

Nonradial entire solutions for Liouville systems

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Abstract

We consider the following system of Liouville equations:

$$\begin{cases} -\Delta u_1 = 2e^{u_1} + \mu e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = \mu e^{u_1} + 2e^{u_2} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_1} < +\infty, \int_{\mathbb{R}^2} e^{u_2} < +\infty. \end{cases}$$

We show the existence of at least $n - \lfloor \frac{n}{3} \rfloor$ global branches of nonradial solutions bifurcating from $u_1(x) = u_2(x) = U(x) = \log \frac{64}{(2 + \mu)(8 + |x|^2)^2}$ at the values $\mu = -2 \frac{n^2 + n - 2}{n^2 + n + 2}$ for any $n \in \mathbb{N}$.

1 Introduction

In this paper we consider the system

$$\begin{cases} -\Delta u_1 = 2e^{u_1} + \mu e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = \mu e^{u_1} + 2e^{u_2} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_1} < +\infty, \int_{\mathbb{R}^2} e^{u_2} < +\infty. \end{cases} \quad (1.1)$$

When $\mu = -1$ we recover the well known Toda system, whose solutions were completely classified in [5]. This type of systems received a growing interest in recent years. We mention the paper [9] for a detailed description of the physical applications and the extensive list of references. We point out that the known results for (1.1) when $\mu \neq -1$ concern the existence (or nonexistence) of *radial* solutions. An important case was considered in [1] and [2] where the authors study the cooperative system

$$\begin{cases} -\Delta u_1 = a_{11}e^{u_1} + a_{12}e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = a_{21}e^{u_1} + a_{22}e^{u_2} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_1} < +\infty, \int_{\mathbb{R}^2} e^{u_2} < +\infty. \end{cases}$$

and under the assumption that the matrix $A = (a_{ij})_{i,j=1,2}$ is symmetric, irreducible and with nonnegative entries they prove that all solutions are radial (see also [8]).

If some of the entries a_{ij} are negative the previous result is not true anymore. Indeed if $\mu = -1$ the classification result of Jost and Wang shows the existence of *nonradial* solutions. It is the only paper where the existence of such solutions appears. Indeed existence results of *nonradial* solutions are a quite difficult task because the system has not a variational structure. Furthermore the lack of compactness due to the scaling and translation invariance causes additional difficulties. In this paper we investigate the nonradial case by using the *bifurcation theory*.

First note that for any $\mu > -2$ problem (1.1) always admits the branch of trivial solutions $(U_{\mu,\delta,y}, U_{\mu,\delta,y})$ where $U_{\mu,\delta,y}$ is the solution of the Liouville equation

$$-\Delta u = (2 + \mu)e^u \quad \text{in } \mathbb{R}^2,$$

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namely

$$U_{\mu,\delta,y} := \log \frac{64\delta}{(2+\mu)(8\delta+|\cdot-y|^2)}. \quad (1.2)$$

For sake of simplicity we consider

$$U_\mu := U_{\mu,1,0}$$

and we look for solutions bifurcating from (U_μ, U_μ) . Note that, using the pair (U_{μ_1}, U_{μ_2}) , our technique allows to consider more general systems like

$$\begin{cases} -\Delta u_1 = 2e^{u_1} + \alpha e^{u_2} & \text{in } \mathbb{R}^2 \\ -\Delta u_2 = \beta e^{u_1} + 2e^{u_2} & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_1} < +\infty, \int_{\mathbb{R}^2} e^{u_2} < +\infty \end{cases}$$

$$\text{with } \alpha = \frac{(2+\mu_2)(\mu_1^2+2\mu_1+2)}{(2+\mu_1)} \text{ and } \beta = \frac{(2+\mu_1)(\mu_2^2+2\mu_2+2)}{(2+\mu_2)}.$$

We let to the interested reader the extension of our results to this case.

The main tool to find solutions bifurcating from (U_μ, U_μ) will be the classical Crandall-Rabinowitz Theorem (see Theorem 2.2). It is well known that this theorem has had enormous applications in many problems in mathematical analysis. However, in our case, its applicability is not at all trivial. This is mainly due to the following reasons,

- The geometrical invariances of the problem induce a lack of compactness
- It is not trivial to prove that the solutions possibly founded in this way are *nonradial*
- The kernel of the natural associated functional is not one dimensional

The first problem will be overcome looking for solutions with suitable invariance. This trick allows to kill the solutions of the linearized system generated by the scaling and translations invariance of the problem and to consider only the degeneracy due to the existence of other nonequivalent solutions. Further since the setting of the problem in \mathbb{R}^2 does not fit in the usual Sobolev spaces we prefer to define the operator in the unit sphere \mathbb{S}^2 . Actually we will consider the system

$$\begin{cases} -\Delta_{\mathbb{S}^2} \phi_1 = 2 \left(e^{\frac{\phi_1+\phi_2}{2}} + e^{\frac{\phi_1-\phi_2}{2}} - 2 \right) & \text{in } \mathbb{S}^2 \\ -\Delta_{\mathbb{S}^2} \phi_2 = 2 \frac{2-\mu}{2+\mu} \left(e^{\frac{\phi_1+\phi_2}{2}} - e^{\frac{\phi_1-\phi_2}{2}} \right) & \text{in } \mathbb{S}^2. \end{cases} \quad (1.3)$$

Here we can use more standard Hilbert spaces and we also avoid problems with the behavior of the solutions at infinity. Moreover using the standard stereographic projection we have that solutions in the sphere \mathbb{S}^2 eventually satisfy system (1.1).

The second obstruction is more delicate. In [4] has been proved that the linearization of (1.1) at (U_μ, U_μ) , namely

$$\begin{cases} -\Delta v_1 = 2e^{U_\mu} v_1 + \mu e^{U_\mu} v_2 & \text{in } \mathbb{R}^2 \\ -\Delta v_2 = \mu e^{U_\mu} v_1 + 2e^{U_\mu} v_2 & \text{in } \mathbb{R}^2 \end{cases} \quad (1.4)$$

admits the sequence of eigenvalues

$$\mu_n = -2 \frac{n^2 + n - 2}{n^2 + n + 2} \quad (1.5)$$

and a corresponding $(2n+4)$ -dimensional kernel. This kernel reduces to a 1-dimensional one if we restrict to the space of the radial function (once we overcome the degeneracy due to the scaling invariance) and this was one of the ideas in [4] to get branches of radial solutions bifurcating by (U_μ, U_μ) . In order to get a one-dimensional *nonradial* kernel the argument is more subtle. In order to give an idea let us argue as follows: let us consider

$$\begin{cases} u_1 = U_\mu + \frac{\Phi_1 + \Phi_2}{2} \\ u_2 = U_\mu + \frac{\Phi_1 - \Phi_2}{2} \end{cases} \quad (1.6)$$

so that (1.1) becomes

$$\begin{cases} -\Delta \Phi_1 = (2+\mu) \left(e^{\frac{\Phi_1+\Phi_2}{2}+U_\mu} + e^{\frac{\Phi_1-\Phi_2}{2}+U_\mu} - 2e^{U_\mu} \right) & \text{in } \mathbb{R}^2 \\ -\Delta \Phi_2 = (2-\mu) \left(e^{\frac{\Phi_1+\Phi_2}{2}+U_\mu} - e^{\frac{\Phi_1-\Phi_2}{2}+U_\mu} \right) & \text{in } \mathbb{R}^2 \end{cases} \quad (1.7)$$

Using polar coordinates let us consider the functions invariant for the following symmetries

$$\mathcal{X} = \{(\Phi_1, \Phi_2) \text{ such that } \Phi_1(\theta, r) = \Phi_1(\theta + \pi, r), \Phi_2(\theta, r) = -\Phi_2(\theta + \pi, r)\}$$

A crucial remark is that the system (1.7) is invariant under the action in \mathcal{X} and so \mathcal{X} is a natural constraint. This invariance with respect to *odd* mappings does not take place directly in (1.1) but it appears as we introduce the new function Φ_1 and Φ_2 in (1.6). Of course radial functions cannot belong to \mathcal{X} and so only nonradial functions are allowed. However the space \mathcal{X} will be not our finale choice because in general the linearization of (1.7) is not 1-dimensional. We need to consider a more complicate space involving more symmetries. This will be discussed in Section 4.

Let us stress that the choice of the symmetries and antisymmetries allows from one side, to get through with the degeneracy induced by dilations and translations and from the other side it reduces the dimension of the kernel. It also allows to find different nonequivalent solutions to problem (1.1). An important role in our results is played by the associated Legendre polynomials $P_n^m(z)$ defined as

$$P_n^m(z) = \frac{(-1)^m}{2^n n!} (1-z^2)^{\frac{m}{2}} \frac{d^{n+m}}{dz^{n+m}} (z^2-1)^n \quad (1.8)$$

for $m, n \in \mathbb{N}$, $m \leq n$. Now we are in position to state our main result,

Theorem 1.1.

Let $n \in \mathbb{N}$ and μ_n be defined in (1.5). The points $(\mu_n, U_{\mu_n}, U_{\mu_n})$ are non-radial multiple bifurcation points for the curve of solutions (μ, U_μ, U_μ) . In particular, for any $m \in \mathbb{N}$ satisfying $\frac{n}{3} < m \leq n$ there exists $\varepsilon_0 > 0$ and, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, a C^1 -family of solutions $(\mu_n(\varepsilon), u_{1,\varepsilon}, u_{2,\varepsilon})$ of (1.1), which satisfy

$$\begin{cases} \mu_n(0) = \mu_n \\ u_{1,\varepsilon}(x) = U_{\mu_n}(x) + \varepsilon P_n^m \left(\frac{8-|x|^2}{8+|x|^2} \right) \cos(m\theta) + \varepsilon(Z_{1,\varepsilon}(x) + Z_{2,\varepsilon}(x)) \\ u_{2,\varepsilon}(x) = U_{\mu_n}(x) - \varepsilon P_n^m \left(\frac{8-|x|^2}{8+|x|^2} \right) \cos(m\theta) + \varepsilon(Z_{1,\varepsilon}(x) - Z_{2,\varepsilon}(x)) \\ Z_{1,0} = Z_{2,0} = 0 \end{cases}, \quad (1.9)$$

where $Z_{1,\varepsilon}, Z_{2,\varepsilon}$ are uniformly bounded functions in $L^\infty(\mathbb{R}^2)$ satisfying

$$\begin{aligned} Z_{1,\varepsilon} \left(\frac{8}{r} e^{i\theta} \right) &= Z_{1,\varepsilon}(re^{i\theta}) & Z_{1,\varepsilon} \left(re^{i(-\theta+\frac{\pi}{m})} \right) &= Z_{1,\varepsilon}(re^{i\theta}) & Z_{1,\varepsilon} \left(re^{i(\theta+\frac{\pi}{m})} \right) &= Z_{1,\varepsilon}(re^{i\theta}) \\ Z_{2,\varepsilon} \left(\frac{8}{r} e^{i\theta} \right) &= (-1)^{n+m} Z_{2,\varepsilon}(re^{i\theta}) & Z_{2,\varepsilon} \left(re^{i(-\theta+\frac{\pi}{m})} \right) &= -Z_{2,\varepsilon}(re^{i\theta}) & Z_{2,\varepsilon} \left(re^{i(\theta+\frac{\pi}{m})} \right) &= -Z_{2,\varepsilon}(re^{i\theta}). \end{aligned} \quad (1.10)$$

Moreover the bifurcation is global and the Rabinowitz alternative holds.

Finally we have the following mass-quantization property

$$\int_{\mathbb{R}^2} e^{u_{1,\varepsilon}} = \int_{\mathbb{R}^2} e^{u_{2,\varepsilon}} = \frac{8\pi}{2 + \mu_n(\varepsilon)}. \quad (1.11)$$

Remark 1.2. From the symmetry properties (1.10) satisfied by $Z_{1,\varepsilon}, Z_{2,\varepsilon}$ we deduce that $u_{1,\varepsilon}, u_{2,\varepsilon}$ verify:

$$\begin{aligned} u_{1,\varepsilon} \left(\frac{8}{r} e^{i\theta} \right) &= \log \frac{r^4}{64} + \begin{cases} u_{1,\varepsilon}(re^{i\theta}) & \text{if } n+m \text{ is even} \\ u_{2,\varepsilon}(re^{i\theta}) & \text{if } n+m \text{ is odd} \end{cases} & u_{2,\varepsilon} \left(\frac{8}{r} e^{i\theta} \right) &= \log \frac{r^4}{64} + \begin{cases} u_{2,\varepsilon}(re^{i\theta}) & \text{if } n+m \text{ is even} \\ u_{1,\varepsilon}(re^{i\theta}) & \text{if } n+m \text{ is odd} \end{cases} \\ u_{1,\varepsilon} \left(re^{i(\theta+\frac{\pi}{m})} \right) &= u_{2,\varepsilon}(re^{i\theta}) & u_{2,\varepsilon} \left(re^{i(\theta+\frac{\pi}{m})} \right) &= u_{1,\varepsilon}(re^{i\theta}) \\ u_{1,\varepsilon} \left(re^{i(-\theta+\frac{\pi}{m})} \right) &= u_{2,\varepsilon}(re^{i\theta}) & u_{2,\varepsilon} \left(re^{i(-\theta+\frac{\pi}{m})} \right) &= u_{1,\varepsilon}(re^{i\theta}) \end{aligned}$$

In particular, $u_{i,\varepsilon} \left(re^{i(\theta+\frac{2\pi}{m})} \right) = u_{i,\varepsilon}(re^{i\theta}) = u_{i,\varepsilon}(re^{-i\theta})$ for both $i = 1, 2$.

By (1.9) and the boundedness of $Z_{1,\varepsilon}, Z_{2,\varepsilon}$ we get the following

Corollary 1.3. We have that

$$u_{1,\varepsilon}(x), u_{2,\varepsilon}(x) \sim -4 \log |x| \quad \text{at } \infty. \quad (1.12)$$

Corollary 1.4. There exists at least $n+1 - \lfloor \frac{n}{3} \rfloor$ nonequivalent solutions bifurcating from $(\mu_n, U_{\mu_n}, U_{\mu_n})$.

We point out that if $n = 2$ (the classical Toda system) from Theorem 1.1 it is possible to recover, at least locally near $(\mu_2, U_{\mu_2}, U_{\mu_2})$ the classification result by Jost and Wang (see Remark 5.1).

The global bifurcation result in Theorem 1.1 is new also in the case of radial solutions. In the paper [4] only a local radial bifurcation result was proved for system (1.1). Further in the radial setting we can separate the branches getting the following result:

Corollary 1.5. *The radial branches bifurcating from $(\mu_n, U_{\mu_n}, U_{\mu_n})$ do not intersect.*

Finally in the appendix we study the behavior of the parameter $\mu_n(\varepsilon)$ in a neighborhood of μ_n . Recall that if $n = 2$ by the results in [5] and [10] we get that the set of solutions is a nondegenerate 8-dimensional manifold. Hence in this case $\mu_2(\varepsilon) = \mu_2$ for any ε small enough. On the other hand we will show that for many integers $m \geq 1$ and $n \geq 3$ this does not occur (see table at the end of appendix). Then the bifurcation diagram in a neighborhood of μ_n looks like very different from that of the classical Toda system.

The paper is organized as follows. In Section 2 we recall some preliminaries and introduce the functional setting in the sphere \mathbb{S}^2 . Section 3 is devoted to the linearization of our functional and in Section 4 we introduce the symmetries described in (1.10). In Section 5 we prove Theorem 1.1 and in the Appendix we study the behavior of the parameter $\mu_n(\varepsilon)$.

2 Preliminaries and functional setting

As mentioned in the introduction we will prove our results in the unit sphere \mathbb{S}^2 .

A useful coordinate system on \mathbb{S}^2 will be the following hybrid between spherical and Cartesian,

$$\mathbb{S}^2 = \left\{ \left(\sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta, z \right) : z \in [-1, 1], \theta \in [0, 2\pi] \right\}.$$

This coordinate system will allow to write the eigenfunctions of $-\Delta_{\mathbb{S}^2}$ in an easier way. So we have this classical result.

Lemma 2.1.

The eigenvalues of the scalar linear problem

$$-\Delta_{\mathbb{S}^2} w = \lambda w \quad \text{in } \mathbb{S}^2 \tag{2.1}$$

are given by $\lambda_n = n(n+1)$.

Each eigenvalue λ_n has multiplicity $2n+1$ and a basis for its eigenspace is given by

$$\{P_n^0(z), P_n^m(z) \cos(m\theta), P_n^m(z) \sin(m\theta)\}_{m=1, \dots, n}. \tag{2.2}$$

(see (1.8) for the definition of $P_n^m(z)$).

Now let us recall the celebrated result of Crandall and Rabinowitz. It will be the main tool of the paper.

Theorem 2.2. ([3], Theorem 1.7)

Let X, Y be Hilbert spaces, $V \subset X$ a neighborhood of 0 and $F : (-2, 2) \times V \rightarrow Y$ a map with the following properties:

- $F(t, 0) = 0$ for any t .
- The partial derivatives $\partial_t F$, $\partial_x F$ and $\partial_{t,x}^2 F$ exist and are continuous.
- $\ker(\partial_x F(0, 0)) = \text{span}(w_0)$ and $Z := \text{R}(\partial_x F(0, 0))^\perp$ have dimension 1
- $\partial_{t,x}^2 F(0, 0)w_0 \notin \text{R}(\partial_x F(0, 0))$.

Then, there exists $\varepsilon_0 > 0$, a neighborhood $U \subset (-2, 2) \times X$ of 0 and continuously differentiable maps $\eta : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}$ and $z : (-\varepsilon_0, \varepsilon_0) \rightarrow Z$ such that

$$\begin{cases} \eta(0) = 0 \\ z(0) = 0 \\ F^{-1}(0) \cap U \setminus ((-2, 2) \times \{0\}) = \{(\eta(\varepsilon), \varepsilon w_0 + \varepsilon z(\varepsilon)); \varepsilon \in (-\varepsilon_0, \varepsilon_0)\} \end{cases}$$

Now let us introduce our functional to apply Theorem (2.2). Let $-\Delta_{\mathbb{S}^2}$ the Laplace-Beltrami operator on \mathbb{S}^2 and $\mathcal{T} : (-2, 2) \times W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2)$ be defined by

$$\mathcal{T} : (\mu, \phi_1, \phi_2) \rightarrow \begin{pmatrix} \Delta_{\mathbb{S}^2} \phi_1 + 2 \left(e^{\frac{\phi_1 + \phi_2}{2}} + e^{\frac{\phi_1 - \phi_2}{2}} - 2 \right) \\ \Delta_{\mathbb{S}^2} \phi_2 + 2 \frac{2 - \mu}{2 + \mu} \left(e^{\frac{\phi_1 + \phi_2}{2}} - e^{\frac{\phi_1 - \phi_2}{2}} \right) \end{pmatrix}. \quad (2.3)$$

It is not difficult to see that \mathcal{T} is C^2 . Next proposition shows that zeros of \mathcal{T} provide solutions to (1.1).

Proposition 2.3.

Let $\Pi : \mathbb{S}^2 \setminus \{0, 0, -1\} \rightarrow \mathbb{R}^2$ be

$$\Pi : \left(\sqrt{1 - z^2} \cos \theta, \sqrt{1 - z^2} \sin \theta, z \right) \rightarrow \left(\sqrt{8 \frac{1 - z}{1 + z}} \cos \theta, \sqrt{8 \frac{1 - z}{1 + z}} \sin \theta \right). \quad (2.4)$$

a stereographic projection. Then, for any (μ, ϕ_1, ϕ_2) solving $\mathcal{T}(\mu, \phi_1, \phi_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the triplet

$$(\mu, u_1, u_2) = \left(\mu, \frac{\phi_1 \circ \Pi^{-1} + \phi_2 \circ \Pi^{-1}}{2} + U_\mu, \frac{\phi_1 \circ \Pi^{-1} - \phi_2 \circ \Pi^{-1}}{2} + U_\mu \right) \quad (2.5)$$

solves (1.1).

Proof.

If (μ, ϕ_1, ϕ_2) is a zero of \mathcal{T} , then by definition it solves (1.3).

Let us now consider $(\Phi_1, \Phi_2) = (\phi_1 \circ \Pi^{-1}, \phi_2 \circ \Pi^{-1})$. Π is a stereographic projection obtained projecting from the south pole $(0, 0, -1)$ to the plane $\{z = \sqrt{8} - 1\}$, therefore it is conformal; its inverse is given by

$$\Pi^{-1} : (x_1, x_2) \rightarrow \left(\frac{2\sqrt{8}x_1}{8 + |x|^2}, \frac{2\sqrt{8}x_2}{8 + |x|^2}, \frac{8 - |x|^2}{8 + |x|^2} \right),$$

hence the conformal factor is $\rho(x) = \frac{32}{(8 + |x|^2)^2} = \frac{2 + \mu}{2} e^{U_\mu(x)}$. Therefore, Φ_1 satisfies in \mathbb{R}^2

$$\begin{aligned} -\Delta \Phi_1 &= \rho \left((-\Delta_{\mathbb{S}^2} \phi_1) \circ \Pi^{-1} \right) \\ &= \frac{2 + \mu}{2} e^{U_\mu} 2 \left(e^{\frac{\phi_1 \circ \Pi^{-1} + \phi_2 \circ \Pi^{-1}}{2}} + e^{\frac{\phi_1 \circ \Pi^{-1} - \phi_2 \circ \Pi^{-1}}{2}} - 2 \right) \\ &= (2 + \mu) \left(e^{\frac{\Phi_1 + \Phi_2}{2} + U_\mu} + e^{\frac{\Phi_1 - \Phi_2}{2} + U_\mu} - 2e^{U_\mu} \right) \end{aligned} \quad (2.6)$$

and similarly Φ_2 satisfies in \mathbb{R}^2

$$-\Delta \Phi_2 = (2 - \mu) \left(e^{\frac{\Phi_1 + \Phi_2}{2} + U_\mu} - e^{\frac{\Phi_1 - \Phi_2}{2} + U_\mu} \right). \quad (2.7)$$

Now, take $(u_1, u_2) := \left(\frac{\Phi_1 + \Phi_2}{2} + U_\mu, \frac{\Phi_1 - \Phi_2}{2} + U_\mu \right)$. By (2.6) and (2.7), u_1 solves

$$-\Delta u_1 = 2e^{u_1} + \mu e^{u_2},$$

and we similarly get $-\Delta u_2 = 2e^{u_2} + \mu e^{u_1}$.

Finally, one can see that the integrability condition in (1.1) is satisfied by re-writing the integral on the sphere via the map Π^{-1} :

$$\int_{\mathbb{R}^2} (e^{u_1} + e^{u_2}) = \int_{\mathbb{R}^2} e^{U_\mu} \left(e^{\frac{\Phi_1 + \Phi_2}{2}} + e^{\frac{\Phi_1 - \Phi_2}{2}} \right) = \frac{2}{2 + \mu} \int_{\mathbb{S}^2} \left(e^{\frac{\phi_1 + \phi_2}{2}} + e^{\frac{\phi_1 - \phi_2}{2}} \right) \leq C e^{\frac{\|\phi_1\|_{L^\infty(\mathbb{S}^2)} + \|\phi_2\|_{L^\infty(\mathbb{S}^2)}}{2}} < +\infty.$$

□

3 The linearized operator

In this section we study the derivative of the operator \mathcal{T} . Next lemma characterizes its kernel.

Lemma 3.1.

The linearized operator of \mathcal{T} in $(\mu, 0, 0)$ is given by

$$\partial_\phi \mathcal{T}(\mu, 0, 0) : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} \Delta_{\mathbb{S}^2} w_1 + 2w_1 \\ \Delta_{\mathbb{S}^2} w_2 + 2\frac{2-\mu}{2+\mu} w_2 \end{pmatrix}. \quad (3.1)$$

Let μ_n be as defined in (1.5).

If $\mu \neq \mu_n$ for any $n \in \mathbb{N}$, then its kernel is 3-dimensional and a basis is given by

$$\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \sin \theta \\ 0 \end{pmatrix} \right\}.$$

If $\mu = \mu_n$ for some $n \in \mathbb{N}$, then its kernel is $2n + 4$ -dimensional and a basis is given by

$$\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \sin \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^0(z) \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \sin(m\theta) \end{pmatrix} \right\}_{m=1, \dots, n}$$

where P_n^m are the associated Legendre polynomials as defined in (1.8).

Proof.

It is immediate to see, by the definition (2.3) of \mathcal{T} , that its derivative in $\phi_1 = \phi_2 = 0$ is given by (3.1).

So the kernel of $\partial_\phi \mathcal{T}(\mu, 0, 0)$ is given by the solutions of

$$\begin{cases} -\Delta_{\mathbb{S}^2} w_1 = 2w_1 & \text{in } \mathbb{S}^2 \\ -\Delta_{\mathbb{S}^2} w_2 = 2\frac{2-\mu}{2+\mu} w_2 & \text{in } \mathbb{S}^2 \end{cases},$$

namely a system of two decoupled scalar linear problem, each of which can be solved using Lemma 2.1.

The first equation corresponds to (2.1) with $\lambda = \lambda_1 = 2$ and it has a three-dimensional kernel given by

$$w_1(\theta, z) = \text{span} \left\{ z, \sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta \right\}.$$

The second equation has non-trivial solutions if and only if $2\frac{2-\mu}{2+\mu} = \lambda_n = n(n+1)$ for some n , namely $\mu = \mu_n$, in which case their expression is given by (2.2) and the dimension of the kernel is $2n + 1$.

By combining the solutions of the two equations we get a 3-dimensional kernel if $\mu \neq \mu_n$ and a $2n + 4$ -dimensional kernel if $\mu = \mu_n$ for some n and the proof is concluded. \square

Next lemma will be crucial to verify the transversality condition in Theorem 2.2.

Lemma 3.2.

Assume $(w_1, w_2) \in \ker \partial_\phi \mathcal{T}(\mu_n, 0, 0)$ and $\partial_{\mu, \phi}^2 \mathcal{T}(\mu_n, 0, 0)(w_1, w_2) \in \mathbb{R}(\partial_\phi \mathcal{T}(\mu_n, 0, 0))$.

Then,

$$\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \sin \theta \\ 0 \end{pmatrix} \right\} \quad (3.2)$$

Proof.

A straightforward computation shows that

$$\partial_{\mu, \phi}^2 \mathcal{T}(\mu_n, 0, 0) : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 8 \\ -\frac{8}{(2+\mu_n)^2} w_2 \end{pmatrix}. \quad (3.3)$$

By the very well-known properties of $-\Delta_{\mathbb{S}^2}$ we have that

$$\mathbb{R}(\partial_\phi \mathcal{T}(\mu_n, 0, 0)) := \left\{ (\psi_1, \psi_2) \in L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2) : \int_{\mathbb{S}^2} (\psi_1 w_1 + \psi_2 w_2) = 0, \forall (w_1, w_2) \in \ker(\partial_\phi \mathcal{T}(\mu_n, 0, 0)) \right\}.$$

If we now take $(w_1, w_2) \in \ker(\partial_\phi \mathcal{T}(\mu_n, 0, 0))$ and we verify the orthogonality condition on $\partial_{\mu, \phi}^2 \mathcal{T}(\mu_n, 0, 0) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, from (3.3) we have $(\psi_1, \psi_2) = \left(0, -\frac{8}{(2 + \mu_n)^2} w_2 \right)$. Hence we get

$$\int_{\mathbb{S}^2} (\psi_1 w_1 + \psi_2 w_2) = -\frac{8}{(2 + \mu_n)^2} \int_{\mathbb{S}^2} (w_2)^2 = 0.$$

Such a condition can only be satisfied if $w_2 = 0$. This leads to consider $\partial_{\mu, \phi}^2 \mathcal{T}(\mu_n, 0, 0) \begin{pmatrix} w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which of course belong to the range of $\partial_\phi \mathcal{T}(\mu_n, 0, 0)$. Since $\begin{pmatrix} w_1 \\ 0 \end{pmatrix} \in \ker \partial_\phi \mathcal{T}(\mu_n, 0, 0)$, thanks to the characterization of the kernel given by Lemma 3.1 we get the claim. \square

4 The role of symmetries

As pointed out in the previous section the kernel of $\partial_\phi \mathcal{T}(\mu_n, 0, 0)$ is never one dimensional. So, in order to apply Theorem 2.2 we need to select a subspace of $W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2)$ where this property holds. It will consist of functions verifying some symmetry and antisymmetry assumptions. So let us introduce the following isometries,

$$\sigma : z \rightarrow -z \qquad \rho_m : \theta \rightarrow \theta + \frac{\pi}{m} \qquad \tau_m : \theta \rightarrow -\theta + \frac{\pi}{m}$$

where $m \geq 1$ is an integer and the following spaces,

$$\mathcal{X}_{n,m} := \{(\phi_1, \phi_2) \in W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2) : \begin{array}{l} \phi_1 \circ \sigma = \phi_1, \quad \phi_1 \circ \rho_m = \phi_1, \quad \phi_1 \circ \tau_m = \phi_1, \\ \phi_2 \circ \sigma = (-1)^{n+m} \phi_2, \quad \phi_2 \circ \rho_m = -\phi_2, \quad \phi_2 \circ \tau_m = -\phi_2 \end{array} \} \quad (4.1)$$

$$\mathcal{Y}_{n,m} := \{(\psi_1, \psi_2) \in L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2) : \begin{array}{l} \psi_1 \circ \sigma = \psi_1, \quad \psi_1 \circ \rho_m = \psi_1, \quad \psi_1 \circ \tau_m = \psi_1, \\ \psi_2 \circ \sigma = (-1)^{n+m} \psi_2, \quad \psi_2 \circ \rho_m = -\psi_2, \quad \psi_2 \circ \tau_m = -\psi_2 \end{array} \} \quad (4.2)$$

Note that in general $\theta + \frac{\pi}{m} \notin [0, 2\pi]$. If this occurs we mean that in (4.1) $\phi_1 \left(\theta + \frac{\pi}{m}, z \right) = \phi_1 \left(\theta + \frac{\pi}{m} - 2\pi, z \right)$ and so the definition is well posed. Same remark applies to τ_m .

Remark 4.1.

It may be interesting to consider the effect produced by the symmetries σ, ρ_m, τ_m on the plane \mathbb{R}^2 via the stereographic projection Π defined in (2.4).

By parameterizing \mathbb{R}^2 using standard polar coordinates (r, θ) we see that the angular variable θ on \mathbb{R}^2 corresponds precisely to the variable θ on \mathbb{S}^2 (hence we allow ourself a slight abuse of notation). Therefore composing Π and ρ_m will still give a rotation whereas composing Π and τ_m will still give an angular reflection:

$$\tilde{\rho}_m := \Pi \circ \rho_m \circ \Pi^{-1} : \theta \rightarrow \theta + \frac{\pi}{m} \qquad \tilde{\tau}_m := \Pi \circ \tau_m \circ \Pi^{-1} : \theta \rightarrow -\theta + \frac{\pi}{m}.$$

On the other hand, $\tilde{\sigma} := \Pi \circ \sigma \circ \Pi^{-1}$ is no longer an isometry on the plane but rather a radial inversion with respect to the sphere of radius $\sqrt{8}$:

$$\tilde{\sigma} : r \rightarrow \frac{8}{r}.$$

If we now consider how the space $\mathcal{X}_{n,m}$ is transformed by such symmetries, namely which symmetries will inherit the solutions on the plane from the corresponding solutions on the sphere, then we get that $(\Phi_1, \Phi_2) := (\phi_1 \circ \Pi^{-1}, \phi_2 \circ \Pi^{-1})$ satisfies

$$\begin{array}{lll} \Phi_1 \left(\frac{8}{r} e^{i\theta} \right) = \Phi_1 (r e^{i\theta}) & \Phi_1 \left(r e^{i(-\theta + \frac{\pi}{m})} \right) = \Phi_1 (r e^{i\theta}) & \Phi_1 \left(r e^{i(\theta + \frac{\pi}{m})} \right) = \Phi_1 (r e^{i\theta}) \\ \Phi_2 \left(\frac{8}{r} e^{i\theta} \right) = (-1)^{n+m} \Phi_2 (r e^{i\theta}) & \Phi_2 \left(r e^{i(-\theta + \frac{\pi}{m})} \right) = -\Phi_2 (r e^{i\theta}) & \Phi_2 \left(r e^{i(\theta + \frac{\pi}{m})} \right) = -\Phi_2 (r e^{i\theta}). \end{array}$$

Next lemma is one of the crucial steps that enable us to apply Theorem 2.2.

Lemma 4.2.

The restriction $\mathcal{T}_{n,m} := \mathcal{T}|_{(-2,2) \times \mathcal{X}_{n,m}}$ maps its domain into $\mathcal{Y}_{n,m}$.

Moreover, if $m > \frac{n}{3}$, the kernel of $\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0)$ is 1-dimensional and it is generated by

$$\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}.$$

Proof. The Laplace-Beltrami operator on \mathbb{S}^2 in the coordinates (θ, z) is given by

$$\Delta_{\mathbb{S}^2} \phi_1(\theta, z) = (1 - z^2) \frac{\partial^2 \phi_1}{\partial z^2} - 2z \frac{\partial \phi_1}{\partial z} + \frac{1}{1 - z^2} \frac{\partial^2 \phi_1}{\partial \theta^2} \quad (4.3)$$

By the previous formula one can easily verify that $\mathcal{T} = \begin{pmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{pmatrix}$ is invariant under the reflection σ , i.e. $\mathcal{T}(\mu, \phi_1 \circ \sigma, \phi_2 \circ \sigma) = \begin{pmatrix} (\mathcal{T}_1(\mu, \phi_1, \phi_2)) \circ \sigma \\ (\mathcal{T}_2(\mu, \phi_1, \phi_2)) \circ \sigma \end{pmatrix}$ and $\mathcal{T}(\mu, \phi_1 \circ \sigma, -\phi_2 \circ \sigma) = \begin{pmatrix} (\mathcal{T}_1(\mu, \phi_1, \phi_2)) \circ \sigma \\ -(\mathcal{T}_2(\mu, \phi_1, \phi_2)) \circ \sigma \end{pmatrix}$. This proves that $\mathcal{T}(\mu, \cdot)$ maps the space

$$\mathcal{X}_\pm := \{(\phi_1, \phi_2) \in W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2) : \phi_1 \circ \sigma = \phi_1, \phi_2 \circ \sigma = \pm \phi_2\}$$

into

$$\mathcal{Y}_\pm := \{(\psi_1, \psi_2) \in L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2) : \psi_1 \circ \sigma = \psi_1, \psi_2 \circ \sigma = \pm \psi_2\}$$

The same argument also works for ρ_m and τ_m as well. This proves that $\mathcal{T}(\mu, \cdot)$ maps $\mathcal{X}_{n,m}$ into $\mathcal{Y}_{n,m}$.

We are left to show that $\ker(\partial_\phi \mathcal{T}(\mu_n, 0)) \cap \mathcal{X}_{n,m} = \text{span} \left\{ \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\}$.

By the property that $P_n^m(-z) = (-1)^{n+m} P_n^m(z)$ we get that the former element belongs to $\mathcal{X}_{n,m}$. Hence the claim follows if we rule out all the other generators of the kernel.

Step 1: if $\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} z \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \cos \theta \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{1-z^2} \sin \theta \\ 0 \end{pmatrix} \right\}$ then $\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} \notin \mathcal{X}_{n,m}$.

First of all, there are no constant multiples of $\begin{pmatrix} z \\ 0 \end{pmatrix}$, since the invariance with respect to σ would imply

$$Cz = C\sigma(z) = -Cz,$$

hence $C = 0$.

Now, assume that $\mathcal{X}^{m,n}$ contains an element of the kind $\begin{pmatrix} w_1 \\ 0 \end{pmatrix}$, with $w_1(\theta, z) = \sqrt{1-z^2}(A \cos \theta + B \sin \theta)$ for some real A, B . Then, we would have $w_1 = w_1 \circ \rho_m$ and the latter can be expressed by

$$w_1(\rho_m(\theta, z)) = \left(\cos\left(\frac{\pi}{m}\right) A + \sin\left(\frac{\pi}{m}\right) B \right) \sqrt{1-z^2} \cos \theta + \left(-\sin\left(\frac{\pi}{m}\right) A + \cos\left(\frac{\pi}{m}\right) B \right) \sqrt{1-z^2} \sin \theta.$$

Therefore, the pair (A, B) must solve the linear system

$$\begin{cases} A = \cos\left(\frac{\pi}{m}\right) A + \sin\left(\frac{\pi}{m}\right) B \\ B = -\sin\left(\frac{\pi}{m}\right) A + \cos\left(\frac{\pi}{m}\right) B \end{cases} \quad (4.4)$$

which has no non-trivial solutions for any $m \geq 1$.

Step 2: if $\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} 0 \\ P_n^0(z) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^l(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^l(z) \sin(m\theta) \end{pmatrix} \right\}_{l=1, \dots, n}$ then $\begin{pmatrix} w_1(\theta, z) \\ w_2(\theta, z) \end{pmatrix} = \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}$.

First we exclude functions of the kind $\begin{pmatrix} 0 \\ w_2 \end{pmatrix}$ with $w_2(\theta, z) = P_n^l(z)(A \cos(l\theta) + B \sin(l\theta))$ for $l = 1, \dots, n, l \neq m$. Since we must have $w_2 = -w_2 \circ \rho_m$ it holds

$$w_2(\rho_m(\theta, z)) = P_n^l(z) \left(\left(\cos\left(\frac{l}{m}\pi\right) A + \sin\left(\frac{l}{m}\pi\right) B \right) \cos(l\theta) + \left(-\sin\left(\frac{l}{m}\pi\right) A + \cos\left(\frac{l}{m}\pi\right) B \right) \sin(l\theta) \right).$$

Therefore, A and B now solve

$$\begin{cases} A = -\cos\left(\frac{l}{m}\pi\right) A - \sin\left(\frac{l}{m}\pi\right) B \\ B = \sin\left(\frac{l}{m}\pi\right) A - \cos\left(\frac{l}{m}\pi\right) B \end{cases},$$

and we have nontrivial solutions if and if only $\frac{l}{m}$ is an odd integer. Since $l \neq m$ and $\frac{l}{m} \leq \frac{n}{m} < 3$ we deduce that $A = B = 0$.

This argument also works, setting $l = B = 0$, to rule out $\begin{pmatrix} 0 \\ P_n^0(z) \end{pmatrix}$.

Finally, we are left with $w_2(x) = P_n^m(z)(A \cos(m\theta) + B \sin(m\theta))$. Here using τ_m in place of ρ_m we have

$$P_n^m(z)(A \cos(m\theta) + B \sin(m\theta)) = w_2(x) = -w_2(\tau_m(x)) = P_n^m(z)(A \cos(m\theta) - B \sin(m\theta)),$$

which implies $B = 0$. So the claim follows. \square

Remark 4.3.

For any $\varphi \in \mathbb{S}^1$ one can construct similar subspaces $\mathcal{X}_{n,m,\varphi} \subset W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2)$ and $\mathcal{Y}_{n,m,\varphi} \subset L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2)$ such that $\mathcal{T}(\mathcal{X}_{n,m,\varphi}) \subset \mathcal{Y}_{n,m,\varphi}$ and the kernel of its derivative in $(\mu_n, 0)$ is generated by $\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta + \varphi) \end{pmatrix}$. This is consistent with the invariance with respect to rotation of the operator \mathcal{T} .

To prove this fact, just argue as in Lemma 4.2 replacing the reflection τ_m with $\tau_{m,\varphi} : \theta \rightarrow -\theta + \frac{\pi - 2\varphi}{m}$.

In particular, for $\varphi = -\frac{\pi}{2}$, we get $P_n^m(z) \sin(m\theta)$. Notice that, in general,

$$P_n^m(z) \cos(m\theta + \varphi) = \cos(\varphi) P_n^m(z) \cos(m\theta) - \sin(\varphi) P_n^m(z) \sin(m\theta)$$

is a combination of the two generators of the kernel.

Of course, since system (1.1) is rotationally invariant, in general this argument does not give rise to new solutions different from the previous ones.

More generally, we may consider a closed group $G \leq O(3)$, a group omomorphism $\varsigma : G \rightarrow \{-1, 1\}$ and subspaces

$$\begin{aligned} \mathcal{X}_G &:= \{(\phi_1, \phi_2) \in W^{2,2}(\mathbb{S}^2) \times W^{2,2}(\mathbb{S}^2); \phi_1 \circ g = \phi_1, \phi_2 \circ g = \varsigma(g)\phi_2, \forall g \in G\}; \\ \mathcal{Y}_G &:= \{(\psi_1, \psi_2) \in L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2); \psi_1 \circ g = \psi_1, \psi_2 \circ g = \varsigma(g)\psi_2, \forall g \in G\}. \end{aligned}$$

By the symmetry properties of \mathcal{T} we have $\mathcal{T}(\mathcal{X}_G) \subset \mathcal{Y}_G$. Therefore, if we can find G, ς such that $\ker(\partial_\phi \mathcal{T}(\mu_n, 0)) \cap \mathcal{X}_G$ is 1-dimensional and satisfies the transversality condition, then the same argument can be used to find solutions in \mathcal{X}_G . The spaces $\mathcal{X}_{n,m,\varphi}, \mathcal{Y}_{n,m,\varphi}$ are obtained taking

$$G = \left\langle \sigma, \rho_{\frac{\pi}{m}}, \tau_{\frac{\pi-2\varphi}{m}} \right\rangle \quad \varsigma(\sigma) = (-1)^{n,m}, \varsigma(\rho_{\frac{\pi}{m}}) = \varsigma\left(\tau_{\frac{\pi-2\varphi}{m}}\right) = -1.$$

By choosing instead $G = O(2)$, namely the group of isometries fixing the z axes, and $\varsigma \equiv 1$ we recover the space of functions depending only on the z variable. Such functions correspond to radially symmetric functions on the plane, therefore in this case we recover the radial branch found in [4].

Lemma 4.4.

The range of the operator $\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0)$ has co-dimension 1 and it is given by

$$\mathcal{R}(\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0)) = \left\{ (\psi_1, \psi_2) \in L^2(\mathbb{S}^2) \times L^2(\mathbb{S}^2) : \int_{\mathbb{S}^2} \psi_2(\theta, z) P_n^m(z) \cos(m\theta) d\theta dz = 0 \right\}$$

Proof.

By the definition of \mathcal{T} and the well-known spectral properties of $-\Delta_{\mathbb{S}^2}$, the range of $\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0)$ coincides with the orthogonal of its kernel.

In Lemma 4.2 we showed that the kernel is 1-dimensional and spanned by $\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}$. From this one can easily derive the characterization of the range and this proves the claim. \square

5 Proof of the main theorem

Proof of Theorem 1.1.

We apply Theorem 2.2 to the operator $\mathcal{T}_{n,m} : (-2, 2) \times \mathcal{X}_{n,m} \rightarrow \mathcal{Y}_{n,m}$. We have to verify that all the hypotheses are satisfied.

It is immediate to see that $\mathcal{T}_{n,m}(\mu, 0, 0) = (0, 0)$ for all μ and that the first and second derivatives exist and are continuous. Lemma 4.2 ensures that $\ker(\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0))$ is 1-dimensional and, by Lemma 4.4, the range has also co-dimension 1. Finally, the transversality condition is ensured by Lemma 3.2.

Therefore, from Theorem 2.2 there exists a branch of non-trivial solutions $(\mu_n(\varepsilon), \phi_{1,\varepsilon}, \phi_{2,\varepsilon})$.

Since the bifurcation evolves in the direction $\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}$, the functions $\phi_{i,\varepsilon}$ will satisfy

$$\begin{pmatrix} \phi_{1,\varepsilon}(\theta, z) \\ \phi_{2,\varepsilon}(\theta, z) \end{pmatrix} = 2\varepsilon \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} + 2\varepsilon \begin{pmatrix} z_{1,\varepsilon}(\theta, z) \\ z_{2,\varepsilon}(\theta, z) \end{pmatrix}$$

with $z_{i,\varepsilon}$ satisfying $z_{i,0} = 0$. Moreover, since $(\phi_{1,\varepsilon}, \phi_{2,\varepsilon}) \in \mathcal{X}_{n,m}$, then also $(z_{1,\varepsilon}, z_{2,\varepsilon})$ belongs to $\mathcal{X}_{n,m}$ so it satisfies the symmetry properties:

$$\begin{aligned} z_{1,\varepsilon}(\theta, -z) &= z_{1,\varepsilon}(\theta, z) & z_{1,\varepsilon}\left(-\theta + \frac{\pi}{m}, z\right) &= z_{1,\varepsilon}(\theta, z) & z_{1,\varepsilon}\left(\theta + \frac{\pi}{m}, z\right) &= z_{1,\varepsilon}(\theta, z) \\ z_{2,\varepsilon}(\theta, -z) &= (-1)^{n+m} z_{2,\varepsilon}(\theta, z) & z_{2,\varepsilon}\left(-\theta + \frac{\pi}{m}, z\right) &= -z_{2,\varepsilon}(\theta, z) & z_{2,\varepsilon}\left(\theta + \frac{\pi}{m}, z\right) &= -z_{2,\varepsilon}(\theta, z) \end{aligned}$$

By Proposition 2.3 we have that the functions $(u_{1,\varepsilon}, u_{2,\varepsilon})$ defined as in (2.5) solve (1.1) and they satisfy all the requested properties. In particular, the symmetries (1.10) satisfied by $(Z_{1,\varepsilon}, Z_{2,\varepsilon}) = (z_{1,\varepsilon} \circ \Pi^{-1}, z_{2,\varepsilon} \circ \Pi^{-1})$ follow from Remark 4.1.

Moreover since $z_{1,\varepsilon}, z_{2,\varepsilon} \in W^{2,2}(\mathbb{S}^2)$ then they are bounded in L^∞ and the same happens for $(Z_{1,\varepsilon}, Z_{2,\varepsilon})$. Hence we get that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ verify

$$\begin{cases} u_{1,\varepsilon}(x) = U_{\mu_n(\varepsilon)}(x) + \varepsilon P_n^m \left(\frac{8 - |x|^2}{8 + |x|^2} \right) \cos(m\theta) + \varepsilon (Z_{1,\varepsilon}(x) + Z_{2,\varepsilon}(x)) \\ u_{2,\varepsilon}(x) = U_{\mu_n(\varepsilon)}(x) - \varepsilon P_n^m \left(\frac{8 - |x|^2}{8 + |x|^2} \right) \cos(m\theta) + \varepsilon (Z_{1,\varepsilon}(x) - Z_{2,\varepsilon}(x)), \end{cases}$$

In order to get (1.9) observe that using the Taylor expansion we get

$$U_{\mu_n(\varepsilon)}(x) = U_{\mu_n}(x) + \log \frac{1}{2 + \mu_n(\varepsilon)} - \log \frac{1}{2 + \mu_n} = U_{\mu_n}(x) - \frac{\mu_n'(0)}{2 + \mu_n} \varepsilon + C(\varepsilon). \quad (5.1)$$

where $C(\varepsilon)$ is a term satisfying $C(\varepsilon) = O(\varepsilon^2)$. In Lemma A.2 in Appendix we prove that $\mu_n'(0) = 0$ and so the claim follows replacing $Z_{1,\varepsilon}$ with $Z_{1,\varepsilon} + \frac{C(\varepsilon)}{\varepsilon}$.

Crandall-Rabinowitz Theorem provides the existence of *local* branches. In order to prove that the bifurcation is global we use a degree argument (see [6]). Since we need to define an operator from a Banach space into itself let us consider $\mathcal{L} : L^\infty(\mathbb{S}^2) \rightarrow L^\infty(\mathbb{S}^2)$ as

$$\mathcal{L} : (\mu, \phi_1, \phi_2) = \begin{pmatrix} \phi_1 - (\Delta_{\mathbb{S}^2} + I)^{-1} \left[2 \left(e^{\frac{\phi_1 + \phi_2}{2}} + e^{\frac{\phi_1 - \phi_2}{2}} - 2 \right) + \phi_1 \right] \\ \phi_2 - (\Delta_{\mathbb{S}^2} + I)^{-1} \left[2 \frac{2 - \mu}{2 + \mu} \left(e^{\frac{\phi_1 + \phi_2}{2}} - e^{\frac{\phi_1 - \phi_2}{2}} \right) + \phi_2 \right] \end{pmatrix}. \quad (5.2)$$

It is immediate to check that Lemma 4.2 holds for the operator \mathcal{L} in the space $\mathcal{X}_{n,m}$. This implies that as the parameter μ crosses μ_n the Morse index of the solution increases by *one*. Hence there is a change in the Leray-Schauder degree $\deg(\mathcal{L}, B_\delta, 0)$ and then classical results in bifurcation theory ensure the existence of a global continuum of solutions to (1.3) that satisfies the Rabinowitz alternative.

Now let us prove (1.11). Integrating (1.3) we get that

$$\int_{\mathbb{S}^2} e^{\frac{\phi_{1,\varepsilon} + \phi_{2,\varepsilon}}{2}} = \int_{\mathbb{S}^2} e^{\frac{\phi_{1,\varepsilon} - \phi_{2,\varepsilon}}{2}} = 4\pi.$$

Then we get

$$\int_{\mathbb{R}^2} e^{u_{1,\varepsilon}} = \int_{\mathbb{R}^2} e^{U_{\mu_n(\varepsilon)}} e^{\frac{\phi_{1,\varepsilon} + \phi_{2,\varepsilon}}{2}} = \frac{2}{2 + \mu_n(\varepsilon)} \int_{\mathbb{S}^2} e^{\frac{\phi_{1,\varepsilon} + \phi_{2,\varepsilon}}{2}} = \frac{8\pi}{2 + \mu_n(\varepsilon)}$$

and the same holds for $\int_{\mathbb{R}^2} e^{u_{2,\varepsilon}}$. This shows (1.11). \square

Proof of Corollary 1.4.

By Theorem 1.1 we get the existence of $n - \lfloor \frac{n}{3} \rfloor$ nonradial nonequivalent solutions and in [4] we got the existence of a radial solution bifurcating by $(\mu_n, U_{\mu_n}, U_{\mu_n})$. So the claim follows. \square

Remark 5.1. *Let us consider the case $n = 2$, namely the very well-known $SU(3)$ Toda system.*

Here, the assumption $m > \frac{n}{3}$ is redundant and Theorem 1.1 can be applied for $m = 1, 2$ to get two branches of non-trivial solutions.

Through Remark 4.3 we get two more independent branches of non-radial solutions; moreover, in [4], Theorem 1.2 the authors find a non-trivial branch of radial solutions.

Therefore, we get five different non-trivial branches of solution. Putting them together with the 3-parameters family of "trivial" solutions $\{U_{\mu_2, \delta, y}\}_{\delta \in \mathbb{R}, y \in \mathbb{R}^2}$ we recover the 8-parameters family of explicit solutions found in [5, 7].

We end this section giving the proof of Corollary 1.5.

Proof of Corollary 1.5.

We consider the operator $\mathcal{L} : L^\infty(\mathbb{S}^2) \rightarrow L^\infty(\mathbb{S}^2)$ as defined in (5.2) restricted to space of functions which depend only on the variable z . Using (4.3) a zero of \mathcal{L} then solves

$$\begin{cases} -(1 - z^2)\partial_{zz}\phi_1(z) + 2z\partial_z\phi_1 = 2\left(e^{\frac{\phi_1 + \phi_2}{2}} + e^{\frac{\phi_1 - \phi_2}{2}} - 2\right) \\ -(1 - z^2)\partial_{zz}\phi_2(z) + 2z\partial_z\phi_2 = 2\frac{2 - \mu}{2 + \mu}\left(e^{\frac{\phi_1 + \phi_2}{2}} - e^{\frac{\phi_1 - \phi_2}{2}}\right), \end{cases}$$

This is a Sturm Liouville problem, and this implies that the solutions can only have simple zeroes.

Near the bifurcation point μ_n we know, from the local bifurcation result, that

$$\phi_{2,\varepsilon}(z) = 2\varepsilon P_n^0(z) + 2\varepsilon z_{2,\varepsilon}(z)$$

and it is not hard to prove that $\phi_{2,\varepsilon}(z)$ has the same number of zeroes of $P_n(z)$, since $P_n(z)$ has only simple zeroes. Then, by standard Sturm Liouville theory, the number of zeroes of $\phi_{2,\varepsilon}(z)$ cannot change along the branch. Since $P_n(z)$ has n zeroes in $(-1, 1)$ this implies that the radial branch bifurcating from μ_n cannot join the radial branch bifurcating from μ_l for $n \neq l$ and so they do not intersect. \square

A Appendix. The derivatives of the parameter $\mu_n(\varepsilon)$

In this appendix we study the behavior of the parameter $\mu_n(\varepsilon)$ in a neighborhood of μ_n . Let us start recalling a known result in bifurcation theory.

Theorem A.1. [6], Chapter I.6

Assume the hypotheses of Theorem 2.2 are satisfied and, moreover, $F \in C^2((-2, 2) \times X, Y)$.

Let $y_0 \in Y$ be any generator of $\mathbb{R}(\partial_x F(0, 0))^\perp$. Then, the derivative $\mu'(0)$ of $\mu(\varepsilon)$ in $\varepsilon = 0$ is given by

$$\mu'(0) = -\frac{1}{2} \frac{\langle \partial_{x,x}^2 F(0, 0)[w_0, w_0], y_0 \rangle}{\|w_0\| \langle \partial_{t,x}^2 F(0, 0)w_0, y_0 \rangle}. \quad (\text{A.1})$$

Assume, in addition, that $\mu'(0) = 0$ and F is of class C^3 . Define as $\mathbb{Q} : y \rightarrow \frac{\langle y, y_0 \rangle}{\|y_0\|^2} y_0$ the projection from Y to $\mathbb{R}(\partial_x F(0, 0))^\perp$ and write as $(\partial_x F(0, 0))^{-1} : \mathbb{R}(\partial_x F(0, 0)) \rightarrow \ker(\partial_x F(0, 0))^\perp$ the inverse of $\partial_x F(0, 0)$ restricted to the complementary of its kernel.

Then, $\mu''(0)$ is given by

$$\mu''(0) = -\frac{1}{3} \frac{\langle \partial_{x,x,x}^3 F(0, 0)[w_0, w_0, w_0] - 3\partial_{x,x}^2 F(0, 0) \left[w_0, (\partial_x F(0, 0))^{-1} (\mathbb{I} - \mathbb{Q}) \partial_{x,x}^2 F(0, 0)[w_0, w_0] \right], y_0 \rangle}{\|w_0\|^2 \langle \partial_{t,x}^2 F(0, 0)w_0, y_0 \rangle}. \quad (\text{A.2})$$

Lemma A.2.

For any $n, m \in \mathbb{N}$ with $1 \leq \frac{n}{m} < 3$, the parameter $\mu_n(\varepsilon)$ defined in Theorem 1.1 satisfies $\mu'_n(0) = 0$.

Proof.

We suffice to apply the formula (A.1) of Theorem 1.1.

We recall that, in our case, $w_0 = \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}$ and the second derivative of $\mathcal{T}_{n,m}$ in 0 is

$$\partial_{\phi,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0) : \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \rightarrow \left(\frac{v_1 w_1 + v_2 w_2}{2}, \frac{n(n+1)}{2} (v_1 w_2 + v_2 w_1) \right). \quad (\text{A.3})$$

Moreover, thanks to Lemma 4.4, the orthogonal of the range is generated by $y_0 = w_0$.

Therefore, (A.1) gives:

$$\begin{aligned} \mu'(0) &= -\frac{1}{2} \frac{\langle \partial_{x,x}^2 F(0, 0)[w_0, w_0], w_0 \rangle}{\|w_0\| \langle \partial_{t,x}^2 F(0, 0)w_0, w_0 \rangle} \\ &= -\frac{\left\langle \partial_{\phi,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0) \left[\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right], \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle}{2\|w_0\| \langle \partial_{\mu,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0)w_0, w_0 \rangle} \\ &= -\frac{\left\langle \begin{pmatrix} (P_n^m(z) \cos(m\theta))^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle}{2\|w_0\| \langle \partial_{\mu,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0)w_0, w_0 \rangle} \\ &= 0 \end{aligned}$$

□

The following functions play a similar role with respect to the Legendre polynomials (1.8):

$$\tilde{P}_n^m(z) = \left(\frac{1-z}{1+z} \right)^{\frac{m}{2}} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\binom{k+n}{k}}{\binom{k+m}{k}} \left(\frac{1-z}{2} \right)^k \quad (\text{A.4})$$

for $n, m \in \mathbb{N}$.

Lemma A.3.

For any $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{R}(\partial_\phi \mathcal{T}(\mu_n, 0, 0))^\perp$, the only solution $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \ker(\partial_\phi \mathcal{T}(\mu_n, 0, 0))^\perp$ of

$$\partial_\phi \mathcal{T}(\mu_n, 0, 0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \Delta_{\mathbb{S}^2} \phi_1 + 2\phi_1 \\ \Delta_{\mathbb{S}^2} \phi_2 + n(n+1)\phi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is given by

$$\phi_i(\theta, z) := \phi_{i,0}(z) + \sum_{l=1}^{+\infty} (\phi_{i,l}^1(z) \cos(l\theta) + \phi_{i,l}^2(z) \sin(l\theta)),$$

where

$$\begin{aligned} \phi_{1,0}(z) &= P_1(z) \left(C_{1,0} - \int_{-1}^z \frac{1}{(1-y^2)(P_1(y))^2} \int_y^1 P_1(x) \psi_{1,0}(x) dx dy \right) \\ \phi_{1,1}^j(z) &= P_1^j(z) \left(C_{1,1}^j - \int_0^z \frac{1}{(1-y^2)(P_1^j(y))^2} \int_y^1 P_1^j(x) \psi_{1,1}^j(x) dx dy \right) \\ (l \geq 2) \quad \phi_{1,l}^j(z) &= -\tilde{P}_1^l(z) \int_{-1}^z \frac{1}{(1-y^2)(\tilde{P}_1^l(y))^2} \int_y^1 \tilde{P}_1^l(x) \psi_{1,l}^j(x) dx dy \\ \phi_{2,0}(z) &= P_n(z) \left(C_{2,0} - \int_0^z \frac{1}{(1-y^2)(P_n(y))^2} \int_y^1 P_n(x) \psi_{n,0}(x) dx dy \right) \end{aligned} \tag{A.5}$$

$$\begin{aligned} (l \leq n) \quad \phi_{2,l}^j(z) &= P_n^l(z) \left(C_{2,l}^j - \int_{-1}^z \frac{1}{(1-y^2)(P_n^l(y))^2} \int_y^1 P_n^l(x) \psi_{2,l}^j(x) dx dy \right) \\ (l \geq n+1) \quad \phi_{2,l}^j(z) &= -\tilde{P}_n^l(z) \int_{-1}^z \frac{1}{(1-y^2)(\tilde{P}_n^l(y))^2} \int_y^1 \tilde{P}_n^l(x) \psi_{2,l}^j(x) dx dy \end{aligned} \tag{A.6}$$

with $C_{i,0}, C_{i,l}^j$ uniquely determined and $\psi_{i,0}, \psi_{i,l}^j$ defined by the Fourier decomposition in θ :

$$\psi_i(\theta, z) = \psi_{i,0}(z) + \sum_{l=1}^{+\infty} (\psi_{i,l}^1(z) \cos(l\theta) + \psi_{i,l}^2(z) \sin(l\theta))$$

Proof.

Since the linearized operator has two decoupled equations, we will suffice to solve $\Delta\phi + n(n+1)\phi = \psi$ for any ψ being orthogonal to the eigenfunctions of the homogeneous problem, which we recall are given by (2.2).

In the system of coordinates (θ, z) the latter problem has the form

$$(1-z^2) \partial_{zz} \phi(\theta, z) - 2z \partial_z \phi(\theta, z) + \frac{1}{1-z^2} \partial_{\theta\theta} \phi(\theta, z) + n(n+1)\phi(\theta, z) = \psi(\theta, z).$$

If we now decompose ϕ, ψ as

$$\begin{aligned} \phi(\theta, z) &= \phi_0(z) + \sum_{l=1}^{+\infty} (\phi_l^1(z) \cos(l\theta) + \phi_l^2(z) \sin(l\theta)) \\ \psi(\theta, z) &= \psi_0(z) + \sum_{l=1}^{+\infty} (\psi_l^1(z) \cos(l\theta) + \psi_l^2(z) \sin(l\theta)), \end{aligned}$$

then each ϕ_l^j solves the o.d.e.

$$(1-z^2) \phi_l^{j''}(z) - 2z \phi_l^{j'}(z) - \frac{l^2}{1-z^2} \phi_l^j(z) + n(n+1)\phi_l^j(z) = \psi_l^j(z).$$

If $l \leq n$ and $\psi_l^j(z) \equiv 0$, the only bounded solution of this equation, up to a multiplicative constant, is $P_n^l(z)$; on the other hand, for $l \geq n+1$, we just get a solution $\tilde{P}_n^l(z)$ defined by (A.4) which is bounded in $\mathbb{S}^2 \setminus \{0, 0, -1\}$.

Now, using the variation of the constants we get

$$\phi_l^j(z) = P_n^l(z) \int_a^z \frac{1}{(1-y^2)(P_n^l(y))^2} \int_b^y P_n^l(x) \psi_l^j(x) dx dy \quad (\text{A.7})$$

for $l \leq n$ and a similar expression, with \tilde{P}_n^l in place of P_n^l . Here a and b are to be determined in such a way that the function is globally defined on the whole sphere, namely up to $y = \pm 1$

It is not hard to see that to get boundedness in $z = \pm 1$ we must have $a = -1$ and $b = 1$.

Arguing similarly we get

$$C_0 := \frac{\int_{-1}^1 (P_n(z))^2 \int_{-1}^z \frac{1}{(1-y^2)(P_n(y))^2} \int_y^1 P_n(x) \psi_0(x) dx dy dz}{\int_{-1}^1 (P_n^l(z))^2 dz} \quad C_l^j := \frac{\int_{-1}^1 (P_n^l(z))^2 \int_0^z \frac{1}{(1-y^2)(P_n^l(y))^2} \int_y^1 P_n^l(x) \psi_l^j(x) dx dy dz}{\int_{-1}^1 (P_n^l(z))^2 dz}. \quad (\text{A.8})$$

By putting together the formulas (A.7) and (A.8) we conclude the proof. \square

Remark A.4.

We will actually need to invert not $\partial_\phi \mathcal{T}(\mu_n, 0, 0)$ but just $\partial_\phi \mathcal{T}_{n,m}(\mu_n, 0, 0)$, namely the restriction of $\partial_\phi \mathcal{T}(\mu_n, 0, 0)$ on the spaces $\mathcal{X}_{n,m}, \mathcal{Y}_{n,m}$ defined by (4.1), (4.2).

Choosing $(\psi_1, \psi_2) \in \mathcal{Y}_{n,m}$ some of the coefficients $\psi_{i,l}^j$ will vanish (hence also $\phi_{i,l}^j$ will). In particular, arguing as in the proof of Lemma 4.2 one can see that $\psi_{i,l}^2 = 0$ for any i, l , $\psi_{1,l}^1 = 0$ unless l is an even multiple of m and $\psi_{1,l}^2 = 0$ unless l is an odd multiple of m (and all the non-vanishing $\psi_{i,0}, \psi_{i,l}^j$ are even or odd in z).

Anyway, the argument to invert the linearized operator of $\mathcal{T}_{n,m}$ still works for the more general \mathcal{T} and for this reason we gave the proof for the more general case.

Remark A.5.

The formulas for $\phi_{i,0}, \phi_{i,l}^j$ in Lemma A.3 are not really well-defined for all z , because the denominators of the fractions may vanish. In fact, the integral (A.5) defining $\phi_{2,0}$ makes no sense for z greater than the first zero z_1 of $P_n(z)$ and similar issues occur in the other formulas.

To fix this, we have to replace the integral with some anti-derivatives of the same function and choosing properly the free constant.

Since P_n has only simple zeros $z_1 < \dots < z_n$, then the limit of (A.5) as z goes to z_1 makes sense. Then, we can define

$$\phi_{2,0} = P_n(z) \left(C_{2,0} - \int_{a_n}^z \frac{dy}{(1-y^2)(P_n(y))^2} \int_y^1 P_n(x) \psi_{2,0}(x) dx \right) \quad \text{if } z_1 < z < z_2$$

with $a_n \in (z_1, z_2)$. This definition is also well-posed as z goes to z_1 and, for a proper choice of a_n , the two definitions can be “glued” smoothly around $z = z_1$. We then repeat the procedure for each interval $(z_2, z_3), \dots, (z_n, 1)$ and we find a smooth solution for all z .

The same argument also fixes other formulas introduced in Lemma A.3, since the zeros of both P_n^l and \tilde{P}_n^l are also simple.

Lemma A.6.

For any $n, m \in \mathbb{N}$ with $1 \leq \frac{n}{m} < 3$ the parameter $\mu_n(\varepsilon)$ defined in Theorem 1.1 satisfies

$$\begin{aligned} \mu''(0) = & C_{m,n} \left(\int_0^1 (P_n^m(z))^4 dz + 2 \int_{-1}^1 z (P_n^m(z))^2 \int_{-1}^z \frac{1}{y^2 (1-y^2)} \int_y^1 x (P_n^m(x))^2 dx dy dz \right. \\ & \left. + \int_{-1}^1 (z+2m) \left(\frac{1-z}{1+z} \right)^m (P_n^m(z))^2 \int_{-1}^z \frac{1}{(y+2m)^2 (1-y^2) \left(\frac{1-y}{1+y} \right)^{2m}} \int_y^1 (x+2m) \left(\frac{1-x}{1+x} \right)^m (P_n^m(x))^2 dx dy dz \right). \end{aligned}$$

with $C_{n,m} := \frac{n(n+1)}{4(n^2+n+1)\pi} \left(\frac{(2n+1)(n-m)!}{(n^2+n+2)(n+m)!} \right)^2 > 0$.

Proof.

Since, by Lemma A.2, we have $\mu'(0) = 0$, then we are in position to apply the formula (A.2) to evaluate $\mu''(0)$. To this purpose, we recall the second and third derivative of \mathcal{T} , given respectively by (A.3) and

$$\partial_{\phi,\phi,\phi}^3 \mathcal{T}_{m,n}(\mu_n, 0, 0) : \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \rightarrow \begin{pmatrix} \frac{u_1 v_1 w_1 + u_1 v_2 w_2 + u_2 v_1 w_2 + u_2 v_2 w_1}{n(n+1)} \\ \frac{2}{4} \frac{u_1 v_1 w_2 + u_1 v_2 w_1 + u_2 v_1 w_1 + u_2 v_2 w_2}{4} \end{pmatrix}.$$

We also need the explicit expression of the inverse operator $(\partial_\phi \mathcal{T}_{m,n}(\mu_n, 0, 0))^{-1} : \mathbb{R}(\partial_\phi \mathcal{T}(\mu_n, 0, 0)) \rightarrow \ker(\partial_\phi \mathcal{T}_{m,n}(\mu_n, 0, 0))^\perp$, which is given by Lemma (A.3).

Concerning the denominator in (A.2), as in the proof of Lemma 3.2 we have

$$\begin{aligned} \|w_0\|^2 \langle \partial_{\mu,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0, 0) w_0, y_0 \rangle &= \left(\int_{\mathbb{S}^2} (|\nabla w_0|^2 + |w_0|^2) \right) \left\langle \begin{pmatrix} 0 \\ -\frac{8}{(2+\mu_n)^2} P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle \\ &= \left(\int_{\mathbb{S}^2} w_0 \cdot (-\Delta_{\mathbb{S}^2} w_0 + w_0) \right) \left(-\frac{8}{(2+\mu_n)^2} \int_{\mathbb{S}^2} (P_n^m(z) \cos(m\theta))^2 d\theta dz \right) \\ &= -\frac{8}{(2+\mu_n)^2} (n(n+1) + 1) \left(\int_{\mathbb{S}^2} |w_0|^2 \right) \left(\int_0^{2\pi} (\cos(m\theta))^2 d\theta \int_{-1}^1 (P_n^m(z))^2 dz \right) \\ &= -\frac{(n^2 + n + 2)^2 (n^2 + n + 1)}{8} \left(\int_0^{2\pi} (\cos(m\theta))^2 d\theta \int_{-1}^1 (P_n^m(z))^2 dz \right)^2 \\ &= -\frac{(n^2 + n + 2)^2 (n^2 + n + 1)}{8} \left(\pi \frac{2(n+m)!}{(2n+1)(n-m)!} \right)^2 \end{aligned}$$

The first term in the numerator of (A.2) is not hard to compute:

$$\begin{aligned} &\left\langle \partial_{\phi,\phi,\phi}^3 \mathcal{T}(\mu_n, 0, 0) \left[\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right], \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 0 \\ \frac{n(n+1)}{4} ((P_n^m(z) \cos(m\theta))^3) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle \\ &= \frac{n(n+1)}{4} \int_0^{2\pi} \cos(m\theta)^4 d\theta \int_{-1}^1 (P_n^m(z))^4 dz \\ &= \frac{3n(n+1)}{8} \pi \int_0^1 (P_n^m(z))^4 dz \end{aligned}$$

To evaluate the other term in the numerator, we first notice that

$$\begin{aligned} \mathbb{Q} \partial_{\phi,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0) [w_0, w_0] &= \frac{\left\langle \partial_{\phi,\phi}^2 \mathcal{T}(\mu_n, 0, 0) \left[\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right], y_0 \right\rangle}{\|y_0\|^2} y_0 \\ &= \frac{1}{\|y_0\|^2} \left\langle \begin{pmatrix} (P_n^m(z) \cos(m\theta))^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle y_0 \\ &= 0; \end{aligned}$$

therefore, we get

$$\begin{aligned} &(\partial_\phi \mathcal{T}(\mu_n, 0, 0))^{-1} (\mathbb{I} - \mathbb{Q}) \partial_{\phi,\phi}^2 \mathcal{T}(\mu_n, 0, 0) [w_0, w_0] \\ &= (\partial_\phi \mathcal{T}(\mu_n, 0, 0))^{-1} \partial_{\phi,\phi}^2 \mathcal{T}(\mu_n, 0, 0) \left[\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right] \\ &= (\partial_\phi \mathcal{T}(\mu_n, 0, 0))^{-1} \begin{pmatrix} (P_n^m(z) \cos(m\theta))^2 \\ 0 \end{pmatrix} \end{aligned}$$

$$= (\partial_\phi \mathcal{T}(\mu_n, 0, 0))^{-1} \begin{pmatrix} (P_n^m(z))^2 \left(\frac{1}{2} + \frac{\cos(2m\theta)}{2} \right) \\ 0 \end{pmatrix}.$$

We can apply Lemma A.3 to the latter. The only non-vanishing coefficients are $\psi_{1,0}(z) = \psi_{1,2m}^1(z) = \frac{1}{2}(P_n^m(z))^2$.

Therefore, we get $\begin{pmatrix} \phi_1(z) \\ 0 \end{pmatrix}$ with

$$\begin{aligned} \phi_1(z) &= P_1(z) \left(C_{1,0} - \frac{1}{2} \int_{-1}^z \frac{1}{(1-y^2)(P_1(y))^2} \int_y^1 P_1(x)(P_n^m(x))^2 dx dy \right) \\ &\quad - \frac{1}{2} \tilde{P}_1^{2m}(z) \int_{-1}^z \frac{1}{(1-y^2) \left(\tilde{P}_1^{2m}(y) \right)^2} \int_y^1 \tilde{P}_1^{2m}(x)(P_n^m(x))^2 dx dy \cos(2m\theta) \end{aligned}$$

Therefore,

$$\begin{aligned} &\left\langle \partial_{\phi,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0, 0) \left[w_0, (\partial_\phi \mathcal{T}_{m,n}(\mu_n, 0, 0))^{-1} (I - Q) \partial_{\phi,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0, 0) [w_0, w_0] \right], y_0 \right\rangle \\ &= \left\langle \partial_{\phi,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0, 0) \left[\begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix}, \begin{pmatrix} \phi_1(z, \theta) \\ 0 \end{pmatrix} \right], y_0 \right\rangle \\ &= \left\langle \begin{pmatrix} 0 \\ \frac{n(n+1)}{2} P_n^m(z) \cos(m\theta) \phi_1(z, \theta) \end{pmatrix}, \begin{pmatrix} 0 \\ P_n^m(z) \cos(m\theta) \end{pmatrix} \right\rangle \\ &= \frac{n(n+1)}{2} \int_{\mathbb{S}^2} (P_n^m(z) \cos(m\theta))^2 \phi_1(z, \theta) d\theta dz \\ &= \frac{n(n+1)}{2} \int_0^{2\pi} d\theta \int_{-1}^1 dz (P_n^m(z) \cos(m\theta))^2 \left(P_1(z) \left(C_{1,0} - \frac{1}{2} \int_{-1}^z \frac{1}{(1-y^2)(P_1(y))^2} \int_y^1 P_1(x)(P_n^m(x))^2 dx dy \right) \right. \\ &\quad \left. - \frac{1}{2} \tilde{P}_1^{2m}(z) \int_{-1}^z \frac{1}{(1-y^2) \left(\tilde{P}_1^{2m}(y) \right)^2} \int_y^1 \tilde{P}_1^{2m}(x)(P_n^m(x))^2 dx dy \cos(2m\theta) \right) \\ &= \frac{n(n+1)}{2} C_{1,0} \int_0^{2\pi} (\cos(m\theta))^2 \int_{-1}^1 P_1(z)(P_n^m(z))^2 dz \\ &\quad - \frac{n(n+1)}{4} \int_0^{2\pi} (\cos(m\theta))^2 d\theta \int_{-1}^1 P_1(z)(P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)(P_1(y))^2} \int_y^1 P_1(x)(P_n^m(x))^2 dx dy dz \\ &\quad - \frac{n(n+1)}{4} \int_0^{2\pi} (\cos(m\theta))^2 \cos(2m\theta) d\theta \int_{-1}^1 \tilde{P}_1^{2m}(z)(P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2) \left(\tilde{P}_1^{2m}(y) \right)^2} \int_y^1 \tilde{P}_1^{2m}(x)(P_n^m(x))^2 dx dy dz \\ &= -\frac{n(n+1)}{4} \pi \int_{-1}^1 z (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)y^2} \int_y^1 x (P_n^m(x))^2 dx dy dz \\ &\quad - \frac{n(n+1)}{8} \pi \int_{-1}^1 (z+2m) \left(\frac{1-z}{1+z} \right)^m (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)(y+2m)^2 \left(\frac{1-y}{1+y} \right)^{2m}} \int_y^1 (x+2m) \left(\frac{1-x}{1+x} \right)^m (P_n^m(x))^2 dx dy dz \end{aligned}$$

We are now in position to apply the formula (A.2) and to prove the Lemma:

$$\begin{aligned} \mu''(0) &= \frac{1}{\|w_0\|^2 \left\langle \partial_{\mu,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0) w_0, y_0 \right\rangle} \left(-\frac{1}{3} \left\langle \partial_{\phi,\phi,\phi}^3 \mathcal{T}_{n,m}(\mu_n, 0, 0) [w_0, w_0, w_0], y_0 \right\rangle \right. \\ &\quad \left. + \left\langle \partial_{\phi,\phi}^2 \mathcal{T}_{n,m}(\mu_n, 0, 0) \left[w_0, (\partial_\phi \mathcal{T}_{m,n}(\mu_n, 0, 0))^{-1} (I - Q) \partial_{\phi,\phi}^2 \mathcal{T}_{m,n}(\mu_n, 0, 0) [w_0, w_0] \right], y_0 \right\rangle \right) \\ &= -\frac{8}{(n^2+n+2)^2 (n^2+n+1)} \left(\frac{(2n+1)(n-m)!}{2\pi(n+m)!} \right)^2 \left(-\frac{1}{3} \frac{3n(n+1)}{8} \pi \int_0^1 (P_n^m(z))^4 dz \right) \end{aligned}$$

$$\begin{aligned}
& - \frac{n(n+1)}{4} \pi \int_{-1}^1 z (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)y^2} \int_y^1 x (P_n^m(x))^2 dx dy dz \\
& - \frac{n(n+1)}{8} \pi \int_{-1}^1 (z+2m) \left(\frac{1-z}{1+z}\right)^m (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)(y+2m)^2 \left(\frac{1-y}{1+y}\right)^{2m}} \int_y^1 (x+2m) \left(\frac{1-x}{1+x}\right)^m (P_n^m(x))^2 dx dy dz \\
& = C_{n,m} \left(\int_0^1 (P_n^m(z))^4 dz + 2 \int_{-1}^1 z (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)y^2} \int_y^1 x (P_n^m(x))^2 dx dy dz \right. \\
& \left. + \int_{-1}^1 (z+2m) \left(\frac{1-z}{1+z}\right)^m (P_n^m(z))^2 \int_{-1}^z \frac{1}{(1-y^2)(y+2m)^2 \left(\frac{1-y}{1+y}\right)^{2m}} \int_y^1 (x+2m) \left(\frac{1-x}{1+x}\right)^m (P_n^m(x))^2 dx dy dz \right)
\end{aligned}$$

□

Remark A.7. We are not able to compute the integrals in (A.9) for any m, n . Of course this is possible in some particular cases like in the table below

n	$\mu''(0) > 0$	$\mu''(0) < 0$	$\mu''(0) > 0$
3	$m = 0, 1$	$m = 2$	$m = 3$
4	$m = 0, 1$	$m = 2, 3$	$m = 4$
5	$m = 0, 1$	$m = 2, 3, 4$	$m = 5$
6	$m = 0, 1$	$m = 2, 3, 4$	$m = 5, 6$
7	$m = 0, 1, 2$	$m = 3, 4, 5$	$m = 6, 7$
8	$m = 0, 1, 2$	$m = 3, 4, 5, 6$	$m = 7, 8$
9	$m = 0, 1, 2, 3$	$m = 4, 5, 6$	$m = 7, 8, 9$
10	$m = 0, 1, 2, 3$	$m = 4, 5, 6, 7$	$m = 8, 9, 10$
11	$m = 0, 1, 2, 3$	$m = 4, 5, 6, 7, 8$	$m = 9, 10, 11$
12	$m = 0, 1, 2, 3, 4$	$m = 5, 6, 7, 8$	$m = 9, 10, 11, 12$
13	$m = 0, 1, 2, 3, 4$	$m = 5, 6, 7, 8, 9$	$m = 10, 11, 12, 13$
14	$m = 0, 1, 2, 3, 4, 5$	$m = 6, 7, 8, 9$	$m = 10, 11, 12, 13, 14$
15	$m = 0, 1, 2, 3, 4, 5$	$m = 6, 7, 8, 9, 10$	$m = 11, 12, 13, 14, 15$
16	$m = 0, \dots, 6$	$m = 7, 8, 9, 10$	$m = 11, \dots, 16$
17	$m = 0, \dots, 7$	$m = 8, 9, 10, 11$	$m = 12, \dots, 17$
18	$m = 0, \dots, 7$	$m = 8, 9, 10, 11$	$m = 12, \dots, 18$
19	$m = 0, \dots, 8$	$m = 9, 10, 11$	$m = 12, \dots, 19$
20	$m = 0, \dots, 9$	$m = 10, 11, 12$	$m = 13, \dots, 20$
21	$m = 0, \dots, 10$	$m = 11, 12$	$m = 13, \dots, 21$
22	$m = 0, \dots, 11$	$m = 12$	$m = 13, \dots, 22$
23	$m = 0, \dots, 23$		
24	$m = 0, \dots, 24$		
25	$m = 0, \dots, 25$		
26	$m = 0, \dots, 26$		
27	$m = 0, \dots, 27$		
28	$m = 0, \dots, 28$		
29	$m = 0, \dots, 29$		
30	$m = 0, \dots, 30$		

The table above indicates that $\mu''(0)$ is always different from zero. However, we do not have a rigorous proof of this.

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