Multi-Objective Potential Games via Hypervolume Maximization

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Abstract

Multi-objective games model strategic interactions where multiple agents simultaneously optimize different conflicting objectives in a noncooperative setting. Pareto Equilibria have been introduced as a fundamental solution concept, ensuring that no player can unilaterally improve on one objective without worsening at least another. While the existence of such equilibria has been well established, selecting a specific, desirable solution remains a nontrivial task. We focus on multi-objective games where each objective admits a corresponding weighted potential function. We also address the corresponding centralized potential problem of optimizing the multiple potential functions simultaneously. Although potential structures have been explored also in the multi-objective context, the problem of efficiently selecting a Pareto solution remains largely unresolved. We focus on the Hypervolume Maximization scalarization method both for each agent's and for the potential problem, which maximizes the improvement of the objectives with respect to a given reference point. We analyze how solving the potential problem through this technique results in computing solutions to the original multi-objective game.

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1 Introduction

Multi-objective games model decision-making environments where multiple autonomous agents aim to simultaneously optimize different conflicting objectives. The main challenge to these settings lies in the interplay between the multi-agent nature of the game and the multi-objective structure of each player's decision problem, which must be addressed simultaneously. To this end, the solution concept of Pareto Equilibria was first introduced in [1], in the context of two-person zero-sum games, and later extended to more general frameworks in [2, 3]. Informally, a Pareto Equilibrium represents a stable solution, according to which each player's strategy is Pareto efficient, given the strategies of the other agents. In other words, no player can unilaterally change their strategy to improve one of their objectives without causing a deterioration in at least one of the others. The existence of Pareto Equilibria in multi-objective games has been extensively studied in the literature (see, e.g., [4, 5]).

Building on the theoretical framework of potential games introduced in [6], we focus on a specific class of multi-objective games characterized by the existence of a weighted potential function. Although potential multi-objective games have been analyzed in [7–9], the challenge of efficiently computing specific Pareto Equilibria according to some specific criteria remains substantially open. We address this issue by combining the structure of multi-objective potential games with a specific scalarization technique that we term Hypervolume Maximization [10, 11]. Inspired by bargaining theory (see [12]), this approach makes it possible to maximize the improvement of all objectives with respect to a given reference point, in the sense of the corresponding hypervolume, which is proportional to the achieved improvements. This is particularly relevant in financial applications such as multi-portfolio selection, where the market index can be selected as reference point in an enhanced index tracking spirit (see e.g. [11, 13–17]). We show that the method is practically viable and that the proposed technique leads to a point exhibiting several favorable properties, which are described in detail in Appendix A. We develop our analysis by applying the Hypervolume Maximization approach to the potential multi-objective problem, showing that it provides a Pareto Equilibrium. We then investigate how the Hypervolume Maximization, when applied to the potential multi-objective problem, reflects also on the individual problems faced by each agent. More in detail, we study how the reference point selected for the potential functions translates to each player's own scalarized problem and reference point. We show that, in general, players incur in a distortion of their desired reference values. This is the price to pay to efficiently compute an equilibrium maximizing the hypervolume, which would otherwise be impractical, as described in the following sections. We then explore a specific class of potential multi-objective games that exhibit more favorable properties: the players do not incur in any distortion of their desired reference values and the solution corresponds to the case of all players cooperating.

The remainder of the paper is organized as follows. In Section 2, we formalize the multi-objective potential game framework. In Section 3, we introduce the hypervolume maximization scalarization and detail its properties. Finally, Section 4 is devoted to some improved results for a specific class of multi-objective potential games.

2 The Multi-Objective Potential Game Framework

We consider N agents/players, each one aiming at optimizing multiple objectives that possibly depend on the choices of the other agents. We comply with the classical framework of complete information, simultaneity and rationality.

Let $x^\nu \in \mathbb{R}^{n_\nu}$ represent the decision variables controlled by player $\nu \in \{1, \dots, N\}$. The total number of variables is therefore $n = \sum_\nu n_\nu$. Each player ν is associated with m objective functions $(\theta_1^\nu, \dots, \theta_m^\nu)$, where $\theta_j^\nu : \mathbb{R}^n \to \mathbb{R}$ denotes the j-th objective of player ν.

Each objective θ_j^ν may depend not only on x^ν, but also on the decision variables of the other players, which are collected in vector $x^{-\nu}$:

$$x^{-\nu} \triangleq \begin{pmatrix} x^1 \\ \vdots \\ x^{\nu-1} \\ x^{\nu+1} \\ \vdots \\ x^N \end{pmatrix} \in \mathbb{R}^{(N-1)n_\nu}.$$

As is customary in the relevant literature, we denote by $x \in \mathbb{R}^n$ the vector formed by all the decision variables and, to emphasize the agent ν's ones within x, we write $(x^\nu, x^{-\nu})$ instead of x, still indicating the vector $x = (x^1, \dots, x^\nu, \dots, x^N)$.

Finally, decision variables x^ν are required to belong to the set $X_\nu \subseteq \mathbb{R}^{n_\nu}$.

Each agent ν tackles the following multi-objective problem composed of m parametric (with respect to other agents' decisions $x^{-\nu}$) objectives:

$$\underset{x^\nu \in X_\nu}{\text{minimize}} \ \left(\theta_1^\nu(x^\nu, x^{-\nu}) \dots, \theta_m^\nu(x^\nu, x^{-\nu})\right) \tag{1}$$

The collection of all agents' multi-objective programs (1) form the Multi-objective Game (MG), that is the problem to

$$\begin{aligned} \text{find } \widehat{x} \in X = \prod_{\nu=1}^N X_\nu : \nu = 1, \dots, N, \ \forall x^\nu \in X_\nu, \\ \exists j_{x^\nu} \in \{1, \dots, m\} \ : \ \theta_{j_{x^\nu}}^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) < \theta_{j_{x^\nu}}^\nu(x^\nu, \widehat{x}^{-\nu}), \text{ or} \\ \forall j \in \{1, \dots, m\} \ : \ \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \leq \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}). \end{aligned} \tag{MG}$$

We term Pareto Equilibrium (PE) any solution to (MG).

A PE therefore corresponds to a stable configuration where each player's strategy is Pareto efficient given the strategies of the others. That is, no unilateral deviation performed by a single agent can lead to a strictly better outcome for one of their objectives without deteriorating at least another one. This is the classical solution concept in the noncooperative interplay among the players.

We focus on a specific class of multi-objective games such that each objective admits an associated potential function shared across all players. The following definition of weighted potential function is widely used in the context of Nash Equilibrium Problems (for general results concerning potential functions, see [6, 18–21] and, related to the multi-objective games context, [7, 8, 22, 23]).

Definition 1 A weighted potential function for objective j is a function $P_j : \mathbb{R}^n \to \mathbb{R}$ such that some weights $w^j \in \mathbb{R}^N_{++}$ exist so that for all $\nu = 1, \dots, N$ and for all $(x^\nu, x^{-\nu}), (y^\nu, x^{-\nu}) \in X$:

$$\theta_j^\nu(x^\nu, x^{-\nu}) - \theta_j^\nu(y^\nu, x^{-\nu}) \ = \ w_\nu^j P_j(x^\nu, x^{-\nu}) - w_\nu^j P_j(y^\nu, x^{-\nu}).$$

Informally, a potential function captures the idea that the goals of all players, with respect to a given objective, are aligned through a common scalar function: any unilateral change in a player's strategy leads to a proportional corresponding change in the potential function. Problem (MG) is referred to as a weighted potential multi-objective game whenever, for every j, a weighted potential function exists according to Definition 1.

Assumption A: We assume (MG) to be a weighted potential game.

In the light of the characterization result [24, Theorem 2.1], and the developments in [8], we take, for all $\nu = 1, \dots N$ and $j = 1, \dots, m$,

$$\theta_j^\nu(x^\nu, x^{-\nu}) = w_\nu^j \phi_j(x^\nu, x^{-\nu}) + \gamma_j^\nu(x^\nu), \quad w_\nu^j > 0, \tag{2}$$

where, for all j, $\phi_j : \mathbb{R}^n \to \mathbb{R}$ and $\gamma_j^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ for all ν. Therefore, we assume that each player's objective is the weighted sum of a shared coupling term and a proprietary term.

The coupling term depends on the strategies of all players, and links together all agents' decisions. Importantly, the coupling term is assumed to be the same for all players, but each player gives this term a specific positive weight. The proprietary term, on the other hand, is solely related to the strategy of each individual player.

We introduce a set of assumptions that play a crucial role in enabling the theoretical developments and characterizations presented in the remainder of the paper.

Assumptions B: for all $\nu = 1, \dots, N$ and for all $j = 1, \dots, m$,

B1 ϕ_j and γ_j^ν are continuously differentiable;
B2 $\phi_j(\cdot, x^{-\nu})$ is convex on X_ν, for every $x^{-\nu} \in \prod_{\lambda \neq \nu} X_\lambda$ and γ_j^ν is convex on X_ν;
B3 X_ν is nonempty compact and convex.

In the rest of the paper, we assume the conditions in **A** and **B** to hold.

We remark that under assumption **B3**, the common feasible set $X \triangleq \prod_{\nu=1}^N X_\nu$ is also nonempty, compact and convex.

The existence of PEs is a well researched issue in the literature see, e.g. [4, 5]. In fact, under Assumptions **B**, it is classically seen to be guaranteed thanks to [25, Theorem 4.1, part 2], which refers to multi-objective optimality for each single agent, and [26, Proposition 1.4.2, Corollary 2.2.5], for what concerns Nash equilibria among agents.

In the same spirit as [6, Lemma 2.7], we report the following uniqueness, up to an additive real constant, result about potential functions, whose proof can be traced back to [27], when, in a weighted potential scenario, players' objectives are given according to (2).

Proposition 1 *The vector* w^j *and*

$$P_j(x) = \phi_j(x) + \sum_{\nu=1}^{N} \frac{1}{w_\nu^j} \gamma_j^\nu(x^\nu) + c_j, \quad c_j \in \mathbb{R}, \tag{3}$$

are, respectively, the unique weights and potential functions for objective j*, up to the additive real constant* c_j*.*

Mimicking the single objective case, we introduce the potential Multi-objective Problem (pMP), that can be viewed as a centralized program for the whole noncooperative system, i.e.,

$$\underset{x \in X}{\text{minimize}} \; \big(P_1(x), \ldots, P_m(x)\big)$$

That is, the problem to

$$\begin{aligned} &\text{find } \widehat{x} \in X : \\ &\forall x \in X, \; \exists j_x \in \{1, \ldots, m\} \; : \; P_{j_x}(\widehat{x}) < P_{j_x}(x), \text{ or} \\ &\qquad\qquad \forall j \in \{1, \ldots, m\} \; : \; P_j(\widehat{x}) \le P_j(x). \end{aligned} \tag{pMP}$$

We term potential Pareto Optimum (pPO) any solution to (pMP). Under Assumptions **A** and **B**, the existence of pPOs is guaranteed thanks to [25, Theorem 4.1, part 2] and the Weierstrass Theorem; moreover, all pPOs turn out to be PEs, while the vice versa does not hold in general (see [27, Example 1]). We remark that (pMP) is a multi-objective problem, and can therefore be addressed by many solution methods, see [25].

3 The Maximum-Hypervolume Approach

In the light of the results in the previous section, we propose to address (pMP), in order to recover PEs. To this end, we rely on the Maximum Hypervolume (Max-H) scalarization, in the same spirit of the bargaining paradigm (see [11, 12, 15, 22]). We defer the description of the main features and desirable properties of the Max-H scalarization approach in the multi-objective context to Appendix A.

In accordance with the Max-H method, we are interested in finding pPOs overperforming some reference values $a \in \mathbb{R}^m$, that is, additionally satisfying the constraints

$$P_j(x) \le a_j, \quad j = 1, \ldots, m.$$

Accordingly, we introduce the set

$$\mathcal{F}_a \triangleq \{x \in X : P_j(x) \leq a_j,\ j = 1, \ldots, m\}.$$

We present the Max-H scalarization applied to (pMP):

$$\underset{x \in \mathcal{F}_a}{\text{maximize}} \prod_{j=1}^{m} (a_j - P_j(x)). \tag{pMax-H$_a$}$$

(pMax-H$_a$) is an optimization problem, whose properties are highlighted in Appendix A. In order to tackle (pMax-H$_a$), one can, in fact, rely on efficient numerical methods. We remark that thanks to Theorem 7, the solutions of (pMax-H$_a$) lead to pPOs, and, thanks to [27], they also turn out to be PEs.

It is worth investigating also the application of the Max-H approach to scalarize each player's multi-objective problem. Consequently, since each player aims at outperforming some reference point $b^\nu \in \mathbb{R}^m$, we introduce the set

$$\mathcal{G}_{\nu,b^\nu}(x^{-\nu}) \triangleq \left\{y^\nu \in X_\nu : \theta_\nu^j(y^\nu, x^{-\nu}) \leq b_j^\nu,\ j = 1, \ldots, m\right\}.$$

Each player ν wishes to

$$\underset{x^\nu \in \mathcal{G}_{\nu,b^\nu}(x^{-\nu})}{\text{maximize}} \prod_{j=1}^{m} \left(b_j^\nu - \theta_\nu^j(x^\nu, x^{-\nu})\right).$$

The collection of the N above parametric programs is the problem to

$$\begin{array}{l} \text{find } \widehat{x} \in \prod_{\nu=1}^{N} \mathcal{G}_{\nu,b^\nu}(\widehat{x}^{-\nu}) : \nu = 1, \ldots, N,\ \forall x^\nu \in \mathcal{G}_{\nu,b^\nu}(\widehat{x}^{-\nu}), \\ \qquad \prod_{j=1}^{m} \left(b_j^\nu - \theta_\nu^j(\widehat{x}^\nu, \widehat{x}^{-\nu})\right) \geq \prod_{j=1}^{m} \left(b_j^\nu - \theta_\nu^j(x^\nu, \widehat{x}^{-\nu})\right). \end{array} \tag{Max-H$_b$G}$$

The main challenge in directly addressing this problem is its numerical tractability, along with the assurance of convergence to an equilibrium. In fact, (Max-H$_b$G) is a Generalized Nash Equilibrium Problem (GNEP) (see [28] and the references therein for a survey about GNEPs). Note that, under standard player Constraint Qualifications, it can be at best reformulated as a quasivariational inequality not belonging to any of the known solvable classes (see [28–32]). On top of this, under our standing assumptions, the pseudo gradient of the game at x, i.e.

$$F(x) = -\begin{pmatrix} \nabla_1 \prod_{j=1}^{m} \left(b_j^1 - \theta_\nu^j(x^1, x^{-1})\right) \\ \vdots \\ \nabla_\nu \prod_{j=1}^{m} \left(b_j^\nu - \theta_\nu^j(x^\nu, x^{-\nu})\right) \end{pmatrix} = -\left[\nabla_\nu \prod_{j=1}^{m} \left(b_j^\nu - \theta_\nu^j(x^\nu, x^{-\nu})\right)\right]_{\nu=1}^{N},$$

where ∇_ν denotes the gradient with respect to x^ν, is not guaranteed to be monotone, as observed in Example 1. Hence, known conditions to ensure the numerical tractability of the problem are not satisfied, and its practical implementation is in jeopardy.

In the following theorem, we show how it is possible to solve (Max-H$_b$G) by addressing (pMax-H$_a$).

Theorem 2 *Let $\widehat{x}$ be a solution of (pMax-H$_a$). Then, it is a solution to (Max-H$_b$G), where*

$$b_j^\nu = w_\nu^j a_j - \sum_{p\neq\nu} \frac{w_\nu^j}{w_p^j} \gamma_j^p(\widehat{x}^p), \quad \forall j = 1, \dots, m,\ \nu = 1, \dots N. \tag{4}$$

Proof For every ν, and any $(x^\nu, \widehat{x}^{-\nu}) \in X$, we have

$$\begin{aligned} a_j - P_j(x^\nu, \widehat{x}^{-\nu}) &= a_j - \left(\phi_j(x^\nu, \widehat{x}^{-\nu}) + \frac{1}{w_\nu^j}\gamma_j^\nu(x^\nu) + \sum_{p\neq\nu} \frac{1}{w_p^j}\gamma_j^p(\widehat{x}^p) \right) \\ &= a_j - \frac{1}{w_\nu^j}\theta_j^\nu(x^\nu, \widehat{x}^{-\nu}) - \sum_{p\neq\nu} \frac{1}{w_p^j}\gamma_j^p(\widehat{x}^p) \qquad (5) \\ &= \frac{1}{w_\nu^j}\left(b_j^\nu - \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}) \right), \end{aligned}$$

for any j, thanks to (4). Fix an arbitrary $x^\nu \in \mathcal{G}_{\nu, b^\nu}(\widehat{x}^{-\nu})$. Thanks to (5), $(x^\nu, \widehat{x}^{-\nu}) \in \mathcal{F}_a$. Hence, for every ν,

$$\begin{aligned} \prod_{j=1}^m \left(b_j^\nu - \theta_j^\nu(x^\nu, \widehat{x}^{-\nu}) \right) &= \prod_{j=1}^m w_\nu^j (a_j - P_j(x^\nu, \widehat{x}^{-\nu})) \leq \prod_{j=1}^m w_\nu^j (a_j - P_j(\widehat{x}^\nu, \widehat{x}^{-\nu})) \\ &= \prod_{j=1}^m \left(b_j^\nu - \theta_j^\nu(\widehat{x}^\nu, \widehat{x}^{-\nu}) \right). \end{aligned}$$

In turn, $\widehat{x}$ solves (Max-H$_b$G). □

To grasp the meaning of (4), consider the case where the players all agree that $\overline{x} \in X$ is a reference point to be improved on. This translates to having

$$\theta_j^\nu(\overline{x}) = w_\nu^j \phi_j(\overline{x}) + \gamma_j^\nu(\overline{x}^\nu), \quad \forall j = 1, \dots, m,\ \nu = 1, \dots N,$$

as reference values b in (Max-H$_b$G). Exploiting Theorem 2, finding a solution $\widehat{x}$ of (pMax-H$_a$) with

$$a_j = P_j(\overline{x}) = \phi_j(\overline{x}) + \sum_{p=1}^N \frac{1}{w_p^j}\gamma_j^p(\overline{x}^p), \tag{6}$$

provides a solution to (Max-H$_b$G) with

$$
\begin{aligned}
b_j^\nu &= w_\nu^j \phi_j(\overline{x}) + \sum_{p=1}^{N} \frac{w_\nu^j}{w_p^j} \gamma_j^p(\overline{x}^p) - \sum_{p\neq\nu} \frac{w_\nu^j}{w_p^j} \gamma_j^p(\widehat{x}^p), \\
&= \underbrace{w_\nu^j \phi_j(\overline{x}) + \gamma_j^\nu(\overline{x}^\nu)}_{\theta_j^\nu(\overline{x})} + \underbrace{\sum_{p\neq\nu} \frac{w_\nu^j}{w_p^j} \left(\gamma_j^p(\overline{x}^p) - \gamma_j^p(\widehat{x}^p)\right)}_{\text{distortion}}.
\end{aligned} \tag{7}
$$

We remark that, as a result of addressing (pMax-H$_a$), the players incur in a distortion (that is quantified) of their chosen reference values $\theta_j^\nu(\overline{x})$. In fact, b_j^ν equals $\theta_j^\nu(\overline{x})$ plus a distortion that depends on the variation in the value of the proprietary terms γ_j^ν of all the other players $p \neq \nu$ between $\widehat{x}$ and $\overline{x}$. The factor $\frac{w_\nu^j}{w_p^j}$ maps such change onto the scale of θ_j^ν.

One of the key features of the Max-H approach (pMax-H$_a$) lies in its practical viability. In the following proposition, we show that whenever P_js are convex, recovering global optima of (pMax-H$_a$) can be done by focusing on stationary points belonging to $\{x \in X : P_j(x) < a_j,\ j = 1, \ldots, m\}$. This is a key feature to solve (pMax-H$_a$) through standard monotonic ascent methods, such as the projected gradient, therefore obtaining pPOs, and then PEs. We remark that, in view of Theorem 2 and the subsequent considerations, this comes at the price of players incurring in a distortion of their chosen reference points $\theta_j^\nu(\overline{x})$.

Proposition 3 *Let P_j be convex for all js. Then, $\prod_{j=1}^m (a_j - P_j(\cdot))$ is pseudoconcave on $\{x \in X : P_j(x) < a_j,\ j = 1, \ldots, m\}$, and stationary points of* (pMax-H$_a$) *belonging to $\{x \in X : P_j(x) < a_j,\ j = 1, \ldots, m\}$ are also global optima.*

Proof The result is essentially a consequence of [33, Theorem 5.17a], following the same line of reasoning in the proof of [10, Theorem 2.1]. □

Another theoretical property worth mentioning is related to the optimal value of (pMax-H$_a$) w.r.t. reference values a.

Theorem 4 *Assume ϕ_j to be convex on X for every j and let $\overline{a} \in \mathbb{R}^m$ be such that $\{y \in X : P_j(y) < \overline{a}_j,\ j = 1, \ldots, m\} \neq \emptyset$. Then, the optimal value function $\psi : \mathbb{R}^m \to \mathbb{R}$, with $\psi(a) \triangleq \max_{\mathcal{F}_a} \prod_{j=1}^m (a_j - P_j(x))$, is continuous at $\overline{a}$ and, in turn, the solution set mapping $\Gamma : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, with $\Gamma(a) \triangleq \arg\max_{\mathcal{F}_a} \prod_{j=1}^m (a_j - P_j(x))$, is outer semicontinuous at $\overline{a}$.*

Proof Observe that P_j is convex on X for every j. The feasible set mapping $\mathcal{F}_a : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is inner semicontinuous at $\overline{a}$: this is due to [34, Theorem 3.1.6] observing that $\{y \in X : P_j(y) < \overline{a}_j,\ j = 1, \ldots, m\} \neq \emptyset$, the set X is convex and the functions P_j are convex. Thus, thanks to [34, Theorem 3.1.1], also continuous at $\overline{a}$. The continuity of ψ at $\overline{a}$ follows in the light of [34, Theorem 4.2.2] noticing that the objective is continuous and X is bounded. Finally, relying on [34, Theorem 3.1.1] again, the outer semicontinuity of Γ is consequence of the continuity of ψ at $\overline{a}$. $\square$

We illustrate the main considerations above by means of the following example.

Example 1 Consider the 2-player (MG), where $n_1 = n_2 = 1$ and $m = 2$. Player one considers

$$\theta^1(x^1, x^2) = \begin{pmatrix} x^1 + \frac{1}{4}(x^2 - x^1 + 1)^2 \\ -x^1 + \frac{1}{4}(x^2 - x^1 + 1)^2 \end{pmatrix}, \qquad X_1 = [0, 1],$$

and player two considers

$$\theta^2(x^1, x^2) = \begin{pmatrix} x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2 \\ -x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2 \end{pmatrix}, \qquad X_2 = [0, 1].$$

Assumptions **A** and **B** are easily seen to be satisfied, with

$$P_1 = x^1 + x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2, \quad P_2 = -x^1 - x^2 + \frac{1}{4}(x^2 - x^1 + 1)^2.$$

In Figure 2, we show the feasible set of the problem, depicting the pPOs as the thick black line, and the point $\overline{x} = (0, 1)$.

The convexity of P_1 and P_2 can easily be verified, and we can therefore solve the Max-H problem (pMax-H$_a$) through (stationay poits-seeking) standard numerical methods thanks to Proposition 3. We consider $\overline{x} = (0, 1)$ as the reference point, and, according to (6), $a = P(\overline{x}) = (2, 0)$ as the reference values.

In Figure 3a, we depict the potential functions' space $P = (P_1, P_2)$. We report $P(X)$ in light gray, the set of all pPOs as the solid black line, the reference values a in green and the nondominated point $P(\widehat{x})$ in blue. Moreover, we represent in pink the area (since it is a hypervolume in $\mathbb{R}^2$) of the rectangle with sides of length $a_1 - P_1$, $a_2 - P_2$ that is maximized through the Max-H approach.

In Figures 3b and 3c, we depict the players' objectives' function spaces. We report $\theta^1(X)$ and $\theta^2(X)$ in light gray, all the pPOs as a solid line, and $\theta^1(\widehat{x})$ and $\theta^2(\widehat{x})$ in blue. We also report b^1 and b^2 (according to (7)) in green, and the corresponding rectangle's area that is maximized through the max-area approach in pink. We also indicate the player-specific reference values $\theta^1(\overline{x})$ and $\theta^2(\overline{x})$ in yellow. As mentioned above, given $\widehat{x}^{-\nu}$, the players do not maximize at $\widehat{x}^{\nu}$ the area w.r.t. their target reference values $\theta^{\nu}(\overline{x})$. In fact, they maximize the area w.r.t. the distorted reference value b^{ν} as obtained by (7).

We also remark that, under our standing assumptions, the pseudo gradient F of (Max-H$_b$G) cannot be expected to be monotone. For the example at hand:

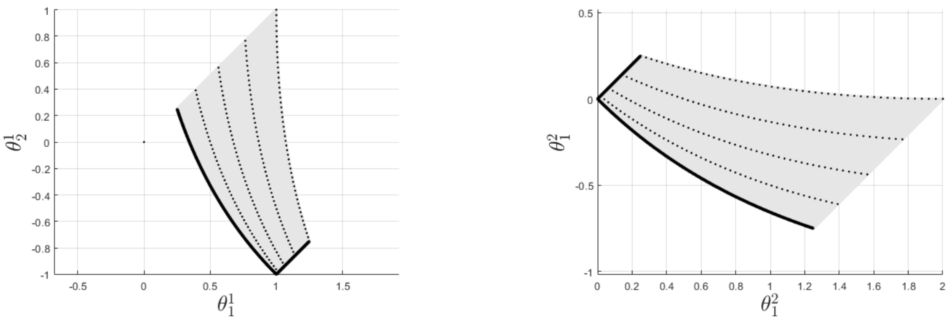
$$JF(x) = -\begin{pmatrix} \dfrac{\partial^2 \left(b_1^1 - \theta_1^1(x^1, x^2)\right)\left(b_2^1 - \theta_2^1(x^1, x^2)\right)}{\partial x^1 \partial x^1} & \dfrac{\partial^2 \left(b_1^1 - \theta_1^1(x^1, x^2)\right)\left(b_2^1 - \theta_2^1(x^1, x^2)\right)}{\partial x^1 \partial x^2} \\ \dfrac{\partial^2 \left(b_1^2 - \theta_1^2(x^1, x^2)\right)\left(b_2^2 - \theta_2^2(x^1, x^2)\right)}{\partial x^2 \partial x^1} & \dfrac{\partial^2 \left(b_1^2 - \theta_1^2(x^1, x^2)\right)\left(b_2^2 - \theta_2^2(x^1, x^2)\right)}{\partial x^2 \partial x^2} \end{pmatrix},$$

taking $\widetilde{x} = (0.2, 0.8) \in \prod_{\nu=1}^{N} \mathcal{G}_{\nu, b^{\nu}}(\widetilde{x}^{-\nu})$, straightforward calculations yield

$$JF(\widetilde{x}) = \frac{2}{25}\begin{pmatrix} 9 & 16 \\ 16 & 9 \end{pmatrix} \not\succeq 0,$$

showing that the pseudo gradient of this game is not monotone at $\widetilde{x}$.

We continue our analysis by comparing the solutions achieved through the weighted potential function optimization approach (pMP) (that is, pPOs) with the solutions of the original multiobjective game (MG) (that is, PEs). In Figures 1a and 1b, we represent all pPOs in both players' objectives space with the thick black line. Moreover, the dotted lines represent $\theta^1(X_1, \overline{x}^2)$ and $\theta^2(\overline{x}^1, X_2)$ for $\overline{x}^1 \in 0, 0.25, 0.5, 0.75$ and $\overline{x}^2 \in 0.25, 0.5, 0.75, 1$. Focusing on player 1 (Figure 1a), the behavior of such curves is the visual proof that any $x^1 \in X_1$ is efficient for player 1's problem for any fixed $\overline{x}^2 \in X_2$. The same argument can be repeated verbatim for player 2. This leads us to conclude that all feasible $x \in X$ are PE. In summary, by addressing (pMP), one operates a selection among PEs. In particular, the pPO $\widehat{x}$ computed through hypervolume maximization is nondominated in each player's objectives' space. □



(a) Player 1's objectives' space

(b) Player 2's objectives' space

Fig. 1: Comparison of the solutions achieved through the weighted potential function optimization approach (pMP) (that is, pPOs in thick black lines) with the solutions of the original multiobjective game (MG) (that is, PEs in light gray), in the objective spaces of the two players (Figures 1a and 1b, respectively). Dotted lines represent $\theta^1(X_1, \overline{x}^2)$ and $\theta^2(\overline{x}^1, X_2)$ for $\overline{x}^1 \in 0, 0.25, 0.5, 0.75$ and $\overline{x}^2 \in 0.25, 0.5, 0.75, 1$.

4 Improved Results for a Class of Games

We identify a class of games (also known as coordination games, see [7]) where each player considers the weighted potential functions as their objectives:

$$\theta_j^\nu(x^\nu, x^{-\nu}) = w_\nu^j P_j(x^\nu, x^{-\nu}).$$

This corresponds to taking, in (2), $\phi_j = P_j$ for all j, and $\gamma_j^\nu \equiv 0$ for all j and ν. Considering this specific structure, we present the following result, whose proof follows immediately from Theorem 2.

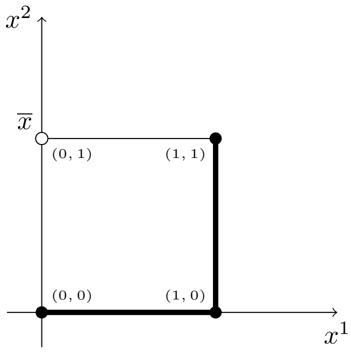


Fig. 2: Visual representation of Example 1: the empty point $\overline{x}$ is the reference, while all the pPOs lie on the thick line

Theorem 5 *Let $\phi_j = P_j$ for all j, and $\gamma_j^\nu \equiv 0$ for all j and ν in (2). If $\widehat{x}$ is a solution of (pMax-H$_a$), then, it is a solution to (Max-H$_b$G), where*

$$b_j^\nu = w_\nu^j a_j, \quad \forall j = 1, \dots, m,\ \nu = 1, \dots N. \tag{8}$$

Consider again $\overline{x} \in X$ as a common reference point for all players. Taking the reference values a as in (6), following the relations in (7), we get $b_j^\nu = \theta_j^\nu(\overline{x})$, that is, the players do not incur in any distortion. In turn, the Max-H approach results in a PE that provides every player ν with the desirable properties outlined in Appendix A with respect to $\theta_j^\nu(\overline{x})$.

We introduce the following multi-objective program involving all objectives of all players:

$$\underset{x \in X}{\text{minimize}} \left(\big(\theta_1^1(x), \dots, \theta_m^1(x)\big), \dots, \big(\theta_1^N(x), \dots, \theta_m^N(x)\big) \right). \tag{9}$$

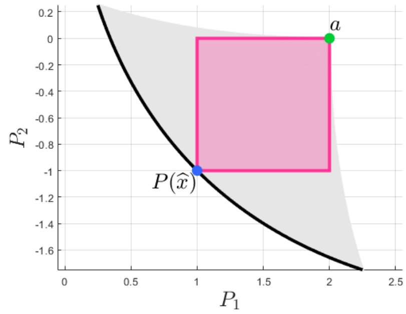
This problem represents the case where all players cooperate, aiming at minimizing all $N \times m$ objectives simultaneously (see [27]).

Theorem 6 *Let $\phi_j = P_j$ for all j, and $\gamma_j^\nu \equiv 0$ for all j and ν in (2). If $\widehat{x}$ is a solution of (pMax-H$_a$) and $\{y \in X : P_j(y) < a_j,\ j = 1, \dots, m\} \neq \emptyset$, then, it is Pareto efficient for (9) such that*

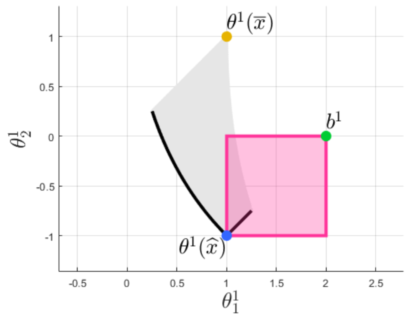
$$\prod_{\nu=1}^N \prod_{j=1}^m (b_j^\nu - \theta_j^\nu(\widehat{x})) \geq \prod_{\nu=1}^N \prod_{j=1}^m (b_j^\nu - \theta_j^\nu(x)), \quad \forall x \in X : (b_j^\nu - \theta_j^\nu(x)) \geq 0,\ \forall j, \nu.$$

Proof Since $\widehat{x}$ is a pPO thanks to Theorem 7, [25, Theorem 4.1, part 2] implies that there exist weights $\pi_j > 0$ such that

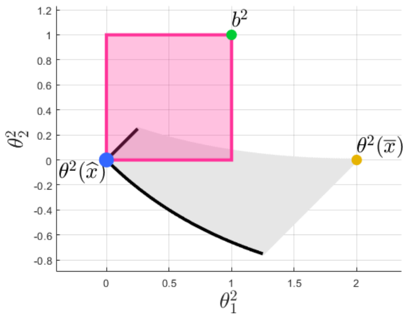
$$\sum_{j=1}^m \pi_j P_j(\widehat{x}) \leq \sum_{j=1}^m \pi_j P_j(x), \quad \forall x \in X.$$



(a) Potential objectives' space



(b) Player 1's objectives' space



(c) Player 2's objectives' space

Fig. 3: Visualization of pPOs (solid line) and the PE $\widehat{x} = (0, 1)$ (blue) in the potential function space (Figure 3a) and in the objective spaces of the two players (Figures 3b and 3c) (gray-shaded area); we also show, for all three figures, the reference value a and b^ν (green), and, for the players' objective space, $\theta^\nu(\overline{x})$ (yellow)

For every $y \in X$, the equality chain

$$\sum_{j=1}^m \pi_j P_j(y) = \sum_{\nu=1}^N \frac{1}{N} \sum_{j=1}^m \pi_j \phi_j(y) = \sum_{\nu=1}^N \frac{1}{N} \sum_{j=1}^m \frac{\pi_j}{w_\nu^j} \theta_j^\nu(y),$$

together with [25, Theorem 4.1, part 1] shows that $\widehat{x}$ is Pareto efficient for (9). Since $\widehat{x}$ solves (pMax-H$_a$), we have

$$\prod_{j=1}^m (a_j - P_j(\widehat{x})) \geq \prod_{j=1}^m (a_j - P_j(x)), \quad \forall x \in \mathcal{F}_a,$$

and then

$$\prod_{j=1}^m (b_j^\nu - \theta_j^\nu(\widehat{x})) \geq \prod_{j=1}^m (b_j^\nu - \theta_j^\nu(x)), \quad \forall x \in \mathcal{F}_a.$$

The claim follows noticing that we have $b_j^\nu - \theta_j^\nu(x) \geq 0$, for all $x \in \mathcal{F}_a$. $\square$

The previous Theorem shows that the solutions to (pMax-H$_a$) yield PEs that are also Pareto efficient for (9). Moreover, $\widehat{x}$ is the Max-H optimum of (9), where the reference values are $\theta_j^{\nu}(\overline{x})$.

Appendix A Max-H Approach Properties

We consider the multi-objective optimization problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \big(f_1(x), \ldots, f_m(x)\big) \\ \text{s.t.} & x \in K, \end{array}$$

where $K \subseteq \mathbb{R}^n$ is a nonempty compact convex set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is the vector of objective functions that are assumed to be convex. A point $\widehat{x} \in K$ is Pareto optimal if $\forall x \in K$: $\exists j_x \in \{1, \ldots, m\}$ such that $f_{j_x}(\widehat{x}) < f_{j_x}(x)$, or $\forall j \in \{1, \ldots, m\}$ we have $f_j(\widehat{x}) \leq f_j(x)$.

Given reference values $b \in \mathbb{R}^m$, we focus on Pareto optimal points $x \in K$ that satisfy $f_j(x) \leq b_j, j = 1, \ldots, m$; therefore we consider the set $\mathcal{S}_b \triangleq \{y \in K : f_j(y) \leq b_j,\, j = 1, \ldots, m\}$

We are interested in solving the multi-objective problem scalarized through the Max-H approach, that is,

$$\underset{x \in \mathcal{S}_b}{\text{maximize}} \ \textstyle\prod_{j=1}^{m}(b_j - f_j(x)). \tag{A1}$$

The Max-H method has a simple geometric interpretation: the objective function is easily seen to be the hypervolume of an hyperrectangle whose sides lenghts along the m dimensions are $b_j - f_j(x)$.

Theorem 7 *Let $\widehat{x}$ be a solution of (A1). If $\big\{y \in K : f_j(y) < b_j,\ j = 1, \ldots, m\big\} \neq \emptyset$, then $\widehat{x}$ is Pareto optimal.*

Proof The claim follows from [10, Proposition D.6 (ii)]. □

Assuming the Slater-type condition in Theorem 7 is sensible because, whenever it does not hold, all feasible points of (A1) turn out to be weak-Pareto optimal (for further details, see [10, Appendix A]).

We list the properties of any optimal solution $\widehat{x}$ of the Max-H approach:

- it is weak Pareto optimal, and, if $\{y \in K : f_j(y) < b_j,\, j = 1, \ldots, m\} \neq \emptyset$, it is Pareto optimal and such that $f_j(\widehat{x}) < b_j,\, j = 1, \ldots, m$;
- focusing for simplicity on the case where $m = 2$, let a decision maker opts for a choice $\widetilde{x} \in \mathcal{S}_b$ improving on $\widehat{x}$ with respect to an objective by a given factor. Then, $\widetilde{x}$ is worse than $\widehat{x}$ with respect to the other objective by *at least the same factor* (see [11]). A more complicated picture emerges in the general case as detailed in Proposition 8 (see [35]);

- among all Pareto-optimal solutions, it dominates the largest subset of solutions that outperform the reference values b. Therefore, it may be interpreted as the solution that maximally improves upon the benchmark;
- it does not depend on the scales of possibly non homogeneous objectives.

Proposition 8 *Let $\widehat{x}$ be a Max-H optimum. Assume $\{y \in K : f_j(y) < b_j,\ j = 1, \ldots, m\} \neq \emptyset$, and let $\widetilde{x}$ be feasible and such that for some $\beta_j > 1$, $b_j - f_j(\widetilde{x}) = \beta_j(b_j - f_j(\widehat{x}))$, for every $j = 1, \ldots, p$, where $p < m$. Then,*

$$\prod_{j=p+1}^{m} (b_j - f_j(\widetilde{x})) \leq \frac{1}{\prod_{j=1}^{p} \beta_j} \prod_{j=p+1}^{m} (b_j - f_j(\widehat{x})).$$

We remark that the Max-H approach turns out to be viable in practice due to the pseudoconcavity of the objective function on $\mathcal{S}_b$ (see [33, Theorem 5.17 a]): global optima can therefore be computed by focusing on stationary points, according to [10, Theorem 2.1].

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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