Analogous computations led to

$$F_4^{(2)} = \frac{ic}{4}\sum_{j,k\in\mathbb{N}}\frac{jk}{j+k}(\bar{W}_j\bar{W}_k - W_jW_k) + \frac{ic}{2}\sum_{j\neq k\in\mathbb{N}}\frac{jk}{j-k}\bar{W}_jW_k. \tag{3.29}$$

□

Remark 3.2.1 *From the definition of G follows moreover*

$$\{P, G\} = H_2.$$

3.2.2 Second Step of BNF

We continue our analysis with a focus on the terms in $\mathcal{F}^6_{\mathbf{kir}}$. The main result we prove is:

Lemma 3.2.4 *The formal Birkhoff normal form, up to homogeneity 6, of H is*

$$H^{(6)} := H_2 + Z_4 + Z_6 + R_{\geq 8},$$

where $Z_6 \in \mathcal{K}^6_{\mathbf{kir}}$ has expression (3.33) *and is formally action preserving and $R_{\geq 8} \in \mathcal{F}^{\geq 8}_{\mathbf{kir}}$.*

Similarly to the first normal form step, we look for a symplectic map $\Phi^{(6)} = e^{\{\cdot,F_6\}}$ generated by $F_6 \in \mathcal{F}^6_{\mathbf{kir}}$, with the aim of removing all the non resonant terms with homogeneity 6 from

$$H^{(6)} := H^{(4)} \circ \Phi^{(6)}.$$

Since we don't need the exact expression of the generating function F_6, we will focus only on the structure of the resonant terms.
We have that

$$\begin{aligned} H^{(6)} &\overset{(3.17)}{=} H_2 + Z_4 + R_{\geq 6} + \{H_2, F_6\} + \sum_{k\geq 2} \frac{\mathrm{ad}_{F_6}}{k!}(H_2) + \sum_{k\geq 1} \frac{\mathrm{ad}_{F_6}}{k!}(Z_4 + R_{\geq 6}) \\ &\overset{(3.18)}{=} H_2 + Z_4 + \{H_2, F_6\} + \tilde{H}_6 + R_{\geq 8}, \end{aligned} \tag{3.30}$$

with

$$\tilde{H}_6 := H_6 + \frac{1}{2}\{\{H_2, F_4\}, F_4\} + \{H_4, F_4\} \in \mathcal{F}^6_{\mathbf{kir}}$$

and

$$R_{\geq 8} = R_{\geq 6} - H_6 - \frac{1}{2}\{\{H_2, F_4\}, F_4\} + \sum_{k\geq 2} \frac{\mathrm{ad}_{F_6}}{k!}(H_2) + \sum_{k\geq 1} \frac{\mathrm{ad}_{F_6}}{k!}(Z_4 + R_{\geq 6}) \in \mathcal{F}^{\geq 8}_{\mathbf{kir}}, \tag{3.31}$$

Similarly to the previous step, we want $\{H_2, F_6\} + \tilde{H}_6 \in \mathcal{K}_{\mathbf{kir}}$, hence the generating function F_6 must solve the homological equation

$$\{H_2, F_6\} + \tilde{H}_6 = Z_6, \tag{3.32}$$

with

$$Z_6 := \Pi_{\mathcal{K}_{\mathbf{kir}}} \left(\frac{1}{2}\{\{H_2, F_4\}, F_4\} + \{H_4, F_4\} \right).$$

Lemma 3.2.5 *The resonant part with degree-6 has expression*

$$Z_6 = \frac{c^2}{2} H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) + \frac{1}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_4^{(2)}, H_2\}, F_4^{(2)}\}) \tag{3.33}$$

and is formally action preserving.

. Let us note that, using homological equation (3.19) we can write

$$\tilde{H}_6 = H_6 + \frac{1}{2}\{Z_4, F_4\} - \frac{1}{2}\{\{H_2, F_4\}, F_4\}, \tag{3.34}$$

we have to analyze all the terms involved in (3.34):

(1) Analysis of H_6:

Lemma 3.2.6 *The resonant terms of H_6 are given by*

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_6) = \frac{c^2}{2} H_2^3 + \frac{3c^2}{2} H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) + \frac{c^2}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3). \tag{3.35}$$

Proof: We have

$$H_6 = \frac{c^2}{2}H_2^3 + \frac{3c^2}{2}H_2^2 P + \frac{3c^2}{2}H_2 P^2 + \frac{c^2}{2}P^3.$$

Since $\{H_2, H_2^3\} = 0$ hence $\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_2^3) = H_2^3$.
By the rules of commutation in (3.14), we have that $\{H_2, H_2^2 P\} = H_2^2\{H_2, P\}$. Thanks to (3.14-viii), $\{H_2, P\}$ is a range element, then we have $H_2^2 P \in \mathcal{R}_{\mathbf{kir}}$. We can conclude that $\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_2^2 P) = 0$.
Finally, we have that $\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_2 P^2) = H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2)$. □

(2) Analysis of $\{Z_4, F_4\}$:
Let us state the following general result

Lemma 3.2.7 *If $f \in \mathcal{K}$ and $g \in \mathcal{R}$ then $\{f, g\} \in \mathcal{R}$.*

Proof: In fact, being f an element of the range, there must be some function h such that $f = \{H_2, h\}$.
Then, using the Jacobi identities and the fact that g is an element of the Ker, one has

$$\{f, g\} = \{\{H_2, h\}, g\} = \{H_2, \{g, h\}\} + \{h, \{H_2, g\}\} = \{H_2, \{g, h\}\}.$$

□

Since $Z_4 \in \mathcal{K}_{\mathbf{kir}}$ and $F_4 \in \mathcal{R}_{\mathbf{kir}}$ thanks to the Lemma 3.5.4, we automatically have that

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{Z_4, F_4\}) = 0. \tag{3.36}$$

(3) Analysis of $-\frac{1}{2}\{\{H_2, F_4\}, F_4\}$:

Lemma 3.2.8

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}\left(-\frac{1}{2}\{\{H_2, F_4\}, F_4\}\right) = -\frac{c^2}{2}H_2^3 - c^2 H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) + \frac{1}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_4^{(2)}, H_2\}, F_4^{(2)}\}) - \frac{c^2}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3), \tag{3.37}$$

where the term $\frac{1}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_4^{(2)}, H_2\}, F_4^{(2)}\})$ is formally action preserving.

Proof:

$$\begin{aligned} &-\{\{H_2, F_4\}, F_4\} \\ =&\{\{F_4, H_2\}, F_4\} \\ =&\{\{F_4^{(1)}, H_2\}, F_4^{(1)}\} + \{\{F_4^{(2)}, H_2\}, F_4^{(1)}\} + \{\{F_4^{(1)}, H_2\}, F_4^{(2)}\} + \{\{F_4^{(2)}, H_2\}, F_4^{(2)}\} \end{aligned} \tag{3.38}$$

Let us investigate one by one the terms involved in (3.38):

$$
\begin{aligned}
& \{\{F_4^{(1)}, H_2\}, F_4^{(1)}\} && (3.39)\\
\overset{(3.25)}{=}\ & c^2\{-H_2 P, H_2 G\} \\
=\ & -c^2 H_2^2\{P, G\} - c^2 P H_2\{H_2, G\} - c^2 H_2 G\{P, H_2\} \\
\overset{(3.26),(3.2.1)}{=}\ & -c^2 H_2^3 - c^2 P^2 H_2 + 2ic^2 H_2 G \sum_{j\in\mathbb{N}} j^2 (W_j - \bar{W}_j) \\
\overset{(3.27)}{=}\ & -c^2 H_2^3 - c^2 P^2 H_2 + c^2 H_2 \sum_{j\in\mathbb{N}} (W_k - \bar{W}_k) \sum_{j\in\mathbb{N}} j^2 (W_j - \bar{W}_j) \\
=\ & -c^2 H_2^3 - c^2 P^2 H_2 + c^2 H_2 \sum_{j,k\in\mathbb{N}} j^2 \left(W_j W_k + \bar{W}_j \bar{W}_k - \bar{W}_j W_k - W_j \bar{W}_k\right).
\end{aligned}
$$

A quick computation shows that

$$
\begin{aligned}
& \Pi_{\mathcal{K}_{\mathbf{kir}}}\left(H_2 \sum_{j,k\in\mathbb{N}} j^2 \left(W_j W_k + \bar{W}_j \bar{W}_k - \bar{W}_j W_k - W_j \bar{W}_k\right)\right) \\
=\ & H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}\left(\sum_{j,k\in\mathbb{N}} j^2 \left(W_j W_k + \bar{W}_j \bar{W}_k - \bar{W}_j W_k - W_j \bar{W}_k\right)\right) \\
=\ & -2H_2 \sum_{j\in\mathbb{N}} j^2 \bar{W}_j W, \\
\overset{(3.23)}{=}\ & -H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2).
\end{aligned}
$$

Since $\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2 H_2) = H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2)$, one has

$$
\Pi_{\mathcal{K}_{\mathbf{kir}}}\left(\{\{F_4^{(1)}, H_2\}, F_4^{(1)}\}\right) = -c^2 H_2^3 - 2c^2 H_2 \Pi_{H_2}(P^2). \tag{3.40}
$$

$$
\begin{aligned}
\{\{F_4^{(2)}, H_2\}, F_4^{(2)}\} =\ & \{F_4^{(2)}, \{H_2, F_4^{(2)}\}\} \\
\overset{(3.25)}{=}\ & \left\{F_4^{(2)}, \frac{c}{2} \sum_{j,k\in\mathbb{N}} jk(W_j W_k + \bar{W}_j \bar{W}_k) + c \sum_{j\neq k\in\mathbb{N}} jk \bar{W}_j W_k\right\} \\
\overset{(3.29)}{=}\ & \left\{\frac{ic}{4} \sum_{j,k\in\mathbb{N}} \frac{jk}{j+k} (\bar{W}_j \bar{W}_k - W_j W_k), \frac{c}{2} \sum_{j,k\in\mathbb{N}} jk W_j W_k\right\} \\
+\ & \left\{\frac{ic}{4} \sum_{j,k\in\mathbb{N}} \frac{jk}{j+k} (\bar{W}_j \bar{W}_k - W_j W_k), c \sum_{j\neq k\in\mathbb{N}} jk \bar{W}_j W_k\right\} \\
+\ & \left\{\frac{ic}{2} \sum_{j\neq k\in\mathbb{N}} \frac{jk}{j-k} \bar{W}_j W_k, \frac{c}{2} \sum_{j,k\in\mathbb{N}} jk(W_j W_k + \bar{W}_j \bar{W}_k)\right\} \\
+\ & \left\{\frac{ic}{2} \sum_{j\neq k\in\mathbb{N}} \frac{jk}{j-k} \bar{W}_j W_k, c \sum_{j\neq k\in\mathbb{N}} jk \bar{W}_j W_k\right\}.
\end{aligned}
$$

Keeping in mind that

$$\begin{aligned}
\{W_jW_k, \bar{W}_{j'}\bar{W}_{k'}\} &= -i[\delta(j,j')(I_j+I_{-j})W_k\bar{W}_{k'}+\delta(k,k')(I_k+I_{-k})W_j\bar{W}_{j'}\\
&\quad +\delta(j,k')(I_j+I_{-j})W_k\bar{W}_{j'}+\delta(k,j')(I_k+I_{-k})W_j\bar{W}_{k'}]\\
\{W_jW_k, W_{j'}\bar{W}_{k'}\} &= -iW_{j'}\left[\delta(k,k')(I_k+I_{-k})W_j+\delta(j,k')(I_j+I_{-j})W_k\right]\\
\{W_j\bar{W}_k, W_{j'}\bar{W}_{k'}\} &= \delta(j,k')(I_j+I_{-j})\bar{W}_kW_{j'}+\delta(k,j')(I_k+I_{-k})W_j\bar{W}_{k'},
\end{aligned}$$

we have that $\{\{F_4^{(2)},H_2\},F_4^{(2)}\}$ is only made by monomials of the form $I_{j_1}W_{j_2}^{\pm}W_{j_3}^{\pm}$ (where $W^-=\bar{W}$). Since

$$\Pi_{H_2}(I_{j_1}W_{j_2}^{\pm}W_{j_3}^{\pm})=\begin{cases}0 & \text{if } j_2\neq j_3\\ I_{j_1}I_{j_2}I_{-j_2} & \text{if } j_2=j_3,\end{cases}$$

we have that $\Pi_{H_2}(\{\{F_4^{(2)},H_2\})$ depends only on the variables I, hence it is formally action preserving.

It remains to analyze the term $\{\{F_4^{(2)},H_2\},F_4^{(1)}\}+\{\{F_4^{(1)},H_2\},F_4^{(2)}\}$: using the Jacobi identity we get

$$\{\{F_4^{(2)},H_2\},F_4^{(1)}\}+\{\{F_4^{(1)},H_2\},F_4^{(2)}\}=2\{F_4^{(2)},\{H_2,F_4^{(1)}\}\}+\{H_2,\{F_4^{(1)},F_4^{(2)}\}\}.$$

By construction, we have that $\{H_2,\{F_4^{(1)},F_4^{(2)}\}\}\in \text{range}(\varphi_{H_2})$, so we will focus on $\{F_4^{(2)},\{H_2,F_4^{(1)}\}\}$:
from (3.19) and (3.25) we have that

$$\begin{aligned}
\{F_4^{(2)},\{H_2,F_4^{(1)}\}\} &\overset{(3.25),(i)}{=} c\{F_4^{(2)},H_2P\}\\
&= cH_2\{F_4^{(2)},P\}+cP\{F_4^{(2)},H_2\}\\
&\overset{(3.25),(ii)}{=} cH_2\{F_4^{(2)},P\}-\frac{c^2}{2}P^3+\frac{c^2}{2}P\Pi_{H_2}(P^2).
\end{aligned}$$

Let us note that

$$cH_2\{F_4^{(2)},P\},\ \frac{c^2}{2}P\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2)\in\mathcal{R}_{\mathbf{kir}}$$

since they are made by terms of the form $I_{j_1}I_{j_2}W_{j_3}^{\pm}$, $j_i\in\mathbb{N}$, therefore

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{F_4^{(2)},\{H_2,F_4^{(1)}\}\})=-\frac{c^2}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3)+\frac{c^2}{2}P\Pi_{H_2}(P^2).$$

From this, follows

$$\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_4^{(2)},H_2\},F_4^{(1)}\}+\{\{F_4^{(1)},H_2\},F_4^{(2)}\})=-c^2\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3). \tag{3.41}$$

Finally, hanks to (3.40) and (3.41), we have (3.37). □

By collecting what we got in (3.35), (3.36) and Lemma 3.2.8 we have Lemma 3.2.5.

3.2.3 Third Step

We focus now on the terms with degree-8 of $H^{(6)}$. Because of the algebraic complexity, the analytical derivation of the 8th-order resonant terms was performed using Wolfram Mathematica. This ensured algebraic accuracy while minimizing the risk of manual algebraic errors. Then main result we are going to prove is the following:

Lemma 3.2.9 *There are infinitely many resonant, non action preserving terms in $H^{(6)}$ with homogeneity 8.*

From (3.30), (3.31) and (3.32) , we have

$$H^{(6)} = H_2 + Z_4 + Z_6 + \tilde{H}_8 + R_{\geq 10},$$

with

$$\tilde{H}_8 := H_8 + \frac{1}{6}\{\{\{H_2, F_4\}F_4\}F_4\} + \frac{1}{2}\{\{H_4, F_4\}, F_4\} + \{H_6, F_4\} + \{Z_4, F_6\} \in \mathcal{F}^8_{\mathbf{kir}} \quad (3.42)$$

and

$$R_{\geq 10} := R_{\geq 8} - \tilde{H}_8 \in \mathcal{F}^{\geq 10}_{\mathbf{kir}}.$$

Remark 3.2.2 *Since Z_4 is an element of Ker and F_6 is a range element, thanks to Lemma 3.5.4 we have that $\{Z_4, F_6\}$ is range element itself and hence, it can be neglected in the analysis of the resonant monomials of* (3.43).

We then focus our analysis on

$$\tilde{\tilde{H}}_8 := H_8 + \frac{1}{6}\{\{\{H_2, F_4\}F_4\}F_4\} + \frac{1}{2}\{\{H_4, F_4\}, F_4\} + \{H_6, F_4\}. \qquad (3.43)$$

Let us analyze the resonant monomials of $\tilde{\tilde{H}}_8$:

Lemma 3.2.10 *The structure of the resonant terms of H_2, Z_4 Z_6 and $\tilde{\tilde{H}}_8$ isn't affected by the value of $c \neq 0$.*

Proof: Since H_2 does not depend on the constant c we have that the set of the degree-2 resonant monomials is independend by the latter.
Recalling the expressions of $F_4^{(1)}$, $F_4^{(2)}$, Z_4 and Z_6 respectively in (3.28), (3.29), (3.20) and (3.33) and defining respectively

$$\begin{aligned}
\mathscr{F}_4^1 &:= \frac{iH_2}{2}\sum_{j\in\mathbb{N}}(\bar{W}_j - W_j) \\
\mathscr{F}_4^2 &:= \frac{i}{4}\sum_{j,k\in\mathbb{N}}\frac{jk}{j+k}(\bar{W}_j\bar{W}_k - W_jW_k) + \frac{i}{2}\sum_{j\neq k\in\mathbb{N}}\frac{jk}{j-k}\bar{W}_jW_k \\
\mathscr{F} &:= \mathscr{F}_4^1 + \mathscr{F}_4^2 \\
\mathscr{Z}_4 &:= -\frac{1}{2}H_2^2 - \frac{1}{2}\Pi_{H_2}(P^2) \\
\mathscr{Z}_6 &:= \frac{1}{2}H_2\Pi_{H_2}(P^2) + \frac{1}{2}\Pi_{H_2}(\{\{\mathscr{F}_4^2, H_2\}, \mathscr{F}_4^2\}),
\end{aligned}$$

we have that $Z_4 = c\mathscr{Z}_4$ and $Z_6 = c^2\mathscr{Z}_6$. This suffices to prove that the distribution of resonant monomilas in Z_4 and Z_6 is not influenced by the value of c, but depends only on $\mathscr{Z}_4$ and $\mathscr{Z}_6$

respectively.
For what concerns the term $\tilde{H}_8$, the argument is the same:
by looking at the expression of H_8 in (3.7) and $\tilde{H}_8$ in (3.42) and setting

$$\begin{aligned}
\mathscr{H}_6 &:= \frac{1}{2}(H_2+P)^3 \\
\mathscr{H}_8 &:= -\frac{1}{2}(H_2+P)^4 \\
\tilde{\mathscr{H}}_8 &:= \mathscr{H}_8 + \frac{1}{6}\{\{\{H_2,\mathscr{F}_4\}\mathscr{F}_4\}\mathscr{F}_4\} + \frac{1}{2}\{\{\mathscr{H}_4,\mathscr{F}_4\},\mathscr{F}_4\} + \{\mathscr{H}_6,\mathscr{F}_4\},
\end{aligned}$$

we note that $\tilde{H}_8 = c^4\tilde{\mathscr{H}}_8$. We can again conclude that the degree-8 resonant monomials depend only on $\tilde{\mathscr{H}}_8$ and not on c. □

Corollary 3.2.1 *The cardinality of resonant non-action preserving monomials of order eight for* (3.5) *is either 0 or infinity.*

Proof: Le us suppose there is resonant quadruple of indexes $\{j_1,\ j_2,\ j_3,\ j_4\}$, i.e.

$$|j_1| \pm |j_2| \pm |j_3| \pm |j_4| = 0$$

and suppose that the associated resonant monomial has non-zero coefficient.
Since the eigenvalues of H_2 are linear with respect to j, one has that also $\{k{\cdot}j_1,\ k{\cdot}j_2,\ k{\cdot}j_3,\ k{\cdot}j_4\}$ is a resonant quadruple of indexes for every $k \in \mathbb{N}$.
Moreover, if we replace $j \mapsto k \cdot j$, $j \in \mathbb{Z}$ we have that Hamiltonian (3.6) become

$$H = kH_2 + k\mathcal{P}((kc)\cdot(H_2+P)).$$

We can conclude using Lemma 3.2.10. □

We will write the (formal) Birkhoff norm form up to homogeneity 8, of H as

$$H_{\mathbf{BNF}} := H \circ \Phi = H_2 + Z_4 + Z_6 + \tilde{Z}_8 + R_{\geq 10},$$

where, as usual, $\tilde{Z}_8$ contains the resonant terms of degree 8.
Finally, in light of Lemma (3.2.9), we split $\tilde{Z}_8$ in

$$\tilde{Z}_8 = Z_8^i + Z_8^e, \tag{3.44}$$

where Z_8^i is action preserving and Z_8^e contains also the variables $\{W_j\}_{j\in\mathbb{N}}$.
We refer to Section (3.5) for a complete description of $\tilde{Z}_8$, Z_8^i, Z_8^e in a finite dimensional context.

3.2.4 Codes for Lemma 3.2.9

In the following sections we report the codes we used to prove Lemma 3.2.9.
We consider the case of initial data $\Omega_{\mathcal{S}}$ with Fourier support restricted to the sites

$$\mathcal{S} := \{1, 2\}.$$

System (3.1) becomes a complex system of 4 ordinary differential equations in the variables $z_{\pm i}$, $\bar{z}_{\pm i}$, $i = 1, 2$. In our codes we will set

$$z_i := \texttt{x[i]},\quad \bar{z}_i := \texttt{y[i]},\quad z_{-i} := \texttt{X[i]},\quad \bar{z}_{-i} := \texttt{Y[i]},$$

in particular one has

$$I_i = \texttt{x[i]y[i]}, \quad W_i = \texttt{x[i]X[i]}, \quad \bar{W} = \texttt{x[i]X[i]}.$$

For the sake of simplicity we set $c = 1$, however, this does not entail any loss of generality thanks to Lemma 3.2.10.

• **Poisson Bracket:**

$$\{f, g\} = i \sum_{j=1,2} \left(\frac{\partial f}{\partial \bar{z}_j} \frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \bar{z}_j} \right)$$

```
Pois[f_, g_] := -I Sum[ D[f, x[i]] D[g, y[i]] + D[f, X[i]] D[g, Y[i]]
            -D[g, x[i]] D[f, y[i]] - D[g, X[i]] D[f, Y[i]], {i, 1, 2}]
```

• The formal Variables

The formal variables H_2 and P defined in (3.3) are:

```
H2 := Sum[ j[i] (x[i] y[i] + X[i] Y[i]), {i, 1, 2}]
```

```
P := Sum[ j[i] (x[i] X[i] + y[i] Y[i]), {i, 1, 2}]
```

• The First Step Hamiltonian Terms

H_4 in (3.7)

```
        H4 := -1/2 (H2 + P)^2
```

Z_4 in (3.20):

```
Z4 := -1/2 H2^2 - Sum[j[i]^2 (x[i] y[i] X[i] Y[i]), {i, 1, 2}]
```

• The Generating Function of the First Step

G in (3.27):

```
G := -I/2  Sum[ (x[i] X[i] - y[i] Y[i]), {i, 1, 2}]
```

$F_4^{(1)}$ in (3.28):

```
F1 := H2*G
```

$F_4^{(2)}$ in (3.29):

```
F2 := I/4 Sum[
        j[i] j[k]/(j[i] + j[k]) (y[i] Y[i] y[k] Y[k] -
        x[i] X[i] x[k] X[k]), {i, 1, 2}, {k, 1, 2}] +
        I/2 Sum[Sum[
        j[i] j[k]/(j[i] - j[k]) (y[i] Y[i] x[k] X[k] -
        y[k] Y[k] x[i] X[i]), {i, 1, k - 1}], {k, 1, 2}]
```

F_4 in (3.24):

```
F4 := F1 + F2
```

• The Second Step Hamiltonian Terms

H_6 in (3.7)

```
H6 := 1/2 (H2 + P)^3
```

$\tilde{H}_6$ in (3.34):

```
h6 := H6 - 1/2 Pois[Pois[H2, F4], F4]
```

• The Third Step Hamiltonian Terms

H_8 in (3.7)

```
H8 := -1/2 (H2 + P)^4
```

$\tilde{\tilde{H}}_8$ in (3.43):

```
h8 := H8 + 1/3 Pois[Pois[H4, F4], F4] + 1/6 Pois[Pois[Z4, F4], F4] +
      Pois[H6, F4]
```

• Study of $\tilde{\tilde{H}}_8$

By looking at (3.16) the only possible resonant monomials of order-8 for the Kirchhoff that are supported in the set of indexes $\mathcal{S}$, are the of the form

$$I_j W_1^2 \bar{W}_2, \quad I_j W_2 \bar{W}_1^2, \quad j = \pm 1, \pm 2.$$

We used Wolfram Mathematica to compute the coefficient of Z_8 corresponding to the monomials of this form. If we consider for example

$$I_1 W_2 \bar{W}_1^2 = \texttt{x[1] y[1] X[2] x[2] y[1] Y[1] y[1] Y[1]]}$$

we have

```
In[1]: =(Coefficient[h8, x[1] y[1] X[2] x[2] y[1] Y[1] y[1] Y[1]] /.
            j[i_] :> i)

Out[1]:= -(3/4)
```

3.3 Generic non-linearity

We now want to consider a generic Kirchhoff equation defined by an Hamiltonian of the form

$$H_\alpha(z,\bar{z}) = H_2 + \mathcal{P}_\alpha\left(H_2 + P\right), \tag{3.45}$$

with $\mathcal{P}$ analytic function with Taylor expansion of the form $\sum_{n\geq 2}\alpha_{2n}y^n$.
The sequence $\alpha = \{\alpha_{2n}\}_{n\in\mathbb{N}}$ is made of free parameters to be determined later on.
We then have an expansion of H in homogeneity degrees of the form

$$H = H_2 + H_{\alpha,4} + H_{\alpha,6} + \ldots + H_{\alpha,2n} + \ldots$$

with

$$H_{\alpha,2n} = \alpha_{2n}(H_2+P)^n \in \mathcal{F}^{2n}_{\mathbf{kir}} \text{ for } n \geq 2. \tag{3.46}$$

We ask if is it possible to choose the coefficients α_{2n} in such a way that, at every step of Formal Birkhoff, the resulting resonant element is also formally action preserving.
Our strategy is to perform a formal Birkhoff normal form algorithm in the same way of the last section, in order to make the resonant terms of degree 4, 6 and 8 explicit with respect of the parameters α. Then we will try to choose α in order to have that all the resonant terms are also formally action preserving.

3.3.1 First step

Of course, α_4 is arbitrary, since the resonant part of $\alpha_4(H_2+P)^2$ is automatically action preserving.

For a given choice of α_4, the generating function F_{4,α_4} of the first Birkhoff step must satisfy the homological equation

$$\{H_2, F_{\alpha,4}\} + H_{\alpha,4} = Z_{\alpha,4} := \Pi_{\mathcal{K}_{\mathbf{kir}}}\left(H_{\alpha,4}\right), \tag{3.47}$$

where, similarly to the expression of Z_4 in (3.20), one has

$$Z_{\alpha,4} = \alpha_4 H_2^2 + 2\alpha_4\sum_{j\in\mathbb{N}} j^2 I_j I_{-j}. \tag{3.48}$$

In analogy with the previous case, the generating function $F_{\alpha,4} \in \mathcal{F}^4_{\mathbf{kir}}$ can be chosen of the form

$$F_{\alpha,4} := F^{(1)}_{\alpha,4} + F^{(2)}_{\alpha,4},$$

with

$$\begin{aligned} F^{(1)}_{\alpha,4} &= H_2 G_\alpha, \\ F^{(2)}_{\alpha,4} &= -\frac{i\alpha_4}{2}\sum_{j,k\in\mathbb{N}}\frac{jk}{j+k}(\overline{W}_j\overline{W}_k - W_jW_k) - \frac{i\alpha_4}{2}\sum_{\substack{j,k\in\mathbb{N}\\ j\neq k}}\frac{jk}{j-k}(\overline{W}_jW_k - W_j\overline{W}_k), \end{aligned} \tag{3.49}$$

and G_α that solves

$$\{H_2, G_\alpha\} = -2\alpha_4 P,$$

namely

$$G_\alpha = -i\alpha_4 \sum_{j\in\mathbb{N}} (\overline{W}_j - W_j). \tag{3.50}$$

3.4 Second Step

After the first step we obtain that the normal form Hamiltonian is

$$H_\alpha^{(4)} = H_2 + Z_{\alpha,4} + H_{\alpha,6} + \frac{1}{2}\{\{H_2, F_{\alpha,4}\}, F_{\alpha,4}\} + \{H_{\alpha,4}, F_{\alpha,4}\} + R_{\alpha,\geq 6},$$

whose terms of degree 6 are

$$\tilde{H}_{\alpha,6} := H_{\alpha,6} + \frac{1}{2}\{\{H_2, F_{\alpha,4}\}, F_{\alpha,4}\} + \{H_{\alpha,4}, F_{\alpha,4}\} \tag{3.51}$$

$$\overset{(3.47)}{=} H_{\alpha,6} - \frac{1}{2}\{\{H_2, F_{\alpha,4}\}, F_{\alpha,4}\} + \{Z_{\alpha,4}, F_{\alpha,4}\}. \tag{3.52}$$

Our aim is to analyze the resonant degree 6 terms $Z_{\alpha,6} := \Pi_{\mathcal{K}_{\mathbf{kir}}}(\tilde{H}_{\alpha,6})$.
Similarly to (3.36), (3.35) and (3.37), one has

$$\begin{aligned}
\Pi_{\mathcal{K}_{\mathbf{kir}}}(H_{\alpha,6}) &= \alpha_6 H_2^3 + 3\alpha_6 H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) + \alpha_6 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3); \\
\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{Z_{\alpha,4}, F_{\alpha,4}\}) &= 0 \\
\Pi_{\mathcal{K}_{\mathbf{kir}}}\Big(-\frac{1}{2}\{\{H_2, F_{\alpha,4}\}, F_{\alpha,4}\}\Big) &= -2\alpha_4^2 H_2^3 - 4\alpha_4^2 H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) \\
&\quad + \frac{1}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_{\alpha,4}^2, H_2\}, F_{\alpha,4}^2\}) - 2\alpha_4^2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3);
\end{aligned}$$

Since the only non action preserving terms are the ones contained in $\Pi_{\mathcal{K}_{\mathbf{kir}}}(P^3)$, in order for $\tilde{H}_{\alpha,6}$ to be formally action preserving, one must have

$$\alpha_6 = 2\alpha_4^2. \tag{3.53}$$

With this choice, we have

$$Z_{\alpha,6} = 2\alpha_4^2 H_2 \Pi_{\mathcal{K}_{\mathbf{kir}}}(P^2) + \frac{1}{2}\Pi_{\mathcal{K}_{\mathbf{kir}}}(\{\{F_{\alpha,4}^2, H_2\}, F_{\alpha,4}^2\}).$$

3.4.1 Third Step

The normal form up degree 6 of H is

$$H_\alpha^{(6)} = H_2 + Z_{\alpha,4} + Z_{\alpha,6} + \tilde{H}_{\alpha,8} + R_{\alpha,\geq 10}$$

with,

$$\tilde{H}_8 := H_{\alpha,8} + \frac{1}{6}\{\{\{H_2, F_{\alpha,4}\}F_{\alpha,4}\}F_{\alpha,4}\} + \frac{1}{2}\{\{H_{\alpha,4}, F_{\alpha,4}\}, F_{\alpha,4}\} + \{H_{\alpha,6}, F_{\alpha,4}\} + \{Z_{\alpha,4}, F_{\alpha,6}\} \in \mathcal{F}_{\mathbf{kir}}^8.$$

Again, since $\{Z_{\alpha,4}, F_{\alpha,6}\} \in \mathcal{R}_{\mathbf{kir}}^8$ we focus on

$$\tilde{\tilde{H}}_8 := H_{\alpha,8} + \frac{1}{6}\{\{\{H_2, F_{\alpha,4}\}F_{\alpha,4}\}F_{\alpha,4}\} + \frac{1}{2}\{\{H_{\alpha,4}, F_{\alpha,4}\}, F_{\alpha,4}\} + \{H_{\alpha,6}, F_{\alpha,4}\} \in \mathcal{F}_{\mathbf{kir}}^8. \tag{3.54}$$

3.4.2 Codes for Theorem 1.2.4

We consider the case of initial data $\Omega_{\mathcal{S}}$ with Fourier support restricted to the sites

$$\mathcal{S} := \{1, 2, 3, 4, 5, 6\}.$$

Similarly to the last section we define `Pois`, `H2` and `P`. we set the first three coefficients of α defining the non linearity as

$$\alpha_4 = \mathtt{a}, \quad \alpha_6 = \mathtt{b}, \quad \alpha_8 = \mathtt{c}.$$

The function G_α in (3.50) is defined by:

```
G := I a Sum[ (x[i] X[i] - y[i] Y[i]), {i, 1, 5}]
```

The function that generates the first step transformation, respectively $F_{\alpha,4}$, $F^{(1)}_{\alpha,4}$ and $F^{(2)}_{\alpha,4}$ in (3.49) are defined by

```
F1 := H2*G
```

```
F2 := (-I a)/2 Sum[
         j[i] j[k]/(j[i] + j[k]) (y[i] Y[i] y[k] Y[k] -
         x[i] X[i] x[k] X[k]), {i, 1, 5}, {k, 1, 5}] +  -I a Sum[
         Sum[j[i] j[k]/(j[i] - j[k]) (y[i] Y[i] x[k] X[k] -
         y[k] Y[k] x[i] X[i]), {i, 1, k - 1}], {k, 1, 5}]
```

```
F4 := F1 + F2
```

The Hamiltonians $H_{\alpha,4}$, $H_{\alpha,6}$, $H_{\alpha,8}$ in (3.46) are defined by

```
H4 := a (H2 + P)^2
```

```
H6 := b (H2 + P)^3
```

```
H8 := c (H2 + P)^4
```

The resonant degree-4 term $Z_{\alpha,4}$ in (3.48) is defined by

```
Z4 := a H2^2 + 2 a Sum[j[i]^2 (x[i] y[i] X[i] Y[i]), {i, 1, 5}]
```

The Hamiltonians $\tilde{H}_{\alpha,6}$ in (3.51) is defined by

```
h6 := H6 - 1/2 Pois[Pois[H2, F4], F4]
```

while $\tilde{\tilde{H}}_{\alpha,8}$ in (3.54) is

```
h8 := H8 + 1/3 Pois[Pois[H4, F4], F4] + 1/6 Pois[Pois[Z4, F4], F4] +
      Pois[H6, F4]
```

If we check the coefficient of a generic resonant degree 6 term

$$\texttt{x[1] X[1] x[1] X[1] y[2] Y[2]} = W_1^2 \bar{W}_2$$

```
In[]:= (Coefficient[h6, x[1] X[1] x[1] X[1] y[2] Y[2]] /. j[i_] :> i)
Out[]=1/4 (-48 a^2 + 24 b)
```

that is equation (3.53).
Assuming hence $\alpha_6 = 2\alpha_4^2$, hence $\texttt{b} = 2\texttt{a}^2$, by checking the coefficient of the resonant degree 8 term

$$\texttt{x[1] X[1] x[1] X[1] x[1] X[1] y[3] Y[3]} = W_1^3 \bar{W}_3$$

one gets

```
In[]:= Coefficient[h8, x[1] X[1] x[1] X[1] x[1] X[1] y[3] Y[3]] /.
            j[i_] :> i)

Out[]=1/36 (-1728 a^3 + 432 c)
```

That is zero only if $-1728\ \texttt{a}^3 + \ 432\texttt{c} = 0$ i.e. $\alpha_8 = 4\alpha_4^3$.
The coefficient of the monomial

$$\texttt{x[1] y[1] x[2] X[2] y[1] Y[1] y[1] Y[1]} = I_1 W_2 \bar{W}_1^2$$

instead, is

```
In[]:=(Coefficient[h8, x[1] y[1] x[2] X[2] y[1] Y[1] y[1] Y[1]] /.
        j[i_] :> i)

Out[]=1/12 (-1080 a^3 + 288 c)
```

then is must hold $\alpha_8 = \frac{15}{4}\alpha_4^3$, that is incompatible with the above condition.

3.5 Dynamics of Specific solutions

Let us now consider the Kirchhoff equation (3.1), defined by the Hamiltonian (3.5) and set, for the sake of simplicity, $c = 1$.
The aim of this section is to study the dynamics of such equation, restricted to a finite and arbitrary set of tangential sites.
This, as we will see, is only possible thanks to the special structure of which the Kirchhoff equation is endowed.
In fact due to the invariance of the Fourier support highlighted in Lemma (1.2.2) we have that the space

$$\Omega_{\mathcal{S}} := \{z \in H^s(\mathbb{T}) : z_k = 0 \text{ if } |k| \notin \mathcal{S}\}$$

is invariant for equation the solutions of (3.1). It follows that if we chose a finite $\mathcal{S}$, equation (3.1) reduces to a finite dimensional Hamiltonian system on the phase space $\Omega_{\mathcal{S}}$, that one can

study from a qualitative point of view.
Throughout the rest of this chapter, we fix

$$\mathcal{S} := \{1,\, 2\}.$$

In the spirit of [59], we prove that there exist specific solutions originating form an initial datum of size ε in $\Omega_{\mathcal{S}}$ that, in Birkhoff coordinates exhibits nontrivial dynamic, namely an energy transfer.
More precisely, we show that such solutions undergoes a variation of order at least ε^8 over a time interval of order ε^{-2}.
This provides a non-integrability result for the formal Birkhoff normal form of Theorem 1.2.3.
Unfortunately, this result cannot be directly transported to equation (3.1).
In contrast to the quintic NLS case, this nonlinear phenomenon does appear to depend on the particular choice of the resonant set, although it can be shown for infinitely many such choices.

Notation.
For positive A and B we adopt the following notations:

- We shall use the notation $A \lesssim B$ (respectively $A \gtrsim B$ to denote $A \leq \mathtt{C}B$ (respectively $A \geq \mathtt{C}B$) where $\mathtt{C}$ is a positive constant depending on parameters fixed once for all. We will emphasize by writing $\lesssim_q$ when the constant $\mathtt{C}$ depends on some other parameter q.
- We will say that A is $O(B)$ if there exists a positive constant $\mathcal{C}$ such that $A = \mathtt{C}B$.
- we will use $A \sim B$ whenever there exist two positive constants $\mathtt{C}_1$ and $\mathtt{C}_2$, depending again on fixed parameters, such that $\mathtt{C}_1 B \leq A \leq \mathtt{C}_2 B$.

Finally, we will denote with $\{w_j\}_{j\in\mathbb{Z}}$ the variables induced by the normal form map Φ in Theorem (1.2.3).

The main result is the following

Theorem 3.5.1 *Let us consider the Birkhoff normal form Hamiltonian $H_{\mathbf{BNF}}$ defined in Theorem* (1.2.3).
There exists $\varepsilon_* > 0$ such that, for every $\varepsilon \leq \varepsilon_*$, there is a set $\mathcal{A}_\varepsilon$ with the following property: for every initial data $w(0) = \{w_j(0)\}_{j\in\mathbb{Z}} \in \mathcal{A}_\varepsilon$ there exists a unique solution of

$$\dot{w}(t) = \frac{\partial H_{\mathbf{BNF}}}{\partial \bar{w}}(w(t), \bar{w}(t)), \tag{3.55}$$

moreover it satisfies

$$\big||w_j(t)|^2 - |w_j(0)|^2\big| \sim t\varepsilon^8, \quad j = \pm 1,\, \pm 2 \tag{3.56}$$

for all $t \in [0, T_*]$ with $T_* \lesssim \varepsilon^{-2}$.

Remark 3.5.1 *The set $\mathcal{A}_\varepsilon$, as we will see in Subsection* (3.5.5) *is made by functions, w with finite Fourier support.*
More precisely, by identifying w with the sequence of its Fourier coefficients $\{w_j\}_{j\in\mathbb{Z}}$, we have

$$\mathcal{U}_\varepsilon \subset \left\{ w = \{w_j\}_{j\in\mathbb{Z}} : |w_1|^2 = |w_{-1}|^2 = \alpha\varepsilon^2,\ |w_2|^2 = |w_{-2}|^2 = \varepsilon^2\left(1 - \frac{\alpha}{2}\right), \right. \\ \left. \alpha \in [0,1], \quad w_j = 0,\ |j| \neq 1,\, 2 \right\}. \tag{3.57}$$

The elements of $\mathcal{A}_\varepsilon$ are then analytic functions, moreover, from (3.57)*, we have that, for every $s \in \mathbb{R}$*

$$\mathcal{A}_\varepsilon \subset B_{\nu(s)\varepsilon}(H_0^s(\mathbb{T},\mathbb{C})), \quad \nu(s) := \begin{cases} \sqrt{2^{s+1}}, & \textit{if} \quad s \geq 1 \\ \sqrt{2^s + 2}, & \textit{if} \quad s \leq 1 \end{cases}$$

3.5.1 Normal Form on Finite Support

For our choice of the Fourier support $\mathcal{S} := \{1, 2\}$, we use the following notations:
we will denote with

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \ldots + \mathcal{H}_{2n} + h.o.t.$$

the Hamiltonian (3.7) restricted to $\Omega_{\mathcal{S}}$.
We will write $\mathcal{Z}_4$, $\mathcal{Z}_6$, $\tilde{\mathcal{Z}}_8$, $\tilde{\mathcal{Z}}_8^i$, $\tilde{\mathcal{Z}}_8^e$ when referring to the restriction on $\Omega_{\mathcal{S}}$ of the quantities Z_4, Z_6, Z_8, Z_8^i, Z_8^e defined respectively in (3.20), (3.33) and (3.44), moreover, we will still denote with Φ the restriction of normal form map Φ of Theorem (1.2.3).

$$\mathcal{H}_2 := I_1 + I_{-1} + 2I_2 + 2I_{-2} \quad \text{and} \quad \mathcal{P} := W_1 + \bar{W}_1 + 2W_2 + 2\bar{W}_2 \tag{3.58}$$

Recalling (3.60), the Birkhoff normal form up to degree 8 of the Hamiltonian is

$$\hat{\mathcal{H}}_{\mathbf{BNF}} := \mathcal{H}_2 + \mathcal{Z}_4 + \mathcal{Z}_6 + \tilde{\mathcal{Z}}_8 + \mathcal{R}_{\geq 10}. \tag{3.59}$$

From (3.20), we have that Z_4 is

$$\mathcal{Z}_4 := \Pi_{H_2}(H_4) = -\frac{1}{2}H_2^2 - \sum_{j=1,2} j^2 I_j I_{-j}$$

or, more explicitly

$$\mathcal{Z}_4 = -\frac{1}{2}I_1^2 - 2I_1I_2 - 2I_2^2 - 2I_1I_{-1} - 2I_2I_{-1} - \frac{1}{2}I_{-1}^2 - 2I_1I_{-2} - 8I_2I_{-2} - 2I_{-1}I_{-2} - 2I_{-2}^2.$$

From (3.33) and (3.24), it follows

$$\begin{aligned} \mathcal{Z}_6 \quad &= \quad \frac{3}{4}I_1^2I_{-1} - \frac{2}{3}I_1I_2I_{-1} + \frac{3}{4}I_1I_{-1}^2 + \frac{16}{3}I_1I_2I_{-2} \\ &\quad +6I_2^2I_{-2} - \frac{2}{3}I_1I_{-1}I_{-2} + \frac{16}{3}I_2I_{-1}I_{-2} + 6I_2I_{-2}^2. \end{aligned}$$

The computations performed on Wolfram Mathematica show that

$$\begin{aligned} \tilde{\mathcal{Z}}_8^e \quad &= \quad -\frac{3}{4}I_1W_2\bar{W}_1^2 + \frac{3}{2}I_2W_2\bar{W}_1^2 - \frac{3}{4}I_{-1}W_2\bar{W}_1^2 - \frac{3}{4}I_1W_1^2\bar{W}_2 \\ &\quad +\frac{3}{2}I_2W_1^2\bar{W}_2 - \frac{3}{4}I_{-1}W_1^2\bar{W}_2 + \frac{3}{2}I_{-2}W_2\bar{W}_1^2 + \frac{3}{2}I_{-2}W_1^2\bar{W}_2 \\ &= \quad -\frac{3}{4}I_1(W_2\bar{W}_1^2 + W_1^2\bar{W}_2) + \frac{3}{2}I_2(W_2\bar{W}_1^2 + W_1^2\bar{W}_2) \\ &\quad -\frac{3}{4}I_{-1}(W_2\bar{W}_1^2 + W_1^2\bar{W}_2) + \frac{3}{2}I_{-2}(W_2\bar{W}_1^2 + W_1^2\bar{W}_2) \\ &= \quad -\frac{3}{2}I_1\mathrm{Re}(W_1^2\bar{W}_2) + 3I_2\mathrm{Re}(W_1^2\bar{W}_2) - \frac{3}{2}I_{-1}\mathrm{Re}(W_1^2\bar{W}_2) + 3I_{-2}\mathrm{Re}(W_1^2\bar{W}_2) \\ &= \quad \left(-\frac{3}{2}I_1 + 3I_2 - \frac{3}{2}I_{-1} + 3I_{-2}\right)\mathrm{Re}(W_1^2\bar{W}_2) \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{Z}_8^i \quad = \quad & -\frac{16}{9}I_{-2}^2 I_{-1}I_1 + \frac{49}{18}I_{-2}I_{-1}^2 I_1 + \frac{1}{8}I_{-1}^3 I_1 + \frac{49}{18}I_{-2}I_{-1}I_1^2 + \frac{3}{8}I_{-1}^2 I_1^2 \\
& + \frac{1}{8}I_{-1}I_1^3 + 2I_{-2}^3 I_2 + \frac{38}{9}I_{-2}^2 I_{-1}I_2 - \frac{16}{9}I_{-2}I_{-1}^2 I_2 + \frac{38}{9}I_{-2}^2 I_1 I_2 \\
& - \frac{128}{9}I_{-2}I_{-1}I_1 I_2 + \frac{49}{18}I_{-1}^2 I_1 I_2 - \frac{16}{9}I_{-2}I_1^2 I_2 + \frac{49}{18}I_{-1}I_1^2 I_2 \\
& + 6I_{-2}^2 I_2^2 + \frac{38}{9}I_{-2}I_{-1}I_2^2 + \frac{38}{9}I_{-2}I_1 I_2^2 - \frac{16}{9}I_{-1}I_1 I_2^2 + 2I_{-2}I_2^3.
\end{aligned}
$$

Finally, on $\Omega_{\mathcal{S}}$, in place of $\| \cdot \|_s$ we will consider the equivalent norm

$$\|w\| := \sqrt{\sum_{|j|=1,2} w_j^2}.$$

In this setting we obtain a finite dimensional version (not only formal) of Theorem 1.2.3:

Lemma 3.5.1 (Birkhoff normal form for the finite dimensional Kirchhoff equation)
Let $\mathcal{H}$ be the Hamiltonian of the Kirchhoff equation restricted to the finite-dimensional phase space $\Omega_{\mathcal{S}}$.
There exist $\varepsilon_1 > 0$, an analytic function $\mathcal{F} : B_{\varepsilon_1} \subset \Omega_{\mathcal{S}} \to \mathbb{C}$ and an analytic, bijective, close-to-identity canonical transformation

$$\Phi : B_{\varepsilon_1} \subset \Omega_{\mathcal{S}} \to \Omega_{\mathcal{S}}$$

of the form $\Phi = e^{\{\cdot, \mathcal{F}\}}$ such that

$$\mathcal{H}_{\mathbf{BNF}} := \mathcal{H} \circ \Phi = \mathcal{H}_2 + \mathcal{Z}_4^i + \mathcal{Z}_6^i + \tilde{\mathcal{Z}}_8 + \mathcal{R}_{\geq 10}, \tag{3.60}$$

where:

- *$\mathcal{Z}_{2k}^i$, for $k = 2, 3$, is a homogeneous polynomial of degree $2k$ depending only on the actions $I = (I_{-2}, I_1, I_1, I_2)$;*
- *$\mathcal{R}_{\geq 10}$ is a real-analytic remainder such that*

$$|\mathcal{R}_{\geq 10}(w)| \lesssim \|w\|^{10}, \quad \forall w \in B_{\varepsilon_1}; \tag{3.61}$$

- *The function $\mathcal{F}$ satisfies*

$$|\mathcal{F}(w)| \lesssim \|w\|^4, \quad \forall w \in B_{\varepsilon_1}; \tag{3.62}$$

- *If we write $J_k = |z_k|^2$, the map Ψ satisfies*

$$J_k \circ \Phi = I_k + \Psi(I_k)$$

  with

$$|\Psi(I_k)| \lesssim \|w\|^4, \quad \forall w \in B_{\varepsilon_1}. \tag{3.63}$$

An inequality identical to (3.63) *is fulfilled by the inverse transformation Φ^{-1}.*

Thanks to the above result we have:

Lemma 3.5.2 *There exist $\varepsilon_2 > 0$ such that, for every initial data $w_0 = \{w_{j,0}\}_{j=\pm1,\pm2}$ with*

$$\varepsilon := \|w_0\| \leq \varepsilon_2,$$

the solutions of

$$\dot{w}(t) = \frac{\partial \mathcal{H}_{\mathbf{BNF}}}{\partial \bar{w}}(w(t), \bar{w}(t)),$$

satisfy

$$|w_j(t)| \lesssim \varepsilon, \quad \forall t \lesssim \varepsilon^{-6}. \tag{3.64}$$

Remark 3.5.2 *In the case $c < 0$, inequality* (A.9) *allows to extend the bound* (3.64) *to all positive times.*
Unfortunately, if $c > 0$ such uniform-in-time estimate is not available and (3.64) *is achieved, by means of the Birkhoff Normal form Theorem* (1.2.3)*, only for times of order ε_2^{-6}.*

3.5.2 Symplectic Polar Coordinates

The aim of this Subsection is to construct a newt of symplectic polar coordinates for $\hat{\mathcal{H}}$, namely:

$$(I, \theta) = (I_{-2}, I_{-1}, I_1, I_2, \theta_{-2}, \theta_{-1}, \theta_1, \theta_2) =: \Phi^2(I, W)$$

given by the map

$$I_j \overset{\Phi^2}{\to} I_j, \quad W_j \overset{\Phi^2}{\to} \sqrt{I_j}\sqrt{I_{-j}}e^{i(\theta_j + \theta_{-j})}. \tag{3.65}$$

We have that the expressions for $\mathcal{Z}_4$, $\mathcal{Z}_6$ and $\tilde{\mathcal{Z}}_8^i$ are left untouched, while

$$\mathcal{Z}_8^e = I_1 I_{-1}\sqrt{I_2}\sqrt{I_{-2}}\left(-\frac{3}{2}I_1 + 3I_2 - \frac{3}{2}I_{-1} + 3I_{-2}\right)\cos(2\theta_1 + 2\theta_{-1} - \theta_2 - \theta_{-2}).$$

Now, by recalling

$$\hat{\mathcal{H}}_{\mathbf{BNF}} := \mathcal{H}_2 + \mathcal{Z}_4 + \mathcal{Z}_6 + \mathcal{Z}_8^i + \mathcal{Z}_8^e,$$

in this new set of variables the Hamiltonian system associated to $\hat{\mathcal{H}}_{\mathbf{BNF}}$ is defined on the phase space

$$\mathbb{T}^4 \times \mathbb{R}^4 \in (\theta_1, \theta_{-1}, \theta_2, \theta_{-2}, I_1, I_{-1}, I_2, I_{-2})$$

by

$$\begin{cases} \dot{\theta}_j = -\dfrac{\partial \hat{\mathcal{H}}_{\mathbf{BNF}}}{\partial I_j}, \quad j = \pm1, \pm2 \\ \dot{I}_j = \dfrac{\partial \hat{\mathcal{H}}_{\mathbf{BNF}}}{\partial \theta_j}, \quad j = \pm1, \pm2. \end{cases} \tag{3.66}$$

Let us note that the quantities

$$I_1 - I_{-1}$$
$$I_2 - I_{-2}$$
$$I_{\pm1} + 2I_{\pm2}$$

are conserved by the dynamic generated by $\mathcal{Z}_8^e$ and, trivially, also for $\mathcal{Z}_4$, $\mathcal{Z}_6$, $\tilde{\mathcal{Z}}_8^i$. It follows:

Lemma 3.5.3 (Integrability)
The system (3.66) *is completely integrabile.*

3.5.3 Action-Angle Coordinates for $\hat{\mathcal{H}}$

We want hence, to construct a new set of action-angles variables for $\hat{\mathcal{H}}$, namely:

$$(\varphi, \mathcal{I}) = (\varphi_1,\, \varphi_{-1},\, \varphi_2,\, \varphi_{-2},\, \mathcal{I}_1,\, \mathcal{I}_{-1},\, \mathcal{I}_2,\, \mathcal{I}_{-2})$$

given by the linear law

$$\begin{pmatrix} \varphi \\ \mathcal{I} \end{pmatrix} = \Phi^3(\theta, I) := \begin{pmatrix} \left(B^{-1}\right)^T & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \theta \\ I \end{pmatrix}, \tag{3.67}$$

where the matrix B and $\left(B^{-1}\right)^T$ are

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad \left(B^{-1}\right)^T = \begin{pmatrix} 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The relation between $\mathcal{I}$ and φ is determined in a way that $\{\mathcal{I}_j, \varphi_k\} = \delta(j,k)$.

$$\begin{cases} \mathcal{I}_1 = I_1 \\ \mathcal{I}_{-1} = -I_1 + I_{-1} \\ \mathcal{I}_2 = \frac{1}{2} I_1 + I_2 \\ \mathcal{I}_{-2} = -I_2 + I_{-2} \end{cases} \quad \Leftrightarrow \quad \begin{cases} I_1 = \mathcal{I}_1 \\ I_{-1} = \mathcal{I}_1 + \mathcal{I}_{-1} \\ I_2 = -\frac{1}{2}\mathcal{I}_1 + \mathcal{I}_2 \\ I_{-2} = -\frac{1}{2}\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_{-2} \end{cases} \tag{3.68}$$

and

$$\begin{cases} \varphi_1 = \theta_1 + \theta_{-1} - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_{-2} \\ \varphi_{-1} = \theta_{-1} \\ \varphi_2 = \theta_2 + \theta_{-2} \\ \varphi_{-2} = \theta_{-2} \end{cases} .$$

In these new variables (3.66) reads

$$\begin{cases} \dot{\varphi}_1 = -\dfrac{\partial \hat{\mathcal{H}}_{\mathbf{BNF}}}{\partial \mathcal{I}_1}, \\ \dot{\mathcal{I}}_1 = \dfrac{\partial \hat{\mathcal{H}}_{\mathbf{BNF}}}{\partial \varphi_1}, \end{cases} \qquad \begin{cases} \dot{\varphi}_j = -\dfrac{\partial \hat{\mathcal{H}}_{\mathbf{BNF}}}{\partial \mathcal{I}_j}, & j = -1,\, \pm 2 \\ \dot{\mathcal{I}}_j = 0, & j = -1,\, \pm 2, \end{cases} \tag{3.69}$$

while Hamiltonian $\hat{\mathcal{H}}_{\mathbf{BNF}}$ reads

$$
\begin{aligned}
\hat{\mathcal{H}}_{\mathbf{BNF}} &= \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\, \mathcal{I}_1,\, \mathcal{I}_{-1},\, \mathcal{I}_2,\, \mathcal{I}_{-2}) \\
&= 2\mathcal{I}_{-2} - 2\mathcal{I}_{-2}^2 + \mathcal{I}_{-1} - 2\mathcal{I}_{-2}\mathcal{I}_{-1} - \frac{1}{2}\mathcal{I}_{-1}^2 + 2\mathcal{I}_{-2}\mathcal{I}_1 - 3\mathcal{I}_{-2}^2\mathcal{I}_1 - \mathcal{I}_{-2}^3\mathcal{I}_1 - \mathcal{I}_{-1}\mathcal{I}_1 \\
&- \frac{10}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1 - \frac{35}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_1 + \frac{3}{4}\mathcal{I}_{-1}^2\mathcal{I}_1 + \frac{65}{18}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_1 + \frac{1}{8}\mathcal{I}_{-1}^3\mathcal{I}_1 \\
&- 2\mathcal{I}_1^2 - \frac{3}{2}\mathcal{I}_{-2}\mathcal{I}_1^2 - 3\mathcal{I}_{-2}^2\mathcal{I}_1^2 + \frac{17}{4}\mathcal{I}_{-1}\mathcal{I}_1^2 + 22\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1^2 - \frac{29}{12}\mathcal{I}_{-1}^2\mathcal{I}_1^2 \\
&+ \frac{10}{3}\mathcal{I}_1^3 + \frac{359}{18}\mathcal{I}_{-2}\mathcal{I}_1^3 - \frac{479}{36}\mathcal{I}_{-1}\mathcal{I}_1^3 - \frac{419}{36}\mathcal{I}_1^4 \\
&+ 4\mathcal{I}_2 - 12\mathcal{I}_{-2}\mathcal{I}_2 + 6\mathcal{I}_{-2}^2\mathcal{I}_2 + 2\mathcal{I}_{-2}^3\mathcal{I}_2 - 4\mathcal{I}_{-1}\mathcal{I}_2 + \frac{16}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_2 \\
&+ \frac{38}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_2 - \frac{16}{9}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_2 + 4\mathcal{I}_1\mathcal{I}_2 - \frac{22}{3}\mathcal{I}_{-2}\mathcal{I}_1\mathcal{I}_2 - \frac{32}{9}\mathcal{I}_{-2}^2\mathcal{I}_1\mathcal{I}_2 \\
&- \frac{20}{3}\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 - 34\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + \frac{65}{9}\mathcal{I}_{-1}^2\mathcal{I}_1\mathcal{I}_2 - 3\mathcal{I}_1^2\mathcal{I}_2 \\
&- \frac{95}{3}\mathcal{I}_{-2}\mathcal{I}_1^2\mathcal{I}_2 + 44\mathcal{I}_{-1}\mathcal{I}_1^2\mathcal{I}_2 + \frac{359}{9}\mathcal{I}_1^3\mathcal{I}_2 - 12\mathcal{I}_2^2 + 18\mathcal{I}_{-2}\mathcal{I}_2^2 + 12\mathcal{I}_{-2}^2\mathcal{I}_2^2 \\
&+ \frac{16}{3}\mathcal{I}_{-1}\mathcal{I}_2^2 + \frac{38}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_2^2 - \frac{16}{9}\mathcal{I}_{-1}^2\mathcal{I}_2^2 - \frac{22}{3}\mathcal{I}_1\mathcal{I}_2^2 - \frac{14}{3}\mathcal{I}_{-2}\mathcal{I}_1\mathcal{I}_2^2 \\
&- 34\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2^2 - \frac{95}{3}\mathcal{I}_1^2\mathcal{I}_2^2 + 12\mathcal{I}_2^3 + 20\mathcal{I}_{-2}\mathcal{I}_2^3 + \frac{76}{9}\mathcal{I}_{-1}\mathcal{I}_2^3 - \frac{28}{9}\mathcal{I}_1\mathcal{I}_2^3 + 10\mathcal{I}_2^4 \\
&+ \Big(3\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1 - \frac{3}{2}\mathcal{I}_{-1}^2\mathcal{I}_1 + 3\mathcal{I}_{-2}\mathcal{I}_1^2 - \frac{15}{2}\mathcal{I}_{-1}\mathcal{I}_1^2 - 6\mathcal{I}_1^3 + 6\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + 6\mathcal{I}_1^2\mathcal{I}_2\Big) \\
&\quad \times \sqrt{-\frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\sqrt{\mathcal{I}_{-2} - \frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\cos(2\varphi_1),
\end{aligned}
\tag{3.70}
$$

with, in particular

$$
\begin{aligned}
\mathcal{H}_2(\mathcal{I}) =& 2\mathcal{I}_{-2} + \mathcal{I}_{-1} + 4\mathcal{I}_2;\\
\\
\mathcal{Z}_4(\mathcal{I}) =& -2\mathcal{I}_{-2}^2 - 2\mathcal{I}_{-2}\mathcal{I}_{-1} - \frac{1}{2}\mathcal{I}_{-1}^2 + 2\mathcal{I}_{-2}\mathcal{I}_1 - \mathcal{I}_{-1}\mathcal{I}_1 - 2\mathcal{I}_1^2 - 12\mathcal{I}_{-2}\mathcal{I}_2 - 4\mathcal{I}_{-1}\mathcal{I}_2\\
&+ 4\mathcal{I}_1\mathcal{I}_2 - 12\mathcal{I}_2^2;\\
\\
\mathcal{Z}_6(\mathcal{I}) =& -3\mathcal{I}_{-2}^2\mathcal{I}_1 - \frac{10}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1 + \frac{3}{4}\mathcal{I}_{-1}^2\mathcal{I}_1 - \frac{3}{2}\mathcal{I}_{-2}\mathcal{I}_1^2 + \frac{17}{4}\mathcal{I}_{-1}\mathcal{I}_1^2 + \frac{10}{3}\mathcal{I}_1^3 + 6\mathcal{I}_{-2}^2\mathcal{I}_2\\
&+ \frac{16}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_2 - \frac{22}{3}\mathcal{I}_{-2}\mathcal{I}_1\mathcal{I}_2 - \frac{20}{3}\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 - 3\mathcal{I}_1^2\mathcal{I}_2 + 18\mathcal{I}_{-2}\mathcal{I}_2^2 + \frac{16}{3}\mathcal{I}_{-1}\mathcal{I}_2^2\\
&- \frac{22}{3}\mathcal{I}_1\mathcal{I}_2^2 + 12\mathcal{I}_2^3;\\
\\
\tilde{\mathcal{Z}}_8(\varphi_1, \mathcal{I}) =& -\mathcal{I}_{-2}^3\mathcal{I}_1 - \frac{35}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_1 + \frac{65}{18}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_1 + \frac{1}{8}\mathcal{I}_{-1}^3\mathcal{I}_1 - 3\mathcal{I}_{-2}^2\mathcal{I}_1^2 + 22\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1^2\\
&- \frac{29}{12}\mathcal{I}_{-1}^2\mathcal{I}_1^2 + \frac{359}{18}\mathcal{I}_{-2}\mathcal{I}_1^3 - \frac{479}{36}\mathcal{I}_{-1}\mathcal{I}_1^3 - \frac{419}{36}\mathcal{I}_1^4 + 2\mathcal{I}_{-2}^3\mathcal{I}_2 + \frac{38}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_2\\
&- \frac{16}{9}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_2 - \frac{32}{9}\mathcal{I}_{-2}^2\mathcal{I}_1\mathcal{I}_2 - 34\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + \frac{65}{9}\mathcal{I}_{-1}^2\mathcal{I}_1\mathcal{I}_2 - \frac{95}{3}\mathcal{I}_{-2}\mathcal{I}_1^2\mathcal{I}_2\\
&+ \frac{359}{9}\mathcal{I}_1^3\mathcal{I}_2 + 12\mathcal{I}_{-2}^2\mathcal{I}_2^2 + \frac{38}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_2^2 - \frac{16}{9}\mathcal{I}_{-1}^2\mathcal{I}_2^2 - \frac{14}{3}\mathcal{I}_{-2}\mathcal{I}_1\mathcal{I}_2^2\\
&- 34\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2^2 - \frac{95}{3}\mathcal{I}_1^2\mathcal{I}_2^2 + 20\mathcal{I}_{-2}\mathcal{I}_2^3 + \frac{76}{9}\mathcal{I}_{-1}\mathcal{I}_2^3 - \frac{28}{9}\mathcal{I}_1\mathcal{I}_2^3 + 10\mathcal{I}_2^4\\
&+ 44\mathcal{I}_{-1}\mathcal{I}_1^2\mathcal{I}_2\\
&+ \Big(3\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1 - \frac{3}{2}\mathcal{I}_{-1}^2\mathcal{I}_1 + 3\mathcal{I}_{-2}\mathcal{I}_1^2 - \frac{15}{2}\mathcal{I}_{-1}\mathcal{I}_1^2 - 6\mathcal{I}_1^3 + 6\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + 6\mathcal{I}_1^2\mathcal{I}_2\Big)\\
&\quad \times \sqrt{-\frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\sqrt{\mathcal{I}_{-2} - \frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\cos(2\varphi_1)\\
\\
\mathcal{Z}_8^e(\mathcal{I}) =& -\mathcal{I}_{-2}^3\mathcal{I}_1 - \frac{35}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_1 + \frac{65}{18}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_1 + \frac{1}{8}\mathcal{I}_{-1}^3\mathcal{I}_1 - 3\mathcal{I}_{-2}^2\mathcal{I}_1^2 + 22\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1^2\\
&- \frac{29}{12}\mathcal{I}_{-1}^2\mathcal{I}_1^2 + \frac{359}{18}\mathcal{I}_{-2}\mathcal{I}_1^3 - \frac{479}{36}\mathcal{I}_{-1}\mathcal{I}_1^3 - \frac{419}{36}\mathcal{I}_1^4 + 2\mathcal{I}_{-2}^3\mathcal{I}_2 + \frac{38}{9}\mathcal{I}_{-2}^2\mathcal{I}_{-1}\mathcal{I}_2\\
&- \frac{16}{9}\mathcal{I}_{-2}\mathcal{I}_{-1}^2\mathcal{I}_2 - \frac{32}{9}\mathcal{I}_{-2}^2\mathcal{I}_1\mathcal{I}_2 - 34\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + \frac{65}{9}\mathcal{I}_{-1}^2\mathcal{I}_1\mathcal{I}_2 - \frac{95}{3}\mathcal{I}_{-2}\mathcal{I}_1^2\mathcal{I}_2\\
&+ \frac{359}{9}\mathcal{I}_1^3\mathcal{I}_2 + 12\mathcal{I}_{-2}^2\mathcal{I}_2^2 + \frac{38}{3}\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_2^2 - \frac{16}{9}\mathcal{I}_{-1}^2\mathcal{I}_2^2 - \frac{14}{3}\mathcal{I}_{-2}\mathcal{I}_1\mathcal{I}_2^2\\
&- 34\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2^2 - \frac{95}{3}\mathcal{I}_1^2\mathcal{I}_2^2 + 20\mathcal{I}_{-2}\mathcal{I}_2^3 + \frac{76}{9}\mathcal{I}_{-1}\mathcal{I}_2^3 - \frac{28}{9}\mathcal{I}_1\mathcal{I}_2^3 + 10\mathcal{I}_2^4\\
&+ 44\mathcal{I}_{-1}\mathcal{I}_1^2\mathcal{I}_2\\
\\
\mathcal{Z}_8^e(\varphi_1, \mathcal{I}) =& \Big(3\mathcal{I}_{-2}\mathcal{I}_{-1}\mathcal{I}_1 - \frac{3}{2}\mathcal{I}_{-1}^2\mathcal{I}_1 + 3\mathcal{I}_{-2}\mathcal{I}_1^2 - \frac{15}{2}\mathcal{I}_{-1}\mathcal{I}_1^2 - 6\mathcal{I}_1^3 + 6\mathcal{I}_{-1}\mathcal{I}_1\mathcal{I}_2 + 6\mathcal{I}_1^2\mathcal{I}_2\Big)\\
&\quad \times \sqrt{-\frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\sqrt{\mathcal{I}_{-2} - \frac{\mathcal{I}_1}{2} + \mathcal{I}_2}\cos(2\varphi_1).
\end{aligned}
\tag{3.71}
$$

(these are computations implemented in Wolfram Mathematica, see Subsection (3.6.1).)

By defining

$$\mathcal{I}_1 = \varepsilon^2\alpha, \quad \mathcal{I}_{-1} = \varepsilon^2\beta, \quad \mathcal{I}_2 = \varepsilon^2\gamma, \quad \mathcal{I}_{-2} = \varepsilon^2\delta,$$

where β, γ, η are fixed parameters and ε is chosen suitably small, one has

$$I_1 = \varepsilon^2\alpha, \quad I_{-1} = \varepsilon^2(\alpha+\beta), \quad I_2 = \varepsilon^2(\gamma - \frac{\alpha}{2}), \quad I_{-2} = \varepsilon^2(\gamma+\delta-\frac{\alpha}{2}) \tag{3.72}$$

such that

$$\max\{0, -\beta\} \le \alpha \le \min\{2\gamma, 2\gamma+2\delta\}$$

For the sake of simplicity we fix now $(\beta,\, \gamma,\, \delta) = (0, 1, 0)$, obtaining a map

$$\Phi^4 : (\varphi_{-2},\, \varphi_{-1},\, \varphi_1,\, \varphi_2,\, \mathcal{I}_{-2},\, \mathcal{I}_{-1},\, \mathcal{I}_1,\, \mathcal{I}_2) \mapsto (\varphi_1,\, \alpha)$$

given by the relations

$$\mathcal{I}_1 = \varepsilon^2\alpha \quad \mathcal{I}_{-1} = 0, \quad \mathcal{I}_2 = \varepsilon^2, \quad \mathcal{I}_{-2} = 0. \tag{3.73}$$

Moreover, by (3.72) we have

$$I_1 = \varepsilon^2\alpha, \quad I_{-1} = \varepsilon^2\alpha, \quad I_2 = \varepsilon^2(1-\frac{\alpha}{2}), \quad I_{-2} = \varepsilon^2(1-\frac{\alpha}{2}) \quad \text{for } 0 \le \alpha \le 2. \tag{3.74}$$

If we now substitute the relations and (3.74) in the terms $\mathcal{H}_2$, $\mathcal{Z}_4$, $\mathcal{Z}_8$, $\mathcal{Z}_8^i$, $\mathcal{Z}_8^e$ written in (3.71) , we have

$$\begin{aligned}
&\mathcal{H}_2(\varphi_1,\, \alpha) = 4\varepsilon^2 \\
&\mathcal{Z}_4(\varphi_1,\, \alpha) = \varepsilon^4\left(-12+4\alpha-2\alpha^2\right) \\
&\mathcal{Z}_6(\varphi_1,\, \alpha) = \varepsilon^6\left(12 - \frac{22}{3}\alpha - 3\alpha^2 + \frac{10}{3}\alpha^3\right) \\
&\mathcal{Z}_8^i(\varphi_1,\, \alpha) = \varepsilon^8\left(10 - \frac{28}{9}\alpha - \frac{95}{3}\alpha^2 + \frac{359}{9}\alpha^3 - \frac{419}{36}\alpha^4\right) \\
&\mathcal{Z}_8^e(\varphi_1,\, \alpha) = \varepsilon^8\left(3\alpha^2(1-\alpha)(2-\alpha)\cos(2\varphi_1)\right) \\
&\tilde{\mathcal{Z}}_8(\varphi_1,\, \alpha) = \varepsilon^8\left(10 - \frac{28}{9}\alpha - \frac{95}{3}\alpha^2 + \frac{359}{9}\alpha^3 - \frac{419}{36}\alpha^4 + 3\alpha^2(1-\alpha)(2-\alpha)\cos(2\varphi_1)\right),
\end{aligned}$$

(these are, again, computations implemented in Wolfram Mathematica.)

Recalling (3.60), we have

$$\begin{aligned}
\hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\, \varepsilon^2\alpha,\, 0,\, \varepsilon^2,\, 0) =& 4\varepsilon^2 - 12\varepsilon^4 + 12\varepsilon^6 + 10\varepsilon^8 \\
&+ \alpha\left(4\varepsilon^4 - \frac{22}{3}\varepsilon^6 - \frac{28}{9}\varepsilon^8\right) \\
&+ \alpha^2\left(-2\varepsilon^4 - 3\varepsilon^6 - \frac{95}{3}\varepsilon^8 + 6\varepsilon^8\cos(2\varphi_1)\right) \\
&+ \alpha^3\left(\frac{10}{3}\varepsilon^6 + \frac{359}{9}\varepsilon^8 - 9\varepsilon^8\cos(2\varphi_1)\right) \\
&+ \alpha^4\left(-\frac{419}{36}\varepsilon^8 + 3\varepsilon^8\cos(2\varphi_1)\right).
\end{aligned}$$

By introducing the rescaled Hamiltonian

$$
\begin{aligned}
\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1,\,\alpha) :=& \varepsilon^{-2}\hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\varepsilon^2\alpha,\,0,\,\varepsilon^2,\,0)\\
=& 4-12\varepsilon^2+12\varepsilon^4+10\varepsilon^6\\
&+\alpha\left(4\varepsilon^2-\frac{22}{3}\varepsilon^4-\frac{28}{9}\varepsilon^6\right)\\
&+\alpha^2\left(-2\varepsilon^2-3\varepsilon^4-\frac{95}{3}\varepsilon^6+6\varepsilon^6\cos(2\varphi_1)\right)\\
&+\alpha^3\left(\frac{10}{3}\varepsilon^4+\frac{359}{9}\varepsilon^6-9\varepsilon^6\cos(2\varphi_1)\right)\\
&+\alpha^4\left(-\frac{419}{36}\varepsilon^6+3\varepsilon^6\cos(2\varphi_1)\right),
\end{aligned}
\tag{3.75}
$$

we have

$$
\begin{cases}
\dot{\varphi}_1 = -\dfrac{\partial\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}}{\partial\alpha},\\[2ex]
\dot{\alpha} = \dfrac{\partial\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}}{\partial\varphi_1},
\end{cases}
\tag{3.76}
$$

3.5.4 Analysis of the Critical Points

The partial derivatives of $\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$ are

$$
\begin{aligned}
\partial_{\varphi_1}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}} &= 6\varepsilon^6\alpha^2(\alpha-2)(1-\alpha)\sin(2\varphi_1)\\
\partial_{\alpha}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}} &= 4\varepsilon^2-4\alpha\varepsilon^2-\frac{22}{3}\varepsilon^4-6\alpha\varepsilon^4+10\alpha^2\varepsilon^4-\frac{28}{9}\varepsilon^6-\frac{190}{3}\alpha\varepsilon^6\\
&+\frac{359}{3}\alpha^2\varepsilon^6-\frac{419}{9}\alpha^3\varepsilon^6+3\varepsilon^6\alpha(4-9\alpha+4\alpha^2)\cos(2\varphi_1)
\end{aligned}
\tag{3.77}
$$

Since $\partial_{\varphi_1}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1,\alpha)=0$ only when $\alpha=0,\,1,\,2$ or $\varphi_1=0,\,\pi/2,\,\pi$, and since the equations $\partial_{\alpha}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1,0)=0$, $\partial_{\alpha}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1,2)=0$ have no real solutions, it follows that in the square $(\varphi_1,\alpha)\in[0,\pi]\times[0,1]$, the Hamiltonian $\hat{\mathcal{H}}$ has at most 11 critical points:

$$
\begin{aligned}
&\omega_j=\left(\frac{\pi}{2},\alpha_j\right),\quad j=1,\ldots,3,\\
&\omega_k^{(1)}=(0,\alpha_k)\,,\quad \omega_k^{(2)}=(\pi,\alpha_k)\,,\quad k=4,\ldots,6\\
&\omega_7^{(1)}=(\varphi_{1,7},1)\,,\quad \omega_7^{(2)}=(\pi-\varphi_{1,7},1)\,.
\end{aligned}
$$

where:
- $\alpha_j,\ j=1,\ldots,3$ are the solutions of

$$
\partial_{\alpha}\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}\left(\frac{\pi}{2},\alpha\right)=0,
$$

namely

$$
4\varepsilon^2-4\alpha\varepsilon^2-\frac{22}{3}\varepsilon^4-6\alpha\varepsilon^4+10\alpha^2\varepsilon^4-\frac{28}{9}\varepsilon^6-\frac{226}{3}\alpha\varepsilon^6+\frac{440}{3}\alpha^2\varepsilon^6-\frac{527}{9}\alpha^3\varepsilon^6=0,
\tag{3.78}
$$

where, $\alpha_1 < \alpha_2 < \alpha_3$.

• $\alpha_k,\ k = 4, \ldots, 6$ are the solutions of

$$\partial_\alpha \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(0, \alpha) = 0,$$

that equals to

$$4\varepsilon^2 - 4\alpha\varepsilon^2 - \frac{22}{3}\varepsilon^4 - 6\alpha\varepsilon^4 + 10\alpha^2\varepsilon^4 - \frac{28}{9}\varepsilon^6 - \frac{154}{3}\alpha\varepsilon^6 + \frac{278}{3}\alpha^2\varepsilon^6 - \frac{311}{9}\alpha^3\varepsilon^6 = 0. \quad (3.79)$$

• $\varphi_{1,7}$ is the solution of

$$\partial_\alpha \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1, 1) = 0,$$

that equals to

$$\frac{10}{3}\varepsilon^6 - \frac{20}{3}\varepsilon^8 + 3\varepsilon^8\cos(2\varphi_1) = 0,$$

namely

$$\varphi_{1,7} = \frac{1}{2}\arccos\left[\frac{10}{9}\left(2 - \frac{1}{\varepsilon^2}\right)\right]. \quad (3.80)$$

In particular, for sufficiently small ε, only three [1] critical points are contained within the square $(\varphi_1, \alpha) \in [0, \pi] \times [0, 1]$, namely ω_1, $\omega_4^{(1)}$ and $\omega_4^{(2)}$. We have moreover that

$$\alpha_k = 1 - a_k(\varepsilon^2), \quad k = 1,\ 4, \quad (3.81)$$

where a_k are positive functions with a zero of order at least 1 at the origin.

- ω_1 is a saddle point since

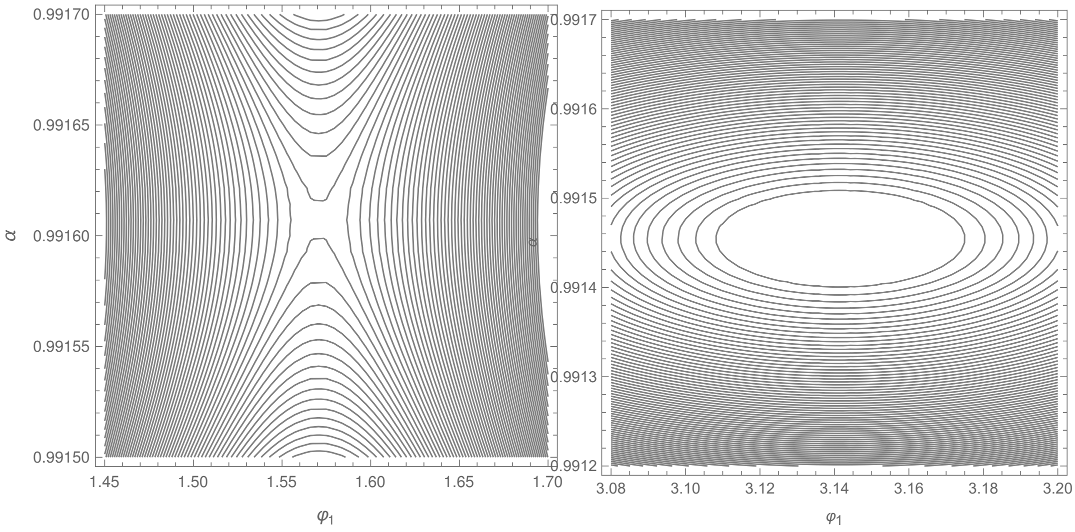
$$D^2\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\omega_1) = \begin{pmatrix} 12\varepsilon^8 a_1(\varepsilon^2) + O(\varepsilon^{12}) & 0 \\ 0 & -4\varepsilon^4 + O(\varepsilon^6) \end{pmatrix}$$

- $\omega_4^{(1)}$ and $\omega_4^{(2)}$ are equilibrium points since

$$D^2\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}\left(\omega_4^{(1)}\right) = D^2\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}\left(\omega_4^{(2)}\right) = \begin{pmatrix} -12\varepsilon^8 a_4(\varepsilon^2) + O(\varepsilon^{12}) & 0 \\ 0 & -4\varepsilon^4 + O(\varepsilon^6) \end{pmatrix}$$

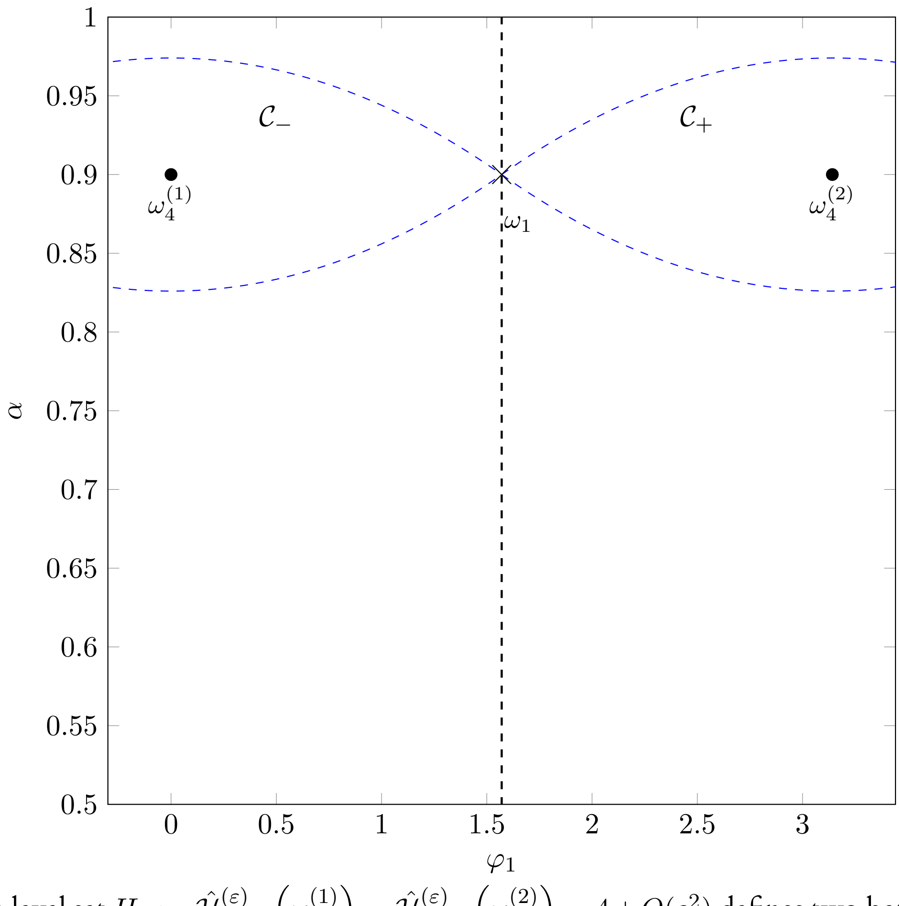
[1] Using the resolution formula for third order equations we have that, for sufficiently small ε, equations (3.78) and (3.79) only have one real solution of the form (3.81), while from (3.80) we have that $\varphi_{1,5}$ exists only if

$$\varepsilon \geq \sqrt{\frac{10}{29}}.$$



(a) Numerical simulations of the level sets around ω_1 of $\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$ with $\varepsilon = 0.1$.

(b) Numerical simulations of the level sets around $\omega_4^{(2)}$ of $\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$ with $\varepsilon = 0.1$.



The level set $H_* := \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}\left(\omega_4^{(1)}\right) = \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}\left(\omega_4^{(2)}\right) = 4 + O(\varepsilon^2)$ defines two heteroclinic orbits $\mathcal{C}_+$ and $\mathcal{C}_-$.

3.5.5 Quantitative Estimate on the Transfer

In this Subsection we prove Theorem (3.5.1). Let us start with a technical result:

Lemma 3.5.4 *Let us consider a Hamiltonian of the form*

$$\mathcal{H}_*(\varphi,\mathcal{I}) := \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\, \mathcal{I}_1,\, 0,\, \varepsilon^2,\, 0) + \mathcal{R}(\varphi,\, \mathcal{I}),$$

with

$$|\partial_\varphi \mathcal{R}(\varphi,\mathcal{I})|\,,\, |\partial_{\mathcal{I}} \mathcal{R}(\varphi,\mathcal{I})| \lesssim \varepsilon^{10}. \tag{3.82}$$

There exist $0 < \varepsilon_3 \ll 1$ *such that, for every* $\varepsilon \leq \varepsilon_3$ *there is a set,* $\mathcal{A}_\varepsilon^{(0)}$*, with the following property: for every solution* $(\varphi(t),\, \mathcal{I}(t))$ *of*

$$\begin{cases} \dot{\varphi} = -\partial_{\mathcal{I}} \mathcal{H}_* \\ \dot{\mathcal{I}} = \partial_\varphi \mathcal{H}_*, \end{cases} \tag{3.83}$$

with initial data $(\varphi(0),\, \mathcal{I}(0)) \in \mathcal{A}_\varepsilon^{(0)}$*, it holds*

$$\partial_\varphi \mathcal{H}_*(\varphi(t),\mathcal{I}(t)), \partial_{\mathcal{I}} \mathcal{H}_*(\varphi(t),\mathcal{I}(t)) > 0, \quad \partial_{\varphi_1} \mathcal{H}_*(\varphi(t),\mathcal{I}(t)) \sim \varepsilon^8, \quad \partial_{\mathcal{I}_1} \mathcal{H}_*(\varphi(t),\mathcal{I}(t)) \sim \varepsilon^2, \tag{3.84}$$

for all $t \lesssim \varepsilon^{-2}$.

Proof: From what we have seen in Subsection (3.5.4) (in particular (3.81)), we have that there exist $\varepsilon_4 \ll 1$ and $\delta_{\varepsilon_4} \sim \varepsilon_4^2$ such that, for $\varepsilon \leq \varepsilon_4$, the derivatives $\partial_{\varphi_1} \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$, $\partial_\alpha \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$ are all strictly positive in

$$\left(\frac{\pi}{2} + \delta_{\varepsilon_4}, \pi - \delta_{\varepsilon_4}\right) \times (\delta_{\varepsilon_4}, 1 - \delta_{\varepsilon_4}).$$

It follows that, for every $\varepsilon \leq \varepsilon_4$,

$$\partial_{\varphi_1} \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1, \alpha) \sim \varepsilon^6, \quad \partial_\alpha \hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}(\varphi_1, \alpha) \sim \varepsilon^2,$$

$$\forall (\varphi_1,\, \alpha) \in \left(\frac{\pi}{2} + \delta_{\varepsilon_4}, \pi - \delta_{\varepsilon_4}\right) \times (\delta_{\varepsilon_4}, 1 - \delta_{\varepsilon_4}).$$

Keeping in mind (3.73) and (3.75), the above relations imply that

$$\begin{gathered} \partial_{\varphi_1} \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\, \mathcal{I}_1,\, 0,\, \varepsilon^2,\, 0) \sim \varepsilon^8, \quad \partial_{\mathcal{I}_1} \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\, \mathcal{I}_1,\, 0,\, \varepsilon^2,\, 0) \sim \varepsilon^2, \\ \forall (\varphi_1,\, \mathcal{I}_1) \in \left(\frac{\pi}{2} + \delta_{\varepsilon_4}, \pi - \delta_{\varepsilon_4}\right) \times \left(\varepsilon^2 \cdot \delta_{\varepsilon_4},\, \varepsilon^2 \cdot (1 - \delta_{\varepsilon_4})\right), \quad \forall \varepsilon \leq \varepsilon_4. \end{gathered} \tag{3.85}$$

Moreover, thanks to the smallness assumption (3.82), a similar estimate to the one displayed in (3.85) holds, also for $|\partial_\varphi \mathcal{H}_*|$, $|\partial_{\mathcal{I}} \mathcal{H}_*|$ when $(\varphi, \mathcal{I})$ belong to the set

$$\tilde{\mathcal{A}}_\varepsilon^{(0)} := \left\{(\varphi,\mathcal{I}) \;:\; (\varphi_1,\, \mathcal{I}_1) \in \left(\frac{\pi}{2} + \delta_{\varepsilon_3}, \pi\delta_{\varepsilon_3}\right) \times \left(\varepsilon^2 \cdot \delta_{\varepsilon_3},\, \varepsilon^2 \cdot (1 - \delta_{\varepsilon_3})\right)\right\}. \tag{3.86}$$

for appropriate choice of $\varepsilon_3 \ll 1$ and $\delta_{\varepsilon_3} \sim \varepsilon_3^2$.
Finally, if we define

$$\mathcal{A}_\varepsilon^{(0)} := \left\{(\varphi,\mathcal{I}) \in \tilde{\mathcal{A}}_\varepsilon^{(0)} : \operatorname{dist}\big((\varphi_1,\mathcal{I}_1), \partial \tilde{\mathcal{A}}_\varepsilon^{(0)}\big) \geq \frac{\varepsilon^2(1 - 2\delta_{\varepsilon_3})}{3}\right\}, \tag{3.87}$$

since

$$\partial_{\varphi_1}(\varphi,\,\mathcal{I})\mathcal{H}_* \sim \varepsilon^8, \quad \partial_{\varphi_1}\mathcal{H}_*(\varphi,\,\mathcal{I}) \sim \varepsilon^2, \quad \text{for} \quad (\varphi,\,\mathcal{I}) \in \tilde{\mathcal{A}}_\varepsilon^{(0)},$$

we have that a solution of (3.83) with initial data in $\mathcal{A}_\varepsilon^{(0)}$, will remain in $\tilde{\mathcal{A}}_\varepsilon^{(0)}$ at least for times $t \lesssim \varepsilon^{-2}$.
This concludes the proof. □

Proof of Theorem (3.5.1)*:*
First, we choose

$$\varepsilon_* := \min\{\varepsilon_0,\,\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3\}, \tag{3.88}$$

for ε_0 in Theorem (A.1.3) ε_1 in Lemma (3.5.1), ε_2 in Lemma (3.5.2) and ε_3 in Lemma (3.5.4).
By recalling the relations (3.73), and keeping in mind Remark (3.5.4), for $\varepsilon \leq \varepsilon_*$ we define

$$\mathcal{A}_\varepsilon^{(1)} := \Big\{(\varphi_1,\,\varphi_{-1},\,\varphi_2,\varphi_{-2},\,\mathcal{I}_1,\,\mathcal{I}_{-1},\,\mathcal{I}_2,\,\mathcal{I}_{-2}) \in \mathcal{A}_\varepsilon^{(0)} \;: \\ \mathcal{I}_{-1} = 0,\,\mathcal{I}_2 = \varepsilon^2,\,\mathcal{I}_{-2} = 0.\Big\}$$

Now, using the maps Φ^2, Φ^3, defined respectively in (3.65) and (3.67), we set

$$\mathcal{A}_\varepsilon^{(2)} := \left(\Phi^3 \circ \Phi^2\right)^{-1}\left(\mathcal{A}_\varepsilon^{(1)}\right).$$

We set

$$\mathcal{A}_\varepsilon := \Big\{w = \{w_j\}_{j\in\mathbb{Z}} :\; w_j \in \mathcal{A}_2,\;\text{if}\, j = \pm 1,\,\pm 2, \quad z_j = 0,\;\text{otherwise}\Big\}.$$

Within this framework, the study of Equation (3.55) with initial data

$$w_0(x) = \frac{1}{\sqrt{2\pi}} \sum_{|j|\in\{1,\,2\}} w_{j,0} e^{ij\cdot x},$$

such that $\{w_{j,0}\}_{j\in\mathbb{Z}} \in \mathcal{A}$, is equivalent to study the finite dimensional system

$$\begin{cases} \dot{\varphi}(t) = -\partial_{\mathcal{I}}\mathcal{H}_{\mathbf{BNF}} \\ \dot{\mathcal{I}}(t) = \partial_\varphi \mathcal{H}_{\mathbf{BNF}}, \end{cases} \tag{3.89}$$

where $\mathcal{H}_{\mathbf{BNF}}$ is the restricted Hamiltonian

$$\mathcal{H}_{\mathbf{BNF}} = \hat{\mathcal{H}}_{\mathbf{BNF}} + \mathcal{R}_{\geq 10}$$

defined in (3.60) and $w = \{w_j\}_{j=\pm 1,\pm 2} \in \mathcal{A}_2^{(\varepsilon)}$.
Let us now adopt the variables $(\varphi,\,\mathcal{I})$ defined in (3.68), we have the following facts:

Claim 1:
Let us suppose $(\varphi(t),\,\mathcal{I}(t))$ is a solution of (3.89) with initial data

$$\mathcal{I}_j(0) \lesssim \varepsilon^2, \quad j = \pm 1, \pm 2,$$

then for all $t \in [0, T]$ with $T \lesssim \varepsilon^{-6}$, we have that

$$|\mathcal{I}_j(t) - \mathcal{I}_j(0)| \lesssim t \cdot \varepsilon^{10}, \quad j = 1,\,\pm 2. \tag{3.90}$$

Proof Claim 1:
Recalling (3.70) we have that $\hat{\mathcal{H}}_{\mathbf{BNF}}$ does not depends on φ_j for $j=1,\pm 2$, hence

$$\begin{aligned}\dot{\mathcal{I}}_j(t) &= \partial_{\varphi_j}\mathcal{H}_{\mathbf{BNF}}\\ &= \partial_{\varphi_j}\hat{\mathcal{H}}_{\mathbf{BNF}} + \partial_{\varphi_j}\mathcal{R}_{\geq 10}\\ &= \partial_{\varphi_j}\mathcal{R}_{\geq 10} \overset{(3.64),(3.61)}{\lesssim} \varepsilon^{10}.\end{aligned}$$

Claim 2:
Let us suppose $(\varphi(t),\,\mathcal{I}(t))$ is a solution of (3.89) with initial data

$$\mathcal{I}_1(0)\leq\varepsilon^2,\quad ,\mathcal{I}_{-1}(0)=0,\quad \mathcal{I}_2(0)=\varepsilon^2,\quad \mathcal{I}_{-2}(0)=0$$

we have

$$\left|\partial_{\varphi_1}\hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,\mathcal{I}_{-1},\,\mathcal{I}_2,\,\mathcal{I}_{-2}) - \partial_{\varphi_1}\hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,0,\,\varepsilon^2,\,0)\right| \lesssim \sqrt{t}\cdot O(\varepsilon^{11}) + t\cdot O(\varepsilon^{15}), \tag{3.91}$$

for all $t\in[0,T]$, with $T\lesssim\varepsilon^{-6}$.

Proof Claim 2:
From (3.90) one has that

$$|\mathcal{I}_{-2}(t)|\lesssim t\cdot\varepsilon^{10},\quad |\mathcal{I}_{-1}(t)|\lesssim t\cdot\varepsilon^{10},\quad |\mathcal{I}_{-1}(t)-\varepsilon^2|\lesssim t\cdot\varepsilon^{10}. \tag{3.92}$$

Let us note that

$$\partial_{\varphi_1}\hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,\mathcal{I}_{-1},\,\mathcal{I}_2,\,\mathcal{I}_{-2}) \overset{(3.70)}{=} \partial_{\varphi_1}\tilde{\mathcal{Z}}_8^{\mathfrak{e}}(\varphi_1,\,\mathcal{I}_1,\,\mathcal{I}_{-1},\,\mathcal{I}_2,\,\mathcal{I}_{-2}).$$

Now, recalling the expression of $\tilde{\mathcal{Z}}_8^{\mathfrak{e}}$ in (3.71), using (3.64) and (3.92), together with the inequality

$$|\sqrt{x_1}-\sqrt{x_1}|\leq\sqrt{|x_1-x_2|},\quad x_1,\,x_2\geq 0,$$

one obtains (3.91).

Let us now prove inequality (3.56) for $j=1$:
Let us first note that the Birkhoff normal form Hamiltonian $\mathcal{H}_{\mathbf{BNF}}(\varphi,\mathcal{I}) = \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi,\mathcal{I}) + \mathcal{R}_{\geq 10}(\varphi,\mathcal{I})$ can be written as

$$\mathcal{H}_{\mathbf{BNF}}(\varphi,\mathcal{I}) = \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,0,\,\varepsilon^2,\,0) + \mathcal{R}(\varphi,\mathcal{I}), \tag{3.93}$$

with

$$\mathcal{R}(\varphi,\mathcal{I}) := \mathcal{R}_{\geq 10}(\varphi,\mathcal{I}) + \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,\mathcal{I}_{-1},\,\mathcal{I}_2,\,\mathcal{I}_{-2}) - \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1,\,\mathcal{I}_1,\,0,\,\varepsilon^2,\,0).$$

Now, if $(\varphi(t),\,\mathcal{I}(t))$ is a solution of (3.89) with initial data in $\mathcal{A}_\varepsilon^{(2)}$, from (3.91) together with (3.61) and (3.90), that

$$|\partial_\varphi\mathcal{R}(\varphi(t),\,\mathcal{I}(t))|\,,\,|\partial_{\mathcal{I}}\mathcal{R}(\varphi(t),\,\mathcal{I}(t))|\lesssim\varepsilon^{10}\quad\text{for}\quad t\lesssim\varepsilon^{-6}.$$

We can then apply Lemma (3.5.4).
Thanks to the latter we have that

$$\dot{\mathcal{I}}_1 > 0 \quad \text{for} \quad t \lesssim \varepsilon^{-2},$$

moreover, recalling that $\mathcal{I}_1 = I_1$, it holds

$$\begin{aligned} |I_1(t) - I_1(0)| &= |\mathcal{I}_1(t) - \mathcal{I}_1(0)| \\ &= \mathcal{I}_1(t) - \mathcal{I}_1(0) \\ &= \int_0^t \dot{\mathcal{I}}_1(s) ds \\ &\overset{(3.93)}{\leq} \int_0^t \partial_{\varphi_1} \hat{\mathcal{H}}_{\mathbf{BNF}}(\varphi_1, \mathcal{I}) + \mathcal{R}(\varphi, \mathcal{I}) ds \\ &\overset{(3.84)}{\gtrsim} t\varepsilon^8. \end{aligned} \tag{3.94}$$

The opposite inequality in (3.56) follows similarly.
For what concerns the remaining cases $k = -1, \pm 2$, let us consider for example $k = 2$ (the others are equivalent):
we recall from (3.68) that

$$I_2 = -\frac{1}{2}\mathcal{I}_1 + \mathcal{I}_2,$$

then

$$\begin{aligned} \dot{I}_2 &= \dot{\mathcal{I}}_2 - \frac{1}{2}\dot{\mathcal{I}}_1 \\ &= \partial_{\varphi_2}\mathcal{H}_{\mathbf{BNF}} - \frac{1}{2}\partial_{\varphi_1}\mathcal{H}_{\mathbf{BNF}} \\ &= \partial_{\varphi_2}\hat{\mathcal{H}}_{\mathbf{BNF}} - \frac{1}{2}\partial_{\varphi_1}\hat{\mathcal{H}}_{\mathbf{BNF}} + \partial_{\varphi_2}\mathcal{R}_{\geq 10} - \frac{1}{2}\partial_{\varphi_1}\mathcal{R}_{\geq 10} \\ &\overset{(3.70)}{=} -\frac{1}{2}\partial_{\varphi_1}\hat{\mathcal{H}}_{\mathbf{BNF}} + \partial_{\varphi_1}\mathcal{R}_{\geq 10} - \frac{1}{2}\partial_{\varphi_2}\mathcal{R}_{\geq 10}. \end{aligned}$$

Inequality (3.56) he proof follows by reasoning as above, using again (3.84). □

3.5.6 Conclusions

We stress that the energy transfer analysis performed in this section is carried out in Birkhoff normal form coordinates.
While the Birkhoff normal form transformation is canonical and preserves the Hamiltonian structure, the lower bound (3.56) on the energy transfer does not trivially transport back to the original Kirchhoff variables.
More precisely, as estimate (3.63) shows, the Birkhoff normal form transformation acts on the linear actions $\{I_k\}_k$ as

$$\mathrm{Id} + O(\varepsilon^4).$$

For this reason pulling back this estimate to the original variables would introduce correction terms of order ε^4 which dominate the bound ε^8 in (3.56) and thus prevent any meaningful conclusion in the original coordinates.
Considering all this, Theorem (3.5.1) should be interpreted as a non-integrability result for the formal Birkhoff normal form, as well as an evidence of energy transfer at the level of the normal form dynamics, leaving open the question of a sharp lower bound in the original coordinates.

3.6 Further Directions

The energy transfer analysis carried out in Subsection 3.5.5 is localized in a region of the phase space that avoids both equilibrium points and secondary invariant tori, and is performed on a truncated four-mode system in Birkhoff normal form.
While this allows us to exhibit explicit orbits along which energy transfer between Fourier modes, two natural questions remain open: whether analogous transfer phenomena persist in the original equation, and how the dynamics behaves near such invariant structures.
As a natural continuation of this work, we plan to investigate the existence and stability of such tori for the Kirchhoff equation in the spirit of [62], where a KAM scheme is developed to construct small-amplitude quasi-periodic solutions in a closely related setting.

3.6.1 Codes For Section 3.5

• The Resonant Terms Z_4, Z_6, Z_8

We define the projector operator $\Pi_{\mathcal{K}}$ as

```
ker[N_] := Coefficient[t*fF[N], t]
```

where

```
fF[N_] := N /. {x[1] -> t*x[1], y[1] -> 1/t*y[1], Y[1] -> 1/t*Y[1],
        X[1] -> t*X[1], x[2] -> t^2*x[2], y[2] -> 1/t^2*y[2],
        Y[2] -> 1/t^2*Y[2], X[2] -> t^2*X[2]}
```

The map Φ^2, defined in (3.65), that gives the symplectic polar coordinates (I, φ) is given by

```
sost2[N_] :=
N /. {x[j_] :> (A[j])^{1/2}*E^{I a[j]},
      y[j_] :> (A[j])^{1/2}*E^{-I a[j]},
      X[j_] :> (A[-j])^{1/2}*E^{I a[-j]},
      Y[j_] :> (A[-j])^{1/2}*E^{-I a[-j]}}
```

Where `A[j]` $= I_j$ are the actions and `a[j]` $= \theta_j$ are the angles.

The map Φ^3 in (3.67), that defines coordinates $(\mathcal{I}, \varphi)$ is given by

```
sost3[N_] :=
N /. {A[1] :> B[1], A[-1] :> B[1] + B[-1], A[2] :> -1/2 B[1] + B[2],
        A[-2] :> -1/2 B[1] + B[2] + B[-2],
        2 I a[-1] + 2 I a[1] - I a[2] - I a[-2] :>
        2 I \[CurlyPhi], -2 I a[-1] - 2 I a[1] + I a[2] +
        I a[-2] :> -2 I \[CurlyPhi]} /.
Exp[2 I \[CurlyPhi]] :> 2*Cos[2 \[CurlyPhi]] - Exp[-2 I \[CurlyPhi]]
```

Where

`B[j]` $= I_j$ and `\[CurlyPhi]` $= \varphi_1$.

Finally, the map Φ^4 in (3.73), that defines coordinates $(\mathcal{I}, \varphi)$ is given by

```
sost4[N_] :=
N /. {B[1] :> \[CurlyEpsilon]^2 \[Alpha], B[-1] :> 0,
        B[2] :> \[CurlyEpsilon]^2, B[-2] :> 0}
```

where

$$\alpha = \texttt{\textbackslash[Alpha]} \quad \text{and} \quad \varepsilon = \texttt{\textbackslash[CurlyEpsilon]}.$$

The resonant terms $\mathcal{Z}_4$, $\mathcal{Z}_6$, $\mathcal{Z}_8$ are

```
Z4 := ker[H4]
```

```
Z6 := ker[h6]
```

```
Z8 := ker[h8]
```

The term $\mathcal{Z}_8^e$ is:

```
N8 := Coefficient[int[Z8], (\[Alpha]^2 \[Beta]^2)/(\[Xi] \[Omega])] +
      Coefficient[int[Z8], (\[Xi] \[Omega])/(\[Alpha]^2 \[Beta]^2)] /.
      j[i_] :> i // Expand
```

while $\mathcal{Z}_8^i$ is:

```
I8:=Z8-N8
```

In action-angle variables the terms $\mathcal{Z}_8^e$, $\mathcal{Z}_8^i$ are

```
In[]:=sost[N8]

Out={3/2 E^(-I a[-2] + 2 I a[-1] + 2 I a[1] - I a[2]) A[-2]^(3/2)
        A[-1] A[1] Sqrt[A[2]] +
        3/2 E^(I a[-2] - 2 I a[-1] - 2 I a[1] + I a[2]) A[-2]^(3/2)
        A[-1] A[1] Sqrt[A[2]] -
        3/4 E^(-I a[-2] + 2 I a[-1] + 2 I a[1] - I a[2]) Sqrt[A[-2]]
        A[-1]^2 A[1] Sqrt[A[2]] -
        3/4 E^(I a[-2] - 2 I a[-1] - 2 I a[1] + I a[2]) Sqrt[A[-2]]
        A[-1]^2 A[1] Sqrt[A[2]] -
        3/4 E^(-I a[-2] + 2 I a[-1] + 2 I a[1] - I a[2]) Sqrt[A[-2]]
        A[-1] A[1]^2 Sqrt[A[2]] -
        3/4 E^(I a[-2] - 2 I a[-1] - 2 I a[1] + I a[2]) Sqrt[A[-2]]
        A[-1] A[1]^2 Sqrt[A[2]] +
        3/2 E^(-I a[-2] + 2 I a[-1] + 2 I a[1] - I a[2]) Sqrt[A[-2]]
        A[-1] A[1] A[2]^(3/2) +
        3/2 E^(I a[-2] - 2 I a[-1] - 2 I a[1] + I a[2]) Sqrt[A[-2]]
        A[-1] A[1] A[2]^(3/2)}
```

```
In[]:=sost[I8]

Out[]={-(16/9) A[-2]^2 A[-1] A[1] + 49/18 A[-2] A[-1]^2 A[1] +
        1/8 A[-1]^3 A[1] + 49/18 A[-2] A[-1] A[1]^2 + 3/8 A[-1]^2 A[1]^2
```

```
        + 1/8 A[-1] A[1]^3 + 2 A[-2]^3 A[2] + 38/9 A[-2]^2 A[-1] A[2]
        - 16/9 A[-2] A[-1]^2 A[2] + 38/9 A[-2]^2 A[1] A[2] -
        128/9 A[-2] A[-1] A[1] A[2] + 49/18 A[-1]^2 A[1] A[2] -
        16/9 A[-2] A[1]^2 A[2] + 49/18 A[-1] A[1]^2 A[2] +
        6 A[-2]^2 A[2]^2 + 38/9 A[-2] A[-1] A[2]^2 +
      38/9 A[-2] A[1] A[2]^2 - 16/9 A[-1] A[1] A[2]^2 + 2 A[-2] A[2]^3}
```

Finally, the partial Hamiltonian $\hat{\mathcal{H}}_{BNF} = \mathcal{H}_2 + \mathcal{Z}_4 + \mathcal{Z}_6 + \tilde{\mathcal{Z}}_8$ is

```
h := H2 + Z4 + Z6 + Z8 /. j[i_] :> i
```

Its expression,(3.70), in the variables $(\mathcal{I}, \varphi_1)$ is

```
In[]:= sost3[sost2 [h]] // Expand

Out[]={2 B[-2] - 2 B[-2]^2 + B[-1] - 2 B[-2] B[-1] - 1/2 B[-1]^2 +
        2 B[-2] B[1] - 3 B[-2]^2 B[1] - B[-2]^3 B[1] - B[-1] B[1] -
        10/3 B[-2] B[-1] B[1] - 35/9 B[-2]^2 B[-1] B[1] +
        3/4 B[-1]^2 B[1] + 65/18 B[-2] B[-1]^2 B[1] + 1/8 B[-1]^3 B[1]
        - 2 B[1]^2 - 3/2 B[-2] B[1]^2 - 3 B[-2]^2 B[1]^2 +
        17/4 B[-1] B[1]^2 + 22 B[-2] B[-1] B[1]^2 - 29/12 B[-1]^2 B[1]^2
        + (10 B[1]^3)/3 + 359/18 B[-2] B[1]^3 - 479/36 B[-1] B[1]^3 - (
        419 B[1]^4)/36 + 4 B[2] - 12 B[-2] B[2] + 6 B[-2]^2 B[2] +
        2 B[-2]^3 B[2] - 4 B[-1] B[2] + 16/3 B[-2] B[-1] B[2] +
        38/9 B[-2]^2 B[-1] B[2] - 16/9 B[-2] B[-1]^2 B[2] + 4 B[1] B[2]
         - 22/3 B[-2] B[1] B[2] - 32/9 B[-2]^2 B[1] B[2] -
        20/3 B[-1] B[1] B[2] - 34 B[-2] B[-1] B[1] B[2] +
        65/9 B[-1]^2 B[1] B[2] - 3 B[1]^2 B[2] - 95/3 B[-2] B[1]^2 B[2]
        + 44 B[-1] B[1]^2 B[2] + 359/9 B[1]^3 B[2] - 12 B[2]^2 +
        18 B[-2] B[2]^2 + 12 B[-2]^2 B[2]^2 + 16/3 B[-1] B[2]^2 +
        38/3 B[-2] B[-1] B[2]^2 - 16/9 B[-1]^2 B[2]^2 - 22/3 B[1] B[2]^2
        - 14/3 B[-2] B[1] B[2]^2 - 34 B[-1] B[1] B[2]^2 -
        95/3 B[1]^2 B[2]^2 + 12 B[2]^3 + 20 B[-2] B[2]^3 +
        76/9 B[-1] B[2]^3 - 28/9 B[1] B[2]^3 + 10 B[2]^4 +
        3 B[-2] B[-1] B[1] Sqrt[-(B[1]/2) + B[2]] Sqrt[
        B[-2] - B[1]/2 + B[2]] Cos[2 \[CurlyPhi]] -
        3/2 B[-1]^2 B[1] Sqrt[-(B[1]/2) + B[2]] Sqrt[B[-2] - B[1]/2 +
        B[2]]Cos[2 \[CurlyPhi]] +
        3 B[-2] B[1]^2 Sqrt[-(B[1]/2) + B[2]] Sqrt[B[-2] - B[1]/2 +
        B[2]]Cos[2 \[CurlyPhi]] -
        15/2 B[-1] B[1]^2 Sqrt[-(B[1]/2) + B[2]] Sqrt[B[-2] - B[1]/2 +
         B[2]] Cos[2 \[CurlyPhi]] - 6 B[1]^3 Sqrt[-(B[1]/2) +
          B[2]] Sqrt[B[-2] - B[1]/2 + B[2]]Cos[2 \[CurlyPhi]] +
        6 B[-1] B[1] B[2] Sqrt[-(B[1]/2) + B[2]] Sqrt[B[-2] - B[1]/2 +
        B[2]]Cos[2 \[CurlyPhi]] +
        6 B[1]^2 B[2] Sqrt[-(B[1]/2) + B[2]] Sqrt[B[-2] - B[1]/2 + B[2]]
        Cos[2 \[CurlyPhi]]}
```

while, the expression (3.75) of $\hat{\mathcal{H}}^{(\varepsilon)}_{\mathbf{BNF}}$, in the variables (α, φ_1) is

```
In[]:= sost4[sost3[sost2[h]]]/ // Expand

Out[]={4 - 12 \[CurlyEpsilon]^2 + 4 \[Alpha] \[CurlyEpsilon]^2 -
        2 \[Alpha]^2 \[CurlyEpsilon]^2 + 12 \[CurlyEpsilon]^4
```

```
 - (22 \[Alpha] \[CurlyEpsilon]^4)/3 -
3 \[Alpha]^2 \[CurlyEpsilon]^4 +
+ (10 \[Alpha]^3 \[CurlyEpsilon]^4)/
3 + 10 \[CurlyEpsilon]^6 - (28 \[Alpha] \[CurlyEpsilon]^6)/9
 - (95 \[Alpha]^2 \[CurlyEpsilon]^6)/3
+ (359 \[Alpha]^3 \[CurlyEpsilon]^6)/9
- (419 \[Alpha]^4 \[CurlyEpsilon]^6)/36 +
6 \[Alpha]^2 \[CurlyEpsilon]^6 Cos[2 \[CurlyPhi]] -
9 \[Alpha]^3 \[CurlyEpsilon]^6 Cos[2 \[CurlyPhi]] +
3 \[Alpha]^4 \[CurlyEpsilon]^6 Cos[2 \[CurlyPhi]]}
```

Chapter 4

Almost Periodic Solutions

4.1 Introduction

This chapter is devoted to prove the existence of almost-periodic solutions for the semilinear Kirchhoff-type equation

$$iu_t = \Lambda u + V * u + \mathbf{f}\left(\int_{\mathbb{T}} (u+\bar{u})\mathfrak{B}[u+\bar{u}]\right)\mathfrak{B}[u+\bar{u}], \tag{4.1}$$

where $\mathbf{f}$ is a real analytic function in an neighborhood of $y=0$ with a zero of order at least 2 in $y=0$ and Λ, V, $\mathfrak{B}$ are Fourier Multiplier such that

1. $V * e^{ikx} = |k|\tilde{V}_k e^{ikx}$ for $\{\tilde{V}_k\}_{k\in\mathbb{Z}} \in \ell^\infty$;
2. $\Lambda = \sqrt{-\Delta}$ and $\Lambda e^{ikx} = |k|e^{ikx}$;
3. $\mathfrak{B}[e^{ik\cdot x}] = \mathfrak{b}_k e^{ik\cdot x}$ for $\{\mathfrak{b}_k\}_{k\in\mathbb{Z}} \in \ell^\infty$.

We suppose moreover that $\{\tilde{V}_k\}_{k\in\mathbb{Z}}$, $\{\mathfrak{b}_k\}_{k\in\mathbb{Z}}$ are symmetric with respect to k, in particular $\tilde{V}_0 = \mathfrak{b}_0 = 0$.

In Subsection 4.1.1 we show how equation (4.1) is related to the Kirchhoff equation (1.1). In Subsection 4.1.2 we study the abstract properties of equation (4.1), while in Section 4.2 we introduce a suitable functional framework, which will be fundamental for solving the homological equation (see Section 4.3).
In particular, we introduce the functional space $\mathfrak{g}_{\boldsymbol{h},s}$ and endow the space of formal power series of Kirchhoff type, $\mathcal{F}_{\mathbf{kir}}$, with a Banach space structure by means of the inhomogeneous weighted Lipschitz norm (4.12).
In section 4.3 we introduce the Diohantine conditions (4.3.1) that allows us to solve the homological equation (4.42).
In Section 4.4 we introduce the projection operators (4.71), which provide a degree decomposition according to the order of vanishing of the Hamiltonian at the torus $\mathcal{T}_{\mathcal{I}}$.
In Section 4.5 we reformulate Theorem 1.2.9 in terms of the twisted-conjugacy Theorem 4.5.1, whose proof is achieved via the Iterative Lemma 4.5.1.
Finally, in Section 4.6 we prove Theorem 1.2.10.

4.1.1 Real Problem

Let us consider a second order semi-linear PDE of the form

$$\psi_{tt} - \Delta\psi + \mathbf{f}\left(\int_{\mathbb{T}^n} \psi\tilde{\mathfrak{B}}\psi\right)\tilde{\mathfrak{B}}\psi + W * \psi = 0, \tag{4.2}$$

where

- $\mathbf{f}$ is an analytic function with a zero of order at least 2 at the origin;
- $W*$ and $\tilde{\mathfrak{B}}$ are the Fourier multipliers

$$W * e^{ikx} = |k|\tilde{W}_k e^{ikx}, \quad \tilde{\mathfrak{B}} e^{ikx} = \tilde{\mathfrak{b}}_k e^{ikx}$$

  with

$$\{\tilde{W}_k\}_{k\in\mathbb{Z}}, \left\{\frac{|\tilde{\mathfrak{b}}_k|}{|k|}\right\}_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}), \quad \text{symmetric with respect to } k.$$

  In particular $W_0,\ \mathfrak{b}_0 = 0$.

This equation can be viewed as a semilinear analog of the Kirchhoff equation.
While the classical Kirchhoff model (1.1) features a non-local nonlinearity involving the gradient norm, our model replaces it with the Fourier multiplier $\tilde{\mathfrak{B}}$.
This formally preserves the original non-local structure of (1.1) and allows us to investigate the dynamics while in a semilinear framework.
The reason of this choice relies on the fact that the Bourgain approach only works in the semilinear case. Indeed, in presence of derivatives on the nonlinearity, the analyticity of $\mathbf{f}$ do not guarantees that the associated Hamiltonian is regular in the sense of definition 4.12 (see Theorem 4.2.3).
Equation (4.2) is an infinite dimensional dynamical system in the variables $(\psi,\eta) = (\psi,\psi_t)$, with Hamiltonian

$$H(\psi,\eta) = \frac{1}{2}\int_{\mathbb{T}} \eta^2 + \frac{1}{2}\int_{\mathbb{T}^n} (D\psi)^2 + \frac{1}{2}\mathbf{F}\left(\int_{\mathbb{T}^n} \psi\tilde{\mathfrak{B}}\psi\right).$$

Here $\mathcal{D} = \sqrt{-\Delta + W*}$.

Adopting the complex coordinates

$$\begin{cases} u = \dfrac{1}{\sqrt{2}}(\mathcal{D}^{\frac{1}{2}}\psi + i\mathcal{D}^{-\frac{1}{2}}\eta) \\ \\ \bar{u} = \dfrac{1}{\sqrt{2}}(\mathcal{D}^{\frac{1}{2}}\psi - i\mathcal{D}^{-\frac{1}{2}}\eta) \end{cases} \iff \begin{cases} \psi = \dfrac{1}{\sqrt{2}}(\mathcal{D}^{-\frac{1}{2}}u + \mathcal{D}^{-\frac{1}{2}}\bar{u}) \\ \\ \eta = \dfrac{1}{i\sqrt{2}}(\mathcal{D}^{\frac{1}{2}}u - \mathcal{D}^{\frac{1}{2}}\bar{u}) \end{cases}$$

equation (4.2) become

$$\begin{cases} u_t = -i\mathcal{D}u - \dfrac{i}{2}\mathbf{f}\left(\dfrac{1}{2}\int_{\mathbb{T}}(u+\bar{u})\mathcal{D}^{-1}\tilde{\mathfrak{B}}[u+\bar{u}]\right)\mathcal{D}^{-1}\tilde{\mathfrak{B}}(u+\bar{u}) \\ \\ \bar{u}_t = i\mathcal{D}\bar{u} + \dfrac{i}{2}\mathbf{f}\left(\dfrac{1}{2}\int_{\mathbb{T}}(u+\bar{u})\mathcal{D}^{-1}\tilde{\mathfrak{B}}[u+\bar{u}]\right)\mathcal{D}^{-1}\tilde{\mathfrak{B}}(u+\bar{u}) \end{cases}. \tag{4.3}$$

By defining

$$\mathfrak{B} := \frac{1}{2}\mathcal{D}^{-1}\tilde{\mathfrak{B}}, \quad \mathfrak{B}\left[e^{ik\cdot x}\right] = \frac{\tilde{\mathfrak{b}}_k}{2\sqrt{k^2 + W_k}}e^{ik\cdot x},$$

and

$$V* := \mathcal{D} - \sqrt{-\Delta}, \quad V * e^{ik\cdot x} = \frac{W_k}{\sqrt{W_k + k^2} + |k|}e^{ik\cdot x},$$

we can write system (4.3)in the more suitable form

$$\begin{cases} u_t = -i\Lambda u - V * u - \dfrac{i}{2}\mathbf{f}\left(\int_{\mathbb{T}}(u+\bar{u})\mathfrak{B}[u+\bar{u}]\right)\mathfrak{B}(u+\bar{u}) \\ \\ \bar{u}_t = i\Lambda \bar{u} + V * \bar{u} + \dfrac{i}{2}\mathbf{f}\left(\int_{\mathbb{T}}(u+\bar{u})\mathfrak{B}[u+\bar{u}]\right)\mathfrak{B}[u+\bar{u}] \end{cases},$$

that, recalling $\Lambda = \sqrt{-\Delta}$, is exactly equation (4.1).

4.1.2 Abstract structure of H

The Hamiltonian

$$H(u,\bar{u}) = \int_{\mathbb{T}} u \cdot \Lambda \bar{u} + \int_{\mathbb{T}} V * |u|^2 + \frac{1}{2}\mathbf{F}\left(\int_{\mathbb{T}}(u+\bar{u})\mathfrak{B}[u+\bar{u}]\right) \tag{4.4}$$

associated to equation (4.1), can be represented, from a purely algebraic point of view, as a polynomial

$$H(u,\bar{u}) = \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} H_{\alpha,\beta} u^{\alpha}\bar{u}^{\beta}$$

in terms of the Fourier coefficients $\{u_j\}_{j\in\mathbb{Z}}$ of u.
It is, indeed, an element of the space $\mathcal{F}$ discussed in Section 3.1.1.
Moreover, since the non-linear part

$$\frac{1}{2}\mathbf{F}\left(\int_{\mathbb{T}}(u+\bar{u})\,\mathfrak{B}[u+\bar{u}]\right)$$

shares the same structure of the Kirchhoff nonlinearity (1.1), in particular we have that H is an element of the sub-space of the Kirchhoff-type power series $\mathcal{F}_{\mathbf{kir}}$ (see (3.1.3)).
We recall that the elements of $\mathcal{F}_{\mathbf{kir}}$ admit an equivalent writing in terms of the variables

$$I_j := u_j\bar{u}_j \quad \text{and} \quad W_j := u_j u_{-j}.$$

In this case, we have

$$\int_{\mathbb{T}^n}((u+\bar{u})\mathfrak{B}[u+\bar{u}]) = \int_{\mathbb{T}^n} u\mathfrak{B}[u] + 2\int_{\mathbb{T}^n} u\mathfrak{B}[\bar{u}] + \int_{\mathbb{T}^n} \bar{u}\mathfrak{B}\left[\bar{u}\right],$$

that in Fourier is

$$\sum_{j\in\mathbb{Z}} \mathfrak{b}_j(u_j u_{-j} + \bar{u}_j\bar{u}_{-j}) + 2\sum_{j\in\mathbb{Z}} \mathfrak{b}_j u_j\bar{u}_j.$$

Defining the quantities

$$P_{\mathfrak{B}} := \sum_{j\in\mathbb{N}} \mathfrak{b}_j(W_j + \bar{W}_j), \qquad H_{\mathfrak{B},2} = \sum_{j\in\mathbb{N}} \mathfrak{b}_j(I_j + I_{-j}). \tag{4.5}$$

we can re-rewrite Hamiltonian (4.4) in the following way:

$$H = \sum_{j\in\mathbb{N}}(j+V_j)(I_j+I_{-j}) + \frac{1}{2}\mathbf{F}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right). \tag{4.6}$$

Using the Taylor expansion of $\mathbf{F}$ at 0 we get

$$H = \Lambda + V + H_4 + H_6 + \ldots + H_{2n} + h.o.t.$$

with

$$H_{2n} = \frac{\mathbf{F^{(n)}}(0)}{n!}(H_{\mathfrak{B},2} + P_{\mathfrak{B}})^n \in \mathcal{F}^{2n}_{\mathbf{kir}}, \quad n \geq 2.$$

We then have an equivalent writing of H in terms of the variables I, W:

$$\begin{aligned} H(I,W) &= \sum_{m,a,b} H_{m,a,b} I^n W^a \bar{W}^b \\ &= \sum_{j\in\mathbb{N}} (j+V_j)(I_j + I_{-j}) + \sum_{m,\,a,\,b} \mathbf{F}_{m,a,b} I^m W^a \bar{W}^b \end{aligned}$$

If we want to write H in function of the u, $\bar{u}$ variables (again formally) we have:

$$H(u) = \sum_{j\in\mathbb{Z}} (|j| + V_j)|u_j|^2 + \sum_{\alpha,\beta\in\mathcal{M}_{\mathbf{kirk}}} \mathbf{F}_{\alpha,\beta} u^\alpha \bar{u}^\beta,$$

where, recalling definition 3.10, $\mathcal{M}_{\mathbf{kirk}}$ is the set of the indexes α, $\beta \in \mathbb{N}^{\mathbb{Z}}_f$, such that

$$\alpha_k = m^+_k + a_k, \quad \alpha_{-k} = m^-_k + a_k, \quad \beta_k = m^+_k + b_k, \quad \beta_{-k} = m^-_k + b_k, \tag{4.7}$$

for $a, b, m^\pm \in \mathbb{N}^{\mathbb{N}}_f$ and $\alpha_0 = \beta_0 = 0$.
We recall moreover that the Hamiltonian (4.4) is invariant under the change $|u_k|^2 \longleftrightarrow |u_{-k}|^2$ or equivalently, in the I, W variables, $I_k \longleftrightarrow I_{-k}$.
This means that, if we write the index $m \in \mathbb{N}^{\mathbb{Z}}_f$ associated to the variable I like $m = (m^+, m^-)$ with $m^\pm \in \mathbb{N}^{\mathbb{N}}_f$, we have the identifications between the coefficients

$$H_{m^+,m^-,a,b} = H_{m^-,m^+,a,b}.$$

4.2 Functional Setting

In this section, we describe the theoretical framework necessary for the study of (1.27).
We recall the definition of formal power series in Section 3.1.1.

4.2.1 Weighted Spaces

We start by choosing a sequence $\{\boldsymbol{h}_j\}_{j\in\mathbb{Z}}$ such that

$$(\mathbf{1}) \qquad \boldsymbol{h}_j = \boldsymbol{h}_{-j} \quad j \in \mathbb{Z} \tag{4.8}$$

$$(\mathbf{2}) \qquad \boldsymbol{h}_{j_1+j_2} \leq \boldsymbol{h}_{j_1} + \boldsymbol{h}_{j_2} \quad j_1, j_2 \in \mathbb{N} \tag{4.9}$$

$$(\mathbf{3}) \qquad \lim_{i\in\mathbb{N}} \frac{\ln\langle i\rangle}{\boldsymbol{h}_i} = 0 \qquad . \tag{4.10}$$

For a fixed $s > 0$ we consider the following scale of Banach spaces

$$\mathfrak{g}_{\boldsymbol{h},s} := \left\{ u := (u_j)_{j\in\mathbb{Z}} \in \ell^2(C) : \|u\|_{\mathfrak{g}_{\boldsymbol{h},s}} = \sup_{j\in\mathbb{Z}} |u_j| e^{s\cdot\boldsymbol{h}_j} \right\}$$

and we endow it with the symplectic structure coming from ℓ^2.
Similar weighted spaces of this kind have been considered in [37] in the context of the NLS equation. However, the norm introduced in [37] is based on the ℓ^1 norm, whereas the norm considered here is based on ℓ^∞. We refer to that paper for a detailed discussion of the consequences that different growth conditions have on the associated weighted spaces. As example of $\mathfrak{g}_{\boldsymbol{h},s}$ one can consider:

1. **Gevrey case:**
   For $\boldsymbol{h}_j = |j|^\theta$ with $0 < \theta < 1$, we get the space of Gevrey functions of index $1/\theta$.

2. **Logarithmic Case:**
   For $\boldsymbol{h}_j = \ln^\tau(2 + |j|)$, $\tau > 2$ we have a specific class of ultra-differentiable functions, that strictly contains the Gevrey functions.

3. **Analytic case:**
   For $\boldsymbol{h}_j = a|j|$, $a \geq 0$ we have a sub-space of the space of analytic functions on the strip

$$\mathbb{T}_\rho = \{(x, y) \in \mathbb{T} \times \mathbb{C} : |y| < a\}.$$

As we anticipated in the Introduction 1, conditions (4.8) and (4.9) are necessary to ensure the monotonicity property of Lemma 4.2.3 while condition (4.10) is fundamental to get the estimate (4.47) for the solution of the homological equation (4.42).
In this regard, since the constant C in (4.47) depends on the structure of $\boldsymbol{h}$, in order to keep the calculations clear we perform the KAM scheme of Section 4.5 in the specific case

$$\boldsymbol{h}_j := \ln(1 + |j|)^{2+\varepsilon},\ \varepsilon > 0.$$

With this choice of $\boldsymbol{h}$ one also obtain the algebra property for the corresponding norm $\mathfrak{g}_{s,\boldsymbol{h}}$.
We stress however that thanks to the Kirchhoff-structure this is not needed, as Lemma 4.2.3 shows.

4.2.2 Space of Hamiltonians

In the case of analytic Hamiltonians, the norm $\mathfrak{g}_{\boldsymbol{h},s}$ induces a Banach-scale structure on the space of formal power series $\mathcal{F}$:
given $H \in \mathcal{F}$ we define its majorant as

$$\underline{H}(u) = \sum_{\alpha.\beta} |H_{\alpha,\beta}| u^\alpha \bar{u}^\beta. \tag{4.11}$$

Definition 4.2.1 (Majorant Analytic Hamiltonian) *For $r > 0$ we denote by $\mathcal{A}_r^{\boldsymbol{h}}(\mathfrak{g}_{\boldsymbol{h},s})$ the subspace of $\mathcal{F}$ formal power series H that are also well defined maps*

$$H : \bar{B}_r(\mathfrak{g}_{\boldsymbol{h},s}) \to \mathbb{R},$$

for which $\underline{H}$ is point-wise absolutely convergent in $B_r(\mathfrak{g}_{\boldsymbol{h},s})$.

Definition 4.2.2 (Regular Hamiltonian)
For $r,\ s > 0$ a Hamiltonian $H \in \mathcal{A}_r(\mathfrak{g}_{\boldsymbol{h},s})$ is **regular** *if its Hamiltonian vector field is bounded, that is*

$$|H|_{r,\boldsymbol{h},s} := \frac{1}{r}\left(\sup_{\|u\|_{\mathfrak{g}_{\boldsymbol{h},s}} \leq r} \|X_{\underline{H}}\|_{\mathfrak{g}_{\boldsymbol{h},s}}\right) \tag{4.12}$$

The space of regular Hamiltonian is denoted by $\mathcal{H}_{r,\boldsymbol{h},s}$.

Let us note that $|\cdot|_{r,\boldsymbol{h},s}$ is a semi-norm on the space of formal power series, since is zero for all the constants. However, since the Hamiltonians of the form (4.4) have a zero at the origin, we will assume $H(0) = 0$, endowing in this way $\mathcal{H}_{r,\boldsymbol{h},s}$ with the structure of Banach scale with norm (4.12).
In the next Subsection we will show that the Poisson brackets and Lie exponential, defined only to a formal level in Section 3.1.1, are actually well-posed within the class of regular Hamiltonians.
In order to do so we need a better characterization of the norm $|\cdot|_{r,\boldsymbol{h},s}$:

Lemma 4.2.1 ([24], Lemma 2.2)
For $H \in \mathcal{H}_{r,\boldsymbol{h},s}$ we have the alternative definition of the norm (4.12)

$$|H|_{r,\boldsymbol{h},s} = \sup_{k\in\mathbb{Z}} \sum_{\alpha,\beta} \beta_k \,|H_{\alpha,\beta}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{(\alpha+\beta-2e_k)}, \tag{4.13}$$

with

$$\mathbf{z}^{(\boldsymbol{h})}_j(r,s) := re^{-s\cdot \boldsymbol{h}_j}. \tag{4.14}$$

Proof: From the definition of $\|\cdot\|_{\mathfrak{g}_{\boldsymbol{h},s}}$ it follows that

$$|u_j| \leq \mathbf{z}^{(\boldsymbol{h})}(r,s), \quad \text{for all} \quad \|u\|_{\mathfrak{g}_{\boldsymbol{h},s}} \leq r. \tag{4.15}$$

Since the majorant $\underline{H}$ is point-wise absolutely convergent, its vector field has explicit expression

$$X^{(j)}_{\underline{H}} = i \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} |H_{\alpha,\beta}|\, \beta_j u^{\alpha} \bar{u}^{\beta-e_j},$$

hence, for every $\|u\|_{\mathfrak{g}_{\boldsymbol{h},s}} \leq r$ we have

$$\begin{aligned}
\left\|X_{\underline{H}}(u)\right\|_{\mathfrak{g}_s} &= \sup_{j\in\mathbb{Z}} \left|X^{(j)}_{\underline{H}}\right| e^{s\cdot\boldsymbol{h}_j} \\
&\leq \sup_{j\in\mathbb{Z}} \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} |H_{\alpha,\beta}|\, \beta_j |u^{\alpha}|\bar{u}^{\beta-e_j}| e^{s\cdot\boldsymbol{h}_j} \\
&\overset{(4.15)}{\leq} \sup_{j\in\mathbb{Z}} \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} |H_{\alpha,\beta}|\, \beta_j |\mathbf{z}^{(\boldsymbol{h})}(r,s)^{\alpha+\beta-e_j} e^{s\cdot\boldsymbol{h}_j} \\
&= \left\|X_{\underline{H}}\left(\mathbf{z}^{(\boldsymbol{h})}(r,s)\right)\right\|_{\mathfrak{g}_s}.
\end{aligned} \tag{4.16}$$

Since $\mathbf{z}^{(\boldsymbol{h})}(r,s) \in B_r\left(\mathfrak{g}_{\boldsymbol{h},s}\right)$, we have that

$$\begin{aligned}
\frac{1}{r}\left(\sup_{\|u\|_{\mathfrak{g}_s}\leq r} \|X_{\underline{H}}\|_{\mathfrak{g}_s}\right) &= \frac{1}{r}\left\|X_{\underline{H}}(\mathbf{z}^{(\boldsymbol{h})}(r,s)\right\|_{\mathfrak{g}_s} \\
&= \frac{1}{r}\sup_{j\in\mathbb{Z}} \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} |H_{\alpha,\beta}|\, \beta_j |\mathbf{z}^{(\boldsymbol{h})}(r,s)^{\alpha+\beta-e_j} e^{s\cdot\boldsymbol{h}_j} \\
&\overset{(4.14)}{=} \sup_{j\in\mathbb{Z}} \sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f} |H_{\alpha,\beta}|\, \beta_j |\mathbf{z}^{(\boldsymbol{h})}(r,s)^{\alpha+\beta-2e_j},
\end{aligned}$$

that is (4.13). □

Remark 4.2.1 *From Lemma 4.2.1 it follows that a formal power series $H \in \mathcal{F}$ for which* (4.13) *is finite, totally converges in the ball $B_r(\mathfrak{g}_{\boldsymbol{h},s})$.*
Indeed one has

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{N}^{\mathbb{Z}}_f}\sup_{\|u\|_{\mathfrak{g}_{\boldsymbol{h},s}}\le r}|H_{\alpha,\beta}|\,|u^{\alpha}\bar{u}^{\beta}| &\overset{(4.15)}{\le} \sum_{\alpha,\beta}|H_{\alpha,\beta}|\mathbf{z}^{(\boldsymbol{h})}(r,s)^{\alpha+\beta}\\ &\overset{(4.14)}{\le} r^2e^{-2s\cdot\boldsymbol{h}_j}\sum_{\alpha,\beta}|H_{\alpha,\beta}|\mathbf{z}^{(\boldsymbol{h})}(r,s)^{\alpha+\beta-2e_j}\\ &\overset{(4.13)}{\le} r^2|H|_{r,\boldsymbol{h},s}.\end{aligned}$$

Hence, both $H(u)$ and $\underline{H}(u)$ are analytic in the open ball $B_r(\mathfrak{g}_{\boldsymbol{h},s})$, moreover, from (4.16) *we have*

$$|H|_{r,\boldsymbol{h},s}=\frac{1}{r}\left\|X_{\underline{H}}(\mathbf{z}^{(\boldsymbol{h})}(r,s))\right\|_{\mathfrak{g}_s}.$$

Remark 4.2.2 *Let us note moreover that, by the reality condition* (3.12)*, one has*

$$\begin{aligned}|H|_{r,\boldsymbol{h},s} &= \sup_{k\in\mathbb{Z}}\sum_{\alpha,\beta}\beta_k\,|H_{\alpha,\beta}|\,\mathbf{z}^{(\boldsymbol{h})}(r,s)^{(\alpha+\beta-2e_k)}\\ &\overset{(3.12)}{=} \sup_{k\in\mathbb{Z}}\sum_{\alpha,\beta}\alpha_k\,|H_{\alpha,\beta}|\,\mathbf{z}^{(\boldsymbol{h})}(r,s)^{(\alpha+\beta-2e_k)}\\ &\overset{(3.12)}{=} \sup_{k\in\mathbb{Z}}\sum_{\alpha,\beta}\frac{(\alpha_k+\beta_k)}{2}\,|H_{\alpha,\beta}|\,\mathbf{z}^{(\boldsymbol{h})}(r,s)^{(\alpha+\beta-2e_k)}\end{aligned}\tag{4.17}$$

.

4.2.3 Lipschitz Norm

In order to keep track of the Lipschitz dependence of the Hamiltonians on the frequency ω we introduce the **inhomogeneous weighted Lipschitz norm** on the space of regular Hamiltonians.
We start by defining the hypercube

$$\mathcal{Q}=\left\{\omega=(\omega_j)_{j\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}:\quad\sup_j|\omega_j-|j||\le\frac{1}{2}\right\}$$

which is endowed with he probability measure μ induced by the standard product measure ν on $\left[-\frac{1}{2},\frac{1}{2}\right]^{\mathbb{Z}}$ from the map

$$\omega:\xi=\{\xi_j\}_{j\in\mathbb{Z}}\in\left[-\frac{1}{2},\frac{1}{2}\right]^{\mathbb{Z}}\to\{|j|+\xi_j\}_{j\in\mathbb{Z}}\in\mathcal{Q},$$

namely

$$\mu(\mathcal{A})=\nu(\omega^{-1}(\mathcal{A})),\quad\mathcal{A}\subseteq\mathcal{Q}.\tag{4.18}$$

Let us consider a set $\mathcal{A} \subset \mathcal{Q}$ of positive measure and let us consider a family of regular Hamiltonian $H(\omega) \in \mathcal{H}_{r,\boldsymbol{h},s}$ parametrized by $\omega \in \mathcal{A}$.
We define the Lipschitz semi-norm of H as

$$|H|^{\mathbf{Lip},\mathcal{A}}_{r,\boldsymbol{h},s} := \sup_{\substack{\omega' \neq \omega \\ \omega,\omega' \in \mathcal{A}}} \frac{|H(\omega) - H(\omega')|_{r,\boldsymbol{h},s}}{|\omega - \omega'|_\infty}. \tag{4.19}$$

Moreover, by setting

$$\Delta_{\omega,\omega'} H := \frac{H(\omega) - H(\omega')}{|\omega - \omega'|_\infty}, \quad \omega,\, \omega' \in \mathcal{A}, \omega \neq \omega', \tag{4.20}$$

the norm (4.19) has the more compact writing

$$|H|^{\mathbf{Lip},\mathcal{A}}_{r,\boldsymbol{h},s} = \sup_{\substack{\omega' \neq \omega \\ \omega,\omega' \in \mathcal{A}}} \left|\Delta_{\omega,\omega'} H\right|_{r,\boldsymbol{h},s}.$$

We finally consider the space of Lipschitz regular Hamiltonians

$$\mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s} := \Big\{ H(\cdot) : \mathcal{A} \to \mathcal{H}_{r,\boldsymbol{h},s} : \sup_{\omega \in \mathcal{A}} |H(\omega)|_{r,\boldsymbol{h},s},\, |H|^{\mathbf{Lip},\mathcal{A}}_{r,\boldsymbol{h},s} < \infty \Big\},$$

moreover for a fixed $\mu > 0$ we endow $\mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$ with the structure of Banach space with the inhomogeneous weighted Lipschitz norm

$$\|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} := \sup_{\omega \in \mathcal{A}} |H(\omega)|_{r,\boldsymbol{h},s} + \mu |H|^{\mathbf{Lip},\mathcal{A}}_{r,\boldsymbol{h},s}. \tag{4.21}$$

4.2.4 Poisson Bracket and Flows

From Section 3.1.1 we recall that the space of formal power series $\mathcal{F}$ is endowed with a Poisson algebra structure via the Poisson bracket

$$\{F, G\} := i \sum_{\alpha^{(i)},\beta^{(i)} \in \mathbb{N}^{\mathbb{Z}}_f} F_{\alpha^{(1)},\beta^{(1)}} G_{\alpha^{(2)},\beta^{(2)}} \sum_{j \in \mathbb{Z}} \big(\beta^{(1)}_j \alpha^{(2)}_j - \alpha^{(1)}_j \beta^{(2)}_j\big) u^{\alpha^{(1)}+\alpha^{(2)}-\mathbf{e}_j} u^{\beta^{(1)}+\beta^{(2)}-\mathbf{e}_j}, \tag{4.22}$$

This also holds beyond the formal setting, when we consider the scale of Regular Hamiltonians $\{\mathcal{H}_{r,\boldsymbol{h},s}\}_{r,s\geq 0}$, as shown by the following result:

Proposition 4.2.1 *let us consider $F,\, G \in \mathcal{H}_{r+\rho,s}$ with $\rho > 0$, we have that $\{F, G\} \in \mathcal{H}_{r,\mathfrak{g}_s}$, moreover*

$$|\{F, G\}|_{r,\boldsymbol{h},s} \leq 8 \max\left\{1, \frac{r}{\rho}\right\} |F|_{r+\rho,\boldsymbol{h},s} |G|_{r+\rho,\boldsymbol{h},s}. \tag{4.23}$$

Analogously, for family of Hamiltonians $F,\, G \in \mathcal{H}^{\mathcal{A}}_{r+\rho,s}$, one has

$$\|\{F, G\}\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} \leq 8 \max\left\{1, \frac{r}{\rho}\right\} \|F\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s} \|G\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}. \tag{4.24}$$

Proof: For

$$F = \sum_{\alpha^{(1)},\beta^{(1)}} F_{\alpha^{(1)},\beta^{(1)}} u^{\alpha^{(1)}} \bar{u}^{\beta^{(1)}}, \qquad G = \sum_{\alpha^{(2)},\beta^{(2)}} G_{\alpha^{(2)},\beta^{(2)}} u^{\alpha^{(2)}} \bar{u}^{\beta^{(2)}},$$

recalling (4.22) we have

$$|\{F,G\}|_{r,\boldsymbol{h},s} \le \sup_k \sum_j \sum_{\alpha^{(1)},\beta^{(1)},\alpha^{(2)},\beta^{(2)}} |F_{\alpha^{(1)},\beta^{(1)}} G_{\alpha^{(2)},\beta^{(2)}}|(\beta_j^{(1)}\alpha_j^{(2)} + \alpha_j^{(1)}\beta_j^{(2)})(\beta_k^{(1)} + \beta_k^{(2)}) \times \\ \times z(r)^{\alpha^{(1)}+\beta^{(1)}+\alpha^{(2)}+\beta^{(2)}-2e_j-2e_k}.$$

Let us analyze the terms of the form

$$A = \sup_k \sum_j \sum_{\alpha^{(1)},\beta^{(1)},\alpha^{(2)},\beta^{(2)}} |F_{\alpha^{(1)},\beta^{(1)}} G_{\alpha^{(2)},\beta^{(2)}}|\beta_j^{(1)}\alpha_j^{(2)}\beta_k^{(1)} z(r)^{(\alpha^{(1)}+\beta^{(1)}+\alpha^{(2)}+\beta^{(2)}-2e_j-2e_k)},$$

we have

$$\begin{aligned} A &\le \frac{1}{2}|G|_{r,s} \sup_k \sum_j \sum_{\alpha^{(1)},\beta^{(1)}} |F_{\alpha^{(1)},\beta^{(1)}}|\beta_j^{(1)}\beta_k^{(1)} z(r)^{(\alpha^{(1)}+\beta^{(1)}-2e_k)} \\ &= \frac{1}{4}|G|_{r,s} \sup_k \sum_j \sum_{\alpha^{(1)},\beta^{(1)}} |F_{\alpha^{(1)},\beta^{(1)}}|(\alpha_j^{(1)}+\beta_j^{(1)})\beta_k^{(1)} z(r)^{(\alpha^{(1)}+\beta^{(1)}-2e_k)} \\ &= \frac{1}{4}|G|_{r,s} \sup_k \sum_j \sum_{\alpha^{(1)},\beta^{(1)}} |F_{\alpha^{(1)},\beta^{(1)}}||\alpha^{(1)}+\beta^{(1)}|\beta_k^{(1)} z(r)^{(\alpha^{(1)}+\beta^{(1)}-2e_k)} \\ &= \frac{1}{4}|G|_{r,s} \sup_k \sum_{\alpha^{(1)},\beta^{(1)}} |F_{\alpha^{(1)},\beta^{(1)}}|\beta_k^{(1)} z(r+\rho)^{(\alpha^{(1)}+\beta^{(1)}-2e_k)}|\alpha^{(1)}+\beta^{(1)}|\left(\frac{r}{r+\rho}\right)^{(|\alpha^{(1)}+\beta^{(1)}|-2)} \\ &\le |G|_{r,s}|F|_{r+\rho,s} \sup_{\alpha^{(1)},\beta^{(1)}\in\mathbb{N}_f^{\mathbb{Z}}} |\alpha^{(1)}+\beta^{(1)}|\left(\frac{r}{r+\rho}\right)^{(|\alpha^{(1)}+\beta^{(1)}|-2)}. \end{aligned}$$

Since $|\alpha^{(1)}+\beta^{(1)}| \ge 2$ we have

$$\sup_{\alpha^{(1)},\beta^{(1)}\in\mathbb{N}_f^{\mathbb{Z}}} |\alpha^{(1)}+\beta^{(1)}|\left(\frac{r}{r+\rho}\right)^{(|\alpha^{(1)}+\beta^{(1)}|-2)} \le \sup_{x\ge 2}\left(\frac{r}{r+\rho}\right)^{(x-2)} \le 2\max\left\{1,\frac{r}{\rho}\right\}.$$

This concludes the first part of the proof. For what concerns (4.24), recalling (4.20) we have

$$\Delta_{\omega,\omega'}\{F,G\} = \{\Delta_{\omega,\omega'}F,G\} + \{F,\Delta_{\omega,\omega'}G\}$$

and hence

$$\|\{F,G\}\|_{r,\boldsymbol{h},s}^{\mathbf{Lip},\mathcal{A}} \overset{(4.23)}{\le} \\ 8\max\left\{1,\frac{r}{\rho}\right\}\sup_{\omega\ne\omega'}\left(|\Delta_{\omega,\omega'}F|_{r,\boldsymbol{h},s}|G|_{r,\boldsymbol{h},s}\} + |F|_{r,\boldsymbol{h},s}|\Delta_{\omega,\omega'}G|_{r,\boldsymbol{h},s}\right). \tag{4.25}$$

Using (4.25) together with (4.23) and the definition (4.21), we get (4.24). □

As a consequence, we have that also the symplectic changes of variables defined in (3.15) are well-defined maps within the scale $\{\mathcal{H}_{r,\boldsymbol{h},s}\}_{r,s\ge 0}$.
In particular the flow of $H \in \mathcal{H}_{r,\boldsymbol{h},s}$ is locally well-posed and generates a symplectic transformation on $\mathfrak{g}_{\boldsymbol{h},s}$:

Proposition 4.2.2 *For fixed $\mu > 0$ and $\mathcal{A} \subseteq \mathcal{Q}$, let us consider $S \in \mathcal{H}^{\mathcal{A}}_{r+\rho,\boldsymbol{h},s}$ with $\rho \geq 0$ and suppose that it holds*

$$\|S\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s} \leq \delta := \frac{\rho}{16e(r+\rho)}. \tag{4.26}$$

Then for every $\omega \in \mathcal{A}$ the time 1-Hamiltonian flow, $\Phi^1_{S(\omega)} : \bar{B}_r(\mathfrak{g}_s) \to \bar{B}_{r+\rho}(\mathfrak{g}_s)$ is well defined and analytic in $B_r(\mathfrak{g}_s)$, symplectic with

$$\sup_{u\in \bar{B}_r(\mathfrak{g}_s)} \left|\Phi^1_{S(\omega)}(u) - u\right|_{\mathfrak{g}_s} \leq (r+\rho)\|S\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s} \leq \frac{\rho}{16e}, \tag{4.27}$$

moreover, for any $H \in \mathcal{H}^{\mathcal{A}}_{r+\rho,\boldsymbol{h},s}$ we have that $H \circ \Phi^1_S = e^{\{\cdot,S\}}H \in \mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$, $\left(e^{\{\cdot,S\}} - \mathrm{Id}\right) H \in \mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$ and

$$\begin{aligned} \left\|e^{\{\cdot,S\}}H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} &\leq 2\|H\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}; && (4.28)\\ \left\|\left(e^{\{\cdot,S\}} - \mathrm{Id}\right)H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} &\leq \delta^{-1}\|S\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}\|H\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}; && (4.29)\\ \left\|\left(e^{\{\cdot,S\}} - \mathrm{Id} - \{\cdot,S\}\right)H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} &\leq \frac{1}{2}\delta^{-2}\left(\|S\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}\right)^2\|H\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}. && (4.30) \end{aligned}$$

More generally, for any $h \in \mathbb{N}$ and any sequence $(c_k)_{k\in\mathbb{N}}$ with $|c_k| \leq \frac{1}{k!}$, we have

$$\left\|\sum_{k\geq h} c_k \mathrm{ad}^k_S(H)\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} \leq 2\|H\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}\left(\frac{\|S\|^{\mu,\mathcal{A}}_{r+\rho,\boldsymbol{h},s}}{2\delta}\right)^h, \tag{4.31}$$

where $\mathrm{ad}_S(H) = \{H,S\}$.

Proof: In the following, for brevity, we sill write $\|\cdot\|_{r,\boldsymbol{h},s}$ instead of $\|\cdot\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}$.
Let us fix $\omega \in \mathcal{A}$ and write S in place of $S(\omega)$, by (4.26) we have that

$$\sup_{u\in B_{r+\rho}(\mathfrak{g}_s)} |X_S|_{g_s} \leq (r+\rho)\|S\|_{r+\rho,\boldsymbol{h},s} < \frac{\rho}{16e}. \tag{4.32}$$

By Lemma B.1. of [24] with $E = \mathfrak{g}_s$, $X = X_S$, $\delta_0 = (r+\rho)\|S\|_{r+\rho,\boldsymbol{h},s}$ and $r_1 = r+\rho$ we have that the Hamiltonian flow of S is well-defined for

$$|t| \leq T := \frac{\rho}{(r+\rho)\|S\|_{r+\rho,\boldsymbol{h},s}} \geq 16e$$

and, for $t = 1$, moreover, using (4.32) we have

$$\sup_{u\in B_r(\mathfrak{g}_s)} \left|\Phi^1_{S(\omega)}(u) - u\right|_{\mathfrak{g}_s} \leq (r+\rho)\|S\|_{r+\rho,\boldsymbol{h},s} \leq \frac{\rho}{16e}$$

From this follows that $H \circ \Phi^1_S = e^{\{S,\cdot\}}H \in \mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$ and $\left(e^{\{S,\cdot\}} - \mathrm{Id}\right) H \in \mathcal{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$.
Finally, (4.28), (4.29) and (4.30) follows all from (4.31) with $c_k = \frac{1}{k!}$ by the definition of Lie series.
Now, for a fixed $k \in \mathbb{N}$ let us construct a sequence of decreasing balls, all contained in $B_{r+\rho}(\mathfrak{g}_s)$: we define

$$r_i := r + \rho\left(1 - \frac{i}{k}\right), \quad i = 0, \dots, k.$$

Let us note that

$$\|S\|_{r_i} \leq \|S\|_{r+\rho} \quad \text{and} \quad 1 + \frac{kr_i}{\rho} \leq k\left(1 + \frac{r}{\rho}\right) \quad \forall i = 0, \dots, k. \tag{4.33}$$

Using k-times (4.23) we have

$$\begin{aligned}
\left\|\mathrm{ad}_s^k(H)\right\|_{r,\boldsymbol{h},s} &= \left\|\left\{S, \mathrm{ad}_s^{k-1}(H)\right\}\right\|_{r,\boldsymbol{h},s} \leq 8\max\left\{1, \frac{rk}{\rho}\right\}\|S\|_{r_{k-1},\boldsymbol{h},s}\left\|\mathrm{ad}_s^{k-1}(H)\right\|_{r_{k-1},\boldsymbol{h},s} \\
&\overset{(4.33)}{\leq} 8\left(1 + \frac{rk}{\rho}\right)\|S\|_{r+\rho,\boldsymbol{h},s}\left\|\mathrm{ad}_s^{k-1}(H)\right\|_{r_{k-1},\boldsymbol{h},s} \\
&\overset{(4.23)}{\leq} 8^k\|H\|_{r+\rho,\boldsymbol{h},s}\prod_{i=0,\dots,k-1}\left(1 + \frac{r_i k}{\rho}\right)\|S\|_{r_i,\boldsymbol{h},s} \\
&\overset{(4.33)}{\leq} \left(8k\left(1 + \frac{r}{\rho}\right)\|S\|_{r_i,\boldsymbol{h},s}\right)^k\|H\|_{r+\rho,\boldsymbol{h},s} \\
&\overset{(4.27)}{\leq} \left(\frac{k\|S\|_{r+\rho,\boldsymbol{h},s}}{2e\delta}\right)^k\|H\|_{r+\rho,\boldsymbol{h},s},
\end{aligned}$$

then, by the fact that $k^k \leq e^k k!$ we have

$$\begin{aligned}
\left\|\sum_{k\geq h} c_k \mathrm{ad}_S^k(H)\right\|_{r,\boldsymbol{h},s} &\leq \sum_{k\geq h} c_k\left\|\mathrm{ad}_S^k(H)\right\|_{r,\boldsymbol{h},s} \\
&\leq \|H\|_{r+\rho,\boldsymbol{h},s}\sum_{k\geq h}\left(\frac{\|S\|_{r+\rho,\boldsymbol{h},s}}{2\delta}\right)^k \\
&= 2\|H\|_{r+\rho,\boldsymbol{h},s}\left(\frac{\|S\|_{r+\rho,\boldsymbol{h},s}}{2\delta}\right)^n.
\end{aligned}$$

Moreover, from (4.27) it follows

$$\left\|\sum_{k\geq h} c_k \mathrm{ad}_S^k(H)\right\|_{r,\boldsymbol{h},s} \leq \frac{1}{2^{h-1}}\|H\|_{r+\rho,\boldsymbol{h},s}.$$

□

4.2.5 Kirchhoff-Type Power Series and I, W variables

Since we are dealing with Hamiltonians that are polynomials of the Kirchhoff type (see (3.1.3)) we want to study how the norm (4.12) behaves on the elements of $\mathcal{F}_{\mathbf{kir}}$.
We start by defining the sub-space

$$\mathbf{H}_{r,\boldsymbol{h},s} := \mathcal{H}_{r,\boldsymbol{h},s} \cap \mathcal{F}_{\mathbf{kir}}$$

and, similarly, for a fixed $\mathcal{A} \subseteq \mathcal{Q}$

$$\mathbf{H}_{r,\boldsymbol{h},s}^{\mathcal{A}} := \left\{H(\cdot) : \mathcal{A} \to \mathbf{H}_{r,\boldsymbol{h},s} : \sup_{\omega\in\mathcal{A}}|H(\omega)|_{r,\boldsymbol{h},s},\ |H|_{r,\boldsymbol{h},s}^{\mathbf{Lip},\mathcal{A}} < \infty\right\},$$

endowed with the norm (4.21).

Recalling that, for a generic monomial in the variable $u = (u_j)_{j\in\mathbb{Z}}$ it holds

$$u^\alpha \bar{u}^\beta = I^m W^a \bar{W}^b, \quad a, b \in \mathbb{N}_f^{\mathbb{N}},\, m \in \mathbb{N}_f^{\mathbb{Z}} \quad \text{such that (4.7) holds,}$$

we can rewrite the semi-norm in (4.17) in terms of the the variables I, W defined in (3.3).

Lemma 4.2.2 *On the elements of $\mathcal{F}_{\mathbf{kir}}$ we have the alternative expression for the norm* (4.13)*:*

$$|H|_{r,\boldsymbol{h},s} = \sup_{k\in\mathbb{N}} \sum_{m,\,a,\,b} \frac{(m_k^+ + m_k^- + a_k + b_k)}{2} |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)}. \tag{4.34}$$

Proof: Keeping in mind the symmetry of the weight $\boldsymbol{h}$ we have

$$\mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)} = \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_{-k})} \quad \forall k \in \mathbb{N}, \tag{4.35}$$

moreover, thanks to (3.11) we got

$$\begin{aligned}
|H|_{r,\boldsymbol{h},s} \quad &= \quad \sup_{k\in\mathbb{Z}} \sum_{m,\,a,\,b} (m_k + b_{|k|})\, |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)} \\
&\overset{(3.11),(4.35)}{=} \sup_{k\in\mathbb{N}} \sum_{m,\,a,\,b} \frac{(m_k^+ + m_k^- + 2b_k)}{2} |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)} \\
&\overset{(3.12),(4.35)}{=} \sup_{k\in\mathbb{N}} \sum_{m,\,a,\,b} \frac{(m_k^+ + m_k^- + 2a_k)}{2} |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)} \\
&\overset{(3.12),(4.35)}{=} \sup_{k\in\mathbb{N}} \sum_{m,\,a,\,b} \frac{(m_k^+ + m_k^- + a_k + b_k)}{2} |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)}.
\end{aligned}$$

□

Remark 4.2.3 *Reasoning as in Remark 4.2.1 we have that if a formal power series of Kirchhoff-type totally converges for $u \in B_r(\mathfrak{g}_{\boldsymbol{h},s})$, then the corresponding formal power series in the I, W variables, namely*

$$H(I, W) = \sum_{m,a,b} H_{m,a,b} I^m W^a \bar{W}^b$$

totally converges for I, $W \in B_{r^2}(\mathfrak{g}_{\boldsymbol{h},2s})$ with estimates,

$$\sum_{m\,a,\,b} |H_{m,a,b}||I^m||W^a|\bar{W}^b| \le \sum_{m\,a,\,b} |H_{m,a,b}|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^+ + m^- + a + b - e_k)} \le r^2 |H|_{r,\boldsymbol{h},s}.$$

Moreover, a formal power series in the variables I, W for which the right-hand side of (4.34) *is finite, totally converges in the ball $B_{r^2}(\mathfrak{g}_{\boldsymbol{h},s})$ and is and element of $\mathcal{H}_{r,\boldsymbol{h},s}$ with norm given by* (4.34).

We now investigate under which conditions a Hamiltonian of the form (4.4) is regular. This reduces to determining for which functions $\mathbf{F}$ the associated vector field $X_{\mathbf{F}}$ is finite on a suitable ball $B_r(\mathfrak{g}_{\mathbf{h},s})$. This question is addressed by the following theorem:

Theorem 4.2.3 *Let us consider an Hamiltonian, H, of the form*

$$H(u) = \mathbf{F}(H_{\mathfrak{B},2} + P_{\mathfrak{B}}),$$

where $\mathbf{F}$ *real function analytic function.*
If $f = F'$ is analytic in a ball of radius r, with norm $|\mathbf{f}|_R$ defined in (1.29) *finite, then $H \in \mathbf{H}_{r,\boldsymbol{h},s}$, provided*

$$4\|\mathfrak{b}\|_\infty C_{\boldsymbol{h}}(s) r^2 \le R, \quad C_{\boldsymbol{h}}(s) := \sum_{j\in\mathbb{N}} e^{-2s\cdot \boldsymbol{h}_j}. \tag{4.36}$$

We have moreover the following bound

$$|H|_{r,\boldsymbol{h},s} \le \frac{(8\|\mathfrak{b}\|_\infty^2 C_{\boldsymbol{h}}(s) r^2)}{R} |\mathbf{f}|_R. \tag{4.37}$$

Proof: Let us first note that for $u \in B_R(\mathfrak{g}_{\boldsymbol{h},s})$ we have

$$\begin{aligned} P_{\mathfrak{B}} &= \sum_{j\in\mathbb{N}} \mathfrak{b}_j (W_j + \bar{W}_j) \le 2 \sum_{j\in\mathbb{N}} |\mathfrak{b}_j||W_j| \\ &\overset{(4.15)}{\le} 2\|u\|_{g_s}^2 \sum_{j\in\mathbb{N}} |\mathfrak{b}_j| e^{-2s\cdot \boldsymbol{h}_j} = 2\|\mathfrak{b}\|_\infty C_{\boldsymbol{h}}(s)\|u\|_{g_s}^2 \end{aligned}$$

the same inequality holds also for $H_{\mathfrak{B},2}$. We then have

$$H_{\mathfrak{B},2} + P_{\mathfrak{B}} \le 4\|\mathfrak{b}\|_\infty C_{\boldsymbol{h}}(s)\|u\|_{g_s}^2. \tag{4.38}$$

Moreover, since also $|\mathbf{F}|_R$ is finite we have that $H = \mathbf{F}(H_{\mathfrak{B},2} + P_{\mathfrak{B}})$ defines an majorant analytic function on the ball $B_r(\mathfrak{g}_{\boldsymbol{h},s})$.
Finally, we can write

$$X_F := \mathbf{f}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right) \sum_{j\in\mathbb{Z}} \mathfrak{b}_j (u_j + \bar{u}_{-j}) e^{ij\cdot x}.$$

Since

$$\begin{aligned} |\mathbf{f}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right)| &\le \sum_{d\ge 2} |f^{(d)}||H_{\mathfrak{B},2} + P_{\mathfrak{B}}|^d \\ &\overset{(4.38)}{\le} \sum_{d\ge 2} |f^{(d)}|(4\|\mathfrak{b}\|_\infty C_{\boldsymbol{h}}(s)\|u\|_{g_s}^2) \\ &\overset{(4.36)}{\le} \frac{(4\|\mathfrak{b}\|_\infty C_{\boldsymbol{h}}(s) r^2)}{R} |\mathbf{f}|_R, \end{aligned}$$

we then have that

$$\begin{aligned} \|X_F\|_{g_s} &\le 2|\mathbf{f}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right)| \cdot \sup_{j\in\mathbb{Z}} \left(|\mathfrak{b}_j||u_j| e^{s\cdot \boldsymbol{h}_j}\right) \\ &\le 2\|\mathfrak{b}_j\|_\infty |\mathbf{f}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right)| \cdot \left(\sup_{j\in\mathbb{Z}} |u_j| e^{s\cdot \boldsymbol{h}_j}\right) \\ &= 2\|\mathfrak{b}\|_\infty \|u\|_{g_s} |\mathbf{f}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right)| \\ &\overset{(4.2.1)}{\le} \|u\|_{g_s} \frac{(8\|\mathfrak{b}\|_\infty^2 C_{\boldsymbol{h}}(s) r^2)}{R} |\mathbf{f}|_R. \end{aligned}$$

From this easily follows

$$
\begin{aligned}
|H|_{r,\boldsymbol{h},s} &= \frac{1}{r}\left(\sup_{\|u\|_{\mathfrak{g}_s}\le r}\|X_{\underline{F}}\|_{\mathfrak{g}_s}\right)\\
&\le \frac{(8\|\mathfrak{b}\|_\infty^2 C_{\boldsymbol{h}}(s)r^2)}{R}|\mathbf{f}|_R\frac{1}{r}\left(\sup_{\|u\|_{\mathfrak{g}_s}\le r}\|u\|_{g_s}\right)\\
&= \frac{(8\|\mathfrak{b}\|_\infty^2 C_{\boldsymbol{h}}(s)r^2)}{R}|\mathbf{f}|_R.
\end{aligned}
$$

□

Remark 4.2.4 *We stress that a crucial ingredient in the proof of Theorem 4.2.3 is the fact that the Hamiltonian depends on the integral term*

$$\int_{\mathbb{T}}(u+\bar{u})\,\mathfrak{B}[u+\bar{u}]$$

and is therefore restricted to Hamiltonians of Kirchhoff type.

In contrast, in the work of Biasco-Procesi-Massetti [24] on the NLS equation, where such a characterization of the Hamiltonian is not available, an analogous result is obtained by exploiting the algebra property of the norm $\|\cdot\|_{s,\boldsymbol{h}}$.

However, in order to ensure this algebra property, one needs to impose additional lower bounds on the growth of the sequence $\{\boldsymbol{h}_j\}$. *For instance, one may require*

$$\boldsymbol{h}_j = \log^{2+\varepsilon}(1+|j|), \qquad \varepsilon>0.$$

Thankfully, in our case, this is not needed.

4.2.6 Monotonicity of the semi-norm with respect of s

A key step in carrying out the KAM algorithm of Section 4.5 is the monotonicity of the norm (4.21) with respect to the parameters r, s, and the set $\mathcal{A}$. We shall restrict our attention to Hamiltonians belonging to **H**. Within this setting we have the following result:

Lemma 4.2.3 *Let us fix* $r, s, \mu > 0$ *and* $\mathcal{A}\subset\mathcal{Q}$. *The following inequalities hold for the norm* (4.21)*:*

(i) For every $r_1\le r_2$ *we have* $\|H\|^{\mu,\mathcal{A}}_{r_1,\boldsymbol{h},s}\le\|H\|^{\mu,\mathcal{A}}_{r_2,\boldsymbol{h},s}$.

(ii) For every $\mu_1\subseteq\mu_2$ *we have* $\|H\|^{\mu_1,\mathcal{A}}_{r,\boldsymbol{h},s}\le\|H\|^{\mu_2,\mathcal{A}}_{r,\boldsymbol{h},s}$.

(iii) For every $\mathcal{A}_1\subseteq\mathcal{A}_2$ *we have* $\|H\|^{\mu,\mathcal{A}_1}_{r,\boldsymbol{h},s}\le\|H\|^{\mu,\mathcal{A}_2}_{r,\boldsymbol{h},s}$.

(iv) For any $\sigma>0$ *we have*

$$\|H\|^{\mathcal{A}}_{r,\boldsymbol{h},s+\sigma}\le\|H\|^{\mathcal{A}}_{r,\boldsymbol{h},s}. \tag{4.39}$$

Proof: While (i), (ii) and (iii) follow trivially from the definition (4.21), (iv) requires a more careful analysis:

we start by proving (4.39) for $|\cdot|_{r,\boldsymbol{h},s}$, namely

$$|H|_{r,\boldsymbol{h},s}\le|H|_{r,\boldsymbol{h},s+\sigma}.$$

Recalling (4.34), we need to show that, for every $m^+,\ m^-,\ a,\ b \in \mathbb{N}_f^{\mathbb{N}}$ and every $k \in \mathrm{supp}(m^+)\cup \mathrm{supp}(m^-) \cup \mathrm{supp}(a) \cup \mathrm{supp}(b)$, one has

$$\begin{aligned} & \frac{(m_k^+ + m_k^- + a_k + b_k)}{2} \, |H_{m,a,b}| \, \mathbf{z}^{(\boldsymbol{h})}(r, s+\sigma)^{2(m^+ + m^- + a + b - e_k)} \\ \leq \;& \frac{(m_k^+ + m_k^- + a_k + b_k)}{2} \, |H_{m,a,b}| \, \mathbf{z}^{(\boldsymbol{h})}(r, s)^{2(m^+ + m^- + a + b - e_k)}. \end{aligned}$$

This equals to verify

$$\exp\left[-2\sigma\left(\sum_{j\in\mathbb{N}}(m_j^+ + m_j^- + a_j + b_j)\boldsymbol{h}_j - \boldsymbol{h}_k\right)\right] \leq 1 \quad \forall k \in \mathbb{N} \tag{4.40}$$

and so

$$\sum_{i\in\mathbb{N}}(m_j^+ + m_j^- + a_j + b_j)\boldsymbol{h}_j - \boldsymbol{h}_k \geq 0 \quad \forall k \in \mathbb{N}. \tag{4.41}$$

but inequality (4.41) is trivially true since $m_k^+ + m_k^- + a_k + b_k \geq 1$.
From this, inequality (4.39) follows easily. □

4.3 Homological Equation

The proof of Theorem 1.2.9 relies on the KAM scheme delineated in Section 4.5. The key step in such scheme is the solution of the so-called Homological equation, namely

$$\{D_\omega, G\} = F, \tag{4.42}$$

where

$$D_\omega = \sum_{j\in\mathbb{Z}} \omega_j |u_j|^2 = \sum_{j\in\mathbb{Z}} \omega_j I_j. \tag{4.43}$$

is a quadratic Hamiltonian.
We know from Lemma 3.1.4 that equation (4.42) is well posed within the space of formal power series $\mathcal{F}$ if F is in the Range of $\{D_\omega, \cdot\}$, namely

$$\mathcal{R}_\omega := \left\{ H \in \mathcal{F} : H_{\alpha,\beta} = 0 \text{ if } \omega\cdot(\alpha-\beta) = 0 \right\}$$

and has solution

$$L_\omega^{-1}(F) := G = \sum_{\alpha,\beta\in\mathbb{N}_f^{\mathbb{Z}}} \frac{F_{\alpha,\beta}}{\omega\cdot(\alpha-\beta)} u^\alpha \bar{u}^\beta.$$

The aim of this section is to show that, for particular values of ω, L_ω^{-1} is indeed a well defined operator on the Banach scale $\{\mathbf{H}_{r,\boldsymbol{h},s}\}_{s>0}$.
In order to keep the resonances

$$\omega(\alpha-\beta), \qquad \alpha,\beta \in \mathbb{N}_f^{\mathbb{Z}},$$

as simple as possible, Bourgain's idea [33] was to impose a strong non-resonance condition. This condition allows one to solve the homological equation not only at a formal level, while at the same time holding on a set of frequencies of positive measure. This motivates the following definition:

Definition 4.3.1 (Diophantine Condition)
For $0 < \gamma < 1$, we say that a vector $\omega \in \mathcal{Q}$ belongs to $\mathtt{D}_\gamma$ if

$$|\omega \cdot \ell| > \gamma \prod_{j\in\mathbb{Z}} \frac{1}{(1+|\ell_j|^2\langle j\rangle^2)}, \; \forall \ell \in \mathbb{Z}^{\mathbb{Z}}, \; \textit{with } |\ell| := \sum_{j\in\mathbb{Z}} |\ell_j| < \infty.$$

Trivially, for every frequency $\omega \in \mathtt{D}_\gamma$, one has that $\omega\cdot(\alpha-\beta) = 0$, only when $\alpha = \beta$. It follows that the resonant monomials are precisely those involving only the linear actions $\{|u_j|^2\}_{j\in\mathbb{Z}}$. Moreover we have that the Diophantine frequencies are typical in $\mathcal{Q}$:

Lemma 4.3.1 *(***[22], Lemma 4.1)**
There exists a positive constant $\mathtt{C}$ such that, for every $0 < \gamma < 1$, it holds

$$\mu\left(\mathcal{Q}\backslash\mathtt{D}_\gamma\right) \le \mathtt{C}\cdot\gamma. \tag{4.44}$$

Where μ is the measure defined in (4.18).

4.3.1 Kirchhof-Type Power Series and Symmetrical Frequencies

If we restrict to the sub-space of Kirchhoff-type power series the operator $\{D_\omega, \cdot\}$ acts on the monomials $I^m W^a \bar{W}^b$ diagonally as

$$\{D_\omega, I^m W^a \bar{W}^b\} = (\omega^+ + \omega^-)(\alpha - b) I^m W^a \bar{W}^b,$$

where $\omega^\pm \in \mathbb{R}^{\mathbb{N}}$ is the vectors of the term of ω with positive (respectively negative) indexes. Since the linear part of Equation (4.1) is a diagonal operator defined by the symmetric sequence $|j| + V_j$ and since also the vectors $\alpha - \beta$ are symmetric for Kirchhoff-type power series, we impose the same symmetry also on the frequencies ω:
we restrict to the set

$$\mathcal{Q}^{\mathbf{Sym}} := \Big\{\omega = \{\omega_j\}_{j\in\mathbb{Z}} \in \mathcal{Q} : \, \omega_j = \omega_{-j}\Big\}.$$

for which the Diophantinte frequencies are

$$\mathtt{D}^{\mathbf{Sym}}_\gamma := \Big\{\omega \in \mathcal{Q}^{\mathbf{Sym}} : \, |\tilde{\omega}\cdot\ell| > \gamma \prod_{j\in\mathbb{N}} \frac{1}{(1+|\ell_j|^2|j|^2)^2}, \; \forall \ell \in \mathbb{Z}^{\mathbb{N}}, \; \text{with } |\ell| := \sum_{j\in\mathbb{N}} |\ell_j| < \infty\Big\}, \tag{4.45}$$

where $\tilde{\omega} \in \mathbb{R}^{\mathbb{N}}$ such that $\tilde{\omega}_j = \omega_j$ for $j \in \mathbb{N}$.
One can prove that $\mathtt{D}^{\mathbf{Sym}}_\gamma$ satisfies an estimate analogous to (4.44), moreover, the operator L_ω^{-1} on $\mathcal{F}_{\mathbf{kir}}$ and for $\omega \in \mathcal{Q}^{\mathbf{Sym}}$ is still well defined, with expression

$$L_\omega^{-1}(F) = \sum_{\substack{m\in\mathbb{N}^{\mathbb{Z}}_f \\ a,b\in\mathbb{N}^{\mathbb{N}}_f}} \frac{F_{m,a,b}}{2\tilde{\omega}\cdot(a-b)} I^m W^a \bar{W}^b, \quad F \in \mathcal{R}_{\mathbf{kir}}(\omega), \tag{4.46}$$

where, similarly to Chapter 3, $\mathcal{R}_{\mathbf{kir}}(\omega) := \mathcal{R}(\omega) \cap \mathcal{F}_{\mathbf{kir}}$.

The aim of this section is to prove the following result:

Lemma 4.3.2 (Resolution of the Homological Equation) *Let r, $s > 0$, $0 < \sigma < 1$ and consider a family of Hamiltonians*

$$\mathtt{D}^{\mathbf{Sym}}_\gamma \ni \omega \to F(\omega) \in \mathbf{H}_{r,\boldsymbol{h},s}.$$

Then, for $G := L_\omega^{-1}(F) \in \mathbf{H}_{r,\boldsymbol{h},s+\sigma}$ with the following estimate:

$$\|G\|_{r,\boldsymbol{h},s+\sigma}^{\gamma,\mathtt{D}_\gamma^{\mathbf{Sym}}} \leq 3\gamma^{-1}e^{C(\sigma)}\|F\|_{r,\boldsymbol{h},s}^{\gamma,\mathtt{D}_\gamma^{\mathbf{Sym}}}, \tag{4.47}$$

where $C(\sigma)$ is given in (4.67).

4.3.2 Decreasing Rearrangement

We start by proving (4.47), for a fixed $\omega \in \mathtt{D}_\gamma^{\mathbf{Sym}}$, namely

$$|G|_{r,\boldsymbol{h},s+\sigma} \leq 3\gamma^{-1}e^{C(\sigma)}|F|_{r,\boldsymbol{h},s}. \tag{4.48}$$

For $j \in \mathbb{N}$ e m^+, m^-, a, $b \in \mathbb{N}_f^{\mathbb{N}}$ such that $|a|$, $|b| \neq 0$ and $m_j^+ + m_j^- + a_j + b_j \neq 0$ one has

$$\begin{aligned} & |G_{m,a,b}|\mathbf{z}_j^{(\boldsymbol{h})}(r,s+\sigma)^{2(m^++m^-+a+b-e_j)} \\ = \; & \frac{|F_{m,a,b}|}{|2\bar{w}\cdot(a-b)|}\mathbf{z}_j^{(\boldsymbol{h})}(r,s+\sigma)^{2(m^++m^-+a+b-e_j)} \\ = \; & \frac{e^{-2\sigma(m^++m^-+a+b-e_k)}}{|2\bar{w}\cdot(a-b)|}|F_{m,a,b}|\mathbf{z}_j^{(\boldsymbol{h})}(r,s)^{2(m^++m^-+a+b-e_j)}, \end{aligned}$$

then, (4.48) holds if

$$\frac{\exp\left[-2\sigma\left(\sum_{i\in\mathbb{N}}\boldsymbol{h}_i(m_i^++m_i^-+a_i+b_i)-\boldsymbol{h}_j\right)\right]}{|2\tilde{\omega}\cdot(a-b)|} \leq 3\gamma^{-1}e^{C(\sigma)}, \quad \forall a,b \in \mathbb{N}_f^{\mathbb{N}}. \tag{4.49}$$

It is crucial to estimate the term

$$\sum_{i\in\mathbb{N}}\boldsymbol{h}_i(m_i^++m_i^-+a_i+b_i)-\boldsymbol{h}_j.$$

We have the following result:

Lemma 4.3.3 ([23] lemma 7.1, [33] (1.13))
Let us consider m^+, m^-, a, $b \in \mathbb{N}_f^{\mathbb{N}}$ with $|a|$, $|b| \geq 1$ and $a \neq b$. If

$$\left|\sum_{i\in\mathbb{N}}(a_i-b_i)|i|\right| \leq 10\sum_{i\in\mathbb{N}}|a_i-b_i|. \tag{4.50}$$

then, for every $j \in \mathbb{Z}$ such that $m_j^+ + m_j^- + a_j + b_j \neq 0$ one has

$$\sum_{i\in\mathbb{N}}|a_i-b_i|\boldsymbol{h}_i \leq 22\left(\sum_{i\in\mathbb{N}}(m_j^++m_j^-+a_j+b_j)\boldsymbol{h}_i-\boldsymbol{h}_j\right) \tag{4.51}$$

To prove Lemma 4.3.3, we rely heavily on notation and results introduced by Bourgain in [33], and later extended by Cong, Li, Shi and Yuan in [35] (see Definition 6.1).
The definitions presented below constitute the main technical tools of the proof. They originate from Bourgain's work [33] in the setting of Gevrey regularity for momentum-preserving Hamiltonians, and were subsequently adapted to a more general framework, including the case

of Sobolev regularity, in [23].

These ideas were originally developed for the nonlinear Schrödinger equation and crucially rely on its quadratic dispersion law.
In the case of the wave equation, which features a linear dispersion law, a related approach was employed by Cong and Yuan, though under stronger non-resonance assumptions. For wave-type equations, it is therefore unclear whether the same strategy can be implemented while retaining the non-resonance conditions (4.3.1).

However, for Kirchhoff-type equations such as (4.1), the situation is different.
The specific structure of the nonlinearity significantly reduces the number of resonances, making it possible to apply the ideas of [33], [24], and [23], while keeping the same non-resonance assumptions (4.3.1).

Definition 4.3.2 (Decreasing Rearrangement) *Let us consider* $v \in \mathbb{N}_f^{\mathbb{N}}$ *and the set* $I = \{1, \dots, |v|\}$. *We denote with* $\hat{n} = \hat{n}(v)$ *the vector* $(\hat{n}_l)_{l \in I}$ *that is the decreasing rearrangement of the sequence*

$$\{n > 1 \ \textit{repeated}\ v_n \textit{times}\} \cup \{1, \ \textit{repeated}\ v_1 \ \textit{times}\},$$

namely, if J *is the maximal index in* $\mathrm{supp}(v)$*, we have*

$$\hat{n}(v) = \left(\underbrace{J, \dots J}_{v_J \ times}, \underbrace{J-1, \dots J-1}_{v_{J-1} \ times}, \dots, \underbrace{1, \dots 1}_{v_1 \ times} \right)$$

For a fixed m^+, m^-, a, $b \in \mathbb{N}_f^{\mathbb{N}}$ with$|a|$, $|b| \geq 1$ and $a \neq b$, we define

$$\hat{n} := \hat{n}(m^+ + m^- + a + b), \quad \hat{m} := \hat{n}(|a-b|),$$

where $|a-b| = (|a_j - b_j|)_{j \in \mathbb{N}}$.
We make the following observations:

1. for every $i \in I$ there exists a coefficient $\sigma_i \in \{-1, 1\}$ if $\hat{n}_l \neq 1$ and $\sigma_i \in \{-1, 0, 1\}$ in the remaining case such that

$$\sum_{i \in I} \sigma_i \hat{n}_i = \sum_{j \geq 1} l(a_l - b_l).$$

2. If we denote by D the cardinality of $\hat{m}$ and with N the one of $\hat{n}$, it holds

$$D \leq N.$$

3. Let $h : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function such that $h(0) = 0$ and let $\sigma_l = \mathrm{sign}(a_{m_l} - b_{m_l})$, we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} h(i)|a_i - b_i| &= \sum_{l \geq 1} h(\hat{m}_l) \\ \sum_{i \in \mathbb{N}} h(i)(a_i - b_i) &= \sum_{l \geq 1} \sigma_l h(\hat{m}_l) \end{aligned} \tag{4.52}$$

Moreover we have

$$\sum_{i\in\mathbb{Z}} |\alpha_i - \beta_i| = D \le N \tag{4.53}$$

4. We have the following

Lemma 4.3.4 *Let $a \neq b \in \mathbb{N}_f^{\mathbb{N}}$, such that $|a|$, $|b| \ge 1$ and (4.50) holds, then*

$$|\hat{m}_1| \le 21 \sum_{l\ge 2} \hat{n}_l \tag{4.54}$$

Proof: If D is the cardinality of $\hat{m}$ then we distinguish two cases:

$D = 1$:
the only possibility is that $|a - b| = 1$ and so must exists a unique $i \in \mathrm{supp}(a - b) \subset \mathbb{N}$ such that $a_i - b_i = 1$(or alternatively -1).
Suppose without loss of generality that $a_i = b_i + 1$, then we have two cases :
if $i < \hat{n}_1$, hence (4.54) follows easily;
if $\hat{n}_1 = i$,then $m_i^+ + m_i^- + a_i + b_i \ge 3$, and so $\hat{n}_2 = \hat{n}_1$ and so $m_1 \le \sum_{i\ge 2} \hat{n}_i$.

$D \ge 2$:
From (4.52) together with we have that

$$|\hat{m}_1| - \left|\sum_{i\ge 2} \sigma_i \hat{m}_i\right| \le \left|\sum_{i\in\mathbb{N}} \sigma_i \hat{m}_i\right| = \left|\sum_{i\in\mathbb{N}} i(a_i - b_i)\right| \le 10 \sum_{i\in\mathbb{N}} |a_i - b_i|. \le 10N$$

then

$$\begin{aligned} |\hat{m}_1| \le 10N + \left|\sum_{i\ge 2} \sigma_i \hat{m}_i\right| &\le 10N + \sum_{i\ge 2} m_i \\ &\le 10N + \sum_{i\ge 2} \hat{n}_i \le 10 + \sum_{i\ge 2} (10 + \hat{n}_i) \\ &\le 10 + 11 \sum_{i\ge 2} \hat{n}_i \le 21 \sum_{i\ge 2} \hat{n}_i \end{aligned}$$

□

Proof of Lemma 4.3.3 Let us proceed by steps:
First of all we have that

$$\sum_{i\in\mathbb{N}} h(i)|a_i - b_i| \le h(\hat{m}_1) + \sum_{l\ge 2} h(\hat{n}_l). \tag{4.55}$$

In fact, using (4.52) and the fact that $m_l \le \hat{n}_l$ we have that

$$\begin{aligned} \sum_{i\in\mathbb{Z}} h(i)|a_i - b_i| &= \sum_{l\ge 1} h(\hat{m}_l) = h(\hat{m}_1) + \sum_{l\ge 2} h(\hat{m}_l) \\ &\le h(\hat{m}_1) + \sum_{l\ge 2} h(n_l) \end{aligned}$$

Using (4.55) together with the sub-additivity of h we have

$$\begin{aligned}
\sum_{i\in\mathbb{N}} |a_i - b_i| h(i) &\leq h(m_1) + \sum_{j\geq 2} h(\hat{n}_l) \\
&\leq h\left(21 \sum_{l\geq 3} \hat{n}_l\right) + \sum_{l\geq 2} h(\hat{n}_l) \\
&\leq 21 \sum_{l\geq 2} h\left(\hat{n}_l\right) + \sum_{l\geq 2} h(\hat{n}_l) \\
&\leq 22 \sum_{l\geq 2} h(\hat{n}_l) \leq 22 \left(\sum_{l\geq 1} h(\hat{n}_l) - h(\hat{n}_1)\right)
\end{aligned}$$

□

4.3.3 Resolution of the Homological equation

The aim of this subsection is to prove Lemma 4.3.2. To this end, we first verify estimate (4.49). We then consider two different cases:

Case 1

$$\left|\sum_{i\in\mathbb{N}} (a_i - b_i) i\right| \geq 10|a - b|.$$

Then, since we got $\bar{w}_j = j + \xi_j$ with $|\xi_j| \leq \dfrac{1}{2}$, it holds

$$|\tilde{\omega} \cdot (a - b)| \geq 10|a - b| - \sup_j |\xi_j||a - b| \geq 9|a - b| \geq 9.$$

Hence, from (4.40) we have

$$\frac{\exp\left[-2\sigma\left(\sum_{i\in\mathbb{N}} \boldsymbol{h}_i(m_i^+ + m_i^- + a_i + b_i) - \boldsymbol{h}_j\right)\right]}{|2\tilde{\omega}\cdot(a-b)|} \leq 1$$

Case 2

$$\left|\sum_{i\in\mathbb{N}} (a_i - b_i) i\right| \leq 10|a - b|.$$

If one between a and b is null (let us suppose without loss of generality $b = 0$) then, thanks to (4.45) we have

$$|\tilde{\omega} \cdot (a - b)| \geq \frac{1}{2} \geq \frac{\gamma}{2}$$

since, $\tilde{\omega} \in \mathcal{Q}^{\mathbf{Sym}}$ implies $\tilde{\omega}_j \geq \frac{1}{2}$ for all j. Then, reasoning as above, inequality (4.49) follows again by (4.40).

If $|a|,\ |b| \geq 1$, thanks to inequality (4.51) of Lemma 4.3.3, we have

$$\begin{aligned}
&\frac{\exp\left[-2\sigma\left(\sum_{i\in\mathbb{N}} \boldsymbol{h}_i(m_i^+ + m_i^- + a_i + b_i) - \boldsymbol{h}_j\right)\right]}{|2\tilde{\omega}\cdot(a-b)|} \\
\overset{(4.51)}{\leq}\ & \frac{\exp\left[-\frac{\sigma}{11}\left(\sum_{i\in\mathbb{N}} \boldsymbol{h}_i|a_i - b_i|\right)\right]}{|2\tilde{\omega}\cdot(a-b)|} \qquad (4.56)\\
\overset{(4.45)}{\leq}\ & \gamma^{-1}\exp\left[\sum_{i\in\mathbb{N}}\Big(-\frac{\sigma}{11}|a_i-b_i|\boldsymbol{h}_i + 2\ln\left(1+|a_i-b_i|^2|j|^2\right)\Big)\right].
\end{aligned}$$

In order to estimate the right-hand side of (4.56) we have the following:

Lemma 4.3.5 *Let us consider the family of functions*

$$f_i^{\boldsymbol{h}}(x) := -\frac{\sigma}{11}x\boldsymbol{h}_i + 2\ln\left(1+x^2|i|^2\right)$$

indexed by $i\in\mathbb{N}$ *then it holds*

$$\sum_{i\in\mathbb{N}} f_i^{\boldsymbol{h}}(|a_i-b_i|) \leq \mathtt{C}(s)$$

with $\mathtt{C}(\sigma) := 2i_\#(5\ln|i_\#| - 3)$*, for a suitable* $i_\#$ *depending on* $\boldsymbol{h}$ *and* σ*.*

Proof: We want to find an uniform bound for the family of functions:

$$f_i^{\boldsymbol{h}}(x) = -\alpha x\boldsymbol{h}_i + 2\ln\left(1+x^2|i|^2\right)$$

with $\alpha := \dfrac{\sigma}{11}$.

Using that $\ln(1+y) \leq 1 + \ln(y)$ for every $y > 0$ we have that

$$f_i^{\boldsymbol{h}}(x) \leq -\alpha x\boldsymbol{h}_i + 2 + 4\ln(x) + 4\ln|i| \quad \text{for all } x\geq 1$$

Keeping in mind that we have

$$\max_{x\geq 1}\Big(-\alpha\boldsymbol{h}_i x + 4\ln(x)\Big) = \begin{cases} -\alpha\boldsymbol{h}_i & \text{if } \alpha\boldsymbol{h}_i \geq 4 \\ -4 + 4\ln\left(\dfrac{4}{\alpha\boldsymbol{h}_i}\right) & \text{if } \alpha\boldsymbol{h}_i < 4 \end{cases}$$

Let us denote by $i_0\in\mathbb{R}_+$ the number for which $\alpha\boldsymbol{h}_i = 2$, i.e.

$$i_0 = \boldsymbol{h}^{-1}\left(\frac{4}{\alpha}\right) \qquad (4.57)$$

We then get

$$\begin{aligned}
\sum_{i\in\mathbb{N}} f_i^{\boldsymbol{h}}(|a_i-b_i|) &= \sum_{\substack{i\in\mathbb{N}\\ a_i-b_i\neq 0}} f_i^{\boldsymbol{h}}(|a_i-b_i|) \qquad (4.58)\\
&\leq \sum_{\substack{i<i_0\\ a_i-b_i\neq 0}}\left(-2 + 4\ln\left(\frac{4}{\alpha}\right) + 4\ln\left(\frac{|i|}{\boldsymbol{h}_i}\right)\right) + \sum_{\substack{i\geq i_0\\ a_i-b_i\neq 0}}\left(-\alpha\boldsymbol{h}_i + 4\ln|i| + 2\right).
\end{aligned}$$

We split (4.58) in

$$
\begin{aligned}
I &:= \sum_{\substack{i<i_0 \\ a_i-b_i\neq 0}} \left(-2+4\ln\left(\frac{4}{\alpha}\right)+4\ln\left(\frac{|i|}{\boldsymbol{h}_i}\right)\right) \\
II &:= \sum_{\substack{i\geq i_0 \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+4\ln|i|+2\right)
\end{aligned}
$$

• I:

$$
\begin{aligned}
\sum_{\substack{i<i_0 \\ a_i-b_i\neq 0}} \left(-2+4\ln\left(\frac{4}{\alpha}\right)+4\ln\left(\frac{|i|}{\boldsymbol{h}_i}\right)\right) &\leq \sum_{\substack{i<i_0 \\ a_i-b_i\neq 0}} \left(-2+4\ln\left(\frac{4}{\alpha}\right)+4\ln\left(\frac{|i_0|}{\boldsymbol{h}_{i_0}}\right)\right) \\
&= 2\sum_{\substack{i<i_0 \\ a_i-b_i\neq 0}} \left(2\ln|i_0|-1\right) \\
&\leq 2|i_0|\left(2\ln|i_0|-1\right)
\end{aligned}
$$

• II:
We can suppose without loss of generality that $|i_0|\geq e$ (we can reduce the size of σ if is not)

$$
\sum_{\substack{i\geq i_0 \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+4\ln|i|+2\right) \leq \sum_{\substack{i\geq i_0 \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+6\ln|i|\right).
$$

Let us consider the index

$$
i^*(\theta) := \inf_{i\in\mathbb{N}} \left(\theta\boldsymbol{h}_{i^*}\geq \ln|i^*|\right), \tag{4.59}
$$

then, setting $i^* := i^*(\alpha/6)$ we have

$$
\begin{aligned}
\sum_{\substack{i\geq i_0 \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+4\ln|i|+2\right) &\leq \sum_{\substack{i\geq i_0 \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+6\ln|i|\right) \\
&\leq \sum_{\substack{i_0\leq i\leq i^* \\ a_i-b_i\neq 0}} \left(-\alpha\boldsymbol{h}_i+6\ln|i|\right) \\
&\leq i^*(6\ln|i^*|-\alpha\boldsymbol{h}_{i_0}) \\
&= 2i^*(3\ln|i^*|-2),
\end{aligned}
$$

where the sum is intended to be zero if $i_0\geq i^*$.
By choosing

$$
i_{\#}(\sigma) := \sup\{i_0, i^*\} \tag{4.60}
$$

we have

$$
\sum_{i\in\mathbb{N}} f_i^{\boldsymbol{h}}(|a_i-b_i|) \leq 2i_{\#}(5\ln|i_{\#}|-3) = \mathtt{C}(\sigma). \tag{4.61}
$$

□

Proof of Lemma 4.3.2

Thanks to Lemma 4.3.5 and inequality (4.56), we have that

$$\sup_{\substack{j\in\mathbb{N}\\ a,b,m^+,m^-\in\mathbb{N}^{\mathbb{N}}_f}} \frac{\exp\left[-2\sigma\left(\sum_{i\in\mathbb{N}}\boldsymbol{h}_i(m_i^+ + m_i^- + a_i + b_i) - \boldsymbol{h}_i\right)\right]}{|2\tilde{\omega}\cdot(a-b)|} \le \gamma^{-1}e^{\mathtt{C}(\sigma)},$$

that, using definition (4.34), yields to

$$|L_\omega^{-1}F(\omega)|_{r,\boldsymbol{h},s+\sigma} \le \gamma^{-1}e^{\mathtt{C}(\sigma)}|F(\omega')|_{r,\boldsymbol{h},s} \quad \forall\omega,\,\omega'\in\mathtt{D}_\gamma^{\mathbf{Sym}}. \tag{4.62}$$

Now, by recalling the definition of Δ in (4.20),

$$\sup_{\substack{\omega,\,\omega'\in\mathtt{D}_\gamma^{\mathbf{Sym}}\\ \omega\neq\omega'}} \left|\Delta_{\omega,\,\omega'}L_\omega^{-1}F\right|_{r,\boldsymbol{h},s+\sigma}.$$

First of all, let us note that

$$\begin{aligned} &\Delta_{\omega,\,\omega'}L_\omega^{-1}F\\ =\ & \sum\frac{\Delta_{\omega,\,\omega'}F_{m,a,b}}{2i\bar{w}\cdot(a-b)}I^mW^a\bar{W}^b + \sum F_{m,a,b}(\omega')\Delta_{\omega,\,\omega'}\left(\frac{1}{2i\tilde{\omega}\cdot(a-b)}\right)I^mW^a\bar{W}^b.\end{aligned}$$

We define

$$\begin{aligned} G_1 &:= \sum\frac{\Delta_{\omega,\,\omega'}F_{m,a,b}}{2i\tilde{\omega}\cdot(a-b)}I^mW^a\bar{W}^b;\\ G_2 &:= \sum F_{m,a,b}(\omega')\Delta_{\omega,\,\omega'}\left(\frac{1}{2i\tilde{\omega}\cdot(a-b)}\right)I^mW^a\bar{W}^b.\end{aligned}$$

Using (4.62), we have that

$$\sup_{\substack{\omega,\,\omega'\in\mathtt{D}_\gamma^{\mathbf{Sym}}\\ \omega\neq\omega'}}|G_1|_{r,\boldsymbol{h},s+\sigma} \le \gamma^{-1}e^{\mathtt{C}(\sigma)}\sup_{\substack{\omega,\,\omega'\in\mathtt{D}_\gamma^{\mathbf{Sym}}\\ \omega\neq\omega'}}\left|\Delta_{\omega,\,\omega'}F\right|_{r,\boldsymbol{h},s}. \tag{4.63}$$

We now have to bound the $\mathcal{H}_{r,\boldsymbol{h},s+\sigma}$-norm of the term $G_2(\omega)$:

$$\begin{aligned} &\left|F_{m,a,b}(\omega')\right|\cdot\left|\Delta_{\omega,\,\omega'}\left(\frac{1}{2i\bar{w}\cdot(a-b)}\right)\right|\mathbf{z}^{(\boldsymbol{h})}(r,s+\sigma)^{2(m^++m^-+a+b-e_k)} \qquad (4.64)\\ \le\ & \sup_{\omega\in\mathtt{D}_\gamma^{\mathbf{Sym}}}|F|_{r,\boldsymbol{h},s}\left[\left|\Delta_{\omega,\,\omega'}\left(\frac{1}{2i\tilde{\omega}\cdot(a-b)}\right)\right|e^{-2\sigma(m^++m^-+a+b-e_k)\boldsymbol{h}}\right]\mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(m^++m^-+a+b-e_k)}.\end{aligned}$$

Now, for $\omega,\,\omega'\in\mathrm{D}_\gamma$ we have

$$\begin{aligned}\left|\Delta_{\omega,\,\omega'}\left(\frac{1}{2i\tilde{\omega}\cdot(a-b)}\right)\right| &= \frac{\left|\frac{1}{2i\bar{w}\cdot(a-b)}-\frac{1}{2i\bar{w}'\cdot(a-b)}\right|}{2\|\omega-\omega'\|_\infty}\\ &\le \frac{1}{4}\frac{|(\omega-\omega')\cdot(a-b)|}{|\omega\cdot(a-b)||\omega'\cdot(a-b)|}\|\omega-\omega'\|_\infty^{-1}\\ &\le \frac{1}{4}\frac{|a-b|}{|\bar{w}\cdot(a-b)||\bar{w}'\cdot(a-b)|}.\end{aligned}$$

Using the non-resonance condition (4.45) we have

$$\begin{aligned}\left|\Delta_{\omega,\omega'}\left(\frac{1}{2i\tilde{\omega}\cdot(a-b)}\right)\right| &\leq |a-b|\gamma^{-2}\prod_{j\in\mathbb{N}}(1+|a_j-b_j|^2\langle j\rangle^2)^2 \\ &\leq \gamma^{-2}\prod_{j\in\mathbb{N}}(1+|a_j-b_j|^2\langle j\rangle^2)^5.\end{aligned}$$

Then, using (4.51) and proceeding like in Lemma (4.3.5) we get that

$$\begin{aligned}&\left|\Delta_{\omega,\omega'}\left(\frac{1}{2i\bar{w}\cdot(a-b)}\right)\right| e^{-2\sigma(m^++m^-+a+b-e_k)\boldsymbol{h}} \\ \leq\;& \gamma^{-2}\exp\left[\sum_{i\in\mathbb{N}}\Big(-\frac{\sigma}{11}|a_i-b_i|\boldsymbol{h}_i+5\ln\big(1+|a_i-b_i|^2\langle i\rangle^2\big)\Big)\right] \qquad (4.65)\\ =\;& \gamma^{-2}\exp\left[5\sum_{i\in\mathbb{N}}\Big(-\frac{2\sigma}{55}|a_i-b_i|\boldsymbol{h}_i+2\ln\big(1+|a_i-b_i|^2\langle i\rangle^2\big)\Big)\right] \\ \leq\;& \gamma^{-2}\exp\left[5\,\mathtt{C}\left(\frac{2\sigma}{5}\right)\right].\end{aligned}$$

Finally, by (4.64) together with (4.65) we get that

$$\sup_{\substack{\omega,\omega'\in\mathrm{D}_\gamma\\ \omega\neq\omega'}}|G_2|_{r,\boldsymbol{h},s+\sigma}\leq\gamma^{-2}e^{5\,\mathtt{C}(\frac{2}{5}\sigma)}\sup_{\omega\in\mathrm{D}_\gamma}|F|_{r,\boldsymbol{h},s}. \qquad (4.66)$$

Now, using (4.63) and (4.66),

$$\begin{aligned}\|G\|^{\gamma,\mathrm{D}_\gamma}_{r,\boldsymbol{h},s+\sigma} &\leq \sup_{\omega\in\mathrm{D}_\gamma}|G|_{r,\boldsymbol{h},s}+\gamma\sup_{\substack{\omega,\omega'\in\mathrm{D}_\gamma\\ \omega\neq\omega'}}|G_1|_{r,\boldsymbol{h},s+\sigma}+\gamma\sup_{\substack{\omega,\omega'\in\mathrm{D}_\gamma\\ \omega\neq\omega'}}|G_2|_{r,\boldsymbol{h},s+\sigma} \\ &\leq 3\gamma^{-1}e^{C(\sigma)}\|F\|^{\gamma,\mathrm{D}_\gamma}_{r,\boldsymbol{h},s},\end{aligned}$$

that is (4.47) with

$$C(\sigma)=\mathtt{C}\left(\frac{2\sigma}{5}\right). \qquad (4.67)$$

□

4.3.4 Logarithmic Weight

We now consider the case of

$$\boldsymbol{h}_i:=\Big[\ln\big(2+|i|\big)\Big]^\tau,\qquad \tau>2.$$

Putting again $\alpha=\dfrac{\sigma}{11}$ we have, by (4.57)

$$i_0=\exp\left[\left(\frac{4}{\alpha}\right)^{1/\tau}\right]-2$$

and, in order to find i^* we have to solve

$$\frac{\alpha}{6}\left[\ln\left(2+|i|\right)\right]^{\tau} \geq \ln\langle i\rangle \quad\Rightarrow\quad \frac{\alpha}{3}\left[\ln\left(2+|i|\right)\right]^{\tau} \geq \ln(2+|i|)$$
$$\Rightarrow\quad \frac{\alpha}{6}\left[\ln\left(2+|i|\right)\right]^{\tau-1} \geq 1$$
$$\Rightarrow\quad |i| \geq \exp\left[\left(\frac{6}{\alpha}\right)^{\frac{1}{\tau-1}}\right] - 2.$$

Recalling (4.59), we can take

$$i* = \exp\left[\left(\frac{6}{\alpha}\right)^{\frac{1}{\tau-1}}\right] - 2.$$

Since $i^* \geq i_0$, from (4.60) we have $i_\# = i^*$. By substituting it in

$$f_i^\tau(x) := -\alpha x\left[\ln\left(2+\langle i\rangle\right)\right]^{\tau} + \ln\left(1+x^2|i|^2\right)$$

and using (4.61), we get

$$\begin{aligned}\sum_{i\in\mathbb{N}} f_i^\tau(|a_i-b_i|) &\leq 2\left\{\exp\left[\left(\frac{3}{\alpha}\right)^{\frac{1}{\tau-1}}\right] - 3\right\}\cdot\left\{5\ln\left\{\exp\left[\left(\frac{3}{\alpha}\right)^{\frac{1}{\tau-1}}\right]-2\right\}-3\right\}\\ &\leq 10\exp\left[\left(\frac{3}{\alpha}\right)^{\frac{1}{\tau-1}}\right]\cdot\left(\frac{3}{\alpha}\right)^{\frac{1}{\tau-1}}.\end{aligned}$$

Then

$$\mathtt{C}(\sigma) = 10\exp\left[\left(\frac{33}{\sigma}\right)^{\frac{1}{\tau-1}}\right]\cdot\left(\frac{33}{\sigma}\right)^{\frac{1}{\tau-1}}$$

and

$$C(\sigma) = 5\mathtt{C}(\frac{2\sigma}{5}) = 50\exp\left[\left(\frac{165}{2\sigma}\right)^{\frac{1}{\tau-1}}\right]\cdot\left(\frac{165}{2\sigma}\right)^{\frac{1}{\tau-1}}. \tag{4.68}$$

4.4 Projection

In this section we introduce the projection operator (4.71), which will play a fundamental role in the KAM scheme developed in Section 4.5. Since our final goal is to construct a symplectic map that conjugates the Hamiltonian in (4.4) to one admitting $\mathcal{T}_{\mathbf{I}}$ as an invariant manifold, it is necessary to distinguish the terms of H that preserve the invariance of $\mathcal{T}_{\mathbf{I}}$ from those that do not.
The idea is to decompose our Hamiltonian as a sum of regular terms with an increasing "order of zero" at $\mathcal{T}_{\mathbf{I}}$.
We start by giving the following definition:

Definition 4.4.1 (Space $\mathscr{H}_{\mathbf{r},\mathbf{h},\mathbf{s}}$)
We define the Banach space

$$\mathscr{H}_{r,\boldsymbol{h},s} := \mathbf{H}_{r,\boldsymbol{h},s}\oplus\mathbb{R} = \{H+c\ :\ H\in\mathbf{H}_{r,\boldsymbol{h},s},\ c\in\mathbb{R}\},$$

endowed with the norm (4.34).
Analogously, we define the Banach space $\mathscr{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$ *of families of Hamiltonians* $H(\omega) \in \mathscr{H}_{r,\boldsymbol{h},s}$ *parametrized by* $\omega \in \mathcal{Q}$*, endowed with the norm* (4.21).

Remark 4.4.1 *It is easy to see that all the theorems stated above for* $\mathbf{H}$ *work also for* $\mathscr{H}$*. Lemma 4.3.2 in particular.*

By fixing

$$\kappa < 1,\, r' := \kappa r < r, \quad \text{and} \quad \mathbf{I} = (\mathbf{I}_j)_{j\in\mathbb{Z}} \text{ such that } \sqrt{\mathbf{I}} \in B_{r'}(\mathfrak{g}_s), \tag{4.69}$$

we look for a direct sum decomposition of the space $\mathscr{H}_{r,\boldsymbol{h},s}$ defined in (4.4.1) of the form

$$\mathscr{H}_{r,\boldsymbol{h},s} := \mathscr{H}^{(-2)}_{r,\boldsymbol{h},s} \oplus \mathscr{H}^{(0)}_{r,\boldsymbol{h},s} \oplus \mathscr{H}^{(\geq 2)}_{r,\boldsymbol{h},s}, \quad H = H^{-2} + H^{(0)} + H^{(\geq 2)}. \tag{4.70}$$

Here, $H^{(0)} \in \mathcal{H}^{(0)}_{r,\boldsymbol{h},s}$ vanishes on $\mathcal{T}_{\mathbf{I}}$, and $H^{(\geq 2)} \in \mathscr{H}^{(\geq 2)}_{r,\boldsymbol{h},s}$ has a zero of order at least 2 on $\mathcal{T}_{\mathbf{I}}$.

We then give the following definition:

Definition 4.4.2 (Projectors)
let us fix $\mathbf{I} \in B_{r'^2}(\mathfrak{g}_{\boldsymbol{h},2s})$, $r' < r$*. For every* $\mathscr{H}_{r,\boldsymbol{h},s}$ *and for* $I \in \mathbf{I} + B_{(r^2-r'^2)}(B_{r^2}(\mathfrak{g}_{\boldsymbol{h},2s}))$ *we define the* **projection operator**

$$\Pi^d : \mathscr{H}_{r,\boldsymbol{h},s} \to \mathscr{H}_{r,\boldsymbol{h},s},\; d = 2q-2,\; q \geq 0$$

as

$$\begin{aligned} \Pi^{-2} H(I, W) &= H(\mathbf{I}, W) \\ \Pi^{2q-2} H(I, W) &= \frac{1}{q!} D^q_I H(\mathbf{I}, W)[I - \mathbf{I}]^q \qquad \text{for } q \geq 1 \end{aligned} \tag{4.71}$$

where

$$D^q_I H(\mathbf{I}, W)[I - \mathbf{I}]^q = \sum_{|\delta| = q} \binom{q}{\delta} \partial^\delta_I H(\mathbf{I}, W)[I - \mathbf{I}]^\delta.$$

For the case of family of regular Hamiltonians $H(\cdot) \in \mathscr{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$*, the definition of the projectors is analogue.*

Let us note that definition (4.71) is possible because, from Remark 4.2.3, we know that every $H \in \mathscr{H}_{r,\boldsymbol{h},s}$, viewed as a function of the variables I and W, is analytic on the open ball $B_{r^2}(\mathfrak{g}_{\boldsymbol{h},2s})$.

The next result shows that the projectors (4.71) are indeed well-posed from $H \in \mathcal{H}_{r,\boldsymbol{h},s} \cap \mathcal{F}_{\mathbf{kir}}$ into itself:

Theorem 4.4.3 *Let us fix* $\mathbf{I}, \kappa$ *as in* (4.69)*,* $\mu > 0$*, a positive measure set* $\mathcal{A} \subseteq \mathcal{Q}$ *and let us consider the quantity*

$$c_\kappa := \begin{cases} \dfrac{1}{\ln \kappa^{-2}} & \text{if } \frac{1}{2} < \kappa^2 < 1 \\[2ex] 2\kappa^2 & \text{if } 0 < \kappa^2 \leq \frac{1}{2}. \end{cases}$$

For every $d = 2q-2$, $q \in \mathbb{N}$ the operator defined in (4.71) *is bounded*

$$\Pi^d : \mathscr{H}^{\mathcal{A}}_{r,\boldsymbol{h},s} \longrightarrow \mathscr{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}.$$

Moreover the following holds:

(i) $\Pi^d\Pi^d = \Pi^s$ *and* $\Pi^d\Pi^{d'} = \Pi^{d'}\Pi^d$ *for every* $d \neq d'$*;*

(ii) *For every* $\kappa_* \leq 1$ *we have*

$$\|\Pi^{2q-2}H\|^{\mu,\mathcal{A}}_{\kappa_* r,\boldsymbol{h},s} \leq \kappa_*^{-2}\left(1+\frac{\kappa_*^2}{\kappa^2}\right)^q c_k^q \|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}; \tag{4.72}$$

(iii) $\Pi^{2q-2}H = 0,\ \forall q \leq q' \Longrightarrow \partial_u^\alpha \partial_{\bar{u}}^\beta H = 0$ *on* $\mathcal{T}_{\mathbf{I}}$ *for* $0 \leq |\alpha|+|\beta| \leq q'$*;*

(iv) *If we define*

$$\Pi^{<d} := \sum_{\substack{j=-2\\ j\in 2\mathbb{Z}}} \Pi^j,$$

then, if $\Pi^{<d_1}F = \Pi^{<d_2}G = 0$*, we have*

$$\Pi^{<d_1+d_2}\{F,G\} = 0;$$

Proof: Let us prove (ii):
We have that, for a fixed $q \geq 0$

$$\begin{aligned}
\Pi^{2q-2}H(I,W,\bar{W}) &= \sum_{\substack{\delta\in\mathbb{N}^{\mathbb{Z}}_f:\\ |\delta|=q}} \binom{q}{\delta}\partial_I^\delta H(\mathbf{I},W,\bar{W}))[I-\mathbf{I}]^\delta \\
&= \sum_{|\delta|=q}(I-\mathbf{I})^\delta \sum_{m\succcurlyeq\delta,a,b}\binom{m}{\delta}H_{m,a,b}\mathbf{I}^{m-\delta}W^a\bar{W}^b.
\end{aligned}$$

Now, having that

$$(I-\mathbf{I})^\delta = \sum_{\gamma\preccurlyeq\delta}\binom{\delta}{\gamma}(-1)^{|\delta-\gamma|}\mathbf{I}^{\delta-\gamma}I^\gamma,$$

we get

$$\Pi^{2q-2}H(I,W,\bar{W}) = \sum_{|\delta|=q}\sum_{m\succcurlyeq\delta,a,b}\sum_{\gamma\preccurlyeq\delta}H_{m,a,b}\binom{m}{\delta}\binom{\delta}{\gamma}(-1)^{|\delta-\gamma|}\mathbf{I}^{m-\gamma}I^\gamma W^a\bar{W}^b.$$

Hence

$$\begin{aligned}
&|\Pi^{2q-2}H|_{\kappa_* r,\boldsymbol{h},s} \\
\overset{(4.15)}{\leq}\ & \sup_{k\in\mathbb{N}}\sum_{|\delta|=q}\sum_{m\succcurlyeq\delta,a,b}\sum_{\gamma\preccurlyeq\delta}\binom{m}{\delta}\binom{\delta}{\gamma}\frac{(m_k^+ + m_k^- + a_k + b_k)}{2}|H_{m,a,b}|\mathbf{I}^{m-\gamma}\mathbf{z}^{(\boldsymbol{h})}(\kappa_* r)^{2(\gamma^+ + \gamma^- + a + b - e_k)}.
\end{aligned}$$

Now, thanks to (4.69) we have

$$\mathbf{I}_k \leq \mathbf{z}^{(\boldsymbol{h})}(\kappa r,s)^2 = \kappa^2\mathbf{z}^{(\boldsymbol{h})}(r,s)$$

and then

$$\mathbf{I}^{m-\gamma} \le \kappa^{2|m|-2|\gamma|}\mathbf{z}^{(\boldsymbol{h})}(r,s)^{2m-2\gamma}. \tag{4.73}$$

From (4.73), we get

$$\begin{aligned}
& |\Pi^{2q-2}H|_{\kappa_* r,\boldsymbol{h},s} \\
\le\ & (\kappa_*)^{-2}\sup_{k\in\mathbb{N}}\sum_{|\delta|=q}\sum_{m\succcurlyeq\delta,a,b}\sum_{\gamma\preccurlyeq\delta}\binom{m}{\delta}\binom{\delta}{\gamma}\times \\
& \kappa^{2|m|}\left(\frac{\kappa_*}{\kappa}\right)^{2|\gamma|}\frac{(m_k^+ + m_k^- + a_k + b_k)}{2}|H_{m,a,b}|\mathbf{z}^{(\boldsymbol{h})}(r)^{2(m+a+b-e_k)} \\
\le\ & (\kappa_*)^{-2}\sum_{|\delta|=q}\sum_{m\succcurlyeq\delta}\sum_{\gamma\preccurlyeq\delta}\binom{m}{\delta}\binom{\delta}{\gamma}\kappa^{2|m|}\left(\frac{\kappa_*}{\kappa}\right)^{2|\gamma|}|H|_{r,\boldsymbol{h},s} \\
=\ & (\kappa_*)^{-2}\sum_{|\delta|=q}\sum_{m\succcurlyeq\delta}\kappa^{2|m|}\left(1+\frac{\kappa_*^2}{\kappa^2}\right)^q|H|_{r,\boldsymbol{h},s}.
\end{aligned} \tag{4.74}$$

In order to estimate (4.74), we use the following result:

Lemma 4.4.1 ([24], Lemma 4.1)

$$\sum_{|\delta|=q}\sum_{m\succcurlyeq\delta}\kappa^{2|m|} \le c_\kappa^q. \tag{4.75}$$

Using (4.75) in (4.74) get

$$|\Pi^{2q-2}H|_{\kappa_* r,\boldsymbol{h},s} \le \kappa_*^{-2}\left(1+\frac{\kappa_*^2}{\kappa^2}\right)^q c_k^q|H|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}.$$

The Lipschitz estimate is analogous. .

The points (i), (ii), (iv) follow, throught a direct computation, from (4.71). □

The operators (4.71) induce the direct sum decomposition

$$\mathscr{H}_{r,\boldsymbol{h},s} = \bigoplus_d \mathscr{H}^{(d)}_{r,\boldsymbol{h},s}$$

where

$$\mathscr{H}^{(d)}_{r,\boldsymbol{h},s} = \{H \in \mathscr{H}_{r,\boldsymbol{h},s} : \Pi^d H = H\}.$$

Similarly for $\mathscr{H}^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}$.
Finally we define the projector

$$\Pi^{\ge 2q-2} := Id - \Pi^{<2q-2}$$

for $d = 2q-2$, $q \in \mathbb{N}$, and the associated subspace

$$\mathscr{H}^{(\ge d)}_{r,\boldsymbol{h},s} := \bigoplus_{d'\ge d}\mathscr{H}^{(d')}_{r,\boldsymbol{h},s} = \{H \in \mathscr{H}_{r,\boldsymbol{h},s} : H^{\ge d} := \Pi^{\ge d}H = H\}.$$

This induces the decomposition considered in (4.70).
Thanks to what stated above, we can give the following definition:

Definition 4.4.4 (Normal Form Hamiltonian)
An Hamiltonian of the form

$$N = D_\omega + \tilde N$$

is in normal form at $\mathcal{T}_{\mathbf{I}}$, if $\tilde N \in \mathscr{H}^{\geq 2}_{r,\boldsymbol{h},s}$.
We denote the affine sub-space of normal form Hamiltonians by $\mathscr{N}_{r,\boldsymbol{h},s}$ $(= \mathscr{N}_{r,\boldsymbol{h},s}(\omega,\mathbf{I}))$.

4.4.1 Projectors and Sub-Spaces

For $\mathcal{D}_\omega$ defined in (4.43) and $\omega \in \mathtt{D}^{\mathbf{Sym}}_\gamma$ let us consider the ker and range of the operator $\{D_\omega,\cdot\}$, respectively

$$\mathcal{K}_{\mathbf{kir}} = (\mathcal{K}_{\mathbf{kir}}(\omega)) = \Big\{ H \in \mathcal{F}_{\mathbf{kir}} \,:\, H = \sum_{m\in\mathbb{N}^{\mathbb{Z}}_f} H_{m,0,0} I^m \Big\}$$
$$\mathcal{R}_{\mathbf{kir}} = (\mathcal{R}_{\mathbf{kir}}(\omega)) = \Big\{ H \in \mathcal{F}_{\mathbf{kir}} \,:\, H = \sum_{\substack{m,a,b\\ a\neq b}} H_{m,a,b} I^m W^a \bar W^b \Big\}. \tag{4.76}$$

Similarly to Chapter 3 we denote by $\Pi_{\mathcal{K}_{\mathbf{kir}}}$, $\Pi_{\mathcal{R}_{\mathbf{kir}}}$ the associated projectors and we define the sub-spaces of $\mathscr{H}_{r,\boldsymbol{h},s}$:

$$\mathscr{H}^{\mathcal{K}_{\mathbf{kir}}}_{r,\boldsymbol{h},s} := \{ H \in \mathscr{H}_{r,\boldsymbol{h},s} \,:\, \Pi_{\mathcal{K}_{\mathbf{kir}}} H = H\}, \quad \mathscr{H}^{\mathcal{R}_{\mathbf{kir}}}_{r,\boldsymbol{h},s} := \{ H \in \mathscr{H}_{r,\boldsymbol{h},s} \,:\, \Pi_{\mathcal{R}_{\mathbf{kir}}} H = H\}. \tag{4.77}$$

By (4.34) we have

$$|\Pi_{\mathcal{K}_{\mathbf{kir}}} H|_{r,\boldsymbol{h},s},\ |\Pi_{\mathcal{R}_{\mathbf{kir}}} H|_{r,\boldsymbol{h},s} \leq |H|_{r,\boldsymbol{h},s}. \tag{4.78}$$

Since the projector Π^d defined in (4.71) commute with $\Pi^{\mathcal{K}}_{\mathbf{kir}}$ and $\Pi^{\mathcal{R}_{\mathbf{kir}}}$, we can define

$$\mathscr{H}^{(d,\mathcal{R}_{\mathbf{kir}})}_{r,\boldsymbol{h},s} := \Big\{ H \in \mathscr{H}_{r,\boldsymbol{h},s} : \Pi^{d,\mathcal{R}_{\mathbf{kir}}} H := \Pi^d \Pi_{\mathcal{R}_{\mathbf{kir}}} H = H \Big\} \tag{4.79}$$

and similarly for $\mathscr{H}^{(\geq d,\mathcal{R}_{\mathbf{kir}})}_{r,\boldsymbol{h},s}$, $\mathscr{H}^{(d,\mathcal{K}_{\mathbf{kir}})}_{r,\boldsymbol{h},s}$, $\mathscr{H}^{(\geq d,\mathcal{K}_{\mathbf{kir}})}_{r,\boldsymbol{h},s}$.
For the special case of $\mathscr{H}^{(0,\mathcal{K}_{\mathbf{kir}})}_{r,\boldsymbol{h},s}$ we have the following:

Lemma 4.4.2 *Every $H \in \mathscr{H}^{(0,\mathcal{K}_{\mathbf{kir}})}_{r,\boldsymbol{h},s}$ has the form*

$$H = \sum_{j\in\mathbb{Z}} \mathbf{m}_j (I_j - \mathbf{I}_j), \quad \mathbf{m} = (\mathbf{m}_j)_{j\in\mathbb{Z}} \,:\, |H|_{r\,\boldsymbol{h},s} = \|\mathbf{m}\|_\infty. \tag{4.80}$$

Moreover, if we define

$$\|\mathbf{m}\|^{\mu,\mathcal{A}}_\infty := \sup_{\mathcal{A}} \|\mathbf{m}\|_\infty + \mu \|\Delta_{\omega,\omega'} \mathbf{m}\|_\infty,$$

then (4.80) *holds also for the Lipschitz norm* $\|\cdot\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}$

Proof: From (4.71) we have

$$\Pi^{0,\mathcal{K}_{\mathbf{kir}}} H = \sum_{m\in\mathbb{N}^{\mathbb{Z}}_f} H_{m,00} \sum_{j\in\mathbb{Z}} m_j \mathbf{I}^{m-e_j} (I_j - \mathbf{I}_j).$$

If $\Pi^{0,\mathcal{K}_{\mathbf{kir}}} H = H$ then we must have $H_{m,a,b} = 0$ if $(m,a,b) \neq (0,0,0)$, $(e_j,0,0)$, $j \in \mathbb{Z}$, hence, by defining
$$H_{0,0,0} = -\sum_{j\in\mathbb{Z}} H_{e_j,0,0} I_j$$
we get
$$H = \sum_{j\in\mathbb{Z}} H_{e_j,0,0}(I_j - \mathbf{I}_j).$$
Defining $H_{e_j,0,0} =: \mathbf{m}_j$ we get the expression (4.80), moreover, recalling (4.34)
$$\begin{aligned} |H|_{r,\boldsymbol{h},s} &= \sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} |\mathbf{m}_j|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(e_j-e_k)} \\ &= \sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} \delta(j,k)\, |\mathbf{m}_j|\, \mathbf{z}^{(\boldsymbol{h})}(r,s)^{2(e_j-e_k)} = \|\mathbf{m}\|_\infty. \end{aligned}$$
The estimate for the Lischitz norm is analogue. □

Lemma 4.4.3 *if $H \in \mathscr{H}^{\mathcal{A}}_{r,\boldsymbol{h},s}$ then*
$$\left\|\Pi^{-2} H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} \leq \|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}. \tag{4.81}$$
Moreover, if $\kappa \leq \frac{1}{\sqrt{2}}$, we have
$$\left\|\Pi^{0} H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}, \left\|\Pi^{(0,\mathcal{K})} H\right\|^{\mu,\mathcal{A}}_{\infty} \leq 3\,\|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}, \quad \left\|\Pi^{\leq 2} H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} \leq 4\,\|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}, \quad \left\|\Pi^{\geq 2} H\right\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s} \leq 5\,\|H\|^{\mu,\mathcal{A}}_{r,\boldsymbol{h},s}. \tag{4.82}$$

Proof: It follows by inequality (4.72) of Theorem 4.4.3, together with (4.79) and (4.78) □

Finally, we have the following

Lemma 4.4.4 *For any $F, G \in \mathscr{H}_{r,\boldsymbol{h},s}$ such that $\Pi^{\leq 0} G = 0$ we have*
$$\Pi^{-2}\{F,G\} = 0,$$
moreover, if $\Pi^{-2} F = 0$ then
$$\Pi^{0}\{F,G\} = 0.$$
Finally, for any $\mathcal{D}_\omega$ defined in (4.43) we have
$$\Pi^{d}\{F,\mathcal{D}_\omega\} = \{F^{(d)}, D_\omega\}.$$

Proof: It follows directly from the definitions (4.22) and (4.71). □

4.5 KAM Scheme

The aim of this section is to prove Theorem 1.2.9.
Before entering the technical part of the argument, we fix some notation:

Notations:
In what follows, we fix the set $\mathcal{A}$ and the parameter μ in (4.21) to be $\mathtt{D}^{\mathbf{Sym}}_\gamma$ and γ, respectively and we fix
$$\boldsymbol{h}_j := \Big[\ln\left(2+|j|\right)\Big]^\tau, \qquad \tau > 2.$$

Accordingly, we omit the dependence on γ, $\mathtt{D}_\gamma^{\mathbf{Sym}}$ and $\boldsymbol{h}$ from the notation and write $\|\cdot\|_{r,s}$ instead of $\|\cdot\|_{r,\boldsymbol{h},s}^{\gamma,\mathtt{D}_\gamma^{\mathbf{Sym}}}$, $\mathscr{H}_{r,s}$ instead of $\mathscr{H}_{r,\boldsymbol{h},s}^{\mathtt{D}_\gamma^{\mathbf{Sym}}}$ and $\mathfrak{g}_s$ instead of $\mathfrak{g}_{\boldsymbol{h},s}$.

Moreover, we write $\mathcal{R}$, $\mathcal{K}$, $\mathscr{H}_{r,s}^{\mathcal{R}}$, and $\mathscr{H}_{r,s}^{\mathcal{K}}$ in place of $\mathcal{R}_{\mathbf{kir}}$, $\mathcal{K}_{\mathbf{kir}}$, $\mathscr{H}_{r,\boldsymbol{h},s}^{\mathcal{R}_{\mathbf{kir}}}$, and $\mathscr{H}_{r,\boldsymbol{h},s}^{\mathcal{K}_{\mathbf{kir}}}$, as defined in (4.76) and (4.77).

Finally, in view of Lemma 4.4.2, for any $H \in \mathscr{H}_{r,s}^{(0,\mathcal{K})}$ of the form (4.80), we write $H \in \ell^\infty$ and set $\|H\|_\infty := \|\mathbf{m}\|_\infty$.

4.5.1 Twisted-Conjugacy

Following the approach of [24], Theorem (1.2.9) will be proved using an abstract "twisted-conjugacy" argument, in the spirit of Herman:
let us fix r_0, $s_0 > 0$ and let us consider

$$\rho,\, r, \sigma > 0: \quad 0 < \rho < \frac{r_0}{2},\, 0 < \sigma < 1,\, 0 < r < \frac{r_0}{2\sqrt{2}}. \tag{4.83}$$

We will prove

Theorem 4.5.1 *For r_0, s_0, ρ, r, σ as in (4.83). There exists $\bar{\varepsilon}$, $\bar{C} > 0$ such that:*
for $\sqrt{\mathbf{I}} \in B_r(\mathfrak{g}_{s_0+\sigma})$, $N_0 \in \mathscr{N}_{r_0,\sigma_0}$ and $H \in D_\omega + \mathscr{H}_{r_0,s_0}$, if

$$(1+\Theta)^3\varepsilon \le \bar{\varepsilon}, \quad \textit{with} \quad \varepsilon := \gamma^{-1}\|H - N_0\|_{r_0,s_0}, \quad \Theta := \gamma^{-1}\|D_\omega - N_0\|_{r_0,s_0}, \tag{4.84}$$

then there exist:

1. *a symplectic diffeomorphism $\Psi : B_{r_0-\rho}(\mathfrak{g}_{s_0+\sigma}) \to B_{r_0}(\mathfrak{g}_{s_0+\sigma})$, close to the identity;*
2. *a unique counter term $\mathbf{M} = \sum_{j\in\mathbb{Z}} \mathbf{m}_j\,(I_j - \mathbf{I}_j)$, Lipschitz depending on $\omega \in \mathtt{D}_\gamma^{\mathbf{Sym}}$, with*

$$\|\mathbf{m}\|_\infty \le \bar{C}\gamma(1+\Theta)\varepsilon; \tag{4.85}$$

3. *a Hamiltonian $N \in \mathscr{N}_{r_0-\rho,s+\sigma}(\mathbf{I},\omega)$, such that*

$$(\mathbf{M} + H)\circ\Psi = N. \tag{4.86}$$

The advantage of introducing such an abstract result is that the problem of proving the existence of an almost-periodic invariant torus can be split into two separate steps.
First, one derives a normal form result that does not rely on any non-degeneracy assumptions, although it contains the most delicate analytical estimates (Lemma 4.5.1).
Second, one shows that the counter-terms appearing in the normal form can be removed by exploiting suitable internal or external parameters, namely

$$V_j(\omega) = \mathbf{M}_j(\omega) + \omega_j - j^2. \tag{4.87}$$

Proof of Theorem 1.2.9
Recalling (4.83), we fix

$$r_0 := 2\sqrt{2}r,\ \rho := r_0 - 2r,\ \sigma := \min\{s, 2\},\ \sigma_0 := s - \sigma \tag{4.88}$$

and we choose $\mathbf{I} : (\mathbf{I}_j)_{j\in\mathbb{Z}}$ such that $\sqrt{\mathbf{I}} \in B_{r(g_s)}$. For $\omega \in \mathtt{D}^{\mathbf{Sym}}_{\gamma}$ we write (4.6) as

$$H = \sum_{j\in\mathbb{N}} (j + V_j)(I_j + I_{-j}) + F = D_\omega + \mathbf{M} + F',$$

where $\mathbf{M} = (\mathbf{m}_j)$ symmetric sequence such that

$$\mathbf{M} = \sum_{j\in\mathbb{Z}} \mathbf{m}_j (I_j - \mathbf{I}_j), \quad \mathbf{m}_j = j - \omega_j + V_j, \quad \text{and} \quad F' = F + \sum_{j\in\mathbb{Z}} \mathbf{I}_j.$$

We want to apply Theorem 4.5.1 to $H = N_0 + P'$ and $N_0 = D_\omega$, we then fix

$$\varepsilon_* := \frac{\bar{\varepsilon}}{8\|\mathfrak{b}\|^2_\infty C_{\boldsymbol{h}}(s_0)} \tag{4.89}$$

where s_0 is defined in (4.88) and $C_{\boldsymbol{h}}$ is the one in Theorem 4.2.3.
For a sufficient small $\bar{\varepsilon}$, (4.37) together with (1.36) ensure us that condition (4.84) are verified, since $\Theta = 0$. Finally, by choosing $V(\omega)$ such that equation (4.87) is verified we obtain that

$$H \circ \Psi = N.$$

Let us note moreover that $V_j(\omega)$ is Lipschitz in ω, since $\mathbf{M}_j(\omega)$ is so. □

4.5.2 Iterative Lemma

In this Subsection we prove Theorem 4.5.1. The proof is based on a quadratic KAM scheme (Theorem 4.5.1), which iteratively construct approximate solutions Ψ_n of (4.86) by linearizing the problem at each step and solving the associate homological equation 4.106. More precisely we apply the following iterative procedure:
Fix r_0, s_0, ρ, r, σ as in (4.83) and consider the sequences $\{\rho_n\}_{n\in\mathbb{N}}$ and $\{\sigma_n\}_{n\in\mathbb{N}}$ of the form:

$$\rho_n := \frac{\rho}{4} 2^{-n}, \qquad \sigma_0 := \frac{\sigma}{8}, \qquad \sigma_n := \frac{3\sigma}{8 R_\theta n^\theta} \quad n \geq 1,\, \theta > 1 \tag{4.90}$$

where $R_\theta = \sum_{n=1}^\infty n^{-\theta}$.
Let us define then

$$r_{n+1} := r_n - 2\rho_n, \quad s_{n+1} := s_n + 2\sigma_n. \tag{4.91}$$

Since

$$\sum_{n=0}^\infty \rho_n = \frac{\rho}{2}, \quad \sum_{n=0}^\infty \sigma_n = \frac{\sigma}{2},$$

then

$$\begin{aligned} r_{n+1} &= r_0 - 2\sum_{j=1}^n \rho_j = r_0 - \rho(1 - 2^{-(n+1)}) \searrow r_0 - \rho =: r_\infty \\ s_{n+1} &= s_0 + 2\sum_{j=1}^n \sigma_j \nearrow s_0 + \sigma =: s_\infty. \end{aligned}$$

Let us note moreover that, for every $r' \geq r_\infty$, $s' \leq s_\infty$

$$\sqrt{\mathbf{I}} \in B_r(\mathfrak{g}_s) \overset{(4.83)}{\Rightarrow} \sqrt{\mathbf{I}} \in B_{\frac{r_0}{2\sqrt{2}}}(\mathfrak{g}_s) \subset B_{\frac{r_\infty}{\sqrt{2}}}(\mathfrak{g}_s) \subset B_{\frac{r'_\infty}{\sqrt{2}}}(\mathfrak{g}_s)$$

and hence the projectors (4.71) are defined in the whole space $\mathscr{H}_{r',s'}$. Moreover, by Lemma 4.4.3 we have, for every $H \in \mathscr{H}_{r',s'}$ with $r' \geq r_\infty$, $s' \leq s_\infty$

$$\left\| \Pi^{(0,\mathcal{K})} H \right\|_\infty = 3\|H\|_{r',s'}.$$

Let us consider an Hamiltonian of the form

$$H_0 = D_\omega + \mathbf{M}_0 + G_0, \quad G_0 \in \mathscr{H}_{r_0,s_0}, \quad \mathbf{M}_0 \in \ell^\infty \tag{4.92}$$

where $\mathbf{M}_0$ is a counter term as in Theorem 4.5.1 of free parameters and let us define

$$\varepsilon_0 := \gamma^{-1} \left(\left\| G^{(0,\mathcal{K})} \right\|_\infty + \left\| G^{(0,\mathcal{R})} \right\|_{r_0,s_0} + \left\| G^{(-2)} \right\|_{r_0,s_0} \right), \quad \Theta_0 := \gamma^{-1} \left\| G^{\geq 2} \right\|_{r_0,s_0} + \varepsilon_0, \tag{4.93}$$

we have the following:

Lemma 4.5.1 *Let* r_0, s_0, ρ, r, σ *as in* (4.83), r_n, s_n, ρ_n, σ_n *as in* (4.91), (4.90), H_0, G_0, $\mathbf{M}_0$ *as in* (4.92) *and* ε_0, Θ_0 *as in* (4.93). *Finally, let* $\sqrt{\mathbf{I}} \in B_r(\mathfrak{g}_s)$. *There exists a constant* $\mathfrak{C} \geq 1$ *sufficiently large such that, if*

$$\varepsilon_0 \leq (1+\Theta_0)^{-3} K^{-2}, \quad K := \mathfrak{C} \left(\frac{r_0}{\rho} \right)^4 \sup_n e^{\Omega_n} e^{-\chi^n (2-\chi)}, \tag{4.94}$$

where

$$\Omega_n := 4n \ln(2) + 1 + 100 \exp \left[\left(\frac{220 R_\theta n^\theta}{\sigma} \right)^{\frac{1}{\tau-1}} \right] \cdot \left(\frac{220 R_\theta n^\theta}{\sigma} \right)^{\frac{1}{\tau-1}},$$

then we can construct iteratively a sequence of generating functions $S_i = S_i^{(-2)} + S_i^{(0)} \in \mathscr{H}_{r_i - \rho_i, s_{i+1}}$ *with* $S_i^{(-2)} \in \mathscr{H}_{r_i, s_i + \sigma_i}$ *and a sequence of counter terms* $\bar{\Lambda}_j \in \ell^\infty$ *such that the following holds for* $n \geq 0$:

(1) *For all i=0,…,n-1 and any* $s_0 \geq s' \geq s_{i+1}$ *the time-1 Hamiltonian flow* Φ_{S_i} *generated by* S_i *satisfies*

$$\sup_{u \in \bar{B}_{r_{i+1}(\mathfrak{g}_{s'})}} \|\Phi_{S_i}(u) - u\|_{\mathfrak{g}_{s'}} \leq \rho 2^{-2i-7} \tag{4.95}$$

Moreover

$$\Psi_n := \Phi_{S_0} \circ \ldots \circ \Phi_{S_{n-1}} \tag{4.96}$$

is a well defined analytic map $\bar{B}_{r_n(\mathfrak{g}_{s'})} \to \bar{B}_{r_0(\mathfrak{g}_{s'})}$ *for every* $s_0 \geq s' \geq s_n$ *with the bound*

$$\sup_{u \in \bar{B}_{r_n(\mathfrak{g}_{s'})}} |\Psi_n(u) - \Psi_{n-1}(u)| \leq \rho 2^{-2n+2}. \tag{4.97}$$

(2) *We define* $\mathcal{L}_0 := 0$ *and, for* $i = 1, \ldots, n$

$$(\mathrm{Id} + \mathcal{L}_n) := e^{\{\cdot, S_{n-1}\}} (\mathrm{Id} + \mathcal{L}_{n-1}), \quad \mathbf{M}_n := \mathbf{M}_{n-1} - \bar{\mathbf{M}}_{n-1}, \quad H_i := e^{\{\cdot, S_{n-1}\}} H_{i-1}, \tag{4.98}$$

where $\mathbf{M}_{i-1}$ *are free parameters and* $\mathcal{L}_i : \ell^\infty \to \mathcal{H}_{r_i,s_i}$ *are linear operators. We have*

$$H_i = D_\omega + G_i + (\mathrm{Id} + \mathcal{L}_i) \mathbf{M}_i, \quad G_i \in \mathcal{H}_{r_i,s_i}. \tag{4.99}$$

Setting, for for $i = 0, \dots, n$

$$\begin{aligned}\varepsilon_i :&= \gamma^{-1}\left(\left\|G_i^{(0,\mathcal{K})}\right\|_\infty + \left\|G_i^{(0,\mathcal{R})}\right\|_{r_i,s_i} + \left\|G_i^{(-2)}\right\|_{r_i,s_i}\right),\\ \Theta_i :&= \gamma^{-1}\left\|G_i^{\geq 2}\right\|_{r_i,s_i} + \varepsilon_i,\end{aligned} \tag{4.100}$$

we have

$$\varepsilon_i \leq \varepsilon_0 e^{-\chi^i+1}, \quad \chi = \frac{3}{2}, \tag{4.101}$$

$$\Theta_n \leq \Theta_0 \sum_{j=0}^{i} 2^{-j} \tag{4.102}$$

and

$$\begin{aligned}\|(\mathcal{L}_n - \mathcal{L}_{n-1})\, h\|_{r_n,s_n} &\leq K\varepsilon_0(1+\Theta_0)2^{-n}\|h\|_\infty,\\ \|\mathcal{L}_n h\|_{r_n,s_n} &\leq K\varepsilon_0(1+\Theta_0)\sum_{j=1}^{n} 2^{-j}\|h\|_\infty\end{aligned} \tag{4.103}$$

for all $h \in \ell^\infty$. Finally, the counter terms satisfy the estimate

$$\left\|\bar{\mathbf{M}}_{i-1}\right\|_\infty \leq \gamma K\varepsilon_{i-1}(1+\Theta_0), \quad i = 1, \dots, n. \tag{4.104}$$

Proof: Throughout this lemma, constant terms in the Hamiltonians H_i will be disregarded, since they do not contribute to the semi-norm $|\cdot|_{r,s}$. Moreover, we adopt the notation $a \lesssim b$ to indicate that there exists a positive constant c, depending only on the fixed parameter θ, such that $a \leq c\,b$.

First Step:
For $n = 0$ Theorem (4.5.1) is trivial since item (1) is empty and , for what concerns item (2), (4.99) and (4.100) follow from (4.92) and (4.93).

n+1-Step:
We need to find a generating function $S_n = S_n^{(-2)} + S_n^{(0)}$ such that it's time-1 Hamiltonian flow $\Phi_n := e^{\{\cdot,S_n\}}$ in order to cancel out the non-quadratic terms which prevent the torus $\mathcal{T}_I$ to be invariant for the Hamiltonian $H_{n+1} := e^{\{\cdot,S_n\}}H_n$

$$\begin{aligned}e^{\{\cdot,S_n\}}H_n &= D_\omega + e^{\{\cdot,S_n\}}(\mathrm{Id}+\mathcal{L}_n)\mathbf{M}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\\ &+ \{G_n^{\leq 0}, S_n\} + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right)(D_\omega + G_n).\end{aligned}$$

By keeping in mind (4.98) we have that

$$\begin{aligned}e^{\{\cdot,S_n\}}H_n &= D_\omega + (\mathrm{Id}+\mathcal{L}_{n+1})\,\mathbf{M}_{n+1}\\ &+ \bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\\ &+ \mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \{G_n^{\leq 0}, S_n\} + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right)(D_\omega + G_n).\end{aligned}$$

We moreover have that

$$\begin{aligned}
\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\} &= \Pi^{\leq 0}\left(\bar{\mathbf{M}}_n + G_n + \mathcal{D}_\omega + G_n^{\geq 2}, S_n\}\right) \\
&\quad + \Pi^{\geq 2}\left(\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\right) \\
&= \Pi^{\leq 0}\left(\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\right) \\
&\quad + G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} \\
&= \Pi^{\leq 0}\left((\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\right) \\
&\quad + \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n\right) - \Pi^{\leq 0}\left(\mathcal{L}_{n+1}\bar{\mathbf{M}}_n\right) \\
&\quad + G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\}.
\end{aligned}$$

Let us note that $(\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n$ is quadratic in $S_n G_n^{\leq 0}$. Hence we can write

$$\begin{aligned}
e^{\{\cdot, S_n\}} H_n &= D_\omega + (\mathrm{Id} + \mathcal{L}_{n+1})\, \mathbf{M}_{n+1} \\
&+ \Pi^{\leq 0}\left((\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\right) \\
&+ G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} + \Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \{G_n^{\leq 0}, S_n\} \\
&+ \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n\right) + \left(e^{\{\cdot, S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) G_n \\
&+ \left(e^{\{\cdot, S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) D_\omega.
\end{aligned} \tag{4.105}$$

We want to eliminate $\Pi^{\leq 0}\left(\bar{\mathbf{M}}_n + G_n + \{S_n, D_\omega + G_n^{\geq 2}\}\right)$ up to terms in $\mathcal{H}^{(-2,\mathcal{K})}$, in other words we have to impose

$$\Pi^{\leq 0}\left((\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n + \{D_\omega + G_n^{\geq 2}, S_n\}\right) = G_n^{(-2,\mathcal{K})}. \tag{4.106}$$

If we project equation (4.106) in the sub-spaces $\mathcal{H}^{-2}$, $\mathcal{H}^{(0,\mathcal{K})}(= \ell^\infty)$, $\mathcal{H}^{(0,\mathcal{R})}$, keeping in mind that

$$\Pi^{(-2)}\{D_\omega, S_n\} = \Pi^{(-2,\mathcal{R})}\{D_\omega, S_n\} = \{D_\omega, S_n^{(-2)}\} \quad \text{and} \quad \Pi^{(0,\mathcal{K})}\{D_\omega, S_n\} = \{D_\omega, S_n^{(0,\mathcal{K})}\} = 0,$$

we have

$$G_n^{(-2,\mathcal{R})} + \Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\mathbf{M}}_n + \left\{D_\omega, S_n^{(-2)}\right\} = 0; \tag{4.107}$$

$$G_n^{(0,\mathcal{K})} + \left(\mathrm{Id} + \Pi^{0,\mathcal{K}}\mathcal{L}_n\right)\bar{\mathbf{M}}_n + \Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, S_n^{(-2)}\right\} = 0; \tag{4.108}$$

$$G_n^{(0,\mathcal{R})} + \Pi^{0,\mathcal{K}}\mathcal{L}_n\bar{\mathbf{M}}_n + \{D_\omega, S_n^{(0)}\} + \Pi^{(0,\mathcal{R})}\left\{G_n^{\geq 2}, S_n^{(-2)}\right\} = 0; \tag{4.109}$$

Existence of $\mathbf{S_n}$, $\bar{\mathbf{M}}_\mathbf{n}$ and relative bounds
By solving (4.107) we have

$$S_n^{(-2)} = L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})} + \Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\mathbf{M}}_n\right). \tag{4.110}$$

Using (4.110) in (4.108) we get

$$\left(\mathrm{Id} + \Pi^{0,\mathcal{K}}\mathcal{L}_n\right)\bar{\mathbf{M}}_n = -G_0^{(0,\mathcal{K})} - \Pi^{(0,\mathcal{K})}\left\{G_0^{\geq 2}, L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})} + \Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\mathbf{M}}_n\right)\right\}. \tag{4.111}$$

Now, by defining

$$A_n : \ell^\infty \to \ell^\infty, \quad A_n\, h := \Pi^{0,\mathcal{K}}\mathcal{L}_n h + \Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, \Pi^{-2,\mathcal{R}}L_\omega^{-1}\left(\mathcal{L}_n h\right)\right\},$$

equation (4.111) become

$$\left(\mathrm{Id} + A_n\right)\bar{\mathbf{M}}_n = -G_0^{(0,\mathcal{K})} - \Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})}\right)\right\}. \tag{4.112}$$

In order to invert the operator $(\mathrm{Id} + A_n)$ we need to estimate the operatorial norm of A_n. To this purpose we have the following:

Lemma 4.5.2 *For every $h \in \ell^\infty$,*

$$\|A_n\, h\|_\infty \leq \frac{1}{2}\|h\|_\infty.$$

Proof:
From (4.94), choosing suitably $\mathfrak{C}$, we have

$$3\mathrm{K}\varepsilon_0(1 + \Theta_0) \leq \frac{1}{4}, \tag{(4.94)}$$

hence, using (4.82) and (4.103)

$$\|\Pi^{0,\mathcal{K}}\mathcal{L}_n h\|_\infty \leq 3\mathrm{K}\varepsilon_0(1 + \Theta_0)\sum_{j=1}^{n} 2^{-j}\|h\|_\infty \leq \frac{1}{4}\|h\|_\infty,$$

For the second term of A_n we have, let us note that, with a suitable choice of $\mathfrak{C}$, from (4.94) it follows

$$288 e^{C(\sigma_n)}\varepsilon_0\mathrm{K}(1 + \Theta_0)\Theta_0 \leq \frac{1}{2}. \tag{4.113}$$

Using (4.113), together with (4.82), (4.3.2) and (4.23), we have

$$\begin{aligned}
\left\|\Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, \Pi^{-2,\mathcal{R}}L_\omega^{-1}\left(\mathcal{L}_n h\right)\right\}\right\|_\infty &\overset{(4.82)}{\leq} 3\left\|\left\{\Pi^{-2,\mathcal{R}}L_\omega^{-1}\left(\mathcal{L}_n h\right), G_n^{\geq 2}\right\}\right\|_{\frac{r_n}{2}, s_n + \sigma_n} \\
&\overset{(4.23)}{\leq} 24\left\|\Pi^{-2,\mathcal{R}}L_\omega^{-1}\left(\mathcal{L}_n h\right)\right\|_{r_n, s_n + \sigma_n}\left\|G_n^{\geq 2}\right\|_{r_n, s_n + \sigma_n} \\
&\overset{(4.82),(4.100)}{\leq} 24\gamma\Theta_n\left\|L_\omega^{-1}\left(\mathcal{L}_n h\right)\right\|_{r_n, s_n + \sigma_n} \\
&\overset{(4.3.2)}{\leq} 72\Theta_n e^{C(\sigma_n)}\left\|\mathcal{L}_n h\right\|_{r_n, s_n} \\
&\overset{(4.103),(4.102)}{\leq} 144 e^{C(\sigma_n)}\mathrm{K}\varepsilon_0\Theta_0(1 + \Theta_0)\sum_{j=1}^{n} 2^{-j}\|h\|_\infty \\
&\overset{(4.94),(4.113)}{\leq} \frac{1}{2}\|h\|_\infty.
\end{aligned}$$

□

Thanks to Lemma (4.5.2), the operator $(\mathrm{Id}+<M_n)$ is invertible with the bound

$$\left\|(\mathrm{Id}+A_n)^{-1}\ h\right\|_\infty \leq 2\|h\|_\infty. \tag{4.114}$$

Finally, solving (4.112), we have

$$\bar{\mathbf{M}}_n = (\mathrm{Id}+A_n)^{-1}\left[-G_0^{(0,\mathcal{K})} - \Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})}\right)\right\}\right],$$

hence

$$\begin{aligned}
\|\bar{\mathbf{M}}_n\|_\infty &\overset{(4.114)}{\leq} 2\left\|G_0^{(0,\mathcal{K})}\right\|_\infty + 2\left\|\Pi^{(0,\mathcal{K})}\left\{G_n^{\geq 2}, L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})}\right)\right\}\right\|_\infty \\
&\overset{(4.100),(4.82)}{\leq} 2\gamma\varepsilon_n + 6\left\|\left\{G_n^{\geq 2}, L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})}\right)\right\}\right\|_{\frac{r_n}{2}, s_n+\sigma_n} \\
&\overset{(4.23)}{\leq} 2\gamma\varepsilon_n + 48\left\|L_\omega^{-1}\left(G_n^{(-2,\mathcal{R})}\right)\right\|_{r_n, s_n+\sigma_n}\|G_n^{\geq 2}\|_{r_n, s_n+\sigma_n} \\
&\overset{(4.3.2)}{\leq} 2\gamma\varepsilon_n + 144\gamma^{-1}e^{C(\sigma_n)}\left\|G_n^{(-2,\mathcal{R})}\right\|_{r_n, s_n}\|G_n^{\geq 2}\|_{r_n, s_n} \\
&\overset{(4.100)}{\leq} 2\gamma\varepsilon_n + 144\gamma e^{C(\sigma_n)}\varepsilon_n\Theta_n \\
&\overset{(4.102)}{\leq} 288\gamma\varepsilon_n e^{C(\sigma_n)}(1+\Theta_0) \\
&\overset{(4.113)}{\leq} \mathtt{K}(1+\Theta_0)\gamma\varepsilon_n,
\end{aligned} \tag{4.115}$$

where we used that $288\gamma\varepsilon_n e^{C(\sigma_n)} \leq \mathtt{K}$ for a suitable choice of $\mathfrak{C}$ in (4.94). For what concerns the term $S_n^{(-2)}$ defined in (4.110), we have

$$\begin{aligned}
\left\|S_n^{(-2)}\right\|_{r_n, s_n+\sigma_n} &\overset{(4.3.2)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\left(\left\|G_0^{(-2,\mathcal{R})}\right\|_{r_n, s_n} + \|\Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\mathbf{M}}_n\|_{r_n, s_n}\right) \\
&\overset{(4.100)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\left(\gamma\varepsilon_n + \|\Pi^{-2,\mathcal{R}}\mathcal{L}_n\bar{\mathbf{M}}_n\|_{r_n, s_n}\right) \\
&\overset{(4.82),(4.103)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\left(\gamma\varepsilon_n + \mathtt{K}\varepsilon_0(1+\Theta_0)\|\bar{\mathbf{M}}_n\|_\infty\right) \\
&\overset{(4.115)}{\leq} 3\varepsilon_n e^{C(\sigma_n)}\left(1+\varepsilon_0\mathtt{K}^2(1+\Theta_0)^2\right) \\
&\overset{(4.94),1}{\leq} 6\varepsilon_n e^{C(\sigma_n)}.
\end{aligned} \tag{4.116}$$

Finally, solving (4.109) we get an explicit expression for $S_n^{(0)}$:

$$S_n^{(0)} = L_\omega^{-1}\left(-G_n^{(0,\mathcal{R})} - \Pi^{0,\mathcal{K}}\mathcal{L}_n\bar{\mathbf{M}}_n - \Pi^{(0,\mathcal{R})}\left\{G_n^{\geq 2}, S_n^{(-2)}\right\}\right),$$

from which we derive

$$\begin{aligned}
\left\|S_n^{(0)}\right\|_{r_n-\rho_n,s_{n+1}} &\overset{(4.3.2)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\Big(\left\|G_n^{(0,\mathcal{R})}\right\|_{r_n-\rho_n,s_n+\sigma_n}+\left\|\Pi^{0,\mathcal{K}}\mathcal{L}_n\bar{\mathbf{M}}_n\right\|_{r_n-\rho_n,s_n+\sigma_n}\\
&\quad+\left\|\Pi^{(0,\mathcal{R})}\left\{G_n^{\geq 2},S_n^{(-2)}\right\}\right\|_{r_n-\rho_n,s_n+\sigma_n}\Big)\\
&\overset{(4.100),(4.115)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\Big(\gamma\varepsilon_n+3\gamma\varepsilon_n\varepsilon_0\mathrm{K}^2(1+\Theta_0)^2\\
&\quad+24\frac{r_n}{\rho_n}\left\|S_n^{(-2)}\right\|_{r_n,s_n+\sigma_n}\left\|G_n^{\geq 2}\right\|_{r_n,s_n+\sigma_n}\Big)\\
&\overset{(4.100),(4.116)}{\leq} 3\gamma^{-1}e^{C(\sigma_n)}\left(\gamma\varepsilon_n+3\gamma\varepsilon_n\varepsilon_0\mathrm{K}^2(1+\Theta_0)^2+144\frac{r_n}{\rho_n}\gamma\Theta_n\varepsilon_n e^{C(\sigma_n)}\right)\\
&\overset{(4.94),(4.91)}{\lesssim} \frac{r_n}{\rho_n}\varepsilon_n e^{2C(\sigma_n)}\left(1+\Theta_0\right) \qquad (4.117)
\end{aligned}$$

where we used, from (4.91), that

$$1\leq\frac{r_n-\rho_n}{\rho_n}\leq\frac{r_0}{\rho_n}\leq\frac{r_0}{\rho}2^{n+2}\quad\forall n.$$

Putting together (4.117) and (4.116) we obtain

$$\begin{aligned}
\left\|S_n\right\|_{r_n-\rho_n,s_{n+1}} &\leq 338\frac{r}{\rho}\varepsilon_n 2^{n+2}e^{2C(\sigma_n)}\left(1+\Theta_0\right)\\
&\overset{(4.101)}{\leq} 338\frac{r}{\rho}2^{n+2}\varepsilon_0 e^{-\chi^n+1}e^{2C(\sigma_n)}\left(1+\Theta_0\right). \qquad (4.118)
\end{aligned}$$

Remark 4.5.1 *By looking at the definition of σ_n in* (4.90) *and the definition of C of Lemma* (4.3.2) *for the logarithmic weight in* (4.68) *we have that*

$$C(\sigma_n) = 50\exp\left[\left(\frac{220R_\theta n^\theta}{\sigma}\right)^{\frac{1}{\tau-1}}\right]\cdot\left(\frac{220R_\theta n^\theta}{\sigma}\right)^{\frac{1}{\tau-1}}$$

then, the last part of (4.118) *can be rewritten as:*

$$\begin{aligned}
&338\frac{r}{\rho}2^{n+2}\varepsilon_0 e^{-\chi^n+1}e^{2C(\sigma_n)}\left(1+\Theta_0\right)\\
&= 2^{12}338\varepsilon_0\left(1+\Theta_0\right)\frac{r}{\rho}2^{-2n-10}\exp\Big[3n\ln(2)-\chi^n+1+2C(\sigma_n)\Big]\\
&= 2^{12}338\varepsilon_0\left(1+\Theta_0\right)\frac{r}{\rho}2^{-2n-10}e^{\Xi_n}e^{-\chi^n},
\end{aligned}$$

where

$$\Xi_n = 3n\ln(2)+1+100\exp\left[\left(\frac{220R_\theta n^\theta}{\sigma}\right)^{\frac{1}{\tau-1}}\right]\cdot\left(\frac{220R_\theta n^\theta}{\sigma}\right)^{\frac{1}{\tau-1}} \qquad (4.119)$$

and so

$$\left\|S_n\right\|_{r_n-\rho_n,s_{n+1}}\leq 2^{12}338\varepsilon_0\left(1+\Theta_0\right)\frac{r}{\rho}2^{-2n-10}e^{\Xi_n}e^{-\chi^n}. \qquad (4.120)$$

Let us moreover that, for $\tau > 2$, depending on θ we have

$$\sup_n (\Xi_n - \chi^n) < +\infty.$$

(from (4.119) *also follows that* $\sup_n(\Xi_n - \chi^n) \geq 1$*), hence, from* (4.94)*, for a suitable* $\mathfrak{C}$*, it follows*

$$2^{12} 338 \frac{r}{\rho} e^{\Xi_n} e^{-\chi^n} \leq \sqrt{\mathrm{K}} \tag{4.121}$$

Thanks to the last remark we then have

$$\|S_n\|_{r_n-\rho_n, s_{n+1}} \leq 2^{-2n-10} \varepsilon_0 (1+\Theta_0)\sqrt{\mathrm{K}}. \tag{4.122}$$

The maps Φ_{S_n} and Ψ_n:

Let us note that

$$\frac{\rho_n}{(r_n - \rho_n)16e} 2^{-n-1} \geq \frac{\rho_n}{r_0 16e} 2^{-n-1} = \frac{\rho}{r_0 16e} 2^{-2n-2} \geq \frac{\rho}{r_0} 2^{-2n-10}$$

and that, thanks to (4.121) (and hence (4.94)), we have

$$\varepsilon_0 (1+\Theta_0)\sqrt{\mathrm{K}} \leq (1+\Theta_0)^{-2} \mathrm{K}^{-\frac{3}{2}} \leq \frac{\rho}{r},$$

then, using (4.122) we have

$$\|S_n\|_{r_n-\rho_n, s_{n+1}} \leq \frac{\rho}{r_0} 2^{-2n-10} \leq \frac{\rho_n}{(r_n-\rho_n)16e} 2^{-n-1}. \tag{4.123}$$

Finally, applying Proposition (4.2.2) with

$$r \rightsquigarrow r_{n+1} = r_n - 2\rho_n, \qquad \rho \rightsquigarrow \rho_n,$$

we get that the hypotheses (4.26) are satisfied and so the time 1-Hamiltonian flow of S_n is a well defined map

$$\Phi_{S_n} : \bar{B}_{r_{n+1}}(\mathfrak{g}_{s'}) \to \bar{B}_{r_n-\rho_n}(\mathfrak{g}_{s'}) \subset \bar{B}_{r_n} \tag{4.124}$$

for every $s' \geq s_{n+1}$.
Moreover, thanks to (4.27) and (4.123) we have that

$$\begin{aligned} \sup_{u \in \bar{B}_{r_{n+1}}(\mathfrak{g}_{s'})} \|\Phi_{S_n}(u) - u\|_{\mathfrak{g}_{s'}} &\leq (r_n - \rho_n)\|S_n\|_{r_n-\rho_n, s_{n+1}} \\ &\leq \frac{r_n - \rho_n}{r_0} \rho 2^{-2n-10} \leq \rho 2^{-2n-10} \end{aligned}$$

and so (4.95) holds for $i = n$.

Finally (4.28) implies

$$\|H_{n+1}\|_{r_{n+1}, s'} = \|e^{\{\cdot, S_n\}} H_n\|_{r_{n+1}, s'} \leq 2\|H_n\|_{r_n-\rho_n, s'} \quad \forall s' \geq s_{n+1}$$

and so H_{n+1} is well-defined and majorant-analytic in the ball $\bar{B}_{r_{n+1}}(\mathfrak{g}_{s'})$.

For what concerns (4.97), thanks to (4.124), we have that Ψ_{n+1} is well-defined $\bar{B}_{r_{n+1}}(\mathfrak{g}_{s'}) \to \bar{B}_{r_0-\rho_0}(\mathfrak{g}_{s'})$ for every $s_0 \geq s' \geq s_{n+1}$, moreover, by defining

$$u_t := (1-t)u + t\Phi_{S_n}(u),\ t \in [0,1]$$

for $u \in \bar{B}_{r_n-\rho_n}(\mathfrak{g}_{s'})$ we have

$$\Psi_{n+1}(u) - \Psi_n(u) = \Psi_n\left(\Phi_{S_n}(u)\right) - \Psi_n(u) = \int_0^1 \Psi_n(u_t)\left[\Phi_{S_n}(u) - u\right]. \tag{4.125}$$

In order to get (4.97) we define, for the sake of simplicity, the following norms:

$$|\cdot|_{r_n} := \sup_{\bar{B}_{r_n}(\mathfrak{g}_{s'})} \|\cdot\|_{\mathfrak{g}_{s'}}, \qquad |\cdot|_{\mathrm{op},r_n} := \sup_{\bar{B}_{r_n}(\mathfrak{g}_{s'})} \|\cdot\|_{\mathrm{op}}, \qquad \Phi_{S_n} := \Phi_n,$$

where $\mathrm{op} := \mathrm{op}(\mathfrak{g}_{s'} \to \mathfrak{g}_{s'})$.
We then have

$$\begin{aligned} |\Psi_{n+1} - \Psi_n|_{r_{n+1}} &\leq \int_0^1 \sup_{\bar{B}_{r_{n+1}}(\mathfrak{g}_{s'})} \left|\Psi_n'\right|_{\mathrm{op}} dt\, |\Phi_n - \mathrm{Id}|_{r_{n+1}} \\ &\leq \int_0^1 \left|\Psi_n'\right|_{\mathrm{op},r_n-\rho_n} dt\, |\Phi_n - \mathrm{Id}|_{r_{n+1}}. \end{aligned}$$

Now, since

$$\Psi_n'(u) = \Phi_0'\left(\Phi_1 \circ \ldots \circ \Phi_{n-1}(u)\right) \cdot \Phi_1'\left(\Phi_2 \circ \ldots \circ \Phi_{n-1}(u)\right) \cdot \ldots \cdot \Phi_{n-1}'(u)$$

and using (4.124), we get

$$\begin{aligned} \left|\Psi_n'\right|_{\mathrm{op},r_n-\rho_n} &= \sup_{\bar{B}_{r_n-\rho_n}(\mathfrak{g}_{s'})} \left|\Phi_0'\left(\Phi_1 \circ \ldots \circ \Phi_{n-1}(u)\right) \cdot \Phi_1'\left(\Phi_2 \circ \ldots \circ \Phi_{n-1}(u)\right) \cdot \ldots \cdot \Phi_{n-1}'(u)\right|_{\mathrm{op}} \\ &\leq \prod_{j=0}^{n-1} |\Phi_j'|_{\mathrm{op},r_{j+1}-\rho_{j+1}} \\ &\leq \prod_{j=0}^{n-1} \left(\frac{1}{\rho_{j+1}} |\Phi_j - \mathrm{Id}|_{r_{j+1}} + 1\right) \\ &\overset{(4.95)}{\leq} \prod_{j=0}^{n-1} (2^{-j-5} + 1) \\ &\leq 2. \end{aligned}$$

Hence,

$$|\Psi_{n+1}(u) - \Psi_n(u)|_{r_{n+1}} \leq 2|\Phi_{S_n} - \mathrm{Id}|_{r_{n+1}} \overset{(4.95)}{\leq} \rho 2^{-2n-6}.$$

This proves (4.97).
Bound on $\mathcal{L}_{\mathbf{n+1}} - \mathcal{L}_{\mathbf{n}}$:
Since

$$\mathcal{L}_{n+1} - \mathcal{L}_n = \left(e^{\{\cdot,S_n\}} - \mathrm{Id}\right) \circ (\mathcal{L}_n + \mathrm{Id}),$$

using (4.103) we get

$$\|(\mathcal{L}_n + \mathrm{Id})\, h\|_{r_{n+1},s_{n+1}} \leq \|(\mathcal{L}_n + \mathrm{Id})\, h\|_{r_n,s_n} \leq 2\|h\|_\infty. \tag{4.126}$$

Hence, by applying Proposition 4.2.2 with $r \rightsquigarrow r_{n+1}$ and $\rho \rightsquigarrow \rho_n$

$$\begin{aligned}
\|(\mathcal{L}_{n+1} - \mathcal{L}_n)\, h\|_{r_{n+1},s_{n+1}} &\leq \left\| \left(e^{\{\cdot, S_n\}} - \mathrm{Id} \right) \circ (\mathcal{L}_n + \mathrm{Id}) h \right\|_{r_{n+1},s_{n+1}} \\
&\overset{(4.29)}{\leq} \frac{16e(r_n - \rho_n)}{\rho_n} \|S_n\|_{r_n - \rho_n, s_{n+1}} \|(\mathcal{L}_n + \mathrm{Id})h\|_{r_n-\rho_n,s_{n+1}} \\
&\leq 32e\frac{r_0}{\rho_n} \|S_n\|_{r_n-\rho_n,s_{n+1}} \|h\|_\infty \\
&\overset{(4.122)}{\leq} e\frac{r_0}{\rho} 2^{-n-1} \varepsilon_0 (1+\Theta_0)\sqrt{\mathtt{K}} \|h\|_\infty \\
&\leq 2^{-n-1}\varepsilon_0(1+\Theta_0)\mathtt{K}\|h\|_\infty, \qquad (4.127)
\end{aligned}$$

where we used that $e\frac{r_0}{\rho} \leq \sqrt{\mathtt{K}}$.
We have that (4.127) proves the first inequality in (4.103), the second follows from the first.

Bounds on $\mathbf{G^{\leq 0}_{n+1}}$ and $\mathbf{G^{\geq 2}_{n+1}}$:

By (4.105) and (4.106) we have that

$$\begin{aligned}
G_{n+1} &= G_n^{(-2,\mathcal{K})} + G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} + \Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \{G_n^{\leq 0}, S_n\} \\
&\quad + \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n \right) + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{S_n, \cdot\} \right) G_n \\
&\quad + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\} \right) D_\omega \\
&= G_n^{(-2,\mathcal{K})} + G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} + \Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \{G_n^{\leq 0}, S_n\} \\
&\quad + \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n \right) + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\} \right) G_n \\
&\quad + \sum_{h=2}^{\infty} \frac{(\mathrm{ad}S_n)^{h-1}}{h!} \{D_\omega, S_n\}.
\end{aligned}$$

Recalling (4.106) we have

$$\begin{aligned}
\Pi^{\leq 0}\{D_\omega, S_n\} &= \{D_\omega, S_n\} = \\
&= -\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n - G_n^{\leq 0} - \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\} + G_n^{(-2,\mathcal{K})}.
\end{aligned}$$

Since $\mathrm{ad}^h S_n \left(G_n^{(-2,\mathcal{K})} \right) = 0$ for every $h \geq 1$, we can conclude that

$$\begin{aligned}
G_{n+1} &= G_n^{(-2,\mathcal{K})} + G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} + \Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \{G_n^{\leq 0}, S_n\} \\
&\quad + \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n \right) + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\} \right) G_n \\
&\quad - \sum_{h=2}^{\infty} \frac{(\mathrm{ad}S_n)^{h-1}}{h!} \left(\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\} \right).
\end{aligned}$$

Let us write

$$G_{n+1} = G_{n+1}^{\leq 0} + G_{n+1}^{\geq 2},$$

where

$$G_{n+1}^{\leq 0} = \Pi^{\leq 0} G_{n+1,\star} + G_n^{(-2,\mathcal{K})} \tag{4.128}$$
$$G_{n+1}^{\geq 2} = G_n^{\geq 2} + \Pi^{\geq 2}\{G_n^{\geq 2}, S_n\} + \Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n + \Pi^{\geq 2}\left(\left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) G_n\right)$$

and

$$\begin{aligned} G_{n+1,\star} &= \{G_n^{\leq 0}, S_n\} + \Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n\right) + \left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) G_n \\ &- \sum_{h=2}^{\infty} \frac{(\mathrm{ad}S_n)^{h-1}}{h!}\left(\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\}\right) \end{aligned}$$

are the terms in G_{n+1} that are quadratic in $\varepsilon_n \sim S_n \sim G_n^{\leq 0}$.
Let us start by giving a bound on $G_{n+1,\star}$:

$$\begin{aligned} \|G_{n+1,\star}\|_{r_{n+1},s_{n+1}} &\leq \left\|\{G_n^{\leq 0}, S_n\}\right\|_{r_{n+1},s_{n+1}} + \left\|\Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n\right)\right\|_{r_{n+1},s_{n+1}} \\ &+ \left\|\left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) G_n\right\|_{r_{n+1},s_{n+1}} \\ &+ \left\|\sum_{h=2}^{\infty} \frac{(\mathrm{ad}S_n)^{h-1}}{h!}\left(\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\}\right)\right\|_{r_{n+1},s_{n+1}} \end{aligned}$$

$$\begin{aligned} \left\|\{G_n^{\leq 0}, S_n\}\right\|_{r_{n+1},s_{n+1}} &\overset{(4.23)}{\leq} 8\max\left\{1, \frac{r_{n+1}}{\rho_n}\right\}\|S_n\|_{r_n-\rho,s_{n+1}}\left\|G_n^{\leq 0}\right\|_{r_n-\rho,s_{n+1}} \\ &\leq 8\frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho,s_{n+1}}\left\|G_n^{\leq 0}\right\|_{r_n-\rho,s_{n+1}} \end{aligned} \tag{4.129}$$

$$\begin{aligned} \left\|\Pi^{\leq 0}\left((\mathcal{L}_{n+1} - \mathcal{L}_n)\bar{\mathbf{M}}_n\right)\right\|_{r_{n+1},s_{n+1}} &\overset{(4.82)}{\leq} 4\left\|(\mathcal{L}_{n+1} - \mathcal{L}_n)\,\bar{\mathbf{M}}_n\right\|_{r_{n+1},s_{n+1}} \\ &\overset{(4.98),(4.125)}{\leq} 128e\frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho_n,s_{n+1}}\left\|\bar{\mathbf{M}}_n\right\|_{\infty} \end{aligned} \tag{4.130}$$

$$\begin{aligned} \left\|\left(e^{\{\cdot,S_n\}} - \mathrm{Id} - \{\cdot, S_n\}\right) G_n\right\|_{r_{n+1},s_{n+1}} &\overset{(4.30)}{\leq} 128e^2\left(\frac{r_n - \rho_n}{\rho_n}\right)^2\left(\|S_n\|_{r_n-\rho_n,s_{n+1}}\right)^2\|G_n\|_{r_n-\rho_n,s_{n+1}} \qquad (4.131) \\ &\qquad (4.132) \\ &\leq 128e^2\left(\frac{r_0}{\rho_n}\right)^2\left(\|S_n\|_{r_n-\rho_n,s_{n+1}}\right)^2\|G_n\|_{r_n-\rho_n,s_{n+1}} \end{aligned}$$

$$\begin{aligned} &\left\|\sum_{h=2}^{\infty} \frac{(\mathrm{ad}S_n)^{h-1}}{h!}\left(\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{S_n, G_n^{\geq 2}\}\right)\right\|_{r_{n+1},s_{n+1}} \\ &\overset{(4.31)}{\leq} \frac{16e(r_n - \rho_n)}{\rho_n}\|S_n\|_{r_n-\rho_n,s_{n+1}}\left\|\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\}\right\|_{r_n-\rho_n,s_{n+1}} \\ &\leq 16e\frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho_n,s_{n+1}}\left\|\Pi^{\leq 0}(\mathrm{Id} + \mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\}\right\|_{r_n-\rho_n,s_{n+1}} \end{aligned}$$

By noting that $\Pi^{\leq 0}\{G_n^{\geq 2}, S_n\} = \{G_n^{\geq 2}, S_n^{(-2)}\}$, we have

$$\begin{aligned}
& \left\|\Pi^{\leq 0}(\mathrm{Id}+\mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \{G_n^{\geq 2}, S_n^{(-2)}\}\right\|_{r_n-\rho_n, s_{n+1}} \\
\leq\ & \left\|\Pi^{\leq 0}(\mathrm{Id}+\mathcal{L}_n)\bar{\mathbf{M}}_n\right\|_{r_n-\rho_n, s_{n+1}} + \left\|G_n^{\leq 0}\right\|_{r_n-\rho_n, s_{n+1}} + \left\|\{G_n^{\geq 2}, S_n^{(-2)}\}\right\|_{r_n-\rho_n, s_{n+1}} \\
\overset{(4.82),(4.126)}{\leq}\ & 8\left\|\bar{\mathbf{M}}_n\right\|_\infty + \left\|G_n^{\leq 0}\right\|_{r_n-\rho_n, s_{n+1}} + \left\|\{G_n^{\geq 2}, S_n^{(-2)}\}\right\|_{r_n-\rho_n, s_{n+1}} \\
\overset{(4.23)}{\leq}\ & 8\left\|\bar{\mathbf{M}}_n\right\|_\infty + \left\|G_n^{\leq 0}\right\|_{r_n-\rho_n, s_{n+1}} + 8\frac{r_0}{\rho_n}\left\|S_n^{(-2)}\right\|_{r_n, s_{n+1}}\left\|G_n^{\geq 2}\right\|_{r_n, s_{n+1}}.
\end{aligned}$$

Then

$$\begin{aligned}
& \left\|\sum_{h=2}^{\infty}\frac{(\mathrm{ad}S_n)^{h-1}}{h!}\Big(\Pi^{\leq 0}(\mathrm{Id}+\mathcal{L}_n)\bar{\mathbf{M}}_n + G_n^{\leq 0} + \Pi^{\leq 0}\{G_n^{\geq 2}, S_n\}\Big)\right\|_{r_{n+1}, s_{n+1}} \qquad (4.133)\\
\leq\ & 128e\frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho_n, s_{n+1}}\left(\left\|\bar{\mathbf{M}}_n\right\|_\infty + \left\|G_n^{\leq 0}\right\|_{r_n-\rho_n, s_{n+1}} + \frac{r_0}{\rho_n}\left\|S_n^{(-2)}\right\|_{r_n, s_{n+1}}\left\|G_n^{\geq 2}\right\|_{r_n, s_{n+1}}\right).
\end{aligned}$$

Putting together (4.129), (4.130), (4.131) and (4.133) we get

$$\begin{aligned}
\|G_{n+1,\star}\|_{r_{n+1}, s_{n+1}} \leq\ & 128e^2\frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho, s_{n+1}}\Big(2\left\|G_n^{\leq 0}\right\|_{r_n-\rho, s_{n+1}} + 2\|\bar{\mathbf{M}}_n\|_\infty \qquad (4.134)\\
+\ & \frac{r_0}{\rho_n}\|S_n\|_{r_n-\rho_n, s_{n+1}}\|G_n\|_{r_n-\rho_n, s_{n+1}} + \frac{r_0}{\rho_n}\left\|S_n^{(-2)}\right\|_{r_n, s_{n+1}}\left\|G_n^{\geq 2}\right\|_{r_n, s_{n+1}}\Big).
\end{aligned}$$

Then, from (4.120), (4.100), (4.115) and (4.134) it follows that

$$\|G_{n+1,\star}\|_{r_{n+1}, s_{n+1}} \leq \gamma\left(\frac{r_0}{\rho}\right)^3 \mathtt{K}(1+\Theta_0)^2\varepsilon_0\varepsilon_n 2^{-n}e^{\Xi_n}e^{-\chi^n}.$$

Finally, using (4.94), (4.82), (4.128), (4.101) we get

$$\begin{aligned}
\varepsilon_{n+1} \overset{(4.100)}{=}\ & \gamma^{-1}(\|G_{n+1}^{(-2)}\|_{r_{n+1}, s_{n+1}} + \|G_{n+1}^{(0,\mathcal{K})}\|_{r_{n+1}, s_{n+1}} + \|G_{n+1}^{(0,\mathcal{R})}\|_{r_{n+1}, s_{n+1}}) \\
\lesssim\ & \left(\frac{r_0}{\rho}\right)^3 \mathtt{K}(1+\Theta_0)^2\varepsilon_0\varepsilon_n 2^{-n}e^{\Xi_n}e^{-\chi^n} \\
\overset{(4.101)}{\lesssim}\ & \left(\frac{r_0}{\rho}\right)^3 \mathtt{K}(1+\Theta_0)^2\varepsilon_0^2 e^{-\chi^{n}+1}2^{-n}e^{\Xi_n}e^{-\chi^n} \\
=\ & \mathtt{K}(1+\Theta_0)^2\varepsilon_0^2 e^{-\chi^{n+1}+1}\left[2^{-n}e^{\Xi_n}e^{-\chi^n(2-\chi)}\right] \\
\overset{(4.94)}{\lesssim}\ & \mathtt{K}^2(1+\Theta_0)^2\varepsilon_0^2 e^{-\chi^{n+1}+1} \\
\overset{(4.94)}{\leq}\ & \varepsilon_0 e^{-\chi^{n+1}+1},
\end{aligned}$$

that is (4.101).

For what concerns (4.102), we have

$$\begin{aligned}
\Theta_{n+1}-\Theta_n &\overset{(4.100)}{=} \gamma^{-1}\left\|G^{\geq 2}_{n+1}\right\|_{r_{n+1},s_{n+1}}-\gamma^{-1}\left\|G^{\geq 2}_{n}\right\|_{r_n,s_n}+\varepsilon_{n+1}-\varepsilon_n\\
&\leq \gamma^{-1}\left(\left\|G^{\geq 2}_{n+1}\right\|_{r_{n+1},s_{n+1}}-\left\|G^{\geq 2}_{n}\right\|_{r_n,s_n}\right)+\varepsilon_{n+1}-\varepsilon_n\\
&\overset{(4.128)}{\leq} \gamma^{-1}\left\|\Pi^{\geq 2}\{G^{\geq 2}_n,S_n\}\right\|_{r_{n+1},s_{n+1}}+\gamma-1\left\|\Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n\right\|_{r_{n+1},s_{n+1}}\\
&\quad+\gamma^{-1}\left\|\Pi^{\geq 2}\left(\left(e^{\{\cdot,S_n\}}-\mathrm{Id}-\{\cdot,S_n\}\right)G_n\right)\right\|_{r_{n+1},s_{n+1}}+\varepsilon_{n+1}.
\end{aligned}$$

By taking the constant $\mathfrak{C}$ in (4.94) sufficiently large and carrying out computations analogous to those above, we prove

$$\begin{aligned}
&\left\|\Pi^{\geq 2}\{G^{\geq 2}_n,S_n\}\right\|_{r_{n+1},s_{n+1}}+\left\|\Pi^{\geq 2}\mathcal{L}_{n+1}\bar{\mathbf{M}}_n\right\|_{r_{n+1},s_{n+1}}\\
&+\left\|\Pi^{\geq 2}\left(\left(e^{\{\cdot,S_n\}}-\mathrm{Id}-\{\cdot,S_n\}\right)G_n\right)\right\|_{r_{n+1},s_{n+1}}+\varepsilon_{n+1}\\
\leq\ &\gamma\Theta_0 2^{-n}.
\end{aligned}$$

From this, it follows

$$\Theta_{n+1}=\Theta_n+\Theta_0 2^{-n}\overset{(4.102)}{\leq}\Theta_0\sum_{j=0}^{n+1}2^{-j},$$

that corresponds to (4.102) for the case $n+1$.

□

From lemma 4.5.1 it follows:

Corollary 4.5.1 *The map* $\Psi:=\lim_{n\to\infty}\Psi_n$*, is well-defined from* $B_{r_\infty}(\mathfrak{g}_{s_\infty})$ *to* $B_{r_0 ty}(\mathfrak{g}_{s_\infty})$*. Moreover the sequence* $\bar{\mathbf{M}}_n$ *is summable and the sequence* $\mathcal{L}_n$ *converges to an operator* $\mathcal{L}:\ell^\infty\to\mathscr{H}_{r_\infty,s_\infty}$*.*

Proof: It follows directly from (4.97), (4.103), and (4.104). □

Now we can prove Theorem 4.5.1.

Proof of Theorem 4.5.1
Recalling $\mathtt{K}$ defined in (4.94), we fix

$$\bar{\varepsilon}:=2^{-15}\mathtt{K},\quad \bar{C}:=2^7\mathtt{K}. \tag{4.135}$$

For $H\in D_\omega+\mathscr{H}_{r_0,s_0}$ of the form $H=D_\omega+G_0$, let us consider the Hamiltonian

$$H_0:=H+\mathbf{M}_0=D_\omega+G_0+\mathbf{M}_0, \tag{4.136}$$

with $\mathbf{m}_0\in\ell^{infty}$. Finally, for $N_0\in\mathscr{N}_{r_0,s_0}$ let us define

$$G:=H-N_0=G_0-(N_0-D_\omega).$$

By (4.84) we have

$$\|G\|_{r_0,s_0}=\gamma\varepsilon, \tag{4.137}$$

moreover, since $N_\omega \in D_\omega + \mathscr{H}^{\geq 2}_{r_0,s_0}$, we have

$$G_0^{\leq 0} = \Pi^{\leq 0}(N_0 - D_\omega + G) = G^{\leq 0}.$$

Then, using (4.81), we get

$$\left\|G_0^{(-2)}\right\|_{r_0,s_0} \leq \|G\|_{r_0,s_0}, \quad \left\|G_0^{(0,\mathcal{K})}\right\|_\infty, \left\|G_0^{(0,\mathcal{R})}\right\|_{r_0,s_0} \leq 3\|G\|_{r_0,s_0},$$

that, together with (4.137), yields to

$$\varepsilon_0 := \left(\left\|G_0^{(-2)}\right\|_{r_0,s_0} + \left\|G_0^{(0,\mathcal{K})}\right\|_\infty + \left\|G_0^{(0,\mathcal{R})}\right\|_{r_0,s_0}\right) \leq 7\varepsilon.$$

Finally, since

$$G_0^{\geq 2} = (N_0 - D_\omega)^{\geq 2} + G^{\geq 2},$$

we also have that

$$\Theta_0 := \gamma^{-1}\left\|G_0^{\geq 2}\right\|_{r_0,s_0} + \varepsilon_0 \overset{(4.4.3)}{\leq} 4\gamma^{-1}(\|N_0 - D_\omega\|_{r_0,s_0} + \|G\|_{r_0,s_0}) + \varepsilon_0 \overset{(4.137),(4.84)}{\leq} 4\Theta + 11\varepsilon.$$

Using $\bar{\varepsilon}$, $\bar{C}$ as in (4.135) we have that

$$\varepsilon_0(1+\Theta_0)^2\mathtt{K}^2 \overset{(4.84),(4.135)}{\leq} 1.$$

The hypotheses (4.94) of Lemma 4.5.1 are satisfied. Hence, from Lemma 4.5.1 together with Corollary 4.5.1 we have that

$$H_0 \circ \Phi = D_\omega + G_\infty + (\mathrm{Id} + \mathcal{L})\left(\mathbf{M}_0 - \sum_{i=0}^{\infty} \bar{\mathbf{M}}_i\right).$$

where $\Psi = \lim_{n\in\mathbb{N}} \Psi_n$ and Ψ_n is defined in (4.96).
Now, setting $\mathbf{M} := \mathbf{M}_0 = \sum_{i=0}^{\infty} \bar{\mathbf{M}}_i$ and recalling (4.136), we have

$$(H + \mathbf{M}) \circ \Psi = D_\omega + G_\infty =: N.$$

Using (4.101) together with (4.100) we have that $G_\infty^{\leq 0} = 0$. Hence $N \in \mathscr{N}_{r_0,s_0}$.
Finally inequality (4.85) come from the choice of $\bar{C}$ in (4.135) together with (4.104). □

4.6 Lower Regularity: Non-Maximal Tori

The purpose of this section is to prove Theorem 1.2.10. Following the strategy introduced in [25], we fix a subset of *tangential sites* in $\mathbb{Z}$ and restrict equation (4.1) to the space of functions whose Fourier support is contained in this set.
The key idea is that a suitable choice of the tangential sites makes it possible to impose very strong Diophantine conditions, while at the same time allowing for lower regularity of the solutions.
As a consequence, however, the resulting solutions are no longer supported on maximal tori.
Similar to the previous Chapter we consider the space $\mathscr{H}^{\gamma,\mathtt{D}_\gamma^{\mathbf{Sym}}}_{r,\boldsymbol{h},s}$ with

$$\boldsymbol{h}_j = \ln(2+|j|)^\tau, \quad \tau > 2. \tag{4.138}$$

We then omit the dependence on γ, $\mathtt{D}_\gamma^{\mathbf{Sym}}$ and $\boldsymbol{h}$ from the notation and write $\|\cdot\|_{r,s}$ in place of $\|\cdot\|^{\gamma,\mathtt{D}_\gamma^{\mathbf{Sym}}}_{r,\boldsymbol{h},s}$, $\mathscr{H}_{r,s}$ instead of $\mathscr{H}^{\mathtt{D}_\gamma^{\mathbf{Sym}}}_{r,\boldsymbol{h},s}$ and $\mathfrak{g}_s$ instead of $\mathfrak{g}_{\boldsymbol{h},s}$.

4.6.1 Invariant Subsets

Let us consider the Hamiltonian

$$H = \sum_{j\in\mathbb{N}} (j + V_j)(I_j + I_{-j}) + \frac{1}{2}\mathbf{F}\left(H_{\mathfrak{B},2} + P_{\mathfrak{B}}\right), \tag{4.139}$$

associated with equation (4.1), where $H_{\mathfrak{B},2}$, $P_{\mathfrak{B}}$ are defined in (4.5).
Let $\mathcal{S} \subset \mathbb{N}$ be an arbitrary subset of $\mathbb{N}$ and consider the subspace

$$\mathfrak{g}_s^{\mathcal{S}} := \left\{u = (u_j)_{j\in\mathbb{Z}} \in \mathfrak{g}_s \ : \ u_k = 0 \text{ for all } |k| \notin \mathcal{S}\right\}.$$

Arguing in complete analogy with Lemma 1.2.2 we obtain the following result for the elements of $\mathscr{H}_{r,s}$:

Lemma 4.6.1 *Let $H \in \mathscr{H}_{r,s}$, for any set $\mathcal{S} \subset \mathbb{N}$, the subspace $\mathfrak{g}_s^{\mathcal{S}}$ is invariant under the flow of H.*

We refer to $\mathcal{S}$ as the set of tangential sites.

In the light of Lemma 4.6.1 we define the subspace

$$\mathscr{H}_{r,s}^{\mathcal{S}} := \{H \in \mathscr{H}_{r,s} : H_{|\mathcal{S}} = H\}$$

of the Hamiltonians in $\mathscr{H}_{r,s}$ depending only on the variables $\mathfrak{g}_s^{\mathcal{S}}$.

In particular, for H in (4.139), we have

$$H_{|\mathcal{S}} = \sum_{j\in\mathcal{S}} (j + V_j)(I_j + I_{-j}) + \frac{1}{2}\mathbf{F}\left(H_{\mathfrak{B},2}^{\mathcal{S}} + P_{\mathfrak{B}}^{\mathcal{S}}\right),$$

where

$$P_{\mathfrak{B}}^{\mathcal{S}} := \sum_{j\in\mathcal{S}} \mathfrak{b}_j(W_j + \bar{W}_j), \qquad H_{\mathfrak{B},2}^{\mathcal{S}} = \sum_{j\in\mathcal{S}} \mathfrak{b}_j(I_j + I_{-j}).$$

We have the following

Lemma 4.6.2 *The Poisson brackets defined in* (4.22)*, the operator L_ω^{-1} defined in* (4.46) *and the projectors defined in* (4.71)*, are well-posed when restricted to $\mathcal{H}_{r,s}^{\mathcal{S}}$.*

Proof: It follows from the definitions (4.46), (4.71) and from the relations (3.14). □

In particular, every result stated above for the space $\mathcal{H}_{r,s}$ holds when restricted to elements of $\mathcal{H}_{r,s}^{\mathcal{S}}$.

Remark 4.6.1 *Lemma 4.6.1 implies that, for any subset $\mathcal{S} \subset \mathbb{N}$ and for any $\sqrt{\mathbf{I}} \in B_r()\mathfrak{g}_s^{\mathcal{S}})$ satisfying the assumptions of Theorem 1.2.9, there exist a potential*

$$V = (V_{\mathcal{S}}, V_{\mathcal{S}^c}),$$

and a symplectic change of variables Φ such that the torus $\mathcal{T}_{\mathbf{I}}$ is a KAM torus for the Hamiltonian $H \circ \Phi$. Here

$$V_{\mathcal{S}} := (V_j)_{|j|\in\mathcal{S}}, \qquad V_{\mathcal{S}^c} := (V_j)_{|j|\notin\mathcal{S}}.$$

Moreover, the compatibility equation (4.87) *depends only on $V_{\mathcal{S}}$, and not on the full potential V.*

4.6.2 Lower Regularity

let us consider the space

$$\mathtt{w}_p := \left\{ u = (u_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) : |u|_p := \sum_{j\in\mathbb{Z}} |u_j| \langle j\rangle^p \leq \infty \right\}$$

where $\langle j\rangle = \max\{1, |j|\}$.
Similarly to the previous subsection, for a subset $\mathcal{S} \subset \mathbb{N}$ we define

$$\mathtt{w}_s^{\mathcal{S}} := \left\{ u = (u_j)_{j\in\mathbb{Z}} \in \mathtt{w}_s \ : \ u_j = 0 \text{ for all } |j| \notin \mathcal{S} \right\}.$$

Similarly to Lemma 4.6.1, also $\mathtt{w}_p^{\mathcal{S}}$ is invariant for the dynamics of H in (4.139).
For $\boldsymbol{h}$ defined as in (4.138) let us consider the function

$$s : \mathbb{N} \to \mathcal{S}, \quad s(i) = \lfloor e^{\boldsymbol{h}_i} \rfloor \tag{4.140}$$

where $\lfloor\cdot\rfloor$ is the floor function. Let us consider moreover its inverse $i(s)$ an the associate set

$$\mathcal{S} = \{s(i), \quad i \in \mathbb{N}\}. \tag{4.141}$$

Finally, by defining

$$\begin{cases} v_i = u_{s(i)} \\ v_{-i} = u_{-s(i)} \end{cases} \tag{4.142}$$

we have the following identification

$$\mathtt{w}_p^{\mathcal{S}} \backsimeq \{v \in \ell^\infty : \sup_{i\in\mathbb{Z}} s^p(|i|)|v_i| \leq \infty\}.$$

Let us note moreover that the right-hand space is equivalent to the space $\mathfrak{g}_p$.

Let us write the hypercube $\mathcal{Q}^{\mathbf{Sym}} = \mathcal{Q}_{\mathcal{S}}^{\mathbf{Sym}} \times \mathcal{Q}_{\mathcal{S}^c}^{\mathbf{Sym}}$, where

$$\mathcal{Q}_{\mathcal{S}}^{\mathbf{Sym}} = \{\omega \in \mathcal{Q}^{\mathbf{Sym}} : \ \omega_j = 0 \text{ if } |j| \notin \mathcal{S}\}$$

and let us consider the following Diophantine conditions:

$$\mathtt{D}_{\gamma,\mathcal{S}} := \left\{ \omega \in \mathcal{Q}_{\mathcal{S}}^{\mathbf{Sym}} \ : \ |\omega\cdot\ell| > \gamma \prod_{s\in\mathcal{S}} \frac{1}{(1+|\ell_s|^2\langle i(s)\rangle^2)^2}, \, \forall \ell \in \mathbb{Z}_f^{\mathbb{N}}, \, \mathrm{supp}(\ell) \subset \mathcal{S} \right\}. \tag{4.143}$$

The aim of this Subsection is to prove that for every choice of $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$ and for any $\sqrt{\mathbf{I}} \in B_r(\mathtt{w}_p^{\mathcal{S}})$, with r satisfying similar smallness assumptions of Theorem 1.2.9, there exists a symplectic map Φ and a potential V, such that $\mathcal{T}_{\mathbf{I}}$ is a KAM torus for $H \circ \Phi$.

In the variables v defined in (4.142), the Hamiltonian (4.139) restricted to $\mathcal{S}$, has the form

$$H_{\mathcal{S}} := H_{|\mathcal{S}} = \sum_{i\in\mathbb{N}} (s(i) + V_{s(i)})(\boldsymbol{\mathcal{I}}_i + \boldsymbol{\mathcal{I}}_{-i}) + P, \tag{4.144}$$

where

$$P = \frac{1}{2}\mathbf{F}\left(\sum_{i\in\mathbb{N}} \mathfrak{b}_{s(i)}(\mathcal{I}_i + \mathcal{I}_{-i} + \mathcal{W}_i + \bar{\mathcal{W}}_i)\right)$$

and $\mathcal{I}_j = |v_j|^2$ and $\mathcal{W}_j = v_j v_{-j}$.
Let us observe that (4.144) has the same form of (4.139), with $V_{\mathcal{S}} = (V_{s(i)})_{i\in\mathbb{N}}$ in place of $V = (V_i)_{i\in\mathbb{N}}$, with $\{\mathfrak{b}_{s(i)}\}_{i\in\mathbb{N}}$ instead of $\{\mathfrak{b}_i\}_{i\in\mathbb{N}}$. The only structural difference is that the quadratic part $\sum_{i\in\mathbb{N}} i(I_i + I_{-i})$ is replaced with

$$\sum_{i\in\mathbb{N}} s(i)(\mathcal{I}_i + \mathcal{I}_{-i}). \tag{4.145}$$

It is trivial that if P inherits the regularity of $\mathbf{F}$, in particular, Theorem 4.2.3 holds and we have that $P \in \mathscr{H}_{r,p}$ for a suitable r.
From this it follows that all the results stated previously in this Chapter can be directly applied to $H_{\mathcal{S}}$ with the only exception of Lemma 4.3.2for which the situation is more delicate.

If we fix the hypercube

$$Q_{\mathcal{S}}^{\mathbf{Sym}} := \left\{\omega = (\omega_i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \quad \sup_i |\omega_i - |s(i)|| \le \frac{1}{2},\, \omega_i = \omega_{-i}\right\}$$

then condition (4.143), in the variables v can be written as

$$\mathtt{D}_{\gamma,\mathcal{S}} = \left\{\omega \in \mathcal{Q}_{\mathcal{S}}^{\mathbf{Sym}} : |\omega \cdot \ell| > \gamma \prod_{i\in\mathbb{N}} \frac{1}{(1+|\ell_i|^2|i|^2)^2},\, \forall \ell \in \mathbb{Z}_f^{\mathbb{N}}\right\},$$

that are essentially the same as (4.45) but in the hypercube $Q_{\mathcal{S}}^{\mathbf{Sym}}$ instead of $\mathcal{Q}^{\mathbf{Sym}}$.
Let us observe that one can easily adapt the proof of Lemma 4.3.3 to our case by substituting the condition

$$\left|\sum_{i\in\mathbb{N}} (a_i - b_i)|s(i)|\right| \le 10 \sum_{i\in\mathbb{N}} |a_i - b_i|,$$

in place of (4.50).
By doing so we get (4.51) also in our. Finally, arguing as in the proof of Lemma 4.3.2 we get same bound

$$\|L_\omega^{-1} F\|_{r,p+\sigma} \le 3\gamma^{-1} e^{C(\sigma)} \|F\|_{r,p}$$

for every $\omega \in \mathtt{D}_{\gamma,\mathcal{S}}$, where $C(\sigma)$ is defined in (4.68).
In fact, the superlinear growth of the eigenvalues of the quadratic part (4.145) simplifies the proofs of Lemmas 4.3.3 and 4.3.2, and could be exploited to obtain a better constant $C(\sigma)$ than the one derived here. However, this refinement is not needed for our purposes.
We can conclude that Theorem 1.2.9 holds also for (4.144).

Proof of Theorem 1.2.10:
For $r > 0$ satisfying the hypotheses of Theorem 1.2.9 le us fix $\sqrt{\mathbf{I}_{\mathcal{S}}} \in B_r(\mathtt{w}_p^{\mathcal{S}})$.
If we denote by $\boldsymbol{\mathcal{I}}_{\mathcal{S}}$ the sequence corresponding to $\mathbf{I}_{\mathcal{S}}$ in the variables (4.142), we have $\sqrt{\boldsymbol{\mathcal{I}}_{\mathcal{S}}} \in B_r(\mathfrak{g}_p)$.
It follows that a direct application of Theorem 1.2.9 on $H_{\mathcal{S}}$ allows to choose a potential $V_{\mathcal{S}}$ and to construct a symplectic map $\Phi_{\mathcal{S}} : \bar{B}_{2r}(\mathfrak{g}_p) \to \bar{B}_{4r}(\mathfrak{g}_p)$ such that

$$\mathcal{T}_{\boldsymbol{\mathcal{I}}_{\mathcal{S}}} = \{v = (v_j)_{j\in\mathbb{N}} : |v_j|^2 = \boldsymbol{\mathcal{I}}_j\}$$

is a KAM torus for $H_{\mathcal{S}} \circ \Phi_{\mathcal{S}}$.
By extending the map $\Phi_{\mathcal{S}}$ to a map $B_{2r}(\mathtt{w}_p^{\mathcal{S}}) \to B_{4r}(\mathtt{w}_p)$ and defining $\Phi : B_{2r}(\mathtt{w}_p) \to B_{4r}(\mathtt{w}_p)$ such that

$$\{\Phi(u)\}_j = \begin{cases} \{\Phi_{\mathcal{S}}(u)\}_j & \text{if } |j| \in \mathcal{S} \\ u_j & \text{if } |j| \notin \mathcal{S}, \end{cases}$$

we have. in conclusion, that $\mathcal{T}_{\mathbf{I}_{\mathcal{S}}}$ is a KAM torus for $H \circ \Phi$. □

Appendix A

A.1 Existence Results

In this Section we collect some of the main results concerning the existence of solutions to equation

$$\partial_{tt}u - \left(\frac{1}{1 + c\int_{\mathbb{T}^n} |\nabla u|^2\, dx}\right)^2 \Delta u = 0, \quad c \in \mathbb{R}. \tag{A.1}$$

We begin with a local well-posedness theorem that has been proved, under different assumptions, by various authors. Although these results concern more general forms of Kirchhoff-type equations, we state them here in the specific case of (A.1).

Theorem A.1.1 (Local Well-Posedness, [1], [48], [81])
Let us fix $s \geq 3/2$. For any pair of initial data $(u(0), u_t(0)) \in H^s(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}(\mathbb{T}^n, \mathbb{R})$ equation (A.1) *admits a unique solution*

$$u \in C^0([0, T], H^s(\mathbb{T}^n, \mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{T}^n, \mathbb{R})),$$

with $T \sim \left(\|u(0)\|_{3/2} + \|v(0)\|_{1/2}\right)^{-2}$.

Equation (A.1) in one of the few example of Kirchhoff-type equation for wich we can obtain global existence of solutions. This strong result can be achieved thanks to the work of S. I. Pokhozhaev, which proved in [82] and [80], that equation (A.1) admits a second-order conservation law:

Theorem A.1.2 (Second Order Conservation Law, [80])
Let us define

$$a(u) := \left|1 + c\int_{\mathbb{T}^n} |\nabla u|^2\right|, \tag{A.2}$$

the quantity

$$K(u) := a(u)\int_{\mathbb{T}^n} |\nabla u_t|^2\, dx + \frac{1}{a(u)}\int_{\mathbb{T}^n} (\Delta u)^2 dx - c\left(\int_{\mathbb{T}^n} \nabla u_t \cdot \nabla u\, dx\right)^2 \tag{A.3}$$

is conserved for the solutions of (A.1)*, namely, if*

$$u \in C^0([0, T], H^2(\mathbb{T}^n, \mathbb{R})) \cap C^1([0, T], H^1(\mathbb{T}^n, \mathbb{R}))$$

is a solution, then

$$\frac{dK(u(t))}{dt} = 0, \quad \forall t \in [0, T]. \tag{A.4}$$

Remark A.1.1 (Second order identity for general Kirchhoff-type equations) *The conservation law* (A.4) *is a consequence of the following second order identity:*
For $s \geq 2$, let us consider a solution

$$u \in C\left([0,T],\, H^s(\mathbb{T}^n,\mathbb{R})\right) \cap C^1\left([0,T],\, H^{s-1}(\mathbb{T}^n,\mathbb{R})\right)$$

of the equation

$$\partial_{tt}u - \mathbf{f}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right) \Delta u = 0, \tag{A.5}$$

with $\mathbf{f} : [0,+\infty) \to \mathbb{R}$ such that

$$\mathbf{f} \in C^2\left([0,+\infty)\right), \quad \mathbf{f}(y) > 0,\, \forall y \in [0,+\infty).$$

$$K_{\mathbf{f}}(u) := \frac{\int_{\mathbb{T}^n} |\nabla \partial_t u|^2\, dx}{\sqrt{\mathbf{f}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)}} + \sqrt{\mathbf{f}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)} \int_{\mathbb{T}^n} (\Delta u)^2 dx$$

$$+ \frac{1}{2}\frac{\mathbf{f}'\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)}{\mathbf{f}^{\frac{3}{2}}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)} \left(\int_{\mathbb{T}^n} \nabla \partial_t u \cdot \nabla u dx\right)^2$$

then one has

$$\frac{dK_{\mathbf{f}}(u(t))}{dt} = \left(\frac{\mathbf{f}'\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)}{\mathbf{f}^{\frac{3}{2}}\left(\int_{\mathbb{T}^n} |\nabla u|^2\, dx\right)}\right)' \left(\int_{\mathbb{T}^n} \nabla \partial_t u \cdot \nabla u\, dx\right)^3. \tag{A.6}$$

In order for $K_{\mathbf{f}}$ to be conserved we then have to find $\mathbf{f}$ such that

$$\left(\frac{\mathbf{f}'(y)}{\mathbf{f}^{\frac{3}{2}}(y)}\right)' = 0. \tag{A.7}$$

By solving the initial value problem (A.7) *we get the special non-linearity in* (A.1).

Remark A.1.2 *The functional K defined in* (A.3) *controls the H^2-norm of u and the H^1-norm of u_t.*
In fact, there exists a constant $\mathtt{K}(T)$ depending on $u(0)$ and quadratically on T, such that

$$\|u\|_2 + \|u_t\|_1 \leq \mathtt{K}_1(T), \quad \forall t \in [0,T]. \tag{A.8}$$

Moreover, if the constant c in (A.1) *is negative, the bound* (A.8) *is also uniform in time, namely, there exists a positive constant $\mathtt{K}_2(u(0))$ depending on $u(0)$, such that*

$$\|u\|_2 + \|u_t\|_1 \leq \mathtt{K}_2(u(0)), \quad \forall t \in [0,T]. \tag{A.9}$$

In fact:

- *If $c < 0$:*
  recalling the expression (A.2) *of $a(u)$, one has*

$$K(u(0)) \overset{(A.4)}{=} K(u(t)) \geq \frac{1}{a(u)} \int_{\mathbb{T}^n} (\Delta u)^2 dx \geq \|u\|_2^2. \tag{A.10}$$

Using (A.10) *we have*

$$\|u\|_1^2 \leq K(u(0)), \tag{A.11}$$

we then have

$$\begin{aligned}
K(u(0)) &= K(u)\\
&\geq a(u)\int_{\mathbb{T}^n} |\nabla u_t|^2\,dx\\
&\overset{(A.2)}{\geq} \left|1-\left|c\int_{\mathbb{T}^n}|\nabla u|^2\right|\right|\int_{\mathbb{T}^n}|\nabla u_t|^2\\
&\geq \left|1-|c|\|u\|_1^2\right|\int_{\mathbb{T}^n}|\nabla u_t|^2\\
&\overset{(A.11)}{\geq} \left|1-|c|K(u(0))\right|\int_{\mathbb{T}^n}|\nabla u_t|^2\\
&= \left|1-|c|K(u(0))\right|\|u_t\|_1^2,
\end{aligned}$$

from which it follows

$$\|u_t\|_1^2 \leq \frac{K(u(0))}{\left|1-|c|K(u(0))\right|}. \tag{A.12}$$

Using (A.10) *together with* (A.12) *we get* (A.9) *for*

$$\mathtt{K}_2(u(0)) = K(u(0))\frac{K(u(0))}{\left|1-|c|K(u(0))\right|}, \tag{A.13}$$

whenever the right-hand-side is well-defined.

- *If* $c>0$*:*
  we have that

$$\begin{aligned}
K(u(0)) &= K(u)\\
&\geq a(u)\int_{\mathbb{T}^n}|\nabla u_t|^2\,dx - c\left(\int_{\mathbb{T}^n}\nabla u_t\cdot\nabla u\,dx\right)^2\\
&\overset{(A.2)}{\geq} \int_{\mathbb{T}^n}|\nabla u_t|^2\,dx + c\int_{\mathbb{T}^n}|\nabla u_t|^2\,dx\int_{\mathbb{T}^n}|\nabla u|^2\,dx - c\left(\int_{\mathbb{T}^n}\nabla u_t\cdot\nabla u\,dx\right)^2\\
&\geq \int_{\mathbb{T}^n}|\nabla u_t|^2\,dx\\
&= \|u_t\|_1^2,
\end{aligned} \tag{A.14}$$

where the last line follows from the Cauchy-Schwartz (C-S) inequality.
Moreover, thanks to (A.14)*, we have*

$$\begin{aligned}
\frac{da(u)}{dt} &\leq 2c\int_{\mathbb{T}^n}|\nabla u\cdot\nabla u_t|\\
&\overset{\mathbf{C-S}}{\leq} 2c\|u_t\|_1\|u\|_1\\
&\overset{(A.14)}{\leq} 2c\|u_t\|_1\sqrt{K(u(0))}\\
&\leq 2c\sqrt{a(u)}\sqrt{K(u(0))},
\end{aligned}$$

from which it follows

$$a(u(t)) \leq a(u(0)) + c^2t^2K(u(0)) \tag{A.15}$$

Finally, from (A.3) *one has*

$$K(u(0)) = K(u) \geq \frac{1}{a(u)} \int_{\mathbb{T}^n} (\Delta u)^2 dx,$$

hence

$$\int_{\mathbb{T}^n} (\Delta u)^2 dx \leq K(u(0))a(u) \overset{(A.15)}{\leq} K(u(0)) \left(a(u(0)) + c^2t^2K(u(0))\right). \tag{A.16}$$

Using (A.14) *together with* (A.16) *we get* (A.8).

By using Theorems (A.1.1) and (A.1.2) together with Remark (A.1.2), we have:

Theorem A.1.3 (Global Well-Posedness, [80])
There exists $\varepsilon_0 > 0$ *such that for every* $s \geq 2$ *and every initial data* $(u_0, v_0) \in H^s(\mathbb{T}^n, \mathbb{R}) \times H^{s-1}(\mathbb{T}^n, \mathbb{R})$ *such that*

$$\varepsilon := \|u_0\|_s + \|v_0\|_{s-1} \leq \varepsilon_0,$$

equation (A.1) *admits a unique solution*

$$u \in C^0([0,T], H^s(\mathbb{T}^n, \mathbb{R})) \cap C^1([0,T], H^{s-1}(\mathbb{T}^n, \mathbb{R})),$$

for every $T > 0$.

Proof: Let us consider two cases: if $c > 0$, the non-linearity in equation (A.1) is well-defined for every values of u. Moreover, the bound (A.8) allows to extend the local existence result of Theorem (A.1.1) to all time intervals, for the case $s = 2$.
Concerning the case of higher regularity Sobolev spaces $H^s \times H^{s-1}$, with $s > 2$:
if u solves (A.1), then u is also a solution of the linear wave equation

$$u_{tt} - a(t)\Delta u = 0, \tag{A.17}$$

with time-dependent coefficient

$$b(t) = \frac{1}{\left(1 + c \int_{\mathbb{T}^n} |\nabla u| \, dx\right)^2}. \tag{A.18}$$

Since the problem is well-posed for all time scales we have that the term $b(t)$ is always well-defined, moreover the existence of solutions for equation (A.17) for all times, is ensured by the general theory of linear hyperbolic equations (see, for instance, [49]).
If we now consider the function $v := \Lambda^s u$ with $\Lambda = \sqrt{-\Delta}$ and $s \geq 0$, since $b(t)$ is only time-depending, v is again a solution of (A.17).
Finally, noting that $\|v\|_2 = \|u\|_{s+2}$, we can conclude for $c > 0$.
If $c < 0$ we have to ensure that the non-linearity (A.18) is well defined. To achieve that, one must impose

$$1 + c \int_{\mathbb{T}^n} |\nabla u(t)| \, dx = 1 + c\|u(t)\|_1^2 \neq 0, \quad \forall t > 0. \tag{A.19}$$

Using (A.9), we have that (A.19) is verified if $u(0)$ satisfies the condition

$$\mathtt{K}_2(u(0)) < \frac{1}{c},$$

where $\mathtt{K}_2(u(0))$ is defined on (A.13).
The proof then follows by reasoning as in the case $c > 0$. □

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