

COALESCING AND BRANCHING SIMPLE SYMMETRIC EXCLUSION PROCESS

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Motivated by kinetically constrained interacting particle systems (KCM), we consider a reversible coalescing and branching simple exclusion process on a general finite graph $G = (V, E)$ dual to the biased voter model on G . Our main goal are tight bounds on its logarithmic Sobolev constant and relaxation time, with particular focus on the delicate slightly supercritical regime in which the equilibrium density of particles tends to zero as $|V| \rightarrow \infty$. Our results allow us to recover very directly and improve to ℓ^p -mixing, $p \geq 2$, and to more general graphs, the mixing time results of Pillai and Smith for the FA-1f (or 1-neighbour) KCM on the discrete d -dimensional torus. In view of applications to the more complex FA- j f KCM, $j > 1$, we also extend part of the analysis to an analogous process with a more general product state space.

1. Introduction. In this work we study a coalescing and branching simple symmetric exclusion process (CBSEP) on a general finite graph $G = (V, E)$. The model was first introduced by Schwartz [35] in 1977 (also see Harris [19]) as follows. Consider a system of particles performing independent continuous time random walks on the vertex set of a (finite or infinite) graph G by jumping along each edge with rate 1, which coalesce when they meet (a particle jumping on top of another one is destroyed) and which branch with rate $\beta > 0$ by creating an additional particle at a empty neighboring vertex. The process is readily seen to be reversible w.r.t. the Bernoulli(p)-product measure with $p = \frac{\beta}{(1+\beta)}$. Initially the model was introduced in order to study the biased voter model [35] (also known as Williams-Bjerknes tumour growth model [41]), which turns out to be its dual additive interacting particle system [18].¹ A further duality in between the two processes in the Sudbury–Lloyd sense [37] has been established since then, which shows that the law of CBSEP at a fixed time can be obtained as a p -thinning of the biased voter model (see [39, Exercise 3.6]). When $\beta = 0$ this model reduces to coalescing random walks, additive dual to the standard voter model, which have both been extensively studied (see e.g. [25, 26]).

When the graph is the d -dimensional hypercubic lattice, the first results were obtained by Bramson and Griffeath [6, 7]. In particular, they showed that the law of CBSEP converges weakly to its unique invariant measure starting from any non-empty set of particles and for any dimension d . Moreover, building on their work, Durrett and Griffeath [13] proved a shape theorem for this process, which easily implies that CBSEP on the discrete torus of side length $L \rightarrow \infty$ exhibits mixing time cutoff (but

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¹In fact, biased voter and Williams-Bjerknes models slightly differ on non-regular graphs. For such graphs CBSEP is the additive dual of the former.

without any control on the critical window). In the case of the regular tree a complete convergence result is due to Louidor, Tessler, Vandenberg-Rodes [27]. In the particular setting of \mathbb{Z} a key observation is that the rightmost (or leftmost) particle performs a biased random walk with explicit constant drift (see Griffeath [18]). For more advanced results see e.g. the work by Sun and Swart [38].

While the main focus of the above-mentioned works was the long-time behavior of the process on *infinite* graphs, our interest will concentrate instead on the mixing time for *finite* graphs. We determine the logarithmic Sobolev constant and relaxation time of the model quite precisely on a wide spectrum of relatively sparse finite graphs and for values of the branching rate β which are $o(1)$ as $|V| \rightarrow \infty$ (see Theorem 1 and Corollary 3.1). For instance, our results imply that for transitive bounded degree graphs the inverse of the logarithmic Sobolev constant and relaxation time when $\beta = 1/|V|$ are, up to a logarithmic correction, equal to the cover time of the graph. We will then use these results to strengthen and extend the findings of Pillai and Smith [31, 32] on the mixing time for the FA-1f kinetically constrained model in the same regime (see Corollary 3.2). Motivated by a different application to the kinetically constrained models FA- j f with $j > 1$ (see [20]), we then investigate a version of the model in which the single vertex state space $\{0, 1\}$ is replaced by an arbitrary finite set and we bound its mixing time (see Theorem 2).

1.1. The CBSEP and g -CBSEP models. Let $G = (V, E)$ be a finite connected graph with n vertices. The degree of $x \in V$ is denoted by d_x . Minimum, maximum, and average degrees in G are denoted by d_{\min}, d_{\max} and d_{avg} , respectively. For any $\omega \in \Omega = \{0, 1\}^V$ and any vertex $x \in V$ we say that x is *filled/empty* or that there is a *particle/hole* at x if $\omega_x = 1/0$. We define $\Omega_+ = \Omega \setminus \{0\}$ to be the event that there exists at least one particle. Similarly, for any edge $e = \{x, y\} \in E$ we refer to $(\omega_x, \omega_y) \in \{0, 1\}^{\{x, y\}}$ as the state of e in ω and write $E_e = \{\omega \in \Omega \mid \omega_x + \omega_y \neq 0\}$ for the event that e is not empty.

Given $p \in (0, 1)$ let $\pi = \otimes_{x \in V} \pi_x$ be the product Bernoulli measure, each vertex being filled with probability p and let $\mu(\cdot) := \pi(\cdot \mid \Omega_+)$. Given an edge $e = \{x, y\}$ we write $\pi_e := \pi_x \otimes \pi_y$ and $\lambda(p) := \pi(E_e) = p(2 - p)$. In the sequel we will always assume for simplicity that p is bounded away from 1 (e.g. $p \leq 1/2$).

CBSEP that will be the main object of this work is the continuous time Markov chain on Ω_+ for which the state of any edge $e \in E$ such that E_e occurs is resampled with rate one w.r.t. $\pi_e(\cdot \mid E_e)$. Thus, any edge containing exactly one particle with rate $(1 - p)/(2 - p)$ moves the particle to the opposite endpoint (the *SEP move*) and with rate $p/(2 - p)$ creates an extra particle at its empty endpoint (the *branching move*). Moreover, any edge containing two particles with rate $2(1 - p)/(2 - p)$ kills one of the two particles chosen uniformly (the *coalescing move*). The chain is readily seen to be reversible w.r.t. μ and ergodic on Ω_+ , because it can reach the configuration with a particle at each vertex. If $c(\omega, \omega')$ denotes the jump rate from ω to ω' , the Dirichlet form $\mathcal{D}(f)$ of the chain has the expression

$$\begin{aligned} \mathcal{D}(f) &= \frac{1}{2} \sum_{\omega, \omega'} \mu(\omega) c(\omega, \omega') (f(\omega') - f(\omega))^2 \\ (1) \quad &= \sum_{e \in E} \mu(\mathbb{1}_{E_e} \text{Var}_e(f \mid E_e)). \end{aligned}$$

It is also not hard to check that CBSEP is equivalent to the coalescing and branching random walks described in the introduction up to a global time-rescaling.

We will also consider a *generalised* version of CBSEP, in the sequel g -CBSEP, defined as follows. We are given a graph G as above together with a finite probability space (S, ρ) . We still write $\rho = \otimes_{x \in V} \rho_x$ for the product probability on $\Omega := V^S$. In the *state space* S we are given a bipartition $S_1 \sqcup S_0 = S$ and we write $p := \rho(S_1) \in (0, 1)$. We define the *projection* $\varphi : \Omega \rightarrow \{0, 1\}^V$ by $\varphi(\omega) = (\mathbb{1}_{\{\omega_x \in S_1\}})_{x \in V}$ and we let $\Omega_+ = \{\omega \in \Omega : \sum_x \varphi_x(\omega) \geq 1\}$. For any edge $e = \{x, y\} \in E$ we also let E_e be the event that there exists a particle at x or at y for $\varphi(\omega)$. In g -CBSEP every edge $e = \{x, y\}$ such that E_e is satisfied is resampled with rate 1 w.r.t. $\rho_x \otimes \rho_y(\cdot | E_e)$. A key property is that its projection chain onto the variables $\varphi(\omega)$ coincides with CBSEP. As for CBSEP, if we let $\rho_+ = \rho(\cdot | \Omega_+)$, the chain is reversible w.r.t. ρ_+ and ergodic on Ω_+ .

The above generalisation was introduced in [20] as a key tool in a sophisticated renormalisation scheme built to determine with very high precision the relaxation time of certain critical kinetically constrained models like the FA-2f KCM (also known as the 2-neighbour model, see Section 1.2). For instance, using Theorems 1 and 2, in [20] we prove that for the FA-2f model in \mathbb{Z}^2 at density p , the relaxation time $T_{\text{rel}}^{\text{FA2f}}$ (see Section 2 for background material on mixing times for reversible continuous time finite Markov chains) satisfies

$$(2) \quad T_{\text{rel}}^{\text{FA2f}} = \exp\left(\frac{\pi^2 + o(1)}{9p}\right), \quad \text{as } p \rightarrow 0.$$

1.2. The FA-1f KCM. We next define another class of models of interest – the j -neighbour or the Fredrickson-Andersen j -facilitated kinetically constrained spin model (FA- j f KCM for short). In the setting of Section 1.1 for CBSEP, these chains evolve as follows. With rate one and independently w.r.t. the other vertices, the state of each vertex $x \in V$ with at least j neighbouring particles is resampled w.r.t. π_x . In this paper we will focus on the simplest case $j = 1$, the case $j = 0$ being trivial. As for CBSEP it is immediate to check that, on Ω_+ , the chain is ergodic with μ as the unique reversible measure and that its Dirichlet form is

$$\mathcal{D}^{\text{FA}}(f) = \sum_x \mu \left(\mathbb{1}_{\{\sum_{\{x,y\} \in E} \omega_y \geq 1\}} p(1-p)(f(\omega^x) - f(\omega))^2 \right),$$

where ω^x denotes the configuration ω flipped at x .

The FA-1f KCM has been extensively studied (see e.g. [5, 9, 10, 14, 15]). Of particular relevance for us are the beautiful works of Pillai and Smith [31, 32] that we present next. For any positive integers d and L set $n = L^d$ and let $\mathbb{Z}_L = \{0, 1, \dots, L-1\}$ be the set of remainders modulo L . The d -dimensional discrete torus with n vertices, \mathbb{T}_n^d in the sequel, is the set \mathbb{Z}_L^d endowed with the graph structure inherited from \mathbb{Z}^d . For the *discrete time* version of FA-1f on \mathbb{T}_n^d with $p = c/n$ [31, 32] provide a rather precise bound for the (total variation) mixing time $T_{\text{mix}}^{\text{FA}}$. Translated into the continuous time setting described above, their results read

$$(3) \quad \begin{aligned} C^{-1}n^2 &\leq T_{\text{mix}}^{\text{FA}} \leq Cn^2 \log^{14}(n) & d = 2 \\ C^{-1}n^2 &\leq T_{\text{mix}}^{\text{FA}} \leq Cn^2 \log(n) & d \geq 3, \end{aligned}$$

where $C > 0$ may depend on d but not on n .

REMARK 1.1. In [31, Section 2] it was argued that $T_{\text{mix}}^{\text{FA}}$ should be lower bounded by the time necessary to get two well-separated particles starting from one. By reversibility this time should correspond to the meeting time T_{meet} of two independent random walks on \mathbb{T}_n^2 with random starting point. In two dimensions

$T_{\text{meet}} = \Theta(n \log(n))$. In particular, in [32, Remark 1.1] it was conjectured that $T_{\text{mix}}^{\text{FA}} = \Omega(p^{-1}n \log(n)) = \Omega(n^2 \log(n))$ in the regime $p = \Theta(1/n)$ and this was recently confirmed by Shapira [36, Theorem 1.3]. As it will be apparent in the proof of Theorem 1(d), this heuristics together with the attractiveness of CBSEP will allow us to prove a lower bound on the logarithmic Sobolev constant and relaxation time of CBSEP on a general graph.

1.2.1. Relationship between CBSEP and FA-1f. Notice that the branching and coalescing moves of CBSEP are exactly the moves allowed in FA-1f. Moreover, the SEP move for the edge $\{x, y\}$ from $(1, 0)$ to $(0, 1)$ can be reconstructed using two consecutive FA-1f moves, the first one filling the hole at y and the second one emptying x . If we also take into account the rate for each move, we easily get the following comparison between the respective Dirichlet forms. There exists an absolute constant $c > 0$ such that

$$(4) \quad c^{-1} \mathcal{D}^{\text{FA}}(f) \leq \mathcal{D}(f) \leq c d_{\max} p^{-1} \mathcal{D}^{\text{FA}}(f), \quad \forall f.$$

Although the two models are clearly closely related, we would like to emphasise that CBSEP has many advantages over FA-1f, making its study simpler. Most notably, CBSEP is *attractive* in the sense that there exists a grand-coupling (see e.g. [24]) which preserves the partial order on Ω given by $\omega \prec \omega'$ iff $\omega_x \leq \omega'_x$ for all $x \in V$ (as it can be readily verified via the construction of Section 4). Furthermore it is also natural to embed on CBSEP a simple random walk, a particularly fruitful feature which we will use in Section 6 and that is challenging to reproduce for FA-1f [5].

2. Preliminaries. In order to state our results we need first to recall some classical material on *mixing times* for finite Markov chains (see e.g. [21, 34]) and on the *resistance distance* on finite graphs (see [11, 40], [24, Ch. 9] and [28, Ch.2]).

Mixing times and logarithmic Sobolev constant. Given a finite positive probability space (Ω, μ) , let $\{\omega(t)\}_{t \geq 0}$ be a continuous time ergodic Markov chain on Ω reversible w.r.t. μ and write $P_\omega^t(\omega') = \mathbb{P}(\omega(t) = \omega' \mid \omega(0) = \omega)$. Let also $h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot)$ be the relative density of the law $P_\omega^t(\cdot)$ w.r.t. μ . The total variation mixing time of the chain, T_{mix} , is defined as

$$T_{\text{mix}} = \inf \left\{ t > 0, \max_{\omega \in \Omega_+} \|P_\omega^t(\cdot) - \mu(\cdot)\|_{\text{TV}} \leq 1/2e \right\},$$

where $\|\cdot\|_{\text{TV}}$ denotes the *total variation distance* defined as

$$\|P_\omega^t(\cdot) - \mu(\cdot)\|_{\text{TV}} = \frac{1}{2} \|h_\omega^t(\cdot) - 1\|_1,$$

where $\|g\|_\alpha^\alpha = \mu(|g|^\alpha)$, $\alpha \geq 1$. The ℓ^2 -mixing time T_2 or, more generally, the ℓ^q -mixing times T_q , $q \geq 1$, are defined by

$$T_q = \inf \left\{ t > 0, \max_{\omega \in \Omega_+} \|h_\omega^t(\cdot) - 1\|_q \leq 1/e \right\}.$$

Clearly $T_{\text{mix}} \leq T_q \forall q > 1$ and it is known that for all $1 < q \leq \infty$ the ℓ^q -convergence profile is determined entirely by that for $q = 2$ (see e.g. [34, Lemma 2.4.6]). Moreover, (see e.g. [34, Corollary 2.2.7]),

$$(5) \quad \frac{1}{2} \alpha^{-1} \leq T_2 \leq \alpha^{-1} \left(1 + \frac{1}{4} \log \log(1/\mu_*)\right),$$

where $\mu_* = \min_{\omega \in \Omega_+} \mu(\omega)$ and α is the *logarithmic Sobolev constant* defined as the inverse of the best constant C in the logarithmic Sobolev inequality valid for any $f : \Omega_+ \rightarrow \mathbb{R}$

$$(6) \quad \text{Ent}(f^2) := \mu(f^2 \log(f^2/\mu(f^2))) \leq C\mathcal{D}(f).$$

Finally, the *relaxation time* T_{rel} is then defined as the best constant C in the Poincaré inequality

$$(7) \quad \text{Var}(f) \leq C\mathcal{D}(f).$$

It is not difficult to prove that $T_{\text{rel}} \leq T_{\text{mix}}$ and that (see e.g. [12, Corollary 2.11])

$$(8) \quad 2T_{\text{rel}} \leq \alpha^{-1} \leq (2 + \log(1/\mu_*)) \times T_{\text{rel}}.$$

Notation warning. In the sequel, unless otherwise indicated, all the quantities introduced above will not carry any additional label when referring to CBSEP. On the contrary, the same quantities referring to other chains, e.g. the FA-1f KCM or g -CBSEP, will always carry an appropriate superscript.

The resistance distance. Given a finite connected simple graph $G = (V, E)$, let \vec{E} denote the set of *ordered pairs* of vertices forming an edge of E . For $\vec{e} = (u, v) \in \vec{E}$ we set $-\vec{e} = (v, u)$. Given an anti-symmetric function θ on \vec{E} (that is $\theta(\vec{e}) = -\theta(-\vec{e})$) and two vertices x, y we say that θ is a *unit flow from x to y* iff $\sum_{v:(u,v) \in \vec{E}} \theta((u, v)) = 0$ for all $u \notin \{x, y\}$ and $\sum_{v:(x,v) \in \vec{E}} \theta((x, v)) = 1$. The *energy of the flow θ* is the quantity $\mathcal{E}(\theta) = \frac{1}{2} \sum_{\vec{e} \in \vec{E}} \theta(\vec{e})^2$ and we set

$$(9) \quad \mathcal{R}_{x,y} = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } x \text{ to } y\}.$$

The Thompson principle states that the infimum in (9) is actually attained at a unique unit flow.

The quantity $\mathcal{R}_{x,y}$ can be interpreted as the effective resistance in the electrical network obtained by replacing the vertices of G with nodes and the edges with unit resistances. In graph theory it is sometimes referred to as the *resistance distance*. It is also connected to the behaviour of the simple random walk on G via the formula $2|E|\mathcal{R}_{x,y} = C_{x,y}$, where $C_{x,y}$ is the expected commute time between x, y . Furthermore, if we let $T_{\text{rel}}^{\text{rw}}$ be the relaxation time of the random walk, the bound $\max_{x,y} \mathcal{R}_{x,y} \leq c\sqrt{T_{\text{rel}}^{\text{rw}}/d_{\min}}$ holds [30, Corollary 1.1] (see also [1, Corollary 6.21] for regular graphs). Finally, by taking the shortest path between x, y and the flow θ which assigns unit flow to each edge of the path, $\mathcal{R}_{x,y} \leq d(x, y)$, where $d(\cdot, \cdot)$ is the graph distance, with equality iff x, y are linked by a unique path. In the sequel and for notation convenience we shall write $\bar{\mathcal{R}}_y$ for the spatial average $\frac{1}{n} \sum_x \mathcal{R}_{x,y}$.

REMARK 2.1. For later use we present bounds on $\max_y \bar{\mathcal{R}}_y$ for certain special graphs. If G is the d -hypercube ($n := |V| = 2^d$) it follows from [33] that $\bar{\mathcal{R}}_y = \Theta(1/\log n)$ for all $y \in V$. If instead G is the regular b -ary tree with $b \geq 2$ then $\max_y \bar{\mathcal{R}}_y = \Theta(\log n)$. If G is a uniform random d -regular graph with $n \rightarrow \infty$ and d independent of n then w.h.p. $T_{\text{rel}}^{\text{rw}} = O(1)$ [8, 16] and therefore w.h.p. $\bar{\mathcal{R}}_y = \Theta(1)$ for all $y \in V$. Finally, if G is the discrete d -dimensional torus $\mathbb{T}_n^d \subset \mathbb{Z}^d$ with n vertices then it follows from [28, Proposition 2.15] that

$$\max_y \bar{\mathcal{R}}_y = \Theta(1) \times \begin{cases} n & \text{if } d = 1, \\ \log(n) & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$$

3. Main results. Our first theorem contains upper and lower bounds for the inverse of the logarithmic Sobolev constant, α^{-1} , and relaxation time, T_{rel} , of CBSEP in the general setting described in the introduction.

Let T_{meet} denote the expected meeting time for two *continuous time* random walks jumping along each edge at rate 1 and started from two uniformly chosen vertices of G . We refer the reader to [1, 22] for the close connections between T_{meet} and $\mathcal{R}_{x,y}$. Let also $T_{\text{mix}}^{\text{rw}}$ denote the mixing time of the *discrete time* lazy simple random walk on G .

THEOREM 1. *Let $p_n \in (0, 1)$ and consider CBSEP with parameter p_n on a sequence of graphs $G = G_n = (V_n, E_n)$ with $|V_n| = n$, maximum degree $d_{\text{max}} = d_{\text{max}}(n)$, minimum degree $d_{\text{min}} = d_{\text{min}}(n)$, and average degree $d_{\text{avg}} = d_{\text{avg}}(n)$.*

(a) *If $p_n = \Omega(1)$, then*

$$(10) \quad \alpha^{-1} \leq O(n)$$

$$(11) \quad T_{\text{rel}} \leq O(1).$$

(b) *If $p_n \rightarrow 0$, then for some absolute constant $c > 0$*

$$(12) \quad \alpha^{-1} \leq c \max \left(\frac{d_{\text{avg}} d_{\text{max}}^2}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \log(n), \left(\max_y \bar{\mathcal{R}}_y \right) n |\log(p_n)| \right)$$

$$(13) \quad T_{\text{rel}} \leq cn \max_y \bar{\mathcal{R}}_y.$$

(c) *There exists an absolute constant $c > 0$ such that for all $p_n \in (0, 1)$*

$$(14) \quad \alpha^{-1} \geq \frac{cn}{d_{\text{avg}}}$$

$$(15) \quad T_{\text{rel}} \geq \frac{1 - \mu(\sum_x \omega_x = 1)}{pd_{\text{avg}}}.$$

(d) *If $p_n = O(1/n)$, then we have the stronger bound*

$$(16) \quad \alpha^{-1} \geq T_{\text{meet}} \Omega(1 + |\log(np_n)|)$$

$$(17) \quad T_{\text{rel}} \geq T_{\text{meet}} \Omega(1).$$

For reader's convenience and in view of our application to the FA-1f KCM we detail the above bounds for the graphs discussed in Remark 2.1 when $p_n = \Theta(1/n)$.

COROLLARY 3.1. *In the setting of Theorem 1 assume that $p_n = \Theta(1/n)$. Then:*

(1) *hypercube:*

$$\Theta \left(\frac{n}{\log n} \right) = 2T_{\text{rel}} \leq \alpha^{-1} \leq O(n),$$

(2) *regular b -ary tree, $b \geq 2$ independent of n :*

$$\Theta(n \log(n)) = 2T_{\text{rel}} \leq \alpha^{-1} \leq O(n \log^2(n)),$$

(3) *uniform random d -regular graph, d independent of n : w.h.p.*

$$\Theta(n) = 2T_{\text{rel}} \leq \alpha^{-1} \leq O(n \log(n)),$$

(4) *discrete torus* \mathbb{T}_n^d :

$$\alpha^{-1} \leq O(1) \times \begin{cases} n^2 \log(n) & d = 1, \\ n \log^2(n) & d = 2, \\ n \log(n) & d \geq 3, \end{cases}$$

and

$$\alpha^{-1} \geq 2T_{\text{rel}} = \Theta(1) \times \begin{cases} n^2 & d = 1, \\ n \log(n) & d = 2, \\ n & d \geq 3. \end{cases}$$

The corollary follows immediately from Theorem 1(b) and (d) together with Remarks 2.1, the well-known results on $T_{\text{mix}}^{\text{rw}}$ for each graph and the fact that (see [2, 11, 22]) for the graphs in Remark 2.1 it holds that

$$T_{\text{meet}} = \Theta(n) \max_y \bar{\mathcal{R}}_y.$$

Using Corollary 3.1 together with (5) and (4), we immediately get the following consequences for the FA-1f KCM to be compared with the r.h.s. of (3).

COROLLARY 3.2. *Consider the FA-1f KCM on $G = \mathbb{T}_n^d$ with parameter $p_n = \Theta(1/n)$ and let $T_{\text{mix}}^{\text{FA}}$ and T_2^{FA} be its mixing time and ℓ^2 -mixing time respectively. Then*

$$(18) \quad \begin{aligned} T_{\text{mix}}^{\text{FA}} &\leq T_2^{\text{FA}} \leq (\alpha^{\text{FA}})^{-1} \log(n) \leq cn \log(n) \alpha^{-1} \\ &\leq O(1) \times \begin{cases} n^3 \log^2(n) & d = 1 \\ n^2 \log^3(n) & d = 2 \\ n^2 \log^2(n) & d \geq 3. \end{cases} \end{aligned}$$

REMARK 3.3. Our results in $d \geq 2$, besides being more directly proved than in [31, 32], hold in the stronger logarithmic Sobolev sense and extend to other graphs, e.g. all the graphs discussed in Corollary 3.1. Furthermore, contrary to the approach followed in [31, 32], our methods can be easily adapted to cover other regimes of p_n . For $d = 1$ the above upper bound on $T_{\text{mix}}^{\text{FA}}$ can be proved to also be sharp up to logarithmic corrections, using the technique discussed in [9, Section 6.2].

Our second theorem concerns the total variation mixing time of the generalised model, g -CBSEP.

Let τ_{cov} denote the cover time of the simple random walk on G (see e.g. [24, Chap. 11] and also [11] for a close connection between the average cover time and the resistance distance), and let

$$T_{\text{cov}}^{\text{rw}} = \inf \left\{ t > 0, \max_{x \in V} \mathbb{P}_x(\tau_{\text{cov}} > t) \leq 1/e \right\}.$$

THEOREM 2. *Consider g -CBSEP on a finite connected graph G with parameter $p = \rho(S_1)$ and let T_{mix} be the mixing time of CBSEP on G with parameter p . Then there exists a universal constant $c > 0$ such that*

$$T_{\text{mix}} \leq T_{\text{mix}}^{g\text{-CBSEP}} \leq c(T_{\text{mix}} + T_{\text{cov}}^{\text{rw}}).$$

The main reason to bound the total variation mixing time of g -CBSEP instead of the ℓ^q -mixing times as for CBSEP is that the scaling of the logarithmic Sobolev constant for g -CBSEP is very different from that of the CBSEP as the following example shows.

EXAMPLE 3.4. Let $G = \mathbb{T}_n^2$, $p_n = 1/n$, $S = \{0, 1, 2\}$, and $\rho(1) = p$, $\rho(0) = \rho(2) = (1-p)/2$. Then,

$$(19) \quad (\alpha^{g\text{-CBSEP}})^{-1} = n^{3/2+o(1)}.$$

In the same setting Corollary 3.1 gives $\alpha^{-1} = O(n \log^2(n))$. To prove (19) it is enough to take as test function in the logarithmic Sobolev inequality for g -CBSEP the indicator that a vertical strip of width $\lfloor \sqrt{n}/2 \rfloor$ of the torus is in state 0.

3.1. Organization of the paper. The remainder of the paper is organized as follows. In Section 4 we introduce natural graphical representations of CBSEP and g -CBSEP. Sections 5 and 6 contain the proofs of Theorems 1 and 2, respectively.

4. Graphical construction. We will describe a graphical construction of g -CBSEP for all initial conditions. The graphical construction of CBSEP can then be immediately deduced by considering the special case $S_1 := \{1\}$ and $S_0 := \{0\}$.

To each edge $e \in E$ we associate a Poisson process of rate $p/(2-p)$ of arrival times $(t_n^e)_{n=1}^\infty$. Similarly, to each oriented edge $\vec{e} \in \vec{E}$ we associate a Poisson process of rate $(1-p)/(2-p)$ of arrival times $(t_n^{\vec{e}})_{n=1}^\infty$. All the above processes are independent as e, \vec{e} vary in E, \vec{E} respectively. Furthermore, for $e \in E$ and $n \geq 1$, we define X_n^e to be independent random variables taking values in S^2 and similarly for $\vec{e} \in \vec{E}$. We assume that for all n and $(u, v) \in \vec{E}$ the law of $X_n^{(u,v)}$ is $\rho_u(\cdot | S_1) \otimes \rho_v(\cdot | S_0)$. Similarly, for $\{u, v\} \in E$ the law of $X_n^{\{u,v\}}$ is $\rho_u(\cdot | S_1) \otimes \rho_v(\cdot | S_1)$. Given an initial configuration $\omega(0) \in \Omega = V^S$ and a realization of the above variables, we define the realization of g -CBSEP $\omega(t)$ as follows.

Fix $t \geq 0$, let t^* be the first arrival time after t and let $\{x, y\}$ be the endpoints of the edge where it occurs. We set $\omega_z(t^*) = \omega_z(t)$ for all $z \in V \setminus \{x, y\}$. If there is no particle at $\{x, y\}$ in $\varphi(\omega(t))$, we set $\omega(t^*) = \omega(t)$. Otherwise, we set $(\omega_x(t^*), \omega_y(t^*)) = X_n^{\{x,y\}}$ if $t^* = t_n^{\{x,y\}}$ and similarly for $(x, y) \in \vec{E}$.

OBSERVATION 4.1. Let $\omega(t)$ and $\omega'(t)$ be constructed using the same Poisson processes $(t_n^e)_{n=1}^\infty$, $(t_n^{\vec{e}})_{n=1}^\infty$ and variables X_n^e , $X_n^{\vec{e}}$ above, but with different initial conditions satisfying $\varphi(\omega(0)) = \varphi(\omega'(0))$. Fix $t \geq 0$ and define the set of updated vertices $\Xi_t = \bigcup e$, where the union is over all $e \in E$ such that there exists $0 \leq t^* \leq t$ and n such that $t^* \in \{t_n^e, t_n^{\vec{e}}, t_n^{-\vec{e}}\}$ and the event E_e occurs for $\omega(t^*)$. Then

- $\varphi(\omega(t)) = \varphi(\omega'(t))$,
- if $x \in \Xi_t$, then $\omega_x(t) = \omega'_x(t)$,
- if $x \in V \setminus \Xi_t$, then $\omega_x(t) = \omega_x(0)$.

In particular, for all $x \in V$ such that there exists $t_x \leq t$ such that $\varphi_x(\omega(t_x)) \neq \varphi_x(\omega(0))$ we have $\omega_x(t) = \omega'_x(t)$.

5. Proof of Theorem 1. For this section we work with CBSEP in the setting of Theorem 1 and abbreviate $p = p_n$. In the sequel c shall denote an absolute constant whose value may change from line to line.

5.1. *Upper bounds – Proof of Theorem 1(a) and 1(b)*. Let us first prove the easy upper bound Theorem 1(a), assuming that $p = \Omega(1)$. We know from [10, Theorem 6.4] that $T_{\text{rel}}^{\text{FA}} = O(1)$. Recalling (4) and the definition of the relaxation time, we get that for CBSEP $T_{\text{rel}} = O(1)$, yielding (11). By (8) this gives $\alpha^{-1} = O(n)$ and concludes the proof of Theorem 1(a).

The rest of this section is dedicated to the proof of the main upper bound – Theorem 1(b). The starting point is the following decomposition of the entropy of any $f : \Omega_+ \rightarrow \mathbb{R}$

$$(20) \quad \text{Ent}(f^2) = \mu \left(\text{Ent}(f^2 | N) \right) + \text{Ent} \left(\mu(f^2 | N) \right),$$

where $N(\omega) = \sum_{x \in V} \omega_x$ is the number of particles. The first term in the r.h.s. above is bounded from above using the logarithmic Sobolev constant of the SEP on G with a fixed number of particles.

PROPOSITION 5.1. *There exists an absolute constant $c > 0$ such that*

$$\mu \left(\text{Ent}(f^2 | N) \right) \leq c \log(n) \frac{d_{\text{avg}} d_{\text{max}}^2}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \mathcal{D}(f),$$

PROOF. Let

$$\mathcal{D}_G^{\text{SEP}}(f) = \frac{1}{2} \sum_{e \in E} \mu \left((f(\omega^e) - f(\omega))^2 \right),$$

where ω^e is the configuration obtained from ω by swapping the states at the endpoints of the edge e , denote the Dirichlet form of the symmetric simple exclusion process on G . Similarly let

$$\mathcal{D}_{K_n}^{\text{BL}}(f) = \frac{1}{2n} \sum_{e \in E(K_n)} \mu \left((f(\omega^e) - f(\omega))^2 \right)$$

be the Dirichlet form of the Bernoulli-Laplace process on the complete graph K_n . The main result of [3, Theorem 1] implies that²

$$\mathcal{D}_{K_n}^{\text{BL}}(f) \leq c \frac{2|E|}{n} \frac{d_{\text{max}}^2}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \mathcal{D}_G^{\text{SEP}}(f).$$

Using $2|E| = \sum_x d_x$ we get, in particular, that

$$\mathcal{D}_{K_n}^{\text{BL}}(f) \leq c \frac{d_{\text{avg}} d_{\text{max}}^2}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \mathcal{D}_G^{\text{SEP}}(f).$$

On other hand, the logarithmic Sobolev constant of the Bernoulli-Laplace process on K_n with $k \in \{1, \dots, n-1\}$ particles is bounded by $c \log n$ uniformly in k [23, Theorem 5]. Hence,

$$\mu \left(\text{Ent}(f^2 | N) \right) \leq c \log(n) \mathcal{D}_{K_n}^{\text{BL}}(f) \leq c \log(n) \frac{d_{\text{avg}} d_{\text{max}}^2}{d_{\text{min}}^2} T_{\text{mix}}^{\text{rw}} \mathcal{D}_G^{\text{SEP}}(f).$$

The proposition then follows using $p \leq 1/2$ and

$$\mathcal{D}_G^{\text{SEP}}(f) \leq \frac{(2-p)}{(1-p)} \mathcal{D}(f). \quad \square$$

²Actually the comparison result proved in [3] is much stronger, since it concerns weighted exchange processes on G and on K_n .

We now examine the term $\text{Ent}(\mu(f^2 | N))$ in the r.h.s. of (20). Let

$$g(k) := \mu(f^2 | N = k)^{1/2}$$

for $k \geq 1$, so that $\text{Ent}(\mu(f^2 | N)) = \text{Ent}_\gamma(g^2)$, where γ is the probability law of N on $\{1, \dots, n\}$. Clearly, γ is $\text{Bin}(n, p)$ conditioned to be positive, so that for any $2 \leq k \leq n$ we have

$$(21) \quad \gamma(k)(1-p)k = \gamma(k-1)p(n-k+1).$$

PROPOSITION 5.2. *There exists an absolute constant $c > 0$ such that*

$$\text{Ent}_\gamma(g^2) \leq c \log(1/p) \times p \sum_y \mu \left([f(\omega^y) - f(\omega)]^2 (1 - \omega_y) \right).$$

PROOF. The proof starts with a logarithmic Sobolev inequality for the γ w.r.t. a suitably chosen reversible death and birth process on $\{1, \dots, n\}$.

LEMMA 5.3. *There exists an absolute constant $c > 0$ such that for any $g : \{1, \dots, n\} \rightarrow \mathbb{R}$*

$$\text{Ent}_\gamma(g^2) \leq c \log(1/p) \times \sum_{k=2}^n \gamma(k) k [g(k) - g(k-1)]^2.$$

Leaving the tedious proof to the [Appendix](#), we move on to bounding the r.h.s. above.

CLAIM 5.4. *For any $2 \leq k \leq n$ we have*

$$(g(k) - g(k-1))^2 \leq \frac{A_k^2}{g^2(k-1) \vee g^2(k)},$$

where

$$A_k = \frac{1}{n-k+1} \sum_y \mu \left((1 - \omega_y) [f^2(\omega) - f^2(\omega^y)] \mid N = k-1 \right).$$

PROOF. We first observe that

$$(22) \quad [g(k) - g(k-1)]^2 \leq \frac{[g^2(k) - g^2(k-1)]^2}{g^2(k) \vee g^2(k-1)}.$$

Next we write

$$\begin{aligned} g^2(k-1) &= \sum_{\omega: N(\omega)=k-1} \frac{\mu(\omega)}{\gamma(k-1)} f^2(\omega) \\ &= \frac{1}{n-k+1} \frac{1}{\gamma(k-1)} \sum_y \sum_{\omega: N(\omega)=k-1} \mu(\omega) (1 - \omega_y) f^2(\omega). \end{aligned}$$

With the change of variable $\eta = \omega^y$ we get that the r.h.s. above is equal to

$$\begin{aligned} &\frac{1}{n-k+1} \frac{1}{\gamma(k-1)} \sum_y \sum_{\omega: N(\omega)=k-1} \mu(\omega) (1 - \omega_y) [f^2(\omega) - f^2(\omega^y)] \\ &\quad + \frac{\gamma(k)(1-p)}{p(n-k+1)\gamma(k-1)} \sum_y \sum_{\eta: N(\eta)=k} \frac{\mu(\eta)}{\gamma(k)} \eta_y f^2(\eta), \end{aligned}$$

the second line being equal to $g^2(k)$ by (21). In conclusion $g^2(k-1) = g^2(k) + A_k$ and the claim follows from (22). \square

CLAIM 5.5. For any $2 \leq k \leq n$ we have

$$A_k^2 \leq \frac{2(g^2(n) + g^2(n-1))}{n-k+1} \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \mid N = k-1 \right).$$

PROOF. Using $a^2 - b^2 = (a-b)(a+b)$ and twice the Cauchy-Schwarz inequality (once w.r.t. to $\mu(\cdot \mid N = k-1, \omega_y = 0)$ and a second time w.r.t. weighted spatial average

$$\text{Av}(g) = \frac{1}{n-k+1} \sum_y \mu((1 - \omega_y) \mid N = k-1) g(y)$$

we get

$$\begin{aligned} A_k^2 &\leq \frac{1}{n-k+1} \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \mid N = k-1 \right) \times \\ &\quad \frac{2}{n-k+1} \sum_y \mu \left([f^2(\omega) + f^2(\omega^y)] (1 - \omega_y) \mid N = k-1 \right). \end{aligned}$$

Inside the second factor in the above r.h.s. the term containing $f^2(\omega)$ is equal to $2\mu(f^2 \mid N = k-1) = 2g^2(k-1)$. Similarly, the term containing $f^2(\omega^y)$, after the change of variable $\eta = \omega^y$ and recalling (21), becomes equal to

$$\frac{2}{n-k+1} \frac{\gamma(k)k(1-p)}{p\gamma(k-1)} \mu(f^2(\eta) \mid N = k) = 2g^2(k). \quad \square$$

Combining Claims 5.4 and 5.5, we get that

$$(23) \quad (g(k) - g(k-1))^2 \leq \frac{2}{n-k+1} \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \mid N = k-1 \right).$$

Using (23) together with (21), we get

$$\begin{aligned} &\sum_{k=2}^n \gamma(k) k [g(k) - g(k-1)]^2 \\ &\leq \sum_{k=2}^n \frac{2k\gamma(k)}{n-k+1} \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \mid N = k-1 \right) \\ &= \frac{2}{1-p} p \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \right). \end{aligned}$$

Using the above bound together with Lemma 5.3 we get the statement of Proposition 5.2. \square

The final step in the proof of (12) is the following comparison between the quantity $p \sum_y \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \right)$ and the Dirichlet form $\mathcal{D}(f)$ using electrical networks. Recall the definition of the resistance distance and of $\max_y \bar{\mathcal{R}}_y$ given in Section 2

PROPOSITION 5.6.

$$p \sum_x \mu \left((f(\omega^x) - f(\omega))^2 (1 - \omega_x) \right) \leq 4n \max_{y \in V} \bar{\mathcal{R}}_y \times \mathcal{D}(f).$$

PROOF. We will identify $\omega \in \{0, 1\}^V$ with its set of particles $\{x \in V : \omega_x = 1\}$ and we set $F_\omega(u) := f(\omega \cup \{u\})$, $u \in V$. For each $\vec{e} = (u, v) \in \vec{E}$ we also write $\nabla_{\vec{e}} F_\omega := F_\omega(v) - F_\omega(u)$. Given $\omega \in \Omega_+$ let $y_\omega \in \omega$ be chosen arbitrarily and let θ_* be the optimal (i.e. with the smallest energy) unit flow from x to y_ω . By applying [28, Lemma 2.9] to the function F_ω and using the Cauchy-Schwarz inequality, we get that for any ω such that $\omega_x = 0$

$$\begin{aligned} (f(\omega^x) - f(\omega))^2 &= (F_\omega(x) - F_\omega(y_\omega))^2 \\ &= \left(\frac{1}{2} \sum_{\vec{e} \in \vec{E}} \theta_*(\vec{e}) \nabla_{\vec{e}} F_\omega \right)^2 \leq \mathcal{E}_{x, y_\omega} \times \frac{1}{2} \sum_{\vec{e} \in \vec{E}} (\nabla_{\vec{e}} F_\omega)^2. \end{aligned}$$

Hence,

$$\sum_x (f(\omega^x) - f(\omega))^2 (1 - \omega_x) \leq n \left(\max_y \bar{\mathcal{R}}_y \right) \times \frac{1}{2} \sum_{\vec{e}} (\nabla_{\vec{e}} F_\omega)^2.$$

We next transform the generic term in the sum above into a Dirichlet form term for CBSEP. For any $\vec{e} = (u, v) \in \vec{E}$ we have

$$\begin{aligned} p\mu(\omega) (\nabla_{\vec{e}} F_\omega)^2 &= \mu(\omega \cup \{u\}) \times \begin{cases} 0 & \{u, v\} \subset \omega \\ p(f(\omega \cup \{v\}) - f(\omega))^2 & u \in \omega \not\equiv v \\ (1-p)(f(\omega \cup \{v\}) - f(\omega \cup \{u\}))^2 & \{u, v\} \cap \omega = \emptyset. \end{cases} \end{aligned}$$

Comparing with the expression of $\mathcal{D}(f)$, (1), we get immediately that

$$\frac{1}{2} \sum_{\omega \in \Omega_+} p\mu(\omega) \sum_{\vec{e} \in \vec{E}} (\nabla_{\vec{e}} F_\omega)^2 \leq 4\mathcal{D}(f). \quad \square$$

This concludes the proof of Theorem (12). Turning to (13), Proposition 5.6 alone is enough to conclude. Indeed, using the two-block argument of [5, Lemma 6.6] (see also Lemma 6.5 and Proposition 6.2 therein) and the well-known fact that the variance w.r.t. a product measure is at most the average of the sum of variances over single spins (see e.g. [4, Chapter 1]), we get

$$\text{Var}(f) \leq cp \sum_x \mu \left((f(\omega^x) - f(\omega))^2 (1 - \omega_x) \right).$$

The desired bound (13) then follows from (4), (7) and Proposition 5.6.

5.2. *Lower bounds – Proof of Theorem 1(c) and 1(d).* Inject $f = \mathbb{1}_{\{N=1\}}$, the indicator of having exactly one particle, in the logarithmic Sobolev inequality (6). For $c > 0$ small enough we have

$$\frac{\text{Ent}(f^2)}{\mathcal{D}(f)} = \frac{\mu(N=1)|\log(\mu(N=1))|}{\frac{2|E|\mu(N=1)}{n} \cdot \frac{p}{2-p}} \geq \frac{|\log \mu(N=1)|}{pd_{\text{avg}}} \geq c \frac{n}{d_{\text{avg}}},$$

since $\mu(N=1) = np(1-p)^{n-1}/(1-(1-p)^n)$. To check the last inequality, one may distinguish the cases np sufficiently large/of order 1/sufficiently small. This proves (14). Using the same function, so that $\text{Var}(f) = \mu(N=1)(1-\mu(N=1))$, we obtain (15) in the same way. This concludes the proof of Theorem 1(c).

The rest of this section is dedicated to the proof of the main lower bound – Theorem 1(d), so we assume that $p_n = O(1/n)$. Let $\lambda_0 > 0$ be the smallest eigenvalue of the restriction to the event that there are at least two particles, $\{N \geq 2\}$, of $-\mathcal{L}$ where \mathcal{L} is the generator of CBSEP. By [17, Lemma 4.2, Equation (1.4)] we have that

$$\begin{aligned} \alpha^{-1} &\geq \lambda_0^{-1} |\log(\mu(N \geq 2))|, \\ T_{\text{rel}} &\geq \lambda_0^{-1} (1 - \mu(N \geq 2)), \end{aligned}$$

the second inequality being easy to check from the definition. It is well known (see e.g. [21, Section 3.4]) that

$$\lambda_0^{-1} \geq \mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau),$$

where τ is the first time when $N = 1$. Putting these together and recalling that $p_n = O(1/n)$, we obtain

$$\begin{aligned} \alpha^{-1} &\geq \mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau) |\log(\mu(N \geq 2))| \geq \mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau) \Omega(1 + |\log(np_n)|), \\ T_{\text{rel}} &\geq \mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau) \mu(N=1) \geq \mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau) \Omega(1). \end{aligned}$$

In turn, again using that $p_n = O(1/n)$, we get

$$\mathbb{E}_{\mu(\cdot|N \geq 2)}(\tau) \geq \mu(N=2|N \geq 2) \mathbb{E}_{\mu(\cdot|N=2)}(\tau) \geq \Omega(1) \mathbb{E}_{\mu(\cdot|N=2)}(\tau).$$

It is not hard to see that CBSEP stochastically dominates a process of coalescing random walks with birth rate 0, which we will call CSEP. Therefore, $\mathbb{E}_{\omega}(\tau) \geq \mathbb{E}_{\omega}^{\text{CSEP}}(\tau)$ for any $\omega \in \Omega_+$. Furthermore, CSEP started with two particles has the law of two independent continuous time random walks which jump along each edge with rate $(1-p)/(2-p)$ and coalesce when they meet. Hence, we obtain (16) and (17), concluding the proof of Theorem 1(d).

6. Mixing time of g -CBSEP – Theorem 2. In this section we prove Theorem 2. The lower bound is an immediate consequence of the fact that the projection chain on the variables $\varphi(\omega)$ coincides with CBSEP. For the upper bound, the key ingredient is the embedding into g -CBSEP of a suitable continuous time simple random walk $(W_t)_{t \geq 0}$ on G via the graphical representation.

Fix $t \geq 0$ and assume that $W_t = u$ for some $u \in V$. Let $t^* = \min\{t_n^{(v,u)} > t\}$, that is the first arrival time attempting to resample an edge of the form (v, u) to $(1, 0)$. Then we set $W_{t^*} = v$. It is clear that the law of $(W_t)_{t \geq 0}$ is that of a random walk jumping to a uniformly chosen neighbour at rate $d_u(1-p)/(2-p)$. Let $\mathbb{P}_v(\cdot)$ denote the law of the random walk starting at v and let σ_{cov} be the its cover time.³

³We use σ_{cov} to distinguish it from the cover time τ_{cov} of the *discrete time* simple random walk on G .

Fix an arbitrary order on V and, given a state $\omega \in \Omega_+$, let $W_0(\omega)$ be the position of the first particle and let $(W_t(\omega))_{t \geq 0}$ be the random walk above starting at $W_0(\omega)$. It is easy to verify that $\varphi(\omega_{W_t(\omega)}(t)) = 1$ for all t and ω , that is, there is always a particle at the position of the random walk. Let

$$\hat{d}(t) := \max_{\substack{\omega, \omega' \in \Omega_+ \\ \varphi(\omega) = \varphi(\omega')}} \|\rho_t^\omega - \rho_t^{\omega'}\|_{TV},$$

where ρ_t^ω denotes the law at time t of g -CBSEP starting at ω . Observation 4.1 then implies

$$(24) \quad \hat{d}(t) \leq \max_{v \in V} \mathbb{P}_v(\sigma_{\text{cov}} > t).$$

Given $\eta \in \{0, 1\}^V$ such that $\sum_x \eta_x \geq 1$ let $\nu^\eta = \rho(\cdot | \varphi(\omega) = \eta)$. It is easy to verify using Observation 4.1 that the law $\rho_t^{\nu^\eta}$ of g -CBSEP with initial law ν^η takes the form

$$\rho_t^{\nu^\eta}(\cdot) = \mu_t^\eta(\nu^{\eta_t}(\cdot)),$$

where the average is taken w.r.t. the law μ_t^η of the CBSEP η_t at time t with parameter $p = \rho(S_1)$ and starting point η . It follows then from the definition of total variation distance that

$$(25) \quad \max_{\eta} \|\rho_t^{\nu^\eta} - \rho_+\|_{TV} = \max_{\eta} \|\mu_t^\eta - \mu\|_{TV},$$

where μ is the reversible measure of CBSEP with parameter p .

In conclusion, by combining (24) and (25) we get

$$\begin{aligned} & \max_{\omega \in \Omega_+} \|\rho_t^\omega - \rho_+\|_{TV} \\ & \leq \max_{\eta: \sum_x \eta_x \geq 1} \left(\max_{\omega: \varphi(\omega) = \eta} \|\rho_t^\omega - \rho_t^{\nu^\eta}\|_{TV} + \|\rho_t^{\nu^\eta} - \rho_+\|_{TV} \right) \\ & \leq \max_{v \in V} \mathbb{P}_v(\sigma_{\text{cov}} > t) + \max_{\eta: \sum_x \eta_x \geq 1} \|\mu_t^\eta - \mu\|_{TV}. \end{aligned}$$

The conclusion of the theorem now follows immediately from a standard comparison between the cover time σ_{cov} of W_t and the discrete time simple random walk on G .

APPENDIX: PROOF OF LEMMA 5.3

Recall that

$$\gamma(k) = \binom{n}{k} \frac{p^k}{(1-p)^k} \frac{(1-p)^n}{1 - (1-p)^n}$$

and consider the birth and death process on $\{1, \dots, n\}$ reversible w.r.t. γ with Dirichlet form

$$\mathcal{D}_\gamma(g) = \sum_{k=2}^n \gamma(k) k [g(k) - g(k-1)]^2,$$

corresponding to the death rates $c(1, 0) = 0$ and

$$\begin{aligned} c(k, k-1) &= k & k &= 2, \dots, n \\ c(k, k+1) &= (n-k) \frac{p}{1-p} & k &= 1, \dots, n-1, \end{aligned}$$

Let $m = \lceil pn \rceil$ and let $i = \max(2, m)$. Using [29, Proposition 4] (see also [42]) the logarithmic Sobolev constant of the above chain is bounded from above, up to an absolute multiplicative constant, by the number $C_* = C_- \vee C_+$, where

$$(26) \quad \begin{aligned} C_+ &= \max_{j \geq i+1} \left(\sum_{k=i+1}^j \frac{1}{\gamma(k)c(k, k-1)} \right) \gamma(N \geq j) |\log(\gamma(N \geq j))|, \\ C_- &= \max_{j \leq i-1} \left(\sum_{k=j}^{i-1} \frac{1}{\gamma(k)c(k, k+1)} \right) \gamma(N \leq j) |\log(\gamma(N \leq j))|. \end{aligned}$$

Assume first that $i = m$ and let us start with C_+ . For $\ell \geq 1$ write $a_\ell = \frac{1}{\gamma(m+\ell)(m+\ell)}$ and $S_k = \sum_{\ell=1}^k a_\ell$. We have

$$\frac{a_{\ell+1}}{a_\ell} = \frac{1-p}{p} \frac{m+\ell}{n-m-\ell} \geq 1,$$

from which it follows that for $0 < \delta < 1$ we have

$$(27) \quad \begin{aligned} \frac{a_{\ell+1}}{a_\ell} &= 1 + \Theta(\ell/m) = e^{\Theta(\ell/m)} & \ell \leq m, \\ \frac{a_{\ell+1}}{a_\ell} &\geq 1 + \frac{\delta}{3} & \ell \geq \delta m. \end{aligned}$$

In particular, for any two integers $s \leq t \leq m$ such that $t - s \geq \min(\sqrt{m}, m/s)$, it holds that for some absolute constant $\beta > 1$

$$(28) \quad \frac{a_t}{a_s} = \prod_{\ell=s}^{t-1} \frac{a_{\ell+1}}{a_\ell} = e^{\Theta((t-s)t/m)} \geq \beta.$$

We first analyse the behaviour of $S_k \gamma(\geq m+k) |\log(\gamma(N \geq m+k))|$ for $k \leq \varepsilon m$ where $\varepsilon > 0$ is a constant chosen sufficiently small depending on β .

LEMMA A.1. *There exists a constant $c > 0$ such that for $\varepsilon > 0$ small enough and $k \leq \varepsilon m$ we have*

$$S_k \gamma(\geq m+k) |\log(\gamma(N \geq m+k))| \leq c$$

PROOF. Define recursively

$$k_0 = 1, \quad k_1 = \lceil \sqrt{m} \rceil, \quad k_{t+1} = k_t + \lceil m/k_t \rceil,$$

and let T be the first index such that $k_T \geq \delta m$. Using (28) together with $a_{\ell+1} \geq a_\ell$, $k_{t+1} - k_t \leq k_t - k_{t-1}$, and $k_t/m \leq \delta$, we claim that for any $2 \leq t \leq T-1$

$$(29) \quad \begin{aligned} \frac{(S_{k_{t+1}} - S_{k_t})}{(S_{k_t} - S_{k_{t-1}})} &= \frac{\sum_{\ell=k_t+1}^{k_{t+1}} a_\ell}{\sum_{\ell=k_{t-1}+1}^{k_t} a_\ell} \geq \beta \frac{k_{t+1} - k_t}{k_t - k_{t-1}} \\ &\geq \beta \left(\frac{k_t}{k_{t-1}} + \frac{k_t}{m} \right)^{-1} \geq \beta \left(\frac{k_t}{k_{t-1}} + \delta \right)^{-1}. \end{aligned}$$

To prove the claim observe that for any positive non-decreasing sequence $(a_j)_{j=1}^\infty$ and integers $m \leq n$

$$\begin{aligned}
\frac{a_{n+1} + \cdots + a_{n+m}}{a_1 + \cdots + a_n} &\geq \min_j \left(\frac{a_{j+m}}{a_j} \right) \left(\frac{a_{n-m+1} + \cdots + a_n}{a_1 + \cdots + a_n} \right) \\
&\geq \min_j \left(\frac{a_{j+m}}{a_j} \right) \left(\frac{\sum_{j=n-m+1}^n a_j}{(n-m)a_{n-m} + \sum_{j=n-m+1}^n a_j} \right) \\
&\geq \min_j \left(\frac{a_{j+m}}{a_j} \right) \frac{m}{n},
\end{aligned}$$

because $\sum_{j=n-m+1}^n a_j \geq a_{n-m}m$.

If now ε, t are chosen small enough and large enough respectively depending on the constant β above, the r.h.s. of (29) is greater than e.g. $\beta^{1/2} > 1$. In other words, the sequence $((S_{k_{t+1}} - S_{k_t}))_{t=t_0}^T, t_0 \gg 1$, is exponentially increasing. Thus, for $k_t \leq k < k_{t+1}, t_0 \leq t < T$, we get that, for some constant c allowed to depend on β and t_0 and to change from line to line, we have

$$\begin{aligned}
S_k &\leq \sum_{s=t_0}^{t+1} (S_{k_s} - S_{k_{s-1}}) + S_{t_0} \leq c(S_{k_{t+1}} - S_{k_t}) + S_{t_0} \\
&\leq c \frac{k_{t+1} - k_t}{m\gamma(m+k_{t+1})} \leq c \frac{k_{t+1} - k_t}{m\gamma(m+k)}.
\end{aligned}$$

If instead $k_t \leq k < k_{t+1}, 0 \leq t < t_0$, we directly have that

$$S_k \leq ka_k \leq c \frac{k_{t+1} - k_t}{m\gamma(m+k)}.$$

Using the bounds

$$\gamma(N \geq m+k) \leq c \frac{m+k}{k} \gamma(m+k), \quad |\log(\gamma(N \geq m+k))| \leq ck^2/m,$$

we finally get that $k_t \leq k < k_{t+1}, t < T$,

$$S_k \gamma(\geq m+k) |\log(\gamma(N \geq m+k))| \leq c \frac{(k_{t+1} - k_t)k_{t+1}}{m} \leq c. \quad \square$$

We next consider the easier case, $k \geq \delta m$. By (27), for c large enough depending on δ and allowed to change from line to line, we have that $S_k \leq S_{k_T} + ca_k \leq ca_k$ and $\gamma(N \geq m+k) \leq c\gamma(m+k)$. Thus, for $k \geq \delta m$ we have that

$$\begin{aligned}
S_k \gamma(N \geq m+k) |\log(\gamma(N \geq m+k))| \\
\leq \frac{c}{m+k} |\log(\gamma(N \geq m+k))| \leq c \log(1/p),
\end{aligned}$$

since for all k we trivially have $\gamma(m+k) \geq p^{m+k}$. In conclusion, we have proved that $C_+ \leq O(\log(1/p))$ if $m \geq 2$. If instead $m = 1$, then the very same computations still give $C_+ \leq O(\log(1/p))$, Lemma A.1 being void.

The bound of C_- follows the same pattern. If $m = O(1)$, the reader may readily check that $C_- = O(1)$ because all terms in (26) are $O(1)$. If instead $m \gg 1$, we still obtain $C_- = O(1)$, concluding the proof of Lemma 5.3.

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