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Simple k-planar graphs are simple $(k+1)$-quasiplanar ☆

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A B S T R A C T

A simple topological graph is k-quasiplanar ($k \geq 2$) if it contains no k pairwise crossing edges, and k-planar if no edge is crossed more than k times. In this paper, we explore the relationship between k-planarity and k-quasiplanarity to show that, for $k \geq 2$, every k-planar simple topological graph can be transformed into a $(k+1)$-quasiplanar simple topological graph.

☆ Preliminary versions of the results presented in this paper appeared at WG 2017 [6] and MFCS 2017 [19].

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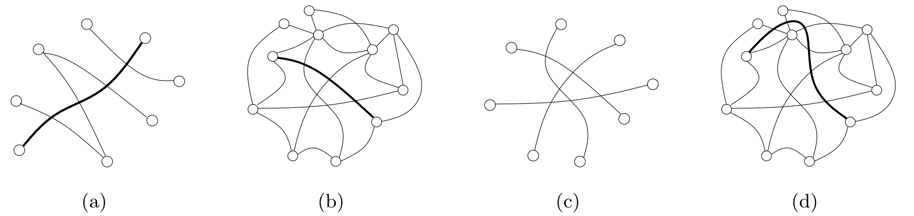


Fig. 1. (a) A crossing configuration that is forbidden in a 3-planar topological graph. (b) A 3-planar topological graph. (c) A crossing configuration that is forbidden in a 4-quasiplanar topological graph. (d) A 4-quasiplanar topological graph obtained from the one of Figure (b) by suitably rerouting the thick edge.

1. Introduction

A *topological graph* is a graph drawn in the plane such that vertices are mapped to distinct points and each edge is mapped to a Jordan arc between its endpoints without passing through any other vertex. In this paper, we only deal with topological graphs containing neither multi-edges nor self-loops. We do not distinguish between the vertices (resp., edges) and the points (resp., arcs) they are mapped to. A topological graph is *simple* if any two edges intersect in at most one point, which is either a common endpoint or a proper crossing.

A topological graph is *k-planar*, for $k \geq 0$, if each edge is crossed at most k times, and *k-quasiplanar*, for $k \geq 2$, if there are no k pairwise crossing edges. A graph is *k-planar* (*k-quasiplanar*) if it is isomorphic to the underlying abstract graph of a k-planar (k-quasiplanar) topological graph.

A graph is *simple k-planar* (*simple k-quasiplanar*) if it is isomorphic to the underlying abstract graph of a simple k-planar (k-quasiplanar) topological graph. By definition, a graph is planar if and only if it is 0-planar (2-quasiplanar). Note that 3-quasiplanar graphs are also called *quasiplanar*. Refer to Fig. 1 for examples.

The k-planar and k-quasiplanar graphs are part of a family of classes of topological graphs that are defined by restrictions on crossings. Informally, these classes are called *beyond planar*, we refer the interested reader to [15]. Further popular classes are defined by the exclusion of (natural or radial) grids [3,21], including fan-planar [20] and fan-crossing free graphs [14].

Primarily, the edge density of graphs has been studied for these classes. By Euler's polyhedron formula, every planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges, and this bound is tight. In fact, for constant $k \geq 0$, every k-planar graph is sparse. Pach and Tóth [25] proved that a k-planar graph with n vertices has at most $4.108\sqrt{k}\,n$ edges.[1] For simple k-planar graphs, where $k \leq 4$, Pach and Tóth [25] also established a finer bound of $(k+3)(n-2)$, and proved that this bound is tight for $k \leq 2$. For $k = 3$ and for $k = 4$, the best known upper bounds on the number of edges are $5.5n - 11$ and $6n - 12$,

[1] The upper bound was stated for k-planar *simple* topological graphs in [25], but the proof extends verbatim to all k-planar topological graphs.

respectively, which are tight up to small additive constants [2,8,22]. A consequence of the result in [2] is that the upper bound for k-planar graphs can be improved to $3.81\sqrt{k}\,n$.

Concerning k-quasiplanar graphs, a 20-year-old conjecture by Pach, Shahrokhi, and Szegedy asserts that for every $k \geq 2$ there is a constant c_k such that every k-quasiplanar graph with n vertices has at most $c_k n$ edges [24]. However, the conjecture has only been settled for $k = 2, 3, 4$. Agarwal et al. [5] were the first to prove that simple 3-quasiplanar graphs have a linear number of edges. This was generalized by Pach et al. [23], who proved that *every* 3-quasiplanar graph on n vertices has at most $65n$ edges. This bound was further improved to $8n-O(1)$ by Ackerman and Tardos [4]. For simple 3-quasiplanar graphs they also proved a bound of $6.5n - 20$, which is tight up to an additive constant. Ackerman [1] proved that 4-quasiplanar graphs have at most a linear number of edges. For $k \geq 5$, several authors have shown super-linear upper bounds on the number of edges in k-quasiplanar graphs (see, e.g., [13,17,18,24,27]). The most recent results are due to Suk and Walczak [26], who proved that every k-quasiplanar simple topological graph on n vertices has at most $c'_k n \log n$ edges, where c'_k depends only on k. For k-quasiplanar topological graphs where two edges can cross in at most t points, they give an upper bound of $2^{\alpha(n)^c} n \log n$, where $\alpha(n)$ is the inverse of the Ackermann function, and c depends only on k and t.

Note that every simple k-planar graph is simple $(k + 1)$-planar, and every simple k-quasiplanar graph is simple $(k+1)$-quasiplanar, by definition. It is not difficult to see that the converse is false in both cases. This naturally defines a hierarchy of k-planarity and a hierarchy of k-quasiplanarity. However, the relation between the hierarchies is not fully understood. For every $k \geq 3$, there are infinitely many simple 3-quasiplanar graphs that are not simple k-planar [7]. Also, it is easy to see that, for $k \geq 1$, every k-planar simple topological graph is $(k + 2)$-quasiplanar. Indeed, if a k-planar simple topological graph G were not $(k+2)$-quasiplanar, it would have $k+2$ pairwise crossing edges, each of which crosses at least $k + 1$ other edges, thus contradicting the hypothesis that G is k-planar.

Contribution. In this paper we focus on simple topological graphs and prove a notable inclusion relationship between the k-planarity and the k-quasiplanarity hierarchies. The proof is constructive and sheds a new light on the structure of k-planar and k-quasiplanar simple topological graphs. We show that every simple k-planar graph is simple $(k + 1)$-quasiplanar for every $k \geq 2$. More precisely, we show that a k-planar simple topological graph, with $k \geq 2$, can be transformed into an isomorphic simple topological graph that contains no $k + 1$ pairwise crossing edges (although an edge may be crossed more than k times). For example, the simple topological graph in Fig. 1b is 3-planar but not 4-quasiplanar. By rerouting an edge, we obtain the simple topological graph in Fig. 1d, which is 4-quasiplanar (but not 3-planar). Note that this result cannot be extended to the case $k = 1$, as a 2-quasiplanar graph is planar.

The proof of our result is based on the following novel methods: *(i)* A general-purpose technique to “untangle” a set of pairwise crossing edges. More precisely, we show how to reroute the edges of a k-planar simple topological graph in such a way that all vertices of a set of $k + 1$ pairwise crossing edges lie in the same connected region of the plane (that is, a face of the arrangement induced by the edges). *(ii)* A global edge rerouting technique, whose main ingredients are a matching argument and a systematic study of the cycles in an auxiliary "conflict" graph, used to remove all forbidden configurations of $(k + 1)$ pairwise crossing edges from a k-planar simple topological graph, provided that these edges are "untangled."

Paper organization. The remainder of the paper is structured as follows. In Section 2 we give some basic terminology, we describe the untangling procedure in *(i)*, and we prove important properties of the resulting topological graphs. Section 3 outlines our general proof strategy. Section 4 shows how to compute a global rerouting as described in *(ii)* that results in a $(k + 1)$-quasiplanar topological graph, for $k \geq 3$. Section 5 proves properties of a rerouted topological graph that are useful to prove both $(k+1)$-quasiplanarity when $k = 2$ and simplicity in Sections 6 and 7, respectively. In particular, Section 6 also contains a more sophisticated argument to compute a suitable global rerouting when $k = 2$. Conclusions and open problems are in Section 8.

2. Basic tools and properties

We first state further basic definitions and notation that will be used throughout the paper. As already stated, we only consider graphs with neither parallel edges nor self-loops. Also, we assume our graphs to be connected, as our results immediately carry over to disconnected graphs. In notation and terminology, we do not distinguish between the vertices (edges) of a topological graph and the points (Jordan arcs) representing them. Recall that a topological graph is simple if any two edges share at most one point, which is either a common endpoint or a proper crossing. A topological graph is *almost simple* if any two edges share at most one internal point, which is a proper crossing (i.e., pairs of adjacent edges may cross but at most once). For a topological graph G, the set $\mathbb{R}^2 \setminus G$ is open, and its connected components are called *faces*. The unique unbounded face is the *outer face*, any bounded face is an *inner face*. Note that the boundary of a face can contain vertices of the graph and crossing points between edges.

For two graphs or topological graphs, G and G', we write $G \simeq G'$ if they are isomorphic or their underlying abstract graphs are isomorphic. A graph G is k-planar (k-quasiplanar) if there exists a k-planar (k-quasiplanar) topological graph G' such that $G \simeq G'$.

Let $G = (V, E)$ be a simple topological graph and let $k \geq 2$ be an integer. A *fan* of G is a set of edges that share a common endpoint. A set $X \subset E$ of k pairwise crossing edges is called a *k-crossing*. Note that the edges in X are pairwise non adjacent since G is a simple topological graph. For a k-crossing X, denote by $\mathrm{V}(X)$ the set of $2k$ endpoints of the k edges in X. The *arrangement of* X, denoted by A_X, is the arrangement of the Jordan arcs in X. A *node* of A_X is either a vertex or a crossing point of two edges in X. A *segment* of A_X is a part of an arc in X between two consecutive nodes (i.e., a maximal uncrossed part of an edge in X). A k-crossing X is *untangled* if in the arrangement A_X all $2k$ vertices in $\mathrm{V}(X)$ are incident to a common face. Otherwise, it is *tangled*. For example,

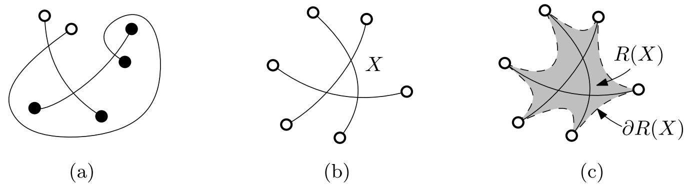


Fig. 2. (a) A tangled 3-crossing; the circled vertices and the solid vertices belong to different faces of the arrangement. (b) An untangled 3-crossing; all vertices belong to the same face of the arrangement (the outer face). (c) The 6-gon spanned by the 3-crossing in (b).

the 3-crossing in Fig. 2a is tangled, whereas the 3-crossing in Fig. 2b is untangled. We observe the following.

Property 1. *Let $G = (V, E)$ be a k-planar simple topological graph and let X be a $(k+1)$-crossing in G. An edge in X cannot be crossed by any other edge in $E \setminus X$. Consequently, for any two distinct $(k+1)$-crossings X and Y in G, we have $X \cap Y = \emptyset$.*

Proof. Each edge e in a $(k+1)$-crossing X crosses each of the remaining k edges in X. Since graph G is k-planar, edge e is not crossed by any other edge in $E \setminus X$. □

In the next subsection we show that tangled $(k+1)$-crossings can always be removed.

2.1. Eliminating tangled $(k+1)$-crossings

The proof of the next lemma describes how to "untangle" all $(k+1)$-crossings in a k-planar simple topological graph. This method is of general interest, as it gives more insights on the structure of k-planar simple topological graphs.

Lemma 1. *Let G be a k-planar simple topological graph. There exists a k-planar simple topological graph G', $G' \simeq G$, without tangled $(k+1)$-crossings.*

Proof. We first show how to untangle a $(k+1)$-crossing X in a k-planar simple topological graph G while neither creating new $(k+1)$-crossings nor introducing new crossings.

Let X be a tangled $(k+1)$-crossing and let A_X be its arrangement. For each face f of A_X, denote by V_f the set of nodes in $\mathrm{V}(X)$ incident to f. Since every vertex in V_X is incident precisely to one arc in X, the set $\mathrm{V}(X)$ is partitioned into subsets V_f over all faces f of A_X.

For every face f of A_X, denote by G_f the subgraph of G consisting of the vertices of V_f, and of all vertices and edges of G that lie in the interior of f. Refer to Fig. 3a for an illustration. By Property 1, every edge in $E \setminus X$ lies in a face of A_X. Consequently, the topological graph $(V, E \setminus X)$ is the disjoint union of the graphs G_f, each of which is k-planar.

For every inner face f, there is a region $D_f \subset f$ homeomorphic to an open disk such that its boundary contains V_f, and all other vertices and edges in G_f lie in $\operatorname{int} D_f$. For

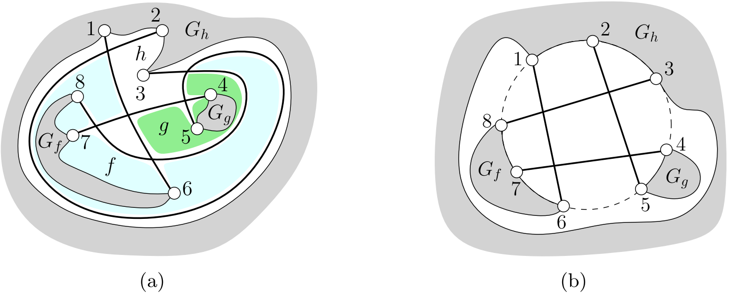


Fig. 3. Illustration of the untangling procedure in the proof of Lemma 1: (a) A 3-planar simple topological graph with a 4-crossing X (thicker edges). (b) The topological graph resulting from the procedure that untangles X.

the outer face h, there is a region D_h homeomorphic to the complement of a closed disk such that its boundary contains V_h, and all other vertices and edges lie in $\operatorname{int} D_h$.

We construct a topological graph G', $G' \simeq G$, as follows (see Fig. 3b). Let C be a circle in the plane. For every face f of A_X (including the outer face), apply a homeomorphism that maps the region D_f to some region in the exterior of C such that a Jordan arc in ∂D_f that contains V_f is mapped into C, and resulting regions are pairwise disjoin. Draw the $k+1$ edges in X as straight-line segments in the interior of C. Each subgraph G_f of G is mapped to a k-planar topological graph G'_f, $G'_f \simeq G_f$. The $(k+1)$-crossing X is mapped to a set X' of $k+1$ edges that is either not a $(k+1)$-crossing or an untangled $(k+1)$-crossing. Since two edges in G' cross only if the corresponding edges cross in G, the topological graph G' is simple and k-planar, and no new $(k+1)$-crossing is created.

Successively apply the above transformation as long as it contains a tangled $(k+1)$-crossing. Since the number of tangled $(k+1)$-crossings decreases, we eventually obtain a k-planar topological graph G', $G' \simeq G$, without tangled $(k+1)$-crossings. □

2.2. *Properties of untangled* $(k+1)$*-crossings*

Let G_0 be a simple k-planar graph. We wish to show that G_0 is simple $(k+1)$-quasiplanar. We may assume that G_0 is *edge-maximal* (i.e., the addition of any edge would yield a graph that is not simple k-planar). Let G be a simple k-planar topological graph such that $G \simeq G_0$. We may assume that G is *crossing minimal* (i.e., G has the minimum number of edge crossings over all k-planar simple topological graphs isomorphic to G_0), and that every $(k+1)$-crossing is untangled by Lemma 1. We may assume, by applying a projective transformation if necessary, that for every $(k+1)$-crossing X, all vertices of $V(X)$ are incident to the outer face of A_X.

Then every (untangled) $(k+1)$-crossing X in G spans a (topological) $2(k+1)$-*gon* in the following sense. All $2(k+1)$ vertices of $\mathrm{V}(X)$ lie on a face f_X of the arrangement A_X induced by the edges of X as drawn in G. Any two vertices of $\mathrm{V}(X)$ that are consecutive along the boundary of f_X can be connected by a Jordan arc that closely follows the boundary of f_X and does not cross any edge in G; see Fig. 2c. Together these arcs form a closed Jordan curve, which partitions the plane into two connected regions: let $\mathrm{R}(X)$ denote the closed region homeomorphic to a disk that contains the edges of X, and let $\partial\,\mathrm{R}(X)$ denote the boundary of $\mathrm{R}(X)$. We think of $\partial\,\mathrm{R}(X)$ as both a closed Jordan curve and as a topological graph that is a $2(k+1)$-cycle. Let $\mathcal{X}$ be the set of all $(k+1)$-crossings of G. By Property 1, we may assume that for every $X, X' \in \mathcal{X}$, $X \neq X'$, the regions $\mathrm{R}(X)$ and $\mathrm{R}(X')$ do not share any interior point. The following observation holds.

Property 2. *For each $X \in \mathcal{X}$, every pair of consecutive vertices of the $2(k+1)$-cycle $\partial\,\mathrm{R}(X)$ are connected by an edge in G, which is crossing-free.*

Proof. Let $X \in \mathcal{X}$, and let $u, v \in V$ be two consecutive vertices of the $2(k+1)$-cycle $\partial\,\mathrm{R}(X)$. We show that uv is an edge in G. Indeed, if uv is not an edge of G, we can augment G by drawing this edge as a crossing-free Jordan arc along $\partial\,\mathrm{R}(X)$ (without violating k-planarity). This contradicts our assumption that G is edge-maximal, and thus proves that uv is an edge in G.

We then show that uv is crossing free in G. Indeed, if it crossed any other edge in G, we could redraw it as a crossing-free Jordan arc along $\partial\,\mathrm{R}(X)$, obtaining a topological graph that is still k-planar but with fewer crossings than G. This contradicts our assumption that G is crossing minimal. □

By Property 2 any two consecutive vertices along the boundary $\partial\,\mathrm{R}(X)$ of a $2(k+1)$-gon $\mathrm{R}(X)$ are connected by an edge e in G. Note that this does not necessarily imply that e is drawn along $\partial\,\mathrm{R}(X)$. It is possible that the cycle formed by the edge e in G and the portion of $\partial\,\mathrm{R}(X)$ connecting the endpoints of e contains other parts of the graph.

Property 3.

(a) Let $X_1, X_2 \in \mathcal{X}$ such that $X_1 \neq X_2$. Then $\mathrm{V}(X_1)$ and $\mathrm{V}(X_2)$ share at most $2k+1$ vertices.
(b) Let X_1, X_2, and X_3 be three pairwise distinct $(k+1)$-crossings in $\mathcal{X}$. Then $\mathrm{V}(X_1)$, $\mathrm{V}(X_2)$, and $\mathrm{V}(X_3)$ share at most two vertices.

Proof. (a) Suppose that $\partial\,\mathrm{R}(X_1)$ and $\partial\,\mathrm{R}(X_2)$ share $2k+2$ vertices. Since $\mathrm{R}(X_1)$ and $\mathrm{R}(X_2)$ are contractible and interior-disjoint, the counterclockwise order of the vertices along $\partial\,\mathrm{R}(X_1)$ and $\partial\,\mathrm{R}(X_2)$, respectively, are reverse to each other. Every edge in X_i, for $i \in \{1, 2\}$, connects antipodal points along $\partial\,\mathrm{R}(X_i)$. Antipodal pairs are invariant under reversal, consequently every edge in X_1 is present in X_2, contradicting our assumption that G is a simple graph.

(b) Suppose that $\partial\,\mathrm{R}(X_1)$, $\partial\,\mathrm{R}(X_2)$, and $\partial\,\mathrm{R}(X_3)$ share three distinct vertices v_1, v_2, v_3. We obtain a plane drawing of $K_{3,3}$ as follows: Place points p_1, p_2, p_3 inside $\mathrm{R}(X_1)$, $\mathrm{R}(X_2)$, $\mathrm{R}(X_3)$, respectively, and connect each of p_1, p_2, p_3 to all of v_1, v_2, v_3. All edges incident to p_i, for $i \in \{1, 2, 3\}$, are drawn as a plane star inside $\mathrm{R}(X_i)$. As the

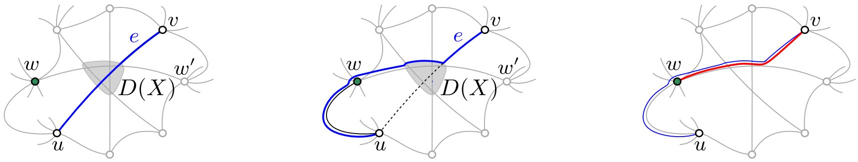


Fig. 4. The rerouting operation for dissolving untangled $(k+1)$-crossings. (a) An untangled $(k+1)$-crossing X. (b) The rerouting of edge uv around the marked vertex w. (c) The additional rerouting of vw.

regions $\mathrm{R}(X_1)$, $\mathrm{R}(X_2)$, and $\mathrm{R}(X_3)$ are interior-disjoint, no two edges cross. As $K_{3,3}$ is nonplanar, we obtain a contradiction. $\square$

3. Edge rerouting operations and proof strategy

We introduce an edge rerouting operation that will be crucial for our proof strategy. Let G be a k-planar simple topological graph in which all $(k+1)$-crossings are untangled (Lemma 1). Let X be a $(k+1)$-crossing in G. Without loss of generality, the vertices in $\mathrm{V}(X)$ lie in the outer face of A_X.

Let $e = uv \in X$ and let $w \in \mathrm{V}(X) \setminus \{u, v\}$ such that u and w are consecutive along $\partial\,\mathrm{R}(X)$. Let $\mathrm{D}(X) \subset \mathrm{R}(X)$ be a region homeomorphic to a disk that encloses all crossing points of X and such that each edge in X crosses the boundary $\partial\,\mathrm{D}(X)$ of $\mathrm{D}(X)$ exactly twice. Let w' be the other endpoint of the unique edge in X that is incident to w. Note that edge uw is in E by Property 2.

The operation *rerouting* $e = uv$ *around* w consists of redrawing e as follows. Refer to Fig. 4b for an illustration. Starting from vertex v, follow the edge e until reaching the first crossing with $\mathrm{D}(X)$; we call this part a *tip* of e. Then, follow the shortest path along $\partial\,\mathrm{D}(X)$ to the crossing of ww' with $\partial\,\mathrm{D}(X)$ closer to w (without crossing ww'). Then, follow edge ww' until vertex w, and go around w until reaching edge uw, (in an orientation that avoids crossing ww'); we call this part the *hook* of e. Finally, complete the new drawing of e by following uw to u; this part is called a *tip* of e (hence edge e has two tips).

Lemma 2. *Let G be a k-planar simple topological graph and let X be an untangled $(k+1)$-crossing in G. Let G', $G' \simeq G$, be the topological graph obtained from G by rerouting an edge $e = uv \in X$ around a vertex $w \in \mathrm{V}(X) \setminus \{u, v\}$ such that u and w are consecutive along $\partial\,\mathrm{R}(X)$. G' has the following properties:*

- *(i) edges e and ww' do not cross;*
- *(ii) the edges that are crossed by e in G' but not in G form a fan at w;*
- *(iii) edge e does not cross any edge more than once.*

Proof. Properties (i) and (ii) immediately follow from the definition of the rerouting operation. We prove property (iii). First note that the tip of e incident to v does not cross any edge in G', since X is a $(k+1)$-crossing and all the crossings of edge e before the rerouting lie in the interior of $\mathrm{D}(X)$. The part of e that follows $\partial\,\mathrm{D}(X)$ crosses all other edges in X, except for ww', exactly once, since the two pairs of crossing points of any two edges of X with $\partial\,\mathrm{D}(X)$ alternate around $\partial\,\mathrm{D}(X)$. The hook of e crosses edges that form a fan at w, and thus do not belong to X, since the only edge of X incident to w is ww', which is not crossed by e. Since these crossings are located in a neighborhood of w, none of the edges incident to w is crossed twice by e. Finally, the tip of e incident to u follows edge uw, which is crossing-free by Property 2. This concludes the proof. □

Homes and home rerouting. In addition, we observe that edge vw, if it exists, can always be (re)drawn inside $\mathrm{R}(X)$ so that it crosses neither uv nor ww' and thus with at most $k-1$ crossings, by following the first three parts of the edge uv, i.e., the tip of uv incident to v, the part of uv that follows $\partial\,\mathrm{D}(X)$, and part of the hook of uv until w; see Fig. 4c. Specifically, if we have rerouted edge uv around w, and edge vw exists, we say that X is a *home* for the edge vw; and we call *home rerouting* the redrawing operation described above.

Global rerouting and full rerouting. In the following we describe our general strategy for transforming a k-planar simple topological graph G into a simple topological graph G', $G' \simeq G$, that is $(k+1)$-quasiplanar. The idea is to appropriately define two injective functions $f : \mathcal{X} \to V$ and $g : \mathcal{X} \to E$, which associate every $(k+1)$-crossing X in G with a vertex $f(X) \in \mathrm{V}(X)$ and with an edge $g(X) \in X$, respectively, such that an endpoint of $g(X)$ and $f(X)$ are consecutive along $\partial\,\mathrm{R}(X)$. Then, we apply the rerouting operation for all pairs $(g(X), f(X))$, i.e., rerouting $g(X)$ around $f(X)$. This operation, which we call *global rerouting*, and denote by (g, f), is well-defined since the $(k+1)$-crossings are pairwise edge-disjoint by Property 1.

After this operation, for each edge $e \in E$ that has a home, we perform a home rerouting operation. Note that every $(k+1)$-crossing in $\mathcal{X}$ is a home for at most one edge, but an edge can have up to two homes (one for each endpoint, no more because f is injective). If an edge has two homes, we pick one of them arbitrarily for the home rerouting operation.

The combined operation consisting of a global rerouting $(g(X), f(X))$ and all possible home reroutings will be called *full rerouting* in the following.

Challenges. There are, however, two potential problems that have to be addressed in order for a global rerouting to eliminate $(k+1)$-crossings. First, Lemma 2 does not guarantee that the topological graph obtained by rerouting a single edge $e = (u, v)$ around a vertex w is simple. Indeed, assuming that u and w are consecutive along $\partial\,\mathrm{R}(X)$, if the edge vw were present in G, then the rerouted edge $e = uv$ would cross such an edge. This crossing between adjacent edges may be solved by a home rerouting operation for vw inside $\mathrm{R}(X)$. However, if vw has a home other than X and we picked this other home to reroute vw inside, this crossing would not be avoided. Furthermore,

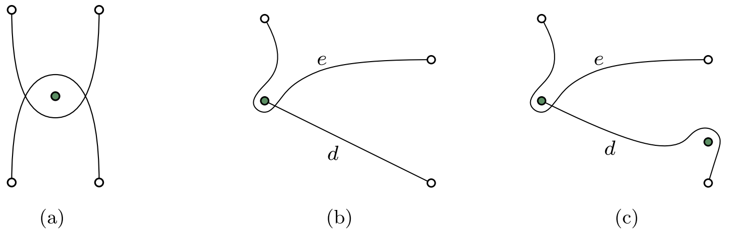


Fig. 5. (a) Two edges rerouted around the same vertex. (b)–(c) Two possible cases in which two edges do not cross before a global rerouting operation but cross afterwards. The vertices used for rerouting are filled green. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

rerouting many edges simultaneously may create new $(k+1)$-crossings. Both problems can be solved by suitably choosing functions f and g.

Outlook. In the next section we start by proving that function f can be chosen to be injective. Note that, given a function f, choosing g to be injective is trivial (by Property 1). We will prove in Section 6 that the injectivity of f and g is sufficient to guarantee that the resulting topological graph does not contain $k+1$ mutually crossing edges, for $k \geq 3$. The case $k = 2$ is more challenging as new 3-crossings may appear after rerouting the edges of g around the vertices of f. To avoid these situations, we modify function f and more carefully define function g, as discussed in Section 6. Finally, we show how to avoid crossings between adjacent edges so as to obtain a simple topological drawing in Section 7.

4. Computing an injective function f

In this section we show the existence of a global rerouting such that no two edges of a k-planar topological graph G, $k \geq 2$, are rerouted around the same vertex (Lemma 5), that is, f is injective. Note that this condition is also necessary for simplicity; see Fig. 5a. We start by defining a bipartite graph composed of the vertices of G and of its $(k+1)$-crossings, and by showing that a matching covering all the $(k+1)$-crossings exists. A bipartite graph with vertex sets A and B is denoted by $H = (A \cup B, \widehat{E})$, where $\widehat{E} \subseteq A \times B$). A *matching from A into B* is a set $M \subseteq E$ such that each vertex in A is incident to exactly one edge in M and each vertex in B is incident to at most one edge in M. For a subset $A' \subseteq A$, we denote by $N(A')$ the set of all vertices in B that are adjacent to a vertex in A'. We recall that, by Hall's theorem, graph H has a matching from A into B if and only if $|N(A')| \geq |A'|$ for every set $A' \subseteq A$.

Let G be a k-planar simple topological graph and let $\mathcal{X}$ be the set of $(k+1)$-crossings of G. We define a bipartite graph $H = (A \cup B, \widehat{E})$ as follows. For each $(k+1)$-crossing $X \in \mathcal{X}$, set A contains a vertex $v(X)$ and set B contains the endpoints of X (that is, $B = \bigcup_{X \in \mathcal{X}} \mathrm{V}(X)$). Also, $\widehat{E}$ contains an edge between a vertex $v(X) \in A$ and a vertex $u \in B$ if and only if $u \in \mathrm{V}(X)$. We have the following.

Lemma 3. *Graph $H = (A \cup B, \widehat{E})$ is a simple bipartite planar graph. Also, each vertex in A has degree $2k+2$.*

Proof. The graph is simple and bipartite by construction. Also, for each $(k+1)$-crossing X, vertex $v(X) \in A$ is incident to the $2k+2$ vertices in B belonging to $\mathrm{V}(X)$. We prove that H is also planar by showing that a planar embedding of H can be obtained from G as follows. First, we remove from G all the vertices and edges that are not in any $(k+1)$-crossing. Then, for each $(k+1)$-crossing X of G, we remove the portion of G in the interior of $\mathrm{D}(X)$ that encloses all crossings among the edges in X and such that each edge in X crosses the boundary of $\mathrm{D}(X)$ exactly twice (as defined in Section 3) and add vertex $v(X)$ inside $\mathrm{D}(X)$. Finally, for each vertex $v \in \mathrm{V}(X)$, let e_v be the edge in X incident to v and let p_v be the intersection point between $\partial\,\mathrm{D}(X)$ and e_v closer to v. We complete the drawing of edge $v(X)v$ by adding an arc between $v(X)$ and p_v in the interior of $\mathrm{D}(X)$ without introducing any crossing. The resulting topological graph is crossing-free. $\square$

Lemma 4. *For every nonempty subset $A' \subseteq A$, we have $|N(A')| \geq |A'| + 2k + 1$.*

Proof. If $|A'| = 1$, then $|A'| + 2k + 1 = 2k + 2$ and a single $(k+1)$-crossing has $2k+2$ vertices. If $|A'| = 2$, then $|A'| + 2k + 1 = 2k + 3$; and two distinct $(k+1)$-crossings jointly have at least $2k+3$ vertices by Property 3(a). Hence, in both cases the statement holds. Consider now the case $|A'| \geq 3$. Let H' be the subgraph of H induced by $A' \cup N(A')$. Since every vertex in A has degree $2k+2$, by Lemma 3 we have $|\,\mathrm{E}(H')| = (2k+2)|A'|$. Also, since H (and thus H') is bipartite planar, by Lemma 3 we have $|\,\mathrm{E}(H')| \leq 2(|A'| + N(A')) - 4$. Thus, $|N(A')| \geq k|A'| + 2 = |A'| + (k-1)|A'| + 2 \geq |A'| + 3k - 3 + 2 \geq |A'| + 2k + 1$, and the statement follows. $\square$

We can now exploit Lemma 4 and Hall's theorem in order to define f as an injective function, which implies that any corresponding global rerouting is such that no two edges are rerouted around the same vertex.

Lemma 5. *Let $G = (V, E)$ be a k-planar simple topological graph, and let $\mathcal{X}$ be the set of $(k+1)$-crossings of G. It is possible to define a global rerouting (g, f) such that f is injective.*

Proof. By Lemma 4 and Hall's theorem, the graph H defined above admits a matching from A into B. For every $(k+1)$-crossing $X \in \mathcal{X}$, let $f(X) \in \mathrm{V}(X)$ be the vertex matched to $v(X)$. It follows that $f : \mathcal{X} \to V$ is injective. The statement follows by choosing $g(X)$, for each $(k+1)$-crossing X, as one of the two edges in X not incident to $f(X)$ and with an endpoint adjacent to $f(X)$ along $\partial\,\mathrm{R}(X)$. $\square$

5. Properties of a rerouted topological graph

Let G' be the topological graph obtained from a k-planar simple topological graph G after applying a full rerouting operation (in which the functions f and g are injective). We study properties of G'. In particular, the edges of G' fall into three categories, depending on how they are represented in G' with respect to G: (1) *nonrerouted edges* have not been rerouted and remain the same as in G; (2) edges that have been rerouted in home rerouting operation we call *safe* (regardless of whether or not they have also been rerouted in the global rerouting operation); and (3) edges that have been rerouted in the global rerouting operation but not in a home rerouting are *critical*. An edge is *rerouted* if it is either safe or critical. Let us start by classifying the new crossings that are introduced by the rerouting algorithm.

Lemma 6. *Consider two edges e_1 and e_2 that cross each other in G' but not in G. After possibly exchanging the roles of e_1 and e_2, one of the following holds:*

(a) e_1 is safe and e_2 is a nonrerouted edge of the home X of e_1 in which e_1 has been rerouted;
(b) e_1 is critical and rerouted around an endpoint of e_2.

Proof. Suppose first that at least one of e_1 and e_2 is critical, say e_1. Since e_1 is critical, there is a $(k+1)$-crossing $X \in \mathcal{X}$ such that $e_1 = g(X) = (u, v)$ is rerouted around the vertex $w = f(X)$ in the global rerouting operation. We can assume that u is adjacent to w along $\partial\,\mathrm{R}(X)$. Recall that the tip of e_1 incident to v is the same in G' as in G, while the tip of e_1 incident to u follows an edge that is crossing-free in G. Also, e_1 does not cross any edge of G' that has been rerouted in X by a home rerouting operation. Then, if e_2 is a nonrerouted edge or a safe edge, e_2 is incident to w and e_1 crosses e_2 with its hook, thus case (b) of the statement applies. If e_2 is critical, then the hook of e_2 does not cross the hook of e_1 because f is injective, and thus we are in case (b) of the statement again, as either the hook of e_1 crosses a tip of e_2 or vice versa.

Suppose now that none of e_1 and e_2 is critical. Since e_1 and e_2 do not cross in G, at least one of them is safe, say e_1. Let $X \in \mathcal{X}$ be the home of e_1 in which e_1 has been rerouted. If e_2 is nonrerouted, then case (a) of the statement applies. On the other hand, e_2 cannot be safe, as otherwise it would be rerouted inside another $2(k+1)$-gon $\mathrm{R}(X')$, which would be interior-disjoint from $\mathrm{R}(X)$, and thus e_1 and e_2 would not cross each other. $\square$

Lemma 7. *Let e be a safe edge in G' and let $X \in \mathcal{X}$ be the home in which e has been rerouted. Assume that $e = (f(X), z)$. The following properties hold:*

(i) e is not part of a $(k+1)$-crossing;
(ii) e does not cross any edge more than once; and

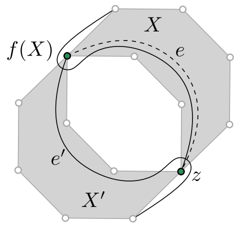


Fig. 6. A safe edge e crosses an adjacent edge e'; see Lemma 7(iii).

(iii) e crosses an adjacent edge e' only if e' is critical, incident to $f(X)$, and rerouted around z, where $z = f(X')$ for some $(k+1)$-crossing $X' \in \mathcal{X} \setminus \{X\}$ with $g(X') = e'$; see Fig. 6 for an illustration.

Proof. By definition, e is the only edge that has X as a home. By Lemma 6 there are only two types of crossings involving e:

(a) inside $\mathrm{R}(X)$ the edge e crosses only the edges of X that have not been rerouted and it does not cross the edge incident to $f(X)$. Since $g(X)$ has been rerouted, e crosses at most $k-1$ such edges. Also, e crosses these edges only once and it is not adjacent to any of them.

(b) e crosses an edge e' that has been rerouted around an endpoint of e. However, by construction e does not cross the edge $g(X)$ that is rerouted around $f(X)$. As f is injective, $g(X)$ is the only edge that is rerouted around $f(X)$. Hence, there is only one more choice for e': to be rerouted around the other endpoint z of e. That is, e' is critical and there is a $(k+1)$-crossing $X' \in \mathcal{X} \setminus \{X\}$ so that $e' = g(X')$ and $f(X') = z$. Since e lies in $\mathrm{R}(X)$, and only the hook of e' enters $\mathrm{R}(X)$, the edges e and e' cross only once.

This proves (ii) and (iii), it remains to prove that e is not part of a $(k+1)$-crossing. Recall that e is crossed by at most $k-1$ edges in X plus at most another edge e' that has been rerouted around an endpoint $z = f(X')$ of e. It follows that if e is part of a $(k+1)$-crossing, then all $k-1$ edges in X that cross e (and that also pairwise cross) also cross e'. But e' does not cross any of these $k-1$ edges in G (because X' and X are disjoint by Property 1), and in G' it is rerouted around z, which is not an endpoint of any of these $k-1$ edges. This proves (i) and completes the proof of the lemma. □

Lemma 8. *No two adjacent critical edges cross in G'.*

Proof. Suppose to the contrary that there are two critical edges $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ that cross in G'. Then by Lemma 6 one edge must have been rerouted around an endpoint of the other. As no edge is rerouted around its own endpoints, we may assume without loss of generality that e_1 has been rerouted around v_2. Then there exists a $(k+1)$-crossing $X \in \mathcal{X}$ such that $e_1 \in X$: namely u is an endpoint of $e_1 = g(X)$ and $v_2 = f(X)$ are vertices of X. Under these conditions, X is a home for e_2. It follows that e_2 is safe, which contradicts our assumption that it is critical. □

The next two lemmas describe properties related to edges that cross in G' but not in G.

Lemma 9. *Every nonrerouted edge e is crossed by at most three critical edges in G'. Further, if e is crossed by exactly three critical edges, then two of them have been rerouted around distinct endpoints of e.*

Proof. Since at most one edge has been rerouted around each vertex, by construction, it suffices to prove that there exists at most one critical edge crossing e that has not been rerouted around an endpoint of e.

For this, note that any edge with this property crosses e also in G, by Lemma 6, and thus it belongs to the same $(k+1)$-crossing as e. Since, by construction, at most one edge per $(k+1)$-crossing is critical, the statement follows. □

Lemma 10. *If G' contains a $(k+1)$-crossing X', then X' contains at most one nonrerouted edge.*

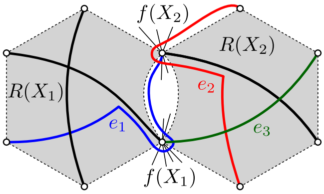
Proof. Assume to the contrary that a $(k+1)$-crossing X' in G' contains at least two nonrerouted edges e_1 and e_2. By Lemma 7(i), every edge in X' is critical or nonrerouted.

We first claim that there exists an edge $e_3 \in X'$ that does not cross e_1 in G. If e_1 has fewer than k crossings in G, then the claim follows by the pigeonhole principle. If e is part of a $(k+1)$-crossing X in G, the claim follows from the fact that the edges of X do not form a $(k+1)$-crossing in G', due to a rerouting of one of its edges. Finally, assume that e_1 has k crossings in G but it is not part of any $(k+1)$-crossing. Then none of the edges crossing e_1 in G can be part of a $(k+1)$-crossing in G, otherwise they each would have k crossings in the $(k+1)$-crossing and an additional crossing with e_1, contradicting the k-planarity of G. Hence none of the edges crossing e_1 in G is critical, they all are nonrerouted, hence X' is a $(k+1)$-crossing in G, contradicting our assumption that e_1 is not part of any $(k+1)$-crossing in G. This completes the proof of the claim.

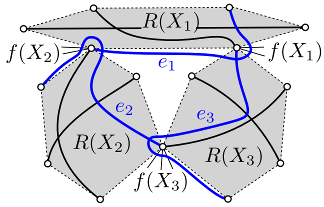
Note that e_3 is critical by Lemma 7(i), which means that e_3 is part of a $(k+1)$-crossing in G containing neither e_1 nor e_2. Hence, e_2 and e_3 do not cross in G by Property 1. We prove that they do not cross in G', either, a contradiction to the assumption that X' is a $(k+1)$-crossing. By Lemma 6(b), all new crossings of e_3 are on its hook; however, since e_1 and e_2 cross in G (which is simple), they do not share any endpoint, and the statement follows. □

6. Proving $(k+1)$-quasiplanarity

Denote by G' the topological graph obtained from a k-planar simple topological graph G by executing a full rerouting operation (assuming that the functions f and g are injective). In this section we first prove that, if $k \geq 3$, then G' does not contain $(k+1)$-crossings. If $k = 2$, the current choice of f and g may not avoid the presence of 3-crossings. Thus, when $k = 2$, we modify f and g to obtain quasiplanarity.



(a) twin



(b) whirl

Fig. 7. The global rerouting may produce 3-crossings in form of twins or whirls.

Lemma 11. *Let G be a k-planar simple topological graph, where $k \geq 3$. The topological graph G' does not contain any $(k+1)$-crossing.*

Proof. Assume for a contradiction that G' contains a $(k+1)$-crossing X'. By Lemma 7(i), X' does not contain any safe edge, and by Lemma 10, X' contains at most one non-rerouted edge.

Suppose that X' contains one such edge e. By Lemma 9, there are at most three critical edges crossing e in G'. If there are less than 3, then the claim follows, as $k \geq 3$. If there are three, say d, h, and l, then we may assume by Lemma 9 that d and h have been rerouted around (distinct) endpoints of e. Thus, d and h do not cross in G, by Property 1, as they belong to different $(k+1)$-crossings. Hence, they can cross in G' only if one of them has been rerouted around an endpoint of the other, by Lemma 6. This is impossible since neither d nor h shares an endpoint with e, as G is simple.

Suppose that X' contains only critical edges. Let e be any edge of X' and assume that it has been rerouted around vertex w. Since at most one edge in X' can be incident to w by Lemma 8 and since $k \geq 3$, there are two edges in X', say d and h, that have been rerouted around distinct endpoints of e. As in the previous case, d and h do not cross. $\square$

In the remainder of this section, we assume that $k = 2$. In this case, Lemma 11 does not hold, as some 3-crossings may still appear after the global rerouting; see Fig. 7 for examples. Next we characterize the 3-crossings in G'. The characterization allows us to avoid these 3-crossings by choosing suitable functions f and g.

Definition 1. Three edges $e_1, e_2, e_3 \in E$ form a *twin* configuration (Fig. 7a) in G' if they are in two distinct 3-crossings $X_1, X_2 \in \mathcal{X}$, where $e_1 = g(X_1)$, $e_2 = g(X_2)$ and $e_3 \in X_2 \setminus \{e_2\}$, such that edge e_1 is incident to $f(X_2)$, edge e_3 is incident to $f(X_1)$ but not to $f(X_2)$, and e_3 is drawn inside $\mathrm{R}(X_2)$.

Definition 2. Three edges $e_1, e_2, e_3 \in E$ form a *whirl* configuration (Fig. 7b) in G' if they are in three pairwise distinct 3-crossings $X_1, X_2, X_3 \in \mathcal{X}$, where $e_1 = g(X_1)$, $e_2 = g(X_2)$, and $e_3 = g(X_3)$, such that edge e_1 is incident to $f(X_2)$, edge e_2 is incident to $f(X_3)$, and edge e_3 is incident to $f(X_1)$.

Lemma 12. *Every 3-crossing in G' forms a twin or a whirl configuration.*

Proof. Let e_1, e_2, and e_3 be three edges that form a 3-crossing in G'. By Lemma 7 we know that none of the three edges is safe. Hence, we may assume without loss of generality that e_1, e_2, and e_3 are either nonrerouted or critical. By Lemma 10, at most one of them is nonrerouted, and thus at least two are critical. We assume that e_1 and e_2 are critical, and distinguish two cases, based on whether e_3 is critical or nonrerouted.

We first consider the case in which e_3 is nonrerouted. Recall that every 3-crossing reroutes at most one critical edge in the global rerouting. For $i \in \{1, 2\}$, let $X_i = \{c_i, d_i, e_i = g(X_i)\}$ be the 3-crossing that triggered the rerouting of e_i around the end-point $f(X_i)$ of c_i.

Then by construction and Lemma 6 the only edges crossed by e_i, for $i \in \{1, 2\}$, in G' are d_i, edges incident to $f(X_i)$, and at most two edges rerouted around an endpoint of e_i.

Since e_1 and e_2 cross and are both critical, one of them has been rerouted around an endpoint of the other. Without loss of generality suppose that e_2 has been rerouted around an endpoint of e_1, that is, e_1 is incident to $f(X_2)$.

Since e_3 crosses both e_1 and e_2 and is nonrerouted, we have the following. On one hand, e_3 is either d_1 or incident to $f(X_1)$; on the other hand, e_3 is either d_2 or incident to $f(X_2)$. Thus, $e_3 \in \{d_1, d_2, (f(X_1), f(X_2))\}$. We claim that $e_3 = d_2$.

To prove the claim, we first argue that $e_3 \neq d_1$. By definition, e_1 and d_1 do not share an endpoint, and d_1 is nonrerouted. The only nonrerouted edges that e_2 crosses in G' are d_2 and edges incident to $f(X_2)$. Since $f(X_2)$ is an endpoint of e_1, it is not an endpoint of d_1. Therefore, e_2 does not cross d_1, which implies $e_3 \neq d_1$.

It remains to prove that $e_3 \neq (f(X_1), f(X_2))$. Suppose, for a contradiction, that $e_3 = (f(X_1), f(X_2))$. This implies that the 3-crossing X_1 is a home for e_3; namely, both $f(X_1)$ and $f(X_2)$ are vertices of X_1 (the former by definition and the latter as an endpoint of e_1), and $f(X_2)$ is incident to e_1. However, this contradicts the assumption that e_3 is nonrerouted. Altogether it follows that $e_3 = d_2$, as claimed, and so e_1, e_2, e_3 form a twin configuration.

We then consider the case in which e_3 is critical. Since only one edge of each 3-crossing in $\mathcal{X}$ is rerouted, e_1, e_2, e_3 come from pairwise distinct 3-crossings X_1, X_2, X_3, with $e_i = g(X_i)$, for $i \in \{1, 2, 3\}$. By Lemma 6 two of these edges cross if and only if one is rerouted around an endpoint of the other. By Lemma 8 the edges e_1, e_2, e_3 are spanned by six pairwise distinct endpoints. Therefore, every rerouting generates at most one crossing among e_1, e_2, e_3 and so every rerouting must generate a crossing between a different pair of segments. It follows that e_1, e_2, e_3 form a whirl configuration (with a suitable permutation of indices). □

In order to suitably select functions f and g so that G' does not contain any twin or whirl configurations, we exploit an auxiliary conflict graph, which we define in the next subsection.

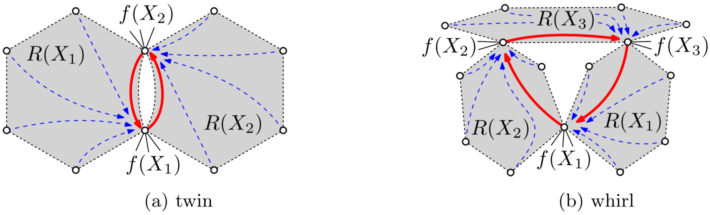


Fig. 8. Twin and whirl configurations induce cycles in the conflict graph.

6.1. Conflict digraph

We define a plane digraph K on the same vertex set V as G that represents the interactions between the 3-crossings in $\mathcal{X}$. The conflict digraph depends on G and on the function $f : \mathcal{X} \to V$, but it does not depend on the function g. For every 3-crossing $X \in \mathcal{X}$, we create five directed edges that are all directed towards $f(X)$ and drawn inside $\mathrm{R}(X)$. These edges start from the five vertices on $\partial\,\mathrm{R}(X)$ other than $f(X)$; see Fig. 8. Note that two vertices in V may be connected by two edges with opposite orientations lying in two different 3-crossings (for instance, in a twin configuration as shown in Fig. 8a). However, K contains neither loops nor parallel edges with the same orientation because f is injective and so every vertex can have incoming edges from at most one 3-crossing.

Property 4. *The following properties hold for digraph K:*

- (i) *K is a directed plane graph.*
- (ii) *At every vertex $v \in V$, the incoming edges in K are consecutive in the cyclic order of incident edges around v.*
- (iii) *If $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, and $e_3 = (v_3, v_1)$ form a whirl configuration in G', then K contains a 3-cycle (v_1, v_2, v_3).*
- (iv) *If $e_1 = g(X_1)$, $e_2 = g(X_2)$, and $e_3 \in X_2$ form a twin configuration in G', then the conflict digraph contains a 2-cycle $(f(X_1), f(X_2))$.*

Proof. (i) Each edge of K lies in a region $\mathrm{R}(X)$, for some $X \in \mathcal{X}$. Since these regions are interior-disjoint, by Property 1, edges from different regions do not cross. All edges in the same region $\mathrm{R}(X)$ are incident to $f(X)$; so they can be drawn in the interior of $\mathrm{R}(X)$ without crossing each other. (ii) For each vertex $v \in V$, there is at most one 3-crossing $X \in \mathcal{X}$ such that $v = f(X)$, since f is injective. Since all incoming edges of v lie in the region $\mathrm{R}(X)$, and all edges lying in $\mathrm{R}(X)$ are directed towards $v = f(X)$, by construction, the statement follows. (iii–iv) Both claims follow directly from the definition of twin and whirl configurations and the definition of K. □

Relations between cycles in K. We observed that K is a plane digraph, where every twin configuration induces a 2-cycle and every whirl configuration induces a 3-cycle.

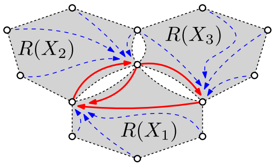


Fig. 9. A (ghost) 3-cycle that contains a 2-cycle.

So in order to prevent the creation of twin and whirl configurations in G', we need to understand the structure of 2- and 3-cycles in the conflict digraph K. In the following paragraphs we introduce some terminology and prove some structural statements about cycles in K.

A cycle in the conflict digraph K is *short* if it has length two or three. For a cycle c in K, let $\mathrm{int}(c)$ denote the interior of c, let $\mathrm{ext}(c)$ denote the exterior of c, let $\mathrm{R}(c)$ denote the compact region bounded by c, and let $\mathrm{V}(c)$ denote the vertex set of c. We use the notation $i \oplus 1 := 1 + (i \bmod k)$ and $i \ominus 1 := 1 + ((k + i - 2) \bmod k)$ to denote successors and predecessors, respectively, in a circular sequence of length k that is indexed $1, \ldots, k$. Let c_1 and c_2 be two cycles in the conflict graph K. We say that c_1 and c_2 are *interior-disjoint* if $\mathrm{int}(c_1) \cap \mathrm{int}(c_2) = \emptyset$. We say that c_1 *contains* c_2 if $\mathrm{R}(c_2) \subseteq \mathrm{R}(c_1)$. In both cases, c_1 and c_2 may share vertices and edges, but they may also be vertex-disjoint. See Fig. 9 for an example.

Lemma 13. *If a vertex $v \in V$ is incident to two interior-disjoint cycles in K, then these cycles have opposite orientations (clockwise vs. counterclockwise). Consequently, every vertex $v \in V$ is incident to at most two interior-disjoint cycles in K.*

Proof. Let v be incident to cycles c_1 and c_2 in K, and assume without loss of generality that c_1 is counterclockwise. For $i \in \{1, 2\}$, the cycle c_i has an edge e_i^{in} directed into v and an edge e_i^{out} directed out of v (possibly $e_1^{\mathrm{in}} = e_2^{\mathrm{in}}$ or $e_1^{\mathrm{out}} = e_2^{\mathrm{out}}$).

By Property 4(ii), the edges directed to (resp., from) v are consecutive in the rotation order of all edges incident to v. The edges e_1^{out} and e_1^{in} (resp., e_2^{out} and e_2^{in}) are also consecutive because the two cycles are interior-disjoint. It follows that the counterclockwise order of the four edges around v is $(e_1^{\mathrm{out}}, e_1^{\mathrm{in}}, e_2^{\mathrm{in}}, e_2^{\mathrm{out}})$. So the cycle c_2 is clockwise, as required. □

Lemma 14. *A short cycle in K is uniquely determined by its vertex set.*

Proof. Recall that between any ordered pair (u, v) of vertices there is at most one directed edge (u, v) in K because such an edge corresponds to a 3-crossing $X \in \mathcal{X}$ with $u, v \in \mathrm{V}(X)$ and $f(X) = v$. As f is injective, there is at most one such 3-crossing.

So the statement is obvious for 2-cycles. Consider two 3-cycles c_1 and c_2 in K with $\mathrm{V}(c_1) = \mathrm{V}(c_2) = \{v_1, v_2, v_3\}$. Without loss of generality, let $c_1 = (v_1, v_2, v_3)$. If c_1 and c_2 share an edge, say (v_1, v_2), then there is a unique way to complete this edge to a directed

3-cycle $(v_1, v_2, v_3) = c_1 = c_2$. Hence suppose that c_1 and c_2 are edge-disjoint, that is, $c_2 = (v_3, v_2, v_1)$.

Let X_i denote the 3-crossing with $f(X_i) = v_i$, for $i \in \{1, 2, 3\}$. All edges directed to v_1 are drawn inside $\mathrm{R}(X_1)$ between vertices of $\mathrm{V}(X_1)$, and both (v_3, v_1), as an edge of c_1, and (v_2, v_1), as an edge of c_2, are edges of K. Therefore, $v_1, v_2, v_3 \in \mathrm{V}(X_1)$. Symmetrically, it follows that $v_1, v_2, v_3 \in \mathrm{V}(X_1) \cap \mathrm{V}(X_2) \cap \mathrm{V}(X_3)$. Three distinct 3-crossings X_1, X_2, X_3 share three distinct vertices, contradicting Property 3(b). It follows that c_2 and c_1 have the same orientation and therefore $c_1 = c_2$. □

Ghosts. We say that a 3-cycle in K is a *ghost* if two of its vertices induce a 2-cycle in K; see, e.g., Fig. 9. Let $\mathcal{C}$ denote the set of all short cycles in K that are not ghosts.

Lemma 15. *Let $c_1, c_2 \in \mathcal{C}$. If there is a vertex of $\mathrm{V}(c_1)$ in $\mathrm{int}(c_2)$, then c_2 contains c_1.*

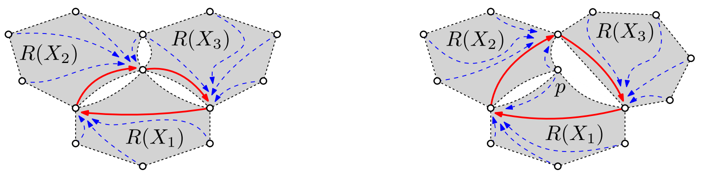
Proof. Suppose to the contrary that there exist short cycles $c_1, c_2 \in \mathcal{C}$ such that there is a vertex $v_1 \in \mathrm{V}(c_1)$ that lies in $\mathrm{int}(c_2)$ but c_2 does not contain c_1. Then some point along c_1 lies in $\mathrm{ext}(c_2)$. Since K is a plane graph, an entire edge of c_1 must lie in $\mathrm{ext}(c_2)$. Denote this edge by (v_2, v_3). Recall that c_1 is short (that is, it has at most three vertices), consequently, $c_1 = (v_1, v_2, v_3)$. Since c_1 has points in both $\mathrm{int}(c_2)$ and $\mathrm{ext}(c_2)$, the two cycles intersect in at least two points. In a plane graph, the intersection of two cycles consists of vertices and edges. Consequently $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \{v_2, v_3\}$. Recall that c_2 is also short, and so it has a directed edge between any two of its vertices. However, (v_2, v_3) lies in $\mathrm{ext}(c_2)$, so the reverse edge (v_3, v_2) is present in c_2. That is, $\{v_2, v_3\}$ induces a 2-cycle in K. Hence c_1 is a ghost, contrary to our assumption $c_1 \in \mathcal{C}$. □

Smooth cycles. Next we define a special type of cycles, called *smooth*, so as to control the interaction between cycles in K.

Definition 3. Let $c = (v_1, \ldots, v_m) \in \mathcal{C}$. Recall that every edge in K lies in a region $\mathrm{R}(X)$, $X \in \mathcal{X}$, and is directed towards $f(X)$. So the cycle c corresponds to a cycle of 3-crossings $(X_1, \ldots, X_m)$, such that $v_i = f(X_i)$ and v_i lies in the common boundary $\partial\,\mathrm{R}(X_i) \cap \partial\,\mathrm{R}(X_{i\oplus 1})$ for $i = 1, \ldots, m$. We say that the 3-crossings $X_1, \ldots, X_m$ are *associated* with c. The cycle c is *smooth* if none of the associated 3-crossings has a vertex in $\mathrm{int}(c)$. For example, the 3-cycle in Fig. 10a is smooth, but the one in Fig. 10b is not.

Note that a smooth cycle in K may contain many vertices of various 3-crossings in its interior; the restrictions apply only to those (two or three) 3-crossings that are associated with the cycle. For instance, there might be many more 3-crossings in the white regions between the 3-crossings in Fig. 10.

Let $\mathcal{C}_\mathrm{s}$ denote the set of all smooth cycles in $\mathcal{C}$, that is, the set of all short nonghost cycles in K that are smooth. In Section 6.2, we show how to choose f such that all cycles in $\mathcal{C}$ are smooth, that is, $\mathcal{C} = \mathcal{C}_\mathrm{s}$.



(a) A smooth 3-cycle. (b) A nonsmooth 3-cycle.

Fig. 10. Examples of smooth and nonsmooth cycles.

Properties of smooth cycles. The following three lemmas formulate important properties of smooth cycles that hold for any injective function f.

Lemma 16. *Let $c \in \mathcal{C}_s$ and let u be a vertex of G that lies in $\operatorname{int}(c)$. Then there is no edge (u, v) in K for any $v \in \mathrm{V}(c)$.*

Proof. Suppose for the sake of a contradiction that (u, v) is an edge of K with $v \in \mathrm{V}(c)$. Let X be the 3-crossing with $f(X) = v$. Every edge directed into v is directed out of some other vertex in $\mathrm{V}(X)$, in particular, $u \in \mathrm{V}(X)$. As X is associated with c, this contradicts the assumption that c is smooth. □

Lemma 17. *Let $c_1, c_2 \in \mathcal{C}_s$ so that $c_1 \neq c_2$ and c_2 contains c_1. Then $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \emptyset$.*

Proof. Suppose to the contrary that there exists a vertex $u \in \mathrm{V}(c_1) \cap \mathrm{V}(c_2)$. We claim that there is no vertex of c_1 that lies in $\operatorname{int}(c_2)$. To see this, consider some $v \in \mathrm{V}(c_1) \cap \operatorname{int}(c_2)$. Then following c_1 from v to u we find an edge (x, y) of K so that x lies in $\operatorname{int}(c_2)$ and $y \in \mathrm{V}(c_2)$. However, such an edge does not exist by Lemma 16. Hence there is no such $v \in \mathrm{V}(c_1) \cap \operatorname{int}(c_2)$. Given that c_2 contains c_1, it follows that $\mathrm{V}(c_1) \subseteq \mathrm{V}(c_2)$.

If c_1 is a 3-cycle, then the claim above implies that so is c_2, and Lemma 14 contradicts our assumption $c_1 \neq c_2$. Hence c_1 is a 2-cycle and c_2 is a 3-cycle. But then c_2 is a ghost, in contradiction to $c_2 \in \mathcal{C}_s$. □

Lemma 18. *Any two cycles in $\mathcal{C}_s$ are interior-disjoint or vertex disjoint.*

Proof. Let $c_1, c_2 \in \mathcal{C}_s$ so that $c_1 \neq c_2$. Suppose, to the contrary, that $\operatorname{int}(c_1) \cap \operatorname{int}(c_2) \neq \emptyset$ and $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) \neq \emptyset$. Without loss of generality, an edge (u_1, u_2) of c_2 lies in $\operatorname{int}(c_1)$.

We may assume that u_1 and u_2 are common vertices of c_1 and c_2. Indeed, if u_1 or u_2 were not common vertices of the cycles, then a vertex of c_2 would lie in the interior of c_1. Then c_1 contains c_2 by Lemma 15, and $\mathrm{V}(c_1) \cap \mathrm{V}(c_2) = \emptyset$ by Lemma 17.

We may further assume that both c_1 and c_2 are 3-cycles. Indeed, if the vertex set of one of them contains that of the other, then one of them is a 3-cycle and the other is a 2-cycle by Lemma 14. Hence the 3-cycle is ghost, contradicting the assumption that both c_1 and c_2 are present in $\mathcal{C}$, $\mathcal{C}_s \subseteq \mathcal{C}$.

Since (u_1, u_2) is a directed edge of c_2 that lies in the interior of c_1, since c_1 is a 3-cycle that has an edge between any two of its vertices, and since u_2 can have incoming edges from at most one 3-crossing (because f is injective), it follows that the edge (u_2, u_1) is present in c_1. This implies that $c_3 = (u_1, u_2)$ is a 2-cycle in K. Therefore $c_3 \in \mathcal{C}$, and both c_1 and c_2 are ghost cycles in $\mathcal{C}_{\mathrm{s}} \subseteq \mathcal{C}$, contradicting the definition of $\mathcal{C}$. Thus c_1 and c_2 are interior-disjoint or vertex disjoint, as claimed. □

6.2. How to choose the function f

As a next step, we show how to define the function f so that in the resulting conflict graph all nonghost cycles are smooth. The idea is to incrementally modify the function f so that the number of vertices that are contained in nonsmooth cycles decreases.

Lemma 19. *Let $G = (V, E)$ be a 2-planar simple topological graph, and let $\mathcal{X}$ be the set of 3-crossings of G. There exists an injective function $f : \mathcal{X} \to V$ such that every short cycle that is not a ghost in the conflict digraph K of G is smooth (that is, $\mathcal{C} = \mathcal{C}_{\mathrm{s}}$).*

Proof. Let $f : \mathcal{X} \to V$ be an arbitrary injective function that maps every 3-crossing $X \in \mathcal{X}$ to a vertex $v \in \mathrm{V}(X)$. Such a function exists by Lemma 5. We repeatedly modify the function f to achieve the desired property.

Following the notation defined above, let K be the conflict digraph of G determined by f, and let $\mathcal{C}$ be the set of short cycles in K that are not ghosts. Also, let $\mathcal{C}_{\mathrm{ns}} \subseteq \mathcal{C}$ denote the subset of cycles in $\mathcal{C}$ that are not smooth. If $\mathcal{C}_{\mathrm{ns}} = \emptyset$, then the proof is complete. As long as $\mathcal{C}_{\mathrm{ns}} \neq \emptyset$, we repeatedly modify f for some vertices in the region $\mathrm{R}(c)$ of a cycle $c \in \mathcal{C}_{\mathrm{ns}}$. This modification of f correspondingly changes the conflict digraph K (and hence the set $\mathcal{C}_{\mathrm{ns}}$). As a measure of progress we maintain that the cardinality of the set V_{ns} decreases, where V_{ns} is the set of vertices that lie in the regions bounded by the cycles in $\mathcal{C}_{\mathrm{ns}}$, that is, $V_{\mathrm{ns}} = V \cap (\bigcup_{c \in \mathcal{C}_{\mathrm{ns}}} \mathrm{R}(c))$.

A cycle $c \in \mathcal{C}_{\mathrm{ns}}$ is *maximal* if there exists no cycle $c' \in \mathcal{C}_{\mathrm{ns}} \setminus \{c\}$ such that c' contains c. Recall that if a cycle $c \in \mathcal{C}_{\mathrm{ns}}$ is not smooth, then there exists a vertex of some associated 3-crossing that lies in the interior of c.

One incremental modification of f. Given an injective function $f : \mathcal{X} \to V$ such that $f(X) \in V(X)$ for every $X \in \mathcal{X}$, a maximal cycle $c \in \mathcal{C}_{\mathrm{ns}}$, an associated 3-crossing $X_1 \in X$, and a vertex $v \in \mathrm{V}(X_1) \cap \mathrm{int}(c)$, we define a new function $f' = F(f, c, X_1, v)$, which is an injective function $f' : \mathcal{X} \to V$ such that $f(X) \in V(X)$ for every $X \in \mathcal{X}$; and in particular $f'(X_1) = v$. Later we will argue how to select c, X_1, and v more carefully so as to guarantee certain properties for f'.

Let $c = (v_1, \ldots, v_m)$ for $m \in \{2, 3\}$ and let $X_1, \ldots, X_m$ denote the associated 3-crossings such that $v \in \mathrm{V}(X_1) \cap \mathrm{int}(c)$.

We define a new injective function $f' : \mathcal{X} \to V$ as follows. We set $f'(X_1) = v$. For all 3-crossings $X \in \mathcal{X}$, $X \neq X_1$, for which $\mathrm{R}(X) \nsubseteq \mathrm{R}(c)$, we set $f'(X) = f(X)$. In particular, $f(X_j) = f(X_j)$ for $j = 2, \ldots, m$ along the cycle c. For all remaining $X \in \mathcal{X}$, where $\mathrm{R}(X) \subseteq \mathrm{R}(c)$, we define $f'(X) \in V \in \mathrm{int}\, c$ using Hall's theorem. For these 3-crossings, Hall's condition is still satisfied by Lemma 4 even if we exclude up to 3 vertices along the cycle c, and we find an injective function f' as in Lemma 5. This completes the definition of $f' = F(f, c, X_1, v)$.

The modified function f' defines a new conflict digraph that we denote by K'. Note that both K and K' have the same vertex set, namely V. Since $f(X_1) \neq f'(X_1) = v$, the cycle c of K is not present in K'. Let $\mathcal{C}'$, $\mathcal{C}'_{\text{ns}}$ and V'_{ns} be defined analogously to $\mathcal{C}$, $\mathcal{C}_{\text{ns}}$ and V_{ns} in K'. We claim that:

(A) every cycle in $\mathcal{C}'_{\text{ns}} \setminus \mathcal{C}_{\text{ns}}$ is contained in c, and
(B) $v_1 \notin \mathrm{R}(c')$ for any cycle $c' \in \mathcal{C}'_{\text{ns}}$.

The combination of (A) and (B) immediately establishes $V'_{\text{ns}} \subsetneq V_{\text{ns}}$, our measure of progress. We call a cycle b *bad* if it violates (A), that is, $b \in \mathcal{C}'_{\text{ns}} \setminus \mathcal{C}_{\text{ns}}$ and c does not contain b.

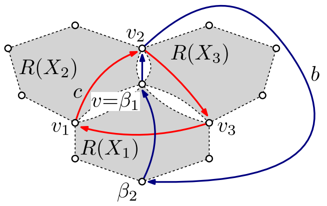
We first show that (A) implies (B). Note that v_1 is a vertex of every cycle $d \in \mathcal{C}_{\text{ns}}$ for which $v_1 \in \mathrm{R}(d)$. To see this, let $d \in \mathcal{C}_{\text{ns}}$ with $v_1 \in \mathrm{R}(d)$. If $v_1 \in \mathrm{int}(d)$, then d contains c by Lemma 15, and so $d = c$ by the maximality of c. Hence $v_1 \in \mathrm{V}(d)$, as claimed. As v_1 has no incoming edge in K', it follows that setting $f(X_1) = v$ destroys *all* cycles in $\mathcal{C}_{\text{ns}}$ that contain v_1. Therefore, if there is a cycle $b \in \mathcal{C}'_{\text{ns}}$ for which $v_1 \in \mathrm{R}(b)$, then b is a new cycle, that is, $b \in \mathcal{C}'_{\text{ns}} \setminus \mathcal{C}_{\text{ns}}$. In fact, in order to contain v_1, the cycle b must be bad: If $v_1 \in \mathrm{R}(b) \subseteq \mathrm{R}(c)$, then v_1 is a vertex of b, which is impossible because v_1 has no incoming edge in K'. By (A), there is no bad cycle, and so no cycle in $\mathcal{C}'_{\text{ns}}$ contains v_1 and (B) holds, as claimed.

To prove (A), we distinguish some cases and argue separately in each case. Before the case distinction, we give a common characterization of bad cycles.

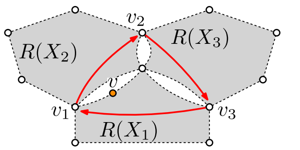
Recall that we do not change $f(X_i)$, for $i \in \{2, \ldots, m\}$, i.e., $f(X_i) = f'(X_i)$, for $i \in \{2, \ldots, m\}$. Therefore, the conflict digraph has the same edges inside $\mathrm{R}(X_i)$, for $i \in \{2, \ldots, m\}$, before and after the modification of f. In particular, as K' is plane, its edges can only cross the edge (v_m, v_1) of c (because it is in K but not in K'), which lies in $\mathrm{R}(X_1)$. All edges of K' inside $\mathrm{R}(X_1)$ are directed to vertex $v \in \mathrm{int}(c)$. Therefore, every edge in K' that crosses (v_m, v_1) has one endpoint in $\mathrm{int}(c)$ and one endpoint in $\mathrm{ext}(c)$.

Characterization of bad cycles. Let b denote a bad cycle in K'; refer to Fig. 11a. Note that f' is unchanged with respect to f for 3-crossings in the exterior of c. Therefore, every cycle in K' that involves only vertices in $c \cup \mathrm{ext}(c)$ is also a cycle in K. This implies that every new cycle in $\mathcal{C}'_{\text{ns}} \setminus \mathcal{C}_{\text{ns}}$ must have a vertex in $\mathrm{int}(c)$, and so b has a vertex $\beta_1 \in \mathrm{int}(c)$.

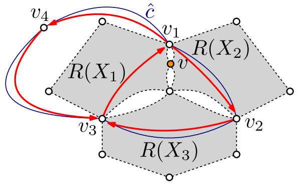
We claim that b also has a vertex $\beta_2 \in \mathrm{ext}(c)$. To see this, suppose to the contrary that $\mathrm{V}(b) \subset \mathrm{R}(c)$. Since b is bad, c does not contain b, that is, $\mathrm{R}(b) \not\subset \mathrm{R}(c)$. Hence b has an edge (u_1, u_2) that passes through $\mathrm{ext}(c)$. As noted above, every edge of K' that crosses (v_m, v_1) has a vertex in $\mathrm{ext}(c)$. Therefore, b does not cross (v_m, v_1) and so



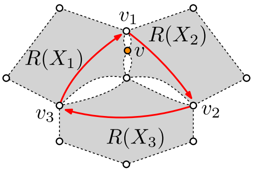
(a) A bad cycle.



(b) Case 1.



(c) Case 2.1.



(d) Case 2.2.

Fig. 11. Illustrations for the proof of Lemma 19.

$u_1, u_2 \in \mathrm{V}(b) \cap \mathrm{V}(c)$ and $b = (u_1, u_2, \beta_1)$. As v_1 has no incoming edge in K', we have $v_1 \notin \mathrm{V}(b)$ and so $m = 3$ and $\{u_1, u_2\} = \{v_2, v_3\}$. Since c is short, it contains an edge between u_1 and u_2, and as the edge (u_1, u_2) passes through $\mathrm{ext}(c)$, the reverse edge (u_2, u_1) is an edge of c. But then $\{u_1, u_2\} = \{v_2, v_3\}$ induce a 2-cycle in K', and b is a ghost, contradicting our assumption that $b \in \mathcal{C}'_{\mathrm{ns}}$. This proves the claim that b has a vertex $\beta_2 \in \mathrm{ext}(c)$.

Given the position of β_1 and β_2, it follows that b crosses c. As noted above, (v_m, v_1) is the only edge of c that can be crossed by an edge of K'. This leaves only four options for b to cross c: the at most three vertices of c and the edge (v_m, v_1). As all edges of K' that cross (v_m, v_1) are directed towards v, the cycle b crosses the edge (v_m, v_1) of c at most once. Moreover, if b crosses (v_m, v_1), then the crossing edge starts from a vertex of X_1 and goes to the vertex v. As b has at most three vertices and due to the position of β_1 and β_2, the cycles b and c can share at most one vertex.

Altogether it follows that a bad cycle b has exactly three vertices: the vertex $\beta_1 = v \in \mathrm{int}(c)$, a vertex $\beta_2 \in \mathrm{ext}(c) \cap \mathrm{V}(X_1)$ that precedes v in b, and a third vertex $\beta_3 \in \{v_2, v_m\}$. Hence $b = (v, v_j, \beta_2)$, for some $j \in \{2, m\}$. (We cannot have $\beta_3 = v_1$ because $v_1 \notin f'(\mathcal{X})$ after the reassignment.) Note that $\beta_3 = v_j$, for $j \in \{2, m\}$, implies that v is a vertex of X_j because (v, v_j) is an edge of b.

Case analysis. In order to prove (A), we distinguish three cases: Case 1, Case 2.1, and Case 2.2 below.

Case 1: There is a maximal cycle $c \in \mathcal{C}_{\mathrm{ns}}$ and a vertex $v \in \mathrm{int}(c)$ such that v is incident to exactly one of $X_1, \ldots, X_m$; refer to Fig. 11b. We may assume that v is incident to X_1 (by cyclically relabeling $X_1, \ldots, X_m$ if necessary). We set $f' = F(f, c, X_1, v)$. By the discussion above, the cycle c is destroyed and no bad cycle is created (because the existence of a bad cycle implies that v is also a vertex of at least one of the other 3-crossing(s) X_j, for $j \in \{2, m\}$).

Case 2: For every maximal cycle $c \in \mathcal{C}_{\text{ns}}$, every vertex of $X_1, \ldots, X_m$ in $\text{int}(c)$ is incident to at least two 3-crossings in $\{X_1, \ldots, X_m\}$. We consider two subcases.

Case 2.1: There are two interior-disjoint maximal 3-cycles in $\mathcal{C}_{\text{ns}}$ that share an edge; refer to Fig. 11c. Denote these two cycles by $c_1 = (v_1, v_2, v_3)$ and $c_2 = (v_1, v_4, v_3)$, and let $X_1, \ldots, X_4$ denote the associated 3-crossings so that $v_i = f(X_i)$, for $i \in \{1, 2, 3, 4\}$. Note that $v_2 \notin \text{V}(X_1)$ because then v_1 and v_2 would induce a 2-cycle in K, which contradicts the fact that c_1 is not a ghost. Analogously, it follows that $v_4 \notin \text{V}(X_1)$. The union of the edges (v_1, v_2), (v_2, v_3), (v_1, v_4), and (v_4, v_3) forms an (undirected) closed Jordan curve $\hat{c}$. On one hand, none of the four edges that form $\hat{c}$ is oriented towards $v_1 = f'(X_1)$, and so the curve $\hat{c}$ lies in the exterior of $\text{R}(X_1)$. On the other hand, the (closed) region $\text{R}(\hat{c})$ bounded by $\hat{c}$ contains the edge (v_3, v_1) in $\text{R}(X_1)$. It follows that $\text{R}(\hat{c}) \supset \text{R}(X_1)$. Consequently, all four vertices in $\text{V}(X_1) \setminus \{v_1, v_3\}$ lie in $\text{int}(c_1) \cup \text{int}(c_2)$. Without loss of generality, we may assume that at least two vertices of $\text{V}(X_1)$ lie in $\text{int}(c_1)$. By Property 3(b), at most one vertex in $\text{int}(c_1)$ is incident to all of X_1, X_2, X_3, there exists a vertex $v \in \text{V}(X_1) \cap \text{int}(c_1)$ incident to either X_2 or X_3 (but not both).

We select $c = c_1$ and set $f' = F(f, c, X_1, v)$. As noted above, any bad cycle is of the form $b = (v, v_j, \beta_2)$, where $j \in \{2, 3\}$ and $\beta_2 \in \text{ext}(c_1) \cap \text{V}(X_1)$.

If $j = 2$, we have $v_2 \in \text{ext}(c_2)$ and $\beta_2 \in \text{ext}(c_1) \cap \text{V}(X_1) \subset \text{int}(c_2) \subset \text{int}(\hat{c})$. In particular, the edge (v_2, β_2) of b crosses c_2, in contradiction to the fact that every edge in K' that crosses c_2 crosses the edge (v_3, v_1) of c_2 and therefore goes to v.

Otherwise, $j = 3$ and the edge (v, v_3) of b together with $v_3 \in \text{V}(X_1)$ and therefore an edge (v_3, v) in K' makes b a ghost, in contradiction to $b \in \mathcal{C}'_{\text{ns}}$.

Therefore, in either case both maximal cycles c_1 and c_2 are destroyed, and no bad cycle is created.

Case 2.2: There are no two interior-disjoint maximal 3-cycles in $\mathcal{C}_{\text{ns}}$ that share an edge; refer to Fig. 11d. Let $c \in \mathcal{C}_{\text{ns}}$ be an arbitrary maximal cycle, where $c = (v_1, \ldots, v_m)$ for $m \in \{2, 3\}$, and let X_1 be an arbitrary associated 3-crossing for which there exists a vertex $v \in \text{V}(X_1) \cap \text{int}(c)$. We set $f' = F(f, c, X_1, v)$.

Assume first that $m = 2$. As noted above, any bad cycle b has exactly three vertices: $b = (v, v_2, \beta_2)$, where $v = f'(X_1)$, $v_2 = f'(X_2)$, and $\beta_2 = f'(X_3)$ for some 3-crossing X_3 in the exterior of c. Note that, by the condition of Case 2 we know that $v \in \text{V}(X_1) \cap \text{V}(X_2)$. Further, $v, v_2 \in \text{V}(X_1) \cap \text{V}(X_2)$ implies that (v, v_2) is a 2-cycle in K'. That is, b is a ghost in K', in contradiction to $b \in \mathcal{C}'_{\text{ns}}$.

Assume next that $m = 3$ (as depicted in Fig. 11d). By the condition of Case 2 we may assume (by cyclically relabeling (X_1, X_2, X_3) if necessary) that $v \in \text{V}(X_1) \cap \text{V}(X_2)$ (and possibly, $v \in \text{V}(X_3)$). As noted above, any bad cycle b has exactly three vertices: $b = (v, v_j, \beta_2)$, where $v = f'(X_1)$, $v_j = f'(X_j)$ for $j \in \{2, 3\}$, and $\beta_2 = f'(X_4)$ for some 3-crossing X_4 in the exterior of c. Assume that b is maximal with these properties.

If $j = 2$, then $c' = (v_1, v_2, \beta_2)$ is a maximal 3-cycle in the original conflict digraph K. We claim that the cycle c' does not contain c. Suppose to the contrary that c' contains c. Then $v, v_3 \in \mathrm{int}(c')$. Hence c' is not smooth, contradicting our assumption that $c \in \mathcal{C}_{\mathrm{ns}}$ is maximal. This proves the claim. It follows that c and c' are interior-disjoint maximal cycles in $\mathcal{C}_{\mathrm{ns}}$ that share the edge (v_1, v_2), contradicting our assumption in Case 2.2.

Otherwise, $j = 3$ and then v is incident to X_3. In this case, $v, v_3 \in \mathrm{V}(X_1) \cap \mathrm{V}(X_3)$, and we create a 2-cycle (v, v_3) in K'. Hence $b = (v, v_3, \beta_2)$ is a ghost, contradicting our assumption $b \in \mathcal{C}'_{\mathrm{ns}}$.

Consequently, there are no bad cycles when $m = 3$ and $j \in \{2, 3\}$.

In all three cases, we have shown that no bad cycle is created, which confirms (A). By (A) and (B), each incremental modification of the initial function f strictly decreases the set V_{ns}. After at most $|V|$ repetitions, we obtain an injective function for which $\mathcal{C}_{\mathrm{ns}} = \emptyset$, as required. □

6.3. How to choose the function g

Let $f : \mathcal{X} \to V$ be a function such that $\mathcal{C} = \mathcal{C}_{\mathrm{s}}$ (that is, all short nonghost cycles in the corresponding conflict digraph K are smooth), which exists by Lemma 19. As a first step, we will use Hall's theorem to show that there is a matching of the cycles in $\mathcal{C}$ to the vertices in V such that every cycle $c \in \mathcal{C}$ is matched to an incident vertex $s(c)$. Then, our plan is to break the cycle c at the 3-crossing $X \in \mathcal{X}$ for which $f(X) = s(c)$, by choosing the value of $g(X)$ appropriately.

For a subset $\mathcal{B} \subseteq \mathcal{C}$, let $\mathrm{V}(\mathcal{B})$ denote the set of all vertices incident to some cycle in $\mathcal{B}$.

Lemma 20. *For every set $\mathcal{B}_0 \subseteq \mathcal{C}$ of pairwise interior-disjoint cycles, $|\mathcal{B}_0| \le |\mathrm{V}(\mathcal{B}_0)|$.*

Proof. We use double counting. Let I be the set of all pairs $(v, c) \in V \times \mathcal{B}_0$ such that v is incident to c. Every cycle is incident to at least two vertices, hence $|I| \ge 2|\mathcal{B}_0|$. By Lemma 13, every vertex is incident to at most two interior-disjoint cycles. Consequently, $|I| \le 2|\mathrm{V}(\mathcal{B}_0)|$. The combination of the upper and lower bounds for $|I|$ yields $|\mathcal{B}_0| \le |\mathrm{V}(\mathcal{B}_0)|$, as claimed. □

Lemma 21. *For every set $\mathcal{B} \subseteq \mathcal{C}$ of cycles, we have $|\mathcal{B}| \le |\mathrm{V}(\mathcal{B})|$.*

Proof. We proceed by induction on the number of cycles in $\mathcal{B}$. In the base case, we have one cycle, which has at least two vertices.

Assume $|\mathcal{B}| \ge 2$, and let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the set of cycles in $\mathcal{B}$ that are maximal for containment. By Lemma 18 the cycles in $\mathcal{B}_0$ are pairwise interior-disjoint, and by Lemma 20, we have $|\mathcal{B}_0| \le |\mathrm{V}(\mathcal{B}_0)|$. Induction for $\mathcal{B} \setminus \mathcal{B}_0$ yields $|\mathcal{B} \setminus \mathcal{B}_0| \le |\mathrm{V}(\mathcal{B} \setminus \mathcal{B}_0)|$. By Lemma 18, the vertex sets $\mathrm{V}(\mathcal{B}_0)$ and $\mathrm{V}(\mathcal{B} \setminus \mathcal{B}_0)$ are disjoint. The combination of the two inequalities yields $|\mathcal{B}| \le |\mathrm{V}(\mathcal{B})|$. □

Lemma 22. *There exists an injective function $s : \mathcal{C} \to V$ that maps every cycle in $\mathcal{C}$ to one of its vertices.*

Proof. Consider the bipartite graph with partite sets $\mathcal{C}$ and V, where the edges represent vertex-cycle incidences. By Hall's theorem and Lemma 21 (Hall's condition), there exists a matching of $\mathcal{C}$ into V, in which each cycle in $\mathcal{C}$ is matched to an incident vertex. □

We are ready to define the function $g : \mathcal{X} \to E$, that maps every 3-crossing $X \in \mathcal{X}$ to one of its edges.

Lemma 23. *Let $f : \mathcal{X} \to V$ be a function obtained by Lemma 19, and let K be the corresponding conflict digraph. There is a function $g : \mathcal{X} \to E$ such that*

- *for every $X \in \mathcal{X}$, $g(X) \in X$ and $g(X)$ is not incident to $f(X)$;*
- *for every 2-cycle $(f(X_1), f(X_2))$ in K, the edges $g(X_1)$ and $g(X_2)$ do not cross in G';*
- *for every 3-cycle $(f(X_1), f(X_2), f(X_3))$ in K, at least two of the edges in $\{g(X_1), g(X_2), g(X_3)\}$ do not cross in G'.*

Proof. By Lemma 22, there is an injective function $s : \mathcal{C} \to V$ that maps every cycle $c \in \mathcal{C}$ to one of its vertices. For each cycle $c \in \mathcal{C}$, vertex $s(c)$ is the endpoint of some directed edge $(q(c), s(c))$ in K. Consequently, there is a 3-crossing $X \in \mathcal{X}$ such that $s(c) = f(X)$ and $q(c) \in \mathrm{V}(X)$. We say that the 3-crossing X is *assigned* to the cycle c. We define $g : \mathcal{X} \to E$ by successively selecting $g(X) \in X$ for every 3-crossing $X \in \mathcal{X}$. We distinguish between two types of 3-crossings, depending on whether or not they are assigned to a 2-cycle of $\mathcal{C}$.

3-crossings that are not assigned to a 2-cycle. For every 3-crossing X that is not assigned to any cycle, choose $g(X)$ to be an arbitrary edge in X that is not incident to the vertex $f(X)$. For every 3-crossing X that is assigned to a 3-cycle $c \in \mathcal{C}$, choose $g(X)$ to be the (unique) edge in X that is incident to neither $q(c)$ nor $s(c)$. If $c = (f(X_1), f(X_2), f(X_3))$ and without loss of generality $s(c) = f(X_2)$, then $g(X_2)$ is not incident to $f(X_1) = q(c)$, consequently $g(X_1)$ is disjoint from $g(X_2)$ in G'. (Note that $g(X_1)$ is not incident to $f(X_2) = s(c)$ because this would induce a 2-cycle $(f(X_1), f(X_2))$ in K, making c a ghost.)

3-crossings assigned to 2-cycles. Consider a 2-cycle $c \in \mathcal{C}$, and let X_1 and X_2 denote the associated 3-crossings so that without loss of generality $s(c) = f(X_1)$. Assume without loss of generality that c is oriented clockwise. We distinguish three cases.

Case 1: $g(X_2)$ has already been selected and $g(X_2)$ is incident to $f(X_1)$. Then let $g(X_1)$ be the unique edge in X_1 incident to $f(X_2)$ (see Fig. 12). We claim that $g(X_1)$ and $g(X_2)$ do not cross in G'. As both edges are rerouted, by Lemma 6 they can only cross in the neighborhood of $f(X_1)$ or $f(X_2)$. Let a_i be the edge of X_i incident to $f(X_i)$,

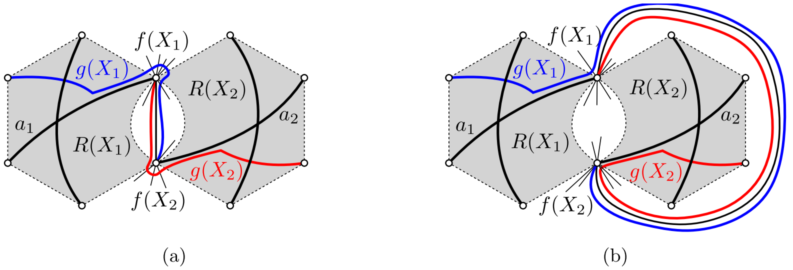


Fig. 12. In Case 1, the edge $g(X_2)$ is incident to $f(X_1)$. We set $g(X_1)$ so that it is incident to $f(X_2)$. The edge $f(X_1)f(X_2)$ may be drawn in various ways in G, two examples are shown above. Regardless of how $f(X_1)f(X_2)$ is drawn, the edge separates $g(X_1)$ and $g(X_2)$ and ensures that they are disjoint.

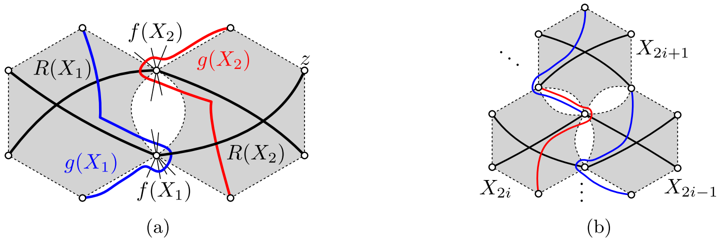


Fig. 13. (a) In Case 2, the edge $g(X_2)$ is not incident to $f(X_1)$. We set $g(X_1)$ so that it is not incident to $f(X_2)$, to ensure that $g(X_1)$ and $g(X_2)$ are disjoint. (b) In Case 3 we face a cycle of 2-cycles. We consistently select edges to be rerouted in even (red edge) and odd (blue edge) 3-crossings so that they are pairwise disjoint in G'.

for $i \in \{1, 2\}$. The edge $g(X_i)$, for $i \in \{1, 2\}$, follows a_i towards the neighborhood of $f(X_i)$ and then crosses the edges incident to $f(X_i)$ following a_i in clockwise order (the orientation of c) until reaching the edge $(f(X_1), f(X_2))$. Then $g(X_i)$ follows $f(X_1)f(X_2)$ to its other endpoint, without crossing the edge. Therefore, the path formed by the edges a_1, $f(X_1)f(X_2)$, and a_2 splits the neighborhoods of $f(X_1)$ and $f(X_2)$ into two components so that $g(X_1)$ and $g(X_2)$ are in different components. Thus $g(X_1)$ and $g(X_2)$ do not cross, as claimed.

Case 2: $g(X_2)$ has already been selected and $g(X_2)$ is not incident to $f(X_1)$. Then let $g(X_1)$ be the unique edge in X_1 incident to neither $f(X_1)$ nor $f(X_2)$ (see Fig. 13a). We claim that $g(X_1)$ and $g(X_2)$ do not cross in G'. As both edges are rerouted, by Lemma 6 they can only cross in the neighborhood of $f(X_1)$ or $f(X_2)$. But as $g(X_1)$ is not incident to $f(X_2)$, there is a neighborhood of $f(X_2)$ that is disjoint from $g(X_1)$, and so $g(X_1)$ and $g(X_2)$ do not cross there. Similarly, there is a neighborhood of $f(X_1)$ that is disjoint from $g(X_2)$, and so $g(X_1)$ and $g(X_2)$ do not cross there, either. It follows that $g(X_1)$ and $g(X_2)$ do not cross in G', as claimed.

Case 3: no 3-crossing X_1 is assigned to a 2-cycle so that $g(X_2)$ has already been selected. Then we are left with 3-crossings that correspond to 2-cycles and form *cycles* $L = (X_1, \dots, X_\ell)$ such that $(f(X_i), f(X_{i\oplus 1}))$ is a 2-cycle in $\mathcal{C}$, for $i = 1, \dots, \ell$. These cycles are interior-disjoint by Lemma 18, and any two consecutive cycles in L have opposite orientations by Lemma 13. It follows that ℓ is even.

Since every 2-cycle in L is smooth, the three vertices $f(X_{i\ominus 1})$, $f(X_i)$, and $f(v_{i\oplus 1})$ are consecutive along $\partial\,\mathrm{R}(X_i)$. For every odd $i \in \{1,\dots,\ell\}$, let $g(X_i)$ be the (unique) edge in X_i incident to $f(X_{i\ominus 1})$ (and incident to neither $f(X_i)$ nor $f(X_{i\oplus 1})$). Similarly, for every even $i \in \{1,\dots,\ell\}$, let $g(X_i)$ be the edge in X_i incident to $f(X_{i\oplus 1})$ (and incident to neither $f(X_i)$ nor $f(X_{i\ominus 1})$). Refer to Fig. 13b for an illustration.

For every odd index $i \in \{1,\dots,\ell\}$, the rerouted edges $g(X_i)$ and $g(X_{i\oplus 1})$ are incident to neither $f(X_{i\oplus 1})$ nor $f(X_i)$. Similarly, for every even index $i \in \{1,\dots,\ell\}$, the rerouted edges $g(X_i)$ and $g(X_{i\oplus 1})$ are incident to $f(X_{i\oplus 1})$ and $f(X_i)$, respectively. In both cases, the rerouted edges $g(X_i)$ and $g(X_{i\oplus 1})$ are disjoint.

Ghost cycles. It remains to consider ghost cycles. Let c_1 be a ghost cycle in K. Without loss of generality, assume that $c_1 = (v_1, v_2, v_3)$, where $v_1 = f(X_1)$, $v_2 = f(X_2)$, $v_3 = f(X_3)$, and $c_2 = (v_1, v_2)$ is a 2-cycle in $\mathcal{C}$. Recall that c_2 is smooth (cf. Lemma 19). By construction, $g(X_1)$ and $g(X_2)$ do not cross in G'. Hence at least two of the edges in $\{g(X_1), g(X_2), g(X_3)\}$ do not cross in G', as required. □

6.4. Putting all the results together

The results in this section can be used to prove that every 2-planar simple topological graph can be transformed into a 3-quasiplanar topological graph by means of a suitable global rerouting operation.

Lemma 24. *Let $G = (V, E)$ be a 2-planar simple topological graph, and let $\mathcal{X}$ be the set of its 3-crossings. There exist functions $f : \mathcal{X} \to V$ and $g : \mathcal{X} \to E$ such that the topological graph G', $G' \simeq G$, obtained from G by applying a full rerouting operation with functions (g, f) is 3-quasiplanar.*

Proof. Lemmas 19 and 23 imply the existence of two functions f and g such that the graph G' obtained from G by applying a global rerouting operation (g, f) contains neither twin nor whirl configurations. Thus, by Lemma 12, G' does not contain 3-crossings and the statement follows. □

7. Obtaining simplicity

Lemmas 11 and 24 imply that, for $k \geq 2$, every k-planar simple topological graph G can be redrawn such that the resulting topological graph G', $G' \simeq G$, contains no $(k+1)$-crossings and no two edges are rerouted around the same vertex. We now show that G' is also simple if $k = 2$. Then we handle the case $k \geq 3$, in which G' may not be simple.

Lemma 25. *For $k = 2$ the topological graph G' in Lemma 24 is simple.*

Proof. Suppose, for a contradiction, that G' is not simple, i.e., either two edges cross at least twice or two adjacent edges cross each other.

Suppose first that there exist two edges e_1 and e_2 that cross at least twice in G'. A safe edge does not cross any edge more than once by Lemma 7(ii). Any two nonrerouted edges cross at most once since G is simple. A critical edge crosses any nonrerouted edge at most once by construction. It remains to consider the case that both e_1 and e_2 are critical, that is, $e_1 = g(X_1)$ and $e_2 = g(X_2)$ for some 3-crossings $X_1, X_2 \in \mathcal{X}$. By Lemma 6, $g(X_1)$ is incident to $f(X_2)$ and $g(X_2)$ is incident to $f(X_1)$. It follows that $(f(X_1), f(X_2))$ is a 2-cycle in the conflict digraph K. By Lemma 19, every 2-cycle in K is smooth, and by Lemma 23, the edges $g(X_1)$ and $g(X_2)$ do not cross in G'. This contradicts our assumption that $e_1 = g(X_1)$ and $e_2 = g(X_2)$ cross twice. We conclude that any two edges in G' cross at most once.

Suppose next that e_1 and e_2 are adjacent and cross at least once in G'. Two adjacent nonrerouted edges do not cross because G is simple. If a safe edge crosses an adjacent edge e', then e' is critical by Lemma 7(iii). Therefore, we may assume that e_1 is critical, that is, $e_1 = g(X_1)$ for some $X_1 \in \mathcal{X}$. As two adjacent critical edges do not cross by Lemma 8, only two cases remain.

Case 1: e_2 is safe. Then by Lemma 7(iii), e_2 is drawn in $\mathrm{R}(X_2)$, for some $X_2 \in \mathcal{X} \setminus \{X_1\}$, e_2 is incident to $f(X_1)$, and e_1 is incident to $f(X_2)$. Since e_1 and e_2 are adjacent, e_2 is also incident to an endpoint of $g(X_1)$. By the injectivity of f, $f(X_1) \neq f(X_2)$, and so $e_2 = f(X_1)f(X_2)$. It follows that $(f(X_1), f(X_2))$ is a 2-cycle in the conflict digraph K. By Lemma 19, every 2-cycle in K is smooth. So e_2 is an edge between two consecutive vertices along $\partial\,\mathrm{R}(X_2)$, in contradiction to X_2 being a home for e_2.

Case 2: e_2 is nonrerouted. By Lemma 6, if $e_1 = g(X_1)$ crosses e_2, then e_2 is incident to $f(X_1)$. So $e_2 = f(X_1)z$, where z is an endpoint of $g(X_1)$ since e_1 and e_2 share an endpoint distinct from $f(X_1)$. If e_2 belongs to $\partial\,\mathrm{R}(X_1)$, then e_1 crosses e_2 neither in G nor in G', contrary to our assumption that the two edges cross each other. Otherwise, the 3-crossing X_1 is a home for e_2. But then e_2 would have been rerouted in a home rerouting operation, contrary to our assumption that e_2 is nonrerouted.

Both cases lead to a contradiction and hence the statement follows. □

We now consider the case $k \geq 3$. We first characterize in the possible configurations that yield pairs of edges that intersect in two or more points in G' (Lemma 26), and then show how to further redraw some of these edges to eliminate multiple intersections without introducing $(k+1)$-crossings (Lemma 27).

Lemma 26. *Let G be a k-planar simple topological graph, where $k \geq 3$, and G' the topological graph obtained from G by executing a full rerouting operation.*

- *If e_1 and e_2 are independent edges that cross more than once in G', then both e_1 and e_2 are critical and they each are rerouted around an endpoint of the other (that is,*

$e_1 = g(X_1)$ and $e_2 = g(X_2)$ for some $(k+1)$-crossings $X_1, X_2 \in \mathcal{X}$, $X_1 \neq X_2$, and $f(e_1)$ is an endpoint of e_2 and $f(e_2)$ is an endpoint of e_1). Furthermore, $f(X_1)$ and $f(X_2)$ are adjacent in neither $\partial\,\mathrm{R}(X_1)$ nor $\partial\,\mathrm{R}(X_2)$.

- *If $e_1 = uv$ and $e_2 = vw$ are adjacent edges that cross in G', then after possibly exchanging the roles of e_1 and e_2, we have that e_1 is critical rerouted around w, and e_2 is safe rerouted in its home which is the $(k+1)$-crossing of e_1. Furthermore, there exists a critical edge e_3 such that e_1 and e_2 have each been rerouted around an endpoint of the other.*

Proof. Assume first that there exist two edges, e_1 and e_2, crossing two or more times in G'. Since G is simple, at least one of them, say e_1, has been rerouted. By Lemma 7(i), neither e_1 nor e_2 is safe, and thus e_1 is critical. Also, if e_2 is nonrerouted, e_1 crosses e_2 at most once by Lemma 2(iii). Thus, we may assume that both e_1 and e_2 are critical. This implies that e_1 and e_2 belong to different $(k+1)$-crossings of G; so, they do not cross in G by Property 1. Hence, by Lemma 6(b), at least one of them has been rerouted around an endpoint of the other, say e_1 around an endpoint of e_2. This introduces a single crossing between e_1 and e_2, namely between the hook of e_1 and a tip of e_2. Thus, the other crossing must be between the hook of e_2 and a tip of e_1. By Lemma 6(b) and by the injectivity of f, no two edges are rerouted around the same vertex. We may assume that $e_1 = g(X_1)$ and $e_2 = g(X_2)$ for some $(k+1)$-crossings $X_1, X_2 \in \mathcal{X}$, $X_1 \neq X_2$; and $f(e_1)$ is an endpoint of e_2 and $f(e_2)$ is an endpoint of e_1. If vertices $f(e_1)$ and $f(e_2)$ are adjacent in $\partial\,\mathrm{R}(X_1)$ (resp., $\partial\,\mathrm{R}(X_2)$), then the tip of e_1 (resp., e_2) follows the edge $f(X_1)f(X_2)$, consequently avoids e_2 (resp, e_1), and so e_1 and e_2 would cross at most once (see Fig. 7b and Fig. 12 for examples).

Assume now that there exist two adjacent edges, $e_1 = uv$ and $e_2 = vw$, that cross in G'. Since G is simple and by Lemma 7(iii), at least one of e_1 and e_2 is critical and rerouted around an endpoint of the other. In particular, we can assume that e_1 is critical and rerouted around the vertex u. But then the $(k+1)$-crossing of e_1 is a home for e_2. In particular, e_2 is safe. By Lemma 7(iii), there exists another $(k+1)$-crossing that is a home for e_2, which implies the existence of a critical edge e_3 incident to u that has been rerouted around v, as claimed. □

Lemma 27. *For every k-planar simple topological graph G, where $k \geq 3$, there exists a $(k+1)$-quasiplanar simple topological graph G^*, $G^* \simeq G$.*

Proof. Let G' be the $(k+1)$-quasiplanar topological graph obtained from G by executing a full rerouting operation (cf. Lemma 11). We may assume that G' is not simple, as otherwise the statement would follow with $G^* = G'$. By Lemma 26, there exists a pair of critical edges that are each rerouted around an endpoint of the other; see Fig. 14a. Let $\mathcal{P}$ be the set of such (unordered) pairs. Note that the pairs in $\mathcal{P}$ are pairwise disjoint since f is injective.

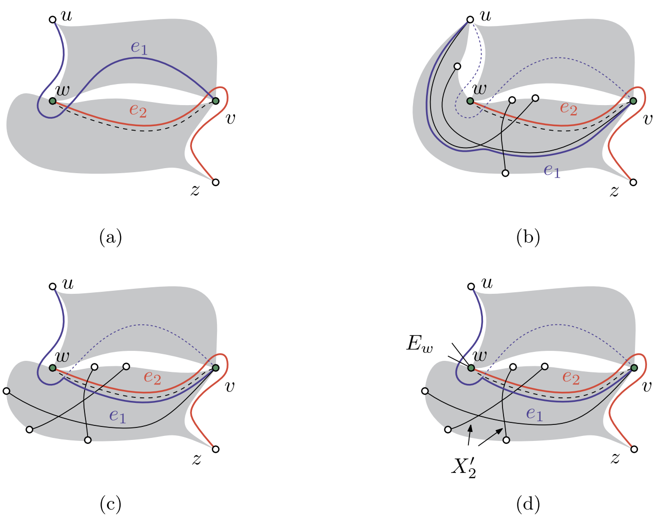


Fig. 14. (a) A double crossing between two edges e_1 and e_2 due to rerouting; the dashed edge vw may be present or not. The configuration is resolved by redrawing the edge e_1 as in (b) if $u \in \mathrm{V}(X_2)$ or (c) if $u \notin \mathrm{V}(X_2)$. (d) Edges crossing e_1 after the transformation.

Rerouting operations. For every pair in $\mathcal{P}$, we reroute one of the two edges as follows. Let $\{e_1, e_2\} \in \mathcal{P}$. We introduce some notation. Assume that $e_1 \in X_1$ and $e_2 \in X_2$ for some $X_1, X_2 \in \mathcal{X}$. Further assume that $e_1 = uv$ is rerouted around vertex w, and $e_2 = wz$ is rerouted around v. Note that v and w are adjacent in neither $\partial\,\mathrm{R}(X_1)$ nor $\partial\,\mathrm{R}(X_2)$ by Lemma 26. Hence u and w are consecutive along $\partial\,\mathrm{R}(X_1)$, and v and z are consecutive along $\partial\,\mathrm{R}(X_2)$.

If $vw \notin E$, then we can arbitrarily choose e_1 or e_2 to be redrawn. Otherwise, edge vw has a home in both X_1 and X_2. If vw has been redrawn in X_1 due to a home rerouting operation, then we redraw e_1, else we redraw e_2. In the following we assume without loss of generality that we redraw e_1.

We distinguish between two cases, based on whether $u \in \mathrm{V}(X_2)$. Assume first that $u \in \mathrm{V}(X_2)$. In this case, both endpoints of $e_1 = uv$ are in $\mathrm{V}(X_d)$ for $d = 1, 2$, but $e_1 \notin X_2$ and X_2 is not a home for e_1 (otherwise e_1 would be safe). Similarly to the home rerouting operation, we redraw e_1 in the region $\mathrm{R}(X_2) \setminus \mathrm{D}(X_2)$, such that it crosses the edges of X_2 at most once, and such that it crosses neither e_2 nor the edge of X_2 incident to u; see Fig. 14b. Note that the new drawing of e_1 does not cross the two possible safe edges in $\mathrm{R}(X_2)$.

Assume next that $u \notin \mathrm{V}(X_2)$. We redraw the portion of e_1 between the two crossings with e_2 by following e_2, crossing neither e_2 nor the possible safe edge vw (if it exists). More precisely, we redraw the tip of e_1 crossed by the hook of e_2 by following the tip of e_2 crossed by the hook of e_1 without crossing it and without crossing vw; see Fig. 14c.

Denote by G^* the topological graph obtained by applying the operation for every pair in $\mathcal{P}$.

Proving that G^* does not contain $(k+1)$-crossings. Since G' is $(k+1)$-quasiplanar by Lemma 11, every $(k+1)$-crossings of G^* involves an edge that has been redrawn. Let e_1 be an edge that has been redrawn from a pair $\{e_1, e_2\} \in \mathcal{P}$, let X_1 and X_2 be the $(k+1)$-crossings of G containing e_1 and e_2, respectively. The edges crossing e_1 in G^* are (see Fig. 14d):

(i) a set $X'_2 \subset X_2$ of edges crossing the tip of e_2 that is used to redraw e_1 and
(ii) a fan E_w of edges incident to the vertex w around which e_1 has been rerouted (and thus these edges cross the hook of e_1).

Since u and w are consecutive around $\mathrm{R}(X_1)$, it follows that e_1 does not cross any edge in X_1. Further, note that X'_2 contains all the edges that cross e_1 in G^* and not in G'. This immediately implies that any $(k+1)$-crossing of G^* that contains e_1 also contains some edge of X'_2, otherwise such a $(k+1)$-crossing would also exist in G'. The edges in X'_2 do not cross edges in E_w, since X_2 does not contain any edge incident to w, other than e_2. Finally, there are at most $k-1$ edges in X'_2, since X_2 contains $k+1$ edges and at least two of them do not cross e_1, namely e_2 and the edge incident to v. Thus, the edges in X'_2 are not involved in any $(k+1)$-crossing with e_1. This implies that G^* is $(k+1)$-quasiplanar.

Proving that G^* is simple. By Lemma 26 and by the choice of the edge $e_1 \in \{e_1, e_2\}$, $\{e_1, e_2\} \in \mathcal{P}$, if G^* contains a pair of independent edges that cross more than once or a pair of adjacent edges cross, then this pair must include an edge that has been rerouted.

Let e_1 be an edge that has been redrawn from a pair $\{e_1, e_2\} \in \mathcal{P}$, with the notation used for the description of the rerouting above. First observe that e_1 does not cross any edge incident to v in G^*: it crosses neither vw nor the edge of X_2 incident to v, by construction. Also, e_1 does not cross any edge incident to u in G^*. In fact, e_1 does not cross uw, since u and w are consecutive along $\partial\,\mathrm{R}(X_1)$; hence, if e crosses an edge incident to u, then this edge belongs to X'_2. However, this implies that $u \in \mathrm{V}(X_2)$, and thus e_1 has been redrawn without crossing the edge of X_2 incident to u.

It remains to prove that e_1 does not cross any edge more than once. First observe that the tip of e_1 incident to u has not been redrawn, since it has been rerouted in G' by following edge uw, which is uncrossed in G since u and w are consecutive along $\partial\,\mathrm{R}(X_1)$. Also, e_1 does not cross any edge in E_w twice, since it crosses all these edges in G', and the only edge crossed twice by e_1 in G' is e_2, by Lemma 26. Hence, if e_1 crosses an edge e_3 twice, then e_3 belongs to X'_2. Since e_1 does not cross any edge of X'_2 in G', by Property 1, we have that both crossings between e_1 and e_3 have been introduced by the redrawing of e_1. However, if $u \in \mathrm{V}(X_2)$, then the redrawing of e_1, which is analogous to the home rerouting operation, does not introduce any double crossing on e_1, by construction. On the other hand, if $u \notin \mathrm{V}(X_2)$, then the part of e_1 that has been redrawn has the same crossings as e_2, which does not cross any edge in X_2 twice. Thus, e_1 does not cross any edge twice. We conclude that G^* is a simple topological graph. $\square$

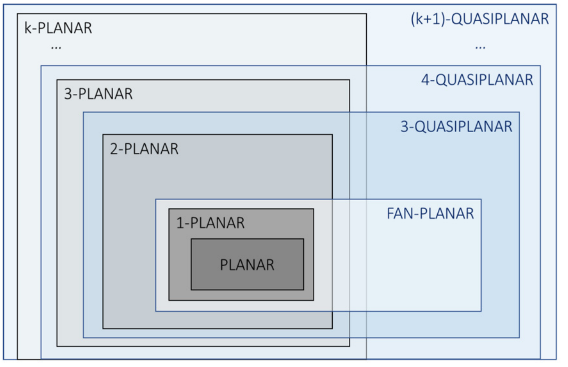


Fig. 15. A Venn diagram showing that every k-planar simple topological graph is a $(k+1)$-quasiplanar simple topological graph, for every $k \geq 2$.

The next theorem summarizes the main result of the paper.

Theorem 1. *Every k-planar simple topological graph is a $(k+1)$-quasiplanar simple topological graph, for every $k \geq 2$.*

Proof. Let G be a k-planar simple topological graph, with $k \geq 2$. We show that there exists a $(k+1)$-quasiplanar simple topological graph G^* such that $G^* \simeq G$. First recall that, by Lemma 1, we can assume that G does not contain any tangled $(k+1)$-crossing. By Lemma 24, for $k = 2$, and by Lemma 11, for $k \geq 3$, there exist functions f and g such that the topological graph $G' \simeq G$ obtained by applying a global rerouting (g, f) to G is $(k+1)$-quasiplanar.

When $k = 2$, the topological graph G' is also simple, as proved in Lemma 25, and the statement follows with $G^* = G'$. For the case $k \geq 3$, instead, G' need not be simple. However, in this case, we can apply an additional redrawing operation which results in a $(k+1)$-quasiplanar simple topological graph G^*, $G^* \simeq G'$, by Lemma 27. $\square$

8. Conclusions and open problems

We have proved that, for any $k \geq 2$, the family of k-planar simple topological graphs is included in the family of $(k+1)$-quasiplanar simple topological graphs, see also Fig. 15 for a diagram illustrating this relationship. This result represents the first nontrivial relationship between the k-planar and the k-quasiplanar graph hierarchies, and contributes to the literature that studies the connection between different families of beyond planar graphs (see, e.g. [10–12,16]). Several interesting problems remain open. Among them:

- We do not know whether our main result extends to nonsimple k-planar graphs for $k \geq 4$: Is every k-planar graph $(k+1)$-quasiplanar? For $k \leq 3$, every k-planar graph is simple k-planar [22, Lemma 1.1]. Hence Theorem 1 readily implies that every k-planar graph is simple $(k+1)$-quasiplanar for $k = 2, 3$.

- For $k \geq 3$, one can also ask whether every k-planar graph is k-quasiplanar. For $k = 2$ the answer is trivially negative, as 2-quasiplanar graphs are precisely the planar graphs. On the other hand, optimal 3-planar graphs are known to be (3-)quasiplanar [9]. We recall that an n-vertex 3-planar graph is optimal if it has $5.5n - 11$ edges [23], and that so far only multi-graphs are known to be in this family, while it is still unknown whether 3-planar graphs with no multi-edges can have $5.5n - 11$ edges (see also [9]). For sufficiently large values of k, one can even investigate whether every k-planar simple topological (sparse) graph G is $f(k)$-quasiplanar, for some function $f(k) = o(k)$.
- One can study *non-inclusion* relationships between the k-planar and the k-quasiplanar graph hierarchies, other than those that are easily derivable from the known edge density results. For example, for any given $k > 3$, can we establish an integer function $h(k)$ such that some $h(k)$-planar graph is not k-quasiplanar?
- A long-standing open problem is to establish the computational complexity of recognizing k-quasiplanar graphs. Is there a polynomial-time algorithm that decides whether a given graph is quasiplanar (or k-quasiplanar for a given constant k)?

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