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THE GRAVITATION OF MARKET PRICES AS A STOCHASTIC PROCESS

Saverio M. Fratini and Alessia Naccarato*
Roma Tre University

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ABSTRACT

The theory of value has been based ever since Adam Smith on the idea that the market prices of commodities, those at which actual trade takes place, gravitate around a central position known as natural prices. This article seeks to develop a statistical idea of the process in question and suggests in particular that market prices can be said to gravitate around natural prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity. A set of possible conditions leading to that result is also presented.

1. INTRODUCTION

In explaining the concept of price he was adopting in *Production of Commodities by Means of Commodities* (1960), Sraffa wrote that '[s]uch classical terms as "necessary price", "natural price" or "price of production" would meet the case, but value and price have been preferred as being shorter and in the present context (which contains no reference to market prices) no more ambiguous' (Sraffa, 1960, p. 9).

In this passage at least, Sraffa thus refers explicitly to the distinction between two different conceptions of price, namely natural price and market price,[1] which has certainly been adopted in the theory of value since Adam Smith[2] but probably for even longer.

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[1] On the idea of normal prices in Sraffa see also Garegnani (2002) and Ravagnani (2002).

[2] See in particular, for a textual analysis of Smith's observations on market prices and the process of gravitation, Aspromourgos (2007) and (2009). As for the idea of a 'natural' or 'normal' position within the classical approach to value and distribution, the articles available

The natural price of a commodity is 'what is sufficient to pay the rent of the land, the wages of the labour, and the profits of the stock employed in raising, preparing, and bringing it to market, according to their natural rates' (Smith, 1976, vol. 2, p. 72—WN I.vii.4). In other words, if the prices of commodities are at their natural levels, then labourers with the same skills, land of the same quality and capital can respectively receive the same rates of wages, rents and profits regardless of the sphere of activity in which they are employed.[3]

The market price is instead '[t]he actual price at which any commodity is commonly sold' and 'is regulated by the proportion between the quantity which is actually brought to market, and the demand of those who are willing to pay the natural price of the commodity', i.e. 'the effectual demand' (Smith, 1976, vol. 2, p. 73—WN I.vii.7, 8).

It is therefore evident that the natural price and the market price of a commodity are different in nature and depend on different forces: (a) the former is a theoretical variable and the latter an actual (observed) magnitude; (b) the former depends—for a given technique—on the 'ordinary or average' rates of wages, rents and profits, which in turn depend on 'the general circumstance of the society' (Smith, 1976, vol. 2, p. 72—WN I.vii.1, 2), whereas the determination of the latter is a market phenomenon based, according to Smith, on the quantity actually brought to market and effectual demand.

Regardless of how 'seemingly independent they appear to be', however, the natural price and the market price 'are necessarily connected' (Smith, 1976, vol. 6, p. 496—LJ(B) 229). This connection is commonly called 'gravitation' in that it is understood as a competitive process that causes the market prices of commodities to 'gravitate' around the natural prices.

Various representations of this process have been developed by scholars through the years. Some are very close to Adam Smith's original conception, while others present important differences.[4] In particular, since the publication of Sraffa's book, the gravitation of market prices has mostly been addressed through the construction of dynamical models where the market prices obtaining at any date are understood not as actual prices but as states of a dynamical system generated by some differential equations

are so many that we cannot provide an exaustive list. We can just metion Garegnani (1984) and Roncaglia (1990) as two authoritative points of view on this topic.

[3] According to Aspromourgos (2007, p. 29), '[i]n latter-day terms, natural price is a notion of opportunity cost: it is the price which just enables payment to the owners of the employed inputs, the remuneration normally available in alternative uses'.

[4] For the comparison of Smith's ideas about gravitation with those of Ricardo and Marx, see in particular Vianello (1989), Salvadori & Signorino (2013) and Signorino (2015).

(or difference equations) starting from a given initial state.[5] Once the market price trajectory is determined, the gravitation consists in its tendency to move towards the natural position. Section 2 will provide a general rapresentation of this mechanical idea of gravitation in the form of a model capable of highlighting its key features.

Although interesting for various reasons, this kind of analysis cannot be considered fully satisfactory because the results obtained are inevitably dependent on the validity of the special assumptions upon which the model is built. As shown in section 3, a different way of understanding gravitation, based on a 'statistical concept of equilibrium', was therefore introduced by Parrinello (1990). His contributions (1998 and 2013) also include considerations on the possibility of different conceptions of the normal state in economics in addition to and beyond the idea of the equilibrium state of a dynamical process borrowed from the classical mechanics.

This article constitutes an attempt to develop Parrinello's ideas. In particular, since the market prices obtaining at a certain date can be treated as random variables, their means after t observations are random variables as well. This leads us to a new conception of the process as set forth in section 4, where it is stated that market prices gravitate around natural prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity. Section 6 then presents a set of possible conditions leading to that result.

2. GRAVITATION AS A DYNAMICAL MODEL: A BRIEF OVERVIEW

As stated above, market prices are the prices at which commodities are actually sold. Unlike the Walrasian mechanism of *tâtonnement*,[6] e.g. classical gravitation is therefore assumed to take place in the real world; it is an actual phenomenon and not merely a theoretical construction.

Since the market phenomena occurring day by day in the real world are so complex as to be unpredictable,[7] it is impossible to pinpoint the forces governing the determination of market prices with the same degree of generality as in the case of natural prices. The classical economists appear in

[5] For basic concepts about dynamical systems and their use in economics, see Frisch (1936).

[6] For some comparisons of classical and neo-classical theories of value and distribution with specific reference to gravitation and price adjustments, see also Ciccone (1999), Garegnani (2002) and Fratini and Levrero (2011).

[7] If this were not so, i.e. if the prices determined for example in a trading session at the Chicago Mercantile Exchange were predictable, no speculative transaction would ever be possible.

particular to have been well aware that market prices cannot be addressed theoretically, 'except for the *sign* of their deviation from their natural or normal levels' (Garegnani, 2002, p. 393).[8]

As in every science, however, a great deal of the real world's complexity can be eliminated by constructing a model in which the central phenomena alone are represented. This is in fact the approach adopted by most scholars in studying gravitation, and the literature thus contains a variety of models. A number of their key features are presented here, but a complete survey lies decidedly beyond the scope of this article.

Let us consider a world with N commodities labelled $n = 1, 2, \ldots, N$, no joint production, no alternative techniques and no natural resources.

The technical coefficients of production are arranged in a matrix A and a (column) vector $\ell \in \mathrm{R}^N_{++}$, in which a_{nj} and ℓ_n respectively represent the amount of commodity j and the amount of labour for the production of each unit of commodity n.[9] The 'final' effectual demand, expressed by the (row) vector $y \in \mathrm{R}^N_{++}$, is taken as given, so that the 'gross' effectual demand is $q = y \cdot (\mathrm{I} - A)^{-1}$. As in Sraffa (1960, p. 10), it is assumed that $q \cdot \ell = 1$.

With the composite commodity y adopted as the numéraire (cf. Sraffa, 1960, p. 11) and given a wage rate (or share) w, the (column) vector[10] of natural prices π is determined together with the natural or ordinary rate of profits r by means of the usual Sraffian system:

$$\pi = (1+r)A \cdot \pi + w\ell \tag{1}$$

$$y \cdot \pi = 1 \tag{2}$$

If an arbitrary vector of market prices $\pi_t = [\pi_{1,t}, \pi_{2,t}, \ldots, \pi_{N,t}]$ is instead taken, with $y \cdot \pi_t = 1$, the rate of profits will not be uniform across sectors.

[8] A similar view is expressed by Salvadori and Signorino (2013): 'Smith devotes much care to determining natural values and to the gravitation process of market magnitudes to their natural counterparts. The same cannot be maintained as regards the question of market price determination' (p. 161); and yet 'while the classical authors extensively investigated long-period, natural values and gravitation, they were more sketchy on market price determination in situations of market disequilibrium' (p.170).

[9] As all the means of production are circulating capital goods and there is no fixed capital in this model, no question connected with the degree of capacity utilisation emerges in the case considered here. See Ciccone (2011) for a study of the possibility of variations in the degree of capacity utilisation during the process of gravitation.

[10] In this article, as a general rule, vectors of quantities are row vectors and vectors of prices are column vectors. Row and column vectors can in any case be distinguished quite easily on the basis of context.

A 'market rate of profits'[11] can then be defined for each sector:

$$r_{n,t} = \frac{\pi_{n,t} - \sum a_{n,j}\pi_{j,t} - \ell_n w}{\sum a_{n,j}\pi_{j,t}} =: r_n(\pi_t);\ \forall n = 1,\ 2,\ \ldots, N. \quad (3)$$

There is thus, in every period, a vector of market rates of profits $r_t(\pi_t) = [r_{1,t}(\pi_t), r_{2,t}(\pi_t), \ldots, r_{N,t}(\pi_t)]$ associated with the market price vector π_t.

Now, as Adam Smith argued, given the effectual demand for commodities, the deviation of market prices from natural prices depends on the quantities actually produced and brought to market.[12] Let $q_t = [q_{1,t}, q_{2,t}, \ldots, q_{N,t}]$, with $q_t \cdot \ell = 1$, be the vector of produced quantities in period t, it is thus possible to write:

$$\pi_t = \pi + \phi(q_t) \quad (4)$$

where $\phi(.)$ is assumed to be a continuous function such that: (i) $\phi(q) = [0, 0, \ldots, 0]$ and (ii) $y \cdot \phi(q_t) = 0 \ \forall\ q_t$: $q_t \cdot \ell = 1$. This means that: (i) market prices correspond to normal prices when the quantities brought to market equal the effective demand; (ii) market relative prices are expressed in terms of the same numéraire as normal relative prices, i.e. $y \cdot \pi_t = y \cdot \pi = 1$.

As regards the quantities produced and brought to market, according to the classical idea of gravitation,[13] they depend on the sectoral rates of

[11] The expression 'market rate of profits' is used by Garegnani (1990, p. 334) to denote the rate of profits obtainable in a sector in a given 'market position', i.e. if the market prices emerging in period t were assumed to persist in the following periods too. This market rate of profits is what is usually regarded in the literature on gravitation as the sectoral rate of profits when the prices are not natural prices. It cannot, however, be viewed as the rate of profits actually obtained in period t. The rate of profits of sector n obtained in period t depends in actual fact both on the market prices π_t, at which the outputs are sold, and on the market prices π_{t-1}, at which the means of production were paid. Therefore: $r_{n,t}(\pi_{t-1}, \pi_t) := (\pi_{n,t} - \Sigma\, a_{n,j} \cdot \pi_{j,t-1} - \ell_n \cdot w) / \Sigma\, a_{n,j} \cdot \pi_{j,t-1}$. See also: Lager (1998) and Bellino (2011, pp. 63–4).

[12] There are several ways of building dynamic gravitation models, here we try to stay as close as possible to Adam Smith's ideas. Cf. also Bellino and Serrano (2011).

[13] In Ricardo's words: [i]t is then the desire, which every capitalist has, of diverting his funds from a less to a more profitable employment that prevents the market price of commodities from continuing for any length of time either much above, or much below their natural price. It is this competition which so adjusts the exchangeable value of commodities, that after paying the wages for the labour necessary to their production, and all other expenses required to put the capital employed in its original state of efficiency, the remaining value or overplus will in each trade be in proportion to the value of the capital employed [Ricardo, 1951, vol. 1, p. 91]. Similar claims can be found in Marx: 'competition levels the rates of profit of the different spheres of production into an average rate of profit [. . .]. This is accomplished by continually transferring capital from one sphere to another, in which the profit happens to stand above the average for the moment. [. . .] These incessant emigrations and immigrations

profits $r(\pi_t)$.[14] We have therefore:

$$q_{t+1} = q_t + \psi[r(\pi_t)] \tag{5}$$

where $\psi(.)$ is assumed to be a continuous function such that: (i) $\psi(r \cdot u) = [0, 0, \ldots, 0]$, where $u = [1, 1, \ldots, 1]$, and (ii) $\psi(r_t) \cdot \ell = 0 \; \forall \; r_t$. In other words, (i) produced quantities are stationary if the rate of profits is uniform across sectors, i.e. $r_{n,t} = r \; \forall \; n = 1, 2, \ldots, N$, and (ii) the total employment of labour does not change during the process[15] but $q_{t+1} \cdot \ell = q_t \cdot \ell = 1$.

Specification of the functional forms $\phi(.)$ and $\psi(.)$ gives a model describing the dynamics of the variables π_t and q_t. Therefore, given an arbitrary initial vector of produced quantities q_0, the system formed by difference equations (4) and (5) makes it possible to determine prices and quantities at every moment of time. In particular, a solution of the system (if it exists) can be denoted as $\pi(q_0,t)$ and $q(q_0,t)$.

Because of the assumption made, the natural prices π and the quantities corresponding to the effectual demand for commodities q constitute an equilibrium (or a stationary state) in the sense of rational mechanics for the dynamical system considered here. In fact, as can be easily proved, when the initial vector of produced quantities is set equal to the effectual demand for commodities, i.e. $q_0 = q$, neither the quantities nor the prices change over time and, in particular, $\pi(q,t) = \pi$ and $q(q,t) = q \; \forall \; t \geq 0$. It becomes possible at this point to study the gravitation of market prices around their natural position in terms of equilibrium (asymptotic) stability. In other words, the natural prices π are regarded as a 'centre of gravitation' for market prices if $\pi(q_0,t) \rightarrow \pi$ as $t \rightarrow \infty$, for every q_0.

There is no need here to go any further along this path, as it has already been widely explored. We shall therefore just recall that while it is possible to construct models in which the natural position is an unstable

of capital, which take place between the different spheres of production, [...] create a tendency to reduce the rate of profit everywhere to the same common and universal level. This movement of capitals is caused primarily by the stand of the market-prices, which lift profits above the level of the universal average in one place and depress them below it in another' [Marx, 1909, vol. 3, p. 243].

[14] The market rates of profits $r_{1,t}(\pi_t)$, $r_{2,t}(\pi_t)$, $\ldots$, $r_{N,t}(\pi_t)$—defined by equation (3)—are here assumed to be the rates that capitalists expect to obtain, in an indefinite future, on their present investments of capital in the various sectors of activity. Needless to say, this assumption is extremely peculiar. It will be eliminated in section 4.

[15] As stressed by Garegnani (1990, p. 332), 'the *aggregate* economic activity (on which the effectual demand of the individual commodities evidently depend) can be taken as given in analysing market prices'. Garegnani therefore assumed the level of aggregate labour employment as constant in his article, just as we are doing here.

equilibrium, there are reasonable assumptions bringing about equilibrium stability. Readers are referred to Boggio (1990) and Bellino (2011) for an overview of these results.[16]

3. PARRINELLO'S CONTRIBUTION TO THE ANALYSIS OF GRAVITATION

The purpose of this article is to put forward an analysis of gravitation differing from the one outlined above. The roots of our approach to gravitation lie in the critical analyses Parrinello developed in variour articles[17] in opposition to its treatment by means of dynamical models and concerning both the idea of market prices as the states of a dynamical process and the conception of the natural or normal position as an equilibrium of that process. Since Parrinello's contribution constitutes the starting point of our analysis, as presented in the next two sections, it should be recalled here before proceeding any further.

Let us start with the first critical argument. When the complexity of the real world is addressed by means of a simplified model, this unquestionably leads to an error of approximation. To evaluate the significance of the error made, it is necessary to know what aspects are being neglected and how important they may prove in explaining the phenomenon in question. In this respect, the main problem with dynamic models of gravitation is the fact that it is impossible to specify with a sufficient degree of generality all the forces and mechanisms that may be involved in the actual determination of market prices and hence impossible to have a complete picture of what is being overlooked.

It is therefore essentially impossible to evaluate the error precisely. It could be extremely serious or insignificant. In both cases, according to Parrinello (1990), there are logical problems:

> [i]f we admit that the disequilibrium model also contains some error of specification, then this model is also unable to explain market prices exactly. A serious cumulation of errors over a sequence of rounds or iterations might make the test not reliable. [...] By contrast, if we believe that the model describes with precision the dynamics of the system and it "proves" that gravitation of market prices toward production prices exists and has the properties required, the method of long period states would become legitimate, but, at the same time, useless. In fact, should this stage be achieved, we would resort directly to the "perfect"

[16] A list of open issues about gravitation is given in Petri (2011). For a critical analysis of gravitation models, see also Sinha and Dupertuis (2009b).

[17] While Parrinello presented his ideas about gravitation in a series of papers, reference will be made in this section above all to those of (1990), (1998) and (2013).

> disequilibrium model and the method of approximation based on attractors should be dismissed as a non necessary approximation. (p. 114)

Parrinello therefore suggests that the gravitation of market prices around the natural position should be addressed from another angle. Instead of tackling gravitation with the tools of the rational mechanics, those used in the neoclassical theory of equilibrium, and studying gravitation as the rest position of a dynamical process, he suggests that the revival of the classical approach can benefit from the adoption of a 'statistical concept of equilibrium' (p. 115) whereby market prices and quantities are regarded as random variables and gravitation as a stochastic process.

In his *Wealth of Nations*, Adam Smith identifies two different kinds of random shocks that may affect the gravitation process. First, there are 'accidental variations in the demand' that can alter the market prices of commodities, as in the case of public mourning leading to a rise in the price of black cloth (cf. Smith, 1976, vol. 2, pp. 76 and 132—WN I.vii.19 and I.x.b.46). Second, there are variations in the quantity obtained by the same employment of productive agents because of the succession good and bad seasons or years: '[t]he same quantity of industry, for example, will, in different years, produce very different quantities of corn, wine, hops, sugar, tobacco, etc' (Smith, 1976, vol. 2, pp. 132 and 4—WN I.x.b.46).

Parrinello therefore includes two random disturbances in his analysis—namely $\mu_{n,t}$ and $\varepsilon_{n,t}$, both assumed to have zero mean and to be normally distributed and serially uncorrelated—in order to take into account the effects of these accidental shocks on the market price and the produced quantity of a certain commodity. Moreover, instead of considering the quantity actually produced in t as related to market prices in $t-1$, Parrinello suggests that reference should be made to the market price *expected* by entrepreneurs for period t.

In particular, referring to a certain commodity n and using the same symbols as in the previous section, Parrinello's gravitation model is made up of the following equations:

$$\text{sign}\ (\pi_{n,t} - \pi_n - \mu_{n,t}) = -\text{sign}\ \left(q_{n,t} - q_n\right) \tag{6}$$

$$\text{sign}\ (q_{n,t} - q_n - \varepsilon_{n,t}) = \text{sign}\ (\pi_{n,t}{}^e - \pi_n). \tag{7}$$

The meaning of the equations is quite simple: (i) the direction of deviations of market prices from the natural level depends both on the sign of the deviation of the quantity brought to market from the effectual demand and on the effect of the random shock on demand; (ii) the sign of the deviation of the quantity brought to market from the effectual demand depends in turn on the sign of the gap ($\pi_{n,t}{}^{e} - \pi_n$) between expected and natural prices. There is, however, also the effect of seasonality.[18]

Since $E[\mu_t] = E[\varepsilon_t] = 0$, equations (6) and (7) entail, as Parrinello points out, that:

$$\text{sign } (E[\pi_{n,t}] - \pi_n) = -\text{sign } (\pi_{n,t}{}^{e} - \pi_n). \tag{8}$$

Within this model, it is therefore possible to obtain $\pi_{n,t}{}^{e} = E[\pi_{n,t}]$—i.e. rational expectations[19]—if and only if $\pi_{n,t}{}^{e} = E[\pi_{n,t}] = \pi_n$. This is the primary conclusion: the theoretical level of price corresponds to what the agents must expect in order to be rational.

This possible reinterpretation of the natural price leads us to Parrinello's other contributions (1998 and 2013) concerning the notions of the normal state in economics and other disciplines. In particular, while the stationary state of a dynamic process can be viewed as a normal state, this is not the only way in which it is conceived in scientific analyses. In actual fact, despite the broad use in economics of the notion of equilibrium borrowed from rational mechanics, various concepts of normal state exists and can be adopted,[20] such as statistical equilibrium.[21]

[18] Notice that, because of the presence of random disturbances, equations (6) and (7) are compatible with the possibility that the deviations of the market from the natural price and of the produced quantity from the effectual demand have the same sign. It is worth pointing out that if investment decisions were based on the market rates of profits defined by equation (3), as in the standard gravitation models, then this sign concordance might occur, with more than two commodities, independently of the random disturbances because, as showed by Steedman (1984), the difference between market and natural price of a commodity and the deviation of its market rate of profits from the natural one can have opposite sign. However, in Parinello's analysis, the actually produced quantity of a commodity does not depend on its market rate of profits, which is based on observed prices, but rather on price expectations, and therefore Steedman's result does not seem immediately relevant for it.

[19] According to a standard definition, agents have rational expectations if their predictions or forecasts do not differ systematically from what actually occurs. The agents' rational expectations about prices must therefore correspond to the expected values of the random variables 'market prices'.

[20] Parrinello (1998) provides us with a 'non-exhaustive taxonomy of normal states': '[f]irst, a normal state can be a *regular state* in which the levels or the rates of change (of different orders) of the economic variables are constant over time: a stationary state, a steady growth and a limit cycle belong to this category. Second, a normal state can be an *attractor*, that is a point or a path to which the economic process tends in the presence of disturbances and which exhibits no endogenous tendency to change. Third, a normal state can be a *consistent state*, in which all individual plans are optimal and mutually consistent (they can be carried out simultaneously). Fourth, a normal state can be a *most probable state*' (p. 259).

[21] Given a collection of alternative *microstates* of a system, with a distribution of probability, a *macrostate* can be defined as a statistical aggregation of microstates. A statistical equilibrium is the most probable macrostate.

Moreover, Parrinello observes that the idea of a normal state plays two foundamental roles in the social sciences. First, it can be used as a 'contrasting alternative' in the selection of causes: normal and abnormal states have different explanations and attention can be focussed either on one or on the other. Second, the deviations of actual states from the normal state can enter into the building of causal-structural models.

According to Parrinello (2016, pp. 142–144), this second role is the one played by the normal state within the usual 'classical mechanism of market adjustment'. The divergence between market and natural prices causes the market rates of profits to differ from the natural one, so that capitalists reallocate their capital across sectors as much as possible in order increase their (expected) income. This mechanism, as Parrninello stresses, involves an agency view of causation. In other words, the cause-effect relationship in question is due to the intervention of free agents pursuing their own interests. The typical questions associated to the agency theory of causation that Parrinello discusses in his article therefore arise (as do others that will, however, be addressed in the following section).

In conclusion, Parrinello also claims (1998, p. 261) that random behaviour at the microlevel can drive to a normal state at the macro level, as in the aforementioned case of statistical equilibrium, so that a precise characterization of individual competitive behaviour is not necessary to this end.

4. MARKET PRICES AND QUANTITIES AS RANDOM VARIABLES

Parrinello's contributions have the unquestionable merit of introducing new elements into the debate on gravitation. In particular, attention will be focussed here on the three following points. First, market prices and the quantities actually produced are to be treated as random variables, which is in fact precisely how magnitudes that cannot be exactly predicted are addressed in scientific analyses. Second, decisions about the quantities to bring to market are based on expectations about future prices and hence about sectoral rates of profits. Third, there is no need to regard the normal state referred to by the theory of natural prices as the equilibrium state of a dynamic process. A statistical conception is also possible (and even recommended).

Let us take the second element as our starting point. In the usual dynamical models of gravitation, as seen in section 2, the quantities actually brought to market in period t depend on the market prices obtaining in period $t - 1$. In a nutshell, this follows from the principle that capitalists tends to invest in the sector where they expect to obtain the highest rate of profits,[22] but under the special assumption that expected sectoral rates of profits correspond to the market rates determined by equation (3) for the market prices of period $t - 1$.

Our intention here is not intend to deny the principle but only to jettison the assumption. Following Parrinello, we regard capitslists' decisions as based on expected prices rather than past prices. In particular, Parrinello defines the expected price of a commodity $\pi_{n,t}^{e}$ (with $n = 1, 2, \ldots, N$) as 'the price which, on average, the producers expect to rule at time t on the basis of their information at time $t - 1$' (Parrinello, 1990, p. 117).[23]

Dropping the special assumption about the correspondence between past and expected market prices give rise to a problem. Under the 'law of unique price' or some other principle,[24] there is just one price for each commodity on the market at a certain date. If insead different agents have different arbitrary beliefs, then there are as many different expected prices for the same commodity, in the same moment, as producers. In this case, it is therefore impossible to write anything resembling either the difference equation (5) or Parrinello's sign condition (7). We are instead forced to admit that since individual decisions are essentially unpredictable and since the quantities of commodities actually produced are the result of many individual decisions, nothing much can be said about them.[25]

This clearly does not mean that the principle that capitalists invest where they expect the highest returns is not at work, but simply that the

[22] The expression 'capital mobility across sectors' is often used in referring to this principle. We prefer to avoid it here because it can be misleading. It suggests, quite erroneously, that the further investment of capital in one sector can correspond to disinvestment in other sectors, i.e. that the total capital is fixed while its allocation changes. This is not so. Since the capital goods employed in one sector are generally different from those employed in another and the market prices vary during this process of investment/disinvestment, both the physical composition and the value of the total capital of the economy change, so that there is in general no balance between further investments and disinvestments.

[23] Parrinello assumes in particular that producers have the same information in $t - 1$, so that $\pi_{n,t}^{e} = E[\pi_{n,t}|I_{t-1}]$, where I_{t-1} denotes the information available at $t - 1$.

[24] As in the case of Adam Smith in particular. As noted by Aspromourgos (2007, pp. 33, 34), when Smith identifies the market price with the actual price at with transactions occur, he refers to 'the most common actual price', since in practice different firms may sell the same commodity at different prices.

[25] In the dynamical models discussed in section 2, given the determinants of natural prices and the quantities initially brought to market q_0, there is only one possible path of quantities: $q(q_0,t)$. There is, in other words, just one possible configuration of the disequilibrium allocation of productive agents amongst industries for every period of time. It is instead assumed here that the gravitation can follow many different paths for the same initial conditions and data. In the absence of specific information about the way in which the quantities actually produced are decided period by period, various different patterns must be regarded as possible.

individual expectations driving the process are the result of capitalists' mental process that cannot be described or addressed with the necessary precision and generality. As Keynes stressed (1936, p. 161), the decisions to do something positive, and in particular to invest, depend 'on spontaneous optimism rather than on a mathematical expectation', they 'can only be taken as a result of animal spirits'.

Therefore, in our analysis, the quantities q_t are random vectors—not only because of seasonality but also for deeper reasons—that take values in the set $Q = \{q_t \in \mathrm{R}^{\mathrm{N}}_{+}: q_t \cdot \ell = 1\}$. Moreover, if it is assumed that producers, in the aggregate, do not make systematic errors, then $E[q_t] = q$.

As regards market prices, the previously introduced equation (4) can be taken as a starting point with the inclusion on its RHS of a vector of random disturbance $\mu_t = [\mu_{1,t}, \mu_{2,t}, \ldots, \mu_{N,t}]$ such that (i) $\mu_{n,t}$ is white noise, $\forall$ $n = 1, 2, \ldots, N$ and (ii) $y \cdot \mu_t = 0$. This gives us:

$$\pi_t = \pi + \phi(q_t) + \mu_t. \tag{9}$$

It is worth noting that as long as the function $\phi(.)$ is not specified—it being simply assumed that $\phi(.)$ can be any function such that i) $\phi(q) = [0, 0, \ldots, 0]$ and ii) $y \cdot \phi(q_t) = 0 \; \forall \; q_t \in Q$—equation (9) requires no assumption stronger than Parrinello's 'sign relation' (6). It is indeed even more general than equation (6) because: (a) it only asserts that deviations of market prices from natural prices depend on the quantities actually produced and on random disturbance, without any particular restriction as regards the sign of those relationships;[26] (b) according to equation (9), as well as equation (4), the market price of each commodity may depend—although it does not necessarily do so—on the vector of produced quantities q_t and not on its quantity $q_{n,t}$ alone (with $n = 1, 2, \ldots, N$).

As a result of equation (9) and the assumptions made, the deviations of market prices from natural prices are random variables.[27] The vector π_t is therefore a random vector which—given the function $\phi(.)$ and the random vectors q_t and μ_t—takes values in $\Pi = \{\pi_t \in \mathrm{R}^{N}_{+}: y \cdot \pi_t = 1\}$, with $E[\pi_t] = \pi$.

Again, little is known about the random vector π_t. In particular, we do not know its distribution function and therefore have no idea of how likely π_t is to remain in a certain neighbourhood of π, or whether this likelihood arises for large enough t. Far more definite results could be obtained,

[26] In particular, Parrinello assumed that, without disturbance, the deviation of prices and the deviation of quantities are opposite in sign. On this point, see also footnote 18.

[27] Smith's idea of gravitation as primarily concerned with the deviations of market prices from natural prices is adopted throughout this article. On this view, the market price of a commodity is thus understood as the natural price plus the deviation, as in equations (4) and (9).

however, by focusing attention on the vector of average market prices after t observations instead of the vector of market prices observed in t.

Let $\bar{\pi}_{n,t}$ be the average of the market prices of commodity n (with $n = 1, 2, \ldots, N$) after t observations, that is:

$$\bar{\pi}_{n,t}=\frac{1}{t}\cdot\sum_{\tau=1}^{t}\pi_{n,\tau} \tag{10}$$

The (column) vector of average market prices after t observations is therefore $\bar{\pi}_t=[\bar{\pi}_{1,t},\bar{\pi}_{2,t},\ldots,\bar{\pi}_{N,t}]$.

It is now possible to put forward some observations on the characteristics of the random vector $\bar{\pi}_t$. In particular, if $|\bar{\pi}_t-\pi|$ is the Euclidean distance between $\bar{\pi}_t$ and π, and $pr(|\bar{\pi}_t-\pi| < \delta)$ the probability that this distance is smaller than a given real number δ, it can be said that natural prices are a centre of gravitation for market prices if this probability tends to 1 as the number of observations t tends to infinity.[28] That is:

$$\lim_{t\to\infty} pr(|\bar{\pi}_t-\pi| < \delta)=1 \tag{11}$$

This means that the average market prices converge in probability to the natural prices or, in more technical terms, that $\bar{\pi}_t$ is a consistent estimator of π. The conditions for this kind of gravitation are discussed in the next section.

5. GRAVITATION AND THE CONSISTENCY OF AVERAGE MARKET PRICES

The random vector of market price $\pi_\tau \in \Pi$, with $\tau = 1, 2, \ldots, t$ was introduced in the previous section and it is known that $E[\pi_\tau] = \pi$. Now, a variance-covariance matrix S for the random vector π_τ can also be defined in the usual way: $E\left[(\pi_\tau-\pi)(\pi_\tau-\pi)^{\mathrm{T}}\right]=S$.

Matrix S is not a diagonal matrix because the market prices ruling in period τ are, in general, correlated among themselves. First, relative prices are considered here in terms of a numéraire commodity and the N prices are therefore not

[28] The idea of convergence in probability used here as ‘gravitation’ around natural prices seems to have some possible connections with the notion of statistical equilibrium mentioned in section 3. In particular, as in the case of statistical equilibrium, even without detailed information concerning the single states (individual behaviour, competitive mechanisms and so on), regularity can be achieved on average due to iteraction and aggregation. It does not seem possible, however, to maintain that our notion of gravitation corresponds exactely to the idea of equilibrium in statistical physics. For the possible application of the idea of statistical equilibrium to economics, see Foley (1994, 2003) and Parrinello (1996).

independent: once a numéraire commodity is adopted, given $N-1$ prices, the Nth can be deduced from them. Second, it is quite possible to imagine that the N commodities include some pairs of complementary or substitutes commodities, whose market prices are therefore nonindependent.

However, in order to simplify the analysis, we made the following assumption:

Assumption: Matrix S is constant over time, it does not change with τ.

The constancy of matrix S is not a very strong assumption. In particular, with this assumption, we are not ruling out autoregresive forms. The prices obtaining in a certain period may very well depend on the prices that emerged in previous periods, even though, according to our assumption, this influence, if it exists, must be recursive: the market prices of period τ are statistically dependent on those of period $\tau-\delta$, with $\delta = 1, 2, \ldots, \tau$, in exactly the same way for every τ. In actual fact, what we are assuming is that the structure of the mechanisms potentially acting on market prices is persistent over time.

In the previous sections we have also defined, by equation (10), an average market price $\bar{\pi}_{n,t}$ for each commodity $n = 1, 2, \ldots, N$, which is the mean of the market prices after t observations, and the vector $\bar{\pi}_t = [\bar{\pi}_{1,t}, \bar{\pi}_{2,t}, \ldots, \bar{\pi}_{N,t}]$. We know that $E[\bar{\pi}_t] = \pi$ and, as a result of the well-known properties of the sample mean (Casella & Berger, 2002, pp. 330–32), $E\left[(\bar{\pi}_\tau - \pi)(\bar{\pi}_\tau - \pi)^{\mathrm{T}}\right] = S_{\bar{\pi}_t} = \frac{1}{t}S$.

It therefore follows from the assumption posited on matrix S that $S_{\bar{\pi}_t}$ is a non-diagonal matrix whose entries tend towards zero as t tends to infinity.

Proposition: Let $\bar{\pi}_t$ be the vector of average market prices after t observations, with $E[\bar{\pi}_t] = \pi$ and $E\left[(\bar{\pi}_\tau - \pi)(\bar{\pi}_\tau - \pi)^{\mathrm{T}}\right] = \frac{1}{t}S$, then $\lim_{t\to\infty} pr(|\bar{\pi}_t - \pi| < \delta) = 1$.

Proof: Let us start by defining the random vector $X := (\bar{\pi}_t - \pi)$ and the function $h(X) = X^{\mathrm{T}}X$. Given a strictly positive real number δ, we define the set $D = \{X \in \mathrm{R}^N\colon\ h(X) \geq \delta\}$ and its complement $\bar{D} = \{X \in \mathrm{R}^N\colon\ h(X) < \delta\}$. Then, if the unknown density function of X is denoted as $g(X)$, the following expectation:

$$\begin{aligned} E[h(X)] &= \int_{R^N} h(X)\cdot g(X)\ dX_1 dX_2 \ldots dX_N \\ &= \int_D h(X)\cdot g(X)\ dX_1 dX_2 \ldots dX_N + \int_{\bar{D}} h(X)\cdot g(X)\ dX_1 dX_2 \ldots dX_N \end{aligned} \tag{13}$$

can be written as the sum of two positive quantities so that it immediately implies:

$$E[h(X)] \geq \int_D h(X) \cdot g(X)\ dX_1 dX_2 \ldots dX_N. \tag{14}$$

Because of the definition of the set D given above, the inequality can be enforced by replacing $h(X)$ with its minimum δ in D:

$$E[h(X)] \geq \delta \int_D g(X)\ dX_1 dX_2 \ldots dX_N = \delta\ pr(X \in D) \tag{15}$$

and rearranging the terms:

$$pr(X \in D) \leq \frac{1}{\delta} E[h(X)] \tag{15'}$$

which implies:

$$pr(X \in \bar{D}) \geq 1 - \frac{1}{\delta} E[h(X)] \tag{16}$$

As a result of equation (16):

$$\lim_{t \to \infty}\ pr(X \in \bar{D}) \geq \lim_{t \to \infty}\ \left(1 - \frac{1}{\delta} E[h(X)]\right). \tag{17}$$

Since $h(X) = X^{\mathrm{T}} X = \sum_{n=1}^{N} \left(\bar{\pi}_{n,t} - \pi_n\right)^2$, then $E[h(X)] = \sum_{n=1}^{N} \sigma^2_{\bar{\pi}_{n,t}} = \frac{1}{t} \sum_{n=1}^{N} \sigma^2_n$, which implies: $\lim_{t \to \infty}\ \frac{1}{\delta} E[h(X)] = 0$.

It follows that:

$$\lim_{t \to \infty}\ pr(X \in \bar{D}) = 1 \tag{18}$$

or in other terms:

$$\lim_{t \to \infty}\ pr(h(X) < \delta) = 1 \tag{18'}$$

Therefore, given that $h(X) < \delta$ entails $|\bar{\pi}_t - \pi| < \delta$, then:

$$\lim_{t \to \infty} pr(|\bar{\pi}_t - \pi| < \delta) = 1. \tag{19}$$

■

6. CONCLUSIONS

As defined by Adam Smith and other classical economists, market prices are those at which trade actually takes place at a certain moment. The complexity of the real world and the exceptional character of the events that may occur at any given moment, therefore, preclude the possibility of a theory of the determination of market prices.

There are, however, two possible approaches, which do not necessarily clash. The first (section 2) involves representing reality by means of a simplified model, thereby radically reducing the complexity of the real world, and studying the determination of market prices within the model. The second (section 4) instead means accepting our ignorance[29] about the forces governing the determination of market prices and regarding them as random variables whose actual values can only be known *ex post*.

The first has so far been adopted by most scholars in analysing the gravitation of market prices around a theoretical central position, i.e. natural prices. Gravitation has therefore been addressed as the stability of the rest position of a dynamical system.

The second is instead adopted in this article, which suggests that the gravitation of market prices should be seen as a stochastic process.[30] We have therefore introduced a new conception of the gravitation of market prices whereby natural prices are a centre of gravitation for market prices if the probability of their means being very close to natural prices after t observations tends to 1 as t tends to infinity.

We believe, in conclusion, that the stochastich approach here proposed may help to solve a series of issues arised with respect to the standard mechanical method. In attempting to outline a list of these, incomplete though it may be, we can start stressing, first, that regarding market prices as random variables, rather than as the state of a dynamical system at a date t, seems to be closer to the idea of Adam Smith and the classical economists, i.e. that market prices, being the prices at which trade actually takes place, can be known only *ex post*, after the exchanges have actually occurred. They can be observed, but not determined.

Second, the differential (or difference) equations that appear in the dynamical models of gravitation are necessarily based on specific assumptions[31] about the adjustment mechanisms of produced quantities, prices and the rates of profits

[29] According to the well-known sentence by Poincarré, '[c]hance is only the measure of our ignorance' because '[f]ortuitous phenomena are, by definition, those whose laws we do not know' (1913, p. 395).

[30] This, clearly, does not means neglecting the existence of forces acting with regularity on market prices, but we recognize that there are also other influences which we are not able to predict with sufficient certainty. Something similar can be said for the rolling of dice. While it is clear that this is influenced by some regularities, such as the laws of gravity, the properties of materials and so on, these are not enough to predict the result of the throw.

[31] Cf. for instance Boggio (1985) in which different assumptions about capitalists' behavior are considered.

obtained in different sectors. Those specific assumptions are not required with the stochastic analysis of gravitation put foreward here because it is not based on adjustments but rather on the tendency of deviations from the mean to balance each other or, to be more precise, on the probability of such a balancing after a large number of observations.

The third point is a consequence of the second. In mechanical analysis the stability of the equilibrium state may (and in general does) depend on the value of the model's parameters. The result presented in section 5 is, on the contrary, absolutely general. It depends on no parameter, but takes place whenever it is assumed that: (i) market prices depend on natural prices and on random deviations, (ii) entrepreneurs as a whole do not make systematic errors about the quantities produced and (iii) the structure of market-price determination (whatever it may be) is persistent over time.

Finally, as seen in the discussion of Parrinello's contributions in section 3, different conceptions of the normal position exist. The equilibrium state of a dynamical model is certainly not the only way in which a normal position can be characterized. In this respect, the stochastic approach presented here is far more open to different interpretations of the normal position and the process of 'gravitation' than the usual mechanical approach.[32] Moreover, the notion of gravitation introduced here appears to be particularly consistent with the view of Smith, Ricardo and Marx[33] that due to competition, actual prices come to approximate natural prices (or production prices) on average over a long enough period of time.

[32] We can incidentally remark that the present idea of 'centre of gravitation' seems to be exempt from some critiques raised against the standard notion, as those in Sinha and Dupertuis (2009a) and (2009b).

[33] Marx, in particular, wrote that 'the fluctuations of supply and demand do not explain anything but the deviations of market-prices from the prices of production. These deviations balance mutually, so that in the course of long periods the *average market-prices* correspond to the prices of production' [Marx, 1909, vol. 3, p. 419, emphasis added]. A similar idea can be found in Adam Smith's analysis, in which, as noted by Aspromourgos, 'the temporary and occasional price' is opposed to 'the average or ordinary price'; and 'equality of natural price and average/ordinary price is an expression of the conviction that competition will indeed prevail upon actual prices, tending to bring them into line with opportunity cost, so that actual prices at least on average approximate natural price' (Aspromourgos, 2007, p. 30, footnote). For the role of average market prices in Ricardo's and Marx's analyses of gravitation, see also Ciccone (1999, p. 65).

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Saverio Maria Fratini
Department of Economics
Roma Tre University
Via S. D'Amico, 77
00145 Rome
Italy
E-mail: saveriomaria.fratini@uniroma3.it

Alessia Naccarato
Department of Economics
Roma Tre University
Via S. D'Amico, 77
00145 Rome
Italy
E-mail: alessia.naccarato@uniroma3.it