

GROUNDSTATES OF THE CHOQUARD EQUATIONS WITH A SIGN-CHANGING SELF-INTERACTION POTENTIAL

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ABSTRACT. We consider a nonlinear Choquard equation

$$-\Delta u + u = (V * |u|^p) |u|^{p-2} u \quad \text{in } \mathbb{R}^N,$$

when the self-interaction potential V is unbounded from below. Under some assumptions on V and on p , covering $p = 2$ and V being the one- or two-dimensional Newton kernel, we prove the existence of a nontrivial groundstate solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ by solving a relaxed problem by a constrained minimization and then proving the convergence of the relaxed solutions to a groundstate of the original equation.

1. INTRODUCTION

We are interested in the nonlinear Choquard equation“

$$(C) \quad -\Delta u + u = (V * |u|^p) |u|^{p-2} u;$$

in the Euclidean space \mathbb{R}^N with $N \in \mathbb{N} = \{1, 2, \dots\}$, where $p \in [1, +\infty)$ is a given exponent and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given self-interaction potential. Solutions to the Choquard equation (C) correspond, *at least formally*, to critical points of the *functional* \mathcal{I} defined for each function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$(1.1) \quad \mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) - \frac{1}{2p} \int_{\mathbb{R}^2} (V * |u|^p) |u|^p.$$

When $p = 2$, $N = 3$ and the self-interaction potential V is *Newton's kernel*, that is, the fundamental solution of the Laplacian on the space \mathbb{R}^3 , the Choquard equation (C) arises in several fields of physics (quantum mechanics [22], one-component plasma [12], self gravitating matter [18]). In the more general setting where $N \in \mathbb{N}_*$, that the

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function $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a *Riesz potential*, that is, for $x \in \mathbb{R}^N \setminus \{0\}$,

$$V(x) = I_\alpha(x) := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}},$$

with $\alpha \in (0, N)$ and that $\frac{N+\alpha}{N} \leq \frac{1}{p} \leq \frac{N+\alpha}{N-2}$, the existence of a groundstate solution, minimizing the functional \mathcal{I} among all solutions and of multiple solutions has been proved [5, 9, 12, 14–16, 19, 25, 28]. Similar results have been obtained when the power nonlinearity $|u|^p$ is replaced by a more general nonlinearity [2, 20]. We also refer the interested reader to the recent survey [21].

In *low dimensions* $N \in \{1, 2\}$, the Newton kernel is not anymore a Riesz potential and is characterized by a *linear or logarithmic growth at infinity*. This means that the above results cannot be transferred readily to the case where V is the one- or two-dimensional Newton kernel and that other ideas and methods are needed. Solutions for the Choquard equation with a low-dimensional Newton kernel have been constructed by ordinary differential equation methods [9].

A key difficulty in order to construct solutions variationally is that the functional \mathcal{I} is *not well-defined on the natural Sobolev space* $H^1(\mathbb{R}^N)$. For the two-dimensional logarithmic Newton kernel, variational methods have been applied successfully in the framework of the Hilbert space of functions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$(1.2) \quad \int_{\mathbb{R}^2} (|\nabla u|^2 + (1 + V^-)|u|^2) < +\infty$$

(see [6, 8, 10, 11, 26, 27]). A delicate point in this approach is that although the Choquard equation (\mathcal{C}) and the associated energy functional \mathcal{I} are *invariant under translations* of the Euclidean space \mathbb{R}^N , the Hilbert space naturally defined by the quadratic form (1.2) is *not* any more *invariant* under translation. At the technical level, this means, for example, that a translated Palais–Smale sequence need not be itself a Palais–Smale sequence. This nonintrinsic character of the method translates in a rigidity of the results, that cannot be readily generalized to similar classes of potentials.

The goal of the present work is to develop a method invariant under translations that would yield more flexible existence results for the Choquard problem (\mathcal{C}) .

Since the main issue in studying variationally (\mathcal{C}) is the unboundedness of the negative part of the potential V^- , we propose to construct solutions to a relaxed variational problem obtained by truncating the potential V and then to obtain solutions of the original problem by passing to the limit on the relaxation parameter.

We obtain the following result:

Theorem 1.1. *Let $N \in \mathbb{N}$ and $p \in [2, +\infty)$. If $p \geq 2$ and if the function $V \in \mathbb{R}^N \rightarrow \mathbb{R}$ is even and satisfies*

$$(V_1) \quad V^+ \in L^q(\mathbb{R}^N) \text{ with } q \in [1, +\infty) \text{ and } \frac{1}{p}(1 - \frac{1}{2q}) \geq \frac{1}{2} - \frac{1}{N},$$

$$(V_2) \quad \sup_{|x-y| \leq 1} |V^-(x) - V^-(y)| < +\infty,$$

$$(V_3) \quad \text{there exists a function } \varphi \in C_c^1(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^N} (V * |\varphi|^p)|\varphi|^p > 0,$$

$$(V_4) \quad \lim_{|x| \rightarrow \infty} V(x) = -\infty,$$

then the Choquard equation (C) has a groundstate weak solution $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} (V * |u|^p) |u|^p > -\infty$.

Here and in the sequel, we have set $V^+ = \max(V, 0)$ and $V^- = \max(-V, 0)$ so that $V = V^+ - V^-$.

The notion of weak solution in Theorem 1.1 that we use comes with suitable integrability assumptions in order to ensure that the equation makes sense.

Definition 1.2. The function $u \in H_{\text{loc}}^1(\mathbb{R}^N)$ is a weak solution to the Choquard equation (C) whenever $(V * |u|^p) |u|^{p-2} u \in H_{\text{loc}}^{-1}(\mathbb{R}^N)$ and, for every test function $\varphi \in H^1(\mathbb{R}^N)$ supported in a compact set, one has

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi + u \varphi = \int_{\mathbb{R}^N} (V * |u|^p) |u|^{p-2} u \varphi.$$

Examples of self-interaction potentials $V : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying the assumptions of Theorem 1.1 are $V(x) = \frac{1}{2\pi} \ln \frac{1}{|x|}$, corresponding when $N = 2$ to the Newtonian kernel, $V(x) = 1 - \frac{|x|}{2}$ corresponding when $N = 1$ to the Newtonian kernel and more exotic examples of the form $V(x) = |x|^\alpha - |x|^\beta$ with $\max\{(N-2)p - 2N, -N\} < \alpha < \beta \leq 1$. Theorem 1.1 also allows for *anisotropic potentials* such as $V(x) = \frac{1}{2\pi} \log \frac{1}{|Ax|}$ with $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear and invertible or $V(x) = \frac{1}{2\pi} \log|x| - \frac{\kappa}{2\pi} \log|Ax|$, with $\kappa \in (0, 1)$; their three-dimensional counterparts have been studied recently [23].

The evenness of the potential V is required to ensure the variational character of the problem. The boundedness assumption (V₁) on the function V ensures that the convolution term takes a well-defined, possibly infinite, value in $[-\infty, +\infty)$. It also implies that $\frac{1}{p} \geq \frac{1}{2} - \frac{1}{N}$. The assumption (V₂) is a *coarse continuity* assumption which governs the large-scale continuity of the function V [24]. This condition plays a role in the estimates on the convolution term.

The condition (V₃) is necessary since any solution u should satisfy the identity

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = \int_{\mathbb{R}^N} (V * |u|^p) |u|^p,$$

obtained by testing the Choquard equation (C) against u (see Lemma 3.2). One way to ensure the condition (V₃) is to require the potential V to be positive in a neighborhood of 0.

The groundstate solution in Theorem 1.1, is constructed as an extremal function for the supremum

$$(1.3) \quad a := \sup \left\{ \int_{\mathbb{R}^N} (V * |u|^p) |u|^p \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = 1 \right\}.$$

Due to the homogeneity of the nonlinear term, one can easily deduce that a multiple of such an extremal is a solution of (C) at the energy level

$$(1.4) \quad b := \inf \left\{ \sup_{t>0} \mathcal{I}(tu) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\};$$

It is also easily seen to be a ground state, in the sense we previously expressed.

The content of the paper is the following: in Section 2 we study the groundstates of a relaxed problem and in Section 3 we prove the convergence of such ground states to ground states for the Choquard equation (C).

2. GROUNDSTATE SOLUTIONS FOR THE RELAXED PROBLEM

In order to construct solutions to the Choquard equation (C), we introduce the parameter $\lambda \in [0, +\infty)$ and we define a *relaxed potential* $V_\lambda : \mathbb{R}^N \rightarrow \mathbb{R}$ defined for each $x \in \mathbb{R}^N$ by

$$(2.1) \quad V_\lambda(x) := \max \{V(x), -\lambda\}.$$

The *relaxed problem* is obtained from the original Choquard equation (C) by replacing the original potential V by the relaxed potential V_λ :

$$(C_\lambda) \quad -\Delta u + u = (V_\lambda * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

A fundamental tool in our analysis is Young's convolution inequality (see for example [4, Theorem 3.9.2; 13, Theorem 4.2]), which plays the same role as the Hardy–Littlewood–Sobolev inequality for the Choquard equation with a Riesz self-interaction potential.

Proposition 2.1 (Young's convolution inequality). *If $V \in L^q(\mathbb{R}^N)$ and $f \in L^r(\mathbb{R}^N)$ with $1 < \frac{1}{q} + \frac{1}{r} \leq 2$, then $V * f \in L^s(\mathbb{R}^N)$ with $\frac{1}{s} = \frac{1}{q} + \frac{1}{r} - 1$ and*

$$\int_{\mathbb{R}^N} |V * f|^s \leq \left(\int_{\mathbb{R}^N} |V|^q \right)^{\frac{s}{q}} \left(\int_{\mathbb{R}^N} |f|^r \right)^{\frac{s}{r}}.$$

In particular, if $f \in L^{\frac{2q}{2q-1}}(\mathbb{R}^N)$, then

$$\left| \int_{\mathbb{R}^N} (V * f)f \right| \leq \left(\int_{\mathbb{R}^N} |V|^q \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |f|^{\frac{2q}{2q-1}} \right)^{2 - \frac{1}{q}}.$$

As a first application of the estimate of Proposition 2.1, we have the well-definiteness of our relaxed variational problem.

Lemma 2.2. *Let $N \in \mathbb{N}$, $p \in (1, +\infty)$ and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be an even measurable function. If $p \geq 2$ and if $V^+ \in L^q(\mathbb{R}^N)$ with $\frac{1}{p}(1 - \frac{1}{2q}) \geq \frac{1}{2} - \frac{1}{N}$, then for every $\lambda \in [0, +\infty)$, the functional $\mathcal{J}_\lambda : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined for each $u \in H^1(\mathbb{R}^N)$ by*

$$\mathcal{J}_\lambda(v) = \int_{\mathbb{R}^N} (V_\lambda * |u|^p)|u|^p$$

is well-defined and continuously differentiable.

Proof of Lemma 2.2. By Young's inequality for convolution (Proposition 2.1), we have

$$\left| \int_{\mathbb{R}^N} (V_\lambda^+ * |u|^p)|u|^p \right| \leq \left(\int_{\mathbb{R}^N} |V^+|^q \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |u|^{\frac{2pq}{2q-1}} \right)^{2 - \frac{1}{q}}.$$

The second factor is controlled by the Sobolev inequality since by our assumption, we have $\frac{1}{2} - \frac{1}{N} \leq \frac{1}{p}(1 - \frac{1}{2q}) \leq \frac{1}{p} \leq \frac{1}{2}$. On the other hand, since $0 \leq V_\lambda^- \leq \lambda$ on \mathbb{R}^N , we have

$$\left| \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p)|u|^p \right| \leq \lambda \left(\int_{\mathbb{R}^N} |u|^p \right)^2.$$

which is controlled by the Sobolev embedding whenever $\frac{1}{2} - \frac{1}{N} \leq \frac{1}{p} \leq \frac{1}{2}$.

The continuous differentiability follows from the same estimates together with the fact that the superposition mapping $u \in H^1(\mathbb{R}^N) \mapsto |u|^{p-2}u \in L^r(\mathbb{R}^N)$ is continuously differentiable whenever $\frac{1}{2} - \frac{1}{N} \leq \frac{p-1}{r} \leq \frac{1}{2}$. \square

We are going to construct a multiple of a solution of the relaxed problem (\mathcal{C}_λ) by showing that the following supremum is achieved:

$$(2.2) \quad a_\lambda := \sup \left\{ \int_{\mathbb{R}^N} (V_\lambda * |u|^p) |u|^p \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) = 1 \right\}.$$

In order to prove this we rely on a Brezis–Lieb inequality for the relaxed potential V_λ .

Lemma 2.3. *Let $N \in \mathbb{N}$, $p, q \in [1, +\infty)$ and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ be an even measurable function. If $V^+ \in L^q(\mathbb{R}^N)$, if the sequence $(u_n)_{n \in \mathbb{N}}$ converges almost everywhere to $u : \mathbb{R}^N \rightarrow \mathbb{R}$ and is bounded in $L^p(\mathbb{R}^N) \cap L^{\frac{2pq}{2q-1}}(\mathbb{R}^N)$, then for every $\lambda \in [0, +\infty)$, we have*

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (V_\lambda * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (V_\lambda * |u_n - u|^p) |u_n - u|^p + 2 \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p) |u_n - u|^p \\ = \int_{\mathbb{R}^N} (V_\lambda * |u|^p) |u|^p. \end{aligned}$$

Lemma 2.3 is a nonlocal version of the classical Brezis–Lieb lemma [7]. Similar identities and inequalities have been proved when the self-interaction potential V has constant sign [1, §5.1; 3; 17; 19; 32, lemma 3.2]. It implies the following interesting Brezis–Lieb inequality

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (V_\lambda * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (V_\lambda * |u_n - u|^p) |u_n - u|^p \leq \int_{\mathbb{R}^N} (V_\lambda * |u|^p) |u|^p.$$

Proof of Lemma 2.3. We treat separately the positive and negative contributions of the self-interaction potential V . For the positive part, we follow essentially the argument for the Riesz potential [19, Lemma 2.4]. Since by assumption the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^{2pq/(q-1)}(\mathbb{R}^N)$ and converges almost everywhere to u , the sequence $(|u_n - u|^p)_{n \in \mathbb{N}}$ converges weakly to 0 in $L^{2q/(2q-1)}(\mathbb{R}^N)$ (see for example [4, Proposition 4.7.2; 31, Proposition 5.4.7]) and by the classical Brezis–Lieb lemma [7, Theorem 1], the sequence $(|u_n|^p - |u_n - u|^p)_{n \in \mathbb{N}}$ converges strongly to the function $|u|^p$ in $L^{2q/(2q-1)}(\mathbb{R}^N)$. By Young’s convolution inequality (Proposition 2.1), the sequence $(V^+ * (|u_n|^p - |u_n - u|^p))_{n \in \mathbb{N}}$ converges strongly to $V^+ * |u|^p$ in $L^{2q}(\mathbb{R}^N)$. Therefore, since the function V^+ is even,

$$(2.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V^+ * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (V^+ * |u_n - u|^p) |u_n - u|^p \\ = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V^+ * (|u_n|^p - |u_n - u|^p)) ((|u_n|^p - |u_n - u|^p) + 2|u_n - u|^p) \\ = \int_{\mathbb{R}^N} (V^+ * |u|^p) |u|^p. \end{aligned}$$

For the negative part, we first rewrite for each $n \in \mathbb{N}$ the integrals as

$$\begin{aligned}
& \int_{\mathbb{R}^N} (V_\lambda^- * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (V_\lambda^- * |u_n - u|^p) |u_n - u|^p - 2 \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p) |u_n - u|^p \\
&= \int_{\mathbb{R}^N} (V_\lambda^- * (|u_n|^p - |u_n - u|^p)) (|u_n|^p - |u_n - u|^p) \\
&\quad + 2 \int_{\mathbb{R}^N} (V_\lambda^- * (|u_n|^p - |u_n - u|^p - |u|^p)) |u_n - u|^p.
\end{aligned}$$

Since the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N)$, by the classical Brezis–Lieb lemma again, the sequence $(|u_n|^p - |u_n - u|^p)_{n \in \mathbb{N}}$ converges strongly to $|u|^p$ in $L^1(\mathbb{R}^N)$, and thus,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda^- * (|u_n|^p - |u_n - u|^p)) (|u_n|^p - |u_n - u|^p) = \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p) |u|^p$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda^- * (|u_n|^p - |u_n - u|^p - |u|^p)) |u_n - u|^p = 0.$$

Hence, we obtain

$$\begin{aligned}
(2.4) \quad & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda^- * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (V_\lambda^- * |u_n - u|^p) |u_n - u|^p - 2 \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p) |u_n - u|^p \\
&= \int_{\mathbb{R}^N} (V_\lambda^- * |u|^p) |u|^p.
\end{aligned}$$

The conclusion follows from the combination of the identities (2.3) and (2.4). \square

Proposition 2.4. *Let $N \in \mathbb{N}$ and $p \in [2, +\infty)$. If the function $V : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the assumptions (V_1) and (V_2) of Theorem 1.1, then for every $\lambda \in [0, +\infty)$ the supremum a_λ is achieved.*

The proof is inspired by the proof of the corresponding property for the Choquard equation with a Riesz potential [19, Proposition 2.2] with the additional difficulty that we have to take care of the convergence of the term involving V^- , for which we have less information (see Lemma 2.3).

Proof of Proposition 2.4. By the assumption (V_3) we have $a_\lambda > 0$. Let $(w_n)_{n \in \mathbb{N}}$ be a sequence in $H^1(\mathbb{R}^N)$ satisfying

$$(2.5) \quad \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) = 1, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda * |w_n|^p) |w_n|^p = a_\lambda.$$

Since the sequence $(w_n)_{n \in \mathbb{N}}$ is bounded in the space $H^1(\mathbb{R}^N)$, it converges weakly, up to a subsequence, to some function $w \in H^1(\mathbb{R}^N)$. We first show that we may assume that $w \not\equiv 0$ on \mathbb{R}^N .

By an inequality of P.-L. Lions [14, Lemma I.2] (see also [19, lemma 2.3; 29, (2.4); 30, lemma 1.21]), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |w_n|^{\frac{2pq}{2q-1}} &\leq C_1 \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_n|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{1}{p}(2 - \frac{1}{q})} \\ &= C_1 \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_n|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{1}{p}(2 - \frac{1}{q})}. \end{aligned}$$

From this, we get, in view of Young's convolution inequality (Proposition 2.1),

$$\left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_n|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{2q-1}{pq}} \geq \frac{\left(\int_{\mathbb{R}^N} (V^+ * |w_n|^p) |w_n|^p \right)^{\frac{q}{2q-1}}}{C_1 \left(\int_{\mathbb{R}^N} |V^+|^q \right)^{\frac{1}{2q-1}}}.$$

Hence we deduce that

$$C_2 \limsup_{n \rightarrow \infty} \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_n|^{\frac{2pq}{2q-1}} \right)^{(2 - \frac{1}{q})(1 - \frac{2q-1}{pq})} \geq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda * |w_n|^p) |w_n|^p = a_\lambda > 0.$$

Since $\frac{1}{p}(1 - \frac{1}{2q}) < \frac{1}{p} \leq \frac{1}{2}$, this implies that there exists a sequence of points $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^N such that

$$(2.6) \quad \liminf_{n \rightarrow +\infty} \int_{B_1(x_n)} |w_n|^{\frac{2pq}{2q-1}} > 0.$$

By replacing for each $n \in \mathbb{N}$ the function w_n by its translation $w_n(\cdot + x_n)$ we can assume that (2.6) holds with $x_n = 0$, and thus by the Rellich–Kondrashov compact embedding theorem that

$$\int_{B_1} |w|^2 = \lim_{n \rightarrow \infty} \int_{B_1} |w_n|^2 > 0,$$

so that $w \not\equiv 0$ on \mathbb{R}^N .

By its weak convergence, the sequence $(w_n)_{n \in \mathbb{N}}$ also satisfies

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) - \int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2);$$

Since the sequence $(w_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$, we have, by the Sobolev embedding theorem and by Lemma 2.3

$$\int_{\mathbb{R}^N} (V * |w|^p) |w|^p \geq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V * |w_n|^p) |w_n|^p - \int_{\mathbb{R}^N} (V * |w_n - w|^p) |w_n - w|^p.$$

By homogeneity, any function $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ verifies

$$\frac{\int_{\mathbb{R}^N} (V_\lambda * |v|^p) |v|^p}{\left(\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) \right)^{\frac{p}{2}}} \leq a_\lambda.$$

Therefore,

$$\begin{aligned}
(2.7) \quad & \frac{\int_{\mathbb{R}^N} (V_\lambda * |w|^p) |w|^p}{\left(\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)\right)^p} \geq \limsup_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} (V_\lambda * |w_n|^p) |w_n|^p - \int_{\mathbb{R}^N} (V_\lambda * |w_n - w|^p) |w_n - w|^p}{\left(\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)\right)^p} \\
& = \limsup_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} (V_\lambda * |w_n|^p) |w_n|^p}{\left(\int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2)\right)^p} \left(\frac{\int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} \right)^p \\
& \quad - \frac{\int_{\mathbb{R}^N} (V_\lambda * |w_n - w|^p) |w_n - w|^p}{\left(\int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2)\right)^p} \\
& \quad \times \left(\frac{\int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} \right)^p
\end{aligned}$$

By the optimizing character of the sequence $(w_n)_{n \in \mathbb{N}}$ and by definition of a_λ , we have

$$\begin{aligned}
(2.8) \quad & \frac{\int_{\mathbb{R}^N} (V_\lambda * |w|^p) |w|^p}{\left(\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)\right)^p} \geq a_\lambda \limsup_{n \rightarrow \infty} \left(\left(\frac{\int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} \right)^p \right. \\
& \quad \left. - \left(\frac{\int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} \right)^p \right) \\
& = a_\lambda \limsup_{n \rightarrow \infty} \left(\left(\frac{\int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} + 1 \right)^p \right. \\
& \quad \left. - \left(\frac{\int_{\mathbb{R}^N} (|\nabla(w_n - w)|^2 + |w_n - w|^2)}{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)} \right)^p \right) \\
& \geq a_\lambda.
\end{aligned}$$

It follows then that the function $u_\lambda = w / \sqrt{\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2)}$ satisfies the required properties. \square

Remark 2.5. In the proof of Proposition 2.4, the last chain of inequalities (2.7)–(2.8) must actually be a chain of equalities, and in particular

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) = \int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2),$$

so that the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly to w in $H^1(\mathbb{R}^N)$ and the function w itself attains a_λ .

3. GROUNDSTATES FOR THE CHOQUARD EQUATION

This section is devoted to the proof of Theorem 1.1 about the existence of groundstates for the Choquard equation (C).

The proof of Theorem 1.1 will use the fact that coarsely continuous functions are large-scale Lipschitz-continuous.

Lemma 3.1. *If $f : \mathbb{R}^N \rightarrow \mathbb{R}$, then for every $x, y \in \mathbb{R}^N$,*

$$|f(x) - f(y)| \leq (|x - y| + 1) \sup \{|f(z) - f(w)| \mid z, w \in \mathbb{R}^N \text{ and } |z - w| \leq 1\}.$$

Proof. We take points x_0, \dots, x_ℓ , with $x_0 = x$, $x_\ell = y$, $\ell \leq |x - y| + 1$, $|x_i - x_{i+1}| \leq 1$ for $i \in \{0, \dots, \ell - 1\}$, and we estimate, by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{i=0}^{\ell-1} |f(x_i) - f(x_{i+1})| \\ &\leq \left(\sum_{i=0}^{\ell-1} |x_i - x_{i+1}| \right) \sup \{ |f(z) - f(w)| \mid z, w \in \mathbb{R}^N \text{ and } |z - w| \leq 1 \}. \\ &\leq (|x - y| + 1) \sup \{ |f(z) - f(w)| \mid z, w \in \mathbb{R}^N \text{ and } |z - w| \leq 1 \}. \quad \square \end{aligned}$$

Definition 1.2 of weak solutions does not allow a priori to test the equation against against a solution. We show that the resulting formula still holds however.

Lemma 3.2 (Testing the equation against a solution). *If $p \geq 2$, if $V \in L^q(\mathbb{R}^N)$ with $\frac{1}{p}(1 - \frac{1}{2q}) \geq \frac{1}{2} - \frac{1}{N}$ and if $u \in H^1(\mathbb{R}^N)$ is a weak solution to the Choquard equation (C), then $\int_{\mathbb{R}^N} (|V| * |u|^p) |u|^p < +\infty$ and*

$$\int_{\mathbb{R}^N} (V * |u|^p) |u|^p = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2).$$

Proof. We consider a function $\eta \in C_c^2(\mathbb{R}^N)$ such that $\eta = 1$ on B_1 , $0 \leq \eta \leq 1$ on \mathbb{R}^N and for each $x \in \mathbb{R}^N$ the function $t \in [0, +\infty) \mapsto \eta(tx)$ is nonincreasing, and we define for each $R > 0$ the function $\eta_R : \mathbb{R}^N \rightarrow \mathbb{R}$ for each $x \in \mathbb{R}^N$ by $\eta_R(x) = \eta(x/R)$. Since $u \in H^1(\mathbb{R}^N)$ and η_R has compact support, the function $\eta_R u$ is an admissible test function of the weak formulation of the Choquard equation (Definition 1.2). Hence, we have

$$\int_{\mathbb{R}^N} (\eta_R |\nabla u|^2 + \eta_R |u|^2 + u \nabla u \cdot \nabla \eta_R) = \int_{\mathbb{R}^N} (V * |u|^p) |u|^p \eta_R.$$

Therefore, since $\eta_R \leq 1$, since $|\nabla \eta_R| \leq \|\nabla \eta\|_{L^\infty} / R$ and since, by combining our assumption with the classical Sobolev embedding, we have $u \in L^{\frac{2pq}{2q-1}}(\mathbb{R}^N)$ so that by Young's convolution inequality (Proposition 2.1) we have $\int_{\mathbb{R}^N} (V^+ * |u|^p) |u|^p < +\infty$, we obtain by Lebesgue's dominated convergence theorem that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\mathbb{R}^N} (\eta_R |\nabla u|^2 + \eta_R |u|^2 + u \nabla u \cdot \nabla \eta_R) - \int_{\mathbb{R}^N} (V^+ * |u|^p) |u|^p \eta_R \\ = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \int_{\mathbb{R}^N} (V^+ * |u|^p) |u|^p. \end{aligned}$$

Hence by Lebesgue's monotone convergence theorem

$$\begin{aligned} \int_{\mathbb{R}^N} (V^- * |u|^p) |u|^p &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}^N} (V^- * |u|^p) |u|^p \eta_R \\ &= \int_{\mathbb{R}^N} (V^+ * |u|^p) |u|^p - \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) < +\infty. \end{aligned}$$

The conclusions then follow. \square

Proof of Theorem 1.1. By Proposition 2.4, for every $\lambda > 0$, there exists a function $w_\lambda \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (|\nabla w_\lambda|^2 + |w_\lambda|^2) = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (V_\lambda * |w_\lambda|^p) |w_\lambda|^p = a_\lambda,$$

where a_λ was defined in (2.2).

Claim 1. *For every $\lambda \in [0, +\infty)$, we have*

$$0 < \lim_{\mu \rightarrow \infty} a_\mu \leq a_\lambda \leq a_0 < +\infty.$$

Proof of the claim. We observe that if $\lambda_1 \leq \lambda_2$, then by the definition of the relaxed potential V_λ in (2.1), we have $V_{\lambda_1} \geq V_{\lambda_2}$ and thus by the definition of a_λ in (2.2), we have $a_{\lambda_1} \geq a_{\lambda_2}$. In particular, we have $a_\lambda \leq a_0$.

Moreover by definition of V_λ in (2.1) again, we have for every $\lambda > 0$ and $\varphi \in C_c^1(\mathbb{R}^N)$.

$$a_\lambda \geq \frac{\int_{\mathbb{R}^N} (V_\lambda * |\varphi|^p) |\varphi|^p}{\left(\int_{\mathbb{R}^N} (|\nabla \varphi|^2 + |\varphi|^2) \right)^p} \geq \frac{\int_{\mathbb{R}^N} (V_\lambda * |\varphi|^p) |\varphi|^p}{\left(\int_{\mathbb{R}^N} (|\nabla \varphi|^2 + |\varphi|^2) \right)^p} > 0.$$

By the assumptions (V_3) and (V_4) , the right-hand side can be chosen to be positive and independent of λ , so that the conclusion follows. \diamond

Claim 2. *There exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, +\infty)$ such that $\lim_{n \rightarrow \infty} \lambda_n = +\infty$, a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathbb{R}^N and a function $w \in H^1(\mathbb{R}^N)$ such that if $w_n = w_{\lambda_n}(\cdot + x_n)$, then*

- (1) $w \neq 0$ in \mathbb{R}^N ,
- (2) the sequence $(w_n)_{n \in \mathbb{N}}$ converges weakly to w in $H^1(\mathbb{R}^N)$,
- (3) the sequence $(w_n)_{n \in \mathbb{N}}$ converges almost everywhere to w in \mathbb{R}^N ,
- (4) the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly to w in $L^p(\mathbb{R}^N)$,
- (5) there exists a constant $\mu \in [0, +\infty)$ such that for every $R > 0$, the sequence $(V_{\lambda_n}^- * |w_n|^p)_{n \in \mathbb{N}}$ converges to $(V^- * |w|^p) + \mu$ in $L^\infty(B_R)$.

Proof of the claim. By Young's convolution inequality (Proposition 2.1), we have

$$(3.1) \quad \int_{\mathbb{R}^N} |w_\lambda|^{\frac{2pq}{2q-1}} \geq \frac{\left(\int_{\mathbb{R}^N} (V^+ * |w_\lambda|^p) |w_\lambda|^p \right)^{\frac{q}{2q-1}}}{\left(\int_{\mathbb{R}^N} |V^+|^q \right)^{\frac{1}{2q-1}}} \geq C a_\lambda^{\frac{q}{2q-1}}.$$

By an inequality of P.-L. Lions [14, Lemma I.2] (see also [19, lemma 2.3; 29, (2.4); 30, lemma 1.21]), we have on the one hand

$$(3.2) \quad \begin{aligned} \int_{\mathbb{R}^N} |w_\lambda|^{\frac{2pq}{2q-1}} &\leq C_1 \int_{\mathbb{R}^N} (|\nabla w_\lambda|^2 + |w_\lambda|^2) \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_\lambda|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{1}{p}(2 - \frac{1}{q})} \\ &= C_1 \left(\sup_{x \in \mathbb{R}^N} \int_{B_1(x)} |w_\lambda|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{1}{p}(2 - \frac{1}{q})}. \end{aligned}$$

By combining (3.1) and (3.2), we deduce that there exists a family $(x_\lambda)_{\lambda \geq 0}$ of points in \mathbb{R}^N such that

$$(3.3) \quad C_1 \liminf_{\lambda \rightarrow \infty} \left(\int_{B_1(x_\lambda)} |w_\lambda|^{\frac{2pq}{2q-1}} \right)^{1 - \frac{1}{p}(2 - \frac{1}{q})} \geq \liminf_{\lambda \rightarrow \infty} a_\lambda^{\frac{q}{2q-1}} > 0,$$

by Claim 1.

There exists thus a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $[0, +\infty)$ such that $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ and the sequence $(w_{\lambda_n}(\cdot + x_{\lambda_n}))_{n \in \mathbb{N}}$ converges weakly to some function $w \in H^1(\mathbb{R}^N)$. This proves (2). By Rellich's compactness theorem, it follows that the sequence $(w_n)_{n \in \mathbb{N}}$ converges to w strongly in $L^r_{\text{loc}}(\mathbb{R}^N)$ whenever $\frac{1}{r} > \frac{1}{2} - \frac{1}{N}$. From the convergence in $L^{\frac{2pq}{2q-1}}_{\text{loc}}(\mathbb{R}^N)$, we deduce in view of (3.3) that

$$(3.4) \quad \int_{B_1} |w|^{\frac{2pq}{2q-1}} = \lim_{n \rightarrow \infty} \int_{B_1} |w_n|^{\frac{2pq}{2q-1}} = \lim_{n \rightarrow \infty} \int_{B_1(x_{\lambda_n})} |w_{\lambda_n}|^{\frac{2pq}{2q-1}} > 0,$$

and (1) follows. The assertion (3) then follows up to extraction of a subsequence from the strong convergence in $L^2_{\text{loc}}(\mathbb{R}^N)$.

In order to prove (4), we first note that

$$(3.5) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n - w|^p &\leq \limsup_{R \rightarrow \infty} 2^{p-1} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R} |w_n|^p + \limsup_{R \rightarrow \infty} 2^{p-1} \int_{\mathbb{R}^N \setminus B_R} |w|^p \\ &\quad + \limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{B_R} |w_n - w|^p. \end{aligned}$$

By Rellich's compactness theorem, we have for every $R \in (0, +\infty)$,

$$(3.6) \quad \limsup_{n \rightarrow \infty} \int_{B_R} |w_n - w|^p = 0.$$

Next, we observe that, by Fatou's lemma,

$$\int_{\mathbb{R}^N} |w|^p \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n|^p < +\infty,$$

so that by Lebesgue's dominated convergence theorem, it follows that

$$(3.7) \quad \limsup_{R \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R} |w|^p = 0.$$

In order to conclude, we observe that for each $n \in \mathbb{N}$, by definition of a_{λ_n} we have

$$\begin{aligned} \int_{\mathbb{R}^N} (V^+ * |w_n|^p) |w_n|^p &\geq \int_{\mathbb{R}^N} (V^+ * |w_n|^p) |w_n|^p - a_{\lambda_n} \\ &= \int_{\mathbb{R}^N} (V_{\lambda_n}^- * |w_n|^p) |w_n|^p \\ &\geq \int_{\mathbb{R}^N \setminus B_R} \left(\int_{B_1} V_{\lambda_n}^-(x-y) |w_n(y)|^p dy \right) |w_n(y)|^p dy \\ &\geq \left(\inf_{\mathbb{R}^N \setminus B_{R+1}} V_{\lambda_n}^- \right) \left(\int_{\mathbb{R}^N \setminus B_R} |w_n|^p \right) \left(\int_{B_1} |w_n|^p \right) \end{aligned}$$

The left-hand side is bounded in view of (V_1) and by Young's convolution inequality (Proposition 2.1). Therefore, we have

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R} |w_n|^p \leq \frac{C_2}{(\inf_{\mathbb{R}^N \setminus B_{R+1}} V^-) \int_{B_1} |w_n|^p}.$$

By the assumption (V_4) and by (3.4), we deduce that

$$(3.8) \quad \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_R} |w_n|^p = 0.$$

Therefore, in view of (3.5), (3.6), (3.7) and (3.8), we have

$$(3.9) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n - w|^p = 0.$$

The assertion (4) follows then.

For the assertion (5), we first observe that for each $n \in \mathbb{N}$,

$$\inf_{B_1} (V_{\lambda_n}^- * |w_n|^p) \leq \frac{\int_{\mathbb{R}^N} (V_{\lambda_n}^- * |w_n|^p) |w_n|^p}{\int_{B_1} |w_n|^p} \leq C_3,$$

in view of (3.4). Therefore, there exists a sequence of points $(x_n)_{n \in \mathbb{N}}$ in B_1 such that

$$\limsup_{n \rightarrow \infty} (V_{\lambda_n}^- * |w_n|^p)(x_n) < +\infty.$$

By assumption (V_2) and by Lemma 3.1, we have in view of the boundedness of the sequence $(w_n)_{n \in \mathbb{N}}$ in $L^p(\mathbb{R}^N)$,

$$0 \leq (V_{\lambda_n}^- * |w_n|^p)(0) \leq (V_{\lambda_n}^- * |w_n|^p)(x_n) + \int_{\mathbb{R}^N} |V(-y) - V(x_n - y)| |w_n(y)|^p dy \leq C_4.$$

This implies by Fatou's lemma that

$$0 \leq (V * |w|^p)(0) \leq \liminf_{n \rightarrow \infty} (V_{\lambda_n}^- * |w_n|^p)(0) < +\infty.$$

The sequence $((V_{\lambda_n}^- * |w_n|^p)(0))_{n \in \mathbb{N}}$ is bounded and, up to the extraction of a subsequence, we can assume that the sequence of real numbers $((V_{\lambda_n}^- * |w_n|^p)(0) - (V^- * |w|^p)(0))_{n \in \mathbb{N}}$ converges to some $\mu \in [0, +\infty)$. Moreover by Lebesgue's monotone convergence theorem, we have,

$$(3.10) \quad (V * |w|^p)(0) = \lim_{n \rightarrow \infty} (V_{\lambda_n}^- * |w|^p)(0).$$

Next, we observe that for every $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$, by assumption (V_2) and by Lemma 3.1

$$\begin{aligned} & |(V_{\lambda_n}^- * |w_n|^p)(x) - (V_{\lambda_n}^- * |w|^p)(x) - (V_{\lambda_n}^- * |w_n|^p)(0) + (V_{\lambda_n}^- * |w|^p)(0)| \\ & \leq \int_{\mathbb{R}^N} |V_{\lambda_n}^-(x - y) - V_{\lambda_n}^-(-y)| \left| |w_n(y)|^p - |w(y)|^p \right| dy \\ & \leq C_5(|x| + 1) \int_{\mathbb{R}^N} \left| |w_n|^p - |w|^p \right|, \end{aligned}$$

and thus by (4) and (3.10), we have

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \frac{|(V_{\lambda_n}^- * |w_n|^p)(x) - (V^- * |w|^p)(x) - \mu|}{1 + |x|} = 0,$$

and the assertion (5) follows. \diamond

Claim 3. *We have*

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} (V * |w|^p) |w|^p = a = \lim_{\lambda \rightarrow \infty} a_\lambda$$

and the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly to w in $H^1(\mathbb{R}^N)$.

Proof of the claim. We first observe that, by the definitions of a in (1.3) and of a_λ in (2.2), in view of the fact that $V_\lambda \geq V$ by definition in (2.1), we have $a_\lambda \geq a$.

By (4) in Claim 2, the Sobolev embedding and the Hölder inequality, the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly to w in $L^{\frac{2pq}{2q-1}}(\mathbb{R}^N)$ and thus by Young's convolution inequality (Proposition 2.1)

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V^+ * |w_n|^p) |w_n|^p = \int_{\mathbb{R}^N} (V^+ * |w|^p) |w|^p.$$

In view of (3) in Claim 2, the sequence $(|w_n|^p)_{n \in \mathbb{N}}$ converges to $|w|^p$ almost everywhere in \mathbb{R}^N . Since $V_\lambda^- \rightarrow V^-$ everywhere in \mathbb{R}^N as $\lambda \rightarrow \infty$, we deduce by an application of Fatou's lemma on $\mathbb{R}^N \times \mathbb{R}^N$ that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V_\lambda^- * |w_n|^p) |w_n|^p \geq \int_{\mathbb{R}^N} (V^- * |w|^p) |w|^p.$$

Hence we have

$$(3.11) \quad \int_{\mathbb{R}^N} (V * |w|^p) |w|^p \geq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (V * |w_n|^p) |w_n|^p = \lim_{n \rightarrow \infty} a_{\lambda_n} = a.$$

On the other hand we have, by definition of a in (1.3),

$$(3.12) \quad \int_{\mathbb{R}^N} (V * |w|^p) |w|^p \leq a \left(\int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) \right)^p \leq a \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2) = a.$$

The claim then follows from (3.11) and (3.12). \diamond

Claim 4. *We have $V * |w|^p \in L_{\text{loc}}^s(\mathbb{R}^N)$ for every $s \in [1, +\infty)$ such that $\frac{1}{s} \geq p(\frac{1}{2} - \frac{1}{N}) + \frac{1}{q} - 1$ and for every $\varphi \in C_c^1(\mathbb{R}^N)$,*

$$a \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi + w \varphi = \int_{\mathbb{R}^N} (V * |w|^p) |w|^{p-2} w \varphi.$$

Proof of the claim. First, for every $\varphi \in C_c^1(\mathbb{R}^N)$,

$$(3.13) \quad a_{\lambda_n} \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \varphi + w_n \varphi = \int_{\mathbb{R}^N} (V_{\lambda_n} * |w_n|^p) |w_n|^{p-2} w_n \varphi$$

By Claim 2 and Sobolev's embedding theorem, the sequence $(w_n)_{n \in \mathbb{N}}$ converges strongly to w in $L^r(\mathbb{R}^N)$ if $\frac{1}{r} \geq \frac{1}{2} - \frac{1}{N}$. By Young's convolution inequality (Proposition 2.1), the sequence $(V^+ * |w_n|^p)_{n \in \mathbb{N}}$ converges strongly to $(V^+ * |w|^p)_{n \in \mathbb{N}}$ in $L_{\text{loc}}^s(\mathbb{R}^N)$ whenever $\frac{1}{s} \geq p(\frac{1}{2} - \frac{1}{N}) + \frac{1}{q} - 1$. On the other hand, since for each $n \in \mathbb{N}$, the function $V_{\lambda_n}^-$ is

bounded and $w_n \in L^p(\mathbb{R}^N)$, the function $V_{\lambda_n}^- * |w_n|^p$ is also bounded and we deduce from Claim 2 that the sequence $(V_{\lambda_n}^- * |w_n|^p)_{n \in \mathbb{N}}$ converges to $\mu + V^- * |w|^p$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$. Therefore, by letting $n \rightarrow \infty$ in (3.13), we obtain

$$(3.14) \quad a \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi + w \varphi = \int_{\mathbb{R}^N} (V * |w|^p) |w|^{p-2} w \varphi + \mu \int_{\mathbb{R}^N} |w|^{p-2} w \varphi.$$

Since $V * |w|^p \in L^s(\mathbb{R}^N)$ whenever $\frac{1}{s} \geq p(\frac{1}{2} - \frac{1}{N}) + \frac{1}{q} - 1$, the identity (3.14) still holds when $\varphi \in H^1(\mathbb{R}^N)$ has compact support by the density of smooth functions in the Sobolev spaces. We conclude the proof of the claim by proving that $\mu = 0$.

By a variant of Lemma 3.2 and by Claim 3, we have

$$\begin{aligned} \int_{\mathbb{R}^N} (V * |w|^p) |w|^p &= a \int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) \\ &= \int_{\mathbb{R}^N} (V * |w|^p) |w|^p - \mu \int_{\mathbb{R}^N} |w|^p, \end{aligned}$$

which can only hold if $\mu = 0$. ◇

We conclude now the proof of Theorem 1.1, by setting $u = a^{\frac{-1}{2p-2}} w$. By Claim 4, the function u satisfies the Choquard equation (C). Let us show that u is a groundstate. By a direct computations, its energy value is

$$\mathcal{I}(u) = \frac{a_\lambda^{-\frac{1}{p-1}}}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + |w|^2) - \frac{a_\lambda^{-\frac{p}{p-1}}}{2p} \int_{\mathbb{R}^N} (V_\lambda * |w|^p) |w|^p = \left(\frac{1}{2} - \frac{1}{2p}\right) a_\lambda^{-\frac{1}{p-1}}.$$

If $v \in H^1(\mathbb{R}^N)$ is a weak solution to the Choquard equation (C), then we have by Lemma 3.2

$$a_\lambda \geq \frac{\int_{\mathbb{R}^N} (V * |v|^p) |v|^p}{\left(\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2)\right)^p} = \left(\frac{2p}{p-1} \mathcal{I}(v)\right)^{p-1},$$

and the conclusion follows. □

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