THE TROPICALIZATION OF THE MODULI SPACE OF CURVES

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Dedicated to Joe Harris on the occasion of his 60th birthday.

Abstract. We show that the skeleton of the Deligne-Mumford-Knudsen moduli stack of stable curves is naturally identified with the moduli space of extended tropical curves, and that this is compatible with the "naive" set-theoretic tropicalization map. The proof passes through general structure results on the skeleton of a toroidal Deligne-Mumford stack. Furthermore, we construct tautological forgetful, clutching, and gluing maps between moduli spaces of extended tropical curves and show that they are compatible with the analogous tautological maps in the algebraic setting.

Contents

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1. Introduction

A number of researchers have introduced and studied the moduli spaces $M_{g,n}^{\mathrm{trop}}$, parametrizing certain metric weighted graphs called *tropical curves*, and exhibited analogies to the Deligne-Mumford-Knudsen moduli stacks of stable pointed curves, $\overline{\mathcal{M}}_{g,n}$, and to the Kontsevich moduli spaces of stable maps [Mik06, Mik07, GM08, GKM09, Koz09, CV10, Koz11, BMV11, Cap11, Cha12, CMV12]. The paper [Cap11] describes, in particular, an order reversing correspondence between the stratification of $M_{g,n}^{\mathrm{trop}}$ and the stratification of $\overline{\mathcal{M}}_{g,n}$, along with a natural compactification $\overline{M}_{g,n}^{\mathrm{trop}}$, the moduli space of *extended* tropical curves, where the correspondence persists. A seminal precursor for all of this work is the paper of Culler and Vogtmann on moduli of graphs and automorphisms of free groups, [CV86], in which a space of metric graphs called "outer space" was introduced.

The analogies between moduli of curves and moduli of graphs go further than the natural stratifications of compactifications. As we show in Section 8, the moduli spaces $\overline{M}_{g,n}^{\mathrm{trop}}$ admit natural maps

$$\pi_{g,n}^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}, \quad i = 1, \ldots, n+1$$

associated to "forgetting the last marked point and stabilizing," analogous to the forgetful maps $\pi_{g,n}$ on the moduli spaces of curves. There are also clutching and gluing maps

$$\kappa_{g_1,n_1,g_2,n_2}^{\mathrm{trop}} : \overline{M}_{g_1,n_1+1}^{\mathrm{trop}} \times \overline{M}_{g_2,n_2+1}^{\mathrm{trop}} \to \overline{M}_{g_1+g_2,n_1+n_2}^{\mathrm{trop}}$$

and

$$\gamma_{g,n}^{\mathrm{trop}} : \overline{M}_{g-1,n+2}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$$

covering the boundary strata of $\overline{M}_{g,n}^{\mathrm{trop}} \smallsetminus M_{g,n}^{\mathrm{trop}}$, analogous to the corresponding clutching and gluing maps κ_{g_1,n_1,g_2,n_2} and $\gamma_{g,n}$ on the moduli spaces of curves. When the various subscripts g, n are evident we suppress them in the notation for these maps.

The main purpose of this paper is to develop these analogies into a rigorous and functorial correspondence. We start with set-theoretic maps from the Berkovich analytic stacks $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ to the tropical moduli spaces $\overline{M}_{g,n}^{\mathrm{trop}}$, described in Definition 1.1.1 below, and use Thuillier's construction of canonical skeletons of toroidal Berkovich spaces [Thu07] to show that these maps are continuous, proper, surjective, and compatible with the tautological forgetful, clutching, and gluing maps. We work extensively with the combinatorial geometry of *extended generalized cone complexes*, as presented in Section 2.5.

To study the skeleton of $\overline{\mathcal{M}}_{g,n}$, we require a mild generalization of Thuillier's construction, presented in Section 6, below; the main technical results are Propositions 6.1.2, 6.1.6 and 6.2.6. Given a proper toroidal Deligne–Mumford stack $\mathcal{X}$ with coarse moduli space X, we functorially construct an extended generalized cone complex, the *skeleton* $\overline{\Sigma}(\mathcal{X})$, which is both a topological closed subspace of the Berkovich analytic space X^{an} associated to X, and also the image of a canonical retraction

$$\mathbf{p}_X : X^{\mathrm{an}} \to \overline{\Sigma}(\mathcal{X}).$$

Composing with the analytic coarse moduli space map we obtain a map $\mathbf{p}_{\mathcal{X}} : \mathcal{X}^{\mathrm{an}} \to \overline{\Sigma}(X)$. The compactified moduli space of tropical curves $\overline{M}_{g,n}^{\mathrm{trop}}$ is similarly an extended generalized cone complex, and one of our primary tasks is to identify the tropical moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$ with the skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$. See Theorem 1.2.1 for a precise statement.

1.1. **The tropicalization map.** There is a natural set theoretic *tropicalization map*

$$\mathrm{Trop} : \overline{\mathcal{M}}_{g,n}^{\mathrm{an}} \to \overline{M}_{g,n}^{\mathrm{trop}},$$

well-known to experts [Tyo10, BPR11, Viv12], defined as follows. A point $[C]$ in $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ is represented, possibly after a field extension, by a stable n-pointed curve C of genus g over the spectrum S of a valuation ring R, with algebraically closed fraction field and valuation denoted val_C. Let $\mathbf{G}$ be the dual graph of the special fiber, as discussed in Section 3.2 below, where each vertex is weighted by the genus of the corresponding irreducible component, and with legs corresponding to the marked points. For each edge e_i in $\mathbf{G}$, choose an étale neighborhood of the corresponding node in which the curve is defined by a local equation $xy = f_i$, with f_i in R.

Definition 1.1.1. The tropicalization of the point $[C] \in \overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ is the stable tropical curve $\mathbf{\Gamma} = (\mathbf{G}, \ell)$, with edge lengths given by

$$\ell(e_i) = \mathrm{val}_C(f_i).$$

See [Viv12, Lemma 2.2.4] for a proof that the tropical curve $\mathbf{\Gamma}$ so defined is independent of the choices of R, C, étale neigborhood, and local defining equation, so the map Trop is well defined.

1.2. **Main results.** Our first main result identifies the map Trop with the projection from $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ to its skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n}^{\mathrm{an}})$.

Theorem 1.2.1. *Let g and n be non-negative integers.*

(1) *There is an isomorphism of generalized cone complexes with integral structure*

$$\Phi_{g,n} : \Sigma(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} M^{\mathrm{trop}}_{g,n}$$

extending uniquely to the compactifications

$$\overline{\Phi}_{g,n} : \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} \overline{M}^{\mathrm{trop}}_{g,n}.$$

(2) *The following diagram is commutative:*

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathbf{p}_{\overline{\mathcal{M}}_{g,n}}"] \arrow[dr, "\mathrm{Trop}"'] & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \arrow[d, "\overline{\Phi}_{g,n}"] \\
& \overline{M}^{\mathrm{trop}}_{g,n}.
\end{tikzcd}$$

In particular the map Trop *is continuous, proper, and surjective.*

The theorem is proven in Section 7.

Our second main result shows that the map Trop is compatible with the tautological forgetful, clutching, and gluing maps.

Theorem 1.2.2. *The following diagrams are commutative.*

The universal curve diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n+1} \arrow[r, "\mathrm{Trop}"] \arrow[d, "\pi^{\mathrm{an}}"'] & \overline{M}^{\mathrm{trop}}_{g,n+1} \arrow[d, "\pi^{\mathrm{trop}}"] \\
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\mathrm{trop}}_{g,n},
\end{tikzcd}$$

the gluing diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g-1,n+2} \arrow[r, "\mathrm{Trop}"] \arrow[d, "\gamma^{\mathrm{an}}"'] & \overline{M}^{\mathrm{trop}}_{g-1,n+2} \arrow[d, "\gamma^{\mathrm{trop}}"] \\
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\mathrm{trop}}_{g,n},
\end{tikzcd}$$

and the clutching diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g_1,n_1+1} \times \overline{\mathcal{M}}^{\mathrm{an}}_{g_2,n_2+1} \arrow[r, "\mathrm{Trop}\times\mathrm{Trop}"] \arrow[d, "\kappa^{\mathrm{an}}"'] & \overline{M}^{\mathrm{trop}}_{g_1,n_1+1} \times \overline{M}^{\mathrm{trop}}_{g_2,n_2+1} \arrow[d, "\kappa^{\mathrm{trop}}"] \\
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\mathrm{trop}}_{g,n}.
\end{tikzcd}$$

Both notation and proofs are provided in Section 8

1.3. **Fans, complexes, skeletons and tropicalization.** There are several combinatorial constructions in the literature relating algebraic varieties to polyhedral cone complexes, and we move somewhat freely among them in this paper. The following is a brief description of the key basic notions, more details will be given in the sequel.

Classical tropicalization studies a subvariety of a torus T over a valued field by looking at its image in $N_{\mathbb{R}}$, the real extension of the lattice of 1-parameter subgroups of the torus, under the coordinatewise valuation map. This basic idea has been generalized in several ways. For algebraic subvarieties of toric varieties, there are extended tropicalization maps to natural partial compactifications on $N_{\mathbb{R}}$ [Kaj08, Pay09a, Rab12]. Similar ideas about extending and compactifying tropicalizations appeared earlier in [Mik07, SS09].

Tropicalization is closely related to several other classical constructions:

1.3.1. *Fans of toric varieties.* A toric variety X with dense torus T corresponds naturally to a fan $\Sigma(X)$ in $N_{\mathbb{R}}$. These appear in [KKMSD73, I.2], where they are called "f.r.p.p. decompositions". See also [Oda88, 1.1] and [Ful93, 1.4]. One key feature of fans, as opposed to abstract cone complexes, is that all of the cones in a fan come with a fixed embedding in an ambient vector space.

1.3.2. *Complexes of toroidal embeddings.* In [KKMSD73, Chapter II], the construction associating a fan to a toric variety is generalized to spaces that look locally sufficiently like toric varieties. To each toroidal embedding without self intersection $U \subset X$, they associate an abstract rational polyhedral cone complex with integral structure, also denoted $\Sigma(X)$. Some authors also refer to these cone complexes as fans [Kat94, Thu07], although they do not come with an embedding in an ambient vector space. For a toroidal embedding $U \subset X$ with self intersections, Thuillier constructs a generalized cone complex, obtained as a colimit of a finite diagram of rational polyhedral cones with integral structure, which we again denote $\Sigma(X)$. See [Thu07, 3.3.2]. Note that both fans and cone complexes associated to toroidal embeddings without self intersection are special cases of Thuillier's construction, so the notation is not ambiguous.

1.3.3. *Extended complexes and skeletons.* Thuillier also introduced natural compactifications of his generalized cone complexes; the more classical $\Sigma(X)$ is an open dense subset of this extended generalized cone complex $\overline{\Sigma}(X)$. The boundary $\overline{\Sigma}(X) \smallsetminus \Sigma(X)$, is sometimes called the "part at infinity", and then $\Sigma(X)$ is referred to as the "finite part" of $\overline{\Sigma}(X)$. See [Thu07, Sections 3.1.2 and 3.3.2]. The extended generalized cone complex $\overline{\Sigma}(X)$ is also called the *skeleton* of the toroidal scheme X. It is an instance of a skeleton of a Berkovich space [Ber99, HL10], and comes with a canonical retraction

$$\mathbf{p} : X^{\beth} \to \overline{\Sigma}(X)$$

such that $\mathbf{p}^{-1}(\Sigma(X)) = X^{\beth} \cap U^{\mathrm{an}}$. Here we use the notation X^{an} for the usual Berkovich analytic space of X, and $X^{\beth}$ is the subset of X^{an} consisting of points over valued fields that extend to Spec of the valuation ring. Notice that if X is proper $X^{\mathrm{an}} = X^{\beth}$, hence $\mathbf{p}$ is a canonical retraction of X^{an} onto the skeleton $\overline{\Sigma}(X)$ that maps U^{an} onto the cone complex $\Sigma(X)$. In Section 6, we extend these constructions to toroidal embeddings of Deligne–Mumford stacks. This generalization is straightforward; no new ideas are needed.

1.3.4. *Logarithmic geometry.* The cone complex of [KKMSD73] is further generalized to logarithmic schemes in [GS11, Appendix B], where the complex associated to a logarithmic scheme X is called the *tropicalization* of X.

1.3.5. *Tropicalization.* Roughly speaking all of these fans, cone complexes, and skeletons are in some sense tropicalizations of the corresponding varieties. Put another way, tropical geometry may be interpreted as the study of skeletons of Berkovich analytifications. The exact relation between compactifications of subvarieties of tori and classical tropicalization is explained by the theory of *geometric tropicalization*, due to Hacking, Keel, and Tevelev [Tev07, HKT09].

We revisit the relations between tropicalization and skeletons of toroidal embeddings in more detail in Sections 5 and 6.

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2. Extended and generalized complexes

2.1. **Cones and extended cones.** Recall that a *polyhedral cone with integral structure*, (σ, M), is a topological space σ, together with a finitely generated abelian group M of continuous real-valued functions on σ, such that the induced map $\sigma \to \mathrm{Hom}(M, \mathbb{R})$ is a homeomorphism onto a strictly convex polyhedral cone in the real vector space dual to M. The cone is *rational* if its image is rational with respect to the dual lattice $\mathrm{Hom}(M, \mathbb{Z})$.

Throughout, all of the cones that we consider are rational polyhedral cones with integral structure, and we refer to them simply as cones. When no confusion seems possible, we write just σ for the cone (σ, M).

Let σ be a cone, and let S_σ be the dual monoid, consisting of linear functions $u \in M$ that are nonnegative on σ. Then σ is canonically identified with the space of monoid homomorphisms

$$\sigma = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0}),$$

where $\mathbb{R}_{\geq 0}$ is taken with its additive monoid structure. The associated *extended cone* is

$$\overline{\sigma} = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0} \sqcup \{\infty\}).$$

A *face* of σ is the subset τ where some linear function $u \in S_\sigma$ vanishes. Each face inherits an integral structure, by restricting the functions in M, and every face of a rational cone is rational. If τ is a face of σ, then the closure of τ in $\overline{\sigma}$ is canonically identified with the extended cone $\overline{\tau}$, and we refer to $\overline{\tau}$ as a face of $\overline{\sigma}$.

2.2. **Cone complexes and extended cone complexes.** A *rational cone complex with integral structure* is a topological space obtained from a finite disjoint union of rational polyhedral cones with integral structure by gluing distinct cones along isomorphic faces in such a way that each cone maps homeomorphically onto its image in the complex. Every face of a cone in the complex is considered as a cone in the complex. See [KKMSD73, II.1] and [Pay09b, Section 2] for further details. All of the polyhedral cone complexes that we consider are rational, with integral structure, so we refer to them simply as *cone complexes*.

Note that cone complexes differ from the fans considered in the theory of toric varieties [Oda88, 1.1], [Ful93, 1.4] in two essential ways. First, unlike a fan, a cone complex does not come with any natural embedding in an ambient vector space. Furthermore, while the intersection of any two cones in a fan is a face of each, the intersection of two cones in a cone complex may be a union of several faces. The latter is similar to the distinction between simplicial complexes and Δ-complexes in cellular topology. See, for instance, [Hat02, Section 2.1].

To each cone complex Σ, we associate the *extended cone complex* $\overline{\Sigma}$ obtained by gluing the extended cones $\overline{\sigma}$ along the extended faces $\overline{\tau}$, whenever τ is a face of σ in Σ.

Write σ_i for the cones in Σ, and σ_i° for the relative interiors. By definition, σ_i° is the interior of σ in $\operatorname{Hom}(M, \mathbb{R})$. It is also the complement of the union of all faces of positive codimension in σ. We write $\overline{\sigma}_i^\circ$ for the complement of the union of all faces of positive codimension in the extended cone $\overline{\sigma}$.

Every point in a cone σ is contained in the relative interior of a unique face, and this generalizes to cone complexes and extended cone complexes in the evident way:

Proposition 2.2.1. *Let Σ be a cone complex. Then $\Sigma = \sqcup \sigma_i^\circ$ and $\overline{\Sigma} = \sqcup \overline{\sigma}_i^\circ$.*

2.3. **Barycentric subdivisions.** Each *extremal ray*, or one-dimensional face, of a cone σ is spanned by a unique primitive generator, i.e. a point whose image in $\operatorname{Hom}(M, \mathbb{R})$ is a primitive lattice point in $\operatorname{Hom}(M, \mathbb{Z})$. The *barycenter* of σ is the ray in its relative interior spanned by the sum of the primitive generators of extremal rays. The iterated stellar subdivision along the barycenters of cones in Σ, from largest to smallest, produces the barycentric subdivision $B(\Sigma)$ of a cone complex Σ. See [KKMSD73, Example III.2.1]. The barycentric subdivision of any cone complex is simplicial, and isomorphic to a fan. See [AMR99, Lemma 8.7].

We define the *barycentric subdivision* $B(\overline{\Sigma})$ *of the extended cone complex* $\overline{\Sigma}$ to be the compact simplicial complex whose cells are the closures in $\overline{\Sigma}$ of the cones in the barycentric subdivision of Σ. Note that the barycentric subdivision $B(\overline{\Sigma})$ of $\overline{\Sigma}$ is not the extended cone complex $\overline{B(\Sigma)}$ of the barycentric subdivision of Σ For instance, if $\Sigma = \sigma = \mathbb{R}^2_{\geq 0}$ is a single quadrant the picture is given in Figure 1.

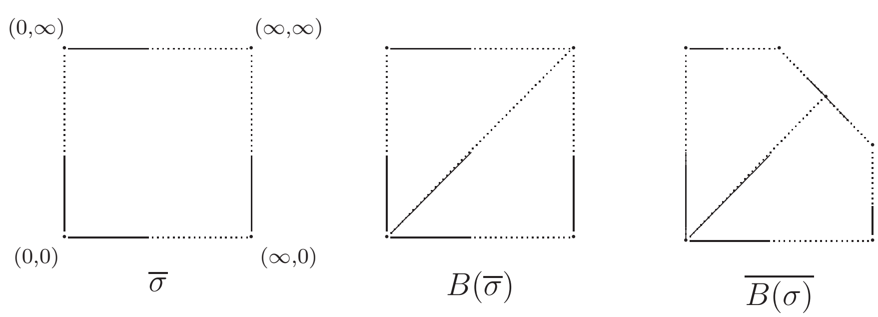


FIGURE 1. Barycentric subdivisions

2.4. **Cone morphisms, face morphisms, and colimits.** A *cone morphism* $\sigma \to \sigma'$ is a continuous map induced by a homomorphism $M' \to M$. In particular, cone morphisms respect the integral structures. A morphism of cone complexes $\Sigma \to \Sigma'$ is a continuous map obtained by gluing cone morphisms. A *face morphism* is a morphism of cone complexes $\Sigma \to \Sigma'$ in which each cone σ in Σ maps isomorphically onto a cone in Σ'.

Note that any cone complex Σ is the colimit of a finite diagram of face morphisms, obtained by gluing its cones along the natural inclusions of faces.

2.5. **Generalized complexes.** In addition to cone complexes, we will consider the following more general objects. A *generalized cone complex* is the colimit of an arbitrary finite diagram consisting of cones σ_i and face morphisms ψ_j,

$$\Sigma = \varinjlim(\sigma_i, \psi_j).$$

A morphism of generalized cone complexes $\Sigma \to \Sigma'$ is a continuous map induced by a cone morphism between suitably chosen diagrams representing Σ and Σ'. Similar objects were named stacky fans in [BMV11, CMV12], but that term is also standard for combinatorial data associated to toric stacks [BCS05]. Generalized cone complexes allow for gluing two faces of the same cone to each other, or for taking the quotient of a cone by a subgroup of its automorphism group. Note that the image of an open cone σ_i° in the generalized cone complex Σ is not necessarily homeomorphic to an open cone. Nevertheless, the space underlying a generalized cone complex has a natural cone complex structure, induced from the barycentric subdivisions of the cones σ_i. We call this cone complex the *barycentric subdivision* $B(\Sigma)$ of Σ.

A generalized cone complex Σ has an associated *generalized extended cone complex* $\overline{\Sigma}$, obtained by taking the colimit of the corresponding diagram of face maps of extended cones. The barycentric subdivision of the cones induces a simplicial complex structure $B(\overline{\Sigma})$ on $\overline{\Sigma}$. Again, this is not the same as the extended complex $\overline{B(\Sigma)}$ of the barycentric subdivision of Σ (Figure 2).

Proposition 2.2.1 does not hold as stated for generalized cone complexes: if $\Sigma = \varinjlim(\tau_i, \phi_j)$ is a generalized cone complex and if τ_1, τ_2 are faces, there may be more than one face map $\phi : \tau_1 \to \tau_2$ over Σ. We may assume all faces of all cones are included in the diagram. We may also replace $\{\tau_i\}$ by a set of representatives $\{\sigma_i\}$ under isomorphisms over Σ, and $\{\phi_j\}$ by the collection $\{\psi_j\}$ of *all* resulting face maps over

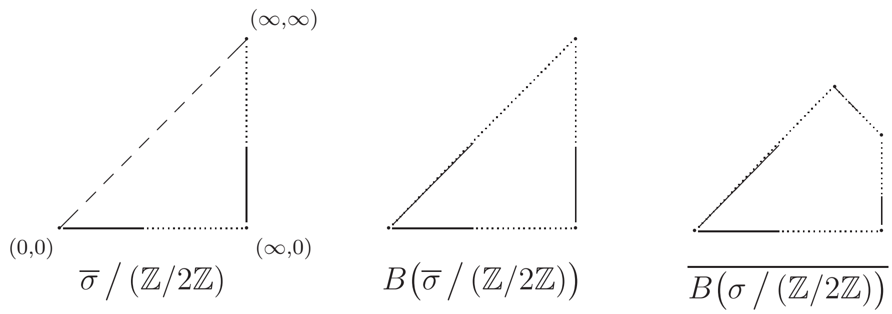


FIGURE 2. The barycentric subdivision of an extended generalized cone complex is *not* the extended cone complex of the barycentric subdivision. The dashed line on the left indicates folding.

Σ between σ_i. Note that a self-face map $\sigma_i \to \sigma_i$ is by definition an isomorphism; the composition of such isomorphisms over Σ is an isomorphism over Σ, so they form a group H_i. Now a point of Σ is in the image of the relative interior σ_i° of a unique cone σ_i, and two points in σ_i° have the same image if and only if they are identified by the diagram, nam ely they are in the same orbit of H_i.

Therefore the correct analogue of Proposition 2.2.1 is the following:

Proposition 2.5.1. *Let Σ be a generalized cone complex. Then $\Sigma = \sqcup \sigma_i^\circ / H_i$ and $\overline{\Sigma} = \sqcup \overline{\sigma}_i^\circ / H_i$, where the union is over all cones of Σ.*

3. Algebraic curves, dual graphs, and moduli

3.1. **Stable curves.** Fix an algebraically closed field k. An *n-pointed nodal curve* $(C; p_1, \ldots, p_n)$ *of genus* g over k is a projective curve C with arithmetic genus $g = g(C)$ over k with only nodes as possible singularities, along with n ordered distinct smooth points $p_i \in C(k)$. The curve is *stable* if it is connected and the automorphism group $\mathrm{Aut}(C, p_i)$ of C fixing the points p_i is finite.

A curve C over an any field is said to be stable if the base change to the algebraic closure is stable.

3.2. **The dual graph of a pointed curve.** The material here can be found with slightly different notation in [ACG11, Cap11]. See also [BMV11].

Recall that to each n-pointed curve $(C; p_1, \ldots, p_n)$ with at most nodes as singularities over an algebraically closed field one assigns its *weighted dual graph*, written somewhat succinctly as

$$\mathbf{G}_C = \mathbf{G} = (V, E, L, h),$$

where

(1) the set of vertices $V = V(\mathbf{G})$ is the set of irreducible components of C;
(2) the set of edges $E = E(\mathbf{G})$ is the set of nodes of C, where an edge $e \in E$ is incident to vertices v_1, v_2 if the corresponding node lies in the intersection of the corresponding components;
(3) the *ordered* set of *legs* of $L = L(\mathbf{G})$ correspond to the marked points, where a marking is incident to the component on which it lies.
(4) the function $h : V \to \mathbb{N}$ is the genus function, where $h(v)$ is the geometric genus of the component corresponding to v.

Note that a node of C that is contained in only one irreducible component corresponds to a loop in G_C.

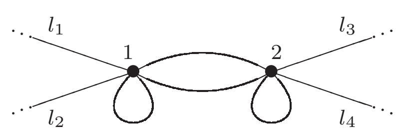


FIGURE 3. A four-legged weighted graph of genus 6

Remark 3.2.1. As customary, the notation suppresses some data, which are nevertheless an essential part of $\mathbf{G}$:

(1) The incidence relations between edges and vertices is omitted.
(2) Consistently with [BM96, ACG11, Cap11] we view an edge $e \in E(\mathbf{G})$ as a pair of distinct half-edges; the effect of this is that if e is a loop, there is a nontrivial graph involution switching the two half edges of e.

When we talk about a *graph* we will always mean a weighted graph, unless we explicitly refer to the *underlying graph* of a weighted graph.

The valence n_v of a vertex $v \in V$ is the total number of incidences of edges and legs at v, where each loop contributes two incidences. The graph is said to be *stable* if it is connected and satisfies the following: for every $v \in V$:

- if $h(v) = 0$ then $n_v \geq 3$; and
- if $h(v) = 1$ then $n_v \geq 1$.

Note that a pointed nodal curve C is stable if and only if the graph $\mathbf{G}_C$ is stable.

The *genus* $g(\mathbf{G})$ of a connected weighted graph is

$$h^1(\mathbf{G},\mathbb{Q}) + \sum_{v\in V} h(v),$$

and for a connected pointed nodal curve C we have

$$g(C) = g(\mathbf{G}_C).$$

In essence this means that the weight $h(v)$ can be imagined as a replacement for $h(v)$ infinitely small loops hidden inside v, or even an arbitrary, infinitely small graph of genus $h(v)$ hidden inside v.

By the *automorphism group* $\mathrm{Aut}(\mathbf{G})$ of $\mathbf{G}$ we mean the set of graph automorphisms preserving the ordering of the legs and the genus function h.

A *weighted graph contraction* $\pi : \mathbf{G} \to \mathbf{G}'$ is a contraction of the underlying graph (composition of edge contractions), canonically endowed with weight function h' given by

$$h'(v') = g(\pi^{-1}v'),$$

for v' in $V(\mathbf{G}')$. Note that weighted graph contractions preserve the genus, and any contraction of a stable weighted graph is stable.

3.3. **Strata of the moduli space of curves.** We consider the *moduli stack* $\overline{\mathcal{M}}_{g,n}$ and its *coarse moduli space* $\overline{M}_{g,n}$. The stack $\overline{\mathcal{M}}_{g,n}$ is smooth and proper, and the boundary $\overline{\mathcal{M}}_{g,n} \smallsetminus \mathcal{M}_{g,n}$ is a normal crossings divisor; this endows $\overline{\mathcal{M}}_{g,n}$ with a natural toroidal structure given by the open embedding $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. See [DM69, ACG11, HM98] for generalites on moduli spaces, and [KKMSD73] for an introduction to toroidal embeddings.

The toroidal structure on $\overline{\mathcal{M}}_{g,n}$ induces a stratification, described as follows. Each stable graph $\mathbf{G}$ of genus g with n legs corresponds to a smooth, locally closed stratum $\mathcal{M}_{\mathbf{G}} \subset \overline{\mathcal{M}}_{g,n}$. The curves parametrized by $\mathcal{M}_{\mathbf{G}}$ are precisely those whose dual graph is isomorphic to $\mathbf{G}$. The codimension of $\mathcal{M}_{\mathbf{G}}$ inside $\overline{\mathcal{M}}_{g,n}$ is the number of edges of $\mathbf{G}$, and $\mathcal{M}_{\mathbf{G}}$ is contained in the closure of $\mathcal{M}_{\mathbf{G}'}$ if and only if there is a graph contraction $\mathbf{G} \to \mathbf{G}'$.

3.4. **Explicit presentation of $\mathcal{M}_{\mathbf{G}}$.** The stratum $\mathcal{M}_{\mathbf{G}}$ parametrizing curves with dual graph isomorphic to $\mathbf{G}$ has the following explicit description in terms of moduli of smooth curves and graph automorphisms, see [ACG11, Section XII.10]:

Recall that the valence of a vertex $v \in V(\mathbf{G})$ is denoted n_v. Consider the moduli space $\widetilde{\mathcal{M}}_{\mathbf{G}} = \prod_v \mathcal{M}_{h(v),n_v}$. The stack $\widetilde{\mathcal{M}}_{\mathbf{G}}$ can be thought of as the moduli stack of "disconnected stable curves", where the universal family $\widetilde{\mathcal{C}}^{dis}_{\mathbf{G}}$ is the disjoint union of the pullbacks of the universal families $\mathcal{C}_{h(v),n_v} \to \mathcal{M}_{h(v),n_v}$. The disconnected curves parametrized by $\widetilde{\mathcal{M}}_{\mathbf{G}}$ have connected components corresponding to $V(\mathbf{G})$, no nodes, and markings in the disjoint union $\widetilde{L} = \sqcup_v \{p^v_1, \ldots, p^v_{n_v}\}$. The data of the graph $\mathbf{G}$ indicate a gluing map $\widetilde{\mathcal{C}}^{dis}_{\mathbf{G}} \to \widetilde{\mathcal{C}}_{\mathbf{G}}$, and $\widetilde{\mathcal{C}}_{\mathbf{G}} \to \widetilde{\mathcal{M}}_{\mathbf{G}}$ is a family of connected curves, with irreducible components identified with $V(\mathbf{G})$, marked points identified with $L(\mathbf{G}) \subset \widetilde{L}$, nodes identified with $E(\mathbf{G})$ and branches of nodes identified with $\widetilde{E} = \widetilde{L} \smallsetminus L(\mathbf{G})$. Indeed, the family of glued curves exhibits $\widetilde{\mathcal{M}}_{\mathbf{G}}$ as the moduli space of curves with graph identified with $\mathbf{G}$. There are sections $\widetilde{\mathcal{M}}_{\mathbf{G}} \to \widetilde{\mathcal{C}}_{\mathbf{G}}$ landing in the nodes, which are images of the sections $\widetilde{\mathcal{M}}_{\mathbf{G}} \to \widetilde{\mathcal{C}}^{dis}_{\mathbf{G}}$ determining the branches.

The group $\mathrm{Aut}(\mathbf{G})$ acts on $\widetilde{\mathcal{C}}_{\mathbf{G}} \to \widetilde{\mathcal{M}}_{\mathbf{G}}$ giving a map $[\widetilde{\mathcal{M}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})] \to \mathcal{M}_{\mathbf{G}}$ such that $[\widetilde{\mathcal{C}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})] \to [\widetilde{\mathcal{M}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})]$ is the pullback of the universal family $\mathcal{C}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}}$.

Proposition 3.4.1. *The quotient stack* $\left[\widetilde{\mathcal{M}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})\right]$ *is canonically isomorphic to* $\mathcal{M}_{\mathbf{G}}$*, and* $\left[\widetilde{\mathcal{C}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})\right]$ *is canonically isomorphic to* $\mathcal{C}_{\mathbf{G}}$*.*

Proof. In [ACG11, Proposition XII.10.11] one obtains a description of the compactification $\left[\,\overline{\widetilde{\mathcal{M}}_{\mathbf{G}}}\,/\,\mathrm{Aut}(\mathbf{G})\,\right]$ as the *normalization* of the closure $\overline{\mathcal{M}_{\mathbf{G}}}$ of $\mathcal{M}_{\mathbf{G}}$ in $\overline{\mathcal{M}}_{g,n}$. Since the open moduli space $\mathcal{M}_{\mathbf{G}}$ is already normal, and since $\left[\widetilde{\mathcal{C}}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})\right]$ is the universal family, the proposition follows. ♣

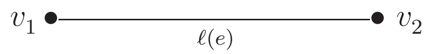
4. Tropical curves and their moduli

4.1. **Tropical curves and extended topical curves.** A *tropical curve* is a metric weighted graph

$$\mathbf{\Gamma} = (\mathbf{G}, \ell) = (V, E, L, h, \ell),$$

where $\ell : E \to \mathbb{R}_{>0}$. When studying moduli of tropical curves we can (and will) restrict our attention to curves whose underlying weighted graph $\mathbf{G}$ is stable, as in every equivalence class of tropical curves there is a unique representative whose underlying weighted graph is stable, see [Cap11]; sometimes such tropical curves are referred to as "stable".

One can realize a tropical curve as an "extended" metric space (keeping the weights on the vertices) by realizing an edge e as an interval of length $\ell(e)$,



and realizing a leg as a copy of $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$ where 0 is attached to its incident vertex:

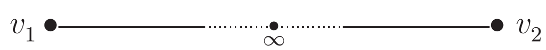
Note that the infinite point on a leg of a tropical curve is a distinguished point which does not correspond to a vertex. Removing these infinite points gives a usual metric graph which is not compact.

We identify $\mathrm{Aut}(\mathbf{\Gamma}) \subset \mathrm{Aut}(\mathbf{G})$ as the subgroup of symmetries preserving the length function ℓ.

An *extended tropical curve* is an *extended metric weighted graph* $\mathbf{\Gamma} = (\mathbf{G}, \ell) = (V, E, L, h, \ell)$, where this time $\ell : E \to \mathbb{R}_{>0} \sqcup \{\infty\}$; we realize an extended tropical curve as an extended metric space by realizing an edge e with $\ell(e) = \infty$ as

$$(\mathbb{R}_{\geq 0} \sqcup \{\infty\}) \cup (\{-\infty\} \sqcup \mathbb{R}_{\leq 0})$$

where the points at infinity are identified:



We again realize a leg of an extended tropical curve as a copy of $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$, with $0 \in \mathbb{R}_{\geq 0} \sqcup \{\infty\}$ attached to its incident vertex.

4.2. **Moduli of tropical curves: fixed weighted graph.** The open cone of dimension $|E|$

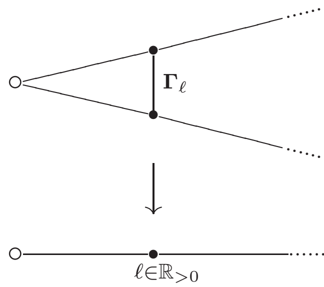
$$\sigma^\circ_{\mathbf{G}} \;=\; (\mathbb{R}_{>0})^E$$

parametrizes tropical curves together with an identification of the underlying graph with $\mathbf{G}$, where each coordinate determines the length of the corresponding edge. There is a natural universal family over $\sigma^\circ_{\mathbf{G}}$ (see for instance [LPP11, Section 5]), so we view it as a fine moduli space for tropical curves whose underlying graphs are identified with $\mathbf{G}$.

Tropical curves whose underlying graphs are isomorphic to $\mathbf{G}$ are parametrized by

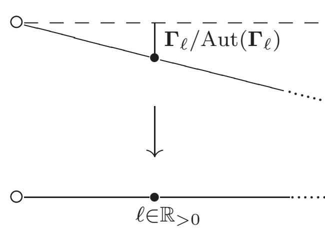
$$M^{\mathrm{trop}}_{\mathbf{G}} = \sigma^\circ_{\mathbf{G}} / \mathrm{Aut}(\mathbf{G}),$$

since the identification of the underlying graph with $\mathbf{G}$ is defined only up to automorphisms of $\mathbf{G}$. Note that $M^{\mathrm{trop}}_{\mathbf{G}}$ is not even homeomorphic to an open cone, in general. However, there is a natural cone complex



Example: $\mathbf{G} =$ $\sigma^\circ_{\mathbf{G}} = \mathbb{R}_{>0}$.

structure on $\sigma_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$, induced by the barycentric subdivision of $\sigma_{\mathbf{G}}$, and $\sigma^\circ_{\mathbf{G}}$ is a union of relative interiors of cones in this complex. Note that in general the universal family of $\sigma^\circ_{\mathbf{G}}$ does not descend to the quotient, but there is a natural complex over $\sigma_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$ whose fiber over a point $[\mathbf{\Gamma}]$ is canonically identified with $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$. See Section 8.2.



$\mathbf{G} =$ 

$M^{\mathrm{trop}}_{\mathbf{G}} = \mathbb{R}_{>0}$.

Similarly, the extended cone $\overline{\sigma}^\circ_{\mathbf{G}} = (\mathbb{R}_{>0} \sqcup \{\infty\})^E$ is a fine moduli space for *extended* tropical curves whose underlying graph is identified with $\mathbf{G}$, and the quotient

$$\overline{M}^{\mathrm{trop}}_{\mathbf{G}} = \overline{\sigma}^\circ_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$$

coarsely parametrizes extended tropical curves whose underlying graph is isomorphic to $\mathbf{G}$.

4.3. **Moduli of tropical curves: varying graphs.** Here we construct the moduli of tropical curves by taking the topological colimit of a natural diagram of cones, in which the arrows are induced by contractions of stable weighted graphs. This approach is not original; it is quite similar, for instance, to the constructions in [Koz11].

As one passes to the boundary of the cone $\sigma^\circ_{\mathbf{G}}$ parametrizing tropical curves whose underlying graph is identified with $\mathbf{G}$, the lengths of some subset of the edges go to zero. The closed cone $\sigma_{\mathbf{G}}$ then parametrizes tropical curves whose underlying graph is identified with a *weighted contraction* of $\mathbf{G}$, as defined in Section 3.2. If $\varpi : \mathbf{G} \to \mathbf{G}'$ is a weighted contraction, then there is a canonical inclusion

$$\jmath_\varpi : \sigma_{\mathbf{G}'} \hookrightarrow \sigma_{\mathbf{G}}$$

identifying $\sigma_{\mathbf{G}'}$ with the face of $\sigma_{\mathbf{G}}$ where all edges contracted by ϖ have length zero.

As a topological space, the coarse moduli space of tropical curves $\overline{M}^{\mathrm{trop}}_{g,n}$ is the colimit of the diagram of cones $\sigma_{\mathbf{G}}$ obtained by gluing the cones $\sigma_{\mathbf{G}}$ along the inclusions $\jmath_\varpi$ for all weighted contractions ϖ:

$$M^{\mathrm{trop}}_{g,n} = \varinjlim \left(\sigma_{\mathbf{G}}, \jmath_\varpi\right). \tag{1}$$

It is therefore canonically a generalized cone complex. Note that every automorphism of a weighted graph $\mathbf{G}$ is a weighted contraction, so the map from $\sigma_{\mathbf{G}}$ to the colimit $M^{\mathrm{trop}}_{g,n}$ factors through $\sigma_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$. Furthermore, two points are identified in the colimit if and only if they are images of two points in some open cone $\sigma^\circ_{\mathbf{G}}$ that differ by an automorphism of $\mathbf{G}$. Therefore, $M^{\mathrm{trop}}_{g,n}$ decomposes as a disjoint union

$$M^{\mathrm{trop}}_{g,n} = \bigsqcup_{\mathbf{G}} M^{\mathrm{trop}}_{\mathbf{G}} = \bigsqcup_{\mathbf{G}} \sigma^\circ_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G}),$$

over isomorphism classes of stable weighted graphs of genus g with n legs. This is not a cell decomposition, but $M^{\mathrm{trop}}_{g,n}$ does carry a natural cone complex structure, induced from the barycentric subdivisions of the cones $\sigma_{\mathbf{G}}$, in which each $M^{\mathrm{trop}}_{\mathbf{G}}$ is a union of relative interiors of cones. There is also a "universal family," whose fiber over a point $[\mathbf{\Gamma}]$ is canonically identified with $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$. Again, see Section 8.2.

Similarly, the coarse moduli space of extended tropical curves $\overline{M}^{\mathrm{trop}}_{g,n}$ is the generalized extended cone complex

$$\overline{M}^{\mathrm{trop}}_{g,n} = \varinjlim \left(\overline{\sigma}_{\mathbf{G}}, \jmath_\varpi\right),$$

which decomposes as a disjoint union

$$\overline{M}^{\mathrm{trop}}_{g,n} = \bigsqcup_{\mathbf{G}} \overline{M}^{\mathrm{trop}}_{\mathbf{G}} = \bigsqcup_{\mathbf{G}} \overline{\sigma}^\circ_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G}).$$

Remark 4.3.1. As in Section 2, while $M^{\mathrm{trop}}_{g,n}$ inherits a cone complex structure from its barycentric subdivision $B(M^{\mathrm{trop}}_{g,n})$, the compactification $\overline{M}^{\mathrm{trop}}_{g,n}$ has a simplicial structure in the form $B(\overline{M}^{\mathrm{trop}}_{g,n})$, which is not the same as the associated extended cone complex $\overline{B(M^{\mathrm{trop}}_{g,n})}$.

5. Thuillier's skeletons of toroidal schemes

Here we recall the basic properties of cone complexes associated to toroidal embeddings, and Thuillier's treatment of their natural compactifications as analytic skeletons.

5.1. **Thuillier's retraction.** We begin by briefly discussing Thuillier's construction of the extended cone complex and retraction associated to a toroidal scheme without self-intersection. Recall that a toroidal scheme is a pair $U \subset X$ that étale locally looks like the inclusion of the dense torus in a toric variety: every point $p \in X$ has an étale neighborhood $\alpha : V \to X$ which admits an étale map $\beta : V \to V_\sigma$ to an affine toric variety, such that $\beta^{-1}T = \alpha^{-1}U$, where $T \subset V_\sigma$ is the dense torus. It is a toroidal embedding without self-intersection if each irreducible component of the boundary divisor $X \smallsetminus U$ is normal, in which case V can be taken to be a Zariski open subset of X. For further details, see [KKMSD73, Thu07].

Remark 5.1.1. Thuillier defines toroidal embeddings in terms of étale charts, whereas in [KKMSD73] they are defined in terms of formal completions. In [KKMSD73, IV.3.II, p. 195] the approaches are shown to be equivalent for toroidal embeddings without self intersection. A short argument of Denef [Den12] shows that the approaches are equivalent in general.

We work over an algebraically closed field k, equipped with the trivial valuation, which sends k^* to zero. The usual Berkovich analytic space associated with a variety X over k is denoted X^{an}. One also associates functorially another nonarchimedean analytic space in the sense of Berkovich, denoted $X^{\beth}$; here $\beth$ is the Hebrew letter *bet*:

Definition 5.1.2. The space $X^{\beth}$ is the compact analytic domain in X^{an} whose K-points, for any valued extension $K|k$ with valuation ring $R \subset K$, are exactly those K-points of X that extend to $\operatorname{Spec} R$.

In particular, we have natural identifications

$$X^{\beth}(K) = X(R),$$

for all such valued extensions. If X is proper, then every K-point of X extends to $\operatorname{Spec} R$, and $X^{\beth}$ is equal to X^{an}. See [Thu07, Sections 1.2, 1.3].

Example 5.1.3. Let X be a toric variety with dense torus T, corresponding to a fan Σ in $N_{\mathbb{R}}$. A K-point x of T extends to a point of X over $\operatorname{Spec} R$ if and only if $\operatorname{Trop}(x)$ is contained in the fan Σ. In other words, $X^{\beth} \cap T^{\mathrm{an}}$ is precisely the preimage of Σ under the classical tropicalization map. The extended tropicalization map takes $X(K)$ into a partial compactification of $N_{\mathbb{R}}$, and the closure of Σ in this partial compactification is the extended cone complex $\overline{\Sigma}$. The preimage of $\overline{\Sigma}$ under the extended tropicalization map is exactly $X^{\beth}$.

Given a toroidal embedding $U \subset X$ over k, Thuillier defines a natural continuous, but not analytic, idempotent self-map $\mathbf{p}_X : X^{\beth} \to X^{\beth}$.

Definition 5.1.4. The *skeleton* $\overline{\Sigma}(X) \subset X^{\beth}$ is the image of the map $\mathbf{p}_X$.

The map $\mathbf{p}_X$ is referred to as the *retraction* of $X^{\beth}$ to its skeleton; we write simply $\mathbf{p}$ when no confusion seems possible.

If $U \subset X$ is a toroidal embedding without self intersection then the image of $U^{\mathrm{an}} \cap X^{\beth}$ is canonically identified with the cone complex $\Sigma(X)$ associated to the toroidal embedding, as constructed in [KKMSD73]. Then $\overline{\Sigma}(X)$ is the closure of $\Sigma(X)$ in $X^{\beth}$, and is canonically identified with the extended cone complex of $\Sigma(X)$. The toroidal structure determines local monomial coordinates on each stratum of X, and the target of the retraction $\overline{\Sigma}(X)$ is the space of monomial valuations in these local coordinates; see [FJ04, Section 1.5.4] for details on monomial valuations. Thuillier shows, furthermore, that $\mathbf{p}$ is naturally a deformation of the identity mapping on $X^{\beth}$, giving a canonical strong deformation retraction of $X^{\beth}$ onto $\overline{\Sigma}(X)$.

Remark 5.1.5. There is a natural order reversing bijection between the strata of the boundary divisor $X \smallsetminus U$ and the cones in $\Sigma(X)$, generalizing the order reversing correspondence between cones in a fan and the boundary strata in the corresponding toric variety, as follows. Let x be a point of $X^{\beth}$ over a valued extension field $K|k$ whose valuation ring is R. Then x is naturally identified with an R-point of X. We write $\overline{x}$ for the reduction of x over the residue field. Then $\mathbf{p}(x)$ is contained in the relative interior σ° of a cone σ if and only if the reduction $\overline{x}$ is in the corresponding locally closed boundary stratum X_σ in X, over the residue field, and x is in U. In other words, the preimage of σ° is the subset of $X^{\beth} \cap U^{\mathrm{an}}$ consisting of points over valued fields whose reduction lies in the corresponding stratum of $X \smallsetminus U$ over the residue field.

Remark 5.1.6. The order reversing bijection between strata in $X \smallsetminus U$ and cones in $\Sigma(X)$ described in Remark 5.1.5, which comes from the reduction map on $X^{\beth}$, should not be confused with the order preserving bijection between strata in $X \smallsetminus U$ and strata in $\overline{\Sigma}(X) \smallsetminus \Sigma(X)$. Quite simply, the preimage under $\mathbf{p}$ of a boundary stratum in $\overline{\Sigma}(X) \smallsetminus \Sigma(X)$ is a stratum in the boundary divisor $(X \smallsetminus U)^{\mathrm{an}}$.

5.2. **Explicit realization of the retraction.** In this section we describe Thuillier's retraction to the extended cone complex more explicitly in local coordinates, for a toroidal embedding $U \subset X$ without self intersection.

The toroidal scheme without self intersections X is covered by Zariski open toric charts

$$V_\sigma \xleftarrow{\ \beta\ } V \xhookrightarrow{\ \alpha\ } X\ .$$

Recall that the cone σ can be described in terms of monoid homomorphisms, as follows. Let M be the group of Cartier divisors on V supported in the complement of U, and let $S_\sigma \subset M$ be the submonoid of such Cartier divisors that are effective. Then the cone σ is the space of monoid homomorphisms to the additive monoid of nonnegative real numbers,

$$\sigma = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0}),$$

equipped with its natural structure as a rational polyhedral cone with integral structure. The associated extended cone $\overline{\sigma}$ is

$$\overline{\sigma} = \mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0} \sqcup \{\infty\}).$$

Let x be a point in $X^{\beth}$. Then x is represented by a point of X over a valuation ring R with valuation val. Let $\overline{x}$ be the reduction of x, which is a point of X over the residue field of R, and let $V \subset X$ be an open subset that contains $\overline{x}$ and has a toric chart $V \to V_\sigma$. Then $\mathbf{p}(x) \in \overline{\sigma}$ is the monoid homomorphism that takes an effective Cartier divisor D on V with support in the complement of U and local equation f at x to

$$\mathbf{p}(x)(D) = \mathrm{val}(f). \tag{2}$$

This is clearly a monoid homomorphism to $\mathbb{R}_{\geq 0} \sqcup \{\infty\}$, and it is nonnegative because D is effective. It is also independent of the choice of chart, the choice of extension field over which x is rational, and the choice of defining equation for D. See [Thu07, Lemma 2.8, Proposition 3.11]. This describes $\mathbf{p}$ as a projection to a natural extended cone complex; we now explain how this cone complex may be seen as a subset of $X^{\beth}$.

First, we may assume that the open set $V \subset X$ is affine. Then V^{an} is the space of valuations on the coordinate ring $k[V]$ that extend the given trivial valuation on k [Ber90, Remark 3.4.2], and $V^{\beth}$ is the subspace of valuations that are nonnegative on all of $k[V]$. Shrinking the toric variety V_σ, if necessary, we may assume that there is a point x in V mapping to a point x_σ in the closed orbit $O_\sigma \subset V_\sigma$; note that the following construction is independent of the choice of x. Since $V \to V_\sigma$ is étale, the completed local ring $\widehat{\mathcal{O}}_{X,x}$ is identified with $\widehat{\mathcal{O}}_{V_\sigma,x_\sigma}$, which is a formal power series ring

$$k[[y_1, ..., y_r]][[S_\sigma]],$$

with exponents in S_σ and coefficients in a formal power series ring in r parameters. Here, r is the dimension of O_σ. For each function $f \in k[V]$, let $\sum_{u\in S_\sigma} a_u(f)z^u$ be the image of f in this power series ring. Then, to each point v in $\overline{\sigma}$, we associate the monomial valuation $\mathrm{val}_v : k[V] \to \mathbb{R} \cup \{\infty\}$ taking f to

$$\mathrm{val}_v(f) = \min\{\langle u, v\rangle \mid a_u(f) \neq 0\}. \tag{3}$$

Since v is in the extened dual cone of S_σ, the valuation val_v is nonnegative on $k[V]$, so this construction realizes $\overline{\sigma}$ as a subset of $V^\beth$. As V ranges over an open cover of X, the subsets $V^\beth$ cover $X^\beth$, and the union of the cones $\overline{\sigma}$, one for each stratum in X, is the extended cone complex $\overline{\Sigma}(X) \subset X^\beth$.

Remark 5.2.1. From this description of $\mathbf{p}$, we see that $\mathbf{p}(x)$ is contained in the relative interior $\overline{\sigma}^\circ$ of $\overline{\sigma}$ as defined in Section 2.2 if and only if $\overline{x}$ is contained in the smallest stratum of V. Also, $\mathbf{p}(x)$ is contained in the boundary $\overline{\sigma} \smallsetminus \sigma$ if and only if x itself is contained in the boundary of V. These two properties of $\mathbf{p}$ determine both the order reversing correspondence between cones of $\overline{\Sigma}(X)$ and strata of X, and the order preserving correspondence between boundary strata in $\overline{\Sigma}(X) \smallsetminus \Sigma(X)$ and boundary strata in $X \smallsetminus U$, discussed in Remarks 5.1.5 and 5.1.6, above.

We recall that Thuillier also constructs a canonical homotopy $H_V : V^\beth \times [0, 1] \to V^\beth$, such that

$$H_V \times \{0\} = \mathrm{id}_V \ : \ V^\beth \to V^\beth,$$

and

$$H_V \times \{1\} = \mathbf{p}_V \ : \ V^\beth \to \overline{\Sigma}_V,$$

giving a strong deformation retraction of $V^\beth$ onto the skeleton $\overline{\Sigma}(V) \subset X^\beth$, and that this construction is functorial for étale morphisms of toroidal schemes. Our goal in Section 6 is to show that a similar construction applies to toroidal Deligne–Mumford stacks.

5.3. **Functoriality.** A morphism $X \to Y$ of toroidal embeddings is *toroidal* if for each $x \in X$ there is a commutative diagram

$$\begin{tikzcd} V_\sigma \arrow[d] & V_X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ V_\tau & V_Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

where the top row is a toric chart at x, the bottom row is a toric chart at $f(x)$, and the arrow $V_\sigma \to V_\tau$ is a *dominant* torus equivariant map of toric varieties; this is a so called toric chart for the morphism $X \to Y$. Toroidal morphisms were introduced in [AK00, Section 1] and coincide with the logarithmically smooth maps of [Kat94].

More generally, we say that a morphism $X \to Y$ as above is *sub-toroidal* if $V_\sigma \to V_\tau$ is only assumed to dominate a torus invariant subvariet of V_τ. A key example is when $X \to Y$ is the normalization of a closed toroidal stratum $X' \subset Y$.

Proposition 5.3.1. *The formation of $\overline{\Sigma}(X)$ is functorial for sub-toroidal morphisms: If $f : X \to Y$ is a sub-toroidal morphism of toroidal embeddings without self intersections, and $f^\beth : X^\beth \to Y^\beth$ is the associated morphism of Berkovich spaces, then $f^\beth$ restricts to a map of generalized extended cone complexes $\overline{\Sigma}(f) : \overline{\Sigma}(X) \to \overline{\Sigma}(Y)$. In particular $\mathbf{p}_Y \circ f^\beth = \overline{\Sigma}(f) \circ \mathbf{p}_X$, and if $Y \to Z$ is another sub-toroidal morphism, then $\overline{\Sigma}(g) \circ \overline{\Sigma}(f) = \overline{\Sigma}(g \circ f)$.*

Proof. We first prove the result for toroidal morphisms, and then indicate the changes necessary for sub-toroidal morphisms.

The result is local, so we may assume there is a toric chart for f which covers X and Y, in which case $\overline{\Sigma}(X) = \overline{\sigma}$ and $\overline{\Sigma}(Y) = \overline{\tau}$. If S_σ and S_τ are the monoids of effective Cartier divisors on X and Y supported away from U_X and U_Y, then we have a pullback homomorphism $S_\tau \to S_\sigma$, which is evidently compatible with composition with a further toric morphism $Y \to Z$. This induces a map $\Sigma(f) : \overline{\sigma} \to \overline{\tau}$ compatible with compositions. By Equation (2) of Section 5.2 we have $\mathbf{p}_Y \circ f^\beth = \overline{\Sigma}(f) \circ \mathbf{p}_X$. By Equation (3) we also have $\overline{\Sigma}(f) = f^\beth|_{\overline{\sigma}}$, as needed.

For sub-toroidal morphisms we only need to replace S_σ by $S'_\sigma := S_\sigma \sqcup \{\infty\}$ and similarly for S'_τ. We define $f^* : S'_\tau \to S'_\sigma$ by declaring that $f^*(\infty) = \infty$ and, if $f(X) \subset D$ for some nonzero divisor $D \in S'_\tau$, then also $f^*D = \infty$. The induced map $\overline{\sigma} \to \overline{\tau}$ maps $\overline{\sigma}$ to an infinite face of $\overline{\tau}$ (see [Thu07, Proposition 2.13]), and the rest of the proof works as stated. ♣

6. Skeletons of toroidal Deligne–Mumford stacks

Here we generalize Thuillier's retraction of the analytification of a toroidal scheme onto its canonical skeleton to the case of toroidal Deligne–Mumford stacks. We follow the construction of [Thu07, Section 3.1.3], where toroidal embeddings with self intersections are treated.

6.1. **Basic construction.** Let $\mathcal{X}$ be a Deligne–Mumford stack over k, with coarse moduli space X. Let $U \subset \mathcal{X}$ an open substack and, for any morphism $V \to \mathcal{X}$, let $U_V \subset V$ be the preimage of U.

Definition 6.1.1. The inclusion $U \subset \mathcal{X}$ is a *toroidal embedding of Deligne–Mumford stacks* if, for every morphism from a scheme $V \to \mathcal{X}$, the inclusion $U_V \subset V$ is a toroidal embedding of schemes.

When U is understood, we refer to $\mathcal{X}$ as a *toroidal Deligne–Mumford stack*. The property of being a toroidal embedding is étale local on schemes, so the inclusion $U \subset \mathcal{X}$ is a toroidal embedding if and only if, for a single étale covering $V \to \mathcal{X}$, the embedding $U_V \subset V$ is toroidal.

Let $\mathcal{X}$ be a toroidal Deligne–Mumford stack with coarse moduli space X, and $V \to \mathcal{X}$ an étale covering by a scheme, where $U_V \subset V$ is a toroidal embedding without self-intersections. We write $V_2 = V \times_{\mathcal{X}} V$. Then $V_2 \rightrightarrows V \to X$ is a right-exact diagram of schemes, and $V_2^\beth \rightrightarrows V^\beth \to X^\beth$ is a right-exact diagram of analytic spaces.

Proposition 6.1.2. *There is a canonical continuous map $H_{\mathcal{X}} : [0,1] \times X^\beth \to X^\beth$ connecting the identity to an idempotent self-map $\mathbf{p}_{\mathcal{X}}$. Writing $\overline{\Sigma}(\mathcal{X})$ for the image of $\mathbf{p}_{\mathcal{X}}$, we have a commutative diagram*

$$\begin{tikzcd}
V_2^\beth \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mathbf{p}_{V_2}"'] & V^\beth \arrow[r] \arrow[d, "\mathbf{p}_V"] & X^\beth \arrow[d, "\mathbf{p}_X"] \\
\overline{\Sigma}(V_2) \arrow[r, shift left] \arrow[r, shift right] & \overline{\Sigma}(V) \arrow[r] & \overline{\Sigma}(\mathcal{X}),
\end{tikzcd}$$

with right exact rows. In particular, $X^\beth$ is contractible and $\overline{\Sigma}(\mathcal{X})$ is the topological colimit of the diagram $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$.

Proof. Since $V_2 \rightrightarrows V \to X$ is right exact it follows that the map $H_X : [0,1] \times X \to X$ making the diagram

$$\begin{tikzcd}
{[0,1] \times V^\beth} \arrow[r] \arrow[d, "H_V"'] & {[0,1] \times X^\beth} \arrow[d, "H_X"] \\
V^\beth \arrow[r] & X^\beth
\end{tikzcd}$$

commutative exists and is unique. To check that the map is independent of the choice of V it suffices to consider a different étale cover by a scheme $V' \to \mathcal{X}$ that factors through $V \to \mathcal{X}$. This induces a map $H'_X : [0,1] \times X^{\beth} \to X^{\beth}$ commuting with $H_{V'}$ as in the preceding diagram, and a map $\mathbf{p}'_X : X^{\beth} \to \overline{\Sigma}(\mathcal{X})$ commuting with $\mathbf{p}_{V'}$ as in the first diagram. By [Thu07, Lemma 3.38] we also have a commutative diagram

$$\begin{tikzcd}
{[0,1] \times V'^{\beth}} \arrow[r] \arrow[d, "H_{V'}"'] & {[0,1] \times V^{\beth}} \arrow[d, "H_V"] \\
V'^{\beth} \arrow[r] & V^{\beth}.
\end{tikzcd}$$

It follows that $H_X = H'_X$, and necessarily $\mathbf{p}'_X = \mathbf{p}_X$. ♣

Remark 6.1.3. One can define $\overline{\Sigma}(\mathcal{X})$ as the topological colimit of the diagram $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$, describe it explicitly as in Proposition 6.2.6 and prove its functorial properties as in Proposition 6.1.6, without recourse to Berkovich analytification, similarly to the work of [KKMSD73] or [GS11]. However Part (2) of Theorem 1.2.1 requires the analytic context.

Remark 6.1.4. Citing a similar argument, Thuillier notes that the skeleton $\overline{\Sigma}(X)$ of a toroidal embedding with self-intersection inherits the structure of a cell complex. We mention one minor gap in the proof of this claim, which is easily corrected. Lemma 3.33 in Thuillier states that every cone of $\Sigma(V)$ maps isomorphically to its image in $\Sigma(X)$, which is not true in general, as seen in Example 6.1.5. Nevertheless, Thuillier's argument does show that cones of the barycentric subdivision of $\Sigma(V)$ map isomorphically to their images in $\Sigma(X)$, so the topological space $\Sigma(X)$ inherits the structure of a cone complex, induced from this barycentric subdivision. The same holds for a toroidal Deligne–Mumford stack $\mathcal{X}$: the skeleton $\overline{\Sigma}(\mathcal{X})$ inherits a simplicial complex structure induced from the barycentric subdivision of $\Sigma(V)$.

Example 6.1.5. Consider $X = \mathbb{A}^3 \smallsetminus \{y = 0\}$ and let U be the complement of the divisor

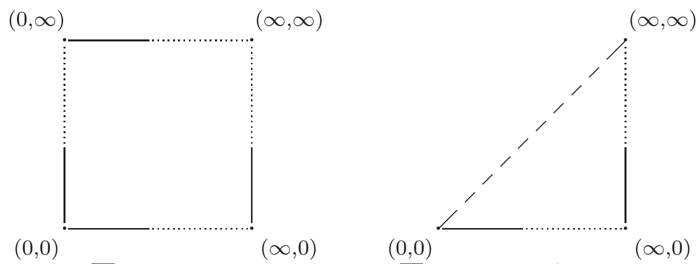
$$D = \{x^2 y - z^2 = 0\}.$$

Since D is a normal crossings divisor, the inclusion $U \subset X$ is a toroidal embedding. Note that D is irreducible but not normal, so X is a toroidal variety with self intersection. Moreover, the preimage of the singular locus in the normalization of D is irreducible. Roughly speaking, this is because the fundamental group of the singular locus acts transitively on the branches of D at the base point. This phenomenon of *monodromy* is discussed systematically in Section 6.2.

We now consider a toric chart on X. Let $V = \mathbb{A}^3 \smallsetminus \{u = 0\}$ and

$$D_V = V(x^2u^2 - z^2) \ \subset \ V,$$

there is a degree 2 étale cover $V \to X$ given by $u^2 = y$. Now V is a toroidal embedding without self intersections. The coordinate change $z_1 = z/u$ gives the equation $x^2 = z_1^2$, so V is isomorphic to $\mathbb{A}^2 \times \mathbb{G}_m$ with its standard toric structure - the toric divisors on V are $\{x = z_1\}$ and $\{x = -z_1\}$. The corresponding cone is $\sigma = \mathbb{R}^2_{\geq 0}$. The group $\mathbb{Z}/2\mathbb{Z}$ acts freely on V, by interchanging the sheets of the étale cover, with quotient X; the involution sends (x, u, z_1) to $(x, -u, -z_1)$. The fiber product $V_2 = V \times_X V$ is therefore $V \times \mathbb{Z}/2\mathbb{Z}$, and the étale equivalence relation $V_2 \rightrightarrows V$ has quotient X. Now the skeleton $\overline{\Sigma}(V_2)$ is $\overline{\sigma} \times \mathbb{Z}/2\mathbb{Z}$, and the equivalence relation $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$ identifies σ with itself by the reflection that switches the two coordinates, since the toric divisors $\{x = z_1\}$ and $\{x = -z_1\}$ are interchanged by the involution. It follows that $\overline{\Sigma}(X)$ is the quotient $\overline{\sigma}/(\mathbb{Z}/2\mathbb{Z})$.



$\overline{\Sigma}(V) = \overline{\sigma}$ $\qquad$ $\overline{\Sigma}(X) = \overline{\sigma} \,/\, (\mathbb{Z}/2\mathbb{Z})$

The image of σ° is not homeomorphic to the relative interior of any cone, but the cones of the barycentric subdivision of $\Sigma(V)$ map isomorphically to their images, giving $\Sigma(X)$ the structure of a cone complex, as in Figure 2 of Section 2.5.

The functorial properties of $\mathbf{p}_X$ also carry over to toroidal stackes:

Proposition 6.1.6. *The construction of $\mathbf{p}_X$ is functorial: if $f : \mathcal{X} \to \mathcal{Y}$ is a sub-toroidal morphism of toroidal Deligne–Mumford stacks, then $f^\beth : X^\beth \to Y^\beth$ restricts to a map of generalized extended cone complexes $\overline{\Sigma}(f) : \overline{\Sigma}(X) \to \overline{\Sigma}(Y)$. In particular $\mathbf{p}_Y \circ f^\beth = \overline{\Sigma}(f) \circ \mathbf{p}_X$, and if $g : \mathcal{Y} \to \mathcal{Z}$ is another sub-toroidal morphism, then $\overline{\Sigma}(g) \circ \overline{\Sigma}(f) = \overline{\Sigma}(g \circ f)$.*

Proof. Let $V \to \mathcal{Y}$ and $W \to V \times_{\mathcal{Y}} \mathcal{X}$ be étale coverings by schemes such that the induced toroidal embeddings are without self intersection.

Then in the diagram

$$\begin{tikzcd}
W^{\beth} \arrow[rr] \arrow[dr] \arrow[dd] & & X^{\beth} \arrow[dr] \arrow[dd] & \\
& V^{\beth} \arrow[rr] \arrow[dd] & & Y^{\beth} \arrow[dd] \\
\overline{\Sigma}(W) \arrow[rr] \arrow[dr] & & \overline{\Sigma}(\mathcal{X}) \arrow[dr, dotted] & \\
& \overline{\Sigma}(V) \arrow[rr] & & \overline{\Sigma}(\mathcal{Y})
\end{tikzcd}$$

the dotted arrow exists since $f^{\beth}(\overline{\Sigma}(\mathcal{X}))$ lies in the image of $\overline{\Sigma}(V)$. It is a morphism of extended generalized cone complexes, since it is covered by the morphism of extended cone complexes $\overline{\Sigma}(W) \to \overline{\Sigma}(V)$. Now all but the right square are already known to be commutative, and the horizontal arrows are surjective, therefore the right square is commutative as well. ♣

Remark 6.1.7. When constructing skeletons of toroidal Deligne–Mumford stacks, it may be tempting to take colimits of diagrams of cone complexes in the category of topological stacks rather than in the category of topological spaces. We avoid this for three reasons. First, the cone complex $\Sigma(\mathcal{X})$ is a subset of X^{an} that lies over the generic point of X. If $\mathcal{X}$ does not have generic stabilizers, then no point of $\Sigma(\mathcal{X})$ has stabilizers, when considered as a point of $\mathcal{X}^{\mathrm{an}}$. Next, the same distinction between the colimit in the category of topological stacks and the colimit in the category of topological spaces appears already for toroidal embeddings of varieties with self intersection, even when the underlying toroidal space has no nontrivial stack structure, as seen in Example 6.1.5. Finally, the colimit of the diagram $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$ in the category of topological stacks depends on the choice of étale cover, while the colimit in the category of topological spaces is independent of all choices. The following example illustrates this possibility.

Example 6.1.8. Let $U \subset X$ be a toroidal scheme, so the embedding of $U \times \mathbf{G}_m$ in $X \times \mathbf{G}_m$ is also toroidal. Fix an integer $n \geq 2$. Then $X \times \mathbf{G}_m$ has a natural étale cover V induced by $z \mapsto z^n$ on $\mathbf{G}_m$. The resulting diagram $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$ realizes $\overline{\Sigma}(X)$ as the quotient of $\overline{\Sigma}(V)$ by the trivial action of a cyclic group of order n. In particular, the colimit in the diagram of topological spaces is $\overline{\Sigma}(V)$, whereas the colimit in the category of topological stacks has a nontrivial stabilizer at every point that depends on the choice of n.

The issue seems to come from the fact that the underlying topological space of a scheme or a Berkovich space should come with a stack structure; possibly points should be replaced by the classifying stacks of appropriate Galois groups. This seems compatible with results of [CMV12]. The simplicial space giving the étale topological type of $X^{\beth}$ and its restriction to $\overline{\Sigma}(X)$ may provide an appropriate formalism.

6.2. **Monodromy of toroidal embeddings.** Let $U \subset \mathcal{X}$ be a toroidal Deligne–Mumford stack. For each étale morphism from a scheme $V \to \mathcal{X}$, let M_V be the group of Cartier divisors on V that are supported on the boundary $V \smallsetminus U_V$, and let $S_V \subset M_V$ be the submonoid of effective divisors. Let M and S be the étale sheaves associated to these presheaves, respectively. In the language of logarithmic geometry, S is the characteristic monoid sheaf associated to the open embedding $U \subset \mathcal{X}$, and M is the characteristic abelian sheaf.

Proposition 6.2.1. *The sheaves S and M are locally constant in the étale topology on each stratum $W \subset \mathcal{X}$.*

Proof. It suffices to check this for M, and it is enough to exhibit an étale cover on W where the sheaf M is constant. Since $\mathcal{X}$ has an étale cover by a toroidal embedding of schemes without self intersections, this follows from the fact that M is constant on each stratum of any toroidal embedding without self-intersection [KKMSD73, Lemma II.1.1, p. 60]. ♣

Fix a stratum $W \subset \mathcal{X}$ and a geometric point w of W. The stalk M_w is the group of étale local germs of Cartier divisors at w supported on $\mathcal{X} \smallsetminus U$, and S_w is the submonoid of germs of effective Cartier divisors. Note that M_w is a finitely generated free abelian group and S_w is a sharp, saturated, and finitely generated submonoid that generates M_w as a group. Hence the dual cone σ_w, the additive submonoid of linear functions on M_w that are nonnegative on S_w, is a strictly convex, full-dimensional, rational polyhedral cone in $\mathrm{Hom}(M_w, \mathbb{R})$. Since M_w is étale locally constant, there is a natural action of $\pi_1^{et}(W, w)$ on M_w that preserves S_w. See [Noo04] for details on étale fundamental groups of Deligne–Mumford stacks.

Definition 6.2.2. The monodromy group H_w is the image of $\pi_1^{et}(W, w)$ in $\mathrm{Aut}(M_w)$.

The action of $\pi_1^{et}(W, w)$ on M_w is determined by the induced permutations of the extremal rays of σ_w. In particular, the monodromy group H_w is finite.

Remark 6.2.3. Note that any two geometric points w and w' in the same stratum $W \subset \mathcal{X}$ have isomorphic monodromy groups, where the isomorphism is well-defined up to conjugation. Similarly, the cones

σ_w and $\sigma_{w'}$ are isomorphic, by isomorphisms that are compatible with the actions of H_w and $H_{w'}$, and well-defined up to conjugation by these actions. In particular, the quotient σ_w/H_w depends only on the stratum W, and not on the point w.

To study the monodromy group H_w at a point w in a stratum $W \subset \mathcal{X}$, we therefore study local charts given by étale covers by toroidal embeddings of schemes without self-intersection, where the monodromy is trivialized.

Definition 6.2.4. An étale morphism from a scheme $V \to \mathcal{X}$ is a *small toric chart* around a point w if

(1) the toroidal embedding $U_V \subset V$ is without self intersections,
(2) there is a unique closed stratum $\widetilde{W} \subset V$, and
(3) the image of $\widetilde{W}$ contains w.

Fix a small toric chart $V \to \mathcal{X}$ and a point $\widetilde{w}$ of $\widetilde{W}$ lifting w. Since V is without self-intersection, the étale sheaves M and S are constant on $\widetilde{W}$, so $\pi_1^{et}(\widetilde{W}, \widetilde{w})$ acts trivially on M_w. The skeleton $\overline{\Sigma}(V)$ is simply the extended cone

$$\overline{\sigma}_V = \mathrm{Hom}(S_V, \mathbb{R}_{\geq 0} \sqcup \{\infty\}).$$

Remark 6.2.5. The monodromy group H_w can be detected from a single small toric chart V around w, as follows. Let $V_2 = V \times_{\mathcal{X}} V$. Consider a point $y \in V_2$ lying over $x_1, x_2 \in \widetilde{W}$, mapping to $w \in W$. Since M is constant on $\widetilde{W}$, we can identify $M_{x_1} \simeq H^0(\widetilde{W}_V, M) \simeq M_{x_2}$. On the other hand pulling back we get $M_{x_2} \simeq M_y \simeq M_{x_1}$. This determines an automorphism of M_w, and every element of H_w occurs in this way.

We now state and prove the main technical result of this section, which says that the skeleton of an arbitrary toroidal embedding of Deligne–Mumford stacks decomposes as a disjoint union of extended open cones, one for each stratum, modulo the action of the respective monodromy groups.

Proposition 6.2.6. *Let $W_1, \ldots, W_s$ be the strata of a toroidal Deligne–Mumford stack $\mathcal{X}$, and let w_i be a point in W_i. Write σ_i for the dual cone of S_{w_i} and H_i for the monodromy group at w_i. Let σ_i° be the relative interior of σ_i. Then we have natural decompositions*

$$\Sigma(\mathcal{X}) \quad = \quad \sigma_1^\circ/H_1 \ \sqcup \cdots \sqcup \ \sigma_s^\circ/H_s,$$

and

$$\overline{\Sigma}(\mathcal{X}) \quad = \quad \overline{\sigma}_1^\circ/H_1 \ \sqcup \cdots \sqcup \ \overline{\sigma}_s^\circ/H_s.$$

Furthermore, if $V_i \to \mathcal{X}$ is a small toric chart around w_i, with $V' = V_1 \sqcup \cdots \sqcup V_s$ and $V_2' = V' \times_{\mathcal{X}} V'$, then the natural map

$$\varinjlim \left(\Sigma(V_2') \rightrightarrows \Sigma(V')\right) \to \Sigma(\mathcal{X})$$

is an isomorphism of generalized cone complexes, and

$$\varinjlim \left(\overline{\Sigma}(V_2') \rightrightarrows \overline{\Sigma}(V')\right) \to \overline{\Sigma}(\mathcal{X})$$

is an isomorphism of extended generalized cone complexes.

Proof. First, note that it suffices to prove the statements for $\overline{\Sigma}(\mathcal{X})$. The decomposition statement for $\Sigma(\mathcal{X})$ follows from the one for $\overline{\Sigma}(\mathcal{X})$, because $\sigma_i^\circ = \overline{\sigma}_i^\circ \cap \Sigma(\mathcal{X})$. Similarly, the isomorphism statement for $\Sigma(\mathcal{X})$ follows from the one for $\overline{\Sigma}(\mathcal{X})$ because $\Sigma(V')$ and $\Sigma(V_2')$ are the preimages of $\Sigma(\mathcal{X})$ in $\overline{\Sigma}(V')$ and $\overline{\Sigma}(V_2')$, respectively.

Since each V_i is a small toric chart, its skeleton is the single extended cone $\overline{\Sigma}(V_i) = \overline{\sigma}_i$, and hence

$$\overline{\Sigma}(V') = \overline{\sigma}_1 \sqcup \cdots \sqcup \overline{\sigma}_s.$$

We write $\mathrm{Im}(\overline{\sigma}_i^\circ)$ for the image of $\overline{\sigma}_i^\circ$ in $\overline{\Sigma}(\mathcal{X})$. First, we show that $\overline{\sigma}_1^\circ \sqcup \cdots \sqcup \overline{\sigma}_s^\circ$ surjects onto $\overline{\Sigma}(\mathcal{X})$. This does not follow from the definition of the skeleton, since $V_1 \sqcup \cdots \sqcup V_s$ need not surject onto $\mathcal{X}$. However, suppose $V^* \to \mathcal{X}$ is a small toric chart around a point in W_i. Then $V^* \times_{\mathcal{X}} V_i$ contains a small toric chart, whose skeleton maps isomorphically by the two projections to $\overline{\sigma}_{V^*}$ and $\overline{\sigma}_i$, see [Thu07, Lemma 3.28 (2)]. Hence the natural map $\overline{\sigma}_{V^*} \to \overline{\Sigma}(\mathcal{X})$ factors through an isomorphism to $\overline{\sigma}_i$. Therefore, we can extend $V_1 \sqcup \cdots \sqcup V_s$ to a cover of $\mathcal{X}$ by small toric charts and conclude that $\overline{\sigma}_1 \sqcup \cdots \sqcup \overline{\sigma}_s$ surjects onto $\overline{\Sigma}(\mathcal{X})$. Finally, each face of $\overline{\sigma}_i$ corresponds to a stratum of $\mathcal{X}$ whose closure contains W_i, so the image of each face of positive codimension in $\overline{\sigma}_i$ is also in the image of a lower dimensional cone, and we conclude that $\overline{\sigma}_1^\circ \sqcup \cdots \sqcup \overline{\sigma}_s^\circ$ surjects onto $\overline{\Sigma}(\mathcal{X})$.

Next, we observe that the images of $\overline{\sigma}_1^\circ, \ldots, \overline{\sigma}_s^\circ$ are disjoint, since a point of $\overline{\sigma}_i^\circ$, considered as a point of $X^\beth$, extends to a point over $\operatorname{Spec} R$ whose reduction lies in W_i. This shows $\overline{\Sigma}(\mathcal{X}) = \mathrm{Im}(\overline{\sigma}_1^\circ) \sqcup \cdots \sqcup \mathrm{Im}(\overline{\sigma}_s^\circ)$.

To prove the decomposition statement, it remains to show $\mathrm{Im}(\overline{\sigma}_i^\circ) = \overline{\sigma}_i^\circ / H_i$. Shrinking V_i if necessary and writing $\widetilde{W}_i \subset V_i$ for the closed stratum, we may assume that the étale map of strata $\widetilde{W}_i \to W_i$ is finite onto its image. Say $W_i' \subset W_i$ is the image of $\widetilde{W}_i$. Since each stratum W_i is smooth, the fundamental group $\pi_1^{et}(W_i', w_i)$ surjects onto $\pi_1^{et}(W_i, w_i)$. The sheaves M and S are trivial on W_i', so every monodromy operator $g \in H_i$ is induced by some geometric point y of $W_i \times_{\mathcal{X}} W_i$ over a pair of points w and w' in W_i' that lie over w_i. Let V_i' be the component of $V_i \times_{\mathcal{X}} V_i$ containing y. Then the projections $\overline{\sigma}_{V'_i} \rightrightarrows \overline{\sigma}_i$ induce the identification $g : \overline{\sigma}_i \xrightarrow{\sim} \overline{\sigma}_i$ Therefore, two points in $\overline{\sigma}_i^\circ$ that differ by an element of H_i have the same image in $\overline{\Sigma}(\mathcal{X})$. Conversely, if v and v' are points in $\overline{\sigma}_i^\circ$ that have the same image in X, then we can consider each as a point of $V_i^\beth$, and consider a point y in $V_i^\beth \times_{\mathcal{X}} V_i^\beth$ lying over v and v'. Then the monodromy operator associated to the reduction of y maps v to v' in $\overline{\sigma}_i^\circ$. This proves the decomposition statement.

We now turn to the isomorphism statement. We have seen that the natural map

$$\varinjlim \left(\overline{\Sigma}(V'_2) \rightrightarrows \overline{\Sigma}(V')\right) \to \overline{\Sigma}(\mathcal{X})$$

is surjective. Let $\mathcal{X}' \subset \mathcal{X}$ be the image of V'. Then

$$\varinjlim \left(\overline{\Sigma}(V'_2) \rightrightarrows \overline{\Sigma}(V')\right) \to \overline{\Sigma}(\mathcal{X}')$$

is an isomorphism. In particular, it is injective. Composing with the inclusions $\overline{\Sigma}(\mathcal{X}') \subset X'^\beth$ and $X'^\beth \subset X^\beth$ shows that the map from the colimit to $\overline{\Sigma}(\mathcal{X})$ is injective, and hence bijective. Being a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. Finally, since all of the maps in the diagram are face maps, the natural extended cone complex structures are preserved, and the homeomorphism is an isomorphism of extended generalized cone complexes. ♣

7. The skeleton of $\overline{\mathcal{M}}_{g,n}$

In this section, we interpret the general construction of the retraction of a toroidal Deligne–Mumford stack onto its canonical skeleton in the special case of $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ and show that $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is naturally identified with the tropical moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$.

7.1. **Versal deformation spaces.** We begin by recalling some facts about deformations of stable curves [ACG11, Chapters XI, XII]. Fix a point p in $\overline{\mathcal{M}}_{g,n}$ corresponding to a stable curve C. Then p has an étale neighborhood $V_p \to \overline{\mathcal{M}}_{g,n}$ in which the locus parametrizing deformations of C in which the node q_i persists is a smooth and irreducible principal divisor D_i with defining equation f_i, and the collection of divisors corresponding to all nodes of C has simple normal crossings. Shrinking V_p if necessary, we may assume that the locus in V_p parametrizing singular curves is the union of these divisors and, for each collection of nodes $\{q_i\}_{i\in I}$, the corresponding intersection

$$W_I = \bigcap_{i\in I} D_i$$

is irreducible. The completion of V_p at p is a formal affine space and the f_i are a subset of a system of formal local coordinates. Furthermore, the dual graph of any curve in the family parametrized by V_p is a contraction of the dual graph $\mathbf{G}$ of C.

The curves parametrized by D_i are exactly those having dual graphs in which the edge e_i corresponding to the node q_i is not contracted. More generally, the locally closed stratum

$$W_I^\circ \subset W_I,$$

consisting of points that are in D_i if and only if $i \in I$, parametrizes those curves whose dual graph is $\mathbf{G}_{/E'}$, the graph in which the edges in E' are contracted and only the edges $\{e_i\}_{i\in I}$ remain, where $E' = \{e_j\}_{j\notin I}$.

Since the defining equation f_i of D_i on V_p measures deformations of the node $q_i \in C$, it has the following interpretation in terms of the local defining equations of the curve at the node. Consider a valuation ring R and a morphism $\phi : \operatorname{Spec} R \to V_p$, corresponding to a curve C_R over $\operatorname{Spec} R$. Assume the closed point in $\operatorname{Spec} R$ maps into the stratum W_I°. Then, for $i \in I$, the node q_i in the special fiber of C_R has an étale neighborhood in C_R with defining equation $xy = f_i$, where we identify f_i with its image in R.

7.2. **Monodromy on $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$.** Let $V \to \overline{\mathcal{M}}_{g,n}$ be a small toric chart around a point p in the stratum $\mathcal{M}_{\mathbf{G}}$, such as the versal deformation spaces discussed above. Then the skeleton $\overline{\Sigma}(V)$ is a single copy of the extended cone $\overline{\sigma}_{\mathbf{G}}$. By Proposition 6.2.6, the image of $\overline{\sigma}_{\mathbf{G}}^\circ$ in $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is the quotient of $\overline{\sigma}_{\mathbf{G}}^\circ$ by the monodromy group $H_{\mathbf{G}}$. Recall that, by definition, $H_{\mathbf{G}}$ is the image of $\pi_1^{et}(\mathcal{M}_{\mathbf{G}}, p)$ in $\operatorname{Aut}(\overline{\sigma}_{\mathbf{G}})$.

Proposition 7.2.1. *The monodromy group $H_{\mathbf{G}}$ is the image of* $\operatorname{Aut}(\mathbf{G})$ *in the set of permutations of* $E(\mathbf{G})$.

Proof. To compute the monodromy group $H_{\mathbf{G}}$, we consider the Galois cover $\widetilde{\mathcal{M}}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}}$, with Galois group $\operatorname{Aut}(\mathbf{G})$, from Section 3.4. The pullbacks of the sheaves M and S are trivial on $\widetilde{M}_{\mathbf{G}}$ because, by construction, the cover $\widetilde{\mathcal{M}}_{\mathbf{G}} \to \mathcal{M}_{\mathbf{G}}$ trivializes the locally constant sheaves of sets on $\mathcal{M}_{\mathbf{G}}$ whose stalk at a point x is the set of nodes of the corresponding curve C_x. By the discussion of versal deformations above, these sets form a group basis for M and a monoid basis for S. The action of $\pi_1^{et}(\mathcal{M}_{\mathbf{G}}, p)$ therefore factors through its quotient $\operatorname{Aut}(\mathbf{G})$, acting in the natural way on $\overline{\sigma}_{\mathbf{G}}$. ♣

Corollary 7.2.2. *The skeleton* $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ *decomposes as a disjoint union*

$$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) = \bigsqcup_{\mathbf{G}} \overline{\sigma}^{\circ}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G}).$$

7.3. **Proof of Theorem 1.2.1.** We have seen that both the skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ and the tropical moduli space $\overline{M}^{\mathrm{trop}}_{g,n}$ decompose naturally as disjoint unions over isomorphism classes of stable graphs of genus g with n legs

$$\overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) = \bigsqcup_{\mathbf{G}} \overline{\sigma}^{\circ}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G}) = \overline{M}^{\mathrm{trop}}_{g,n}.$$

We now show that these bijections induce an isomorphism of extended generalized cone complexes and are compatible with the naive set theoretic tropicalization map from Definition 1.1.1.

Choose a small toric chart $V_{\mathbf{G}} \to \overline{\mathcal{M}}_{g,n}$ around a point in each stratum $\mathcal{M}_{\mathbf{G}}$. Let

$$V = \bigsqcup_{\mathbf{G}} V_{\mathbf{G}},$$

with its étale map $V \to \overline{\mathcal{M}}_{g,n}$. Then $\overline{\Sigma}(V) = \bigsqcup_{\mathbf{G}} \overline{\sigma}_{\mathbf{G}}$ and, by Proposition 6.2.6, the skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$ is naturally identified with the colimit of the diagram $\overline{\Sigma}(V_2) \rightrightarrows \overline{\Sigma}(V)$, where $V_2 = V \times_{\overline{\mathcal{M}}_{g,n}} V$. By Proposition 2.5.1, we can replace this diagram with one in which each cone $\overline{\sigma}_{\mathbf{G}}$ appears exactly once. By Proposition 7.2.1, the self-maps $\overline{\sigma}_{\mathbf{G}} \to \overline{\sigma}_{\mathbf{G}}$ in this diagram are exactly those induced by an automorphism of $\mathbf{G}$. Furthermore, by the discussion of versal deformations in Section 7.1, the closure in $\overline{\mathcal{M}}_{g,n}$ of any stratum corresponding to a contraction of $\mathbf{G}$ contains $\mathcal{M}_{\mathbf{G}}$, so the proper inclusions of faces $\jmath : \overline{\sigma}_{\mathbf{G}'} \to \overline{\sigma}_{\mathbf{G}}$ in this diagram are exactly those corresponding to graph contractions $\varpi : \mathbf{G} \to \mathbf{G}'$.

The same diagram of extended cones is considered in Section 4.3, where its colimit is identified with $\overline{M}^{\mathrm{trop}}_{g,n}$, giving an isomorphism $\overline{\Phi}_{g,n}$ of extended generalized cone complexes, which restricts to an isomorphism of generalized cone complexes $\Phi_{g,n} : \Sigma(\overline{\mathcal{M}}_{g,n}) \to M^{\mathrm{trop}}_{g,n}$, as required.

It remains to check that this identification agrees with the naive set theoretic tropicalization map. Suppose $C = C_p$ is a curve over a valued field K that extends to a curve C_R over $\operatorname{Spec} R$, and let $\mathbf{G}$ be the dual graph of the special fiber. Then the point p has an étale neighborhood in $\overline{\mathcal{M}}_{g,n}$ in which each node q_i of the reduction of C is defined by an equation $xy = f_i$, with f_i in R. The naive set theoretic tropicalization map takes p to the metric graph with underlying $\mathbf{G}$ in which the length of the edge e_i corresponding to q_i is $\mathrm{val}_C(f_i)$; see

Definition 1.1.1. By the discussion of versal deformations in Section 7.1, the divisors $D_i = (f_i)$ also give a basis for the monoid S_p. The explicit description of the retraction to the skeleton, from Section 5.2, then shows that this retraction takes p to the same metric graph, and the theorem follows. ♣

Remark 7.3.1. We note that our proof of Part (1) of Theorem 1.2.1 is based on the observation that $\overline{\Sigma}(X)$ and $\overline{M}_{g,n}^{\mathrm{trop}}$ are put together in the same way from the same extended cones, and does not require the analytic interpretation of $\overline{\Sigma}(X)$ as a skeleton.

8. Tropical tautological maps

8.1. **Curves and tropical curves: the analogy of strata.** As discussed in Section 3.3, the strata in $\overline{\mathcal{M}}_{g,n}$ correspond to stable graphs $\mathbf{G}$, and the codimension of the stratum $\mathcal{M}_{\mathbf{G}}$ is the number of edges in $\mathbf{G}$. Furthermore, $\mathcal{M}_{\mathbf{G}}$ is contained in $\overline{\mathcal{M}_{\mathbf{G}'}}$ if and only if there is a graph contraction $\mathbf{G} \to \mathbf{G}'$.

The natural stratification of the tropical moduli space $M_{g,n}^{\mathrm{trop}}$ is similar, but the inclusions are reversed, as seen in Section 4.3. The stratum $M_{\mathbf{G}}^{\mathrm{trop}}$ parametrizing stable tropical curves with underlying graph $\mathbf{G}$ is contained in $\overline{M_{\mathbf{G}'}^{\mathrm{trop}}}$ if and only if there is a graph contraction $\mathbf{G}' \to \mathbf{G}$. As the orders are reversed, dimension and codimension are also interchanged; the dimension of $M_{\mathbf{G}}^{\mathrm{trop}}$ is equal to the number of edges in $\mathbf{G}$; see [Cap11, Thm. 4.7].

This order reversing correspondence between stratifications may be seen as a consequence of Theorem 1.2.1. The tropical moduli space is the finite part of the skeleton of the moduli space of curves, and there is a natural order reversing correspondence between strata in a toroidal space and cones in the associated complex. See Remark 5.1.5.

8.2. **Tropical forgetful maps and their sections.** Assume as usual $2g - 2 + n > 0$. In the algebraic situation there is a natural *forgetful* morphism

$$\pi = \pi_{g,n} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

obtained functorially by forgetting the last marked point and replacing the curve by its stabilization, if necessary. It was shown by Knudsen that this exhibits $\overline{\mathcal{M}}_{g,n+1}$ as the universal curve over $\overline{\mathcal{M}}_{g,n}$. On the level of coarse moduli spaces, we have that the fiber of $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ over the point $[(C; p_1, \ldots, p_n)]$ is the quotient $C/\mathrm{Aut}(C; p_1, \ldots, p_n)$.

The forgetful map $\pi_{g,n}$ has n tautological sections, $\sigma_1, \ldots, \sigma_n$, corresponding to the marked points. Knudsen identified the image of σ_i

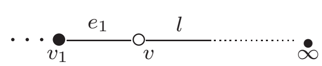
as the locus in $\overline{\mathcal{M}}_{g,n+1}$ where the marked points p_i and p_{n+1} lie on a smooth rational component meeting the rest of the curve in a unique point, and containing no other marked point.

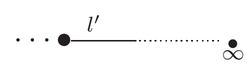
Let us construct a natural forgetful map in the tropical setting.

$$\pi_{g,n}^{\mathrm{trop}} = \pi^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \longrightarrow \overline{M}_{g,n}^{\mathrm{trop}}.$$

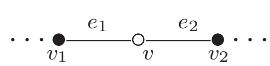
Given a tropical curve $\mathbf{\Gamma} \in \overline{M}_{g,n+1}^{\mathrm{trop}}$, denote by v the vertex where the $n+1$-st leg of $\mathbf{\Gamma}$ is attached; let us remove this leg and denote by $\mathbf{\Gamma}_*$ the resulting tropical curve. If $\mathbf{\Gamma}_*$ is not stable, then $h(v) = 0$ and the valence of v in $\mathbf{\Gamma}_*$ is 2; it is clear that one of the two following cases occurs. Case (1): adjacent to v there are a leg l and an edge e_1, whose second endpoint we denote by v_1. Case (2): adjacent to v there are two edges e_1, e_2, the second endpoint of which we denote by v_1 and v_2 respectively. In these cases we replace $\mathbf{\Gamma}_*$ by a stable tropical curve, $\widehat{\mathbf{\Gamma}_*}$, as follows.

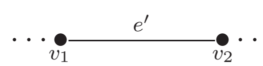
In case (1) we remove e_1 and v, and reattach a leg l' at v_1, as in the following picture.

$\mathbf{\Gamma}_* =$ 

$\widehat{\mathbf{\Gamma}_*} =$ 

In case (2) we define $\widehat{\mathbf{\Gamma}_*}$ to be the graph obtained by removing v, e_1, e_2 from $\mathbf{\Gamma}_*$, and adding an edge e' with endpoints v_1, v_2 and length equal to $\ell(e_1) + \ell(e_2)$:

$\mathbf{\Gamma}_* =$ 

$\widehat{\mathbf{\Gamma}_*} =$ 

Notice that in both cases we have a canonical point (not a vertex), p_v, on $\widehat{\mathbf{\Gamma}_*}$ corresponding to v. Indeed, in case (1), if the length of e_1 is finite, p_v is the point on l' at distance $\ell(e_1)$ from v_1. If $\ell(e_1)$ is infinite, then p_v is the infinity point on the new leg l'.

In case (2), if the edge e_1 (say) has finite length, then the point p_v is the point of e' at distance $\ell(e_1)$ from v_1. If $\ell(e_1) = \ell(e_2) = \infty$, then p_v is defined to be the infinity point on the new edge e'.

We thus obtain a continuous cellular map, $\pi^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$ sending $\mathbf{\Gamma}$ to $\mathbf{\Gamma}_*$ if stable, and to $\widehat{\mathbf{\Gamma}_*}$ otherwise.

Let us now show that the tropical forgetful map realizes $\overline{M}_{g,n+1}^{\mathrm{trop}}$ as the universal curve over $\overline{M}_{g,n}^{\mathrm{trop}}$.

Proposition 8.2.1. *Let* $\mathbf{\Gamma} \in \overline{M}_{g,n}^{\mathrm{trop}}$ *and let* $F_{\mathbf{\Gamma}}$ *be the fiber of* $\pi^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$ *over* $\mathbf{\Gamma}$. *Then* $F_{\mathbf{\Gamma}}$ *is homeomorphic to* $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$. *Moreover, if* $\mathbf{\Gamma} \in M_{g,n}^{\mathrm{trop}}$ *then* $F_{\mathbf{\Gamma}}$ *is isometric to* $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$.

Proof. We have a map $F_{\mathbf{\Gamma}} \to \mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$ by sending a tropical curve with $n+1$ legs (the last of which adjacent to the vertex v) to the point p_v corresponding to v. To obtain the inverse, we identify $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma})$ to the space of points $p \in \mathbf{\Gamma}$ up to isometries preserving the weights on the vertices. Then by attaching a leg at p we obtain a tropical curve, $\mathbf{\Gamma}_p$, with $n+1$ marked points. More precisely, we have the following possibilities. If p is a vertex of $\mathbf{\Gamma}$ then we simply add a leg adjacent to p. If p is not a vertex of $\mathbf{\Gamma}$, then we declare p to be a vertex of weight zero, and attach a leg at it; the new vertex p has thus valency 3, and hence the tropical curve $\mathbf{\Gamma}_p$ is stable.

It is clear that as p varies in its $\mathrm{Aut}(\mathbf{\Gamma})$-orbit, the isomorphism class of $\mathbf{\Gamma}_p$ does not change. So the above construction descends to a map $\mathbf{\Gamma}/\mathrm{Aut}(\mathbf{\Gamma}) \to F_{\mathbf{\Gamma}}$, which is the inverse of the map defined before.

It is clear that this map is a homeomorphism, and an isometry if all edges of $\mathbf{\Gamma}$ have finite length. ♣

Remark 8.2.2. It would be interesting to develop the theory on a stack level in such a way that the fiber is exactly the curve.

Example 8.2.3. Consider $\overline{M}_{1,1}^{\mathrm{trop}}$. It has two strata, one of dimension zero and one of dimension one. The dimension zero stratum corresponds to the (unique) curve $\mathbf{\Gamma}_0$ with one vertex v of weight 1, no edges, and a leg attached to v. Thus $\mathrm{Aut}(\mathbf{\Gamma}_0) = 0$.

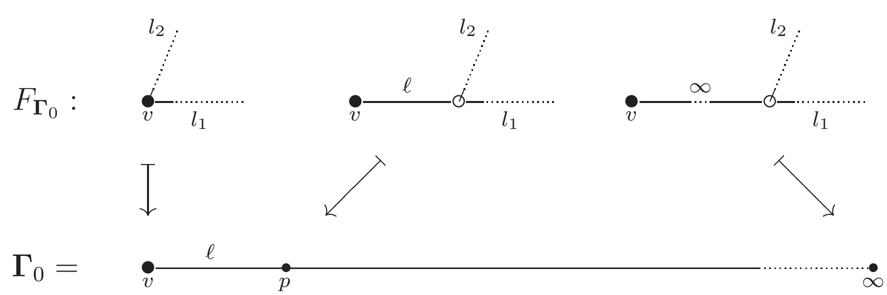


FIGURE 4. Fiber of π^{trop} over the smallest stratum of $\overline{M}_{1,1}^{\mathrm{trop}}$

Figure 4 represents $F_{\mathbf{\Gamma}_0} \subset \overline{M}_{1,2}^{\mathrm{trop}}$ and its isometry with $\mathbf{\Gamma}_0$; at the top we have the three types of curves parametrized by $F_{\mathbf{\Gamma}_0}$, and at the bottom corresponding the point of $\mathbf{\Gamma}_0$. Notice that the curves on the right and on the left are unique, whereas in the middle they vary with $\ell \in \mathbb{R}_{>0}$. The one-dimensional stratum of $\overline{M}_{1,1}^{\text{trop}}$ is a copy of $\mathbb{R}_{>0} \sqcup \{\infty\}$; it parametrizes curves $\mathbf{\Gamma}_d$ whose graph has one vertex of weight zero, one loop-edge of length $d \in \mathbb{R}_{>0} \cup \{\infty\}$, and one leg. Then $\mathrm{Aut}(\mathbf{\Gamma}_d) = \mathbb{Z}/2\mathbb{Z}$ where the involution corresponds to switching the orientation on the loop-edge. The quotient $\mathbf{\Gamma}_d/\mathrm{Aut}(\mathbf{\Gamma}_d)$ is drawn below

The following Figure 5 represents at the top one-dimensional families of curves of $F_{\mathbf{\Gamma}_d}$; the curves on the left vary with $0 < \ell < \frac{d}{2}$, while on the right with $\ell' \in \mathbb{R}_{>0}$. The middle row represents the three remaining points of $F_{\mathbf{\Gamma}_d}$.

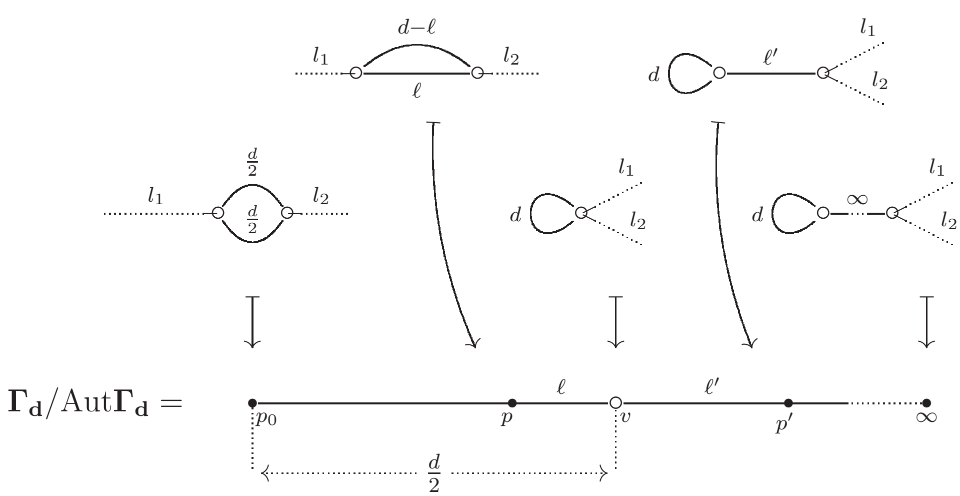


FIGURE 5. Fiber of π^{trop} over $[\mathbf{\Gamma}_d] \in \overline{M}_{1.1}^{\text{trop}}$ with $d > 0$.

Finally, Figure 6 depicts the forgetful map from $\overline{M}_{1,2}^{\text{trop}}$ to $\overline{M}_{1,1}^{\text{trop}}$.

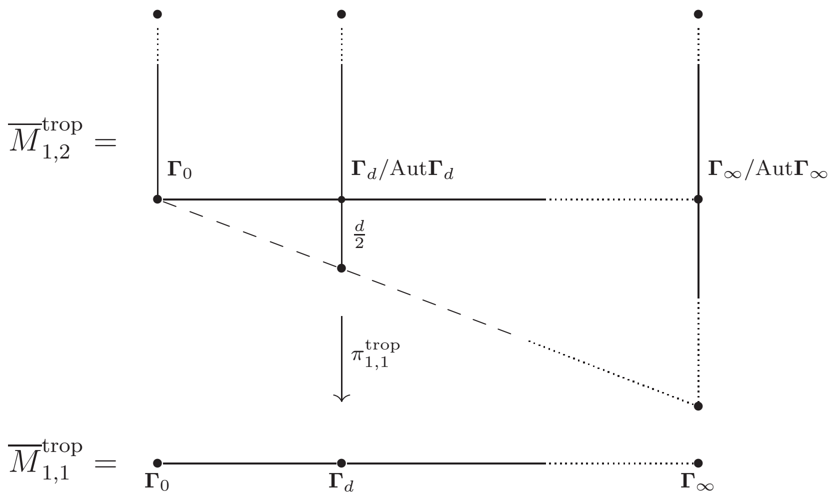


FIGURE 6. The forgetful map $\pi_{1,1}^{\mathrm{trop}}$.

Proof of the commutativity of the first diagram of Theorem 1.2.2. Having defined the map π^{trop} we can consider the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}^{\mathrm{an}} \arrow[r, "\mathrm{Trop}"] \arrow[d, "\pi^{\mathrm{an}}"'] & \overline{M}_{g,n+1}^{\mathrm{trop}} \arrow[d, "\pi^{\mathrm{trop}}"] \\ \overline{\mathcal{M}}_{g,n}^{\mathrm{an}} \arrow[r, "\mathrm{Trop}"] & \overline{M}_{g,n}^{\mathrm{trop}} \end{tikzcd}$$

where π^{an} is the morphism canonically associated to the algebraic forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ (explicitly described below). Let $[C] \in \overline{\mathcal{M}}_{g,n+1}^{\mathrm{an}}$, so that $[C]$ is represented by a pair

$$(\mathrm{val}_C : K \longrightarrow \mathbb{R} \sqcup \{\infty\}, \quad \mu_C : \mathrm{Spec}\, R \longrightarrow \overline{\mathcal{M}}_{g,n+1})$$

(recall that val_C is a valuation of the field $K \supset k$ extending the trivial valuation on k, and $R \subset K$ is its ring of integers). The morphism μ_C corresponds to a family of stable curves, $C \to \mathrm{Spec}\, R$; we write C_s and C_K for its special and generic fiber. Now set $[C'] := \pi^{\mathrm{an}}([C]) \in \overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$; this point is is represented by the pair

$$(\mathrm{val}_C : K \longrightarrow \mathbb{R} \sqcup \{\infty\}, \quad \pi \circ \mu_C : \mathrm{Spec}\, R \longrightarrow \overline{\mathcal{M}}_{g,n}).$$

It is clear that the special fiber C'_s of $C' \to \mathrm{Spec}\, R$ is equal to $\pi(C_s)$.

Denote by $\mathbf{G}$ the dual graph of C_s. Recall that we have $\mathrm{Trop}([C]) = (\mathbf{G}, \ell_C)$ where the length function ℓ_C is determined by the valuation

val_C, and by the local geometry of the family $C \to \operatorname{Spec} R$ at the nodes of its special fiber, C_s; see Definition 1.1.1. Similarly, writing $\mathbf{G}' = \mathbf{G}_{C'_s}$, we have

$$\mathrm{Trop}(\pi^{\mathrm{an}}([C])) = \mathrm{Trop}([C']) = (\mathbf{G}', \ell_{C'}).$$

Now, by our description of the map π^{trop}, it is clear that the graph underlying $\pi^{\mathrm{trop}}(\mathrm{Trop}([C])) = \pi^{\mathrm{trop}}(\mathbf{G}, \ell_C)$ is equal to the dual graph of the algebraic curve $\pi(C_s)$; on the other hand $\pi(C_s) = C'_s$. We conclude that the graph underlying $\mathrm{Trop}(\pi^{\mathrm{an}}([C]))$ and $\pi^{\mathrm{trop}}(\mathrm{Trop}([C]))$ is the same. It remains to prove that the length functions of these two points are the same. Let us write

$$\pi^{\mathrm{trop}}(\mathrm{Trop}([C])) = \pi^{\mathrm{trop}}(\mathbf{G}, \ell_C) = (\mathbf{G}', \widetilde{\ell}\,)$$

where $\widetilde{\ell}$ is determined by π^{trop}, as explained before the statement of Proposition 8.2.1. To show that $\ell_{C'} = \widetilde{\ell}$, notice that they depend on the same valuation, namely val_C; hence we have to analyse the total spaces of the families locally at the nodes of their special fibers.

If the curve C_s remains stable after removing its (n+1)-st marked point, then the total space of the family $C' \to \operatorname{Spec} R$ (regardless of its marked points) is exactly the same as that of $C \to \operatorname{Spec} R$, and the dual graph of C'_s is obtained from $\mathbf{G}$ by removing one leg; so the edges are the same and $\ell_{C'}$ and $\widetilde{\ell}$ are both equal to ℓ_C.

Now suppose C_s is not stable after the removal of its last marked point. The situation is identical to the one we had in the tropical setting, when defining the map π^{trop}; as on that occasion, we now distinguish two cases. In case (1) the removal of the last marked point from C_s creates a "one-pointed rational tail", i.e. a smooth rational component, E, attached to the rest of C_s at only one node, and having only one marked point on it. In the family $C' \to \operatorname{Spec} R$ the component E is contracted to a smooth point of C'_s, and the local geometry of C' near the rest of C'_s is the same. So, the graph $\mathbf{G}'$ has one fewer edge than $\mathbf{G}$ and both $\ell_{C'}$ and $\widetilde{\ell}$ coincide with the restriction of ℓ_C to the edges of $\mathbf{G}'$.

The remaining case (2) is more interesting. Here the removal of the last marked point creates an "unpointed exceptional component", i.e. a smooth rational component, E, with no marked points and such that

$$E \cap \overline{C_s \smallsetminus E} = \{q_1, q_2\},$$

with q_1 and q_2 nodes of C_s. Let $xy = f_i$ be the local equation of C at q_i. Then, denoting by $e_i \in E(\mathbf{G})$ the edge corresponding to q_i, we have

$$\ell_C(e_i) = \mathrm{val}_C(f_i), \qquad i = 1, 2. \tag{4}$$

The curve C'_s is obtained from C_s by collapsing E to a node; its dual graph is obtained from the dual graph of C_s by removing the last leg (adjacent to the vertex, v, corresponding to E), removing v, and "merging" e_1 and e_2 into a unique edge e'. Now, the total space of C', locally at the node of C'_s corresponding to e', has equation $xy = f_1 f_2$ and hence

$$\ell_{C'}(e') = \mathrm{val}_C(f_1 f_2) = \mathrm{val}_C(f_1) + \mathrm{val}_C(f_2).$$

On the other hand, by definition of π^{trop}, we have

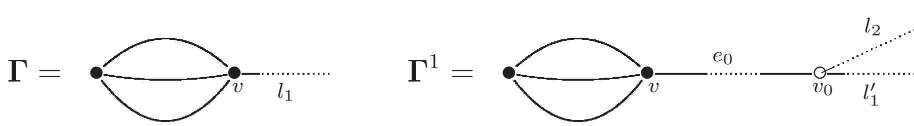
$$\widetilde{\ell}(e') = \ell_C(e_1) + \ell_C(e_2) = \mathrm{val}_C(f_1) + \mathrm{val}_C(f_2)$$

by (4). Hence $\ell_{C'}(e') = \widetilde{\ell}(e')$. Of course, all the remaining edges of $\mathbf{G}'$ are naturally identified with edges of $\mathbf{G}$, and the values of $\ell_{C'}$ and $\widetilde{\ell}$ on them is equal to the value of ℓ_C. The proof of the commutativity of the first diagram in Theorem 1.2.2 is complete. ♣

We now proceed to define the "tautological sections" of the forgetful maps, in analogy with the algebraic case. Let $\mathbf{\Gamma} = (V, E, L, h, \ell)$ be a tropical curve in $\overline{M}_{g,n}^{\mathrm{trop}}$. For $i \in \{1, \ldots, n\}$ we define the tropical curve

$$\mathbf{\Gamma}^i = (V^i, E^i, L^i, h^i, \ell^i)$$

as follows. Let $l_i \in L$ be the i-th leg of $\mathbf{\Gamma}$ and $v \in V$ its endpoint. $\mathbf{\Gamma}^i$ is obtained by attaching an edge e_0 at v whose second endpoint we denote by v_0. We set $V^i = V \cup \{v_0\}$ and $E^i = E \cup \{e_0\}$; the weight function h^i is the extension of h such that $h^i(v_0) = 0$; the length function ℓ^i is the extension of ℓ such that $\ell^i(e_0) = \infty$. Finally we remove the leg l_i and attach two legs at v_0, denoted by l'_i and l_{n+1}; summarizing $L^i = L \smallsetminus \{l_i\} \cup \{l'_i, l_{n+1}\}$. Here is an picture with $i = n = 1$:



Now we can state

Proposition 8.2.4. *The tropical forgetful map* $\pi^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$ *admits* n *continuous sections* $\sigma_i^{\mathrm{trop}} : \overline{M}_{g,n}^{\mathrm{trop}} \to: \overline{M}_{g,n+1}^{\mathrm{trop}}$, *with* $\sigma_i^{\mathrm{trop}}(\mathbf{\Gamma}) :=$

$\mathbf{\Gamma}^i$ *for every* $\mathbf{\Gamma} \in \overline{M}_{g,n}^{\mathrm{trop}}$. *The diagram*

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}^{\mathrm{an}} \arrow[r, "\mathrm{Trop}"] \arrow[d, "\sigma_i^{\mathrm{an}}"'] & \overline{M}_{g,n}^{\mathrm{trop}} \arrow[d, "\sigma_i^{\mathrm{trop}}"] \\
\overline{\mathcal{M}}_{g,n+1}^{\mathrm{an}} \arrow[r, "\mathrm{Trop}"] & \overline{M}_{g,n+1}^{\mathrm{trop}}
\end{tikzcd}$$

is commutative.

Proof. It is clear that $\mathbf{\Gamma}^i \in \overline{M}_{g,n+1}^{\mathrm{trop}}$ and that the map σ_i^{trop} is continuous. We need to prove that $\pi^{\mathrm{trop}}(\mathbf{\Gamma}^i) = \mathbf{\Gamma}$. Indeed removing the last leaf from $\mathbf{\Gamma}^i$ gives a tropical curve $(\mathbf{\Gamma}^i)_*$ which is not stable, as the vertex v_0 has valency 2. Hence $\pi^{\mathrm{trop}}(\mathbf{\Gamma}^i) = \widehat{(\mathbf{\Gamma}^i)_*}$; as $\widehat{(\mathbf{\Gamma}^i)_*} = \mathbf{\Gamma}$ the first statement is proven. The proof of commutativity is identical to the proof of commutativity of the clutching diagram below. ♣

8.3. **Tropical clutching maps.** In the algebro-geometric setting, if $g = g_1 + g_2$ and $n = n_1 + n_2$, always assuming $2g_i - 2 + n_i > 0$, we have the so-called *clutching* maps $\kappa = \kappa_{g_1,n_1,g_2,n_2}$

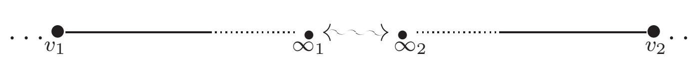
$$\begin{array}{ccccl}
\overline{\mathcal{M}}_{g_1,n_1+1} & \times & \overline{\mathcal{M}}_{g_2,n_2+1} & \xrightarrow{\kappa} & \overline{\mathcal{M}}_{g,n} \\
(C_1; p_1^1, \ldots, p_{n_1+1}^1) & , & (C_2; p_1^2, \ldots, p_{n_2+1}^2) & \mapsto & (C; p_1, \ldots, p_n).
\end{array}$$

These are obtained by gluing C_1 with C_2 by identifying $p_{n_1+1}^1 = p_{n_2+1}^2$ in such a way that in C the intersection of C_1 with C_2 consists of exactly one (separating) node.

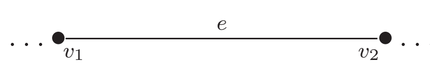
We now construct the analogous maps in the tropical setting, always keeping the numerical assumptions of the algebraic case. To define the tropical clutching map,

$$\begin{array}{rcll}
\kappa^{\mathrm{trop}} : \overline{M}_{g_1,n_1+1}^{\mathrm{trop}} & \times \overline{M}_{g_2,n_2+1}^{\mathrm{trop}} & \longrightarrow & \overline{M}_{g,n}^{\mathrm{trop}} \\
\mathbf{\Gamma}_1 & , \quad \mathbf{\Gamma}_2 & \mapsto & \mathbf{\Gamma}
\end{array}$$

we attach the last leg of $\mathbf{\Gamma}_1$ (adjacent to the vertex v_1) to the last leg of $\mathbf{\Gamma}_2$ (adjacent to v_2) by identifying their infinite points



to form an edge e of $\mathbf{\Gamma}$ with endpoints v_1 and v_2:



The length of e is, quite naturally, set to be equal to ∞. Since the new edge e is a bridge (i.e. a disconnecting edge) of the underlying graph, we have that the genus of $\mathbf{\Gamma}$ is equal to the sum of the genera of $\mathbf{\Gamma}_1$

and $\mathbf{\Gamma}_2$. Hence we have that the image of κ^{trop} lies in $\overline{M}^{\mathrm{trop}}_{g,n}$. Observe that this image is entirely contained in the locus of extended tropical curves having at least one bridge of infinite length.

Proof of the commutativity of the clutching diagram of Theorem 1.2.2. We begin by reviewing the map κ^{an} of our diagram:

$$\begin{tikzcd} \overline{\mathcal{M}}^{\mathrm{an}}_{g_1,n_1+1} \times \overline{\mathcal{M}}^{\mathrm{an}}_{g_2,n_2+1} \arrow[r, "\mathrm{Trop}\times\mathrm{Trop}"] \arrow[d, "\kappa^{\mathrm{an}}"'] & \overline{M}^{\mathrm{trop}}_{g_1,n_1+1} \times \overline{M}^{\mathrm{trop}}_{g_2,n_2+1} \arrow[d, "\kappa^{\mathrm{trop}}"] \\ \overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\mathrm{trop}}_{g,n}. \end{tikzcd}$$

The map κ^{an} is defined by functoriality of analytification, and can be understood as follows. Fix again an algebraically closed field K with valuation $\mathrm{val} : K \to \mathbb{R} \sqcup \{\infty\}$, valuation ring R and special point s. Consider a K-point $[C^1, C^2] \in \overline{\mathcal{M}}^{\mathrm{an}}_{g_1,n_1+1} \times \overline{\mathcal{M}}^{\mathrm{an}}_{g_2,n_2+1}$. The point is simply a morphism $\operatorname{Spec} K \to \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$; since the moduli space is proper it extends to a morphism we denote

$$\mu_1 \times \mu_2 : \operatorname{Spec} R \to \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}.$$

For $i = 1, 2$ the two projections provide us with K valued points $[C^i] \in \overline{\mathcal{M}}^{\mathrm{an}}_{g_i,n_i+1}$ represented by

$$\mu_i : \operatorname{Spec} R \to \overline{\mathcal{M}}_{g_i,n_i+1},$$

giving two stable pointed curves $C^i \to \operatorname{Spec} R$.

Write $\kappa^{\mathrm{an}}([C^1, C^2]) = [C] \in \overline{\mathcal{M}}^{\mathrm{an}}_{g,n}$; it is represented by the composition

$$\kappa \circ (\mu_1 \times \mu_2) : \operatorname{Spec} R \to \overline{\mathcal{M}}_{g,n},$$

in effect gluing the two families of curves $C_i \to \operatorname{Spec} R$ along the two sections $\sigma_{n_i+1} : \operatorname{Spec} R \to C_i$.

Now, $\mathrm{Trop}\times\mathrm{Trop}([C^1, C^2]) = \big((\mathbf{G}_1, \ell_1), (\mathbf{G}_2, \ell_2)\big)$ with $\mathbf{G}_i = \mathbf{G}_{C^i_s}$ and ℓ_i defined in Definition 1.1.1. Next

$$\kappa^{\mathrm{trop}}(\mathrm{Trop} \times \mathrm{Trop}([C^1, C^2])) = (\mathbf{G}, \widetilde{\ell})$$

where, according to our description above, $\mathbf{G}$ is obtained from $\mathbf{G}_1$ and $\mathbf{G}_2$ by merging their respective last legs into one edge, denoted by e (which is thus a bridge of $\mathbf{G}$). The definition of $\widetilde{\ell}$ is as follows:

$$\widetilde{\ell}(\widetilde{e}) = \begin{cases} \ell_i(\widetilde{e}) & \text{if } \widetilde{e} \in E(\mathbf{G}_i), \quad i = 1, 2. \\ +\infty & \text{otherwise i.e. if } \widetilde{e} = e. \end{cases}$$

Consider now $[C]$ and its associated family, $C \to \operatorname{Spec} R$. We have a diagram

$$\begin{tikzcd} C^1 \sqcup C^2 \arrow[r, "\eta"] \arrow[dr] & C \arrow[d] \\ & \operatorname{Spec} R \end{tikzcd}$$

where the map η glues together the last marked points of C^1 and C^2; let us write $C^* = C^1 \sqcup C^2$ for simplicity.

The special fiber of C is $C_s = \kappa([C_s^1, C_s^2])$ and hence its dual graph is equal to $\mathbf{G}$.

Let us look at the local geometry at a node of C_s. Pick the node corresponding to the new edge e, then the generic fiber C_K has a node specializing to it (the node corresponding to the gluing of the last marked point of C_K^1 with the last marked point of C_K^2) and hence the local equation at this node is $xy = 0$. Therefore $\ell_C(e) = \infty = \widetilde{\ell}(e)$, as required.

Consider now a node corresponding to an edge $\widetilde{e} \neq e$; without loss of generality this edge $\widetilde{e}$ corresponds to a node of C_s^1, at which the local equation of C^1 is $xy = f$ with $f \in R$. This also serves as a local equation of C^* at the corresponding node, and since it is disjoint from σ_{n_1+1}, also a local equation of C. Therefore

$$\ell_C(\widetilde{e}) = \operatorname{val}(f) = \widetilde{\ell}(\widetilde{e})$$

and we are done. ♣

8.4. **Tropical gluing maps.** In the algebraic setting, for $g > 0$ there is a map

$$\gamma : \overline{\mathcal{M}}_{g-1,n_1+2} \to \overline{\mathcal{M}}_{g,n}$$

obtained by gluing the last two marked points. We now define the tropical gluing maps

$$\gamma^{\text{trop}} : \overline{M}_{g-1,n+2}^{\text{trop}} \longrightarrow \overline{M}_{g,n}^{\text{trop}}$$

(always assuming $g > 0$). The procedure is similar to the definition of the tropical clutching map; γ^{trop} maps a tropical curve $\mathbf{\Gamma}$ with $n+2$ legs to the tropical curve $\mathbf{\Gamma}'$ obtained by attaching the last two legs of $\mathbf{\Gamma}$, so as to form an edge e' of infinite length for $\mathbf{\Gamma}'$. It is clear that $\mathbf{\Gamma}'$ has now only n legs, and its genus is gone up by one, as the new edge e' is not a bridge of $\mathbf{\Gamma}'$.

Proof of the commutativity of the gluing diagram of Theorem 1.2.2. The diagram whose commutativity we must prove is the following.

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\rm an}_{g-1,n+2} \arrow[r, "\mathrm{Trop}"] \arrow[d, "\gamma^{\rm an}"'] & \overline{M}^{\rm trop}_{g-1,n+2} \arrow[d, "\gamma^{\rm trop}"] \\
\overline{\mathcal{M}}^{\rm an}_{g,n} \arrow[r, "\mathrm{Trop}"] & \overline{M}^{\rm trop}_{g,n} .
\end{tikzcd}$$

The proof follows the same pattern used to prove the commutativity of the first diagram in the theorem. Let $[C] \in \overline{\mathcal{M}}^{\rm an}_{g-1,n+2}$ be represented by the pair

$$(\mathrm{val}_C : K \longrightarrow \mathbb{R} \sqcup \{\infty\}, \quad \mu_C : \operatorname{Spec} R \longrightarrow \overline{\mathcal{M}}_{g-1,n+2}).$$

Denote by $\mathbf{G}$ the dual graph of C_s. Now set $\gamma^{\rm an}([C]) = [C'] \in \overline{\mathcal{M}}^{\rm an}_{g,n}$, represented by the pair

$$(\mathrm{val}_C : K \longrightarrow \mathbb{R} \sqcup \{\infty\}, \quad \gamma \circ \mu_C : \operatorname{Spec} R \longrightarrow \overline{\mathcal{M}}_{g,n}).$$

The special fiber C'_s of $C' \to \operatorname{Spec} R$ is equal to $\gamma(C_s)$. It is clear that the graph underlying $\mathrm{Trop}(\gamma^{\rm an}([C]))$ and the graph underlying $\gamma^{\rm trop}(\mathrm{Trop}([C]))$ are isomorphic to the dual graph of $\gamma(C_s)$, denoted by $\mathbf{G}'$. It remains to show the length functions on $E(\mathbf{G}')$ of $\mathrm{Trop}(\gamma^{\rm an}([C]))$ and $\gamma^{\rm trop}(\mathrm{Trop}([C]))$ coincide.

Recall that $\mathbf{G}'$ is obtained from $\mathbf{G}$ by adding a new edge, e', joining the endpoints of the last two legs (and removing these two legs). The length of e' in the tropical curve $\gamma^{\rm trop}(\mathrm{Trop}([C]))$ is set to be equal to ∞, whereas the length of every other edge $e \in E(\mathbf{G}') \smallsetminus \{e'\} = E(\mathbf{G})$ is $\ell_C(e)$.

Now consider $\mathrm{Trop}(\gamma^{\rm an}([C])) = (\mathbf{G}', \ell_{C'})$; recall that $\ell_{C'}$ depends on the local geometry of $C' \to \operatorname{Spec} R$ near the nodes of C'_s. Consider the node q' corresponding to the new edge e'; the generic fiber of $C' \to \operatorname{Spec} R$ also has a node specializing to q', therefore the local equation of C' at q' is $xy = 0$. Hence

$$\ell_{C'}(e') = \mathrm{val}_C(0) = \infty$$

just as in $\gamma^{\rm trop}(\mathrm{Trop}([C]))$. Locally at every other node of C'_s we have that C and C' are isomorphic, hence on the corresponding edges of $\mathbf{G}'$ we have $\ell_{C'} = \ell_C$. The proof is now complete ♣.

8.5. **Functorial interpretation of the maps.** We have defined tropical forgetful, clutching and gluing maps, as well as sections, using the modular meaning of $\overline{M}^{\rm trop}_{g,n}$. Theorem 1.2.1 allows us to interpret these maps in terms of the functorial properties of the maps $\mathbf{p}$.

First note that all the algebraic tautological maps are sub-toroidal: the map π is toroidal since it is a family of nodal curves, see [AK00, 2.6]. The section σ_i is an isomorphism onto a toroidal substack, and the clutching and gluing maps factor through an étale covering of degree one or two followed by a normalization map, by [ACG11, Proposition XII.10.11], so they are indeed sub-toroidal. By Proposition 6.1.6 we have a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n+1} \arrow[r, "\mathbf{p}"] \arrow[d, "\pi^{\mathrm{an}}"'] & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n+1}) \arrow[d, "\overline{\Sigma}(\pi)"] \\
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathbf{p}"] & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n})
\end{tikzcd}$$

and similarly for the maps σ_i, γ, κ. Theorem 1.2.1 extends this to a commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n+1} \arrow[r, "\mathbf{p}"'] \arrow[d, "\pi^{\mathrm{an}}"'] \arrow[rr, bend left, "\mathrm{Trop}"] & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n+1}) \arrow[r, "\sim", "\overline{\Phi}"'] \arrow[d, "\overline{\Sigma}(\pi)"] & \overline{M}^{\mathrm{trop}}_{g,n+1} \arrow[d, dotted] \\
\overline{\mathcal{M}}^{\mathrm{an}}_{g,n} \arrow[r, "\mathbf{p}"] \arrow[rr, bend right, "\mathrm{Trop}"'] & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \arrow[r, "\overline{\Phi}", "\sim"'] & \overline{M}^{\mathrm{trop}}_{g,n+1}.
\end{tikzcd}$$

Since the two arrows designated by $\overline{\Phi}$ are isomorphisms, there is necessarily a unique arrow $\overline{\Phi} \circ \overline{\Sigma}(\pi) \circ \overline{\Phi}^{-1}$ making the diagram commutative; it therefore must coincide with the map π^{trop} we defined above. The same holds for the maps σ_i, γ, and κ.

8.6. **Variations on the tropical gluing and clutching maps.** In the algebro-geometric situation, the clutching and gluing maps together cover the entire boundary of $\overline{\mathcal{M}}_{g,n}$, since the result of desingularizing a node while adding its two branches as marked points is either the disjoint union of two stable curves with suitable genera g_1, g_2, and suitable n_1+1 and n_2+1 marked points, or one curve of genus $g-1$ with $n+2$ marked points. The situation is quite different in the tropical setting, indeed we have the following fact.

Lemma 8.6.1. *In $\overline{M}^{\mathrm{trop}}_{g,n}$ the union of the image of $\gamma^{\mathrm{trop}}_{g,n}$ with the images of all the clutching maps $\kappa^{\mathrm{trop}}_{g_1,n_1,g_2,n_2}$ is equal to $\overline{M}^{\mathrm{trop}}_{g,n} \smallsetminus M^{\mathrm{trop}}_{g,n}$, i.e. to the locus of tropical curves having at least one edge of infinite length.*

Proof. We just need to prove that a point of $\mathbf{\Gamma} \in \overline{M}_{g,n}^{\rm trop} \smallsetminus M_{g,n}^{\rm trop}$ lies in the image of a clutching or gluing map. Let e be an edge of $\mathbf{\Gamma}$ having infinite length, write v_1 and v_2 for its (possibly equal) endpoints. Let $\mathbf{\Gamma}'$ be the tropical curve obtained by removing e and attaching a leg l_1 at v_1 and a leg l_2 at v_2. If e is not a bridge, $\mathbf{\Gamma}'$ is easily seen to be a stable tropical curve of genus $g-1$ with $n+2$ marked ponts (which we order so that l_1 and l_2 are the last ones); it is clear that the image of $\mathbf{\Gamma}'$ via the gluing map is $\mathbf{\Gamma}$.

If e is a bridge, then $\mathbf{\Gamma}' = \mathbf{\Gamma}_1 \sqcup \mathbf{\Gamma}_2$, with $\mathbf{\Gamma}_i$ containing the vertex v_i; it is clear that $\mathbf{\Gamma}_i$ is a stable tropical curve whose last leg we set equal to l_i, for $i = 1, 2$. Then $\mathbf{\Gamma}$ is equal to $\kappa^{\rm trop}(\mathbf{\Gamma}_1, \mathbf{\Gamma}_2)$. ♣

The locus of smooth algebraic curves in $\overline{\mathcal{M}}_{g,n}$ corresponds in the tropical moduli space to the smallest stratum, that is the single point $\bullet_{g,n} \in \overline{M}_{g,n}^{\rm trop}$ parametrizing the tropical curve whose graph has a unique vertex of weight g (and no edges). Hence the boundary of $\overline{\mathcal{M}}_{g,n}$ corresponds the open subset $\overline{M}_{g,n}^{\rm trop} \smallsetminus \{\bullet_{g,n}\}$.

In this sense, a tropical counterpart of the fact that the algebraic and gluing maps cover the boundary of $\overline{\mathcal{M}}_{g,n}$ should be that some generalized tropical gluing and clutching maps cover $\overline{M}_{g,n}^{\rm trop} \smallsetminus \{\bullet_{g,n}\}$.

We shall now define a generalization of the previously defined gluing and clutching maps having that goal in mind.

We denote $\mathbb{R}_+ = \mathbb{R}_{>0} \sqcup \{\infty\}$. For every pair $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ We have a new tropical gluing map $\gamma^{\rm trop}[x, y]$

$$\gamma^{\rm trop}[x, y] : \overline{M}_{g-1,n+2}^{\rm trop} \to \overline{M}_{g,n}^{\rm trop}$$

constructed as follows. Denote by l_{n+1} and l_{n+2} the last two legs of $\mathbf{\Gamma}' \in \overline{M}_{g-1,n+2}^{\rm trop}$, and by v_{n+1} and v_{n+2} the vertex they are adjacent to. As in the definition of γ in the previous section, we send $\mathbf{\Gamma}'$ to a curve in $\mathbf{\Gamma} \in \overline{M}_{g,n}^{\rm trop}$ by merging l_{n+1} and l_{n+2} into one edge e of Γ. The difference is that now the new edge will have length equal to $\ell(e) = x + y$. This is obtained by fixing on l_{n+1} a point p_{n+1} of distance x from v_{n+1}, and "clipping off" the remaining infinite line; similarly, we fix a point p_{n+2} of distance y from v_{n+2}, on l_{n+2} and disregard the rest of the leg; then we glue p_{n+1} to p_{n+2} obtaining an edge between v_{n+1} and v_{n+2} of length $x + y$. It is clear that $\gamma^{\rm trop}[x, y]$ is continuous. Observe that $\gamma^{\rm trop}[x, y]$ depends only on $x + y$ (and hence it is symmetric) and that the "more natural" gluing map $\gamma^{\rm trop}$ defined before is obtained as

$$\gamma^{\rm trop} = \gamma^{\rm trop}[\infty, x] = \gamma^{\rm trop}[\infty, \infty].$$

Summarizing, we have defined a continuous family of maps

$$\gamma^{\mathrm{trop}}[\,,\,] : \overline{M}^{\mathrm{trop}}_{g-1,n+2} \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \overline{M}^{\mathrm{trop}}_{g,n}.$$

In a completely analogous way we define the generalized clutching maps $\kappa^{\mathrm{trop}}[\,,\,] = \kappa^{\mathrm{trop}}_{g_1,n_1,g_2,n_2}[\,,\,]$:

$$\kappa^{\mathrm{trop}}[\,,\,] : \overline{M}^{\mathrm{trop}}_{g_1,n_1+1} \times \overline{M}^{\mathrm{trop}}_{g_2,n_2+1} \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \overline{M}^{\mathrm{trop}}_{g,n}.$$

As before, $\kappa^{\mathrm{trop}}[x,y] = \kappa^{\mathrm{trop}}[x,y]$ and the original clutching map is recovered as

$$\kappa^{\mathrm{trop}} = \kappa^{\mathrm{trop}}[x,\infty] = \kappa^{\mathrm{trop}}[\infty,\infty].$$

Remark 8.6.2. It is clear that the union of the image of $\gamma^{\mathrm{trop}}_{g-1,n+2}[\,,\,]$ with the images of the maps $\kappa^{\mathrm{trop}}_{g_1,n_1,g_2,n_2}[\,,\,]$ is equal to $\overline{M}^{\mathrm{trop}}_{g,n} \smallsetminus \{\bullet_{g,n}\}$.

Remark 8.6.3. It would be interesting to find a lifting of these generalized clutching and gluing maps to Berkovich analytic spaces.

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