

Ground state solutions for a nonlinear Choquard equation

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Abstract

We discuss the existence of ground state solutions for the Choquard equation

$$-\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N.$$

We prove the existence of solutions under general hypotheses, investigating in particular the case of a homogeneous nonlinearity $F(u) = \frac{|u|^p}{p}$. The cases $N = 2$ and $N \geq 3$ are treated differently in some steps. The solutions are found through a variational mountain-pass strategy. The results presented are contained in the papers [8, 2].

1 Introduction

We investigate the existence of solutions for nonlinear Choquard equations of the form

$$-\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where Δ is the standard Euclidean laplacian, $*$ indicates the convolution, $F \in C^1(\mathbb{R}, \mathbb{R})$ is a smooth nonlinearity and $I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R}$ is, for $\alpha \in (0, N)$, the Riesz potential:

$$I_\alpha(x) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{\frac{N}{2}} 2^\alpha} \frac{1}{|x|^{N-\alpha}}. \quad (1.2)$$

Problem (1.1) can be seen as a non-local counterpart of the very well-known scalar field equation

$$-\Delta u + u = G'(u) \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

which can be formally recovered from (1.1) by letting α go to 0 and setting $G = \frac{F^2}{2}$.

Problem (1.3) has been widely studied since many years. General existence results were provided in [4] when $N \geq 3$ and [3] (when $N = 2$) under mild

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hypotheses on G .

Anyway, the argument from both [4] and [3] does not seem to be suitable to attack problem (1.3): roughly speaking, the authors use a constrained minimization technique and then a dilation to get rid of the Lagrangian multiplier, which does not work in our case because of the scaling properties of the Riesz potential (1.2).

We study the problem (1.1) variationally: its solutions are critical points of the following energy functional on $H^1(\mathbb{R}^N)$:

$$\mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F'(u). \quad (1.4)$$

In particular, we look for solutions at a mountain-pass level b defined by

$$b := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)), \quad (1.5)$$

with

$$\Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)); \gamma(0) = 0, \mathcal{I}(\gamma(1)) < 0 \}.$$

In particular, we by-pass the issue of Palais-Smale sequences by a *scaling trick* introduced in [5], which basically allows us to consider Palais-Smale sequences also asymptotically satisfying the Pohožaev identity

$$\mathcal{P}(u) := \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} |u|^2 - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) = 0, \quad (1.6)$$

for which convergence is easier to be proved.

We can show existence of solutions under general hypotheses, in the same spirit of [4, 3]. In the particular yet very important case of a power-type nonlinearity $F(u) = \frac{|u|^p}{p}$ such hypotheses are equivalent to $1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2}$, which in [7] is shown to be also a necessary condition. This shows that the hypotheses we make are somehow natural.

We also show that the mountain-pass type solution is also a ground state, namely an energy-minimizing solution: it satisfies

$$\mathcal{I}(u) = c := \inf \{ \mathcal{I}(v) : v \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ solves (1.1)} \}. \quad (1.7)$$

We first show the existence of mountain-pass solutions in Section 2 and then in Section 3 we prove that they are actually ground states. Such results were originally presented in [8] for the dimension $N \geq 3$ and in [2] for the case $N = 2$.

2 Existence of mountain-pass solutions

We show here existence of a solution for (1.1) under general hypotheses on F . First of all, we want to exclude the trivial case of an identically vanishing F :

(F_0) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) \neq 0$.

Then, we also need some growth assumptions which give a well-posed variational formulation, namely a energy functional \mathcal{I} being well-defined on $H^1(\mathbb{R}^N)$.

Such assumptions are different depending whether the dimension is two or it is greater, since the limiting-case embeddings in Sobolev spaces are different: in the higher-dimensional case, we impose a power-type growth whereas in \mathbb{R}^2 we require one of exponential type:

($N \geq 3$) (F_1) There exists $C > 0$ such that $|F'(s)| \leq C \left(|s|^{\frac{\alpha}{N}} + |s|^{\frac{\alpha+2}{N-2}} \right)$ for any $s > 0$

($N = 2$) (F'_1) For any $\theta > 0$ there exists $C_\theta > 0$ such that $|F'(s)| \leq C_\theta \min \{1, |s|^{\frac{\alpha}{2}}\} e^{\theta|s|^2}$ for any $s > 0$.

It is not hard to see that (F_1), combined with Sobolev and Hardy-Littlewood-Sobolev inequality, implies the finiteness of the term $\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)$, hence the well-posedness and smoothness of the functional \mathcal{I} defined by (1.4). In dimension two we need, in place of Sobolev's inequality, a special form of the Moser-Trudinger inequality on the whole plane, which was given in [1]:

$$\forall \beta \in (0, 4\pi) \exists C_\beta > 0 \text{ such that } \int_{\mathbb{R}^2} |\nabla u|^2 \leq 1 \Rightarrow \int_{\mathbb{R}^2} \min \{1, u^2\} e^{\beta u^2} \leq C_\beta \int_{\mathbb{R}^2} |u|^2 \quad (2.1)$$

The last hypotheses we need is a sort of *sub-criticality* with respect to the critical power in Hardy-Littlewood-Sobolev inequality. Again, we state the condition differently depending on the dimension, since in dimension 2 there is no critical Sobolev exponent:

($N \geq 3$) (F_2) $\lim_{s \rightarrow 0} \frac{F(s)}{|s|^{1+\frac{\alpha}{N}}} = \lim_{s \rightarrow +\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$

($N = 2$) (F'_2) $\lim_{s \rightarrow 0} \frac{F(s)}{|s|^{1+\frac{\alpha}{N}}} = 0$

Precisely, the result we present is the following:

Theorem 2.1.

Assume F satisfies (F_0), (F_1), (F_2) if $N \geq 3$ and (F_0), (F'_1), (F'_2) if $N = 2$. Then, the problem (1.1) has a non-trivial solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$.

We start by showing the existence of a Pohožaev-Palais-Smale sequence. We argue as in [5] to get the asymptotical Pohožaev identity.

Lemma 2.2.

Assume F satisfies (F_0), (F_1) (or, in case $N = 2$, (F_0), (F'_1)). Then, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $H^1(\mathbb{R}^N)$ such that:

$$\mathcal{I}(u_n) \xrightarrow{n \rightarrow +\infty} b \quad \mathcal{I}'(u_n) \xrightarrow{n \rightarrow +\infty} 0 \text{ in } H^1(\mathbb{R}^N)' \quad \mathcal{P}(u_n) \xrightarrow{n \rightarrow +\infty} 0$$

Proof.

We divide the proof in three steps: first we show that the mountain-pass level (1.5) is not degenerate and then we apply a variant of the mountain-pass principle.

Step 1: $b > 0$ We suffice to show that $\Gamma \neq \emptyset$, namely that there exists some $u_0 \in H^1(\mathbb{R}^N)$ with $\mathcal{I}(u_0) < 0$.

By (F_0) , we can choose s_0 such that $F(s_0) \neq 0$, therefore if we take a smooth v_0 approximating $s_0 \mathbf{1}_{B_1}$ we easily get $\int_{\mathbb{R}^N} (I_\alpha * F(v_0))F(v_0) > 0$.

If now we consider $v_t = v_0 \left(\frac{\cdot}{t}\right)$, we get

$$\mathcal{I}(v_t) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla v_0|^2 + \frac{t^N}{2} \int_{\mathbb{R}^N} |v_0|^2 - \frac{t^{N+\alpha}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(v_0))F(v_0), \quad (2.2)$$

which is negative for large t , so we can take $u_0 = v_t$ with $t \gg 1$.

Step 2: $b < +\infty$ We need to show that for any $\gamma \in \Gamma$ there exists t_γ such that $\mathcal{I}(\gamma(t_\gamma)) \geq \varepsilon > 0$.

If $\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \leq \delta \ll 1$, then by assumption (F_2) and H-L-S and Sobolev's inequality we get

$$\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) \leq C \left(\left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{N+\alpha}{N-2}} + \left(\int_{\mathbb{R}^N} |u|^2 \right)^{1+\frac{\alpha}{N}} \right) \leq \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2),$$

which means $\mathcal{I}(u) \geq \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)$, and the same can be proved similarly when $N = 2$.

Now, for any fixed $\gamma \in \Gamma$ we can take t_γ such that $\int_{\mathbb{R}^2} (|\nabla \gamma(t_\gamma)|^2 + |\gamma(t_\gamma)|^2) = \delta$ and we get $\mathcal{I}(\gamma(t_\gamma)) \geq \frac{\delta}{4} =: \varepsilon$.

Step 3: Conclusion Consider the functional $\tilde{\mathcal{I}} : \mathbb{R} \times H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$\tilde{\mathcal{I}}(\sigma, v) := \mathcal{I}(v(e^{-\sigma} \cdot)) = \frac{e^{(N-2)\sigma}}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{e^{N\sigma}}{2} \int_{\mathbb{R}^N} |v|^2 - \frac{e^{(N+\alpha)\sigma}}{2} \int_{\mathbb{R}^N} (I_\alpha * F(v))F(v).$$

By applying to $\tilde{\mathcal{I}}$ the standard min-max principle (see [9] for instance) we get a sequence $(\sigma_n, v_n)_{n \in \mathbb{N}}$ with $\tilde{\mathcal{I}}(\sigma_n, v_n) \rightarrow b$ and $\tilde{\mathcal{I}}(\sigma_n, v_n)' \rightarrow 0$, which is equivalent to what the Lemma required.

□

To prove Theorem 2.1 we need to show the convergence of the Pohožaev-Palais-Smale sequence we just found. Here we need the sub-criticality assumption $(F_2), (F_2')$

Lemma 2.3.

Assume F satisfies $(F_1), (F_2)$ (or, in case $N = 2$, $(F_1'), (F_2')$) and $(u_n)_{n \in \mathbb{N}}$ satisfies

$$\mathcal{I}(u_n) \text{ is bounded} \quad \mathcal{I}'(u_n) \xrightarrow{n \rightarrow +\infty} 0 \text{ in } H^1(\mathbb{R}^N)' \quad \mathcal{P}(u_n) \xrightarrow{n \rightarrow +\infty} 0.$$

Then, up to subsequences,

- either $u_n \xrightarrow{n \rightarrow +\infty} 0$ strongly in $H^1(\mathbb{R}^N)$
- or $u_n(\cdot - x_n) \xrightarrow{n \rightarrow +\infty} u$ weakly for some $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^N and $u \in H^1(\mathbb{R}^N) \setminus \{0\}$.

Proof.

Assume the first alternative does not occur. Then, we show it weakly converges to some $u \neq 0$.

Step 1: $(u_n)_{n \in \mathbb{N}}$ is bounded It follows by just writing

$$\frac{\alpha + 2}{2(N + \alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{\alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} |u_n|^2 = \mathcal{I}(u_n) - \frac{\mathcal{P}(u_n)}{N + \alpha} \xrightarrow{n \rightarrow +\infty} b$$

Step 2: $\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |u_n|^p \geq \frac{1}{C}$ By using the asymptotic Pohožaev identity it is not hard to see that

$$\inf_n \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) > 0. \text{ Moreover, } (F_2) \text{ implies, for any } \epsilon > 0, p \in \left(2, \frac{2N}{N-2}\right),$$

$$|F(s)|^{\frac{2N}{N+\alpha}} \leq \epsilon \left(|s|^2 + |s|^{\frac{2N}{N-2}}\right) + C_\epsilon |s|^p,$$

therefore, by the following inequality from [6]

$$\int_{\mathbb{R}^N} |u_n|^p \leq C \left(\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \right) \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |u_n|^p \right)^{1 - \frac{2}{p}},$$

we get

$$\begin{aligned} & \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |u_n|^p \right)^{1 - \frac{2}{p}} \geq \frac{1}{C} \frac{\int_{\mathbb{R}^N} |u_n|^p}{\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2)} \\ & \geq \frac{1}{C_\epsilon} \left(\int_{\mathbb{R}^N} |F(u_n)|^{\frac{2N}{N+\alpha}} - \epsilon \int_{\mathbb{R}^N} \left(|u|^2 + |u|^{\frac{2N}{N-2}} \right) \right) \\ & \geq \frac{1}{C'_\epsilon} \left(\left(\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) \right)^{\frac{N}{N+\alpha}} - C_\epsilon \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \right) \geq \frac{1}{C}. \end{aligned}$$

and a similar estimate holds true in the case $N = 2$.

Step 3: $u_n(\cdot - x_n)$ converges We choose x_n such that $\liminf_{n \rightarrow +\infty} \int_{B_1} |u_n(\cdot - x_n)|^p > 0$, its weak limit (which exists because Step 1 ensures boundedness) must be some $u \neq 0$. By Sobolev embeddings, one can show that $(I_\alpha * F(u_n)) F'(u_n) \xrightarrow{n \rightarrow +\infty} (I_\alpha * F(u)) F'(u)$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. This easily yields that u solves (1.1)

□

Proof of Theorem 2.1.

By Lemma 2.2, \mathcal{I} admits a Pohožaev-Palais-Smale sequence $(u_n)_{n \in \mathbb{N}}$ at the energy level b . We apply Lemma 2.3 to the latter sequence: if the first alternative occurred, then we would have $\mathcal{I}(u_n) \xrightarrow{n \rightarrow +\infty} \mathcal{I}(0) = 0$, contradicting Lemma 2.3. Therefore, the second alternative must occur and in particular $u \not\equiv 0$ solves (1.1). \square

We conclude this section by showing that Theorem 2.1 is actually sharp in the case of a power nonlinearity $F(u) = \frac{|u|^p}{p}$; in other words, we give a non-existence result for all the values p not matching the assumptions of Theorem 2.1. To show non-existence, we use a Pohožaev identity, which is a classical property of solutions of (1.1).

Proposition 2.4.

Any solution u of (1.1) satisfies the Pohožaev identity (1.6).

Theorem 2.5.

If $F(u) = \frac{|u|^p}{p}$ then problem (1.1) admits a non-trivial solution if and only if $p \in \left(1 + \frac{\alpha}{N}, \frac{N + \alpha}{N - 2}\right)$, with the latter condition to be read as $p > 1 + \frac{\alpha}{2}$ if $N = 2$.

Proof.

If $p \in \left(1 + \frac{\alpha}{N}, \frac{N + \alpha}{N - 2}\right)$ then one can easily see that $F(u) = \frac{|u|^p}{p}$ satisfies $(F_0), (F_1), (F_2)$, hence the existence of non-trivial solutions follows from Theorem 2.1.

Conversely, assume p is outside that range and u solves (1.1), By testing both sides of against u we get

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p.$$

Moreover, u satisfies the Pohožaev identity (1.6), which has the form

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 - \frac{N+\alpha}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p = 0.$$

A linear combination of the two formulas gives

$$\left(\frac{N-2}{2} - \frac{N+\alpha}{2p}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + \left(\frac{N}{2} - \frac{N+\alpha}{2p}\right) \int_{\mathbb{R}^N} |u|^2,$$

which implies $u \equiv 0$ if $p \leq 1 + \frac{\alpha}{N}$ or $p \geq \frac{N+\alpha}{N-2}$. \square

3 From solutions to groundstates

In the last part of this paper we show that the mountain pass solutions given by Theorem 2.1 are actually energy-minimizing, in the sense of (1.7).

Theorem 3.1.

The mountain-pass solution found in Theorem 2.1 is actually a ground state, namely its energy level is given by (1.7).

The previous Theorem can be easily proved by constructing, for any solution v of (1.1), a path $\gamma_v \in \Gamma$ which attains its maximum energy on v .

Lemma 3.2.

Assume F satisfies (F_1) and $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ solves (1.1). Then, there exists a path $\gamma_v \in \Gamma$ such that:

$$\gamma_v(0) = 0 \quad \gamma_v\left(\frac{1}{2}\right) = v \quad \mathcal{I}(\gamma_v(t)) < \mathcal{I}(v) \text{ for any } t \neq \frac{1}{2} \quad \mathcal{I}(\gamma_v(t)) < 0$$

Proof.

Fix a non-trivial solution v of (1.1) and consider the path $\bar{\gamma}_v =: [0, +\infty) \rightarrow$

$$H^1(\mathbb{R}^N) \text{ defined by } \bar{\gamma}_v(t) = \begin{cases} v\left(\frac{\cdot}{t}\right) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

Along the path, the energy is given by (2.2), which is negative for $t \gg 1$. Moreover, due to the Pohožaev identity (1.6) we can also write

$$\mathcal{I}(\bar{\gamma}_v(t)) = \left(\frac{t^{N-2}}{2} - \frac{N-2}{2(N+\alpha)}t^{N+\alpha}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + \left(\frac{t^N}{2} - \frac{N}{2(N+\alpha)}t^{N+\alpha}\right) \int_{\mathbb{R}^N} |u|^2,$$

which has its maximum in $t = 1$. Therefore, up to a rescaling of t , this path has all the required properties.

Anyway, being

$$\int_{\mathbb{R}^N} (|\nabla \bar{\gamma}_v(t)|^2 + |\bar{\gamma}_v(t)|^2) = t^{N-2} \int_{\mathbb{R}^N} |\nabla v|^2 + t^N \int_{\mathbb{R}^N} |v|^2,$$

$\bar{\gamma}_v$ is continuous at $t = 0$ only if $N \geq 3$, so in the case $N = 2$ we need a modification for t close to 0.

$$\text{If } N = 2 \text{ we take } \bar{\gamma}_v(t) = \begin{cases} v\left(\frac{\cdot}{t}\right) & \text{if } t > t_0 \\ \frac{t}{t_0} v\left(\frac{\cdot}{t_0}\right) & \text{if } t \leq t_0 \end{cases} \text{ for some suitable } t_0 \ll 1.$$

We only need to verify that $\mathcal{I}(\bar{\gamma}_v(t)) \leq \mathcal{I}(\bar{\gamma}_v(1))$ for $t \leq t_0$.

Using the assumption (F'_1) and Moser-Trudinger's (2.1) and Hardy-Littlewood-Sobolev inequalities we get

$$\int_{\mathbb{R}^2} (I_\alpha * F(\bar{\gamma}_v(t))) F(\bar{\gamma}_v(t)) \leq C \left(\frac{\int_{\mathbb{R}^2} |\bar{\gamma}_v(t)|^2}{\int_{\mathbb{R}^2} |\nabla \bar{\gamma}_v(t)|^2} \right)^{1+\frac{\alpha}{2}} = Ct_0^{2+\alpha} \left(\int_{\mathbb{R}^2} |v|^2 \right)^{1+\frac{\alpha}{2}},$$

therefore using again Pohožaev identity we get, for t_0 small enough,

$$\mathcal{I}(\bar{\gamma}_v(t)) \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{t_0^2}{2} \int_{\mathbb{R}^2} |v|^2 + Ct_0^{2+\alpha} \left(\int_{\mathbb{R}^2} |v|^2 \right)^{1+\frac{\alpha}{2}}$$

$$= \mathcal{I}(v) + \left(\frac{t_0^2}{2} - \frac{\alpha}{2(2+\alpha)} \right) \int_{\mathbb{R}^2} |v|^2 + Ct_0^{2+\alpha} \left(\int_{\mathbb{R}^2} |v|^2 \right)^{1+\frac{\alpha}{2}} < \mathcal{I}(v)$$

and the proof is complete. \square

Proof of Theorem 3.1.

Let u be the mountain-pass solution found in Theorem 2.1. By the lower-semicontinuity of the norm we find $\mathcal{I}(u) \leq b$, whereas the definition (1.7) of ground state yields $\mathcal{I}(u) \geq c$.

Now, take another solution $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ and apply Lemma 3.2: we get

$$\mathcal{I}(v) = \sup_{t \in [0,1]} \mathcal{I}(\gamma_v(t)) \geq \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)) = b.$$

Being v arbitrary, we get $c \geq b$, hence $c \leq \mathcal{I}(u) \leq b \leq c$, therefore $\mathcal{I}(u) = b = c$. \square

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