# A general existence result for stationary solutions to the Keller-Segel system

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#### Abstract

We consider the following Liouville-type PDE, which is related to stationary solutions of the Keller-Segel's model for chemotaxis:

$$\left\{ \begin{array}{ll} -\Delta u + \beta u = \rho \left( \frac{e^u}{\int_{\Omega} e^u} - \frac{1}{|\Omega|} \right) & \text{in } \Omega \\ \partial_{\nu} u = 0 & \text{on } \partial \Omega \end{array} \right. ,$$

where  $\Omega \subset \mathbb{R}^2$  is a smooth bounded domain and  $\beta, \rho$  are real parameters. We prove existence of solutions under some algebraic conditions involving  $\beta, \rho$ . In particular, if  $\Omega$  is not simply connected, then we can find solution for a generic choice of the parameters. We use variational and Morse-theoretical methods.

## 1 Introduction

We are interesting in the study of the following partial differential equation:

$$\begin{cases}
-\Delta u + \beta u = \rho \left( \frac{e^u}{\int_{\Omega} e^u} - \frac{1}{|\Omega|} \right) & \text{in } \Omega \\
\partial_{\nu} u = 0 & \text{on } \partial\Omega
\end{cases} . \tag{$P_{\beta,\rho}$}$$

Here,  $\Omega \subset \mathbb{R}^2$  is a smooth bounded open domain in the plane,  $\beta$  and  $\rho$  are real parameters and  $|\Omega|$  is the Lebesgue measure of  $\Omega$ .

Problem  $(P_{\beta,\rho})$  is related to a model introduced by Keller and Segel in [22] to study *chemotaxis* in biology, namely the movement of organisms according to the presence of chemicals in the environment.

In particular,  $(P_{\beta,\rho})$  models stationary solutions in Keller and Segel's model.

In the case  $\beta > 0$ , solutions to  $(P_{\beta,\rho})$  have been found via a mountain-pass argument in [30], whereas families of blowing-up solutions have been constructed in [29, 16, 1, 9, 10].

Here we allow the parameter  $\beta$  to have any sign. We will tackle problem  $(P_{\beta,\rho})$  variationally; in fact, solutions are all and only the critical points of the energy functional

$$\mathcal{J}_{\beta,\rho}(u) = \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 + \beta u^2 \right) - \rho \log \int_{\Omega} e^u.$$
 (1.1)

Since both  $(P_{\beta,\rho})$  and (1.1) are invariant under addition of constants, it will not be restrictive to look for solutions to  $(P_{\beta,\rho})$  satisfying  $\int_{\Omega} u = 0$ ; equivalently, we will consider  $\mathcal{J}_{\beta,\rho}$  not on its natural

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domain  $H^1(\Omega)$ , but rather on its subspace

$$\overline{H}^1(\Omega) := \left\{ u \in H^1(\Omega) : \int_{\Omega} u = 0 \right\}.$$

In particular, we will study the topology and the homology of very low energy sub-levels

$$\mathcal{J}_{\beta,\rho}^{c} := \left\{ u \in \overline{H}^{1}(\Omega) : \mathcal{J}_{\beta,\rho}(u) \leq c \right\}$$

with  $c=-L\ll 0$ , then will deduce existence of solutions using Morse theory. This argument has been introduced in [17] for a fourth-order problem in Riemannian geometry and it has become a rather classical tool in the study of Liouville-type equation (see for instance the surveys [26, 25]). With respect to most previous results, the main new difficulties are given by the Neumann boundary conditions, rather than Dirichlet, and the presence of the extra linear term  $\beta u$ .

Neumann conditions may cause concentration on  $\partial\Omega$ , which was excluded in the case of Dirichlet conditions, whereas concentration on the interior is similar in the two cases. The main difference when concentration occurs at the boundary is due to the fact that, roughly speaking, a shrinking ball centered at a point  $p \in \partial\Omega$  is asymptotically half of a shrinking ball contained in  $\Omega$ .

The argument to fix such an issue is inspired by [28], which deals with a similar problem on higher-dimensional manifolds with boundary.

Another issue may be given by linear part  $-\Delta + \beta$  not being positive definite, if  $\beta < 0$ , which is a new feature in second order Liouville equations.

This naturally leads to consider the projection of u into positive and negative sub-spaces of the operator  $-\Delta + \beta$ . Precisely, we take an orthonormal frame  $\{\varphi_i\}_{i\in\mathbb{N}}$  of eigenfunctions of  $-\Delta$  with associated positive non-decreasing eigenvalues  $\{\lambda_i\}_{i\in\mathbb{N}}$  (counted with multiplicity), so that

$$u = \sum_{i=1}^{+\infty} \left( \int_{\Omega} u \varphi_i \right) \varphi_i \qquad \Rightarrow \qquad -\Delta u = \sum_{i=1}^{+\infty} \lambda_i \left( \int_{\Omega} u \varphi_i \right) \varphi_i.$$

Now, if  $-\beta$  is not an eigenvalue of  $-\Delta$ , then  $-\lambda_{I+1} < \beta < -\lambda_I$  for some  $I \in \mathbb{N}$ , therefore we can define the projection  $\Pi_I$  on a finite dimensional subspace, on which orthogonal  $-\Delta + \beta$  is positive definite:

$$\Pi_I u := \left( \int_{\Omega} u \varphi_1, \dots, \int_{\Omega} u \varphi_I \right) \in \mathbb{R}^I. \tag{1.2}$$

Arguing as in [17], we can show that, if  $\mathcal{J}_{\beta,\rho}(u) \ll 0$ , then either  $\frac{e^u}{\int_{\Omega} e^u}$  is concentrated around a finite number of points or  $\Pi_I u$  is large (or both occur).

To express this alternative we will use the join, which has been used in the variational study of Liouville system of two equations, where one has an alternative between the concentration of each component (see [6, 4, 20, 7, 5]).

Given two topological spaces X and Y, its join  $X \times Y$  is defined as the product between the two spaces and the unit interval, with identifications at each endpoint. We set

$$X \star Y := \frac{X \times Y \times [0, 1]}{\sim},\tag{1.3}$$

with  $\sim$  being defined by

$$(x, y, 0) \sim (x, y', 0) \quad \forall x \in X, \forall y, y' \in Y, \qquad (x, y, 1) \sim (x', y, 1) \quad \forall x, x' \in X, \forall y \in Y.$$

We will suitably choose X and Y as objects to model each alternative. When t = 0, the whole Y is collapsed, which means only the first alternative occurs, similarly at t = 1 only Y is left hence we have the second alternative; if 0 < t < 1 we see both spaces because both alternatives occur.

To be in position to apply such methods, we need some compactness assumptions on the energy functional  $\mathcal{J}_{\beta,\rho}$ .

Unfortunately, Palais-Smale condition is not known to hold for problem  $(P_{\beta,\rho})$ , nor for similar Liouville-type PDEs. Anyway, such problem can be by-passed thanks to a deformation lemma by [23] and some compactness of solutions to  $(P_{\beta,\rho})$  holding locally uniformly in  $\rho$ .

As in most known results, such a result holds true provided  $\rho$  is not an integer multiple of  $4\pi$ . This time we also need  $-\beta$  not to be an eigenvalue of  $-\Delta$ , in order for the linear operator to be non-degenerate (see Section 2).

Finally, we need to verify that the solution found using these tools is not the trivial one  $u \equiv 0$ . This is equivalent to evaluate the Morse index of the trivial solution.

We will get non-trivial solutions under some algebraic condition involving the parameters  $\beta$ ,  $\rho$  and the eigenvalues  $\lambda_i$ . In the case when  $\Omega$  is multiply connected, we are allowed to take more cases, since the topology of low sublevels  $\mathcal{J}_{\beta,\rho}^{-L}$  is more involved. Precisely, the main result of this paper is the following:

#### Theorem 1.1.

Assume  $\beta \neq -\lambda_i \neq \beta - \frac{\rho}{|\Omega|}$  for any  $i \in \mathbb{N}$  and  $\rho \notin 4\pi\mathbb{N}$  and let  $I \leq J, K$  be non-negative integers such that

$$4K\pi < \rho < 4(K+1)\pi \qquad -\lambda_{I+1} < \beta < -\lambda_{I} \qquad -\lambda_{J+1} < \beta - \frac{\rho}{|\Omega|} < -\lambda_{J}.$$

If  $\Omega$  is simply connected and  $2K + I \neq J$ , then the problem  $(P_{\beta,\rho})$  has non-trivial solutions. If  $\Omega$  is not simply connected and  $(K,I) \neq (0,J)$ , then the problem  $(P_{\beta,\rho})$  has non-trivial solutions.

#### Remark 1.2.

Since Theorem 1.1 is proved via Morse theory, then the same arguments would also give multiplicity of solutions, provided the energy functional  $\mathcal{J}_{\beta,\rho}$  is a Morse function, as was done with similar problems in [14, 3, 4, 19, 5, 15]. However, it is not clear under which conditions on  $\Omega$  this occurs, although we suspect it is a somehow generic conditions.

One easily see that, assuming  $u \equiv 0$  to be the only solution to  $(P_{\beta,\rho})$ ,  $\mathcal{J}_{\beta,\rho}$  is Morse if and only if  $\beta$  and  $\rho$  satisfy an algebraic relation (see Proposition 4.1); this fact implies that the solution found in Theorem 1.1 is not trivial.

The same arguments as Theorem 1.1 are also useful, with minor modifications, to find a solution

$$\begin{cases} -\Delta u + \beta u = \rho \left( \frac{he^u}{\int_{\Sigma} he^u} - \frac{1}{|\Sigma|} \right) & \text{in } \Sigma \\ \partial_{\nu} u = 0 & \text{on } \partial \Sigma \end{cases},$$

with  $\Sigma$  being a compact surface with boundary and  $h \in C^{\infty}(\Sigma)$  being strictly positive. Here, if h is not constant, we can also cover the case in which Theorem 1.1 gave a trivial solution. Moreover, by arguing as in the previous references, one can show that  $\mathcal{J}_{\beta,\rho}$  satisfies the Morse property under a generic assumption on h and/or on the metric g on  $\Sigma$ .

The plan of this paper is the following. Section 2 is devoted to the study of compactness properties of the equation  $(P_{\beta,\rho})$ ; Section 3, which is divided in three sub-sections, deals with the analysis of energy sublevels of  $\mathcal{J}_{\beta,\rho}$ , and finally in Section 4 Theorem (1.1) is proved.

We will denote as d(x,y) the distance between two points  $x,y\in\overline{\Omega}$  and, similarly, for  $\Omega',\Omega''\subset\Omega$ ,

$$d(x, \Omega') := \inf\{d(x, y) : y \in \Omega'\}$$
 
$$d(\Omega', \Omega'') := \inf\{d(x, y) : x \in \Omega', y \in \Omega''\}.$$

We will denote as  $B_r(x)$  the open ball of radius r centered at p. The symbol  $\int_{\Omega'} f := \frac{1}{|\Omega'|} \int_{\Omega'} f$ will stand for the average of  $f \in L^1(\Omega')$  on some  $\Omega' \subset \Omega$ .

The letter C will denote large constant which may vary among different formulas and lines.

#### 2 Compactness issues

This section is devoted to the proof of the following concentration-compactness result with quantization of blow-up limits:

#### Proposition 2.1.

Let  $(u_n)_{n\in\mathbb{N}}$  be a sequence of solutions to  $(P_{\beta,\rho})$  with  $\rho_n \underset{n\to+\infty}{\to} \rho$  and  $-\beta \neq \lambda_i$  for all i's. Then, up to sub-sequences, one of the following alternatives occur:

(Compactness)  $(u_n)_{n\in\mathbb{N}}$  is compact in  $\overline{H}^1(\Omega)$ ;

(Concentration) The blow-up set S, defined by

$$S := \left\{ x \in \overline{\Omega} : \exists x_n \underset{n \to +\infty}{\to} x \text{ such that } u_n(x_n) \underset{n \to +\infty}{\to} +\infty \right\},\,$$

is non-empty and finite.

Moreover,  $\rho_n \frac{e^{u_n}}{\int_{\Omega} e^{u_n}} \xrightarrow[n \to +\infty]{} \sum_{x \in S} \sigma(x) \delta_x$  as measures, with

$$\sigma(x) := \lim_{r \to 0} \lim_{n \to +\infty} \rho_n \frac{\int_{B_r(x)} e^{u_n}}{\int_{\Omega} e^{u_n}} = \begin{cases} 4\pi & \text{if } x \in \mathcal{S} \cap \partial\Omega\\ 8\pi & \text{if } x \in \mathcal{S} \cap \Omega \end{cases} . \tag{2.1}$$

In particular, if  $\rho \notin 4\pi \mathbb{N}$  then (Compactness) occurs.

When Compactness occurs, a standard consequence is the following: since the set of solutions is compact, then its energy is uniformly bounded from above, hence the whole space  $\overline{H}^1(\Omega)$  can be retracted on a suitable sublevel containing all solutions.

#### Corollary 2.2.

Assume  $\rho \notin 4\pi \mathbb{N}$  and  $-\beta \neq \lambda_i$  for all i's.

Then, there exists L>0 such that  $\mathcal{J}_{\beta,\rho}^L$  is a deformation retract of  $\overline{H}^1(\Omega)$ . In particular, it is contractible.

Proof.

Since we can write the energy functional as  $\mathcal{J}_{\beta,\rho}(u) = \frac{\|u\|_{\overline{H}^1(\Omega)}^2}{2} - K_1(u) - \lambda K_2(u)$ , we are in position to apply Proposition 1.1 in [23]. From Proposition 2.1, there are no solutions  $u_n$  to  $(P_{\beta,\rho})$  with  $L \leq \mathcal{J}_{\beta,\rho_n} \leq L+1$ , if L is large enough; therefore, arguing as in [23],  $\mathcal{J}_{\beta,\rho}^L$  is a deformation retract of  $\mathcal{J}_{\beta,\rho}^{L+1}$ . Being L arbitrary, we find that  $\mathcal{J}_{\beta,\rho}^L$  is a retract of the whole space  $\overline{H}^1(\Omega)$ .  $\square$ 

Proposition 2.1 is rather classical for Liouville-type equations like  $(P_{\beta,\rho})$ . It was first given by [11] and, in the case of Neumann conditions, in [30]. With respect to the latter reference, the presence of the extra term  $\beta u$  in the linear part, which may cause the maximum principle to fail, can be dealt by just moving it to the right-hand side.

A key tool is a "minimal mass" lemma. The proof in [30] also works in the case  $f_n \not\equiv 0$ .

**Lemma 2.3.** ([30], Lemma 3.2)

Let  $(v_n)_{n\in\mathbb{N}}$  be a sequence of solutions to

$$\begin{cases}
-\Delta u_n = \rho_n \left( \frac{e^{u_n}}{\int_{\Omega} e^{u_n}} - \frac{1}{|\Omega|} \right) + f_n & \text{in } \Omega \\
\partial_{\nu} u_n = 0 & \text{on } \partial\Omega
\end{cases}$$
(2.2)

with  $(f_n)_{n\in\mathbb{N}}$  bounded in  $L^q(\Omega)$  for some q>1 with  $\int_{\Omega} f_n = 0 \ \forall n\in\mathbb{N}$ . Then, there exists  $\sigma_0 = \sigma_0(\Omega) > 0$  such that if  $\limsup_{r\to 0} \rho_n \frac{\int_{B_r(x)} e^{v_n}}{\int_{\Omega} e^{v_n}} \leq \sigma_0$  for all  $x\in\overline{\Omega}$ , then  $(u_n)_{n\in\mathbb{N}}$  is compact in  $\overline{H}^1(\Omega)$ .

Roughly speaking, the idea to prove Proposition 2.1 will be the following. If *Concentration* occurs, then we have blow-up at a finite number of points, thanks to Lemma 2.3. Then, the local mass at each blow-up point  $x \in \mathcal{S}$  is found via a Pohožaev identity based on the asymptotic behavior of solution, and this fact excludes the presence of a residual mass.

Proof of Proposition 2.1.

Let  $(u_n)_{n\in\mathbb{N}}$  be a sequence of solutions with  $\sup_{\Omega} u_n \leq C$ . Then, Jensen's inequality gives  $\int_{\Omega} e^{u_n} \geq |\Omega| e^{\int_{\Omega} u_n} = |\Omega|$ , therefore

$$|-\Delta u_n + \beta u_n| \le \rho_n \left( \left| \frac{e^{u_n}}{\int_{\Omega} e^{u_n}} \right| + \frac{1}{|\Omega|} \right) \le (\rho + 1) \left( \frac{e^C}{|\Omega|} + \frac{1}{|\Omega|} \right)$$

is uniformly bounded, hence by standard regularity  $u_n$  is bounded in  $W^{2,2}(\Omega)$  and compact in  $H^1(\Omega)$ .

Suppose now  $\sup_{\Omega} u_n \underset{n \to +\infty}{\to} +\infty$ , namely  $S \neq \emptyset$ . This time, we just have  $\|-\Delta u_n + \beta u_n\|_{L^1(\Omega)} \leq 2(\rho+1)$ ; since  $-\beta$  is not an eigenvalue of  $-\Delta$ , this gives  $\|\nabla u_n\|_{L^q(\Omega)} + \|u_n\|_{L^q(\Omega)} \leq C$  for any q < 2. Therefore,  $u_n$  will solve (2.2) with  $f_n = -\beta u_n \in L^q(\Omega)$ , hence we can apply Lemma 2.3 to get

$$|\mathcal{S}|\sigma_0 \le \sum_{x \in \mathcal{S}_i} \sigma(x) \le \rho.$$

This means that  $\mathcal{S}$  is finite and we easily get  $\rho_n \frac{e^{u_n}}{\int_{\Omega} e^{u_n}} \xrightarrow[n \to +\infty]{} \sum_{x \in \mathcal{S}} \sigma(x) \delta_x + f$  for some  $f \in L^1(\Omega) \cap L^{\infty}_{loc}(\Omega \setminus \mathcal{S})$ , while  $u_n \xrightarrow[n \to +\infty]{} \sum_{x \in \mathcal{S}} \sigma(x) G_x + w$  in  $W^{1,q}(\Omega) \cap C^{1,\alpha}_{loc}(\Omega \setminus \mathcal{S})$  for q < 2,  $\alpha < 1$ , with  $G_x$  and w solving, respectively.

$$\begin{cases}
-\Delta G_x + \beta G_x = \delta_x - \frac{1}{|\Omega|} & \text{in } \Omega \\
\partial_{\nu} G_x = 0 & \text{on } \partial\Omega
\end{cases} \qquad \begin{cases}
-\Delta w + \beta w = f - \oint_{\Omega} f & \text{in } \Omega \\
\partial_{\nu} w = 0 & \text{on } \partial\Omega
\end{cases}.$$

We need to show that  $f \equiv 0$ , which will be done arguing as in [8] (Lemmas 2.2 and 2.3). If  $f \not\equiv 0$ , then one easily sees that  $f = Ve^w$ , with  $V := \underbrace{\frac{\rho}{\lim\limits_{x \to +\infty} \int_{\Omega} e^{u_n}}}_{\neq +\infty} e^{\sum_{x \in \mathcal{S}} \sigma(x)G_x} \in L^{\infty}_{\text{loc}}(\Omega \setminus \mathcal{S})$  whereas,

due to the behavior of  $G_x$ ,  $V \sim |\cdot -x|^{-\frac{\sigma(x)}{2\pi}}$  if  $x \in \mathcal{S} \cap \Omega$  and  $V \sim |\cdot -x|^{-\frac{\sigma(x)}{\pi}}$  if  $x \in \mathcal{S} \cap \partial \Omega$ . Now, since  $-\Delta w \geq -\beta w - \int_{\Omega} f \in L^q(\Omega)$ , then w is bounded from below, namely  $V \leq Cf \in L^1(\Omega)$ , which in particular means  $\sigma(x) < 4\pi$  for any  $x \in \mathcal{S}$ . This contradicts (2.1), which can be deduced arguing as in [30] (Lemma 3.4), hence it must be  $f \equiv 0$ . Finally, if *Concentration* occurs then

$$\rho = \lim_{n \to +\infty} \int_{\Omega} \rho_n \frac{e^{u_n}}{\int_{\Omega} e^{u_n}} = \lim_{n \to +\infty} \int_{\Omega} \sum_{x \in \mathcal{S}} \sigma(x) \delta_x = \sum_{x \in \mathcal{S}} \sigma(x) = \sum_{x \in \mathcal{S} \cap \partial \Omega} 4\pi + \sum_{x \in \mathcal{S} \cap \Omega} 8\pi \in 4\pi \mathbb{N}.$$

# 3 Analysis of sublevels

In this Section, which is the largest of the paper, we will study topologically the energy sublevels of  $\mathcal{J}_{\beta,\rho}$ .

In the first sub-section, we will introduce a topological space which will be later "compared" to energy sublevels, and we will compute some of its homology groups. Then, we will construct maps from this topological space to low sublevels and vice-versa and we will deduce that  $\mathcal{J}_{\beta,\rho}^{-L}$  has non-trivial homology.

# 3.1 The space $(\Omega_{\partial})_{K,I}$ and its homology

Let us introduce a set of barycenters on  $\overline{\Omega}$ , namely a set of finitely-supported probability measures on  $\overline{\Omega}$ .

With respect to most previous works, we will not give a constraint on the cardinality of the support, basically because points in  $\Omega$  and  $\partial\Omega$  have to be treated differently, for reasons which will be discussed in the forthcoming sub-sections. Roughly speaking, points in the interior will count twice as much as points in the boundary.

$$(\Omega_{\partial})_{K} := \bigcup_{l=0}^{\left\lfloor \frac{K}{2} \right\rfloor} \Omega_{l,K-2l} \qquad \Omega_{l,m} := \left\{ \sum_{k=1}^{l} t_{k} \delta_{x_{k}} + \sum_{k'=1}^{m} t'_{k'} \delta_{x'_{k'}}; \sum_{k=1}^{l} t_{k} + \sum_{k'=1}^{m} t'_{k'} = 1, \ x_{k} \in \Omega, x'_{k'} \in \partial\Omega \right\}$$

On such spaces, we will consider the distance induced by the Lip' norm, that is the norm on the space of signed measures induced by duality with Lipschitz functions:

$$\|\mu\|_{\operatorname{Lip}'\left(\overline{\Omega}\right)} := \sup_{h \in \operatorname{Lip}\left(\overline{\Omega}\right), \|h\|_{\operatorname{Lip}(\overline{\Omega})} \le 1} \left| \int_{\Omega} h \mathrm{d}\mu \right|.$$

As a first result, we see that such barycenters spaces are Euclidean deformation retracts. The proof has been given in [28] in the case when  $\Omega$  is replaced by a 4-dimensional compact manifold with boundary, but the same proof holds in any dimension and in particular for planar domains.

**Lemma 3.1.** ([28], Lemma 4.10)

There exist  $\epsilon_0 > 0$  and a continuous retraction

$$\widetilde{\Psi}: \left\{ \mu \in \mathcal{M}\left(\overline{\Omega}\right): \, d_{\operatorname{Lip}'\left(\overline{\Omega}\right)}(\mu, (\Omega_{\partial})_K) < \epsilon_0 \right\} \to (\Omega_{\partial})_K.$$

Among all the "layers" which compose  $(\Omega_{\partial})_K$ , a special role will be played by the first one, consisting of measures supported on  $\partial\Omega$ . In [28] it was shown that it is a deformation retract within  $(\Omega_{\partial})_K$ ; again, their proof is also valid in our case.

**Lemma 3.2.** ([28], Proposition 4.5)

For any  $K \geq 1$  the set  $\Omega_{0,K} = (\partial \Omega)_K \subset (\Omega_{\partial})_K$  is a deformation retract of some its open neighborhood U in  $(\Omega_{\partial})_K$ .

Since  $\partial\Omega$  is homotopically equivalent to a disjoint union of g circles, we can use a result from [15] to compute the homology of  $\Omega_{0,K}$ .

**Lemma 3.3.** ([15], Proposition 5.1)

The homology groups of  $\Omega_{0,K} = (\partial \Omega)_K$  are given by

$$\widetilde{H}_q((\partial\Omega)_m) = \begin{cases} \mathbb{Z}^{\binom{g+q-K+1}{g}\binom{g}{2K-q-1}} & \max\{K-1, 2K-g-1\} \le q \le 2K-1 \\ 0 & q < \max\{K-1, 2K-g-1\}, q > 2K-1 \end{cases}$$

The space  $(\Omega_{\partial})_K$  will be used in the analysis of sublevels to express the fact that, if  $\mathcal{J}_{\beta,\rho}(u) \ll 0$ , then u may concentrates at a finite number of points.

Anyway, it  $\mathcal{J}_{\beta,\rho}$  is very low, it may also happen that the projection  $\Pi_I$  on the space of negative eigenvalues for  $-\Delta + \beta$ , defined by (1.2), is very large in norm. Since  $\Pi_I u \in \mathbb{R}^I$ , this naturally leads to consider, after a normalization, the sphere  $\mathbb{S}^{I-1}$  to deal with phenomena.

As anticipated in the introduction, the alternative between concentration and large  $\Pi_I$  will be modeled by the join (1.3), therefore we will be interested in the following space:

$$(\Omega_{\partial})_{K,I} = (\Omega_{\partial})_K \star \mathbb{S}^{I-1}. \tag{3.1}$$

#### Remark 3.4.

In [17], the authors used a space of the kind  $\frac{X \times \mathbb{B}^{I-1}}{\sim}$ , with  $\mathbb{B}^{I-1}$  indicating the unit ball in  $\mathbb{R}^{I-1}$  and  $\sim$  defined by  $(x,y) \sim (x',y)$  for any  $x,x' \in X$  and  $y \in \mathbb{S}^{I-1}$ .

Actually, this space is homeomorphic to the join  $X \star \mathbb{S}^{I-1}$ , with one possible homeomorphism given by the map

$$X\star\mathbb{S}^{I-1}\ni(x,y,t)\qquad\longleftrightarrow\qquad(x,ty)\in\frac{X\times\mathbb{B}^{I-1}}{2}.$$

Since we are interested in the homology of  $(\Omega_{\partial})_{K,I}$ , we will use a well-known result concerning the homology of a join.

**Lemma 3.5.** ([18], Theorem 3.21)

Let X and Y be two CW-complexes and  $X \star Y$  their join as defined by (1.3). Then, its homology group are

$$\widetilde{H}_q(X \star Y) = \bigoplus_{q'=0}^q \widetilde{H}_{q'}(X) \otimes \widetilde{H}_{q-q'-1}(Y),$$

where  $\widetilde{H}_q$  denotes the reduced homology groups:  $H_q(X) = \left\{ \begin{array}{ll} \widetilde{H}_q(X) \oplus \mathbb{Z} & \text{if } q = 0 \\ \widetilde{H}_q(X) & \text{if } q \geq 1 \end{array} \right.$ 

We are now able to get some information on the homology on  $(\Omega_{\partial})_{K,I}$ . In particular, we will compute its maximal dimensional homology group, which is not trivial.

#### Proposition 3.6.

Let g be the genus of  $\Omega$ .

The homology groups of  $(\Omega_{\partial})_{K,I}$  satisfy

$$\widetilde{H}_{2K+I-1}((\Omega_{\partial})_{K,I}) = \mathbb{Z}^{\binom{K+g}{g}}.$$

Proof.

We start with the case I=0. If K=1, then  $(\Omega_{\partial})_1=\partial\Omega$  so its homology is computed immediately; otherwise, we write  $(\Omega_{\partial})_K=U\cup V$ , with U as in Lemma 3.2 and  $V:=(\Omega_{\partial})_K\setminus(\partial\Omega)_K$ . The space

 $V = \bigcup_{l=0}^{\lfloor \frac{n}{2} \rfloor} \Omega_{l,K-2l}$  is a stratified set whose maximal dimension equals 2K-3, and the same holds

true for  $U \cap V$ , hence  $\widetilde{H}_q(U \cap V) = H(U \cap V) = 0$  for any  $q \geq 2K - 2$ . Therefore, the Mayer-Vietoris exact sequence gives:

$$0 = \widetilde{H}_{2K-1}(U \cap V) \to \widetilde{H}_{2K-1}(U) \oplus \widetilde{H}_{2K-1}(V) \to \widetilde{H}_{2K-1}((\Omega_{\partial})_K) \to \widetilde{H}_{2K-2}(U \cap V) = 0,$$

that is

$$\widetilde{H}_{2K-1}((\Omega_{\partial})_K) = \widetilde{H}_{2K-1}(U) \oplus H_{2K-1}(V) = \widetilde{H}_{2K-1}((\partial\Omega)_K) = Z^{\binom{K+g}{g}}.$$

Finally, if  $I \geq 1$ , then Lemma 3.5 gives

$$\begin{split} \widetilde{H}_{2K+I-1}((\Omega_{\partial})_{K,I}) &= \widetilde{H}_{2K+I-1}\left((\Omega_{\partial})_{K} \star \mathbb{S}^{I-1}\right) \\ &= \bigoplus_{q'=0}^{2K+I-1} \widetilde{H}_{q'}((\Omega_{\partial})_{K}) \otimes \widetilde{H}_{2K+I-q'-2}\left(\mathbb{S}^{I-1}\right) \\ &= H_{2K-1}((\Omega_{\partial})_{K}) \\ &= \mathbb{Z}^{\binom{K+g}{g}}. \end{split}$$

#### Remark 3.7.

As pointed out by the referee, one can compute the Euler characteristic of  $(\Omega_{\partial})_K$  using the results in [21, 2]:

$$\chi((\Omega_{\partial})_K) = \chi\left(\Omega_{\lfloor \frac{K}{2} \rfloor, 0}\right) = 1 - \frac{1}{\lfloor \frac{K}{2} \rfloor!} \prod_{k=1}^{\lfloor \frac{K}{2} \rfloor} (k - \chi(\Omega)) = 1 - \binom{\lfloor \frac{K}{2} \rfloor + g - 1}{g - 1}.$$

# The map $\Phi^{\Lambda}: (\Omega_{\partial})_{K,I} \to \mathcal{J}_{\beta,\rho}^{-L}$

In this Subsection we will build a map from the space  $(\Omega_{\partial})_{K,I}$ , whose properties have just been

discussed, into a suitably low energy sublevel  $J_{\beta,\rho}^{-L}$ . Precisely, we will construct a family  $\Phi^{\Lambda}$  of maps with  $\mathcal{J}_{\beta,\rho}\left(\Phi^{\Lambda}\right) \underset{\Lambda \to +\infty}{\to} -\infty$  uniformly, so that for

 $\Lambda$  large enough the image of  $\mathcal{J}_{\beta,\rho}$  is contained in  $\mathcal{J}_{\beta,\rho}^{-L}$ . The choice of L, hence of  $\Lambda$ , will be made in the following subsection.

Consistently with the previous discussion, the family of test functions defined by  $\Phi^{\Lambda}$  will have the following property: as  $\Lambda$  goes to 0, either it concentrates at a finite number of points, according to the definition of  $(\Omega_{\partial})_K$  (if  $t \neq 1$ ), or its projection  $\Pi_I$  will be large (if  $t \neq 0$ ).

#### Proposition 3.8.

Let  $(\Omega_{\partial})_{K,I}$  be defined by (3.1) and let  $\Phi^{\Lambda}: (\Omega_{\partial})_{K,I} \to \overline{H}^{1}(\Omega)$  be defined, for  $L \gg 0$ , in the following way:

$$\zeta = (\mu, \varsigma, t) = \left(\sum_{k} t_{k} \delta_{x_{k}}, (\varsigma_{1}, \dots, \varsigma_{I}), t\right) \qquad \longmapsto \qquad \Phi^{\Lambda}(\zeta) := \phi^{\Lambda(1-t)} - \oint_{\Omega} \phi^{\Lambda(1-t)} + \psi^{\Lambda t}$$

$$\phi^{\Lambda(1-t)} = \phi^{\Lambda(1-t)}(\mu) \qquad := \qquad \log \sum_{k} \frac{t_{k}}{(1 + (\Lambda(1-t))^{2}|\cdot -x_{k}|^{2})^{2}}$$

$$\psi^{\Lambda t} = \psi^{\Lambda t}(\varsigma) \qquad := \qquad \sqrt{\log^{+}(\Lambda t)} \sum_{i=1}^{I} \varsigma_{i} \varphi_{i}$$

If  $\rho > 4K\pi$  and  $\beta < -\lambda_I$ , then  $\mathcal{J}_{\beta,\rho}\left(\Phi^{\Lambda}(\zeta)\right) \underset{\Lambda \to +\infty}{\to} -\infty$  independently on  $\zeta$ .

To prove this result, we will estimate separately the three parts defining  $\mathcal{J}_{\beta,\rho}$ : the Dirichlet integral, the  $L^2$  norm and the nonlinear term. Each estimate is contained in a separate lemma.

#### Lemma 3.9.

Let  $\Phi^{\Lambda}$  be as in Proposition 3.8.

Then,

$$\int_{\Omega} \left| \nabla \Phi^{\Lambda}(\zeta) \right|^{2} \leq 16K\pi \log^{+}(\Lambda(1-t)) + \lambda_{I} \log^{+}(\Lambda t) + C\sqrt{\log^{+}(\Lambda t)}.$$

Proof.

Since, by definition, we have

$$\left|\nabla\Phi^{\Lambda}(\zeta)\right|^{2} = \left|\nabla\phi^{\Lambda(1-t)}\right|^{2} + 2\nabla\phi^{\Lambda(1-t)}\cdot\nabla\psi^{\Lambda t} + \left|\nabla\psi^{\Lambda t}\right|^{2},$$

we will suffice to show the following estimates:

$$\int_{\Omega} \left| \nabla \phi^{\Lambda(1-t)} \right|^2 \leq 16K\pi \log^+(\Lambda(1-t)) + C; \tag{3.2}$$

$$\int_{\Omega} \nabla \phi^{\Lambda(1-t)} \cdot \nabla \psi^{\Lambda t} \leq C \sqrt{\log^{+}(\Lambda t)}; \tag{3.3}$$

$$\int_{\Omega} \left| \nabla \psi^{\Lambda t} \right|^2 \leq \lambda_I \log^+(\Lambda t). \tag{3.4}$$

The first estimate can be obtained similarly as [24] (Proposition 4.2), the main difference being that we have to take care of the points  $x_k$  lying on  $\partial\Omega$ . The estimate is trivial if  $\Lambda(1-t)$  is bounded from above, otherwise we get:

$$\begin{split} \left| \nabla \phi^{\Lambda(1-t)} \right| &= \left| \frac{\sum_{k} \frac{-4t_{k}(\Lambda(1-t))^{2}(\cdot - x_{k})}{(1+(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2})^{3}}}{\sum_{k} \frac{t_{k}}{(1+(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2})^{2}}} \right| \\ &\leq \frac{\sum_{k} \frac{4t_{k}(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2}}{(1+(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2})^{3}}}{\sum_{k} \frac{t_{k}}{(1+(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2})}} \\ &\leq \max_{k} \frac{4(\Lambda(1-t))^{2}|\cdot - x_{k}|}{1+(\Lambda(1-t))^{2}|\cdot - x_{k}|^{2}} \\ &\leq \min\left\{4\Lambda(1-t), \frac{4}{\min_{k}|\cdot - x_{k}|}\right\}. \end{split}$$

Now, we divide  $\Omega$  in some regions  $\Omega_k$  depending on which point  $x_k$  is the closest:

$$\Omega_k := \left\{ x \in \Omega : |x - x_k| = \min_{k'} |x - x_{k'}| \right\};$$

therefore, we get

$$\begin{split} \int_{\Omega} \left| \nabla \phi^{\Lambda(1-t)} \right|^2 & \leq & \sum_{k} \int_{\Omega_{k}} \left| \nabla \phi^{\Lambda(1-t)} \right|^2 \\ & \leq & \sum_{k} \int_{\Omega_{k} \backslash B_{\frac{1}{\Lambda(1-t)}}} \frac{16}{|\cdot -x_{k}|^2} + \sum_{k} \int_{B_{\frac{1}{\Lambda(1-t)}}} 16(\Lambda(1-t))^2 \\ & \leq & 16 \sum_{k} \int_{\Omega_{k} \backslash B_{\frac{1}{\Lambda(1-t)}}} \frac{1}{|\cdot -x_{k}|^2} + C. \end{split}$$

To evaluate the last integral we distinguish the cases  $x_k \in \Omega$  and  $x_k \in \partial \Omega$ . In the former, we are basically integrating the function  $\frac{1}{|\cdot|^2}$  on an annulus whose internal radius  $\frac{1}{\Lambda(1-t)}$  is shrinking, plus negligible terms, hence its asymptotical value will be  $2\pi \log(\Lambda(1-t))$ . On the other hand, if  $x_k \in \partial \Omega$ , then we are actually integrating on a domain asymptotically resembling a half-annulus

with its internal radius shrinking, therefore we only get half of before, namely  $\pi \log(\Lambda(1-t))$ . Following these considerations, we get (3.2):

$$\left|\nabla\phi^{\Lambda(1-t)}\right| \leq 16\left(\sum_{x_k\in\Omega} (2\pi\log(\Lambda(1-t)) + C) + \sum_{x_k\in\partial\Omega} \pi(\log(\Lambda(1-t)) + C)\right) + C$$

$$\leq 16K\log(\Lambda(1-t)) + C.$$

Concerning (3.3), by the construction of  $\psi^{\Lambda}$  we get

$$\int_{\Omega} \nabla \phi^{\Lambda(1-t)} \cdot \nabla \psi^{\Lambda t} = \sqrt{\log^{+}(\Lambda t)} \sum_{i=1}^{I} s_{i} \int_{\Omega} \nabla \phi^{\Lambda(1-t)} \cdot \nabla \varphi_{i}$$

$$= \sqrt{\log^{+}(\Lambda t)} \sum_{i=1}^{I} s_{i} \lambda_{i} \int_{\Omega} \left( \phi^{\Lambda(1-t)} - \int_{\Omega} \phi^{\Lambda(1-t)} \right) \varphi_{i}$$

$$\leq C \sqrt{\log^{+}(\Lambda t)} \sum_{i=1}^{I} s_{i} \lambda_{i} \sqrt{\int_{\Omega} \left( \phi^{\Lambda(1-t)} - \int_{\Omega} \phi^{\Lambda(1-t)} \right)^{2}} \sqrt{\int_{\Omega} \varphi_{i}^{2}}$$

$$\leq C \sqrt{\log^{+}(\Lambda t)} \sqrt{\int_{\Omega} \left( \phi^{\Lambda(1-t)} - \int_{\Omega} \phi^{\Lambda(1-t)} \right)^{2}},$$

therefore we suffice to show that the last integral is uniformly bounded. To this purpose, we first estimate the average of  $\phi^{\Lambda(1-t)}$ :

$$\int_{\Omega} \phi^{\Lambda(1-t)} = \int_{\Omega} \log \frac{1}{(1 + (\Lambda(1-t))^2 \min_k |\cdot -x_k|^2)^2} + O(1)$$

$$= \int_{\Omega \setminus \bigcup_k B_{\frac{1}{\Lambda(1-t)}}} \log \frac{1}{(\Lambda(1-t)\min_k |\cdot -x_k|)^4} + O(1)$$

$$= -4 \log^+(\Lambda(1-t)) + O(1). \tag{3.5}$$

Now, (3.3) will follow by estimating  $\phi^{\Lambda(1-t)} + 4\log^+(\Lambda(1-t))$  in  $L^2(\Omega)$ , which can be done similarly as before:

$$\int_{\Omega} \left( \phi^{\Lambda(1-t)} + 4 \log^{+}(\Lambda(1-t)) \right)^{2} = \int_{\Omega} \left( \log \sum_{k} \frac{t_{k} \max\{1, \Lambda(1-t)\}}{(1 + (\Lambda(1-t))^{2} | \cdot -x_{k}|^{2})^{2}} \right)^{2}$$

$$\int_{\Omega} \log \left( \sum_{k} \frac{t_{k}}{| \cdot -x_{k}|^{4}} \right)^{2}$$

$$\leq \int_{\Omega} \log \frac{1}{\min_{k} | \cdot -x_{k}|^{8}}$$

$$\leq C. \tag{3.6}$$

Finally, (3.4) follows easily by the properties of the  $\phi_i$ 's:

$$\int_{\Omega} |\nabla \psi^{\Lambda t}|^{2} = \log^{+}(\Lambda t) \int_{\Omega} \left| \sum_{i=1}^{I} \varsigma_{i} \nabla \varphi_{i} \right|^{2}$$

$$= \log^{+}(\Lambda t) \sum_{i=1}^{I} \varsigma_{i}^{2} \int_{\Omega} |\nabla \varphi_{i}|^{2}$$

$$= \log^{+}(\Lambda t) \sum_{i=1}^{I} \varsigma_{i}^{2} \lambda_{i}$$

$$\leq \log^{+}(\Lambda t) \lambda_{I}.$$

#### Lemma 3.10.

Let  $\Phi^{\Lambda}$  be as in Proposition 3.8.

$$\int_{\Omega} \Phi^{\Lambda}(\zeta)^{2} \leq \log^{+}(\Lambda t) + C\sqrt{\log^{+}(\Lambda t)}.$$

Proof.

By expanding the square of the sum and using (3.5), (3.6), we have

$$\begin{split} \int_{\Omega} \Phi^{\Lambda}(\zeta)^2 &= \int_{\Omega} \left( \phi^{\Lambda(1-t)} - \oint_{\Omega} \phi^{\Lambda(1-t)} \right)^2 + 2 \int_{\Omega} \left( \phi^{\Lambda(1-t)} - \oint_{\Omega} \phi^{\Lambda(1-t)} \right) \psi^{\Lambda t} + \int_{\Omega} \left( \psi^{\Lambda t} \right)^2 \\ &\leq \int_{\Omega} \left( \phi^{\Lambda(1-t)} - \oint_{\Omega} \phi^{\Lambda(1-t)} \right)^2 + 2 \sqrt{\int_{\Omega} \left( \phi^{\Lambda(1-t)} - \oint_{\Omega} \phi^{\Lambda(1-t)} \right)^2} \sqrt{\int_{\Omega} \left( \psi^{\Lambda t} \right)^2} + \int_{\Omega} \left( \psi^{\Lambda t} \right)^2 \\ &\leq C \left( 1 + \sqrt{\int_{\Omega} \left( \psi^{\Lambda t} \right)^2} \right) + \int_{\Omega} \left( \psi^{\Lambda t} \right)^2. \end{split}$$

Therefore, we only need a suitable estimate for  $\psi^{\Lambda t}$ , which in turn follows from the very definition of the  $\varphi_i$ 's:

$$\int_{\Omega} (\psi^{\Lambda t})^2 = \log^+(\Lambda t) \int_{\Omega} \left( \sum_{i=1}^{I} \varsigma_i \varphi_i \right)^2 = \log^+(\Lambda t) \sum_{i=1}^{I} \varsigma_i^2 \int_{\Omega} \varphi_i^2 = \log^+(\Lambda t) \sum_{i=1}^{I} \varsigma_i^2 \le \log^+(\Lambda t).$$

By combining the two estimates the proof is complete.

### Lemma 3.11.

Let  $\Phi^{\Lambda}$  be as in Proposition 3.8. Then,

$$\log \int_{\Omega} e^{\Phi^{\Lambda}(\zeta)} \ge 2\log^{+}(\Lambda(1-t)) - C\sqrt{\log^{+}(\Lambda t)}.$$

Proof.

We first notice that, since  $\psi^{\Lambda t}$  belongs to a finite-dimensional space, all of its norms are equivalent, and in particular  $L^2$  and  $L^{\infty}$ , therefore by Lemma (3.10),

$$\left|\psi^{\Lambda t}\right| \le \left\|\psi^{\Lambda t}\right\|_{L^{\infty}(\Omega)} \le C \left\|\psi^{\Lambda t}\right\|_{L^{2}(\Omega)} \le C \sqrt{\log^{+}(\Lambda t)}.$$

Moreover, in view of the asymptotical behavior (3.5) of the average of  $\phi_{\lambda(1-t)}$ , we are reduce to show that  $\log \int_{\Omega} e^{\phi^{\Lambda(1-t)}} \ge -2\log^+(\Lambda(1-t)) - C$ ; this follows from the simple calculations:

$$\int_{\Omega} e^{\phi^{\Lambda(1-t)}} = \sum_{k} t_{k} \int_{\Omega} \frac{1}{(1+(\Lambda(1-t))^{2}|\cdot -x_{k}|^{2})^{2}} \\
\geq \sum_{k} t_{k} \int_{B_{\frac{1}{\Lambda(1-t)}}(x_{k})} \frac{1}{(1+(\Lambda(1-t))^{2}|\cdot -x_{k}|^{2})^{2}} \\
\geq \sum_{k} t_{k} \int_{B_{\frac{1}{\Lambda(1-t)}}(x_{k})} \frac{1}{2}$$

$$\geq \frac{C}{\max\{1,\Lambda(1-t)\}^2}.$$

By putting together these three lemmas, Proposition (3.8) may be proved easily.

Proof of Proposition 3.8.

By Lemmas 3.9, 3.10, 3.11, we get

$$\mathcal{J}_{\beta,\rho}\left(\Phi^{\Lambda}(\zeta)\right) = \frac{1}{2} \int_{\Omega} \left|\nabla\Phi^{\Lambda}(\zeta)\right|^{2} + \frac{\beta}{2} \int_{\Omega} \Phi^{\Lambda}(\zeta)^{2} - \rho \log \int_{\Omega} e^{\Phi^{\Lambda}(\zeta)}$$

$$\leq \left(8K\pi - 2\rho\right) \log^{+}(\Lambda(1-t)) + \frac{\lambda_{I} + \beta}{2} \log^{+}(\Lambda t) + C\sqrt{\log^{+}(\Lambda t)}$$

$$\leq -\min\left\{2\rho - 8K\pi, -\frac{\lambda_{I} + \beta}{2}\right\} \max\left\{\log^{+}(\Lambda(1-t)), \log^{+}(\Lambda t)\right\} + C\sqrt{\log^{+}(\Lambda t)}$$

$$\leq -\min\left\{2\rho - 8K\pi, -\frac{\lambda_{I} + \beta}{2}\right\} \log \frac{\Lambda}{2} + C\sqrt{\log \Lambda}$$

$$\xrightarrow{\Lambda \to +\infty} -\infty,$$

uniformly on  $\zeta \in (\Omega_{\partial})_{K,I}$ .

# The map $\Psi: \mathcal{J}_{\beta,\rho}^{-L} \to (\Omega_{\partial})_{K,I}$

We will now show the existence of "counterpart" to the map  $\Phi$  defined in the previous subsection. Precisely, we will build a map  $\Psi$  from a low sub-level  $\mathcal{J}_{\beta,\rho}^{-L}$  to  $(\Omega_{\partial})_{K,I}$  which is somehow compatible with  $\Phi$ , in the sense that their composition is homotopically equivalent to the identity on  $(\Omega_{\partial})_{K,I}$ .

#### Proposition 3.12.

Let  $(\Omega_{\partial})_{K,I}$  be defined by (3.1). If  $\rho < 4(K+1)\pi$  and  $\beta > -\lambda_{I+1}$ , then there exists  $L \gg 0$  and a map  $\Psi : \mathcal{J}_{\beta,\rho}^{-L} \to (\Omega_{\partial})_{K,I}$ .

Moreover, if  $\rho > 4K\pi$  and  $\beta < -\lambda_I$ , then there exists a map  $\Phi : (\Omega_{\partial})_{K,I} \xrightarrow{\Gamma} \mathcal{J}_{\beta,\rho}^{-L}$  such that the composition  $\Psi \circ \Phi$  is homotopically equivalent to the identity on  $(\Omega_{\partial})_{K,I}$ .

The existence of such maps  $\Phi$  and  $\Psi$  easily gives, via the functorial properties of homology, the following information on the homology groups of energy sublevels.

#### Corollary 3.13.

Under the assumptions of Propositions 3.8 and 3.12, the homology groups of the sublevel  $\mathcal{J}_{\beta,\rho}^{-L}$  satisfy

$$\mathbb{Z}^{\binom{K+g}{g}} \hookrightarrow \widetilde{H}_{2K+I-1}\left(\mathcal{J}_{\beta,\rho}^{-L}\right).$$

The main tool to prove Proposition 3.12 is a so-called *improved* Moser-Trudinger inequality. Roughly speaking, such inequalities state that, under some *spreading* conditions on u, the best constant in the classical Moser-Trudinger inequality can be improved.

We recall here the well-known Moser-Trudinger inequalities, in two forms depending whether we consider only compactly supported function or also function which may touch the boundary. We stress that, in the two cases, the constant multiplying the Dirichlet integral is different.

**Proposition 3.14.** ([27], Theorem 2, [13], Corollary 2.5)

There exists C > 0 such that for any  $u \in \overline{H}^1(\Omega)$ 

$$\log \int_{\Omega} e^u \le \frac{1}{8\pi} \int_{\Omega} |\nabla u|^2 + C. \tag{3.7}$$

If instead  $u \in H_0^1(\Omega)$ , then

$$\log \int_{\Omega} e^{u} \le \frac{1}{16\pi} \int_{\Omega} |\nabla u|^{2} + C. \tag{3.8}$$

We will now prove the improved Moser-Trudinger inequality, a classical result in variational Liouvilletype problems (see [17, 24, 12, 6, 4, 5]).

Basically, if u is somehow spread in some regions, then the constant  $8\pi$  in (3.7) can almost be multiplied by an integer number. This time, the integer will depend not only on the number of regions but also on how many of them touch the boundary; moreover, we need to take account of the negative projection  $\Pi_I$ .

To prove such a result, we will take cutoff functions on the regions where u is spread and apply to each cutoff either (3.7) or (3.8). We will also use some splitting in Fourier modes, which we need to deal with  $\Pi_I$ .

#### Lemma 3.15.

Let  $\delta > 0$ ,  $\{\Omega_{1k}\}_{k=1}^l$ ,  $\{\Omega_{2k'}\}_{k'=1}^m$  and  $u \in \overline{H}^1(\Omega)$  satisfying

$$d(\Omega_{ik}, \Omega_{i'k'}) \ge 2\delta \quad \forall (i, k) \ne (i', k'); \qquad d(\Omega_{1k}, \partial\Omega) \ge \delta \quad \forall k = 1, \dots, l;$$

$$\frac{\int_{\Omega_{ik}} e^u}{\int_{\Omega} e^u} \ge \delta \quad \forall i, k; \qquad \|\Pi_I u\| \le 1.$$

Then, for any  $\varepsilon > 0$  there exists  $C = C(\varepsilon, \delta, \beta, I, l, m) > 0$  such that

$$\log \int_{\Omega} e^{u} \le \frac{1+\varepsilon}{8\pi(2l+m)} \int_{\Omega} \left( |\nabla u|^{2} + \beta u^{2} \right) + C.$$

Proof.

First of all, for any i, k we take cutoff functions  $\eta_{ik} \in \text{Lip}(\overline{\Omega})$  satisfying

$$0 \le \eta_{ik} \le 1$$
 in  $\overline{\Omega}_{ik}$   $\eta_{ik}|_{\Omega_{ik}} \equiv 1$   $\operatorname{spt}(\eta_{ik}) \subset B_{\delta}(\Omega_{ik})$   $|\nabla \eta_{ik}| \le \frac{1}{\delta}$  in  $B_{\delta}(\Omega_{ik}) \setminus \Omega_{ik}$ .

Then, we split  $u = u_1 + u_2 + u_3$  via truncation in Fourier modes:

$$u_1 = \sum_{i=1}^{I} \left( \int_{\Omega} u \varphi_i \right) \varphi_i \qquad \qquad u_2 = \sum_{i=I+1}^{N_{\varepsilon}-1} \left( \int_{\Omega} u \varphi_i \right) \varphi_i \qquad \qquad u_3 = \sum_{i=N_{\varepsilon}}^{+\infty} \left( \int_{\Omega} u \varphi_i \right) \varphi_i,$$

with  $N_{\varepsilon}$  so large that

$$\frac{1}{\lambda_N} \left( 1 + \frac{1}{\varepsilon} \right) \frac{1}{\delta^2} \le \varepsilon \qquad \lambda_{N_{\varepsilon}} \le (1 + \varepsilon)(\lambda_{N_{\varepsilon}} + \beta). \tag{3.9}$$

By applying (3.7) to each  $\eta_{1k}u - \overline{\eta_{1k}u}$  we get:

$$\log \int_{\Omega} e^{u} \leq \log \int_{\Omega_{1k}} e^{u} + \log \frac{1}{\delta} 
\leq \log \int_{\Omega} e^{\eta_{1k}u} + \log \frac{1}{\delta} 
\leq \|\eta_{1k}u_{1}\|_{L^{\infty}(\Omega)} + \|\eta_{1k}u_{2}\|_{L^{\infty}(\Omega)} + \log \int_{\Omega} e^{\eta_{1k}u_{3}} + \log \frac{1}{\delta} 
\leq \|u_{1}\|_{L^{\infty}(\Omega)} + \|u_{2}\|_{L^{\infty}(\Omega)} + \overline{\eta_{1k}u_{3}} + \frac{1}{8\pi} \int_{\Omega} |\nabla(\eta_{2k}u_{3})|^{2} + C.$$

Similarly, since  $\eta_{2k}u \in H_0^1(\Omega)$ , we can apply (3.8):

$$\log \int_{\Omega} e^{u} \leq \log \int_{\Omega_{2k}} e^{u} + \log \frac{1}{\delta}$$

$$\leq \log \int_{\Omega} e^{\eta_{2k}u} + \log \frac{1}{\delta}$$

$$\leq \|\eta_{2k}u_1\|_{L^{\infty}(\Omega)} + \|\eta_{2k}u_2\|_{L^{\infty}(\Omega)} + \log \int_{\Omega} e^{\eta_{2k}u_3} + \log \frac{1}{\delta}$$

$$\leq \|u_1\|_{L^{\infty}(\Omega)} + \|u_2\|_{L^{\infty}(\Omega)} + \frac{1}{16\pi} \int_{\Omega} |\nabla(\eta_{1k}u_3)|^2 + C.$$

The terms involving  $u_1$  and  $u_2$  can be estimated because, since each belongs to a finite-dimensional space, all of its norms are equivalent on the respective space. We can use the  $L^2$  norm for  $u_1$ , which is uniformly bounded by hypotheses:

$$||u_1||_{L^{\infty}(\Omega)} \le C||u_1||_{L^2(\Omega)} = C||\Pi_I u|| \le C.$$

As for  $u_2$ , since we got rid of low Fourier coefficients, we can choose as a norm  $\sqrt{\int_{\Omega} (|\nabla u_2|^2 + \beta u_2^2)}$ ; since we took an orthogonal decomposition, we get

$$||u_{2}||_{L^{\infty}(\Omega)} \leq C\sqrt{\int_{\Omega} (|\nabla u_{2}|^{2} + \beta u_{2}^{2})}$$

$$\leq \varepsilon \int_{\Omega} (|\nabla u_{2}|^{2} + \beta u_{2}^{2}) + C$$

$$\leq \varepsilon \int_{\Omega} (|\nabla u_{2}|^{2} + \beta u_{2}^{2}) + \varepsilon \int_{\Omega} (|\nabla u_{3}|^{2} + \beta u_{3}^{2}) + \varepsilon \int_{\Omega} |\nabla u_{1}|^{2} + C$$

$$= \varepsilon \int_{\Omega} (|\nabla u|^{2} + \beta u^{2}) - \varepsilon \beta ||\Pi_{I}u||^{2} + C$$

$$\leq \varepsilon \int_{\Omega} (|\nabla u|^{2} + \beta u^{2}) + C.$$

We then estimate the average of  $\eta_{1k}u_3$  via Poincaré-Wirtinger inequality:

$$|\overline{\eta_{1k}u_3}| \le ||u_3||_{L^1(\Omega)} \le ||\nabla u_3||_{L^2(\Omega)} \le \varepsilon \int_{\Omega} |\nabla u_3|^2 + C.$$

Concerning the last term, we expand the square and use the properties of the  $\eta_{ik}$ 's:

$$\int_{\Omega} |\nabla(\eta_{ik}u_3)|^2 = \int_{\Omega} |\eta_{ik}\nabla u_3 + u_3\nabla\eta_{ik}|^2 
\leq (1+\varepsilon)\int_{\Omega} \eta_{ik}^2 |\nabla u_3|^2 + \left(1+\frac{1}{\varepsilon}\right)\int_{\Omega} u_3^2 |\nabla\eta_{ik}|^2 
\leq (1+\varepsilon)\int_{B_{\delta}(\Omega_{ik})} |\nabla u_3|^2 + \left(1+\frac{1}{\varepsilon}\right)\frac{1}{\delta^2}\int_{B_{\delta}(\Omega_{ik})} u_3^2.$$

Since by hypothesis  $B_{\delta}(x_{ik}) \cap B_{\delta}(x_{i'k'}) = \emptyset$  for  $(i,k) \neq (i',k')$ , then putting together all these estimates and summing on i = 1, 2 and all k's we get

$$(2l+m)\int_{\Omega} e^{u} \leq (2l+m)\varepsilon \int_{\Omega} \left( |\nabla u|^{2} + \beta u^{2} \right) + m\varepsilon \int_{\Omega} |\nabla u_{3}|^{2} + \frac{1+\varepsilon}{8\pi} \int_{\Omega} |\nabla u_{3}|^{2} + \frac{1}{8\pi} \left( 1 + \frac{1}{\varepsilon} \right) \frac{1}{\delta^{2}} \int_{\Omega} u_{3}^{2} + C.$$

At this point, we need the conditions (3.9) defining  $u_3$ : the former gives

$$\left(1 + \frac{1}{\varepsilon}\right) \frac{1}{\delta^2} \int_{\Omega} u_3^2 \le \frac{1}{\lambda_{N_{\varepsilon}}} \left(1 + \frac{1}{\varepsilon}\right) \frac{1}{\delta^2} \int_{\Omega} |\nabla u_3|^2 \le \varepsilon \int_{\Omega} |\nabla u_3|^2.$$

on the other hand, the latter implies

$$\int_{\Omega} |\nabla u_3|^2 = \sum_{j=N_{\varepsilon}}^{+\infty} \lambda_i \left( \int_{\Omega} u \varphi_i \right)^2$$

$$\leq (1+\varepsilon) \sum_{j=N_{\varepsilon}}^{+\infty} (\lambda_{i} + \beta) \left( \int_{\Omega} u \varphi_{i} \right)^{2}$$

$$= (1+\varepsilon) \int_{\Omega} (|\nabla u_{3}|^{2} + \beta u_{3}^{2})$$

$$\leq (1+\varepsilon) \int_{\Omega} (|\nabla u_{3}|^{2} + \beta u_{3}^{2}) + (1+\varepsilon) \int_{\Omega} |\nabla u_{1}|^{2} + (1+\varepsilon) \int_{\Omega} (|\nabla u_{2}|^{2} + \beta u_{2}^{2})$$

$$\leq (1+\varepsilon) \int_{\Omega} (|\nabla u|^{2} + \beta u^{2}) + C.$$

Therefore, we get

$$(2l+m)\int_{\Omega} e^{u} \leq (2l+m)\varepsilon \int_{\Omega} \left( |\nabla u|^{2} + \beta u^{2} \right) + \left( \frac{1+\varepsilon}{8\pi} + \frac{\varepsilon}{8\pi} + m\varepsilon \right) \int_{\Omega} |\nabla u_{3}|^{2} + C$$

$$\leq \left( (2l+m)\varepsilon + (1+\varepsilon) \left( \frac{1+\varepsilon}{8\pi} + \frac{\varepsilon}{8\pi} + m\varepsilon \right) \right) \int_{\Omega} \left( |\nabla u|^{2} + \beta u^{2} \right) + C,$$

which, up to re-labeling  $\varepsilon$ , concludes the proof.

We need a few more technical steps to prove Proposition 3.12.

First of all, we need a covering lemma basically saying that, if concentration does not occur, then one has spreading in the sense of Lemma 3.15.

#### Lemma 3.16.

Let  $f \in L^1(\Omega)$  be non-negative a.e., satisfying  $\int_{\Omega} f = 1$  and such that, for any  $\varepsilon > 0$  and  $x_{11}, \ldots, x_{1l} \in \Omega, x_{21}, \ldots, x_{2m} \in \partial \Omega$  with  $2l + m \leq K$  for some  $K \in \mathbb{N}$ ,

$$\int_{\bigcup_{i,k} B_{\varepsilon}(x_{ik})} f < 1 - \varepsilon.$$

Then, there exist  $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\varepsilon,\Omega), \widetilde{r} = \widetilde{r}(\varepsilon,\Omega) > 0$  and  $\widetilde{x}_{11},\ldots,\widetilde{x}_{1\widetilde{l}},\widetilde{x}_{21},\ldots,\widetilde{x}_{2\widetilde{m}}$  satisfying

$$2\widetilde{l} + \widetilde{m} \ge K + 1; \qquad d(\widetilde{x}_{1k}, \partial\Omega) \ge \widetilde{r} \quad \forall k = 1, \dots, \widetilde{l}$$
$$|\widetilde{x}_{ik} - \widetilde{x}_{i'k'}| \ge 4\widetilde{r} \quad \forall (i, k) \ne (i', k') \qquad \int_{B_{\widetilde{r}}(\widetilde{x}_{ik})} f \ge \widetilde{\varepsilon} \quad \forall i, k$$

Proof.

We will mostly argue as in [17] (Lemma 2.3) and [24] (Lemma 3.3), with minor modifications. Fix  $\widetilde{r} := \frac{\varepsilon}{6}$  and take the finite cover of  $\overline{\Omega}$  given by  $\{B_{\widetilde{\varepsilon}}(y_n)\}_{n=1}^N$  for some  $L = L_{\widetilde{r},\overline{\Omega}}$ , then set  $\widetilde{\varepsilon} := \frac{\varepsilon}{L}$ 

One easily sees that there exists some n such that  $\int_{B_{\widetilde{\varepsilon}}(y_n)} f \geq \widetilde{\varepsilon}$ ; up to re-labeling, we can assume

that this hold true if and only if  $n \leq N'$ , for some  $N' \leq N$ . Now, we choose recursively the points  $\{\widetilde{y}_j\} \subset \{y_n\}_{n=1}^{N'}$ : we set  $\widetilde{y}_1 := y_1$  and

$$\Omega_1 := \left\{ \bigcup_{n=1}^{N'} B_{\widetilde{r}}(y_n) : |y_n - \widetilde{y}_1| < 4\widetilde{r} \right\} \subset B_{5\widetilde{r}}(\widetilde{y}_1).$$

If there is some  $l_0$  such that  $|y_{n_0} - \widetilde{y}_1| \ge 4\widetilde{r}$ , then we set  $\widetilde{y}_2 = y_{n_0}$  and

$$\Omega_2 := \left\{ \bigcup_{n=2}^{N'} B_{\widetilde{r}}(y_n) : |y_n - \widetilde{y}_2| < 4\widetilde{r} \right\} \subset B_{5\widetilde{r}}(\widetilde{y}_2).$$

Inductively, we find a finite number of points  $\tilde{y}_j$  and closed set  $\Omega_j$ .

Among the  $\widetilde{y}_j$ 's, some of them will be at a distance less than  $\delta$  from  $\partial\Omega$ ; we denote the number

of such points as  $\widetilde{m}$  and the number of the other  $\widetilde{y}_j$ 's as  $\widetilde{l}$ , then we denote the former points as  $\widetilde{x}_{1k}$  and the latter as  $\widetilde{x}_{2k'}$ , so that  $\{\widetilde{y}_j\}_j = \{\widetilde{x}_{11}, \dots, \widetilde{x}_{1\tilde{l}}, \widetilde{x}_{21}, \dots, \widetilde{x}_{2\tilde{m}}\}$  and we call  $\Omega_{ik}$  the set  $\Omega_j$ corresponding to the point  $\tilde{x}_{ik}$ . To complete the proof, we only need to show that  $2\tilde{l} + \tilde{m} > K$ , since we already verified that the other required properties are satisfied.

Assume by contradiction that  $2l + \widetilde{m} \leq K$ , set  $x_{1k} := \widetilde{x}_{1k}$  for  $k = 1, \dots, l$  and take, for  $k = 1, \dots, \widetilde{m}$ , some  $x_{2k'} \in \partial \Omega$  such that  $d(x_{2k'}, \tilde{x}_{2k}) \leq \tilde{r}$ . Then, by hypothesis,

$$\int_{\Omega \setminus \bigcup_{i \mid k} B_{\varepsilon}(x_{ik})} f \ge \varepsilon.$$

However, due to our construction,

$$\bigcup_{n=1}^{N'} B_{\widetilde{r}}(y_n) \subset \bigcup_{i,k} \Omega_{ik} \subset \bigcup_{i,k} B_{5\widetilde{r}}(\widetilde{x}_{ik}) \subset \bigcup_{i,k} B_{\varepsilon}(x_{ik}),$$

which leads to a contradiction:

$$\int_{\Omega \setminus \bigcup_{i,k} B_{\varepsilon}(x_{ik})} f \le \int_{\Omega \setminus \bigcup_{n=1}^{N'} B_{\widetilde{r}}(y_n)} f \le \int_{\bigcup_{n=1}^{N'} B_{\widetilde{r}}(y_n)} f \le (N - N')\widetilde{\varepsilon} < \varepsilon$$

Now we see that either concentration at a finite number of points or large  $\|\Pi_I\|$  must occur in very low sublevels. In fact, if it does not, then by the previous lemma one has spreading and small  $\Pi_I$ , hence by the improved Moser-Trudinger inequality the energy  $\mathcal{J}_{\beta,\rho}$  cannot be too low. This is explained in details by the following lemma.

#### Lemma 3.17.

For any  $\varepsilon > 0$  there exists  $L = L(\varepsilon) > 0$  such that, if  $J_{\beta,\rho}(u) \leq -L$  and  $\|\Pi_I u\| > 1$ , then there exists  $x_{11}, \ldots, x_{1l} \in \Omega, x_{21}, \ldots, x_{2m} \in \partial \Omega$  with  $2l + m \leq K$  such that

$$\frac{\int_{\bigcup_{i,k} B_{\varepsilon}(x_{ik})} e^{u}}{\int_{\Omega} e^{u}} \ge 1 - \varepsilon.$$

Assume that the statement is not true. Then, there exists  $\varepsilon > 0$  and  $(u_n)_{n \in \mathbb{N}}$  such that  $\|\Pi_I u_n\| \le 1$ ,  $J_{\beta,\rho}(u_n) \underset{n\to+\infty}{\to} -\infty$  and

$$\frac{\int_{\bigcup_{i,k} B_{\varepsilon}(x_{ik})} e^{u_n}}{\int_{\Omega} e^{u_n}} < 1 - \varepsilon.$$

for any  $x_{11}, \ldots, x_{1l} \in \Omega$ ,  $x_{21}, \ldots, x_{2m} \in \partial \Omega$  satisfying  $2l + m \leq K$ . We apply lemma 3.16 to  $f = \frac{e^{u_n}}{\int_{\Omega} e^{u_n}}$  and we find  $\widetilde{\varepsilon}, \widetilde{r}$ , not depending on n, and  $\widetilde{x}_{11}, \ldots, \widetilde{x}_{1\widetilde{l}}, \widetilde{x}_{21}, \ldots, \widetilde{x}_{2\widetilde{m}}$ as in the lemma.

One can easily see that Lemma 3.15 can be applied to  $\Omega_{ik} = B_{\widetilde{r}}(\widetilde{x}_{ik})$  with  $\delta = \min\{\widetilde{\varepsilon}, \widetilde{r}\}$  and  $\varepsilon' = \frac{4\pi}{g} \left( 2\widetilde{l} + \widetilde{m} \right) - 1$ . This leads to the following contradiction:

$$-\infty \underset{n \to +\infty}{\longleftarrow} J_{\beta,\rho}(u_n) = \frac{4\pi \left(2\widetilde{l} + \widetilde{m}\right)}{1 + \varepsilon'} \left(\frac{1 + \varepsilon'}{8\pi \left(2\widetilde{l} + \widetilde{m}\right)} \int_{\Omega} \left(|\nabla u_n|^2 + \beta u_n^2\right) - \rho \log \int_{\Omega} e^{u_n}\right) \ge -C$$

Since  $\frac{e^u}{\int_{\Omega} e^u}$  tends to concentrates in very low sublevels, provided  $\Pi_I$  is not too large, then it will be very close to an element of  $(\Omega_{\partial})_K$ . This will be essential to later use the retraction  $\widetilde{\Psi}$  defined in Lemma 3.1.

## Lemma 3.18.

For any  $\varepsilon > 0$  there exists  $L = L(\varepsilon) > 0$  such that any  $u \in \mathcal{J}_{\beta,\rho}^{-L}$  satisfies either of the following condition:

$$d_{\operatorname{Lip}'(\overline{\Omega})}\left(\frac{e^u}{\int_{\Omega} e^u}, (\Omega_{\partial})_K\right) \le \varepsilon \qquad or \qquad \|\Pi_I u\| > 1.$$

Proof.

Fix  $\varepsilon > 0$ , apply Lemma 3.17 with  $\frac{\varepsilon}{3}$  and take  $L = L\left(\frac{\varepsilon}{3}\right)$  as in the lemma. For any  $u \in \mathcal{J}_{\beta,\rho}^{-L}$  satisfying  $\|\Pi_I u\| \le 1$ , take  $\mu(u) = \sum_i t_{ik} \delta_{x_{ik}}$  with  $x_{ik}$  as in the lemma and  $t_{ik}$  defined by

$$t_k := \int_{B_{\frac{\varepsilon}{3}}(x_{ik}) \setminus \bigcup_{k' \le k-1 \text{ or } i' \le i-1} B_{\frac{\varepsilon}{3}}(x_{i'k'})} f(u) + \frac{1}{2l+m} \int_{\Omega \setminus \bigcup_{i',k'} B_{\frac{\varepsilon}{3}}(x_{i'k'})} f(u) \qquad f(u) = \frac{e^u}{\int_{\Omega} e^u}.$$

To conclude the proof, we suffice to show that

$$\left| \int_{\Omega} (hf(u) - h d\mu(u)) \right| \le \varepsilon ||h||_{Lip(\overline{\Omega})}.$$

We split the integral between the union of the balls of radius  $\frac{\varepsilon}{3}$  and its complement: on the latter,

$$\left| \int_{\Omega \setminus \bigcup_{ik} B_{\frac{\varepsilon}{2}}(x_{ik})} (hf(u) - h d\mu(u)) \right| = \left| \int_{\Omega \setminus \bigcup_{ik} B_{\frac{\varepsilon}{2}}(x_{ik})} hf(u) \right| \le \|h\|_{L^{\infty}(\Omega)} \int_{\Omega \setminus \bigcup_{ik} B_{\frac{\varepsilon}{2}}(x_{ik})} f(u) \le \frac{\varepsilon}{3} \|h\|_{\operatorname{Lip}(\overline{\Omega})}.$$

On the union of balls, we have:

$$\left| \int_{\bigcup_{ik} B_{\frac{\pi}{3}}(x_{ik})} (hf(u) - h d\mu(u)) \right|$$

$$= \left| \int_{\bigcup_{ik} B_{\frac{\pi}{3}}(x_{ik})} hf(u) - \sum_{ik} \left( \int_{B_{\frac{\pi}{3}}(x_{ik}) \setminus \bigcup_{k' \leq k-1 \text{ or } i' \leq i-1} B_{\frac{\pi}{3}}(x_{i'k'})} f(u) + \frac{1}{2l+m} \int_{\Omega \setminus \bigcup_{i',k'} B_{\frac{\pi}{3}}(x_{i'k'})} f(u) \right) h(x_{ik}) \right|$$

$$= \left| \sum_{ik} \left( \int_{B_{\frac{\pi}{3}}(x_{ik}) \setminus \bigcup_{k' \leq k-1 \text{ or } i' \leq i-1} B_{\frac{\pi}{3}}(x_{i'k'})} f(u)(h-h(x_{ik})) - \frac{h(x_{ik})}{2l+m} \int_{\Omega \setminus \bigcup_{i',k'} B_{\frac{\pi}{3}}(x_{i'k'})} f(u) \right) \right|$$

$$\leq \|\nabla h\|_{L^{\infty}(\Omega)} \sum_{ik} \int_{B_{\frac{\pi}{3}}(x_{ik}) \setminus \bigcup_{k' \leq k-1 \text{ or } i' \leq i-1} B_{\frac{\pi}{3}}(x_{i'k'})} f(u)| \cdot -x_{ik}| + \|h\|_{L^{\infty}(\Omega)} \int_{\Omega \setminus \bigcup_{i',k'} B_{\frac{\pi}{3}}(x_{i'k'})} f(u)$$

$$\leq \frac{\varepsilon}{3} \|\nabla h\|_{L^{\infty}(\Omega)} \int_{\bigcup_{i',k'} B_{\frac{\pi}{3}}(x_{i'k'})} f(u) + \frac{\varepsilon}{3} \|h\|_{L^{\infty}(\Omega)}$$

$$\leq \frac{\varepsilon}{3} \|\nabla h\|_{L^{\infty}(\Omega)} + \frac{\varepsilon}{3} \|h\|_{L^{\infty}(\Omega)}$$

$$\leq \frac{\varepsilon}{3} \|h\|_{L^{\infty}(\Omega)} + \frac{\varepsilon}{3} \|h\|_{L^{\infty}(\Omega)}$$

The proof is now complete.

We are now in condition to prove the main result of this subsection.

We will construct  $\Psi: \mathcal{J}_{\beta,\rho}^{-L} \to (\Omega_{\partial})_{K,I}$  in the following way. The element in  $(\Omega_{\partial})_K$  will be given by the retraction  $\widetilde{\Psi}$ , while the element in  $\mathbb{S}^{I-1}$  is just the normalization of  $\Pi_I u \in \mathbb{R}^I$ . The choice of the third parameter in the join will be more delicate, especially in the homotopy, because we need to be sure that everything is well-defined outside the endpoints of the interval.

Proof of Proposition 3.12.

Take  $\varepsilon_0$  as in Lemma 3.1 and  $L = L(\varepsilon_0)$  as in Lemma 3.17. We define the map  $\Psi : \mathcal{J}_{\beta,\rho}^{-L} \to (\Omega_{\partial})_{K,I}$ 

$$\Psi(u) = (\mu(u), \varsigma(u), t(u)) := \left(\widetilde{\Psi}\left(\frac{e^u}{\int_{\Omega} e^u}\right), \frac{\Pi_I u}{\|\Pi_I u\|}, \min\{1, \|\Pi_I u\|\}\right).$$

We need to verify that it is well-posed, namely that  $\mu(u)$  is well-defined if  $t \neq 1$  and  $\varsigma(u)$  is welldefined if  $t \neq 0$ .

Assume  $t \neq 1$ : this means  $\|\Pi_I u\| < 1$  so, since  $\mathcal{J}_{\beta,\rho}(u) \leq -L$ , Lemma 3.18 will give  $d_{\operatorname{Lip}'(\overline{\Omega})}\left(\frac{e^u}{\int_{-e^u}}, (\Omega_{\partial})_K\right) \leq -L$ 

 $\varepsilon_0$ ; hence, Lemma 3.1 ensures that  $\widetilde{\Psi}$  is well-defined, hence  $\mu(u)$  is.

On the other hand, if  $t \neq 0$ , then  $\Pi_I u \neq 0$ , hence one can define  $\frac{\Pi_I u}{\|\Pi_I u\|}$ 

As for second part of the lemma, consider the map  $\Phi := \Phi^{\Lambda_0}$  as defined in Proposition 3.8, with  $\Lambda_0 \gg 1$  so large that  $\Phi^{\Lambda_0}((\Omega_{\partial})_{K,I}) \subset J_{\beta,\rho}^{-L}$ .

To get a homotopical equivalence, we let  $\Lambda$  go to  $+\infty$ . One immediately sees that  $\frac{e^{\phi^{\Lambda(1-t)}(\mu)}}{\int_{\Omega} e^{\phi^{\Lambda(1-t)}(\mu)}} \xrightarrow{\Lambda \to +\infty}$  $\mu$  for any  $\mu \in (\Omega_{\partial})_K$  and  $t \neq 1$ , hence being  $e^{\psi^{\Lambda t}}$  negligible with respect to  $\int_{\Omega} e^{\phi^{\Lambda(1-t)}}$  (see proof of Lemma 3.11) one also has  $\frac{e^{\Phi^{\Lambda}(\zeta)}}{\int_{\Omega} e^{\Phi^{\Lambda}(\zeta)}} \xrightarrow{\Lambda \to +\infty} \mu$ . Similarly, since  $\phi^{\Lambda(1-t)} - \int_{\Omega} \phi^{\Lambda(1-t)}$  is bounded in  $L^2(\Omega)$ , its projection will be negligible with respect to  $\psi^{\Lambda t}$ , therefore  $\Pi_I \Phi^{\Lambda} \xrightarrow{\Lambda \to +\infty} \zeta$  as long as  $t \neq 0$ .

The scalar parameter t in the join will be more delicate to handle, because by the proof of Lemma 3.10 one gets  $t\left(\Phi^{\Lambda}(\zeta)\right) \sim \min\left\{1, \log^+(\Lambda t)\right\}$ ; moreover, it is forced to be either 0 or 1 if either element in the join is not defined.

Therefore, before letting  $\Lambda$  go to  $+\infty$ , we need to properly rescale such a parameter, taking into account when it is allowed to be different from 0 and/or 1. To this purpose, we notice that, since

 $\frac{e^{\phi^{\Lambda(1-t)}(\mu)}}{\int_{\Omega} e^{\phi^{\Lambda(1-t)}(\mu)}}$  gets closer to  $(\Omega_{\partial})_{K,I}$  as  $\Lambda(1-t)$  is larger, we can assume that  $\mu\left(\Phi^{\Lambda}(\zeta)\right)$  is well-

defined for  $\Lambda(1-t) \geq \frac{\Lambda_0}{3}$  and similarly that  $t\left(\Phi^{\Lambda}(\zeta)\right)$  is well-defined for  $\Lambda t \geq \frac{\Lambda_0}{3}$ . Therefore, we will construct an intermediate parameter t', which we set to be either 0 or 1 if t is outside the previous range, which fills the whole (0,1) as  $\Lambda$  goes to  $+\infty$ : as a first step, we fix  $\Lambda_0$  and interpolate linearly between  $t(\Phi^{\Lambda}(\zeta))$  and t', and everything is well defined because  $\Lambda = \Lambda_0$  is fixed; then, we pass to the limit as  $\Lambda \to +\infty$  and, by the previous considerations, it is still well-posed and in the limit we recover the identity map.

Precisely, a continuous homotopical equivalence between  $\Psi \circ \Phi^{\Lambda_0}$  and  $\mathrm{Id}_{(\Omega_{\partial})_{K,I}}$  is given by

$$F(\zeta,s) = \begin{cases} \left(\mu\left(\Phi^{\Lambda_0}(\zeta)\right), \varsigma\left(\Phi^{\Lambda_0}(\zeta)\right), (1-2s)t\left(\Phi^{\Lambda_0}(\zeta)\right) + 2st'(t,1)\right) & \text{if } 0 \leq s < \frac{1}{2} \\ \left(\mu\left(\Phi^{\frac{\Lambda_0}{2-2s}}(\zeta)\right), \varsigma\left(\Phi^{\frac{\Lambda_0}{2-2s}}(\zeta)\right), t'(t,2-2s)\right) & \text{if } \frac{1}{2} \leq s < 1 \\ \zeta & \text{if } s = 1 \end{cases}$$

with 
$$t'(t,r) := \begin{cases} 0 & \text{if } t < \frac{r}{3} \\ \frac{3t-r}{3-2r} & \text{if } \frac{r}{3} \le t \le 1 - \frac{r}{3} \\ 1 & \text{if } t > 1 - \frac{r}{3} \end{cases}$$

#### Remark 3.19.

All the result shown in this section hold true, also when K and/or I equals zero. In each case, the

space  $(\Omega_{\partial})_{K,I}$  is replaced by

$$X = \begin{cases} (\Omega_{\partial})_K & \text{if } I \neq K = 0\\ \mathbb{S}^{I-1} & \text{if } K \neq I = 0\\ \emptyset & \text{if } I = K = 0 \end{cases}.$$

One can easily see that all the proofs are still valid in all these cases. When I = 0, in Proposition 3.8 we just consider  $\Phi^{\Lambda}(\mu, -, 0)$  and when K = 0 we take  $\Phi^{\Lambda}(-, \varsigma, 1)$ ; in Proposition 3.12 we just set  $F(\zeta, s) = \Phi \circ \Phi^{\frac{\Lambda}{1-s}}$ .

If I = K = 0, then Propositions 3.8 and 3.12 make no sense, but Lemma 3.15 applied with l = 0, m = 1,  $\Omega_{21} = \Omega$  implies that  $\mathcal{J}_{\beta,\rho}$  is coercive, namely  $\mathcal{J}_{\beta,\rho}^{-L} = \emptyset$  for L large. For this reason, the proof of Theorem 1.1 can be adapted also to this case.

# Proof of the main result

We need one last lemma concerning the Morse property of the functional  $\mathcal{J}_{\beta,\rho}$ .

Assume  $(P_{\beta,\rho})$  has no non-trivial solutions and  $\beta - \frac{\rho}{|\Omega|} \neq -\lambda_j$  for any  $j \in \mathbb{N}$ .

Then  $\mathcal{J}_{\beta,\rho}$  is a Morse functional and the Morse index J of the trivial solution  $u \equiv 0$  is such that  $\lambda_{J+1} < \beta - \frac{\rho}{|\Omega|} < \lambda_J$ .

Proof.

One immediately sees that the second derivative of  $\mathcal{J}_{\beta,\rho}$  is given by

$$\mathcal{J}_{\beta,\rho}(u)[v,w] = \int_{\Omega} (\nabla v \cdot \nabla w) + \beta \int_{\Omega} vw - \rho \frac{\int_{\Omega} vwe^{u} \int_{\Omega} e^{u} - \int_{\Omega} ve^{u} \int_{\Omega} we^{u}}{\left(\int_{\Omega} e^{u}\right)^{2}},$$

hence in  $u \equiv 0$  its quadratic form is

$$\mathcal{J}_{\beta,\rho}(0)[v,v] = \int_{\Omega} |\nabla v|^2 + \left(\beta - \frac{\rho}{|\Omega|}\right) \int_{\Omega} v^2.$$

Assume the only solution to  $(P_{\beta,\rho})$  is the trivial one. Then, the  $\mathcal{J}_{\beta,\rho}$  is a Morse functional if and only if the previous quadratic form is nondegenerate. One immediately sees that this depends on the relative position of  $\frac{\rho}{|\Omega|} - \beta$  and the  $\lambda_j(\Omega)$ 's as we required.

Now we are finally in position to prove the main result of the paper.

Proof of Theorem 1.1.

Assume, by contradiction, that  $u \equiv 0$  is the only solution to  $(P_{\beta,\rho})$ . By Lemma 4.1,  $\mathcal{J}_{\beta,\rho}$  is a Morse functional and the Morse index of the solution is J, therefore by Morse theory the relative homology of sublevels satisfies  $H_q\left(\mathcal{J}_{\beta,\rho}^L,\mathcal{J}_{\beta,\rho}^{-L}\right)=\left\{\begin{array}{ll}\mathbb{Z} & \text{if } q=J\\ 0 & \text{if } q\neq J\end{array}\right\}$ , for any L > 0.

Moreover, by Corollary 2.2,  $\mathcal{J}_{\beta,\rho}^L$  is contractible if L is large enough; therefore, the exactness of the sequence (see [18], Theorem 2.13 and Proposition 2.22)

$$\cdots \to \widetilde{H}_q\left(\mathcal{J}_{\beta,\rho}^{-L}\right) \to \widetilde{H}_q\left(\mathcal{J}_{\beta,\rho}^{L}\right) \to H_q\left(\mathcal{J}_{\beta,\rho}^{L},\mathcal{J}_{\beta,\rho}^{-L}\right) \to \widetilde{H}_{q-1}\left(\mathcal{J}_{\beta,\rho}^{-L}\right) \to \widetilde{H}_{q-1}\left(\mathcal{J}_{\beta,\rho}^{L}\right) \to \cdots$$

yields

$$\widetilde{H}_q\left(\mathcal{J}_{\beta,\rho}^{-L}\right) = H_{q+1}\left(\mathcal{J}_{\beta,\rho}^L,\mathcal{J}_{\beta,\rho}^{-L}\right) = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } q = J-1 \\ 0 & \text{if } q \neq J-1 \end{array} \right.,$$

which contradicts Proposition 3.6. In fact, if  $2K+I \neq J$ , Corollary 3.13 gives a non-trivial homology group for  $q = 2K + I - 1 \neq J - 1$ ; moreover, if  $\Omega$  is not simply connected and K > 0, then even when 2K + I = J we get a bigger homology group:  $\mathbb{Z} \subsetneq \mathbb{Z}^{\binom{K+g}{g}} \hookrightarrow H_{J-1}\left(\mathcal{J}_{\beta,\rho}^{-L}\right)$ 

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