

Long time stability of small finite gap solutions of the cubic Nonlinear Schrödinger equation on \mathbb{T}^2

A. Maspero ^{*}, M. Procesi [†]

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Abstract

In this paper we study long time stability of a class of nontrivial, quasi-periodic solutions depending on one spacial variable of the cubic defocusing non-linear Schrödinger equation on the two dimensional torus. We prove that these quasi-periodic solutions are orbitally stable for finite but long times, provided that their Fourier support and their frequency vector satisfy some complicated but explicit condition, which we show holds true for most solutions.

The proof is based on a normal form result. More precisely we expand the Hamiltonian in a neighborhood of a quasi-periodic solution, we reduce its quadratic part to diagonal constant coefficients through a KAM scheme, and finally we remove its cubic terms with a step of nonlinear Birkhoff normal form. The main difficulty is to impose second and third order Melnikov conditions; this is done by combining the techniques of reduction in order of pseudo-differential operators with the algebraic analysis of resonant quadratic Hamiltonians.

Contents

1	Introduction and main result	2
1.1	The stability problem	2
1.2	Scheme of the Proof	5
2	A Birkhoff normal form result	7
3	Functional setting	14
3.1	Quadratic Hamiltonians	15
3.2	Quasi-Töplitz structure	18
4	Preparation of the Hamiltonian	19
4.1	The Birkhoff map for the one dimensional cubic NLS	19
4.2	Adapted variables	20
5	Normal form of the quadratic terms	25
5.1	Proof of Theorems 5.1 and 5.3	26
5.1.1	Step 1: descent method.	27
5.1.2	Step 2: removal of Birkhoff non resonant monomials at order ε	32
5.1.3	Step 3: diagonalization of the Birkhoff resonant terms	34
6	Non-degeneracy of the Hamiltonian $\tilde{D}(\lambda, \varepsilon)$	39
7	KAM algorithm for the reducibility	45
8	Birkhoff normal form for the cubic terms	52
9	Proof of Theorem 1.3	53

^{*}International School for Advanced Studies (SISSA), Via Bonomea 265, 34136, Trieste, Italy

Email: alberto.maspero@sissa.it

[†]Dipartimento di Matematica e Fisica Università degli Studi Roma Tre, L.go S. Leonardo Murialdo 1, 00146 Roma, Italy

Email: procesi@mat.uniroma3.it

A Proof of Lemma 2.5	54
B Proof from Section 3	55
B.1 Proof of Proposition 3.13	55
B.2 Proof of Lemma 3.14	56
B.3 Proof of Lemma 3.18	59
B.4 Proof of Lemma 3.19	60
C Proof of Proposition 7.7	60
D A crash course on polynomial rings and algebraic extensions	64

1 Introduction and main result

1.1 The stability problem

In this paper we study long time stability of a class of nontrivial, quasi-periodic solutions depending on one spatial variable of the cubic defocusing non-linear Schrödinger equation (NLS) on the two dimensional torus \mathbb{T}^2 :

$$i\partial_t v = -\Delta v + |v|^2 v, \quad (x, y) \in \mathbb{T}^2. \quad (1.1)$$

In particular we prove that there exists a family of such solutions which are orbitally stable in $H^p(\mathbb{T}^2)$ for finite but long times, for any $p > 1$. This means, essentially, that if we take an initial datum close enough in $H^p(\mathbb{T}^2)$ to one of these solutions, then we stay in a neighborhood of the *orbit* of the solution for long time, in the $H^p(\mathbb{T}^2)$ topology.

Solutions of (1.1) depending on only one-variable are completely characterized; indeed the restriction of (1.1) to the subspace of functions depending only on one-variable, say x , is the 1-dimensional defocusing NLS (dNLS)

$$i\partial_t q = -\partial_{xx} q + |q|^2 q \quad x \in \mathbb{T}, \quad (1.2)$$

which is a well known integrable system [ZS71, ZM74]. The dynamics of dNLS is completely understood: the phase space is foliated in invariant tori of finite and infinite dimensions and the dynamics on each torus is either quasiperiodic (namely it is a combination of periodic motions with a finite number of different frequencies) or almost periodic (a superimposition of periodic motions with infinitely many different frequencies).

Actually even more is true: Grébert and Kappeler [GK14] showed that there exists a globally defined map $\Phi : L^2(\mathbb{T}) \rightarrow \ell^2 \times \ell^2$, $q \mapsto (z_m, \bar{z}_m)_{m \in \mathbb{Z}}$, the *Birkhoff map*, which introduces *Birkhoff coordinates*, namely complex conjugates canonical coordinates $(z_m, \bar{z}_m)_{m \in \mathbb{Z}}$, with the property that the dNLS Hamiltonian, once expressed in such coordinates, is a real analytic function depending only on the actions $I_m := |z_m|^2$. As a consequence, in Birkhoff coordinates the flow (1.2) is conjugated to an infinite chain of nonlinearly coupled oscillators:

$$i\dot{z}_m = \alpha_m^{\text{dNLS}}(I) z_m \quad \forall m \in \mathbb{Z}, \quad (1.3)$$

where the $\alpha_m^{\text{dNLS}}(I)$ are frequencies depending only on the actions $(I_m)_{m \in \mathbb{Z}}$.

Then, having fixed an arbitrary number $d \in \mathbb{N}$, an ordered set $\mathcal{S}_0 := (\mathfrak{m}_1, \dots, \mathfrak{m}_d) \subset \mathbb{Z}$ of modes and a vector $I_m = (I_{m_i})_{1 \leq i \leq d} \subseteq \mathbb{R}_{>0}^d$, the set

$$\mathbb{T}^d \equiv \mathbb{T}^d(\mathcal{S}_0, I_m) := \{(z_m)_{m \in \mathbb{Z}} : |z_{\mathfrak{m}_i}|^2 = I_{m_i} \text{ for } 1 \leq i \leq d, \quad z_m = 0 \text{ if } m \notin \mathcal{S}_0\} \quad (1.4)$$

is an invariant torus of (1.2) of dimension d which is supported on the set \mathcal{S}_0 . We say that $q(t)$ is a *finite gap solution* of (1.2) if it is supported (in Birkhoff coordinates) on a finite dimensional torus, i.e. $\forall t \quad \Phi(q(t)) \subseteq \mathbb{T}^d(\mathcal{S}_0, I_m)$, for some set \mathcal{S}_0 of cardinality $d < \infty$ and vector I_m . Any finite gap solution is quasiperiodic in time, $q(t) \equiv q(\omega t)$, with a frequency vector $\omega \in \mathbb{R}^d$ which is specified by the values of the actions I_m and thus it is associated to the torus itself. In particular, if ω is nonresonant, than the orbit of the finite gap solution is dense on the torus.

Clearly any finite gap solution of (1.2) is also a solution of (1.1). Of course if we take an initial datum of (1.1) that is close to the initial datum of a finite gap solution but *not y-independent* we expect the dynamics to be very complicated, but to stay close to the one of the integrable subspace \mathbb{T}^d for long times.

The purpose of this work is to prove, roughly speaking, that “many” tori $\mathbb{T}^d(\mathcal{S}_0, I_m)$ are orbitally stable

for long times: if one chooses an initial datum which is δ -close to one of these tori, then the solution of (1.1) stays close to the torus for times of order δ^2 . In order to make this statement precise we need to introduce some notations. For any $p > 1$ we denote $H^p(\mathbb{T}^n)$ the usual Sobolev space of functions whose Fourier coefficients have finite norm

$$\|v\|_{H^p(\mathbb{T}^n)} := \left(\sum_{\vec{j} \in \mathbb{Z}^n} \langle \vec{j} \rangle^{2p} |v_{\vec{j}}|^2 \right)^{1/2},$$

where $\langle \vec{j} \rangle := \sqrt{1 + |\vec{j}|^2}$. From now on we shall systematically identify $H^p(\mathbb{T})$ with the closed subspace of $H^p(\mathbb{T}^2)$ of functions depending only on the x variable. Consequently $\Phi^{-1}(\mathbb{T}^d)$ is a closed torus of $H^p(\mathbb{T}) \subset H^p(\mathbb{T}^2)$.

We define now the notion of closeness to a torus \mathbb{T}^d which we will use in the following:

Definition 1.1. *We say that $v \in H^p(\mathbb{T}^2)$ is δ -close to the torus $\mathbb{T}^d = \mathbb{T}^d(\mathcal{S}_0, I_m)$ if*

$$\text{dist}(v, \Phi^{-1}(\mathbb{T}^d))_{H^p(\mathbb{T}^2)} \leq \delta \quad \text{and} \quad ||z_m|^2 - I_m| \leq \delta^2 \quad \forall m \in \mathcal{S}_0, \quad (1.5)$$

where

$$z \equiv (z_m, \bar{z}_m)_{m \in \mathbb{Z}} := \Phi(v(x, 0)).$$

Namely we ask that the distance between v and the preimage $\Phi^{-1}(\mathbb{T}^d)$ is of order δ , and furthermore that the Birkhoff actions of $v(\cdot, 0)$ with indexes in \mathcal{S}_0 are δ^2 -close to the corresponding actions I_m of the torus.

With this definition of δ -closeness, we can now define δ -orbital stability:

Definition 1.2. *A torus $\mathbb{T}^d(\mathcal{S}_0, I_m)$ is said to be (δ, K) -orbitally stable for $|t| < T$ if for any initial datum $v_0 \in H^p(\mathbb{T}^2)$ which is δ -close to $\mathbb{T}^d(\mathcal{S}_0, I_m)$, then the solution $v(t, x)$ of (1.1) stays $K\delta$ -close to $\mathbb{T}^d(\mathcal{S}_0, I_m)$ for any $|t| \leq T$.*

We state now our main theorem. Up to our knowledge, it is the first result of stability for quasi-periodic solutions in higher dimensional setting.

Theorem 1.3. *Fix $p > 1$. For any generic choice of support sites \mathcal{S}_0 there exist $\varepsilon_*, T_*, \mathbf{k}_*, \mathbf{K}_* > 0$ and for any $0 < \varepsilon < \varepsilon_*$, there exists a positive measure Cantor-like set $\mathcal{I} \in (0, \varepsilon)^d$ such that the following holds true. For any $I_m \in \mathcal{I}$, any $\delta < \mathbf{k}_* \sqrt{\varepsilon}$, the torus $\mathbb{T}^d(\mathcal{S}_0, I_m)$ is (δ, \mathbf{K}_*) -orbitally stable for $|t| < T_*/\delta^2$.*

In order to make the theorem precise, we need to specify what we mean by *generic set* \mathcal{S}_0 . The point is that we cannot prove Theorem 1.3 for any arbitrary torus, but we need to impose some restrictions both on the *Birkhoff support* \mathcal{S}_0 of the torus, and on the values of the *Birkhoff actions* I_m . Concerning the Birkhoff support \mathcal{S}_0 , we need to select it in a complicated but explicit way, see Definition 2.3–2.4. These sets are generic in the following sense (see Lemma 2.5): given any $R \gg 1$ if we choose the elements of \mathcal{S}_0 randomly in $[-R, R]$, then the probability of choosing a “good” set (i.e. one for which we can prove our Theorem) goes to one as R goes to infinity. It is possible that the conditions we give on the support can be weakened, but this produces serious technical difficulties. In any case we suspect that some selection of the support of the torus is necessary for our stability result.

Once we choose the support, we need also to select the Birkhoff actions I_m of the torus. This selection is inevitable, since we can produce a positive measure set of actions I_m such that the torus $\mathbb{T}^d(\mathcal{S}_0, I_m)$ is linearly hyperbolic, hence *linearly unstable* in $H^p(\mathbb{T}^2)$ (see Remark 5.13). Clearly we want to rule out such a behavior. Thus we also impose conditions on the actions which are quite explicit and give rise to a Cantor-like set of positive measure.

Theorem 1.3 is essentially a result on the *H^p -orbital stability* of many small finite gap solutions. Indeed, consider a finite gap solution $q(t)$ with $\Phi(q(t)) \in \mathbb{T}^d(\mathcal{S}_0, I_m)$, and \mathcal{S}_0, I_m satisfying the conditions of the theorem. The fact that the actions belong to $(0, \varepsilon)^d$ implies that any finite gap q supported on $\mathbb{T}^d(\mathcal{S}_0, I_m)$ is small in size, and more precisely one has the bound $\|q\|_{L^2(\mathbb{T})} \leq \mathbf{d}\sqrt{\varepsilon}$ (see formula (4.4)). Then Theorem 1.3 says essentially that for any initial datum $v_0 \in H^p(\mathbb{T}^2)$ sufficiently close to $q(0)$, the solution $v(t)$ stays close to the torus supporting the orbit of $q(t)$ for long times (remark that we cannot hope to control directly the quantity $\|v(t) - q(t)\|_{H^p(\mathbb{T}^2)}$ for long times).

Our work should be compared with the results of Faou, Gauckler, Lubich [FGL13] and Procesi, Procesi [PP15], which both deal with stability issues of nontrivial solutions of (1.1). In [FGL13], the authors consider only 1-gap solutions, but are able to prove orbital stability for times of order δ^{-N} for arbitrary $N > 0$. On the other hand, in [PP15] the authors prove only linear stability, namely that the torus is

stable for times of order δ^{-1} , but for a much larger class of tori which depend on both variables x and y . Note however that the class of solutions considered in [PP15] does not contain \mathfrak{d} -gap solutions for $\mathfrak{d} > 1$. For a more detailed comparison, see Remark 2.13 and Remark 2.14 below.

The proof of Theorem 1.3 is based on a Birkhoff normal form result. To develop such a normal form, first we introduce *adapted coordinates*, which are canonical coordinates $(\mathcal{Y}, \theta, a, \bar{a})$ which describe a neighborhood of the invariant torus $\mathbb{T}^{\mathfrak{d}}(\mathcal{S}_0, I_m)$ and such that $\mathcal{Y} = 0, a = \bar{a} = 0$ is the torus itself. In the classical Hamiltonian language, the \mathcal{Y} are the variables tangential to the torus, while the a, \bar{a} are the normal ones. In such variables the NLS Hamiltonian takes the form

$$\omega \cdot \mathcal{Y} + \sum_{\bar{j}} |\bar{j}|^2 |a_{\bar{j}}|^2 + \mathcal{H}^{(\geq 0)}(\mathcal{Y}, \theta, a, \bar{a}) \quad (1.6)$$

where ω are the frequencies associated to the torus, and $\mathcal{H}^{(\geq 0)}$ is a perturbation term of size ε (recall that ε is the size of the actions) and with a zero of order 2 in the variables $(\mathcal{Y}, a, \bar{a})$. Then the stability of the torus $\mathbb{T}^{\mathfrak{d}}$ is equivalent to the stability of the zero solution for the Hamiltonian equations of (1.6). The classical methods used in this problem consist in developing a normal form theory in which one reduces to diagonal, θ -independent form the terms of degree two in (a, \bar{a}) , and removes iteratively the nonresonant terms of higher and higher degree in $(\mathcal{Y}, a, \bar{a})$. Here we do this up to order three; namely after reducing to constant coefficients the quadratic part of the Hamiltonian, we eliminate also the cubic terms, and we are left with an Hamiltonian (see (2.33)) in which the coupling term has a vector field whose Sobolev norm is $\sim \delta^3$; this leads to the time of stability of Theorem 1.3.

Before closing this introduction, we want to put our work in some historical context. It is well known since the work of Bourgain [Bou93] that (1.1) is globally in time well posed in the Sobolev space $H^p(\mathbb{T}^d)$ for any $p > 0$ and $d = 1, 2$. Since then, the problem of describing qualitatively the dynamics of NLS solutions has been widely studied, and the results obtained so far can be essentially divided into three groups: in the first group of papers, the authors study stability of periodic or quasiperiodic solutions of NLS on \mathbb{T} or \mathbb{T}^2 for long but finite time [Zhi01, Bam99, Bou00, FGL13, GH07a, GH07b, GP15, Wil15]. In the second group, the authors study instability phenomena, showing that (on a possibly very large time scale) the solution goes arbitrary distant from the initial datum [CKS⁺10, Han14, Gua14, HP15, GK15, GHP16]. Finally, the third group of papers concerns with the existence of quasi-periodic solutions of (1.1) for any time [EK10, GXY, Wan16, PX13, PP15, PP16].

We start with the results in the first group. In dimension $d = 1$, it is known since the work of Zhidkov [Zhi01, Sect. 3.3] that plane waves solution of (1.2) (namely solutions of the form $v(t, x) = Ae^{i(mx - \omega t)}$ with $\omega = m^2 + A^2$) are H^1 orbitally stable for any time. Remark that plane waves solutions are exactly 1-gap solutions of (1.2) (see e.g. [GK02, Lemma 3.5]), thus their orbit is an invariant torus of dimension $\mathfrak{d} = 1$. Orbital stability of \mathfrak{d} -finite gap solutions with $\mathfrak{d} > 1$ was studied by Bambusi [Bam99], who proved H^1 -orbital stability for exponentially long times, namely times of order $\sim e^{T_0 \delta^{-1/d}}$. Furthermore the author does not impose any condition on the choice of the Birkhoff support; this is due to the fact that resonances in dimension 1 have a much simpler structure.

In [Bou00], Bourgain proved H^s orbital stability of small finite-gap solutions (with s sufficiently large) for times of order δ^{-N} , with an arbitrary N . Finally Cong, Liu and Yuan [CLY16] prove a similar stability result but for the KAM tori of a 1-d NLS with external parameters.

In dimension $d = 2$, much less is known. Faou, Gauckler and Lubich [FGL13] proved that plane waves solutions (i.e. 1-gap solutions) of (1.1) are H^s orbitally stable for s large enough and for times of order δ^{-N} , for arbitrary N . Note that [FGL13] consider *large* 1-gap solutions and control the distance from the torus for times much longer than ours. The reason is that for such solutions the authors know the exact formula for the tangential and normal frequencies and therefore they can impose Melnikov conditions at any order and perform arbitrary many steps of normal form (see Remark 2.13 for more details).

We now describe the results of the second group, concerning instability phenomena for (1.1). While the integrability of (1.2) prevents any phenomena of growth of Sobolev norms in dimension 1, the situation in dimension $d > 1$ is much more complicated. Bourgain [Bou96] gives upper bounds on the solutions of (1.1) of the form

$$\|v(t)\|_{H^s(\mathbb{T}^2)} \leq t^{2(s-1)^+} \|v(0)\|_{H^s(\mathbb{T}^2)}, \quad t > 0, \quad (1.7)$$

for s sufficiently large (see also [Sta97, Soh11, CKO12, PTV17] for more recent results on upper bounds of the form (1.7) for nonlinear Schrödinger equations and [Bou99, Del10, MR17, BGMRb, Mon17] for linear

time-dependent Schrödinger equations). Estimate 1.7 leaves open the question whether it is possible to construct a solution whose Sobolev norm is unbounded. In the last few years many efforts have been done in this direction. Colliander et al. [CKS⁺10] proved, for any $\epsilon, K > 0$ and $s > 1$, the existence of a initial datum $u(0) \in H^s(\mathbb{T}^2)$ and a time $T > 0$ s.t. $\|u_0\|_{H^s(\mathbb{T}^2)} < \epsilon$ and $\|u(T)\|_{H^s(\mathbb{T}^2)} \geq K$. This result was later refined by Guardia and Kaloshin [GK15], Carles and Faou [CF12] and extended to different Schrödinger equations [HP15, Gua14, GHP16]. All these results deal with the instability of the zero solution. Hani [Han14] studied instability of 1-gap solutions, and proved that for any $0 < s < 1$, $\delta \ll 1, K \gg 1$, there exists a solution of (1.1) with $\|v(0) - Ae^{imx}\|_{H^s(\mathbb{T}^2)} \leq \delta$ and $\|v(T)\|_{H^s(\mathbb{T}^2)} \geq K$ for some later $T > 0$. Note that the instability happens in a topology different from the one of [FGL13]. The important question of the existence of solutions of (1.1) with unbounded Sobolev norms, i.e. s.t.

$$\overline{\lim}_{t \rightarrow \infty} \|v(t)\|_{H^s(\mathbb{T}^2)} = +\infty$$

is still an open problem, and it has been solved only in the product space $\mathbb{R} \times \mathbb{T}^2$ [HPTV15].

Finally, we describe the results of the third group, concerning existence of quasi-periodic solution for NLS. While in dimension $d = 1$ the existence of KAM quasi-periodic solution is nowadays well understood in several perturbed dNLS (we mention just the latest contributions [FP15, BKM16]; see reference therein), in dimension $d > 1$ the situation is much more complicated and it has been addressed only recently by Eliasson and Kuksin [EK10], Geng, Xu and You [GXY], Wang [Wan16] and Procesi with collaborators [PX13, PP15, PP16]. In particular in these last papers the authors proved that, *for most choices* of tangential sites, there exist families of small quasi-periodic solutions of (1.1) (depending on both x and y) supported essentially on the tangential sites. Such families of solutions give rise to both linearly stable and unstable KAM tori, but the question of nonlinear stability (or instability!) of such tori is still open (see Remark 2.14 for more comments).

1.2 Scheme of the Proof

It is worth to add some words about our strategy. The proof consists in three main steps.

The first step is to introduce *canonical* adapted coordinates in a neighborhood of a finite dimensional invariant torus.

The second step consists in a *reducibility* argument. We consider the part of $\mathcal{H}^{(\geq 0)}$ which is quadratic in (a, \bar{a}) , call it $\mathcal{H}^{(0)}$, and we reduce it to a diagonal form with constant coefficients.

The final step consists in applying one step of non linear Birkhoff Normal Form.

Step one: adapted coordinates. The question of introducing canonical coordinates in a neighborhood of a solution has been addressed in several papers, see e.g. [Kuk88, BB15, BM16]. Here however the difficulty is that we need to keep track of how such change of variables affects the constants of motion (namely the mass and momentum). Here we take advantage of the fact that mass and momentum are left unchanged by the Birkhoff map of dNLS, thus first we introduce Birkhoff coordinates on the invariant subspace of y -independent functions, and then pass to action-angle coordinates the sites where the finite gap is supported. Finally we extend this transformation as the identity to all the phase space. Actually for later development, it is necessary to know that the Birkhoff coordinates are majorant analytic; this was proved in [Mas17]. In such coordinates the Hamiltonian of NLS has the form (1.6).

Step two: reducibility. This is the most technical part of our argument. It is known that reducibility requires (i) to impose the so called *second order Melnikov* conditions and (ii) to perform a convergent KAM scheme. Both these procedures are particularly delicate in higher spacial dimensions, and they have been achieved for Schrödinger equations only in some special cases [EK09, PX13, CHP15, PP16, GP16, BGMRa].

Imposing second order Melnikov conditions means requiring lower bounds for expression such as

$$|\omega \cdot \ell + |\vec{j}_1|^2 \pm |\vec{j}_2|^2|. \quad (1.8)$$

Here there are two problems. First, since ω is an integer valued vector up to corrections of order ϵ , the quantity (1.8) can be of order ϵ and hence comparable with the size of the perturbation. Since these expressions appear as *denominators* in any diagonalization scheme our problem is *not* perturbative. Moreover, since we are working in dimension higher than one, it is well known that in order to perform a reduction one needs to know the asymptotics of the eigenvalues associated to the Hamiltonian vector field of $\sum |\vec{j}|^2 |a_{\vec{j}}|^2 + \mathcal{H}^{(0)}$.

In order to solve these problems we combine the techniques of reduction in order of pseudo-differential operators with the algebraic analysis of resonant quadratic Hamiltonians. More precisely, we construct

a change of variables which is *not* close to identity and which conjugates (1.6) to the form

$$\omega \cdot \mathcal{Y} + \sum_{\bar{j}} \tilde{\Omega}_{\bar{j}} |a_{\bar{j}}|^2 + \tilde{\mathcal{H}}^{(0)}(\theta, a, \bar{a}) + h.o.t. \quad (1.9)$$

Here $\tilde{\Omega}_{\bar{j}} - |\bar{j}|^2$ is of size ε , while $\tilde{\mathcal{H}}^{(0)}(\theta, a, \bar{a})$ is a perturbation of size ε^2 and *smoothing* in the following sense: the linear operator associated to its Hamiltonian vector field is the sum of a 2-smoothing term plus a term which gains 2-derivatives only in the x direction and is independent of y . This information allows us to extract the asymptotics of the frequencies required for the reduction, just as the Töplitz-Lipschitz matrices in [EK10].

In order to pass from (1.6) to (1.9) we first exploit the fact that the Hamiltonian vector field of (1.6) is of the form

$$2|q(\omega t, x)|^2 a(x, y) + q^2(\omega t, x) \bar{a}(x, y)$$

up to a finite rank remainder. Due to this special form we can extend to the two dimensional setting the techniques of reduction in orders developed in the one dimensional setting [PT01, IPT05, BBM14].

After this procedure the perturbation is still of *size* ε , but it is now smoothing. At this point we apply the strategy of [PP12], namely we explicitly construct a not close to identity change of variables which diagonalizes the resonant non-perturbative terms of the Hamiltonian (these changes of variables are similar to those used in the problems of almost reducibility, see [Eli01]). This construction requires that we select the sites \mathcal{S}_0 .

After this procedure, the Hamiltonian (1.6) is transformed into the Hamiltonian (1.9), and the relevant quantity to bound from below is now

$$\left| \omega \cdot \ell + \tilde{\Omega}_{\bar{j}_1} \pm \tilde{\Omega}_{\bar{j}_2} \right|, \quad (1.10)$$

where the normal frequencies have been corrected by some algebraic functions of the actions of size ε . Here the main difficulty is to prove that the terms of order ε in such expressions are not identically zero; to show this we use the algebraic techniques of irreducible polynomials, following [PPN]; here it is fundamental to exploit the fact that the mass and momentum are preserved.

At this point we perform a KAM reducibility scheme which puts the quadratic part of (1.9) to constant coefficients, conjugating it to the Hamiltonian

$$\omega \cdot \mathcal{Y} + \sum_{\bar{j}} \Omega_{\bar{j}} |a_{\bar{j}}|^2 + \mathcal{H}^{(1)}(\theta, a, \bar{a}) + h.o.t. \quad (1.11)$$

where $\mathcal{H}^{(1)}$ is cubic in (a, \bar{a}) .

Step three: Birkhoff normal form. Finally we perform one step of nonlinear Birkhoff normal form to remove $\mathcal{H}^{(1)}$ from (1.11). This is nowadays quite standard (see e.g. [BG06]), provided that (i) one is able to impose *third order* Melnikov conditions, i.e. to give lower bounds to expressions of the form

$$|\omega \cdot \ell + \Omega_{\bar{j}_1} \pm \Omega_{\bar{j}_2} \pm \Omega_{\bar{j}_3}|, \quad (1.12)$$

and (ii) one proves that the Hamiltonian (1.11) is majorant analytic.

Concerning the estimate of (1.12), we again use algebraic techniques to prove that the terms of order ε in such expressions are not identically zero, and the asymptotics of the eigenvalues in order to get quantitative bounds. Concerning the majorant analyticity of the Hamiltonian, we prove that each change of variables performed so far preserves this property; it is here that one needs the majorant analyticity of the Birkhoff map.

In conclusion we construct a nonlinear change of variables which conjugates (1.11) to

$$\omega \cdot \mathcal{Y} + \sum_{\bar{j}} \Omega_{\bar{j}} |a_{\bar{j}}|^2 + \mathcal{R}^{(\geq 2)}(\mathcal{Y}, \theta, a, \bar{a}), \quad (1.13)$$

where $\mathcal{R}^{(\geq 2)}$ contains only terms which are at least of order four in (a, \bar{a}) , or linear in \mathcal{Y} and quadratic in (a, \bar{a}) , or quadratic in \mathcal{Y} . As a consequence, its Hamiltonian vector field is of size $\sim \delta^3$, which implies the stability of zero for times of order δ^{-2} , thus proving our main theorem.

A final comment: It is a natural question whether it is possible to perform more steps of nonlinear Birkhoff normal form, removing from (1.13) monomials of higher and higher order, and obtaining a longer time

of stability. Performing these steps requires to be able to impose n^{th} order Melnikov conditions of the form

$$|\omega \cdot \ell + \Omega_{\bar{j}_1} \pm \dots \pm \Omega_{\bar{j}_n}| . \quad (1.14)$$

As before, one should verify that these expressions do not vanish identically except in resonant cases; this is what we are not able to prove so far. Indeed, the only information that we have is on the corrections of order ε to $\Omega_{\bar{j}}$, and one can produce examples where some linear combinations of them vanish identically already at order 4.

Structure of the paper: Section 2 contains some preparation in order to state precisely our result on normal form; furthermore we precise the notion of genericity of \mathcal{S}_0 . In Section 3 we define the class of smoothing Hamiltonians and study their properties. In Section 4 we construct the adapted coordinates $(\mathcal{Y}, \theta, a, \bar{a})$ and show that in these coordinates the Hamiltonian has the form (1.6). In Section 5 we begin the step of reducibility, and construct the change of coordinates not close to the identity which conjugates (1.6) to (1.9). In Section 6 we prove that the terms of order ε in expressions of the form (1.10) and (1.12) are not identically zero. Quantitative lower bounds for these expressions are proved in Appendix C. In Section 7 we conclude the reducibility step by performing a KAM scheme, conjugating (1.9) to (1.11). In Section 8 we perform the step of nonlinear Birkhoff normal form, conjugating (1.11) to (1.13). Finally in Section 9 we study the dynamics of (1.13) and prove the stability of zero for long times.

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2 A Birkhoff normal form result

In this section we state our results on the Birkhoff normal form (see Theorem 2.9 and Theorem 2.10) and describe the genericity conditions that we need to impose on \mathcal{S}_0 . In order to do this, we need some preparation.

Constants of motion. NLS on \mathbb{T}^2 has three constants of motion that we will constantly use. They are the Hamiltonian

$$H_{\text{NLS}}(v) := \int_{\mathbb{T}^2} |\nabla v(x, y)|^2 dx dy + \frac{1}{2} \int_{\mathbb{T}^2} |v(x, y)|^4 dx dy , \quad (2.1)$$

the mass

$$M(v) := \int_{\mathbb{T}^2} |v(x, y)|^2 dx dy , \quad (2.2)$$

and the (vector valued) momentum

$$P(v) := i \int_{\mathbb{T}^2} \bar{v}(x, y) \cdot \nabla v(x, y) dx dy . \quad (2.3)$$

Mass shift. Since the mass is a constant of motion, we make a trivial phase shift and consider the equivalent Hamiltonian

$$H(u) := \int_{\mathbb{T}^2} |\nabla u(x, y)|^2 dx dy + \frac{1}{2} \int_{\mathbb{T}^2} |u(x, y)|^4 dx dy - M(u)^2 \quad (2.4)$$

corresponding to the Hamilton equations

$$i\partial_t u = -\Delta u + |u|^2 u - 2M(u)u , \quad (x, y) \in \mathbb{T}^2 . \quad (2.5)$$

Clearly the solutions of (2.5) differ from the solutions of (1.1) only by a phase shift¹. If we pass u to the Fourier coefficients

$$u(x, y, t) := \sum_{\vec{j}=(m,n) \in \mathbb{Z}^2} u_{\vec{j}}(t) e^{i(mx+ny)}$$

the Hamiltonian (2.4) takes the form

$$H(u) = \sum_{\vec{j}=(m,n) \in \mathbb{Z}^2} |\vec{j}|^2 |u_{\vec{j}}|^2 - \frac{1}{2} \sum_{\vec{j}} |u_{\vec{j}}|^4 + \frac{1}{2} \sum_{\substack{\vec{j}_i \in \mathbb{Z}^2 \\ \vec{j}_1 - \vec{j}_2 + \vec{j}_3 - \vec{j}_4 = 0}}^* u_{\vec{j}_1} \bar{u}_{\vec{j}_2} u_{\vec{j}_3} \bar{u}_{\vec{j}_4} \quad (2.6)$$

where the \sum^* means the sum over the quadruples \vec{j}_i such that $\{\vec{j}_1, \vec{j}_3\} \neq \{\vec{j}_2, \vec{j}_4\}$, and it is a consequence of having removed the mass in (2.4).

The Hamiltonian (2.6) is analytic in the phase space

$$h^p(\mathbb{Z}^2) := \{u = (u_{\vec{j}})_{\vec{j} \in \mathbb{Z}^2} : \sum_{\vec{j} \in \mathbb{Z}^2} |u_{\vec{j}}|^2 \langle \vec{j} \rangle^{2p} := |u|_{h^p}^2 < \infty\} .$$

which we endow with the standard symplectic form

$$i \sum_{\vec{j} \in \mathbb{Z}^2} du_{\vec{j}} \wedge d\bar{u}_{\vec{j}} . \quad (2.7)$$

Finite gap solutions of NLS. As we already pointed out in the introduction, the subspace of functions depending only on the variable x is an invariant subspace. In Fourier coordinates, such subspace is identified by having Fourier coefficients $u_{(m,n)} = 0$ if $n \neq 0$. and it is clear by the structure of (2.6) that such a subspace is invariant for the dynamics. Actually, the Hamiltonian (2.6) restricted on such a subspace is nothing else than the Hamiltonian of the 1-dimensional defocusing NLS (dNLS) with the mass shift, namely

$$H_{\text{dm}}(q) := H_{\text{dnls}}(q) - M(q)^2 = \int_{\mathbb{T}} |\nabla q(x)|^2 dx + \frac{1}{2} \int_{\mathbb{T}} |q(x)|^4 dx - M(q)^2 , \quad (2.8)$$

whose equations of motion are

$$i\partial_t q = -\partial_{xx} q + |q|^2 q - 2M(q)q , \quad x \in \mathbb{T} . \quad (2.9)$$

Since dNLS is integrable and the mass M is an integral of motion for dNLS, the Birkhoff map conjugates (2.9) to a system of equations of the form (1.3), where the $(\alpha_m^{\text{dnls}}(I))_{m \in \mathbb{Z}}$'s are replaced by new frequencies $(\alpha_m(I))_{m \in \mathbb{Z}}$, see subsec. 4.1 for more details and references.

A consequence of this fact is that any torus $\mathbb{T}^{\text{d}}(\mathcal{S}_0, I_{\mathfrak{m}})$ of the form (1.4) is also an invariant torus for the massless dNLS equation (2.9), and the dynamics which is induced on it is quasi-periodic with frequency vector $\alpha_{\mathfrak{m}}(I_{\mathfrak{m}}) \equiv (\alpha_{\mathfrak{m}_1}(I_{\mathfrak{m}}), \dots, \alpha_{\mathfrak{m}_d}(I_{\mathfrak{m}}))$. Since the map $I_{\mathfrak{m}} \mapsto \alpha_{\mathfrak{m}}(I_{\mathfrak{m}})$ is highly nonlinear, for technical reasons it is more convenient to parametrize the vector $\omega := \alpha_{\mathfrak{m}}(I_{\mathfrak{m}})$ in a linear way; therefore we define

$$\omega^{(0)} := (\mathfrak{m}_1^2, \dots, \mathfrak{m}_d^2) \quad (2.10)$$

and for ε sufficiently small and $\lambda \in [\frac{1}{2}, 1]^{\text{d}}$, we set

$$\alpha_{\mathfrak{m}}(I_{\mathfrak{m}}) \equiv \omega^{(0)} - \varepsilon \lambda =: \omega(\lambda) . \quad (2.11)$$

As discussed in subsec. 4.1, **Se \mathcal{O}_0 e' $[1/2, 1]^{\text{d}}$ lo toglierei dovunque... senno there exists a positive measure compact domain \mathcal{O}_0 where** the action-to-frequency map can be inverted, obtaining a map $\lambda \rightarrow (I_{\mathfrak{m}_i}(\lambda, \varepsilon))_{1 \leq i \leq d}$ s.t.

$$I_{\mathfrak{m}_i}(\lambda, \varepsilon) = \varepsilon \lambda_i + O(\varepsilon^2) , \quad 1 \leq i \leq d . \quad (2.12)$$

In the following we will use $\omega \equiv \omega(\lambda)$ as the vector of frequencies of the finite gap solution. Such vector is chosen to be non-resonant so that the orbit of the finite gap solution is dense on $\mathbb{T}^{\text{d}}(\mathcal{S}_0, I_{\mathfrak{m}})$.

¹In order to show the equivalence we consider any solution $u(x, t)$ of (2.5) and consider the invertible map

$$u \mapsto v = u e^{-2iM(u)t} \quad \text{with inverse} \quad v \mapsto u = v e^{2iM(v)t}$$

then

$$iv_t = iu_t e^{-2iM(u)t} + 2M(u)u e^{-2iM(u)t} = (-\Delta u + |u|^2 u - 2M(u)u) e^{-2iM(u)t} + 2M(u)u e^{-2iM(u)t} = -\Delta v + |v|^2 v .$$

Adapted coordinates around a finite gap solution. We now introduce local coordinates in $h^p(\mathbb{Z}^2)$ adapted to the finite dimensional tori (1.4). Remark that, while in Birkhoff coordinates such tori have a very simple form, in the original coordinates the representation is not so trivial, and they have the form

$$q(\lambda; \theta, x) = \sqrt{\varepsilon} \left(\sum_{i=1}^{\mathfrak{d}} \sqrt{\lambda_i} e^{i\theta_i + i\mathfrak{m}_i x} + \varepsilon p(\lambda, \varepsilon; \theta, x) \right) \quad (2.13)$$

for some real analytic function $p(\lambda, \varepsilon; \theta, x)$.

To introduce coordinates around such tori we first apply the Birkhoff map Φ on $(u_{(m,0)})_{m \in \mathbb{Z}}$ leaving the remaining $(u_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}}$ unchanged, and we set $(z_m)_{m \in \mathbb{Z}} := \Phi((u_{(m,0)})_{m \in \mathbb{Z}})$. Next we pass the z_m with $m \in \mathcal{S}_0$ to symplectic action-angle variables defined close to the torus, setting

$$z_{\mathfrak{m}_i} = \sqrt{I_{\mathfrak{m}_i}(\lambda) + \mathcal{Y}_i} e^{i\theta_i}, \quad 1 \leq i \leq \mathfrak{d}, \quad z_m = a_{(m,0)}, \quad m \in \mathbb{Z} \setminus \mathcal{S}_0, \quad u_{(m,n)} = a_{(m,n)}, \quad n \neq 0. \quad (2.14)$$

In such a way we introduce coordinates $\mathcal{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_{\mathfrak{d}}) \in \mathbb{R}^{\mathfrak{d}}$, $\theta = (\theta_1, \dots, \theta_{\mathfrak{d}}) \in \mathbb{T}^{\mathfrak{d}}$, $a = (a_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$ such that $\mathcal{Y} = 0$, $a = 0$ describe torus $\mathbb{T}^{\mathfrak{d}}(\mathcal{S}_0, I_{\mathfrak{m}}(\lambda))$. Clearly here $\mathbb{Z}^2 \setminus \mathcal{S}_0 \equiv \mathbb{Z}^2 \setminus (\mathcal{S}_0 \times \{0\})$. We will use systematically such a notation.

We denote by $\Lambda : (\mathcal{Y}, \theta, a) \mapsto (u_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2}$ the map

$$u_{(m,n)} = a_{(m,n)}, \quad n \neq 0, \quad (u_{(m,0)})_{m \in \mathbb{Z}} = \Phi^{-1}((\sqrt{I_{\mathfrak{m}_i}(\lambda) + \mathcal{Y}_i} e^{i\theta_i})_{i=1, \dots, \mathfrak{d}}, (a_{(m,0)})_{m \in \mathbb{Z} \setminus \mathcal{S}_0}). \quad (2.15)$$

Since the Birkhoff map Φ is symplectic, the symplectic form (2.7) in the variables (\mathcal{Y}, θ, a) is given by

$$\sum_{i=1}^{\mathfrak{d}} d\mathcal{Y}_i \wedge d\theta_i + i \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} da_{\bar{j}} \wedge d\bar{a}_{\bar{j}}. \quad (2.16)$$

Next we describe the phase space and its topology. We fix once and for all a real $p > 1$ and define the phase space $\mathbb{C}^{\mathfrak{d}} \times \mathbb{T}^{\mathfrak{d}} \times \mathfrak{h}^p$ where

$$\mathfrak{h}^p \equiv \mathfrak{h}^p(\mathcal{S}_0) := \{\mathbf{a} = ((a_{\bar{j}}, \bar{a}_{\bar{j}}))_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \in h^p(\mathbb{Z}^2 \setminus \mathcal{S}_0), \quad \|a\| \equiv |a|_{h^p} < \infty\}.$$

Finally we define the complex domain

$$D(s, r) := \mathbb{T}_s^{\mathfrak{d}} \times D(r)$$

where

$$D(r) := \left\{ \mathcal{Y} \in \mathbb{C}^{\mathfrak{d}} : |\mathcal{Y}|_1 := \sum_{i=1}^{\mathfrak{d}} |\mathcal{Y}_i| < r^2, \quad \mathbf{a} \in \mathfrak{h}^p : \|\mathbf{a}\| < r \right\},$$

$$\mathbb{T}_s^{\mathfrak{d}} := \{\theta \in \mathbb{C}^{\mathfrak{d}} : \operatorname{Re}(\theta) \in \mathbb{T}^{\mathfrak{d}}, \quad |\operatorname{Im}(\theta)| < s\}.$$

We will show in Sec. 4 that for $r < \sqrt{\varepsilon}$ the map (2.14) is well defined and analytic from $D(s, r)$ to a neighborhood of the torus $\mathbb{T}^{\mathfrak{d}}(\mathcal{S}_0, I_{\mathfrak{m}}(\lambda))$.

Norm on vector fields. We describe now how to measure the norm of Hamiltonian vector fields. Recall that a Hamiltonian is a *real valued function* $\mathbb{C}^{\mathfrak{d}} \times \mathbb{T}_s^{\mathfrak{d}} \times \mathfrak{h}^p \rightarrow \mathbb{R}$. Given a Hamiltonian function $F(\mathcal{Y}, \theta, \mathbf{a})$ we associate to it its Hamiltonian vector field

$$X_F := (\partial_{\theta} F, -\partial_{\mathcal{Y}} F, -i\partial_{\bar{a}} F, i\partial_a F). \quad (2.17)$$

More in general consider vector fields which are functions from

$$\mathbb{C}^{\mathfrak{d}} \times \mathbb{T}_s^{\mathfrak{d}} \times \mathfrak{h}^p \rightarrow \mathbb{C}^{\mathfrak{d}} \times \mathbb{C}^{\mathfrak{d}} \times \mathfrak{h}^p : (\mathcal{Y}, \theta, \mathbf{a}) \rightarrow (X^{(\mathcal{Y})}, X^{(\theta)}, X^{(a)}, X^{(\bar{a})})$$

which are analytic in $D(s, r)$. On the vector field we use as norm

$$\|X\|_r := |X^{(\theta)}|_{\infty} + \frac{|X^{(\mathcal{Y})}|_1}{r^2} + \frac{\|X^{(a)}\|}{r} + \frac{\|X^{(\bar{a})}\|}{r}. \quad (2.18)$$

Next we introduce the notion of *majorant analytic* Hamiltonians and vector fields. We write a real valued Hamiltonian h in Taylor-Fourier series which is well defined and pointwise absolutely convergent:

$$h(\mathcal{Y}, \theta, \mathbf{a}) = \sum_{\alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0}, \ell \in \mathbb{Z}^{\mathfrak{d}}, l \in \mathbb{N}^{\mathfrak{d}}} h_{\alpha, \beta, l, \ell} e^{i\ell \cdot \theta} \mathcal{Y}^l a^{\alpha} \bar{a}^{\beta}, \quad h_{\alpha, \beta, l, \ell} = \bar{h}_{\beta, \alpha, l, -\ell}. \quad (2.19)$$

Correspondingly we expand vector fields in Taylor Fourier series (again well defined and pointwise absolutely convergent):

$$X^{(w)}(\mathcal{Y}, \theta, \mathbf{a}) = \sum_{\alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0}, \ell \in \mathbb{Z}^d, l \in \mathbb{N}^d} X_{\alpha, \beta, l, \ell}^{(w)} e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta,$$

where w denotes the components θ_i, \mathcal{Y}_i or $a_{\bar{j}}, \bar{a}_{\bar{j}}$. To a vector field we associate its *majorant*

$$\underline{X}_s^{(w)}[\mathcal{Y}, \mathbf{a}] := \sum_{\ell \in \mathbb{Z}^d, l \in \mathbb{N}^d, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0}} |X_{\alpha, \beta, l, \ell}^{(w)}| e^{s|\ell|} \mathcal{Y}^l a^\alpha \bar{a}^\beta. \quad (2.20)$$

Then we have the following

Definition 2.1. A vector field $X : D(s, r) \rightarrow \mathbb{C}^d \times \mathbb{C}^d \times \mathfrak{h}^p$ will be said to be majorant analytic in $D(s, r)$ if \underline{X}_s defines an analytic vector field $D(r) \rightarrow \mathbb{C}^d \times \mathbb{C}^d \times \mathfrak{h}^p$.

Since Hamiltonian functions are defined modulo constants, we give the following definition of *majorant analytic* Hamiltonian and its norm:

Definition 2.2. A real valued Hamiltonian H will be said to be majorant analytic in $D(s, r)$ if its Hamiltonian vector field X_H is majorant analytic in $D(s, r)$. We define its norm by

$$|H|_{s,r} := \sup_{(\mathcal{Y}, \mathbf{a}) \in D(r)} \|\underline{(X_H)}_s(\mathcal{Y}, \mathbf{a})\|_r. \quad (2.21)$$

Note that the norm $|\cdot|_{s,r}$ controls the norm of the vector field X_H defined in (2.17) in the domain $\mathbb{T}_{s'}^d \times D(r')$ for all $s' < s, r' < r$.

By convention we define the *scaling degree* of a monomial $e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta$ as

$$\deg(l, \alpha, \beta) := 2|l| + |\alpha| + |\beta| - 2, \quad (2.22)$$

and define the *projection* on the homogeneous components of *scaling degree* d as

$$H^{(d)} := \sum_{\substack{\ell \in \mathbb{Z}^d, l \in \mathbb{N}^d, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0} \\ 2|l| + |\alpha| + |\beta| = d + 2}} H_{\alpha, \beta, l, \ell} e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta,$$

similarly for $H^{(\leq d)}$ and $H^{(\geq d)}$.

Lipschitz families of Hamiltonians. In the following we will consider Hamiltonians $h(\lambda; \mathcal{Y}, \theta, \mathbf{a}) \equiv h(\lambda)$ depending on an external parameter $\lambda \in \mathcal{O}$, where \mathcal{O} is some compact set. Thus we define the *weighted Lipschitz* norm:

$$|h|_{s,r}^{\mathcal{O}} := \sup_{\lambda \in \mathcal{O}} |h(\lambda, \cdot)|_{s,r} + \sup_{\lambda_1 \neq \lambda_2 \in \mathcal{O}} \frac{|h(\lambda_1, \cdot) - h(\lambda_2, \cdot)|_{s,r}}{|\lambda_1 - \lambda_2|}. \quad (2.23)$$

It will be convenient to define the Lipschitz norm also for maps $f : \mathcal{O} \rightarrow E$ with values in a Banach space E , whose norm we denote simply by $|\cdot|_E$. We pose

$$|f|_E^{\mathcal{O}} := \sup_{\lambda \in \mathcal{O}} |f(\lambda)|_E + \sup_{\lambda_1 \neq \lambda_2 \in \mathcal{O}} \frac{|f(\lambda_1) - f(\lambda_2)|_E}{|\lambda_1 - \lambda_2|}. \quad (2.24)$$

The Hamiltonian and the constants of motion in the adapted coordinates. When expressed in the coordinates $(\mathcal{Y}, \theta, \mathbf{a})$ defined in (2.14), the Hamiltonian H of (2.6) takes the form

$$\mathcal{H}(\mathcal{Y}, \theta, \mathbf{a}) := \omega \cdot \mathcal{Y} + \sum_{\bar{j} = (m, n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0} |\bar{j}|^2 |a_{\bar{j}}|^2 + \mathcal{H}^{(\geq 0)}(\mathcal{Y}, \theta, \mathbf{a}) \quad (2.25)$$

where $\mathcal{H}^{(\geq 0)}$ has scaling degree greater than or equal to zero and $|\mathcal{H}^{(\geq 0)}|_{s_0, r_0} \leq C\varepsilon$ for some $s_0, r_0 > 0$, see Section 4 for the details. In these coordinates the mass M and the momentum P become

$$\mathcal{M}(\mathcal{Y}, \theta, \mathbf{a}) = \sum_{i=1}^d \mathcal{Y}_i + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} |a_{\bar{j}}|^2, \quad \mathcal{P}(\mathcal{Y}, \theta, \mathbf{a}) = \begin{bmatrix} \mathcal{P}_x \\ \mathcal{P}_y \end{bmatrix} = \sum_i \begin{bmatrix} \mathbf{m}_i \\ 0 \end{bmatrix} \mathcal{Y}_i + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \bar{j} |a_{\bar{j}}|^2. \quad (2.26)$$

Genericity condition. As we mentioned in the introduction, we need to impose some restrictions on the (Birkhoff) support \mathcal{S}_0 of the finite gap solutions. Indeed, we ask \mathcal{S}_0 to fulfill some arithmetic conditions which we now describe.

Definition 2.3. Order \mathcal{S}_0 so that $\mathfrak{m}_1 < \mathfrak{m}_2 < \dots < \mathfrak{m}_d$. For every $n \in \mathbb{Z}$ let

$$\mathcal{S}_{0n} := \{(\mathfrak{m}_i, n) : 1 \leq i \leq d\} .$$

For any $1 \leq i < j \leq d$ let

$$\mathcal{C}_{i,j}^\pm := \{(m, n) \in \mathbb{Z}^2 : (m - \mathfrak{m}_i)(m - \mathfrak{m}_j) + n^2 = 0, \quad \pm n > 0\} . \quad (2.27)$$

Finally denote

$$\mathcal{S} := \bigcup_{n \in \mathbb{Z} \setminus \{0\}} \mathcal{S}_{0n} , \quad \mathcal{C} := \bigcup_{i < j} \mathcal{C}_{i,j} , \quad \mathcal{C}_{i,j} := \mathcal{C}_{i,j}^+ \cup \mathcal{C}_{i,j}^- .$$

Definition 2.4 (Arithmetic genericity). We say that \mathcal{S}_0 is generic if

$$\mathcal{S} \cap \mathcal{C} = \emptyset , \quad \mathcal{C}_{i,j} \cap \mathcal{C}_{i',j'} = \emptyset , \quad \forall \{i, j\} \neq \{i', j'\} . \quad (2.28)$$

Given $L \in \mathbb{N}$, we say that \mathcal{S}_0 is L -generic if it is generic and moreover

$$\sum \ell_i \mathfrak{m}_i \neq 0 \quad \forall 0 < |\ell| \leq L . \quad (2.29)$$

The following lemma explains in which sense the ‘‘good’’ sets are generic:

Lemma 2.5. Fix any $L \in \mathbb{N}$. There are infinitely many choices of L -generic sets. More precisely for $R \gg 1$ let B_R be the set of all ordered sets $\mathcal{S} = (\mathfrak{m}_1, \dots, \mathfrak{m}_d)$ such that $\max(|\mathfrak{m}_i|) \leq R$. Then denoting by G_R the L -generic sets in B_R (i.e. those which satisfy Definition 2.4,) we have

$$\lim_{R \rightarrow \infty} \frac{|G_R|}{|B_R|} = 1 .$$

The proof of the lemma is postponed in Appendix A.

Let us motivate the genericity conditions. One of the main problems in developing perturbation theory for NLS on \mathbb{T}^2 is the presence of rectangle-resonances, namely quadruple of integers $\vec{j}_1, \vec{j}_2, \vec{j}_3, \vec{j}_4 \in \mathbb{Z}^2$ such that

$$\vec{j}_1 - \vec{j}_2 + \vec{j}_3 + \vec{j}_4 = 0 , \quad |\vec{j}_1|^2 - |\vec{j}_2|^2 + |\vec{j}_3|^2 - |\vec{j}_4|^2 = 0 . \quad (2.30)$$

It is easy to check that $\vec{j}_1, \vec{j}_2, \vec{j}_3, \vec{j}_4$ fulfill (2.30) if and only if they form the vertex of a rectangle in \mathbb{Z}^2 , hence the name rectangle-resonance. In principle we would like to avoid all the resonances of the form (2.30) when two \vec{j}_i 's are chosen in \mathcal{S}_0 and two are chosen outside \mathcal{S}_0 . One realizes immediately that this is not possible: indeed any two \vec{j}_i 's chosen in \mathcal{S}_{0n} , $n \neq 0$, will form a *horizontal rectangle* with two sites in \mathcal{S}_0 . Similarly, any two \vec{j}_i 's chosen in $\mathcal{C}_{i,j}$, $1 \leq i < j \leq d$, will form a *rotated rectangle* with two sites in \mathcal{S}_0 (from the very definition of \mathcal{S}_{0n} and \mathcal{C}).

Then the genericity condition states that there are no intersection at integer points outside \mathcal{S}_0 between an horizontal rectangle and a rotated rectangle or between two rotated rectangles.

Remark 2.6. If $d > 2$, then there are always rotated rectangles. The reason is the following: if $\mathfrak{m}_i - \mathfrak{m}_k$ is an even number, then the point $(\frac{\mathfrak{m}_i + \mathfrak{m}_k}{2}, \frac{\mathfrak{m}_k - \mathfrak{m}_i}{2})$ has integer coordinates and it forms a right angle with $(\mathfrak{m}_i, 0)$, $(\mathfrak{m}_k, 0)$, i.e. it belongs to \mathcal{C}_{ik} . Clearly if the cardinality of \mathcal{S}_0 is at least 3, there are at least two points whose distance is an even number.

Admissible monomials. We set

$$\begin{aligned} \tilde{\mathcal{M}} &:= \sum_i \mathcal{Y}_i + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C} \cup \mathcal{S}_0)} |a_j|^2 \\ \tilde{\mathcal{P}}_x &:= \sum_i \mathfrak{m}_i \mathcal{Y}_i + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{S}_0 \cup \mathcal{C})} m |a_{(m,n)}|^2 + \sum_{\substack{i < k \\ (m,n) \in \mathcal{C}_{i,k}^+}} (m - \mathfrak{m}_i) (|a_{(m,n)}|^2 - |a_{(\mathfrak{m}_i + \mathfrak{m}_j - m, -n)}|^2) \\ \tilde{\mathcal{P}}_y &:= \sum_{(m,n) \in \mathbb{Z}^2} n |a_{(m,n)}|^2 \end{aligned} \quad (2.31)$$

and give the following

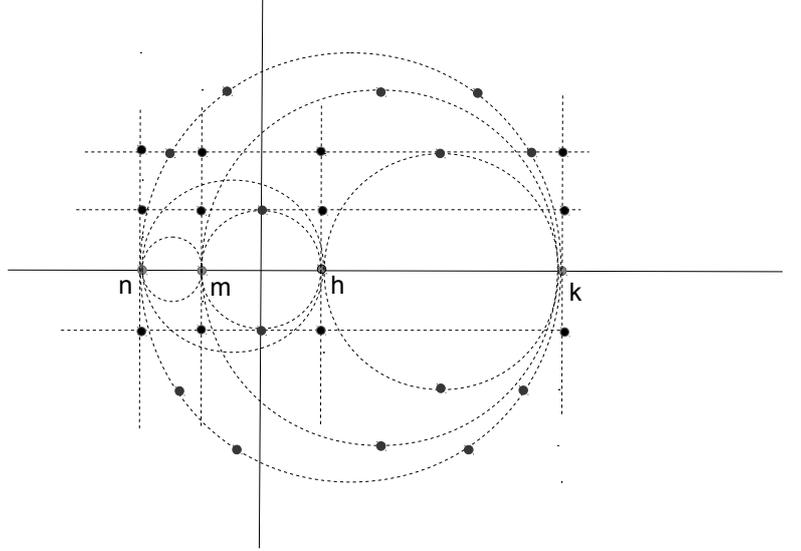


Figure 1: The dots on the circles are the $\mathcal{C}_{i,j}$, the ones on the straight lines are $\mathcal{S}_{0,n}$. The arithmetic genericity condition requires that all the intersection points of two dotted curves are non-integer.

Definition 2.7 (admissible). Given $\mathbf{j} = (\vec{j}_1, \dots, \vec{j}_b) \in (\mathbb{Z}^2 \setminus \mathcal{S}_0)^b$, $\ell \in \mathbb{Z}^d$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_b) \in \{-1, 0, 1\}^b$, we say that $(\mathbf{j}, \ell, \boldsymbol{\sigma})$ is admissible, and denote $(\mathbf{j}, \ell, \boldsymbol{\sigma}) \in \mathfrak{A}_b$, if the monomial $\mathbf{m} = e^{i\theta \cdot \ell} a_{\vec{j}_1}^{\sigma_1} \dots a_{\vec{j}_b}^{\sigma_b}$ Poisson commutes with $\tilde{\mathcal{M}}, \tilde{\mathcal{P}}_x, \tilde{\mathcal{P}}_y$. Here we use the convention that $a_{\vec{j}}^+ = a_{\vec{j}}, a_{\vec{j}}^- = \bar{a}_{\vec{j}}, a_{\vec{j}}^0 = 1$.

Definition 2.8 (action preserving). Given $\mathbf{j} = (\vec{j}_1, \dots, \vec{j}_b) \in (\mathbb{Z}^2 \setminus \mathcal{S}_0)^b$, $\ell \in \mathbb{Z}^d$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_b) \in \{-1, 0, 1\}^b$, we say that $(\mathbf{j}, \ell, \boldsymbol{\sigma})$ is action preserving and denote $(\mathbf{j}, \ell, \boldsymbol{\sigma}) \in \mathfrak{K}_b$, if $\ell = 0$ and the monomial $\mathbf{m} = a_{\vec{j}_1}^{\sigma_1} \dots a_{\vec{j}_b}^{\sigma_b}$ Poisson commutes with the actions $|a_{\vec{j}}|^2$ for all $\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0$.

Birkhoff normal form around a finite gap solution. We are finally ready to state our result on Birkhoff normal form:

Theorem 2.9. *There exists $L > 0$ (depending only on d), such that for a L -generic choice of the set \mathcal{S}_0 (in the sense of Definition 2.4) there exists a $\gamma_\star > 0$ such that the following holds true. Given arbitrary $s_0 > 0$, and $0 < \gamma \leq \gamma_\star$, there exists $\varepsilon_\star = \varepsilon_\star(\gamma) > 0$ and for any $0 < \varepsilon \leq \varepsilon_\star$, $r_0 \ll \sqrt{\varepsilon}$, there exist a compact domain $\mathcal{O}_1 \subseteq \mathcal{O}_0 \subseteq [1/2, 1]^d$, Lipschitz functions $(\Omega_{\vec{j}})_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$ defined on \mathcal{O}_1 and real number $\tau > 0$ s.t. the set*

$$\mathcal{C} := \left\{ \lambda \in \mathcal{O}_1 : |\omega \cdot \ell + \sigma_1 \Omega_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \Omega_{\vec{j}_2}(\lambda, \varepsilon) + \sigma_3 \Omega_{\vec{j}_3}(\lambda, \varepsilon)| \geq \varepsilon \frac{\gamma}{\langle \ell \rangle^\tau}, \forall (\mathbf{j}, \ell, \boldsymbol{\sigma}) \in \mathfrak{A}_3 \setminus \mathfrak{K}_3 \right\} \quad (2.32)$$

has positive measure. Moreover, for each $\lambda \in \mathcal{C}$, there exists a symplectic change of variables \mathcal{T} , majorant analytic together with its inverse, s.t. \mathcal{T} and \mathcal{T}^{-1} map $D(s/32, \varrho_0 r) \rightarrow D(s, r)$ for all $r \leq r_0$, $s_0/64 \leq s \leq s_0$ (here $\varrho_0 > 0$ is a parameter depending on $s_0, \max(|\mathbf{m}_k|^p)$). Moreover

$$(\mathcal{H} \circ \mathcal{T})(\mathcal{Y}, \theta, a, \bar{a}) = \omega \cdot \mathcal{Y} + \sum_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\vec{j}} |a_{\vec{j}}|^2 + \mathcal{R}^{(\geq 2)}(\mathcal{Y}, \theta, a, \bar{a}) \quad (2.33)$$

where $\mathcal{R}^{(\geq 2)}$ contains just monomials of scaling degree at least 2 and fulfills

$$\left| \mathcal{R}^{(\geq 2)} \right|_{s/32, \varrho_0 r} \leq C_{\mathcal{R}} r^2 \quad (2.34)$$

for some positive constant $C_{\mathcal{R}}$ independent of ε .

The mass \mathcal{M} and the momentum \mathcal{P} (defined in (2.26)) fulfill

$$\mathcal{M} \circ \mathcal{T} = \tilde{\mathcal{M}}, \quad \mathcal{P} \circ \mathcal{T} = \tilde{\mathcal{P}},$$

where $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}}$ are defined in (2.31).

We are able to describe quite precisely the asymptotics of the frequencies $\Omega_{\vec{j}}$ of Theorem 2.9. As happens often in higher dimensional settings, such asymptotics depend on the fact that $\vec{j} \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C} \cup \mathcal{S}_0)$ or to the sets \mathcal{S} , \mathcal{C} .

Theorem 2.10. *Under the same assumptions as Theorem 2.9, for any $\varepsilon \leq \varepsilon_*$, $\lambda \in \mathcal{O}_1$ the frequencies $\Omega_{\vec{j}} \equiv \Omega_{\vec{j}}(\lambda, \varepsilon)$, $\vec{j} = (m, n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0$, have the following asymptotics:*

$$\Omega_{\vec{j}}(\lambda, \varepsilon) = m^2 + \frac{\varpi_m(\lambda, \varepsilon)}{\langle m \rangle}, \quad n = 0, \quad (2.35)$$

$$\Omega_{\vec{j}}(\lambda, \varepsilon) = \tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon) + \frac{\Theta_m(\lambda, \varepsilon)}{\langle m \rangle^2} + \frac{\Theta_{m,n}(\lambda, \varepsilon)}{\langle m \rangle^2 + \langle n \rangle^2}, \quad n \neq 0 \quad (2.36)$$

where

$$\tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon) := \begin{cases} m^2 + n^2, & \vec{j} = (m, n) \notin \mathcal{S} \cup \mathcal{C}, \\ \varepsilon \mu_i(\lambda) + n^2, & \vec{j} = (\mathbf{m}_i, n), \\ m^2 + n^2 - \mathbf{m}_i^2 + \varepsilon \mu_{i,k}^+(\lambda), & \vec{j} = (m, n) \in \mathcal{C}_{i,k}, n > 0 \\ m^2 + n^2 - \mathbf{m}_k^2 - \varepsilon \mu_{i,k}^-(\lambda), & \vec{j} = (m, n) \in \mathcal{C}_{i,k}, n < 0 \end{cases} \quad (2.37)$$

Here the $(\mu_i(\lambda))_{1 \leq i \leq \mathbf{d}}$ are the roots of the polynomial

$$P(t, \lambda) := \prod_{i=1}^{\mathbf{d}} (t + \lambda_i) - 2 \sum_{i=1}^{\mathbf{d}} \lambda_i \prod_{k \neq i} (t + \lambda_k), \quad (2.38)$$

while for any $1 \leq i < k \leq \mathbf{d}$ fixed, the $\mu_{i,k}^+(\lambda), \mu_{i,k}^-(\lambda)$ are the roots of the polynomial

$$Q(t, \lambda_i, \lambda_k) := t^2 - (\lambda_i - \lambda_k)t + 3\lambda_i \lambda_k. \quad (2.39)$$

Finally $\mu_i, \mu_{i,k}^\pm, (\varpi_m(\lambda, \varepsilon))_{m \in \mathbb{Z}}, (\Theta_m(\lambda, \varepsilon))_{m \in \mathbb{Z}}$ and $(\Theta_{m,n}(\lambda, \varepsilon))_{(m,n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$ fulfill

$$\begin{aligned} & \sum_{1 \leq i \leq \mathbf{d}} |\mu_i(\cdot)|^{\mathcal{O}_1} + \sum_{\substack{1 \leq i < k \leq \mathbf{d} \\ \pm}} |\mu_{i,k}^\pm(\cdot)|^{\mathcal{O}_1} \\ & + \sup_{\varepsilon \leq \varepsilon_*} \frac{1}{\varepsilon^2} \left(\sup_{m \in \mathbb{Z}} |\varpi_m(\cdot, \varepsilon)|^{\mathcal{O}_1} + \sup_{m \in \mathbb{Z}} |\Theta_m(\cdot, \varepsilon)|^{\mathcal{O}_1} + \sup_{(m,n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0} |\Theta_{m,n}(\cdot, \varepsilon)|^{\mathcal{O}_1} \right) \leq \mathbf{M}_0 \end{aligned} \quad (2.40)$$

for some positive constant \mathbf{M}_0 .

We conclude this section with a list of remarks:

Remark 2.11. *The $(\mu_i(\lambda))_{1 \leq i \leq \mathbf{d}}$ and the $(\mu_{i,k}^\pm(\lambda))_{1 \leq i < k \leq \mathbf{d}}$ are distinct nonzero algebraic functions which are homogeneous of degree 1. They depend only on the number \mathbf{d} of tangential sites, hence not on the single sites $(\mathbf{m}_i)_{1 \leq i \leq \mathbf{d}}$.*

Remark 2.12. *The asymptotic expansion of the normal frequencies (2.36) does not contain any constant term. The reason is that we canceled such term when we subtracted the quantity $M(u)^2$ from the Hamiltonian, see (2.4). Of course if we had not removed $M(u)^2$, we would have had a constant correction to the frequencies, equal to $\|q(\omega t, \cdot)\|_{L^2}$. Since $q(\omega t, x)$ is a solution of (1.1), it enjoys mass conservation, thus $\|q(\omega t, \cdot)\|_{L^2} = \|q(0, \cdot)\|_{L^2}$ is independent of time.*

Remark 2.13. *In [FGL13] the authors prove that plane waves solutions of (1.1) are H^p -orbital stable for times of order δ^{-N} , with N arbitrary chosen, provided p is large enough. The reason is the following: plane waves solution have a very specific expression, given by $Ae^{i(m \cdot x - \omega t)}$. This simple formula allows to compute exactly the expressions for the tangential frequency ω and the normal frequencies $\Omega_{\vec{j}}$ as a function of A , indeed one has*

$$\omega = m^2 + A^2, \quad \Omega_{\vec{j}} = \sqrt{|\vec{j}|^4 + 2|\vec{j}|^2 A^2}.$$

Such formulas are much more explicit than the mere asymptotic expansions that we find in (2.36), and allow the authors of [FGL13] to impose Melnikov conditions of the form

$$|\sigma_1 \Omega_{\vec{j}_1} + \dots + \sigma_n \Omega_{\vec{j}_n}| \geq \frac{\gamma}{\mu_3(\vec{j}_1, \dots, \vec{j}_n)^\alpha}$$

at any order n (here $\mu_3(\vec{j}_1, \dots, \vec{j}_n)$ is the third largest integer among $|\vec{j}_1|, \dots, |\vec{j}_n|$). As a consequence they can perform an arbitrary number of steps of Birkhoff normal form.

On the contrary we have a weaker control on the asymptotics of the frequencies, since we know their exact expression only at order ε and not at higher orders in ε ; this allows us to impose Melnikov conditions up to order three, see (2.32).

Remark 2.14. In [PP15, PP16] the authors show that for a generic choice of tangential sites $S'_0 \subseteq \mathbb{Z}^2$, it is possible to construct families of small quasi-periodic solutions of (1.1) which are (essentially) supported on S'_0 . Such solutions give rise to finite dimensional KAM tori which depend on both variables x and y and which are linearly orbitally stable (namely stable for times $|t| \sim \delta^{-1}$). While we borrow ideas and techniques from such papers, the construction that we perform requires extra-care. Indeed the condition of genericity for S'_0 used in [PP15, PP16] does not allow to choose $S'_0 \subset \mathbb{Z} \times \{0\}$, as we do here. Furthermore in [PP15, PP16] the asymptotics of the normal frequencies $\Omega_{\vec{j}}$ are less explicit than formula (2.36), therefore it is not clear if it is possible, in the setup of [PP15, PP16], to impose the third order Melnikov conditions (2.32) and obtain nonlinear stability of the KAM tori.

3 Functional setting

In this section we introduce the technical apparatus that we will use in the following.

Given $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus S_0}$, to the monomial $e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta$ we associate various numbers. First we denote by

$$\eta(\alpha, \beta) := \sum_{\vec{j} \in \mathbb{Z}^2 \setminus S_0} (\alpha_{\vec{j}} - \beta_{\vec{j}}), \quad \eta(\ell) := \sum_{i=1}^d \ell_i. \quad (3.1)$$

The second quantity that we associate to $e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta$ are the momentum $\pi(\alpha, \beta) = (\pi_x, \pi_y)$ and $\pi(\ell)$ defined by

$$\pi(\alpha, \beta) = \begin{bmatrix} \pi_x(\alpha, \beta) \\ \pi_y(\alpha, \beta) \end{bmatrix} = \sum_{\vec{j}=(m,n) \in \mathbb{Z}^2 \setminus S_0} \begin{bmatrix} m \\ n \end{bmatrix} (\alpha_{\vec{j}} - \beta_{\vec{j}}), \quad \pi(\ell) = \sum_{i=1}^d \mathfrak{m}_i \ell_i. \quad (3.2)$$

Given a monomial $e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta$ we have the commutation rules

$$\begin{aligned} \{\mathcal{M}, e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta\} &= i(\eta(\alpha, \beta) + \eta(\ell)) e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta \\ \{\mathcal{P}_x, e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta\} &= i(\pi_x(\alpha, \beta) + \pi(\ell)) e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta, \quad \{\mathcal{P}_y, e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta\} = i\pi_y(\alpha, \beta) e^{i\theta \cdot \ell} \mathcal{Y}^l a^\alpha \bar{a}^\beta \end{aligned}$$

Remark 3.1. A function \mathcal{F} commutes with the mass \mathcal{M} and the momentum \mathcal{P} defined in (2.26) if and only if the following selection rules on its coefficients hold:

$$\begin{aligned} \{\mathcal{F}, \mathcal{M}\} = 0 &\Leftrightarrow \mathcal{F}_{\alpha, \beta, \ell} (\eta(\alpha, \beta) + \eta(\ell)) = 0 \\ \{\mathcal{F}, \mathcal{P}_x\} = 0 &\Leftrightarrow \mathcal{F}_{\alpha, \beta, \ell} (\pi_x(\alpha, \beta) + \pi(\ell)) = 0, \quad \{\mathcal{F}, \mathcal{P}_y\} = 0 \Leftrightarrow \mathcal{F}_{\alpha, \beta, \ell} (\pi_y(\alpha, \beta)) = 0 \end{aligned}$$

where $\eta(\alpha, \beta), \eta(\ell)$ are defined in (3.1) and $\pi(\alpha, \beta), \pi(\ell)$ are defined in (3.2).

From now on we shall always assume that our Hamiltonians commute with \mathcal{M}, \mathcal{P} , so the selection rules of Remark 3.1 hold.

Definition 3.2. We will denote by $\mathcal{A}_{s,r}$ the subspace of real valued functions in the closure of the monomials ($e^{i\ell \cdot \theta} \mathcal{Y}^l a^\alpha \bar{a}^\beta$) which Poisson commute with \mathcal{M} and \mathcal{P} , with respect to the norm $|\cdot|_{s,r}$ (defined in (2.21)). Such Hamiltonians will be called regular. Given a compact set $\mathcal{O} \subset \mathbb{R}^d$, we denote by $\mathcal{A}_{s,r}^\mathcal{O}$ the Banach space of Lipschitz maps $\mathcal{O} \rightarrow \mathcal{A}_{s,r}$ with finite norm $|\cdot|_{s,r}^\mathcal{O}$ (defined in (2.23)).

In the next proposition we describe some basic properties of the norm $|\cdot|_{s,r}$ which will be used repeatedly in the paper.

Proposition 3.3. For every $s, r > 0$ the following holds true:

- (i) Degree decomposition: given a Hamiltonian $h \in \mathcal{A}_{s,R}^\mathcal{O}$ which is homogeneous of degree d , then $h \in \mathcal{A}_{s,r}^\mathcal{O}$ for all $r > 0$ and one has

$$|h|_{s,r}^\mathcal{O} \leq \left(\frac{r}{R}\right)^d |h|_{s,R}^\mathcal{O}. \quad (3.3)$$

The same estimate holds also if d is the minimal degree and $r \leq R$.

- (ii) Changes of variables: Let $h, f \in \mathcal{A}_{s,r}^\mathcal{O}$. For any $0 < s' < s$ and $0 < r' < r$, let $\delta := \min\left(1 - \frac{r'}{r}, s - s'\right)$. If $\delta^{-1}|f|_{s,r}^\mathcal{O} < \varrho$ sufficiently small then the Hamiltonian vector field X_f defines a close to identity canonical change of variables \mathcal{T}_f such that

$$|h \circ \mathcal{T}_f|_{s',r'}^\mathcal{O} \leq (1 + C\varrho)|h|_{s,r}^\mathcal{O}, \quad \forall 0 < s' < s, \quad 0 < r' < r.$$

- (iii) Remainder estimates: Let $f, g \in \mathcal{A}_{s,r}^\mathcal{O}$ of minimal scaling degree respectively \mathbf{d}_f and \mathbf{d}_g and define the function

$$\widetilde{\mathcal{V}}_i(f; h) = \sum_{l=i}^{\infty} \frac{(-\text{ad}f)^l}{l!} h, \quad \text{ad}(f)h := \{h, f\}. \quad (3.4)$$

Then $\widetilde{\mathcal{V}}_i(f; g)$ is of minimal scaling degree $\mathbf{d}_f i + \mathbf{d}_g$ and we have the bound

$$|\widetilde{\mathcal{V}}_i(f; h)|_{s',r'}^\mathcal{O} \leq C(s)\delta^{-i} (|f|_{s,r}^\mathcal{O})^i |g|_{s,r}^\mathcal{O}, \quad \forall 0 < s' < s, \quad 0 < r' < r.$$

Note that the same holds if we substitute in (3.4) the sequence $\{\frac{1}{l!}\}$ with any sequence $\{b_l\}$ such that $|b_l| \leq \frac{1}{l!} \quad \forall l$.

The proof of the proposition, being quite technical, is postponed in Appendix B.

3.1 Quadratic Hamiltonians

Inside $\mathcal{A}_{s,r}$ are the polynomial subspaces, i.e. spaces of fixed scaling degree which we shall denote by $\mathcal{A}_s^{(d)}$. By (3.3), the polynomials in such spaces are analytic for all $r > 0$.

Here we want to study a special subspace of $\mathcal{A}_s^{(0)}$ defined as follows:

Definition 3.4. We denote by $\mathcal{Q}_s^\mathcal{O}$ the subspace of $\mathcal{A}_s^{(0)}$ which contains Hamiltonians quadratic in \mathbf{a} .

In particular $\mathcal{Q}_s^\mathcal{O}$ is the subspace of real valued quadratic functions in the closure of the monomials of the form $e^{i\ell\theta} a^\alpha \bar{a}^\beta$ with $|\alpha| + |\beta| = 2$. Now we study some properties of $\mathcal{Q}_s^\mathcal{O}$. For such Hamiltonians, by (3.3), the norm $|\cdot|_{s,r}$ is independent from the domain $D(r)$, hence we always suppose that $r = 1$ and we shall drop it:

$$|h|_{s,r}^\mathcal{O} = |h|_{s,1}^\mathcal{O} \equiv |h|_s^\mathcal{O}.$$

We decompose any quadratic Hamiltonian $H \in \mathcal{Q}_s^\mathcal{O}$, $s > 0$, in three components:

$$H(\lambda; \theta, a, \bar{a}) = H^{\text{line}}(\lambda; \theta, a, \bar{a}) + H^{\text{diag}}(\lambda; \theta, a, \bar{a}) + H^{\text{out}}(\lambda; \theta, a, \bar{a}). \quad (3.5)$$

Each component contains monomials supported in different regions of \mathbb{Z}^2 , which now we describe in more details.

- (i) $H^{\text{line}}(\lambda; \theta, a, \bar{a})$ contains only monomials with $|\alpha| + |\beta| = 2$ and $\text{supp } \alpha \cup \text{supp } \beta \subseteq \mathbb{Z} \times \{0\}$:

$$H^{\text{line}}(\lambda; \theta, a, \bar{a}) = \sum_{m_1, m_2 \in \mathbb{Z}} H_{m_1, m_2}^-(\lambda; \theta) a_{(m_1, 0)} \bar{a}_{(m_2, 0)} + 2\text{Re}(H_{m_1, m_2}^+(\lambda; \theta) a_{(m_1, 0)} a_{(m_2, 0)}),$$

$$\text{with } H_{m_1, m_2}^- = \overline{H_{m_2, m_1}^-}, \quad H_{m_1, m_2}^+ = H_{m_2, m_1}^+.$$

- (ii) $H^{\text{diag}}(\lambda; \theta, a, \bar{a})$ contains monomials with $|\alpha| = |\beta| = 1$ and $\text{supp } \alpha \cup \text{supp } \beta \subseteq \mathbb{Z}^2 \setminus \mathbb{Z}$ (namely is the diagonal part of the representative matrix):

$$H^{\text{diag}}(\lambda; \theta, a, \bar{a}) = \sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ n \in \mathbb{Z} \setminus \{0\}}} H_{m_1, m_2, n}^-(\lambda; \theta) a_{(m_1, n)} \bar{a}_{(m_2, n)}, \quad H_{m_1, m_2, n}^- = \overline{H_{m_2, m_1, n}^-}$$

- (iii) $H^{\text{out}}(\lambda; \theta, a, \bar{a})$ contains monomials with $|\alpha| = 2$ or $|\beta| = 2$ and $\text{supp } \alpha \cup \text{supp } \beta \subseteq \mathbb{Z}^2 \setminus \mathbb{Z}$ (namely is the out-diagonal part of the representative matrix):

$$H^{\text{out}}(\lambda; \theta, a, \bar{a}) = 2\text{Re}\left(\sum_{\substack{m_1, m_2 \in \mathbb{Z} \\ n > 0}} H_{m_1, m_2, n}^+(\lambda; \theta) a_{(m_1, n)} a_{(m_2, -n)}\right)$$

Note that the sum is over only one index n , this is due to the conservation of \mathcal{P}_y . Furthermore the condition $n > 0$ is used in order to avoid repetitions. It is easy to see that $H^{\text{line}}, H^{\text{diag}}, H^{\text{out}}$ are in $\mathcal{Q}_s^{\mathcal{O}}$ (see Proposition B.1(i)). We will denote $\mathcal{Q}_s^{\mathcal{O}, \text{diag}}$ (respectively $\mathcal{Q}_s^{\mathcal{O}, \text{out}}, \mathcal{Q}_s^{\mathcal{O}, \text{line}}$) the relative subspaces.

Remark 3.5. *The following commutation rules hold:*

- (i) $\{H^{\text{diag}}, K^{\text{diag}}\} \in \mathcal{Q}_s^{\mathcal{O}, \text{diag}}, \{H^{\text{diag}}, K^{\text{out}}\} \in \mathcal{Q}_s^{\mathcal{O}, \text{out}}, \{H^{\text{out}}, K^{\text{out}}\} \in \mathcal{Q}_s^{\mathcal{O}, \text{diag}}$
- (ii) $\{H^{\text{diag}}, K^{\text{line}}\} = \{H^{\text{out}}, K^{\text{line}}\} = 0$.

Consider a quadratic Hamiltonian $H(\theta, a, \bar{a})$ as above and expand its coefficients in Fourier series

$$H_{m_1, m_2, n}^{\pm}(\lambda; \theta) = \sum_{\ell \in \mathbb{Z}^d} H_{m_1, m_2, n}^{\pm, \ell}(\lambda) e^{i\theta \cdot \ell}. \quad (3.6)$$

In such a way, to any Hamiltonian $H \in \mathcal{Q}_s$ we can associate a sequence $\{H_{m_1, m_2, n}^{\sigma, \ell}\}$ as above. On the contrary, a sequence $\{H_{m_1, m_2, n}^{\sigma, \ell}\}$ corresponds to a Hamiltonian only if it fulfills the constraints which arise from the fact that the corresponding Hamiltonian should be real.

Given a sequence $\{H_{m_1, m_2, n}^{\sigma, \ell}\}$, we define the norm

$$|\{H_{m_1, m_2, n}^{\sigma, \ell}\}|_s := \sup_{\|\mathbf{a}\| \leq 1} \left\| \left\{ \sum_{m_1, \ell} e^{|\ell|s} |H_{m_1, m_2, n}^{-, \ell} a_{(m_1, n)} + e^{|\ell|s} |\overline{H}_{m_2, m_1, n}^{+, \ell} \bar{a}_{(m_1, -n)} \right\} \right\|. \quad (3.7)$$

Then $|H|_s \equiv |\{H_{m_1, m_2, n}^{\sigma, \ell}\}|_s$. The same for the corresponding Lipschitz norm.

Remark 3.6. *By mass and momentum conservation*

$$H_{m_1, m_2, n}^{\sigma, \ell}(\lambda) \neq 0 \quad \text{only if} \quad \pi(\ell) + m_1 + \sigma m_2 = 0, \quad \eta(\ell) + 1 + \sigma = 0, \quad (3.8)$$

where σ is either $+1$ or -1 .

Remark 3.7. *In particular, for a monomial $e^{i\theta \cdot \ell} a_{(m_1, n_1)} \bar{a}_{(m_2, n_2)}$ mass and momentum are*

$$\eta(\ell) = 0, \quad \pi(\ell) + m_1 - m_2 = 0, \quad n_1 - n_2 = 0,$$

while for $e^{i\theta \cdot \ell} \bar{a}_{(m_1, n_1)} \bar{a}_{(m_2, n_2)}$ they are

$$\eta(\ell) - 2 = 0, \quad \pi(\ell) - m_1 - m_2 = 0, \quad n_1 + n_2 = 0.$$

Remark 3.8. *A quadratic Hamiltonian in $\mathcal{A}_{s, r}^{\mathcal{O}}$ can be canonically identified with a majorant bounded matrix $M(\theta)$ by setting $H = \frac{1}{2}(\mathbf{a}, J^{-1}M(\theta)\mathbf{a})$, where $J^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is easily seen that $|H|_s$ is equivalent to the operator norm on $\mathcal{L}(\mathfrak{h}^p, \mathfrak{h}^p)$ of the matrix $\sum_{\ell} \underline{M}(\ell) e^{|\ell|s}$, hence we have the algebra property*

$$|\{H, K\}|_s \leq C_0 |H|_s |K|_s. \quad (3.9)$$

Next we introduce the concept of *order* of a quadratic Hamiltonian.

Definition 3.9 (Order of a Hamiltonian). *Given $\mathbf{N} = (N_1, N_2) \in \mathbb{Z}^2$, we say that a quadratic Hamiltonian with coefficients $\{H_{m_1, m_2, n}^{\pm, \ell}\}$ has order \mathbf{N} iff there exists $s > 0$ s.t. the sequence $\{(\langle m_1 \rangle^{-N_1} + \langle n \rangle^{-N_2}) H_{m_1, m_2, n}^{\pm, \ell}\}$ has a finite $|\cdot|_s^{\mathcal{O}}$ norm. We will denote*

$$[H]_{s, \mathbf{N}}^{\mathcal{O}} := |\{(\langle m_1 \rangle^{-N_1} + \langle n \rangle^{-N_2}) H_{m_1, m_2, n}^{\pm, \ell}\}|_s^{\mathcal{O}}$$

We denote by $\mathcal{Q}_{s, \mathbf{N}}^{\mathcal{O}}$ the subset of Hamiltonians in $\mathcal{Q}_s^{\mathcal{O}}$ of order \mathbf{N} . In the same way given a linear map

$$\mathbf{a} \rightarrow \mathbf{L}\mathbf{a}, \quad (\mathbf{L}\mathbf{a})_{(m, n)} = \sum_{m_1} \left(L_{m_1, m, n}^{-}(\lambda; \theta) a_{(m_1, n)} + L_{m_1, m, n}^{+}(\lambda; \theta) \bar{a}_{(m_1, -n)} \right)$$

we set

$$Y := \sum_{m_1, m_2, n} (\langle m_1 \rangle^{-N_1} + \langle n \rangle^{-N_2}) \left(L_{m_1, m_2, n}^{-}(\lambda; \theta) a_{(m_1, n)} + L_{m_1, m_2, n}^{+}(\lambda; \theta) \bar{a}_{(m_1, -n)} \right) \partial_{a_{(m_2, n)}} + c.c.$$

and define

$$[L]_{s, \mathbf{N}} := \sup_{(\mathbf{y}, \mathbf{a}) \in D(r)} \|\mathbf{Y}_s\|_r. \quad (3.10)$$

Same for the Lipschitz norm.

Remark 3.10. Note that in the previous definition we could have used the index m_2 instead of m_1 or we could have used a symmetric weight (say $\langle m_1 \rangle + \langle m_2 \rangle$) in the definition of order. In fact such definitions are equivalent due to momentum conservation. Our definition is constructed so that, if $\mathbf{N} = (N, N)$, then a linear operator L with $[L]_{s, \mathbf{N}} < \infty$ maps $\mathfrak{h}^{p+N} \rightarrow \mathfrak{h}^p$.

Remark 3.11. Let $\mathbf{N} \in \mathbb{N}^2$, then clearly

$$[H]_{s, \mathbf{N}}^{\mathcal{O}} \leq |H|_s^{\mathcal{O}} \leq [H]_{s, -\mathbf{N}}^{\mathcal{O}}.$$

Remark 3.12. For any $\mathbf{N} \in \mathbb{N}^2$ with non negative components we have

$$[A]_{s, -(N_1, 0)}^{\mathcal{O}}, [A]_{s, -(0, N_2)}^{\mathcal{O}} \leq [A]_{s, -\mathbf{N}}^{\mathcal{O}} \leq [A]_{s, -(N_1, 0)}^{\mathcal{O}} + [A]_{s, -(0, N_2)}^{\mathcal{O}} \quad (3.11)$$

Lemma 3.13 (Bony decomposition). Set $c^{-1} := 2 \max_{1 \leq i \leq d} \langle \mathfrak{m}_i \rangle$ and write

$$\begin{aligned} H &= H^B + H^R, \\ H^B &:= \sum_{\substack{\sigma \in \pm, \ell \in \mathbb{Z}^d, m_1, m_2, n: \\ m_1 + \sigma m_2 + \pi(\ell) = 0 \\ |\ell| \leq c \langle m_1 \rangle}} H_{m_1, m_2, n}^{\sigma, \ell}(\lambda) e^{i\theta \cdot \ell} a_{(m_1, n)}^{\sigma} a_{(m_2, -\sigma n)}^{\sigma}, \\ H^R &:= \sum_{\substack{\sigma \in \pm, \ell \in \mathbb{Z}^d, m_1, m_2, n: \\ m_1 + \sigma m_2 + \pi(\ell) = 0 \\ |\ell| > c \langle m_1 \rangle}} H_{m_1, m_2, n}^{\sigma, \ell}(\lambda) e^{i\theta \cdot \ell} a_{(m_1, n)}^{\sigma} a_{(m_2, -\sigma n)}^{\sigma}. \end{aligned}$$

For any $0 < s' < s$ set

$$\delta := s - s'. \quad (3.12)$$

We have that for all $k > 0$

$$[H^R]_{s', -(N_1+k, N_2)}^{\mathcal{O}} \leq k! \delta^{-k} (1 + c^{-k}) [H]_{s, -(N_1, N_2)}^{\mathcal{O}}, \quad (3.13)$$

namely the part H^R is infinitely smoothing in the x direction.

The proof of the Lemma is postponed in Appendix B. We now discuss some algebra properties of the norm $[\cdot]_{s, \mathbf{N}}^{\mathcal{O}}$.

Lemma 3.14. Fix $\mathbf{N}, \mathbf{M} \in \mathbb{N}^2$. Let $F \in \mathcal{Q}_{s, -\mathbf{N}}^{\mathcal{O}}, G \in \mathcal{Q}_{s, -\mathbf{M}}^{\mathcal{O}}$. The following holds true:

(i) For any $0 < s' < s$ one has $\{F, G\} \in \mathcal{Q}_{s', -\mathbf{N}-\mathbf{M}}^{\mathcal{O}}$ with the quantitative estimate

$$[\{F, G\}]_{s', -\mathbf{N}-\mathbf{M}}^{\mathcal{O}} \leq C_{\mathbf{N}, \mathbf{M}} ([F]_{s', -\mathbf{N}}^{\mathcal{O}} [G]_{s', -\mathbf{M}}^{\mathcal{O}} + \delta^{-N_1 - M_1} |F|_s^{\mathcal{O}} |G|_s^{\mathcal{O}}). \quad (3.14)$$

Here $C = C_{\mathbf{N}, \mathbf{M}} > 0$ does not depend on s', s and δ is defined as in (3.12).

(ii) Let $G \in \mathcal{Q}_s^{\mathcal{O}}$ and let F fulfill

$$[F]_{s, -\mathbf{N}}^{\mathcal{O}} < \eta := \frac{1}{C_0}, \quad (3.15)$$

where here (and everywhere below) C_0 is the algebra constant of formula (3.9). Then the Hamiltonian flow \mathcal{T}_F is a smoothing perturbation of the identity and $\forall 0 < s' < s$

$$[G \circ \mathcal{T}_F - G]_{s', -\mathbf{N}}^{\mathcal{O}} \leq C_{\mathbf{N}} \frac{|G|_s^{\mathcal{O}} (|F|_{s, -\mathbf{N}}^{\mathcal{O}} + \delta^{-N_1} |F|_s^{\mathcal{O}})}{1 - \eta^{-1} |F|_s^{\mathcal{O}}}. \quad (3.16)$$

(iii) Let $G \in \mathcal{Q}_s^{\mathcal{O}}$. Then $\forall \mathbf{i} \in \mathbb{N}$ the operator $\widetilde{\mathcal{G}}_{\mathbf{i}}(F; G)$ is of order $-\mathbf{iN}$ and $\forall 0 < s' < s$ one has

$$[\widetilde{\mathcal{G}}_{\mathbf{i}}(F; G)]_{s', -\mathbf{iN}}^{\mathcal{O}} \leq C_{\mathbf{i}, \mathbf{N}} \frac{|G|_s^{\mathcal{O}} (\delta^{-N_1} |F|_{s, -\mathbf{N}}^{\mathcal{O}})^{\mathbf{i}}}{1 - \eta^{-1} |F|_s^{\mathcal{O}}}.$$

Note that the same holds if we substitute in (3.4) the sequence $\{\frac{1}{\mathfrak{n}}\}$ with any sequence $\{b_l\}$ such that $\forall l$ one has $|b_l| \leq \frac{1}{\mathfrak{n}}$.

The proof of the lemma is postponed in Appendix B.

Remark 3.15. Consider the Poisson bracket $\{F, G\}$ of two quadratic Hamiltonians $F \in \mathcal{Q}_{s, -\mathbf{N}}, G \in \mathcal{Q}_{s, -\mathbf{M}}$. Then

$$\{F, G\} = \{F^B, G^B\} + \{F, G^R\} + \{F^R, G^B\}$$

and the last two terms $\{F, G^R\}, \{F^R, G^B\}$ are infinitely smoothing.

3.2 Quasi-Töplitz structure

Consider the term $\mathcal{H}^{(\geq 0)}$ in (2.25) and denote by $\mathcal{H}^{(0)}$ its component of scaling degree 0, which is given explicitly in (4.28). It turns out that $\mathcal{H}^{(0)}$ is quadratic in \mathbf{a} and its coefficients are independent of n , for $n \neq 0$. We would like to preserve such property, but during the process of normal form that we perform, we cannot avoid generating coefficients which are depending on n . It turns out that at each step of the normal form procedure (and later on during the KAM iteration), we will have Hamiltonians whose coefficients have this specific form:

$$H_{m_1, m_2, n} = H_{m_1, m_2}^{\text{hor}} + H_{m_1, m_2, n}^{\text{mix}}$$

where $\{H_{m_1, m_2}^{\text{hor}}\}$ is independent of n , while $\{H_{m_1, m_2, n}^{\text{mix}}\}$ depends on n but has order $(-2, -2)$. In the rest of the section we will show that the transformations which we will perform preserve such structure. To do so, we need some more notations:

Definition 3.16. We say that a quadratic Hamiltonian $H \in \mathcal{Q}_s^{\mathcal{O}}$ is horizontal if its coefficients $\{H_{m_1, m_2, n}^{\pm, \ell}\}_{n \neq 0}$ are independent of n , i.e. $H_{m_1, m_2, n}^{\pm, \ell} \equiv H_{m_1, m_2}^{\pm, \ell}$ for all $n \neq 0$. We will denote by $\mathcal{Q}_{s, M}^{\text{hor}}$ the subspace of $\mathcal{Q}_s^{\mathcal{O}}$ of horizontal Hamiltonians of order $(M, 0)$. We will denote the corresponding norm as $[\cdot]_{s, M}$. If $M = 0$ we will write just $\mathcal{Q}_s^{\text{hor}}$.

Note that $\mathcal{Q}_{s, M}^{\text{hor}}$ depends on \mathcal{O} . If we need to evidence such dependence we will write $\mathcal{Q}_{s, M}^{\mathcal{O}, \text{hor}}$. However, in most of our algorithms \mathcal{O} is fixed, hence we avoid to write it explicitly. Note that the condition $H \in \mathcal{Q}_s^{\mathcal{O}, \text{hor}}$ does not impose any restriction on the component H^{line} .

Remark 3.17. $\mathcal{Q}_{s, M}^{\text{hor}}$ is closed under Poisson bracket and composition with flows: if $F, G \in \mathcal{Q}_s^{\text{hor}}$, then for any $0 < s' < s$,

$$\{F, G\} \in \mathcal{Q}_{s'}^{\text{hor}}, \quad \widetilde{\mathcal{V}}_i(F; G) \in \mathcal{Q}_{s'}^{\text{hor}}, \quad G \circ \mathcal{T}_F \in \mathcal{Q}_{s'}^{\text{hor}}.$$

In the following we want to study the flow generated by an Hamiltonian of the form $F = F^{\text{hor}} + F^{\text{mix}}$, where $F^{\text{hor}} \in \mathcal{Q}_{s, -1}^{\text{hor}}$ or $F^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\text{hor}}$ while $F^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}$ is of order $-\bar{2} := (-2, -2)$.

Lemma 3.18. For $a = 1, 2$, consider $F = F^{\text{hor}} + F^{\text{mix}}$ with $F^{\text{hor}} \in \mathcal{Q}_{s, -a}^{\text{hor}}$, $F^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\mathcal{O}}$, and $G = G^{\text{hor}} + G^{\text{mix}}$, $G^{\text{hor}} \in \mathcal{Q}_s^{\text{hor}}$, $G^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\mathcal{O}}$. Set² $\eta = \min(C_0^{-1}, C_{\bar{2}}^{-1})$, and assume that $\eta^{-1} |F|_s^{\mathcal{O}} < \frac{1}{2}$. The following holds:

(i) For every $i \in \mathbb{N}$ the Hamiltonian $\widetilde{\mathcal{V}}_i(F; G)$ equals $\widetilde{\mathcal{V}}_i(F; G)^{\text{hor}} + \widetilde{\mathcal{V}}_i(F; G)^{\text{mix}}$ with $\widetilde{\mathcal{V}}_i(F; G)^{\text{hor}} = \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}}) \in \mathcal{Q}_{s', -ia}^{\text{hor}}$ and $\widetilde{\mathcal{V}}_i(F; G)^{\text{mix}} \in \mathcal{Q}_{s', -\bar{2}}^{\mathcal{O}}$ for all $0 < s' < s$. Furthermore one has the quantitative estimate

$$|\widetilde{\mathcal{V}}_i(F; G)^{\text{hor}}|_{s', -ia}^{\mathcal{O}} \leq C_{a, i} \frac{|G^{\text{hor}}|_s^{\mathcal{O}} (\delta^{-a} |F^{\text{hor}}|_{s, -a}^{\mathcal{O}})^i}{1 - \eta^{-1} |F|_s^{\mathcal{O}}},$$

for $i \geq 0$, here δ is defined in (3.12). Similarly for $i \geq 1$ we have

$$|\widetilde{\mathcal{V}}_i(F; G)^{\text{mix}}|_{s', -\bar{2}}^{\mathcal{O}} \leq \left(|G^{\text{mix}}|_{s, -\bar{2}}^{\mathcal{O}} |F|_s^{\mathcal{O}} + |G|_s^{\mathcal{O}} |F^{\text{mix}}|_{s, -\bar{2}}^{\mathcal{O}} + |G|_s^{\mathcal{O}} |F|_s^{\mathcal{O}} \right) \frac{2\delta^{-2} C_{\bar{2}} \left(\eta^{-1} |F|_s^{\mathcal{O}} \right)^{i-1}}{1 - \eta^{-1} |F|_s^{\mathcal{O}}}$$

Note that the same holds if we substitute in (3.4) the sequence $\{\frac{1}{i!}\}$ with any sequence $\{b_i\}$ such that $\forall l$ one has $|b_l| \leq \frac{1}{l!}$.

(ii) The Hamiltonian vector field X_F generates a well defined flow Φ_F^τ for $\tau \leq 1$. Moreover

$$\sup_{(\mathcal{Y}, \theta, \mathbf{a}) \in D(s, r)} \|(\Phi_F^\tau - \text{Id})(\mathbf{a}, \mathcal{Y}, \theta)\|_r \leq \tau \eta^{-1} |F|_s^{\mathcal{O}}.$$

The lemma is proved in Appendix B.

Lemma 3.19. Consider $F = F^{\text{hor}} + F^{\text{mix}}$ and $G = G^{\text{hor}} + G^{\text{mix}}$ such that $F^{\text{hor}}, G^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\text{hor}}$, $F^{\text{mix}}, G^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\mathcal{O}}$. Set³ $\eta = \min(C_0^{-1}, C_{\bar{2}}^{-1})$, and assume that $\eta^{-1} |F|_s^{\mathcal{O}} < \frac{1}{2}$. The following holds:

²here C_0 is the algebra constant in (3.9)

³here C_0 is the algebra constant in (3.9)

(i) For every $i \in \mathbb{N}$ the Hamiltonian $\overline{\mathcal{H}}_i(F; G)$ equals $\overline{\mathcal{H}}_i(F; G)^{\text{hor}} + \overline{\mathcal{H}}_i(F; G)^{\text{mix}}$ with $\overline{\mathcal{H}}_i(F; G)^{\text{hor}} = \overline{\mathcal{H}}_i(F^{\text{hor}}; G^{\text{hor}}) \in \mathcal{Q}_{s, -2}^{\text{hor}}$ and $\overline{\mathcal{H}}_i(F; G)^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\text{O}}$. Furthermore one has the quantitative estimates

$$[\overline{\mathcal{H}}_i(F; G)^{\text{hor}}]_{s, -2}^{\text{O}} \leq \frac{|G^{\text{hor}}|_{s, -2}^{\text{O}} (\eta^{-1} [F^{\text{hor}}]_{s, -2}^{\text{O}})^i}{1 - \eta^{-1} |F^{\text{hor}}|_{s, -2}^{\text{O}}},$$

for $i \geq 0$. Similarly for $i \geq 1$ we have

$$[\overline{\mathcal{H}}_i(F; G)^{\text{mix}}]_{s, -\bar{2}}^{\text{O}} \leq \frac{[G^{\text{mix}}]_{s, -\bar{2}}^{\text{O}} (\eta^{-1} [F]_{s, -2}^{\text{O}})^i}{1 - \eta^{-1} (|F|_{s, -2}^{\text{O}})} + \frac{[G^{\text{hor}}]_{s, -2}^{\text{O}} [F^{\text{mix}}]_{s, -\bar{2}} (\eta^{-1} [F]_{s, -2}^{\text{O}})^{i-1}}{1 - \eta^{-1} (|F|_{s, -2}^{\text{O}})}$$

Note that the same holds if we substitute in (3.4) the sequence $\{\frac{1}{n}\}$ with any sequence $\{b_l\}$ such that $\forall l$ one has $|b_l| \leq \frac{1}{l}$.

The lemma is proved in Appendix B.

Note that the difference between Lemma 3.18 and Lemma 3.19 is that in this last one the quantitative estimates *do not* lose regularity in s and do not gain in order.

4 Preparation of the Hamiltonian

The aim of this section is to write the Hamiltonian (2.4), the mass M (2.2) and the momentum P (2.3) in the adapted variables (2.14) around the tori (1.4).

4.1 The Birkhoff map for the one dimensional cubic NLS

First we gather some properties of the Birkhoff map for the dNLS (1.2). The main references for this subsection are the book [GK14] and the paper [Mas17].

We introduce the Fourier-Lebesgue spaces for $p, s \geq 0$:

$$h^{p, s} \equiv h^{p, s}(\mathbb{Z}) := \left\{ (q_m, \bar{q}_m)_{m \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) : \|q\|_{p, s}^2 := \sum_{m \in \mathbb{Z}} e^{2s|m|} \langle m \rangle^{2p} |q_m|^2 < \infty \right\} \quad (4.1)$$

We shall denote by $B^{p, s}(\rho)$ the ball of radius ρ and center 0 in the topology of $h^{p, s}$.

Theorem 4.1. *There exists $\rho_* > 0$ and a real analytic, symplectic, majorant analytic map $\Phi : B^{0, 0}(\rho_*) \rightarrow \ell^2 \times \ell^2$ with $d\Phi(0, 0) = \mathbb{I}$ s.t. $\forall p, s \geq 0$ the following is true.*

(i) *The restriction of Φ to $B^{p, s}(\rho_*)$ gives rise to a real analytic map $\underline{\Phi} : B^{p, s}(\rho_*) \rightarrow h^{p, s}$. Moreover there exists $C > 0$ s.t. for all $0 < \rho < \rho_*$*

$$\sup_{\|q\|_{p, s} \leq \rho} \|(\Phi - \mathbb{I})(q, \bar{q})\|_{p, s} \leq C \rho^3. \quad (4.2)$$

The same estimate holds for $\underline{\Phi}^{-1} - \mathbb{I}$, with a different constant.

(ii) *For $p \geq 1$, Φ introduces local Birkhoff coordinates for dNLS. More precisely the integrals of motion of dNLS are real analytic functions of the actions $I_j = |z_j|^2$. In particular, the Hamiltonian H_{dNLS} , the mass $M(q) := \int_{\mathbb{T}} |q(x)|^2 dx$ and the momentum $P(q) := \int_{\mathbb{T}} \bar{q}(x) i \partial_x q(x) dx$ have the form*

$$(H_{\text{dNLS}} \circ \Phi^{-1})(z, \bar{z}) \equiv h_{\text{dNLS}}(|z_m|^2)_{m \in \mathbb{Z}} \quad (4.3)$$

$$(M \circ \Phi^{-1})(z, \bar{z}) = \sum_{m \in \mathbb{Z}} |z_m|^2, \quad (4.4)$$

$$(P \circ \Phi^{-1})(z, \bar{z}) = \sum_{m \in \mathbb{Z}} m |z_m|^2. \quad (4.5)$$

(iii) *For $p \geq 2$, define the dNLS action-to-frequency map $I \mapsto \alpha_m^{\text{dNLS}}(I)$, where $\alpha_m^{\text{dNLS}}(I) := \frac{\partial h_{\text{dNLS}}}{\partial I_m}$, $\forall m \in \mathbb{Z}$. Then, in a neighborhood of $I = 0$, one has the asymptotic expansion*

$$\alpha_m^{\text{dNLS}}(I) = m^2 + 2 \sum_i I_i - I_m + \frac{\varpi_m(I)}{\langle m \rangle}, \quad m \in \mathbb{Z} \quad (4.6)$$

where $\varpi_m(I)$ is at least quadratic in I and $\sup_m |\varpi_m(I)| < \infty$.

Proof. Item (i) is the main content of [Mas17], where it is proved that the Birkhoff map is majorant analytic between some Fourier Lebesgue-spaces. Item (ii) is proved in [GK14, KP96]. Item (iii) in [KL12, KST15]. \square

Consider now the mass shift Hamiltonian H_{dm} defined in (2.8). By Theorem 4.1, H_{dm} is integrable and one has

$$(H_{\text{dm}} \circ \Phi^{-1})(z, \bar{z}) = h_{\text{dnls}}((|z_m|^2)_{m \in \mathbb{Z}}) - \left(\sum_m |z_m|^2 \right)^2 =: h_{\text{dm}}((|z_m|^2)_{m \in \mathbb{Z}}) .$$

We define the frequencies $\alpha_m(I) := \frac{\partial h_{\text{dm}}}{\partial I_m}$ which by (4.4) are given by

$$\alpha_m(I) := \alpha_m^{\text{dnls}}(I) - 2 \sum_i I_i , \quad (4.7)$$

and by (4.6) one has the following expansion in a neighborhood of $I = 0$:

$$\alpha_m(I) = m^2 - I_m + \frac{\varpi_m(I)}{\langle m \rangle} , \quad m \in \mathbb{Z} . \quad (4.8)$$

Now with $\mathcal{S}_0 = (\mathfrak{m}_1, \dots, \mathfrak{m}_d)$ we consider the map

$$I_{\mathfrak{m}} \equiv (I_{\mathfrak{m}_1}, \dots, I_{\mathfrak{m}_d}) \rightarrow (\alpha_{\mathfrak{m}_1}(I_{\mathfrak{m}}), \dots, \alpha_{\mathfrak{m}_d}(I_{\mathfrak{m}})) \equiv \alpha_{\mathfrak{m}}(I_{\mathfrak{m}}) ; \quad (4.9)$$

by Theorem 4.1 (iii) this map is generically a diffeomorphism⁴. As we have already mentioned, we prefer to parametrize the frequencies with a vector of parameters $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathcal{O}_0 \equiv [1/2, 1]^d$, using the fact that, by a standard application of the implicit function theorem, for ϵ sufficiently small there exists an analytic map $\mathcal{O}_0 \ni \lambda \rightarrow I_{\mathfrak{m}}(\lambda, \epsilon) \in \mathbb{R}_{>0}^d$ of the form (2.12) such that

$$\alpha_{\mathfrak{m}}(I_{\mathfrak{m}}(\lambda, \epsilon)) = \omega^{(0)} - \epsilon \lambda =: \omega(\lambda) ,$$

where $\omega^{(0)}$ is defined in (2.10). We will call $\omega(\lambda) \equiv \alpha_{\mathfrak{m}}(I_{\mathfrak{m}}(\lambda, \epsilon))$ the *tangential frequencies*, while $(\Omega_m(\lambda))_{m \in \mathbb{Z} \setminus \mathcal{S}_0}$, where

$$\Omega_m(\lambda) := \alpha_m(I_{\mathfrak{m}}(\lambda, \epsilon)) , \quad m \notin \mathcal{S}_0 \quad (4.10)$$

the *normal frequencies*; by (4.8) the normal frequencies have the following asymptotic expansion as $|m| \rightarrow \infty$:

$$\Omega_m(\lambda) = m^2 + \frac{\varpi_m(I_{\mathfrak{m}}(\lambda, \epsilon))}{\langle m \rangle} , \quad \text{with} \quad \sup_{\lambda \in \mathcal{O}_0} \sup_{m \in \mathbb{Z}} |\varpi_m(I_{\mathfrak{m}}(\lambda, \epsilon))| + |\partial_\lambda \varpi_m(I_{\mathfrak{m}}(\lambda, \epsilon))| \leq C \epsilon^2 . \quad (4.11)$$

4.2 Adapted variables

In this section we introduce adapted local variables in a neighborhood of the torus $\mathbb{T}^d(\mathcal{S}_0, \lambda) \equiv \mathbb{T}^d(\mathcal{S}_0, I_{\mathfrak{m}}(\lambda, \epsilon))$ defined in (1.4). We parametrize the torus $\mathbb{T}^d(\mathcal{S}_0, \lambda)$ as $z^{\text{fg}}(\lambda; \theta) = (z_m^{\text{fg}}(\lambda; \theta))_{m \in \mathbb{Z}}$ with

$$z_{\mathfrak{m}_i}^{\text{fg}}(\lambda; \theta) = \sqrt{I_{\mathfrak{m}_i}(\lambda)} e^{i\theta_i} , \quad \text{for } 1 \leq i \leq d , \quad z_m^{\text{fg}} = 0 , \quad \text{for } m \notin \mathcal{S}_0 ; \quad (4.12)$$

we denote by $q^{\text{fg}}(\lambda; \theta)$ its preimage through the Birkhoff map⁵:

$$q^{\text{fg}}(\lambda; \theta) \equiv (q_m^{\text{fg}}(\lambda; \theta))_{m \in \mathbb{Z}} \equiv \Phi^{-1}(z^{\text{fg}}(\lambda; \theta)) . \quad (4.13)$$

The explicit expression of $q^{\text{fg}}(\lambda; \theta)$ in the x variable was given in (2.13). We study now the analytic properties of the map $\Lambda : (\mathcal{Y}, \theta, \mathbf{a}) \mapsto (u_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2}$ defined in (2.15). First remark that by (4.13) the set $\Phi^{-1}(\mathbb{T}^d(\mathcal{S}_0, \lambda))$ is described in the $(\mathcal{Y}, \theta, \mathbf{a})$ coordinates by $\mathcal{Y} = 0, \mathbf{a} = 0$, i.e.

$$\Phi^{-1}(\mathbb{T}^d(\mathcal{S}_0, \lambda)) \equiv \{\Lambda(0, \theta, 0) : \theta \in \mathbb{T}^d\} .$$

We show also that Λ maps $D(s, r)$ into the set \mathcal{V}_δ of functions which are δ -close to the torus \mathbb{T}^d , which we recall is

$$\mathcal{V}_\delta := \left\{ v \in H^p(\mathbb{T}^2) : \text{dist}_{H^p(\mathbb{T}^2)} \left(v, \Phi^{-1}(\mathbb{T}^d(\mathcal{S}_0, \lambda)) \right) < \delta , \right. \\ \left. \left| \left| \Phi_{\mathfrak{m}_i}((v_{(m,0)})_{m \in \mathbb{Z}}) \right|^2 - I_{\mathfrak{m}_i}(\lambda) \right| \leq \delta^2 \text{ for } 1 \leq i \leq d \right\} . \quad (4.14)$$

Note that we are constantly identifying a function with its Fourier coefficients.

⁴generically in the sense outside a set of measure 0

⁵Abusing notation, from now on we will often write $q = \Phi^{-1}(z)$ in place of $(q, \bar{q}) = \Phi^{-1}(z, \bar{z})$.

Proposition 4.2. Fix an arbitrary $p > 1$ and the set \mathcal{S}_0 . For any $s > 0$, there exist $\varepsilon_0(s) > 0, c_*(s), c_*(s) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(s)$ the following is true.

(i) There exists $0 < r_* \leq \sqrt{\varepsilon}/4$ s.t. for any $0 < r \leq r_*$ the change of coordinates $\Lambda: D(s, r) \rightarrow \Lambda(D(s, r)), (\mathcal{Y}, \theta, \mathbf{a}) \mapsto (u_{\vec{j}})_{\vec{j} \in \mathbb{Z}^2}$ is majorant analytic, and the symplectic form in the variables $(\mathcal{Y}, \theta, \mathbf{a})$ is given by (2.16).

(ii) For $0 \leq r \leq r_*$, $\mathcal{V}_{c_*r} \subseteq \Lambda(D(s, r)) \subseteq \mathcal{V}_{c_*r}$.

(iii) The Hamiltonian (2.4) in the variables $(\mathcal{Y}, \theta, \mathbf{a})$ takes the form

$$\mathcal{H}(\lambda; \mathcal{Y}, \theta, \mathbf{a}) = \mathcal{N}^{(0)} + \mathcal{H}^{(0)}(\lambda; \theta, \mathbf{a}) + \mathcal{H}^{(1)}(\lambda; \theta, \mathbf{a}) + \mathcal{H}^{(2)}(\lambda; \theta, \mathcal{Y}, \mathbf{a}) \quad (4.15)$$

where

$$\mathcal{N}^{(0)} := \omega(\lambda) \cdot \mathcal{Y} + \mathcal{D}, \quad (4.16)$$

$$\mathcal{D} := \sum_{i=1}^d \omega_{\mathbf{m}_i}(\lambda) \mathcal{Y}_i + \sum_{\vec{j}=(m,n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\vec{j}}^{(0)} |a_{\vec{j}}|^2 \quad (4.17)$$

and the normal frequencies $\Omega_{\vec{j}}^{(0)}$ are defined by

$$\Omega_{\vec{j}}^{(0)} := \begin{cases} |\vec{j}|^2 & \text{if } \vec{j} = (m, n) \text{ with } n \neq 0 \\ \Omega_m(\lambda) & \text{if } \vec{j} = (m, 0) \end{cases}, \quad (4.18)$$

where $\Omega_m(\lambda)$ is defined in (4.10). Moreover $\mathcal{H}^{(0)} \in \mathcal{Q}_s^{\text{hor}}$, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(\geq 2)}$ belong to $\mathcal{A}_{s, r_*}^{\mathcal{O}}$ with the quantitative bounds

$$|\mathcal{H}^{(0)}|_s^{\mathcal{O}} \leq C\varepsilon, \quad |\mathcal{H}^{(1)}|_{s, r}^{\mathcal{O}} \leq C\sqrt{\varepsilon}r, \quad |\mathcal{H}^{(\geq 2)}|_{s, r}^{\mathcal{O}} \leq Cr^2, \quad \forall 0 < r \leq r_*, \quad (4.19)$$

where C is independent of ε, r .

(iv) The mass M and the momentum P (defined in (2.2) and (2.3)) in the variables $(\mathcal{Y}, \theta, \mathbf{a})$ take the form (2.26).

We split the proof of Proposition 4.2 in several steps. First we prove item (i) and (ii).

Proof of Proposition 4.2 (i) and (ii). (i) The map Λ is the identity on $\mathbb{Z}^2 \setminus \mathbb{Z}$ (see (2.15)), thus we need only to study the analytic properties of the map $(\mathcal{Y}, \theta, (a_{(m,0)})_{m \in \mathbb{Z} \setminus \mathcal{S}_0}) \rightarrow (u_{(m,0)})_{m \in \mathbb{Z}}$. Such a map is the composition of the Birkhoff map Φ^{-1} and the map Υ which passes the special sites \mathcal{S}_0 to action-angle coordinates:

$$((\mathcal{Y}_i, \theta_i)_{1 \leq i \leq d}, (a_{(m,0)})_{m \in \mathbb{Z} \setminus \mathcal{S}_0}) \xrightarrow{\Upsilon} ((z_{\mathbf{m}_1}, \dots, z_{\mathbf{m}_d}), (z_m)_{m \in \mathbb{Z} \setminus \mathcal{S}_0}) \xrightarrow{\Phi^{-1}} (u_{(m,0)})_{m \in \mathbb{Z}} \quad (4.20)$$

$$z_{\mathbf{m}_j} = \sqrt{I_j(\lambda) + \mathcal{Y}_j} e^{i\theta_j}, \quad \mathbf{m}_j \in \mathcal{S}_0 \quad (4.21)$$

$$z_m = a_{(m,0)}, \quad m \in \mathbb{Z} \setminus \mathcal{S}_0. \quad (4.22)$$

By Theorem 4.1 the Birkhoff map Φ and its inverse Φ^{-1} are majorant analytic canonical diffeomorphism from a small ball around the origin in $h^p(\mathbb{Z})$ to $h^p(\mathbb{Z})$ for any $p \geq 0$. In particular there exists $0 < R \leq \rho_*$ (where ρ_* is the domain on majorant analyticity of Φ , see Theorem 4.1) such that $\Phi^{-1}: B(R) \times B(R) \rightarrow B(2R) \times B(2R)$, $B(R)$ being a ball of center 0 and radius R in the topology of $h^p(\mathbb{Z})$.

For a given $s > 0$, fix ε, r_* so that

$$16r_*^2 \leq \varepsilon \leq C_* e^{-2s} R^2, \quad C_*^{-1} = 48d \max_i |\mathbf{m}_i|^{2p}. \quad (4.23)$$

Consider now the map Υ . It is proved in [BBP13, Lemma 7.6] that the first condition in (4.23) implies that Υ is majorant analytic and $\Upsilon_s: D(s, \sqrt{\varepsilon}/4) \rightarrow B(C\sqrt{\varepsilon}) \times B(C\sqrt{\varepsilon})$, where $C > 0$ can be chosen uniformly in ε for $\varepsilon \leq \varepsilon_0$, while the second condition in (4.23) ensures that the image of Υ falls in the domain of majorant analyticity of Φ^{-1} . As a consequence the map $\Phi^{-1} \circ \Upsilon$ is majorant analytic. Recall now that Λ is simply obtained by extending $\Phi^{-1} \circ \Upsilon$ as the identity on $\mathbb{Z}^2 \setminus \mathbb{Z}$, therefore it is majorant analytic as a map $D(s, \sqrt{\varepsilon}/4) \rightarrow B(C\sqrt{\varepsilon}) \times B(C\sqrt{\varepsilon})$, where now we denoted by $B(R)$ the ball in the

topology of $h^p(\mathbb{Z}^2)$. The fact that Λ transforms the standard symplectic form into (2.16) is a direct computation, using the symplecticity of the Birkhoff map Φ .

(ii) First we show that there exists a constant $c_* > 0$ s.t. $\Lambda(D(s, r)) \subseteq \mathcal{V}_{c_* r}$. Thus let $u = \Lambda(\mathcal{Y}, \theta, \mathbf{a})$ with $(\mathcal{Y}, \theta, \mathbf{a}) \in D(s, r)$. Recall that $q^{\text{fg}}(\lambda; \theta) = \Phi^{-1}(\Upsilon(0, \theta, 0))$. One has that

$$\begin{aligned} \text{dist}_{H^p(\mathbb{T}^2)}\left(u, \Phi^{-1}\left(\mathbb{T}^d(\mathcal{S}_0, \lambda)\right)\right) &\leq \|u - q^{\text{fg}}(\lambda; \theta)\|_{h^p(\mathbb{Z}^2)} = \|u\|_{h^p(\mathbb{Z}^2 \setminus \mathbb{Z})} + \|u - q^{\text{fg}}(\theta)\|_{h^p(\mathbb{Z})} \\ &= \|(a_{(m,n)})_{n \neq 0}\|_{h^p(\mathbb{Z}^2 \setminus \mathbb{Z})} + \|\Phi^{-1}(\Upsilon(\mathcal{Y}, \theta, \mathbf{a})) - \Phi^{-1}(\Upsilon(0, \theta, 0))\|_{h^p(\mathbb{Z})} \\ &\stackrel{\text{Thm.4.1}}{\leq} r + C\|\Upsilon(\mathcal{Y}, \theta, \mathbf{a}) - \Upsilon(0, \theta, 0)\|_{h^p(\mathbb{Z})} \\ &\leq r + C\|(a_{(m,0)})_{m \in \mathbb{Z}}\|_{h^p(\mathbb{Z} \setminus \mathcal{S}_0)} + C\left(\sum_{i=1}^d m_i^{2p} \left|\sqrt{I_{m_i}(\lambda) + \mathcal{Y}_i} - \sqrt{I_{m_i}(\lambda)}\right|^2\right)^{1/2} \\ &\leq C' \left(r + \frac{r^2}{\sqrt{\varepsilon}}\right) \stackrel{(4.23)}{\leq} c_* r, \end{aligned}$$

which proves one condition. Then, since

$$(u_{(m,0)})_{m \in \mathbb{Z}} = \Phi^{-1}\left(\left(\sqrt{I_{m_i}(\lambda) + \mathcal{Y}_i} e^{i\theta_i}\right)_{1 \leq i \leq d}, (a_{(m,0)})_{m \in \mathbb{Z} \setminus \mathcal{S}_0}\right) \quad (4.24)$$

one has $\Phi_{m_i}((u_{(m,0)})_{m \in \mathbb{Z}}) = \sqrt{I_{m_i}(\lambda) + \mathcal{Y}_i} e^{i\theta_i}$, which clearly implies that

$$\left|\Phi_{m_i}((u_{(m,0)})_{m \in \mathbb{Z}})\right|^2 - I_{m_i}(\lambda) \leq r^2.$$

Thus $\Lambda(D(s, r)) \subseteq \mathcal{V}_{c_* r}$.

Now we show the converse, namely that $\exists c_* > 0$ s.t. $\mathcal{V}_{c_* r} \subseteq \Lambda(D(s, r))$. Then take $\delta = c_* r$, $u \in \mathcal{V}_\delta$, and we show that $u = \Lambda(\mathcal{Y}, \theta, \mathbf{a})$ for some $(\mathcal{Y}, \theta, \mathbf{a}) \in D(s, c_*^{-1} \delta)$. First we put $(a_{(m,n)})_{n \neq 0} := (u_{(m,n)})_{n \neq 0}$; the condition $\text{dist}\left(u, \Phi^{-1}\left(\mathbb{T}^d(\mathcal{S}_0, \lambda)\right)\right) \leq \delta$ implies immediately that $\|(a_{(m,n)})_{n \neq 0}\|_{h^p(\mathbb{Z}^2 \setminus \mathbb{Z})} \leq \delta$.

Consider now $(u_{(m,0)})_{m \in \mathbb{Z}}$, which are the Fourier coefficients of $u(\cdot, 0)$. Denote $\theta_* := \text{argmin}_\theta \|u - q^{\text{fg}}(\lambda; \theta_*)\|$; such minimum exists since \mathbb{T}^d is compact and $\theta \mapsto q^{\text{fg}}(\lambda; \theta)$ is continuous. Then

$$\|(u_{(m,0)})_{m \in \mathbb{Z}}\|_{h^p(\mathbb{Z})} \leq \|(u_{(m,0)})_{m \in \mathbb{Z}} - q^{\text{fg}}(\lambda; \theta_*)\|_{h^p(\mathbb{Z})} + \|q^{\text{fg}}(\lambda; \theta_*)\|_{h^p(\mathbb{Z})} \leq \delta + C\sqrt{\varepsilon} \leq R \leq \rho_* ,$$

where ρ_* is the size of the domain of majorant analyticity of the Birkhoff map Φ , see Theorem 4.1. Hence the vector $z \equiv (z_m)_{m \in \mathbb{Z}} := \Phi((u_{(m,0)})_{m \in \mathbb{Z}})$ is well defined; we pose $a_{(m,0)} := z_m$ for any $m \notin \mathcal{S}_0$. By the analyticity of Φ^{-1} and the fact that $d\Phi^{-1}(0) = \mathbb{I}$, we bound

$$\begin{aligned} \delta &= \|u - q^{\text{fg}}(\lambda; \theta_*)\|_{h^p(\mathbb{Z})} = \|\Phi^{-1}(z) - \Phi^{-1}(z^{\text{fg}}(\lambda; \theta_*))\|_{h^p(\mathbb{Z})} \geq c_* \|z - z^{\text{fg}}(\lambda; \theta_*)\|_{h^p(\mathbb{Z})} \\ &= c_* \|z - z^{\text{fg}}(\lambda; \theta_*)\|_{h^p(\mathcal{S}_0)} + c_* \|z\|_{h^p(\mathbb{Z} \setminus \mathcal{S}_0)}. \end{aligned}$$

Hence we obtain that $\|(a_{(m,0)})_{m \notin \mathcal{S}_0}\|_{h^p(\mathbb{Z} \setminus \mathcal{S}_0)} \leq c_*^{-1} \delta$. Finally write $z_{m_i} = \sqrt{I_{m_i}(\lambda) + \mathcal{Y}_i} e^{i\theta_i}$ for some \mathcal{Y} and θ . Then the second condition in (4.14) implies $\delta^2 \geq |z_{m_i}|^2 - I_{m_i}(\lambda) \geq |\mathcal{Y}_i|$. Therefore we have shown that $u = \Lambda(\mathcal{Y}, \theta, \mathbf{a})$ for some $(\mathcal{Y}, \theta, \mathbf{a}) \in D(s, c_*^{-1} \delta)$. \square

We begin now the proof of item (iii). Thus we consider expression (2.4) and apply in two steps the change of coordinates Φ^{-1} and Υ of (4.20). To begin with, we start from the Hamiltonian in Fourier coordinates (2.6), and set

$$q_m := u_{(m,0)} \quad \text{if } m \in \mathbb{Z}, \quad a_{\vec{j}} := u_{\vec{j}} \quad \text{if } \vec{j} = (m, n) \in \mathbb{Z}^2, n \neq 0.$$

We rewrite the Hamiltonian accordingly in increasing degree in a , obtaining

$$\begin{aligned}
H(q, a) &= \sum_{m \in \mathbb{Z}} m^2 |q_m|^2 - \frac{1}{2} \sum_{m \in \mathbb{Z}} |q_m|^4 + \frac{1}{2} \sum_{\substack{m_i \in \mathbb{Z} \\ m_1 - m_2 + m_3 - m_4 = 0}}^* q_{m_1} \bar{q}_{m_2} q_{m_3} \bar{q}_{m_4} \\
&+ \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} |\bar{j}|^2 |a_{\bar{j}}|^2 \\
&+ 2 \sum_{\substack{\bar{j}_i = (m_i, n_i), i=3,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_3 - n_4 = 0}}^* q_{m_1} \bar{q}_{m_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} + \frac{1}{2} \sum_{\substack{\bar{j}_i = (m_i, n_i), i=2,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_2 + n_4 = 0}} (q_{m_1} \bar{a}_{\bar{j}_2} q_{m_3} \bar{a}_{\bar{j}_4} + \bar{q}_{m_1} a_{\bar{j}_2} \bar{q}_{m_3} a_{\bar{j}_4}) \\
&+ \sum_{\substack{\bar{j}_i = (m_i, n_i), i=2,3,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ -n_2 + n_3 - n_4 = 0}} q_{m_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} + \sum_{\substack{\bar{j}_i = (m_i, n_i), i=1,2,3, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_1 - n_2 + n_3 = 0}} a_{\bar{j}_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{q}_{m_4} \\
&+ \frac{1}{2} \sum_{\substack{\bar{j}_i = (m_i, n_i), i=1,2,3,4, n_i \neq 0 \\ \bar{j}_1 - \bar{j}_2 + \bar{j}_3 - \bar{j}_4 = 0}}^* a_{\bar{j}_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} - \frac{1}{2} \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} |a_{\bar{j}}|^4
\end{aligned} \tag{4.25}$$

Step 1: First we introduce Birkhoff coordinates on the line $\mathbb{Z} \times \{0\}$: We set

$$((z_m)_{m \in \mathbb{Z}}, (a_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}}) \mapsto ((q_m)_{m \in \mathbb{Z}}, (a_{\bar{j}})_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}}), \quad (q_m)_{m \in \mathbb{Z}} = \Phi^{-1}((z_m)_{m \in \mathbb{Z}}). \tag{4.26}$$

Since the first line of (4.25) is the dNLS with mass shift, when we apply Φ^{-1} we obtain

$$\begin{aligned}
H'(z, a) &= h_{\text{dm}}(|z_m|^2)_{m \in \mathbb{Z}} + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} |\bar{j}|^2 |a_{\bar{j}}|^2 \\
&+ 2 \sum_{\substack{\bar{j}_i = (m_i, n_i), i=3,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_3 - n_4 = 0}}^* q_{m_1} \bar{q}_{m_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} + \frac{1}{2} \sum_{\substack{\bar{j}_i = (m_i, n_i), i=2,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_2 + n_4 = 0}} (q_{m_1} \bar{a}_{\bar{j}_2} q_{m_3} \bar{a}_{\bar{j}_4} + \bar{q}_{m_1} a_{\bar{j}_2} \bar{q}_{m_3} a_{\bar{j}_4}) \\
&+ \sum_{\substack{\bar{j}_i = (m_i, n_i), i=2,3,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ -n_2 + n_3 - n_4 = 0}} q_{m_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} + \sum_{\substack{\bar{j}_i = (m_i, n_i), i=1,2,3, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_1 - n_2 + n_3 = 0}} a_{\bar{j}_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{q}_{m_4} \\
&+ \frac{1}{2} \sum_{\substack{\bar{j}_i = (m_i, n_i), i=1,2,3,4, n_i \neq 0 \\ \bar{j}_1 - \bar{j}_2 + \bar{j}_3 - \bar{j}_4 = 0}}^* a_{\bar{j}_1} \bar{a}_{\bar{j}_2} a_{\bar{j}_3} \bar{a}_{\bar{j}_4} - \frac{1}{2} \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} |a_{\bar{j}}|^4
\end{aligned} \tag{4.27}$$

where in the last three lines we think $(q_m)_{m \in \mathbb{Z}} = \Phi^{-1}((z_m)_{m \in \mathbb{Z}})$ as a function of the Birkhoff variables z .

Step 2: We go to action-angle coordinates only on the set $\mathcal{S}_0 = (m_1, \dots, m_d) \subset \mathbb{Z}$ and rename z_m for $m \notin \mathcal{S}_0$ as $a_{(m,0)}$, see (4.21) and (4.22). We think each q_m as a function

$$q_m = q_m(\lambda; \mathcal{Y}, \theta, \mathbf{a}) \equiv \Phi_m^{-1}((\sqrt{I_{m_i}(\lambda)} + \mathcal{Y}_i e^{i\theta_i})_{i=1, \dots, d}, (a_{(m,0)})_{m \in \mathbb{Z} \setminus \mathcal{S}_0}).$$

Next we Taylor expand the Hamiltonian (4.27) around the finite-gap torus corresponding to $(\mathcal{Y}, \theta, \mathbf{a}) = (0, \theta, 0)$. First one has (up to an irrelevant constant)

$$\begin{aligned}
h_{\text{dm}}(|z_m|^2)_{m \in \mathbb{Z}} &= h_{\text{dm}}(I_{m_1}(\lambda) + \mathcal{Y}_1, \dots, I_{m_d}(\lambda) + \mathcal{Y}_d, (|a_{(m,0)}|^2)_{m \in \mathbb{Z} \setminus \mathcal{S}_0}) \\
&= \omega(\lambda) \cdot \mathcal{Y} + \sum_{m \in \mathbb{Z} \setminus \mathcal{S}_0} \Omega_m(\lambda) |a_{(m,0)}|^2 + \mathcal{O}(\mathcal{Y}^2, \mathcal{Y} |\mathbf{a}|^2, |\mathbf{a}|^4),
\end{aligned}$$

where the $\Omega_m(\lambda)$ are defined in (4.10). Now we consider the terms which contain q_m . Expanding q_m in Taylor series we have

$$\begin{aligned}
q_m &= q_m(\lambda; 0, \theta, 0) + \sum_{i=1}^d \frac{\partial q_m}{\partial \mathcal{Y}_i}(\lambda; 0, \theta, 0) \mathcal{Y}_i + \sum_{m_1 \in \mathbb{Z} \setminus \mathcal{S}_0} \frac{\partial q_m}{\partial a_{(m_1,0)}}(\lambda; 0, \theta, 0) a_{(m_1,0)} \\
&+ \sum_{m_1 \in \mathbb{Z} \setminus \mathcal{S}_0} \frac{\partial q_m}{\partial \bar{a}_{(m_1,0)}}(\lambda; 0, \theta, 0) \bar{a}_{(m_1,0)} + \mathcal{O}(\mathcal{Y}^2, \mathcal{Y} \mathbf{a}, \mathbf{a}^2)
\end{aligned}$$

Remark that

$$q_m(\lambda; 0, \theta, 0) = \Phi_m^{-1}((\sqrt{I_{m_i}(\lambda)} e^{i\theta_i})_{i=1,\dots,d}, 0) = q_m^{\text{fg}}(\lambda; \theta) .$$

Inserting such formula in the Hamiltonian (4.27) and collecting the terms in increasing scaling degree we obtain the new Hamiltonian

$$\mathcal{H}(\lambda; \mathcal{Y}, \theta, \mathbf{a}) = \mathcal{N}^{(0)} + \mathcal{H}^{(0)}(\lambda; \theta, \mathbf{a}) + \mathcal{H}^{(1)}(\lambda; \theta, \mathbf{a}) + \mathcal{H}^{(2)}(\lambda; \theta, \mathcal{Y}, \mathbf{a})$$

where $\mathcal{N}^{(0)}$ is defined in (4.16). Correspondingly $\mathcal{H}^{(0)}$ is the second and third line of (4.15), namely all the remaining terms of degree two in \mathbf{a} :

$$\begin{aligned} \mathcal{H}^{(0)} := & 2 \sum_{\substack{\vec{j}_i = (m_i, n_i), i=3,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_3 - n_4 = 0}}^* q_{m_1}^{\text{fg}}(\lambda; \theta) \bar{q}_{m_2}^{\text{fg}}(\lambda; \theta) a_{\vec{j}_3} \bar{a}_{\vec{j}_4} \\ & + \frac{1}{2} \sum_{\substack{\vec{j}_i = (m_i, n_i), i=2,4, n_i \neq 0 \\ m_1 - m_2 + m_3 - m_4 = 0 \\ n_2 + n_4 = 0}} (q_{m_1}^{\text{fg}}(\lambda; \theta) \bar{a}_{\vec{j}_2} q_{m_3}^{\text{fg}}(\lambda; \theta) \bar{a}_{\vec{j}_4} + \bar{q}_{m_1}^{\text{fg}}(\lambda; \theta) a_{\vec{j}_2} q_{m_3}^{\text{fg}}(\lambda; \theta) a_{\vec{j}_4}) . \end{aligned} \quad (4.28)$$

$\mathcal{H}^{(1)}$ are the terms with scaling degree 1:

$$\mathcal{H}^{(1)} := \sum_{\substack{\ell \in \mathbb{Z}^d, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2} \setminus \mathcal{S}_0 \\ |\alpha| + |\beta| = 3}} H_{\alpha, \beta, \ell}(\lambda) e^{i\theta \cdot \ell} a^\alpha \bar{a}^\beta \quad (4.29)$$

in particular such Hamiltonian is cubic in \mathbf{a} and it does not contain any monomial of the form $e^{i\theta \cdot \ell} \mathcal{Y}^\ell a^\alpha \bar{a}^\beta$ with $|\ell| = |\alpha + \beta| = 1$.

Finally $\mathcal{H}^{(\geq 2)}$ contains all the rest i.e. all the terms with scaling degree ≥ 2 .

In the next lemma we estimate the norms of the Hamiltonians $\mathcal{H}^{(0)}$, $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(\geq 2)}$:

Lemma 4.3. *Fix $s > 0$. There exists $\varepsilon_0 > 0$ and for any $\varepsilon \leq \varepsilon_0$, there exists $r_* \leq \sqrt{\varepsilon}/4$ s.t. $\mathcal{H}^{(0)} \in \mathcal{Q}_s^{\text{hor}}$ while $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(\geq 2)}$ belong to $\mathcal{A}_{s, r_*}^{\mathcal{O}}$. Finally the following bounds hold:*

$$|\mathcal{H}^{(0)}|_s^{\mathcal{O}} \leq C\varepsilon, \quad |\mathcal{H}^{(1)}|_{s, r}^{\mathcal{O}} \leq C\sqrt{\varepsilon}r, \quad |\mathcal{H}^{(\geq 2)}|_{s, r}^{\mathcal{O}} \leq Cr^2, \quad \forall 0 < r \leq r_*, \quad (4.30)$$

where C is independent of ε, r .

Proof. We start from the Hamiltonian (2.6). The terms of order four in (2.6) belong to $\mathcal{A}_{s, r}$ for all $s, r \geq 0$. Indeed we have that

$$H_4 := \sum_{\vec{j}_1 + \vec{j}_2 = \vec{j}_3 + \vec{j}_4} u_{\vec{j}_1} u_{\vec{j}_2} \bar{u}_{\vec{j}_3} \bar{u}_{\vec{j}_4} \Rightarrow (X_{H_4})_{\vec{j}} = (u \star u \star \bar{u})_{\vec{j}} \Rightarrow (\underline{X}_{H_4})_{\vec{j}} = (u \star u \star \bar{u})_{\vec{j}} \Rightarrow |(\underline{X}_{H_4})_{\vec{j}}| \leq (v \star v \star \bar{v})_{\vec{j}}$$

with $v_{\vec{j}} = |u_{\vec{j}}|$. Then we repeat the same argument for $M(u) = \int_{\mathbb{T}^2} |u|^2$ (see the definition of the Hamiltonian (2.4)) and the result follows by the algebra property of our space. This shows that the starting Hamiltonian (2.6) is majorant analytic. Moreover one has

$$\sup_{|u|_{\mathfrak{h}^p(\mathbb{Z}^2)} \leq \rho} \|\underline{X}_{H_4}(u, \bar{u})\|_\rho \leq \rho^2 . \quad (4.31)$$

Next consider the map Λ of (2.15). By Proposition 4.2(i), Λ is majorant analytic and maps $D(s, \sqrt{\varepsilon}/4) \rightarrow B(C_1\sqrt{\varepsilon}) \times B(C_1\sqrt{\varepsilon})$. Thus the pullback of X_{H_4} through Λ , which we denote Y , is a majorant analytic vector field $D(s, r) \rightarrow \mathbb{C}^d \times \mathbb{C}^d \times \mathfrak{h}^p, \forall 0 < r \leq r_*$. Hence $\mathcal{H}^{(\geq 0)} := H_4 \circ \Lambda$ is majorant analytic and

$$\left| \mathcal{H}^{(\geq 0)} \right|_{s, \sqrt{\varepsilon}/4} \leq \sup_{(\mathcal{Y}, \mathbf{a}) \in D(\sqrt{\varepsilon}/4)} \|\underline{Y}_s(\mathcal{Y}, \mathbf{a})\|_{\sqrt{\varepsilon}/4} \leq \sup_{|u|_{\mathfrak{h}^p(\mathbb{Z}^2)} \leq C_1\sqrt{\varepsilon}} \|\underline{X}_{H_4}(u)\|_{C_1\sqrt{\varepsilon}} \stackrel{(4.31)}{\leq} C_1\varepsilon . \quad (4.32)$$

As a consequence each homogeneous component of $\mathcal{H}^{(\geq 0)}$ is a majorant analytic Hamiltonian; in particular let $\mathcal{H}^{(d)}$ the homogeneous component of $\mathcal{H}^{(\geq 0)}$ of scaling degree d ; then

$$\left| \mathcal{H}^{(d)} \right|_{s, r} \leq \left(\frac{4r}{\sqrt{\varepsilon}} \right)^d \left| \mathcal{H}^{(d)} \right|_{s, \sqrt{\varepsilon}/4} \stackrel{(3.3)}{\leq} \left(\frac{4r}{\sqrt{\varepsilon}} \right)^d \left| \mathcal{H}^{(\geq 0)} \right|_{s, \sqrt{\varepsilon}/4} \stackrel{(4.32)}{\leq} \left(\frac{4r}{\sqrt{\varepsilon}} \right)^d C_1\varepsilon .$$

The estimate for the Lipschitz norm $|\mathcal{H}^{(d)}|_{s, r}^{\mathcal{O}}$ is analogous, and the lemma follows. \square

Proof of Proposition 4.2 (iii) and (iv). Item (iii) follows from (4.15) and Lemma 4.3. Item (iv) is proved similarly by performing **Step 1** and **Step 2** on the mass M and momentum P , we skip the details. \square

⁶note that there is in fact NO dependence on s since there are no angle variables at this point. Furthermore, there is NO dependence on $\lambda \in \mathcal{O}$

5 Normal form of the quadratic terms

In this section we consider only the quadratic part of the Hamiltonian (4.15), namely

$$\mathcal{N}^{(0)} + \mathcal{H}^{(0)} \equiv \omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{(0)} , \quad (5.1)$$

where \mathcal{D} is defined in (4.17) and $\mathcal{H}^{(0)}$ in (4.28). Our aim is to reduce such Hamiltonian to a diagonal form by a convergent KAM procedure by requiring some (potentially rather complicated but relatively explicit) Diophantine condition on λ . The problem of the KAM algorithm is that it requires to impose the so called *second Melnikov conditions* on the frequencies, which do not hold for $\mathcal{N}^{(0)}$.

Thus, first we must put (5.1) in a normal form which is suitable in order to start a KAM algorithm to obtain the reducibility result. Following ideas of [PP12], we perform a finite number of normal form transformations whose effect is to put the Hamiltonian (5.1) in a form suitable to the application of a KAM scheme.

In this section we will constantly use the notation $a \ll b$ with the meaning $a \leq Cb$ for some positive constant C independent of ε and r .

The aim of the section is to prove the following result:

Theorem 5.1. *Fix $p > 1$ and $s_0 > 0$. For a generic choice of the set \mathcal{S}_0 (in the sense of Definition 2.4), there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$, $\exists r_0 \ll \sqrt{\varepsilon}$ s.t. the following holds true. There exists a compact domain $\mathcal{O}_1 \subseteq \mathcal{O}_0 \subseteq [1/2, 1]^d$ such that for any $\lambda \in \mathcal{O}_1$ there exists an invertible symplectic change of variables $\mathcal{T}^{(0)}$:*

$$\mathbf{a} \mapsto \mathcal{L}(\lambda, \varepsilon, \theta)\mathbf{a}, \quad \mathcal{Y} \mapsto \mathcal{Y} + i(\mathbf{a}, \mathcal{Q}(\lambda, \varepsilon, \theta)\mathbf{a}), \quad \theta \mapsto \theta$$

well defined and majorant analytic together with its inverse, such that $\mathcal{T}^{(0)}, (\mathcal{T}^{(0)})^{-1}: D(s/8, \varrho r) \rightarrow D(s, r)$ for all $0 < r \leq r_0$, $s_0/64 \leq s \leq s_0$ (here $\varrho > 0$ depends on $s_0, \max(|\mathbf{m}_k|^p)$) and

$$(\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{(0)}) \circ \mathcal{T}^{(0)}(\mathcal{Y}, \theta, a, \bar{a}) = \omega \cdot \mathcal{Y} + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} \tilde{\Omega}_{\bar{j}} |a_{\bar{j}}|^2 + \sum_{m \in \mathbb{Z} \setminus \mathcal{S}_0} \Omega_m |a_{(m,0)}|^2 + \tilde{\mathcal{H}}^{(0)} + \tilde{\mathcal{H}}^{(1)} + \tilde{\mathcal{H}}^{(\geq 2)}. \quad (5.2)$$

Here the frequencies $\tilde{\Omega}_{\bar{j}}$ are defined in (2.37), and Ω_m in (4.10). Furthermore $\tilde{\mathcal{H}}^{(0)}$ has scaling degree 0 and has the form

$$\tilde{\mathcal{H}}^{(0)} = \tilde{\mathcal{H}}^{(0, \text{hor})} + \tilde{\mathcal{H}}^{(0, \text{mix})}, \quad [\tilde{\mathcal{H}}^{(0, \text{hor})}]_{s/8, -2}^{\mathcal{O}_1} + [\tilde{\mathcal{H}}^{(0, \text{mix})}]_{s/8, -\bar{2}}^{\mathcal{O}_1} \leq \varepsilon^2.$$

Similarly $\tilde{\mathcal{H}}^{(1)}$ has scaling degree 1 and does not depend on \mathcal{Y} , $\tilde{\mathcal{H}}^{(\geq 2)}$ has scaling degree 2, and

$$\left| \tilde{\mathcal{H}}^{(1)} \right|_{s/8, \varrho r}^{\mathcal{O}_1} \leq \sqrt{\varepsilon} r, \quad \left| \tilde{\mathcal{H}}^{(\geq 2)} \right|_{s/8, \varrho r}^{\mathcal{O}_1} \leq r^2.$$

The mass \mathcal{M} and the momentum \mathcal{P} of (2.26) in the new coordinates are given by

$$\mathcal{M} \circ \mathcal{T}^{(0)} = \tilde{\mathcal{M}}, \quad \mathcal{P} \circ \mathcal{T}^{(0)} = \tilde{\mathcal{P}}$$

defined in (2.31).

Remark 5.2. *In the new variables, the selection rules of Remark 3.1 become*

$$\begin{aligned} \{\mathcal{H}, \tilde{\mathcal{M}}\} = 0 &\Leftrightarrow \mathcal{H}_{\alpha, \beta, \ell}(\tilde{\eta}(\alpha, \beta) + \eta(\ell)) = 0 \\ \{\mathcal{H}, \tilde{\mathcal{P}}_x\} = 0 &\Leftrightarrow \mathcal{H}_{\alpha, \beta, \ell}(\tilde{\pi}_x(\alpha, \beta) + \pi(\ell)) = 0 \\ \{\mathcal{H}, \tilde{\mathcal{P}}_y\} = 0 &\Leftrightarrow \mathcal{H}_{\alpha, \beta, \ell}(\pi_y(\alpha, \beta)) = 0 \end{aligned}$$

where $\eta(\ell)$ is defined in (3.1), $\pi(\ell)$ in (3.2), while

$$\begin{aligned} \tilde{\eta}(\alpha, \beta) &:= \sum_{\bar{j} \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C} \cup \mathcal{S}_0)} (\alpha_{\bar{j}} - \beta_{\bar{j}}), \\ \tilde{\pi}_x(\alpha, \beta) &:= \sum_{\substack{\bar{j} = (m, n) \\ \bar{j} \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{S}_0 \cup \mathcal{C})}} m(\alpha_{\bar{j}} - \beta_{\bar{j}}) + \sum_{\substack{i < k \\ \bar{j} = (m, n) \in \mathcal{C}_{i, k}^+}} (m - \mathbf{m}_i)(\alpha_{\bar{j}} - \beta_{\bar{j}}) + \sum_{\substack{i < k \\ \bar{j} = (m, n) \in \mathcal{C}_{i, k}^-}} (m - \mathbf{m}_k)(\alpha_{\bar{j}} - \beta_{\bar{j}}). \end{aligned}$$

As far as it concerns the structure of the transformation $\mathcal{T}^{(0)}$, we have the following result:

Theorem 5.3. *We have that \mathcal{L}, \mathcal{Q} are block diagonal in the y -Fourier modes. More precisely the matrix $\mathcal{L} = \text{diag}_{n \in \mathbb{N}}(L_n)$, with each L_n acting on the sequence $(a_{(m,n)}, a_{(m,-n)})_{m \in \mathbb{Z}}$. The L_n satisfy the following properties: $L_0 = \text{Id}$, while L_n with $n \neq 0$ is the composition of four maps: $L_n = L_n^{(D)} \circ L_n^{(B)} \circ R_n \circ U_n$ where*

- *Setting $L^{(D)} = \text{diag}_{n \in \mathbb{N}}(L_n^{(D)})$, we have $L^{(D)} - \text{Id} = L^{(D,\text{hor})} + L^{(D,\text{mix})}$, with $L_n^{(D,\text{hor})}$ independent of n , for all $n \neq 0$, and*

$$[L^{(D,\text{hor})}]_{s/4, -2}^{\mathcal{O}_0}, \quad [L^{(D,\text{mix})}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \ll \varepsilon,$$

see formula (3.10).

- *Setting $L^{(B)} = \text{diag}_{n \in \mathbb{N}}(L_n^{(B)})$, we have $L^{(B)} - \text{Id} = L^{(B,\text{hor})} + L^{(B,\text{mix})}$, with $L_n^{(B,\text{hor})}$ independent of n , for all $n \neq 0$, and*

$$[L^{(B,\text{hor})}]_{s/8, -2}^{\mathcal{O}_0}, \quad [L^{(B,\text{mix})}]_{s/8, -\bar{2}}^{\mathcal{O}_0} \ll \varepsilon.$$

- *R_n is a finite dimensional phase shift given explicitly by formula (5.51), moreover if*

$$\mathcal{C} \cap \{(m, n), (m, -n)\}_{m \in \mathbb{Z}} = \emptyset \tag{5.3}$$

then R_n is independent of n .

- *U_n acts non trivially only on $(\mathcal{S} \cup \mathcal{C}) \cap \{(m, n), (m, -n)\}_{m \in \mathbb{Z}}$ it is invertible and depends analytically on $\lambda \in \mathcal{O}_1$ together with its inverse. Moreover if (5.3) holds then U_n is independent of n and orthogonal.*

The matrix \mathcal{Q} has the same structure of \mathcal{L} .

The rest of the section is devoted to the proof of Theorem 5.1 and Theorem 5.3.

5.1 Proof of Theorems 5.1 and 5.3

We prove our statement in three steps, where we construct one by one the changes of variables in Theorem 5.3 justifying their role in the diagonalization process.

Step 1. The change of coordinates $\mathcal{L}^{(D)}$ puts (5.1) into the form

$$\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{\text{hor}} + \mathcal{H}^{\text{mix}}$$

so that $\mathcal{H}^{\text{hor}} \in \mathcal{Q}_{s', -2}^{\mathcal{O}_0, \text{hor}}$ and $\mathcal{H}^{\text{mix}} \in \mathcal{Q}_{s', -\bar{2}}^{\mathcal{O}_0}$. Furthermore we isolate the terms of order ε .

Step 2. The change of coordinates $\mathcal{L}^{(B)}$ removes from the terms at order ε in $\mathcal{H}^{\text{hor}} + \mathcal{H}^{\text{mix}}$ all the monomials which are Birkhoff non-resonant, in the sense of Definition 5.5 below.

Step 3. The changes of coordinates \mathcal{R}, \mathcal{U} are not close to the identity and are used to put the Birkhoff resonant terms at order ε in diagonal form.

We start by decomposing \mathcal{D} and $\mathcal{H}^{(0)}$ in their line, diagonal and out-diagonal components as defined in Section 3.1. Note that $\mathcal{D} = \mathcal{D}^{\text{line}} + \mathcal{D}^{\text{diag}}$ (it does not have an out-diagonal component), while

$$\mathcal{H}^{(0)}(\lambda; \theta, a, \bar{a}) = \mathcal{H}^{\text{diag}}(\lambda; \theta, a, \bar{a}) + \mathcal{H}^{\text{out}}(\lambda; \theta, a, \bar{a}),$$

where

$$\mathcal{H}^{\text{diag}}(\lambda; \theta, a, \bar{a}) \equiv \sum_{m_1, m_2, n \in \mathbb{Z}} H_{m_1, m_2}^-(\lambda; \theta) a_{(m_1, n)} \bar{a}_{(m_2, n)} \tag{5.4}$$

$$= 2 \sum_{\substack{m_1 - m_2 + m_3 - m_4 = 0 \\ m_1 \neq m_2, n \neq 0}} q_{m_3}^{\text{fg}}(\lambda; \theta) \bar{q}_{m_4}^{\text{fg}}(\lambda; \theta) a_{(m_1, n)} \bar{a}_{(m_2, n)}$$

$$\mathcal{H}^{\text{out}}(\lambda; \theta, a, \bar{a}) \equiv 2 \sum_{m_1, m_2, n > 0} \text{Re} (H_{m_1, m_2}^+(\lambda; \theta) a_{(m_1, n)} a_{(m_2, -n)}) \tag{5.5}$$

$$= \sum_{\substack{m_1 + m_2 = m_3 + m_4 \\ n > 0}} (q_{m_3}^{\text{fg}}(\lambda; \theta) q_{m_4}^{\text{fg}}(\lambda; \theta) \bar{a}_{(m_1, n)} \bar{a}_{(m_2, -n)} + \bar{q}_{m_1}^{\text{fg}}(\lambda; \theta) \bar{q}_{m_2}^{\text{fg}}(\lambda; \theta) a_{(m_1, n)} a_{(m_2, -n)});$$

here $(q_m^{\text{fg}}(\lambda; \theta))_{m \in \mathbb{Z}} = \Phi^{-1}(z^{\text{fg}}(\lambda; \theta))$. Note that in (5.5) we are summing only on $n > 0$, and not on all n as in (4.28), hence the coefficient is changed accordingly. Furthermore note that $\mathcal{H}^{(0)}$ does not have a line component.

5.1.1 Step 1: descent method.

From now on we will constantly write $a < b$ with the meaning $a \leq Cb$, with a constant C independent of ε, r .

We prove the following:

Lemma 5.4. *There exists an invertible symplectic transformation $\mathcal{T}^{(D)} : D(s/4, r/2) \rightarrow D(s, r) \forall s_0/64 \leq s \leq s_0, 0 < r \leq r_0$, which transforms the Hamiltonian (5.1) in the following form:*

$$(\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{(0)}) \circ \mathcal{T}^{(D)} = \omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}_1 + \mathcal{H}_2, \quad (5.6)$$

where

(i) the map $\mathcal{T}^{(D)}$ is the time-1 flow of a quadratic Hamiltonian $\chi_1 = \chi_1^{\text{hor}} + \chi_1^{\text{mix}}$ such that $[\chi_1^{\text{hor}}]_{3s/4, -1}^{\mathcal{O}_0} + [\chi_1^{\text{mix}}]_{3s/4, -2}^{\mathcal{O}_0} \leq \varepsilon$.

(ii) $\mathcal{H}_1 = \mathcal{H}_1^{\text{hor}} + \mathcal{H}_1^{\text{mix}} \in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0}$ and $[\mathcal{H}_1^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [\mathcal{H}_1^{\text{mix}}]_{s/4, -2}^{\mathcal{O}_0} \leq \varepsilon$. Explicitly

$$\mathcal{H}_1^{\text{hor}}(\lambda; \theta, a, \bar{a}) = 2\varepsilon \sum_{\substack{m_1, m_2, n \in \mathbb{Z} \setminus \{0\} \\ |m_1| \neq |m_2|}} \frac{-(\omega^{(0)} \cdot \partial_\theta)^2 Q_{m_1, m_2}^-(\lambda; \theta)}{(m_1^2 - m_2^2)^2} a_{(m_1, n)} \bar{a}_{(m_2, n)} \quad (5.7)$$

$$+ 2\varepsilon \sum_{m_1, n \neq 0} Q_{m_1}^-(\lambda; \theta) a_{(m_1, n)} \bar{a}_{(-m_1, n)} \quad (5.8)$$

$$\mathcal{H}_1^{\text{mix}}(\lambda; \theta, a, \bar{a}) = 2\varepsilon \sum_{m_1, m_2, n > 0} \text{Re} \left(\frac{i\omega^{(0)} \cdot \partial_\theta Q_{m_1, m_2}^+(\lambda; \theta)}{m_1^2 + m_2^2 + 2n^2} a_{(m_1, n)} a_{(m_2, -n)} \right), \quad (5.9)$$

where

$$Q_{m_1, m_2}^-(\lambda; \theta) := \sum_{i \neq j: m_i - m_j = m_2 - m_1} \sqrt{\lambda_i \lambda_j} e^{i(\theta_i - \theta_j)}, \quad (5.10)$$

$$Q_{m_1}^-(\lambda; \theta) := \sum_{i \neq j: m_i - m_j = -2m_1} \sqrt{\lambda_i \lambda_j} e^{i(\theta_i - \theta_j)}, \quad (5.11)$$

$$Q_{m_1, m_2}^+(\lambda; \theta) := \sum_{i, j: m_i + m_j = m_2 + m_1} \sqrt{\lambda_i \lambda_j} e^{-i(\theta_i + \theta_j)}. \quad (5.12)$$

Note that $Q_{m_1, m_2}^\sigma(\lambda; \theta) = 0$ if there exists no couple $m_i, m_j \in \mathcal{S}_0$ such that $m_i + \sigma m_j = m_1 + \sigma m_2$.

(iii) $\mathcal{H}_2 = \mathcal{H}_2^{\text{hor}} + \mathcal{H}_2^{\text{mix}} \in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0}$ and $[\mathcal{H}_2^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [\mathcal{H}_2^{\text{mix}}]_{s/4, -2}^{\mathcal{O}_0} \leq \varepsilon^2$.

(iv) One has $\mathcal{M} \circ \mathcal{T}^{(D)} = \mathcal{M}$ and $\mathcal{P} \circ \mathcal{T}^{(D)} = \mathcal{P}$.

Proof. Decompose $\mathcal{H}^{(0)} = \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}}$ where $\mathcal{H}^{\text{diag}}$ and \mathcal{H}^{out} defined in (5.4) respectively (5.5) are both horizontal. We recall that $\mathcal{H}^{(0)}$ does not have a line component. We use the method of the Lie series, constructing the symplectic map $\mathcal{T}^{(D)}$ as the time-1 flow of a Hamiltonian χ_1 of the form

$$\begin{aligned} \chi_1 &= \chi_1^{\text{diag}} + \chi_1^{\text{out}}, \\ \chi_1^{\text{diag}} &:= \sum_{m_1, m_2, n \neq 0} \chi_{m_1, m_2, n}^-(\lambda; \theta) a_{(m_1, n)} \bar{a}_{(m_2, n)}, \quad \chi_{m_1, m_2, n}^- = \overline{\chi_{m_2, m_1, n}^-} \\ \chi_1^{\text{out}} &:= 2 \sum_{m_1, m_2, n > 0} \text{Re}(\chi_{m_1, m_2, n}^+(\lambda; \theta) a_{(m_1, n)} a_{(m_2, -n)}). \end{aligned} \quad (5.13)$$

By construction $\chi_1^{\text{line}} = 0$, so that χ_1 Poisson commutes with any Hamiltonian $\mathcal{F}^{\text{line}}$. This implies that $\mathcal{T}^{(D)}$ preserves the line component $\mathcal{D}^{\text{line}}$. Using formula (5.1) we have that

$$\begin{aligned} (\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{(0)}) \circ \mathcal{T}^{(D)} &= \omega \cdot \mathcal{Y} + \mathcal{D} \\ &+ \{\mathcal{D}^{\text{diag}}, \chi_1\} + \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}} + \{\omega \cdot \mathcal{Y}, \chi_1\} + \frac{1}{2} \{\mathcal{H}^{\text{out}}, \chi_1^{\text{diag}}\} \end{aligned} \quad (5.14)$$

$$+ \{\mathcal{H}^{\text{diag}}, \chi_1\} + \frac{1}{2} \{\{\mathcal{D}^{\text{diag}} + \omega \cdot \mathcal{Y}, \chi_1\}, \chi_1\} + \frac{1}{2} \{\mathcal{H}^{\text{out}}, \chi_1^{\text{diag}}\} \quad (5.15)$$

$$+ \{\mathcal{H}^{\text{out}}, \chi_1^{\text{out}}\} + \widetilde{\mathcal{T}}_2(\chi_1; \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}}) + \widetilde{\mathcal{T}}_3(\chi_1; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}) \quad (5.16)$$

Our aim is to reduce the *order* of $\mathcal{H}^{(0)}$, to this purpose we set

$$\{\mathcal{D}^{\text{diag}}, \chi_1\} + \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}} + \{\omega \cdot \mathcal{Y}, \chi_1\} + \frac{1}{2}\{\mathcal{H}^{\text{out}}, \chi_1^{\text{diag}}\} = \mathcal{Z} + \mathcal{R} \quad (5.17)$$

where \mathcal{Z} is in the kernel of $\text{ad}(\mathcal{D}^{\text{diag}})$ while $\mathcal{R} \in \mathcal{Q}_{s/2, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/2, -\bar{2}}^{\mathcal{O}_0}$. Note that $\chi_1, \mathcal{Z}, \mathcal{R}$ are all to be determined. Writing (5.17) in Fourier-Taylor components, explicitly we obtain for the diagonal part

$$i(m_1^2 - m_2^2)\chi_{m_1, m_2, n}^- + H_{m_1, m_2}^- + \omega \cdot \partial_\theta \chi_{m_1, m_2, n}^- = \delta(|m_1|, |m_2|)\mathcal{Z}_{m_1, m_2}^- + \mathcal{R}_{m_1, m_2}^-$$

and for the out-diagonal part

$$i(m_1^2 + m_2^2 + 2n^2)\chi_{m_1, m_2, n}^+ + H_{m_1, m_2}^+ + \omega \cdot \partial_\theta \chi_{m_1, m_2, n}^+ + \frac{1}{2}\{\mathcal{H}^{\text{out}}, \chi_1^{\text{diag}}\}_{m_1, m_2}^+ = \mathcal{R}_{m_1, m_2, n}^+,$$

note that the last summand in the left hand side depends only on χ^- . A simple computation using the fact that the diagonal coefficients of the matrix H^{diag} are zero, i.e. $H_{m, m}^- = 0$, shows that a solution is given by

$$\begin{aligned} \chi_{m_1, m_2, n}^- &:= \begin{cases} i \frac{H_{m_1, m_2}^-}{m_1^2 - m_2^2} - \frac{\omega \cdot \partial_\theta H_{m_1, m_2}^-}{(m_1^2 - m_2^2)^2}, & |m_1| \neq |m_2|, \\ 0, & |m_1| = |m_2| \end{cases}, \\ \chi_{m_1, m_2, n}^+ &:= i \frac{H_{m_1, m_2}^+ + \frac{1}{2}\{\mathcal{H}^{\text{out}}, \chi_1^{\text{diag}}\}_{m_1, m_2}^+}{(m_1^2 + m_2^2 + 2n^2)}, \\ \mathcal{Z} &= 0, \\ \mathcal{R}_{m_1, m_2}^- &= \begin{cases} H_{m_1, -m_1}^-, & m_1 = -m_2 \\ -\frac{(\omega \cdot \partial_\theta)^2 H_{m_1, m_2}^-}{(m_1^2 - m_2^2)^2}, & |m_1| \neq |m_2|, \\ 0 & \text{otherwise} \end{cases}, \quad \mathcal{R}_{m_1, m_2, n}^+ = \omega \cdot \partial_\theta \chi_{m_1, m_2, n}^+. \end{aligned} \quad (5.18)$$

By construction χ_1^{diag} and $\mathcal{R}^{\text{diag}}$ are horizontal since their coefficients *do not* depend on n . We show now that

$$\{\mathcal{M}, \chi_1\} = 0, \quad \{\mathcal{P}, \chi_1\} = 0.$$

This follows immediately by remarking that $\mathcal{H}^{\text{diag}}$ and \mathcal{H}^{out} commute with \mathcal{M} and \mathcal{P} and hence they satisfy the selection rules of Remark 3.1. By the explicit formula for χ_1 it follows that the same selection rules hold for χ_1 . This shows that item (iv) holds.

We pass now to the quantitative estimates. We define

$$\chi_1^{\text{hor}} := (\chi_1)^{\text{diag}}, \quad \chi_1^{\text{mix}} := (\chi_1)^{\text{out}}.$$

In order to estimate χ_1^{hor} , we split it in two components by writing

$$\begin{aligned} \chi_1^{\text{hor}} &:= \chi_{1, -1}^{\text{hor}} + \chi_{1, -2}^{\text{hor}}, \\ (\chi_{1, -1}^{\text{hor}})_{m_1, m_2} &:= \begin{cases} i \frac{H_{m_1, m_2}^-}{m_1^2 - m_2^2}, & |m_1| \neq |m_2|, \\ 0, & |m_1| = |m_2| \end{cases}, \quad (\chi_{1, -2}^{\text{hor}})_{m_1, m_2} := \begin{cases} -\frac{\omega \cdot \partial_\theta H_{m_1, m_2}^-}{(m_1^2 - m_2^2)^2}, & |m_1| \neq |m_2| \\ 0, & |m_1| = |m_2|. \end{cases} \end{aligned}$$

We prove that $\chi_{1, -1}^{\text{hor}} \in \mathcal{Q}_{s, -1}^{\mathcal{O}_0, \text{hor}}$, $\chi_{1, -2}^{\text{hor}} \in \mathcal{Q}_{3s/4, -2}^{\mathcal{O}_0, \text{hor}}$, $\chi_1^{\text{mix}} \in \mathcal{Q}_{3s/4, -\bar{2}}^{\mathcal{O}_0}$ with quantitative estimates

$$[\chi_{1, -1}^{\text{hor}}]_{s, -1}^{\mathcal{O}_0} + [\chi_{1, -2}^{\text{hor}}]_{3s/4, -2}^{\mathcal{O}_0} + [\chi_1^{\text{mix}}]_{3s/4, -\bar{2}}^{\mathcal{O}_0} \leq (|\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0} + |\mathcal{H}^{\text{out}}|_s^{\mathcal{O}_0}) \leq \varepsilon. \quad (5.19)$$

We begin with $\chi_{1, -1}^{\text{hor}}$. By definition $[\chi_{1, -1}^{\text{hor}}]_{s, -1}^{\mathcal{O}_0}$ is the $|\cdot|_s^{\mathcal{O}_0}$ norm of the Hamiltonian with Fourier-Taylor

coefficients $\left\{ \langle m_1 \rangle \left| \frac{H_{m_1, m_2}^{-, \ell}}{m_1^2 - m_2^2} \right| \right\}$. Such coefficients are estimated by

$$\langle m_1 \rangle (\chi_{1, -1}^{\text{hor}})_{m_1, m_2, \ell}^{-, \ell} |_{\mathbb{C}}^{\mathcal{O}_0} \leq \frac{\langle m_1 \rangle}{|m_1^2 - m_2^2|} |H_{m_1, m_2}^{-, \ell}|_{\mathbb{C}}^{\mathcal{O}_0} \leq |H_{m_1, m_2}^{-, \ell}|_{\mathbb{C}}^{\mathcal{O}_0}$$

since $\frac{\langle m_1 \rangle}{|m_1^2 - m_2^2|} \leq 1$ for $|m_1| \neq |m_2|$. Thus, by Proposition B.1 (iii), $[\chi_{1,-1}^{\text{hor}}]_{s,-1}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0}$. Consider now $\chi_{1,-2}^{\text{hor}}$. By construction

$$\left| \langle m_1 \rangle^2 (\chi_{1,-2}^{\text{hor}})^{-,\ell} \right|_{\mathbb{C}}^{\mathcal{O}_0} < \frac{\langle m_1 \rangle^2 |\ell|}{|m_1^2 - m_2^2|^2} |H_{m_1, m_2}^{-,\ell}|_{\mathbb{C}}^{\mathcal{O}_0} < |\ell| |H_{m_1, m_2}^{-,\ell}|_{\mathbb{C}}^{\mathcal{O}_0},$$

which implies that $[\chi_{1,-2}^{\text{hor}}]_{3s/4,-2}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0}$.

Finally consider χ_1^{mix} . By definition, $[\chi_1^{\text{mix}}]_{s,-\vec{2}}^{\mathcal{O}_0}$ is the $|\cdot|_s^{\mathcal{O}_0}$ norm of the Hamiltonian with Fourier-Taylor coefficients $\{(\langle m_1 \rangle^2 + \langle n \rangle^2) |\chi_{m_1, m_2, n}^{+,\ell}|\}$. We denote $G_{m_1, m_2}^{+,\ell} := \frac{1}{2}(\{\mathcal{H}^{\text{out}}, \chi^{\text{diag}}\})_{m_1, m_2}^{+,\ell}$ and get

$$\begin{aligned} \left| (\langle m_1 \rangle^2 + \langle n \rangle^2) \chi_{m_1, m_2, n}^{+,\ell} \right|_{\mathbb{C}}^{\mathcal{O}_0} &\leq \frac{(\langle m_1 \rangle^2 + \langle n \rangle^2)}{|m_1^2 + m_2^2 + 2n^2|} (|H_{m_1, m_2}^{+,\ell}|_{\mathbb{C}}^{\mathcal{O}_0} + |G_{m_1, m_2}^{+,\ell}|_{\mathbb{C}}^{\mathcal{O}_0}) \\ &< (|H_{m_1, m_2}^{+,\ell}|_{\mathbb{C}}^{\mathcal{O}_0} + |G_{m_1, m_2}^{+,\ell}|_{\mathbb{C}}^{\mathcal{O}_0}) \end{aligned}$$

since $\frac{(\langle m_1 \rangle^2 + \langle n \rangle^2)}{m_1^2 + m_2^2 + 2n^2} \leq 1$. Once again we get by Proposition B.1 (iii) that

$$[\chi_1^{\text{mix}}]_{3s/4,-\vec{2}}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{out}}|_s^{\mathcal{O}_0} (1 + 2|\chi_1^{\text{diag}}|_{3s/4}^{\mathcal{O}_0}) \leq |\mathcal{H}^{\text{out}}|_s^{\mathcal{O}_0} (1 + |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0}).$$

We have thus proven (5.19).

We show now that line (5.14) belongs to $\mathcal{Q}_{s/2,-2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/2,-\vec{2}}^{\mathcal{O}_0}$, i.e. it is either horizontal and of order -2 or of order $-\vec{2}$. By the homological equation (5.17), line (5.14) equals

$$\begin{aligned} \mathcal{R} &= \mathcal{R}^{\text{diag}} + \mathcal{R}^{\text{out}} = \mathcal{R}_1^{\text{diag}} + \mathcal{R}_2^{\text{diag}} + \mathcal{R}^{\text{out}}, \\ \mathcal{R}_1^{\text{diag}} &= \{\delta(m_1, -m_2) H_{m_1, -m_1}^-\}, \quad \mathcal{R}_2^{\text{diag}} = \omega \cdot \partial_\theta \chi_{1,-2}^{\text{hor}}, \\ \mathcal{R}^{\text{out}} &= \omega \cdot \partial_\theta \chi_1^{\text{mix}} \equiv \{\omega \cdot \partial_\theta \chi_{m_1, m_2, n}^+\} \end{aligned}$$

First we estimate $\mathcal{R}_1^{\text{diag}}$. By momentum conservation, see Remark 3.1, its coefficients are not-zero only if $2m_1 = -\pi(\ell)$ so $[\mathcal{R}_1^{\text{diag}}]_{3s/4,-2}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0}$.

Consider now $\mathcal{R}_2^{\text{diag}}$. One has $[\mathcal{R}_2^{\text{diag}}]_{s/2,-2}^{\mathcal{O}_0} \leq [\chi_{1,-2}^{\text{hor}}]_{3s/4,-2}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0}$.

Finally consider \mathcal{R}^{out} . One has

$$[\mathcal{R}^{\text{out}}]_{s/2,-\vec{2}}^{\mathcal{O}_0} \leq [\chi_1^{\text{mix}}]_{3s/4,-\vec{2}}^{\mathcal{O}_0} \leq |\mathcal{H}^{\text{out}}|_s^{\mathcal{O}_0}.$$

Collecting all the estimates we have that

$$[\mathcal{R}^{\text{diag}}]_{s/2,-2}^{\mathcal{O}_0} + [\mathcal{R}^{\text{out}}]_{s/2,-\vec{2}}^{\mathcal{O}_0} \leq |\mathcal{H}^{(0)}|_s^{\mathcal{O}_0} < \varepsilon. \quad (5.20)$$

Thus we proved that line (5.14) belongs to $\mathcal{Q}_{s/2,-2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/2,-\vec{2}}^{\mathcal{O}_0}$.

Let us now prove the same for (5.15). The first term to consider is

$$\{\mathcal{H}^{\text{diag}}, \chi_1\} = \{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\} + \{\mathcal{H}^{\text{diag}}, \chi_{1,-2}^{\text{hor}}\} + \{\mathcal{H}^{\text{diag}}, \chi_1^{\text{mix}}\}, \quad (5.21)$$

where we used that $\chi_1 = \chi_{1,-1}^{\text{hor}} + \chi_{1,-2}^{\text{hor}} + \chi_1^{\text{mix}}$. We will prove that $\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\} + \{\mathcal{H}^{\text{diag}}, \chi_{1,-2}^{\text{hor}}\} \in \mathcal{Q}_{s/2,-2}^{\mathcal{O}_0, \text{hor}}$ while $\{\mathcal{H}^{\text{diag}}, \chi_1^{\text{mix}}\} \in \mathcal{Q}_{s/2,-\vec{2}}^{\mathcal{O}_0}$ with quantitative estimates

$$[\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\}]_{s/2,-2}^{\mathcal{O}_0} + [\{\mathcal{H}^{\text{diag}}, \chi_{1,-2}^{\text{hor}}\}]_{s/2,-2}^{\mathcal{O}_0} + [\{\mathcal{H}^{\text{diag}}, \chi_1^{\text{mix}}\}]_{s/2,-\vec{2}}^{\mathcal{O}_0} \leq (|\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0})^2 < \varepsilon^2 \quad (5.22)$$

First note that since both $\mathcal{H}^{\text{diag}}$ and χ_1^{diag} are horizontal, $\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\} + \{\mathcal{H}^{\text{diag}}, \chi_{1,-2}^{\text{hor}}\}$ is horizontal as well. We pass to quantitative estimates. By Lemma 3.18 we have

$$[\{\mathcal{H}^{\text{diag}}, \chi_{1,-2}^{\text{hor}}\}]_{s/2,-2}^{\mathcal{O}_0} \leq |\chi_{1,-2}^{\text{hor}}|_{3s/4,-2}^{\mathcal{O}_0} |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0} < \varepsilon^2. \quad (5.23)$$

Now $\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\}$ is apparently only of order -1 ; however a direct computation shows that it is in fact of order -2 :

$$\begin{aligned}
(\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\})_{m_1, m_2}^\ell &= - \sum_{\substack{m_3, \ell_1, \ell_2 \\ \ell_1 + \ell_2 = \ell, m_1 - m_3 = \pi(\ell_1) \\ m_3 - m_2 = \pi(\ell_2) \\ |m_3| \neq |m_2|}} \frac{H_{m_1, m_3}^{-, \ell_1} H_{m_3, m_2}^{-, \ell_2}}{(m_3^2 - m_2^2)} + \sum_{\substack{m_4, \ell_1, \ell_2 \\ \ell_1 + \ell_2 = \ell, m_1 - m_4 = \pi(\ell_2) \\ m_4 - m_2 = \pi(\ell_1) \\ |m_1| \neq |m_4|}} \frac{H_{m_1, m_4}^{-, \ell_2} H_{m_4, m_2}^{-, \ell_1}}{(m_1^2 - m_4^2)} \\
&= - \sum_{\substack{m_3, \ell_1, \ell_2 \\ \ell_1 + \ell_2 = \ell, m_1 - m_3 = \pi(\ell_1) \\ m_3 - m_2 = \pi(\ell_2) \\ |m_3| \neq |m_2|, |m_1| \neq |m_1 - m_3 + m_2|}} \frac{H_{m_1, m_3}^{-, \ell_1} H_{m_3, m_2}^{-, \ell_2}}{(m_3^2 - m_2^2)} - \frac{H_{m_1, m_1 - m_3 + m_2}^{-, \ell_2} H_{m_1 - m_3 + m_2, m_2}^{-, \ell_1}}{(m_1^2 - (m_1 - m_3 + m_2)^2)} \\
&= - \sum_{\substack{m_3, \ell_1, \ell_2 \\ \ell_1 + \ell_2 = \ell, m_1 - m_3 = \pi(\ell_1) \\ m_3 - m_2 = \pi(\ell_2) \\ |m_3| \neq |m_2|, |m_1| \neq |m_1 - m_3 + m_2|}} H_{m_1, m_3}^{-, \ell_1} H_{m_3, m_2}^{-, \ell_2} \left(\frac{1}{(m_3^2 - m_2^2)} - \frac{1}{(m_1^2 - (m_1 - m_3 + m_2)^2)} \right) \\
&= - \sum_{\substack{m_3, \ell_1, \ell_2 \\ \ell_1 + \ell_2 = \ell, m_1 - m_3 = \pi(\ell_1) \\ m_3 - m_2 = \pi(\ell_2) \\ |m_3| \neq |m_2|, |m_1| \neq |m_1 - m_3 + m_2|}} H_{m_1, m_3}^{-, \ell_1} H_{m_3, m_2}^{-, \ell_2} \left(\frac{2(m_1 - m_3)}{(m_3 - m_2)(m_3 + m_2)(2m_1 - m_3 + m_2)} \right)
\end{aligned}$$

Note that the conditions $|m_3| \neq |m_2|$ and $|m_1| \neq |m_4|$ come from the definition of χ_1^{diag} . By Remark 3.15 we may assume $|\ell|, |\ell_i| \leq c\langle m_1 \rangle$ (where $c^{-1} = 2 \max_{1 \leq i \leq d} (|m_i|)$) since otherwise we have an infinitely smoothing term. We have

$$m_3 + m_2 = 2m_1 + m_3 - m_1 + m_2 - m_1 = 2m_1 - \pi(\ell_1) - \pi(\ell)$$

thus using also that $|\pi(\ell_1) + \pi(\ell)| \leq |m_1|$ and $|\pi(\ell_2)| \leq |m_1|$ one deduce the estimate

$$\langle m_1 \rangle^2 \frac{2|m_1 - m_3|}{|m_3 - m_2||m_3 + m_2||2m_1 - m_3 + m_2|} \leq \frac{2|\pi(\ell_1)|\langle m_1 \rangle^2}{|\pi(\ell_2)||2m_1 - \pi(\ell_1) - \pi(\ell)||2m_1 - \pi(\ell_2)|} \leq 2|\pi(\ell_1)|.$$

This implies that $\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{diag}}\} \in \mathcal{Q}_{3s/4, -2}^{\mathcal{O}_0, \text{hor}}$ with quantitative estimate

$$[\{\mathcal{H}^{\text{diag}}, \chi_{1,-1}^{\text{hor}}\}]_{3s/4, -2}^{\mathcal{O}_0} \leq (|\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0})^2 \leq \varepsilon^2. \quad (5.24)$$

Finally consider $\{\mathcal{H}^{\text{diag}}, \chi_1^{\text{mix}}\}$. By Lemma B.20(i) one has

$$[\{\mathcal{H}^{\text{diag}}, \chi_1^{\text{mix}}\}]_{s/2, -\bar{2}}^{\mathcal{O}_0} \leq |\chi_1^{\text{mix}}|_{3s/4, -\bar{2}}^{\mathcal{O}_0} |\mathcal{H}^{\text{diag}}|_s^{\mathcal{O}_0} \leq \varepsilon^2. \quad (5.25)$$

Thus estimate (5.22) follows from (5.23), (5.24), (5.25).

Consider now the last two terms of line (5.15). Substituting (5.17) into

$$\begin{aligned}
\frac{1}{2} \{ \{\mathcal{D}^{\text{diag}} + \omega \cdot \mathcal{Y}, \chi_1\}, \chi_1 \} + \frac{1}{2} \{ \mathcal{H}^{\text{out}}, \chi_1^{\text{diag}} \} &= \frac{1}{2} \{ -\mathcal{H}^{\text{diag}} - \frac{1}{2} \{ \mathcal{H}^{\text{out}}, \chi_1^{\text{diag}} \} + \mathcal{R}, \chi_1 \} - \frac{1}{2} \{ \mathcal{H}^{\text{out}}, \chi_1^{\text{mix}} \} \\
&= \frac{1}{2} \{ \chi_1, \mathcal{H}^{\text{diag}} \} + \frac{1}{4} \{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1 \} \\
&\quad + \frac{1}{2} \{ \mathcal{R}, \chi_1 \} + \frac{1}{2} \{ \chi_1^{\text{mix}}, \mathcal{H}^{\text{out}} \}.
\end{aligned} \quad (5.26)$$

$$+ \frac{1}{2} \{ \mathcal{R}, \chi_1 \} + \frac{1}{2} \{ \chi_1^{\text{mix}}, \mathcal{H}^{\text{out}} \}. \quad (5.27)$$

The first term of (5.26) is the same as (5.21), thus it is estimated by (5.22).

Consider now the second term in (5.26). Using that \mathcal{H}^{out} is horizontal one decompose

$$\{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1 \} = \{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1^{\text{diag}} \} + \{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1^{\text{out}} \} \in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$$

with quantitative estimates

$$[\{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1^{\text{diag}} \}]_{s/4, -2}^{\mathcal{O}_0} + [\{ \{ \chi_1^{\text{diag}}, \mathcal{H}^{\text{out}} \}, \chi_1^{\text{out}} \}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq (|\mathcal{H}|_s^{\mathcal{O}_0})^3 \leq \varepsilon^3. \quad (5.28)$$

Next consider the term $\{ \mathcal{R}, \chi_1 \}$. By construction $\mathcal{R} = \mathcal{R}^{\text{diag}} + \mathcal{R}^{\text{out}}$, $\chi_1 = \chi_1^{\text{diag}} + \chi_1^{\text{out}}$ with $\mathcal{R}^{\text{diag}}, \chi_1^{\text{diag}}$ horizontal. So

$$\{ \mathcal{R}, \chi_1 \} = \{ \mathcal{R}^{\text{diag}}, \chi_1^{\text{diag}} \} + \{ \mathcal{R}^{\text{diag}}, \chi_1^{\text{out}} \} + \{ \mathcal{R}^{\text{out}}, \chi_1^{\text{diag}} \} + \{ \mathcal{R}^{\text{out}}, \chi_1^{\text{out}} \}$$

and the first term belongs to $\mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}}$, while the last three belong to $\mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$, and (5.19), (5.20) give the quantitative estimates

$$[\{\mathcal{R}^{\text{diag}}, \chi_1^{\text{diag}}\}]_{s/4, -2}^{\mathcal{O}_0} + [\{\mathcal{R}^{\text{diag}}, \chi_1^{\text{out}}\}]_{s/4, -\bar{2}}^{\mathcal{O}_0} + [\{\mathcal{R}^{\text{out}}, \chi_1^{\text{diag}}\}]_{s/4, -\bar{2}}^{\mathcal{O}_0} + [\{\mathcal{R}^{\text{out}}, \chi_1^{\text{out}}\}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2. \quad (5.29)$$

Finally we study $\{\mathcal{H}^{\text{out}}, \chi_1^{\text{mix}}\}$, which is easily seen to belong to $\mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$ with quantitative estimate

$$[\{\mathcal{H}^{\text{out}}, \chi_1^{\text{mix}}\}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2. \quad (5.30)$$

Altogether we have proved that (5.15) = (5.15)^{hor} + (5.15)^{mix} $\in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$ with estimates

$$[(5.15)^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [(5.15)^{\text{mix}}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2. \quad (5.31)$$

In order to study line (5.16) we apply Lemma 3.18 to $\widetilde{\mathcal{V}}_2(\chi_1; \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}})$. We have

$$[\widetilde{\mathcal{V}}_2(\chi_1; \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}})^{\text{hor}}]_{s/2, -2}^{\mathcal{O}_0} \leq \varepsilon^2, \quad [\widetilde{\mathcal{V}}_2(\chi_1; \mathcal{H}^{\text{diag}} + \mathcal{H}^{\text{out}})^{\text{mix}}]_{s/2, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2.$$

Now consider $\widetilde{\mathcal{V}}_3(\chi_1; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}})$. Write it as

$$\begin{aligned} \widetilde{\mathcal{V}}_3(\chi_1; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}) &= \sum_{k \geq 3} \frac{\text{ad}(\chi_1)^k [\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}]}{k!} = \sum_{k \geq 3} \frac{\text{ad}(\chi_1)^{k-1} [\{\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}, \chi_1\}]}{k!} \\ &= \sum_{k \geq 3} \frac{\text{ad}(\chi_1)^{k-1} [-\mathcal{H}^{\text{diag}} - \mathcal{H}^{\text{out}} + \frac{1}{2}\{\chi_1^{\text{diag}}, \mathcal{H}^{\text{out}}\} + \mathcal{Z} + \mathcal{R}]}{k!}. \end{aligned}$$

Again we apply Lemma 3.18 to obtain

$$[(\widetilde{\mathcal{V}}_3(\chi_1; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}))^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [(\widetilde{\mathcal{V}}_3(\chi_1; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}))^{\text{mix}}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2.$$

So far we have proved that

$$(\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}} + \mathcal{H}^{(0)}) \circ \mathcal{T}^{(D)} = \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}} + \mathcal{R} + \widetilde{\mathcal{H}}$$

with $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{\text{hor}} + \widetilde{\mathcal{H}}^{\text{mix}} \in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$ fulfilling

$$[\widetilde{\mathcal{H}}^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [\widetilde{\mathcal{H}}^{\text{mix}}]_{s/4, -\bar{2}}^{\mathcal{O}_0} \leq \varepsilon^2.$$

The next step is to extract from the so obtained Hamiltonian the terms which are exactly of order ε . Obviously such terms can be contained only in \mathcal{R} , so we extract from \mathcal{R} the monomials of order exactly ε , which we denote by \mathcal{H}_1 :

$$\mathcal{H}_1 := \partial_\varepsilon \mathcal{R}|_{\varepsilon=0}.$$

By formula (2.13), it follows that

$$q_m^{\text{fg}}(\theta) = \begin{cases} \sqrt{\varepsilon} \sqrt{\lambda_i} e^{i\theta_i} + \varepsilon^{3/2} p_m(\theta), & m = \mathfrak{m}_i \in \mathcal{S}_0 \\ \varepsilon^{3/2} p_m(\theta), & m \notin \mathcal{S}_0 \end{cases}, \quad (5.32)$$

thus

$$H_{m_1, m_2}^-(\lambda; \theta) = 2 \sum_{m_3 - m_4 = m_2 - m_1}^* q_{m_3}^{\text{fg}}(\theta) \bar{q}_{m_4}^{\text{fg}}(\theta) \quad (5.33)$$

$$= 2\varepsilon \sum_{i \neq j: \mathfrak{m}_i - \mathfrak{m}_j = m_2 - m_1} \sqrt{\lambda_i \lambda_j} e^{i(\theta_i - \theta_j)} + \mathcal{O}(\varepsilon^2) \quad (5.34)$$

and

$$\begin{aligned} H_{m_1, m_2}^+(\lambda; \theta) &= \sum_{m_3 + m_4 = m_2 + m_1} \bar{q}_{m_3}^{\text{fg}}(\theta) q_{m_4}^{\text{fg}}(\theta) \\ &= \varepsilon \sum_{i, j: \mathfrak{m}_i + \mathfrak{m}_j = m_2 + m_1} \sqrt{\lambda_i \lambda_j} e^{-i(\theta_i + \theta_j)} + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (5.35)$$

Now we substitute such kernels into the expression of \mathcal{R} (5.18) and separate the terms of order ε which define \mathcal{H}_1 . All the other terms define \mathcal{H}_2 . \square

5.1.2 Step 2: removal of Birkhoff non resonant monomials at order ε

We begin with the following definition:

Definition 5.5. A monomial of the form $e^{i\theta \cdot \ell} a^\alpha \bar{a}^\beta$, $|\alpha| + |\beta| = 2$ will be said to be Birkhoff resonant iff

$$\omega^{(0)} \cdot \ell + \Omega^{(0)} \cdot (\alpha - \beta) = 0 .$$

In the next lemma we describe the monomials in the Hamiltonian \mathcal{H}_1 (defined in (5.6)) which are Birkhoff resonant.

Lemma 5.6. The following holds true:

(i) Consider a monomial $e^{i\theta \cdot \ell} a_{(m_1, n)} \bar{a}_{(m_2, n)}$, $|m_1| \neq |m_2|$, $|\ell| = 2$. Then the monomial is Birkhoff resonant iff there exist $1 \leq i, j \leq \mathbf{d}$ such that

$$m_1 = \mathbf{m}_j , \quad m_2 = \mathbf{m}_i , \quad \ell = \mathbf{e}_i - \mathbf{e}_j . \quad (5.36)$$

Such monomials have support in $\bigcup_{n \in \mathbb{Z}} \mathcal{S}_{0n}$. Thus, the support of such monomials forms a horizontal rectangle with two points in \mathcal{S}_0 .

(ii) Consider a monomial $e^{i\theta \cdot \ell} a_{(m_1, n)} \bar{a}_{(-m_1, n)}$, $|\ell| = 2$. Then the monomial is Birkhoff resonant iff there exist $1 \leq i, j \leq \mathbf{d}$, $i \neq j$, such that $\mathbf{m}_i = -\mathbf{m}_j$.

(iii) Consider a monomial $e^{i\theta \cdot \ell} \bar{a}_{(m_1, n)} a_{(m_2, -n)}$, $|\ell| = 2$. Then the monomial is Birkhoff resonant iff there exist $1 \leq i, j \leq \mathbf{d}$ such that

$$m_2 = \mathbf{m}_i + \mathbf{m}_j - m_1 , \quad (m_1 - \mathbf{m}_i)(m_1 - \mathbf{m}_j) + n^2 = 0 , \quad \ell = \mathbf{e}_i + \mathbf{e}_j . \quad (5.37)$$

Such monomials have support in $\bigcup_{i < j} \mathcal{C}_{ij}$. Thus, their support form a rotated rectangle with two point in \mathcal{S}_0 .

Proof. (i) Obviously if (5.36) holds, then the monomial is Birkhoff resonant. Assume now that the monomial is Birkhoff resonant. We will use the conservation of mass and momentum as described in Remark 3.7. By conservation of mass $\sum_i \ell_i = 0$, which together with $|\ell| = 2$ implies $\ell = \mathbf{e}_i - \mathbf{e}_j$, $i \neq j$. Now by conservation of momentum $\pi(\ell) + m_1 - m_2 = 0$, which shows that $\mathbf{m}_i - \mathbf{m}_j + m_1 - m_2 = 0$. Thus

$$\begin{aligned} 0 &= \omega^{(0)} \cdot \ell + \Omega^{(0)} \cdot (\alpha - \beta) = \omega^{(0)} \cdot (\mathbf{e}_i - \mathbf{e}_j) + m_1^2 - m_2^2 = \mathbf{m}_i^2 - \mathbf{m}_j^2 + m_1^2 - m_2^2 \\ &= 2(\mathbf{m}_i - \mathbf{m}_j)(\mathbf{m}_j - m_1) . \end{aligned}$$

If $\mathbf{m}_j = \mathbf{m}_i$, then by momentum conservation $m_1 = m_2$, which contradicts $|m_1| \neq |m_2|$. Thus $m_1 = \mathbf{m}_j$, $m_2 = \mathbf{m}_i$.

(ii) By conservation of mass and momentum $\ell = \mathbf{e}_i - \mathbf{e}_j$, $i \neq j$, $\mathbf{m}_i - \mathbf{m}_j + 2m_1 = 0$. Then

$$0 = \omega^{(0)} \cdot \ell + \Omega^{(0)} \cdot (\alpha - \beta) = \mathbf{m}_i^2 - \mathbf{m}_j^2 = (\mathbf{m}_i - \mathbf{m}_j)(\mathbf{m}_j + \mathbf{m}_i) .$$

Since $i \neq j$, it follows that $\mathbf{m}_j + \mathbf{m}_i = 0$.

(iii) Once again if (5.37) holds, than the monomial is Birkhoff resonant. Assume now to have a Birkhoff resonant monomial. By conservation of mass $\sum_i \ell_i = 2$, which together with $|\ell| = 2$ implies $\ell = \mathbf{e}_i + \mathbf{e}_j$. Now the conservation of momentum reads $\pi(\ell) - m_1 - m_2 = 0$, hence one has $\mathbf{m}_i + \mathbf{m}_j - m_1 - m_2 = 0$. Thus

$$\begin{aligned} 0 &= \omega^{(0)} \cdot \ell + \Omega^{(0)} \cdot (\alpha - \beta) = \omega^{(0)} \cdot (\mathbf{e}_i + \mathbf{e}_j) - m_1^2 - m_2^2 - 2n^2 = \mathbf{m}_i^2 + \mathbf{m}_j^2 - m_1^2 - m_2^2 - 2n^2 \\ &= -2[(m_1 - \mathbf{m}_i)(m_1 - \mathbf{m}_j) + n^2] , \end{aligned}$$

which shows the claimed condition. \square

Remark 5.7. By the condition of arithmetic genericity of \mathcal{S}_0 (see Definition 2.4), one has $\mathbf{m}_i \neq -\mathbf{m}_j \forall i, j$. Thus the monomials described in Lemma 5.6 (ii) are always Birkhoff non resonant.

In the next lemma we perform a canonical transformation which removes all the Birkhoff non resonant monomials.

Lemma 5.8. *There exists an invertible symplectic transformation $\mathcal{T}^{(B)} : D(s/8, r/4) \rightarrow D(s/4, r/2)$ $\forall s_0/64 \leq s \leq s_0, 0 < r \leq r_0$ which transforms the Hamiltonian (5.6) in the following form:*

$$(\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}_1 + \mathcal{H}_2) \circ \mathcal{T}^{(B)} = \omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Z}_1 + \widehat{\mathcal{H}}_2, \quad (5.38)$$

where

(i) the map $\mathcal{T}^{(B)}$ is the time-1 flow of a quadratic Hamiltonian $\chi_2 = \chi_2^{\text{hor}} + \chi_2^{\text{mix}}$ such that $[\chi_2^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [\chi_2^{\text{mix}}]_{s/4, -2}^{\mathcal{O}_0} \leq \varepsilon$.

(ii) \mathcal{Z}_1 is the Birkhof resonant part of \mathcal{H}_1 and has the following form: $\mathcal{Z}_1 = \mathcal{Z}_1^{\text{hor}} + \mathcal{Z}_1^{\text{mix}}$, where

$$\begin{aligned} \mathcal{Z}_1^{\text{hor}} &= \mathcal{Z}_1^{\text{diag}} := 2\varepsilon \sum_{n \neq 0} \sum_{i \neq j} \sqrt{\lambda_i \lambda_j} e^{i(\theta_i - \theta_j)} a_{(\mathbf{m}_j, n)} \bar{a}_{(\mathbf{m}_i, n)} \\ \mathcal{Z}_1^{\text{mix}} &= \mathcal{Z}_1^{\text{out}} := 4\varepsilon \sum_{i < j} \sum_{(m, n) \in \mathcal{C}_{i, j}^+} \sqrt{\lambda_i \lambda_j} \operatorname{Re}(e^{i(\theta_i + \theta_j)} \bar{a}_{(m, n)} \bar{a}_{(\mathbf{m}_i + \mathbf{m}_j - m, -n)}), \end{aligned}$$

where $\mathcal{C}_{i, j}^+$ is defined in (2.27). Note that $\mathcal{Z}_1^{\text{hor}}$ is both horizontal and diagonal while $\mathcal{Z}_1^{\text{mix}}$ is out-diagonal and supported only on the finite set $\mathcal{C}_{i, j}$.

(iii) $\widehat{\mathcal{H}}_2 = \widehat{\mathcal{H}}_2^{\text{hor}} + \widehat{\mathcal{H}}_2^{\text{mix}} \in \mathcal{Q}_{s/8, -2}^{\mathcal{O}_0, \text{hor}} + \mathcal{Q}_{s/8, -2}^{\mathcal{O}_0}$ and $[\widehat{\mathcal{H}}_2^{\text{hor}}]_{s/8, -2}^{\mathcal{O}_0} + [\widehat{\mathcal{H}}_2^{\text{mix}}]_{s/8, -2}^{\mathcal{O}_0} \leq \varepsilon^2$.

(iv) One has $\mathcal{M} \circ \mathcal{T}^{(B)} = \mathcal{M}$ and $\mathcal{P} \circ \mathcal{T}^{(B)} = \mathcal{P}$.

Proof. Once again we use the method of the Lie series. Thus we look for $\mathcal{T}^{(B)}$ as the time-1 flow map of an Hamiltonian $\chi_2 = \chi_2^{\text{diag}} + \chi_2^{\text{out}}$ to be determined. As in the previous step χ_2 Poisson commutes with $\mathcal{D}^{\text{line}}$. Then one has

$$\begin{aligned} (\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}_1 + \mathcal{H}_2) \circ \mathcal{T}^{(B)} &= \omega \cdot \mathcal{Y} + \mathcal{D} \\ &\quad + \{\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}, \chi_2\} + \mathcal{H}_1 \end{aligned} \quad (5.39)$$

$$+ \widetilde{\mathcal{G}}_2(\chi_2; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}) + \widetilde{\mathcal{V}}_1(\chi_2; \mathcal{H}_1) \quad (5.40)$$

$$+ \mathcal{H}_2 \circ \mathcal{T}^{(B)} \quad (5.41)$$

This time we fix χ_2 in order to remove the Birkhoff non resonant terms of \mathcal{H}_1 , so we solve the homological equation with $\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}$ (on the contrary of (5.17)). The homological equation is

$$\{\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}, \chi_2\} + \mathcal{H}_1 = \mathcal{Z}_1$$

for some χ_2, \mathcal{Z}_1 to be determined. We claim that we may divide $\chi_2 = \chi_2^{\text{hor}} + \chi_2^{\text{mix}}$ so that we can solve the horizontal part of the equation and the not-horizontal separately, i.e.

$$\{\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}, \chi_2^{\text{hor}}\} + \mathcal{H}_1^{\text{hor}} = \mathcal{Z}_1^{\text{hor}}, \quad \{\omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}, \chi_2^{\text{mix}}\} + \mathcal{H}_1^{\text{mix}} = \mathcal{Z}_1^{\text{mix}}, \quad (5.42)$$

with

$$[\chi_2^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} + [\chi_2^{\text{mix}}]_{s/4, -2}^{\mathcal{O}_0} \leq \varepsilon.$$

In the first equation of (5.42), remarking that $\mathcal{H}_1^{\text{hor}}$ is also a diagonal Hamiltonian, we make the ansatz that $\mathcal{Z}_1^{\text{hor}}$ and χ_2^{hor} are diagonal as well. Passing to Taylor-Fourier expansion we get for the coefficients $\{\chi_{m_1, m_2, n}^{-, \ell}\}$ of χ_2^{hor} the equation

$$i(\omega \cdot \ell + m_1^2 - m_2^2) \chi_{m_1, m_2, n}^{-, \ell} + (\mathcal{H}_1^{\text{hor}})_{m_1, m_2}^{-, \ell} = (\mathcal{Z}_1^{\text{hor}})_{m_1, m_2}^{-, \ell}. \quad (5.43)$$

Now by the explicit expression of $\mathcal{H}_1^{\text{hor}}$ given in (5.7) we have that the Taylor-Fourier support of $\mathcal{H}_1^{\text{hor}}$ is the set

$$\operatorname{supp}(\mathcal{H}_1^{\text{hor}}) := \left\{ ((m_1, n), (m_2, n), \ell) : \exists 1 \leq i, j \leq \mathbf{d}, i \neq j \text{ s.t. } m_2 - m_1 = \mathbf{m}_i - \mathbf{m}_j, \ell = \mathbf{e}_i - \mathbf{e}_j \right\}.$$

By Lemma 5.6(i) – (ii), all the monomials with $m_1 \neq m_2$ are Birkhoff non resonant, thus using also that $|\ell| = 2$

$$|\omega \cdot \ell + m_1^2 - m_2^2| \geq |\omega^{(0)} \cdot \ell + m_1^2 - m_2^2| - \mathcal{O}(\varepsilon) > 1/2, \quad (5.44)$$

and moreover such divisor does not depend on n . We obtain

$$(\mathcal{Z}_1^{\text{hor}})_{m_1, m_2}^{-, \ell} = \begin{cases} 2\varepsilon\sqrt{\lambda_i\lambda_j}, & m_1 = \mathbf{m}_j, m_2 = \mathbf{m}_i, \ell = \mathbf{e}_i - \mathbf{e}_j \\ 0 & \text{otherwise} \end{cases} \quad (5.45)$$

and

$$\chi_{m_1, m_2, n}^{-, \ell} = \frac{(\mathcal{Z}_1^{\text{hor}})_{m_1, m_2}^{-, \ell} - (\mathcal{H}_1^{\text{hor}})_{m_1, m_2}^{-, \ell}}{i(\omega \cdot \ell + m_1^2 - m_2^2)}.$$

It is clear that $\chi_2^{\text{hor}} \in \mathcal{Q}_{s/4, -2}^{\mathcal{O}_0, \text{hor}}$ is horizontal and using also the estimate (5.44) one gets

$$[\chi_2^{\text{hor}}]_{s/4, -2}^{\mathcal{O}_0} < \varepsilon.$$

Consider now the second equation in (5.42). By Lemma 5.6 (iii) the resonant monomials are those fulfilling (5.37). Thus, using also the rectangle condition $|(\mathbf{m}_j, 0)|^2 - |(m_1, n)|^2 + |(\mathbf{m}_i, 0)|^2 - |(m_2, -n)|^2 = 0$, we define for $n > 0$

$$(\mathcal{Z}_1^{\text{mix}})_{m_1, m_2, n}^{+, \ell} = \begin{cases} 2\varepsilon\sqrt{\lambda_i\lambda_j}, & (m_1 - \mathbf{m}_i)(m_1 - \mathbf{m}_j) + n^2 = 0, \ell = -\mathbf{e}_i - \mathbf{e}_j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_{m_1, m_2, n}^{+, \ell} = \frac{(\mathcal{Z}_1^{\text{mix}})_{m_1, m_2, n}^{-, \ell} - (\mathcal{H}_1^{\text{mix}})_{m_1, m_2, n}^{-, \ell}}{i(\omega \cdot \ell + m_1^2 + m_2^2 + 2n^2)}.$$

As in (5.44), we have the estimate of the small divisors

$$|\omega \cdot \ell + m_1^2 + m_2^2 + 2n^2| \geq \frac{1}{2}$$

which implies that $\chi_2^{\text{mix}} \in \mathcal{Q}_{s/4, -\bar{2}}^{\mathcal{O}_0}$ and

$$[\chi_2^{\text{mix}}]_{s/4, -\bar{2}}^{\mathcal{O}_0} < \varepsilon.$$

Now remark that \mathcal{H}_1 commutes with \mathcal{M} and \mathcal{P} and hence it satisfies the selection rules of Remark 3.1. By the explicit formula for χ_2 it follows that the same selection rules hold for χ_2 , hence $\{\mathcal{M}, \chi_2\} = 0$ and $\{\mathcal{P}, \chi_2\} = 0$ and item (iv) follows.

Now we analyze line (5.40). First one has that, since $\chi_2 = \chi_2^{\text{hor}} + \chi_2^{\text{mix}}$, we can apply Lemma 3.18 to get

$$[\overline{\mathcal{G}}_1(\chi_2; \mathcal{H}_1)^{\text{hor}}]_{s/8, -2}^{\mathcal{O}_0} < \varepsilon^2, \quad [\overline{\mathcal{G}}_1(\chi_2; \mathcal{H}_1)^{\text{mix}}]_{s/8, -\bar{2}}^{\mathcal{O}_0} < \varepsilon^2.$$

Using that $\{\chi_2, \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}\} = \mathcal{H}_1 - \mathcal{Z}_1$, one obtains

$$\overline{\mathcal{G}}_2(\chi_2; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}) = \sum_{k \geq 2} \frac{\text{ad}(\chi_2)^{k-1}[\{\chi_2, \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}\}]}{k!} = \sum_{k \geq 1} \frac{\text{ad}(\chi_2)^k[\mathcal{H}_1 - \mathcal{Z}_1]}{(k+1)!}.$$

Since $\mathcal{H}_1 - \mathcal{Z}_1 = (\mathcal{H}_1 - \mathcal{Z}_1)^{\text{hor}} + (\mathcal{H}_1 - \mathcal{Z}_1)^{\text{mix}}$ we apply Lemma 3.18 to get

$$[(\overline{\mathcal{G}}_2(\chi_3; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}))^{\text{hor}}]_{s/8, -2}^{\mathcal{O}_0} + [(\overline{\mathcal{G}}_2(\chi_3; \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{diag}}))^{\text{mix}}]_{s/8, -\bar{2}}^{\mathcal{O}_0} < \varepsilon^2.$$

Now consider line (5.41). Using again Lemma 3.18 (with $\mathbf{i} = 0$) we get $\widehat{\mathcal{H}}_2 := \mathcal{H}_2 \circ \mathcal{T}^{(B)} = \widehat{\mathcal{H}}_2^{\text{hor}} + \widehat{\mathcal{H}}_2^{\text{mix}}$ with the claimed estimates. \square

5.1.3 Step 3: diagonalization of the Birkhoff resonant terms

In the final step we consider the resonant Hamiltonian in normal form $\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Z}_1$ and we diagonalize it through a transformation which is not close to the identity.

Remark 5.9. Due to our genericity condition we have that $(m, n) \in \cup_{i < k} \mathcal{C}_{i, k}$ implies that $m \notin \mathcal{S}_0$. Moreover given (m, n) with $n > 0$ there exists at most one couple (i, k) , $i < k$, such that $(m, n) \in \mathcal{C}_{i, k}^+$. In the same way given (m, n) with $n < 0$ there exists at most one couple (i, k) , $i < k$, such that $(m, n) \in \mathcal{C}_{i, k}^-$ and consequently $(\mathbf{m}_i + \mathbf{m}_k - m, -n) \in \mathcal{C}_{i, k}^+$.

We now perform a phase shift which removes the dependence on the angles in \mathcal{Z}_1 .

Lemma 5.10. *Consider the Hamiltonian (5.38). For all $s_0/2 \leq s \leq s_0, 0 < r \leq r_0$, there exists an invertible symplectic change of variables $\mathcal{R} : D(s/8, e^{-s_0}r/4) \rightarrow D(s/8, r/4) : (\mathcal{Y}^{(+)}, \theta, \mathbf{b}) \mapsto (\mathcal{Y}, \theta, \mathbf{a})$ s.t.*

$$(\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Z}_1) \circ \mathcal{R} = \omega \cdot \mathcal{Y}^{(+)} + \widehat{\mathcal{D}} + \widehat{\mathcal{Z}}_1 \quad (5.46)$$

where

$$\widehat{\mathcal{D}} := \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})} \Omega_{(m,n)}^{(0)} |b_{(m,n)}|^2 \quad (5.47)$$

$$\widehat{\mathcal{Z}}_1 := \sum_{\substack{1 \leq i \leq d \\ n \in \mathbb{Z} \setminus \{0\}}} (\mathbf{m}_i^2 - \omega_i + n^2) |b_{(\mathbf{m}_i, n)}|^2 + 2\varepsilon \sum_{\substack{1 \leq i, k \leq d, i \neq k \\ n \in \mathbb{Z} \setminus \{0\}}} \sqrt{\lambda_i \lambda_k} b_{(\mathbf{m}_i, n)} \bar{b}_{(\mathbf{m}_k, n)} \quad (5.48)$$

$$+ \sum_{i < k, (m,n) \in \mathcal{C}_{i,k}^+} \left((m^2 + n^2 - \omega_i) |b_{(m,n)}|^2 + ((\mathbf{m}_i + \mathbf{m}_k - m)^2 + n^2 - \omega_k) |b_{(\mathbf{m}_i + \mathbf{m}_k - m, -n)}|^2 \right) \quad (5.49)$$

$$+ 4\varepsilon \sqrt{\lambda_i \lambda_k} \operatorname{Re}(b_{(m,n)} b_{(\mathbf{m}_i + \mathbf{m}_k - m, -n)}) \quad (5.50)$$

Furthermore the following is true:

(i) \mathcal{R} is the identity on the variables in $\mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C})$, and on $\mathcal{S} \cup \mathcal{C}$ is a phase shift.

(ii) $\mathcal{M} \circ \mathcal{R} = \widetilde{\mathcal{M}}$, $\mathcal{P} \circ \mathcal{R} = \widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{P}}$ are defined in (2.31).

Proof. We define the symplectic ⁷ transformation $\mathcal{R} : (\mathcal{Y}^{(+)}, \theta, \mathbf{b}) \mapsto (\mathcal{Y}, \theta, \mathbf{a})$ as

$$\mathcal{Y}_i = \mathcal{Y}_i^{(+)} - \sum_{n \neq 0} |b_{(\mathbf{m}_i, n)}|^2 - \sum_{i < k} \sum_{\substack{j \in \mathcal{C}_{i,k}^+ \\ j \in \mathcal{C}_{i,k}^-}} |b_j|^2 - \sum_{k < i} \sum_{\substack{j \in \mathcal{C}_{k,i}^- \\ j \in \mathcal{C}_{k,i}^+}} |b_j|^2,$$

$$a_{(m,n)} = \begin{cases} e^{i\theta_i} b_{(\mathbf{m}_i, n)} & m = \mathbf{m}_i, \quad n \neq 0 \\ e^{i\theta_i} b_{(m,n)} & (m,n) \in \mathcal{C}_{i,k}^+, \quad i < k \\ e^{i\theta_k} b_{(m,n)} & (m,n) \in \mathcal{C}_{i,k}^-, \quad i < k \\ b_{(m,n)} & \text{otherwise} \end{cases} \quad (5.51)$$

It's a simple computation to show that \mathcal{R} conjugates $\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Z}_1$, the mass \mathcal{M} and the momentum \mathcal{P} to the claimed functions. \square

Note that after this change of variables the dynamics of the action-angles (θ, \mathcal{Y}) , of $\widehat{\mathcal{D}}$ and of $\widehat{\mathcal{Z}}_1$ is decoupled.

Lemma 5.11. *There exists $\varrho > 0$ depending on $s_0, \max |\mathbf{m}_i|^p$ and an open set \mathcal{O}_1 such that for all $s_0/64 \leq s \leq s_0, 0 < r \leq r_0$ and for any $\lambda \in \mathcal{O}_1$ there exists a linear invertible symplectic transformation $\mathcal{U} : D(s/8, \varrho r) \rightarrow D(s/8, e^{-s_0}r/4)$ of the $(\mathbf{a}, \mathcal{Y}, \theta) \mapsto (U\mathbf{a}, \mathcal{Y}, \theta)$ which depends analytically on λ and transforms the Hamiltonian (5.38) in the following form (we are calling the variables \mathcal{Y}, \mathbf{a} again):*

$$(\omega \cdot \mathcal{Y}^{(+)} + \widehat{\mathcal{D}} + \widehat{\mathcal{Z}}_1) \circ \mathcal{U} = \omega \cdot \mathcal{Y} + \widehat{\mathcal{D}} + \widehat{\mathcal{Z}}_1 \circ \mathcal{U} = \omega \cdot \mathcal{Y} + \widetilde{\mathcal{D}}(\lambda, \varepsilon) \quad (5.52)$$

where

(i) $U = \operatorname{diag}(U_n)$ where U_n acts non trivially only on $(\mathcal{S} \cup \mathcal{C}) \cap \{(m,n), (m,-n)\}_{m \in \mathbb{Z}}$ and is the identity elsewhere. The U_n depend analytically on λ and satisfy bounds of the form

$$\|U_n\|^{\operatorname{op}}, \|(U_n)^{-1}\|^{\operatorname{op}} \leq 1.$$

⁷ a simple computation shows that the symplectic form in the new variables is given by

$$d\mathcal{Y}^{(+)} \wedge d\theta + i \sum_{(m,n) \in \mathbb{Z}^2 \setminus \mathcal{S}_0} db_{(m,n)} \wedge d\bar{b}_{(m,n)}$$

(ii) $\tilde{\mathcal{D}}(\lambda, \varepsilon)$ is the diagonal Hamiltonian

$$\tilde{\mathcal{D}}(\lambda, \varepsilon) := \sum_{\tilde{j} \in \mathbb{Z}^2 \setminus \mathbb{Z}} \tilde{\Omega}_{\tilde{j}}(\lambda, \varepsilon) |a_{\tilde{j}}|^2 + \sum_{m \in \mathbb{Z} \setminus \mathcal{S}_0} \Omega_m(\lambda) |a_{(m,0)}|^2 \quad (5.53)$$

where the normal frequencies $\tilde{\Omega}_{\tilde{j}}(\lambda, \varepsilon)$ are defined in (2.37) and $\Omega_m(\lambda)$ in (4.10).

(iii) $\tilde{\mathcal{H}}_2 := \hat{\mathcal{H}}_2 \circ \mathcal{R} \circ \mathcal{U} = \tilde{\mathcal{H}}_2^{\text{hor}} + \tilde{\mathcal{H}}_2^{\text{mix}} \in \mathcal{Q}_{s/8, -2}^{\mathcal{O}_1, \text{hor}} + \mathcal{Q}_{s/8, -\bar{\mathbf{z}}}^{\mathcal{O}_1}$ with the bounds $[\tilde{\mathcal{H}}_2^{\text{hor}}]_{s/8, -2}^{\mathcal{O}_1} + [\tilde{\mathcal{H}}_2^{\text{hor}}]_{s/8, -\bar{\mathbf{z}}}^{\mathcal{O}_1} < \varepsilon^2$.

(iv) One has $\tilde{\mathcal{M}} \circ \mathcal{U} = \tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}} \circ \mathcal{U} = \tilde{\mathcal{P}}$.

Before proving the Lemma, we discuss some basic fact on normal forms for quadratic Hamiltonians. Let $\mathbf{z} = (z, \bar{z})$ be a finite dimensional phase space -say of dimension $2k$ - with respect to the Poisson form $idz \wedge d\bar{z}$. Let $\dot{\mathbf{z}} = iA\mathbf{z}$ be a linear Hamiltonian system corresponding to the real Hamiltonian

$$\mathcal{Q} = \frac{1}{2}(\mathbf{z}, J^{-1}A\mathbf{z}) \in \mathbb{R}, \quad A^T J = -JA, \quad EAE = -\bar{A}, \quad (5.54)$$

where

$$J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}$$

and (\cdot, \cdot) is the real scalar product. Assume that the eigenvalues of the matrix A are distinct, real and different from zero, say $\pm a_1, \dots, \pm a_k$. Then by the standard theory of quadratic normal forms there exists a symplectic matrix U which diagonalizes A and preserves the real structure:

$$U^{-1}AU = D = \text{diag}(a_1, \dots, a_k, -a_1, \dots, -a_k), \quad U^T J U = J, \quad E U E = \bar{U}.$$

Consequently $\mathcal{U} : \mathbf{w} \mapsto U\mathbf{w} = \mathbf{z}$ is canonical with $\mathbf{w} = (w, \bar{w})$ and

$$\mathcal{Q} \circ \mathcal{U} = \sum_{i=1}^k a_i |w_i|^2.$$

Finally, since the eigenvalues of A are distinct, U depends analytically on the matrix elements of A .

We now specialize this normal form result to *block diagonal* Hamiltonians. Recall that a Lagrangian subspace is a subspace which coincides with its symplectic orthogonal, i.e. $W = W^\perp$. In particular it has dimension k .

Let $\mathbf{z} = (z^{(1)}, z^{(2)})$ with

$$z^{(1)} = (z_{j_1}^{\sigma_1}, z_{j_2}^{\sigma_2}, \dots, z_{j_k}^{\sigma_k}), \quad z^{(2)} = (z_{j_1}^{-\sigma_1}, z_{j_2}^{-\sigma_2}, \dots, z_{j_k}^{-\sigma_k}) = \bar{z}^{(1)} \quad \text{where } z_j^+ = z_j, z_j^- = \bar{z}_j$$

be a Lagrangian decomposition, namely with $\text{Span}(z_{j_1}^{\sigma_1}, z_{j_2}^{\sigma_2}, \dots, z_{j_k}^{\sigma_k})$ a Lagrangian subspace. We write all our matrices in terms of this decomposition; so for example setting $\Sigma = \text{diag}(\sigma_i 1)$ we have

$$J = \begin{pmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

Assume that $A = \text{diag}(A^{(1)}, -\bar{A}^{(1)})$ is block diagonal w.r.t the decomposition. Then U is block diagonal as well, namely $U = \text{diag}(U^{(1)}, \bar{U}^{(1)})$. Now the fact that U is symplectic reads in the block decomposition that $U^{(1)}$ is *orthogonal* w.r.t. Σ i.e. $(U^{(1)})^T \Sigma \bar{U}^{(1)} = \Sigma$. We are interested in a simple consequence of this fact. Define

$$\mathcal{Q}_0 = \frac{1}{2}(\mathbf{z}, J^{-1} \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \mathbf{z}) = \text{Re}(z^{(1)}, \Sigma \bar{z}^{(1)}). \quad (5.55)$$

Then using the orthogonality condition $(U^{(1)})^T \Sigma \bar{U}^{(1)} = \Sigma$ we have

$$\mathcal{Q}_0 = \text{Re}(z^{(1)}, (U^{(1)})^T \Sigma \bar{U}^{(1)} \bar{z}^{(1)}) = \text{Re}(U^{(1)} z^{(1)}, \Sigma \bar{U}^{(1)} \bar{z}^{(1)}) = \mathcal{Q}_0 \circ \mathcal{U}$$

i.e. for any block diagonal symplectic \mathcal{U} we have $\mathcal{Q}_0 \circ \mathcal{U} = \mathcal{Q}_0$.

We wish to apply this theory to $\hat{\mathcal{Z}}_1$. We shall show that $\hat{\mathcal{Z}}_1$ is the sum of non-interacting quadratic Hamiltonians of two types.

(Type I) The first type has dimension $2\mathbf{d}$ with Hamiltonian

$$\mathcal{Q}_1 = \mathbb{K} \sum_{i=1}^{\mathbf{d}} |z_i|^2 + (z, M\bar{z}) \quad (5.56)$$

with \mathbb{K} some real number and $M(\lambda) := (M_{ij}(\lambda))_{i,j}$ given for any $1 \leq i, j \leq \mathbf{d}$ by

$$M_{ij}(\lambda) := \begin{cases} \lambda_i, & i = j \\ 2\sqrt{\lambda_i \lambda_j}, & i \neq j \end{cases}. \quad (5.57)$$

It is easily seen that this Hamiltonian is block diagonal w.r.t the Lagrangian decomposition with $z^{(1)} = z$, moreover the first summand in (5.56) is the invariant Hamiltonian \mathcal{Q}_0 while the second summand corresponds to a Hamiltonian as in (5.54) with matrix $A = \text{diag}(M, M)$.

(Type II) The second type of Hamiltonian has dimension 4 and is block diagonal w.r.t the Lagrangian decomposition $z^{(1)} = (z_1, \bar{z}_2)$, with Hamiltonian

$$\mathcal{Q}_2 = \mathbb{K}(|z_1|^2 - |z_2|^2) + \frac{1}{2}(\mathbf{z}, J^{-1}B\mathbf{z}), \quad B = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix}, \quad N \in GL_2(\mathbb{R}) \quad (5.58)$$

As in the previous case, the first summand in (5.58) is the invariant Hamiltonian \mathcal{Q}_0 -recall that now $\Sigma = \text{diag}(1, -1)$ - while the second summand corresponds to a Hamiltonian as in (5.54) with matrix $A = \text{diag}(N, N)$. We shall show that this type of Hamiltonian appears for each $\vec{j} = (m, n) \in \mathcal{C}_{i,k}^+$ by identifying $z_1 = z_{\vec{j}}, z_2 = z_{\bar{\vec{j}}}$ where $\vec{j} = (\mathbf{m}_i + \mathbf{m}_k - m, -n)$. In this case $\mathbb{K} = m^2 - \mathbf{m}_i^2 + n^2$ while

$$N \equiv N(\lambda_i, \lambda_k) := \begin{pmatrix} \lambda_i & 2\sqrt{\lambda_i \lambda_k} \\ -2\sqrt{\lambda_i \lambda_k} & -\lambda_k \end{pmatrix} \quad (5.59)$$

Lemma 5.12. (i) *The characteristic polynomial of $M(\lambda)$, $P(t, \lambda) = \det(t\mathbb{I} - M(\lambda))$ coincides with (2.38) and is irreducible over $\mathbb{Z}[t, \lambda_1, \dots, \lambda_{\mathbf{d}}]$. Consequently the eigenvalues of M , which we denote by $\mu_i(\lambda)$ are distinct algebraic functions of λ homogeneous of degree one.*

(ii) *Consider any open domain contained in a single connected component where all the eigenvalues $\mu_i(\lambda)$ are distinct. In any of such domain there exists an orthogonal matrix $U_1 \in O_{\mathbf{d}}(\mathbb{R})$, depending analytically on λ , such that*

$$\mathcal{U}_1 : \mathbf{z} \rightarrow (U_1 z, \bar{U}_1 \bar{z}) = (w, \bar{w})$$

is symplectic and for any \mathbb{K} we have

$$\mathcal{Q}_1 \circ \mathcal{U}_1 = \sum_{i=1}^{\mathbf{d}} (\mathbb{K} + \mu_i(\lambda)) |w_i|^2, \quad \mathcal{Q}_0 \circ \mathcal{U}_1 = \sum_{i=1}^{\mathbf{d}} |z_i|^2 \circ \mathcal{U}_1 = \sum_{i=1}^{\mathbf{d}} |w_i|^2.$$

(iii) *The characteristic polynomial of $N(\lambda_i, \lambda_k)$, $Q(t, \lambda_i, \lambda_k) = \det(t\mathbb{I} - N(\lambda_i, \lambda_k))$ coincides with (2.39) and is irreducible over $\mathbb{Z}[t, \lambda_1, \dots, \lambda_{\mathbf{d}}]$. Consequently the eigenvalues of N , which we denote by $\mu_{i,k}^{\pm}(\lambda)$ are distinct algebraic functions of λ_i, λ_k , homogeneous of degree one.*

(iv) *There exist open connected domains in which $\mu_{i,k}^{\pm}(\lambda)$ are real and distinct. In any of such domain there exists a matrix $U_2 \in O_{\mathbf{d}}(\mathbb{R})$ such that*

$$\mathcal{U}_2 : \mathbf{z} = (z^{(1)}, \bar{z}^{(1)}) \rightarrow (U_2 z^{(1)}, \bar{U}_2 \bar{z}^{(1)}) = (w_1, \bar{w}_2, \bar{w}_1, w_2)$$

is symplectic and for any \mathbb{K} we have

$$\mathcal{Q}_2 \circ \mathcal{U}_2 = (K + \mu_{i,k}^+(\lambda)) |w_1|^2 - (K + \mu_{i,k}^-(\lambda)) |w_2|^2, \quad \mathcal{Q}_0 \circ \mathcal{U}_2 = (|z_1|^2 - |z_2|^2) \circ \mathcal{U}_2 = |w_1|^2 - |w_2|^2.$$

Proof. (i) The fact that $\det(t\mathbb{I} - M(\lambda))$ coincides with (2.38) is a direct computation. In order to prove that $P(t, \lambda)$ is irreducible over $\mathbb{Z}[t, \lambda_1, \dots, \lambda_{\mathbf{d}}] \subset \mathbb{Q}(\lambda_1, \dots, \lambda_{\mathbf{d}})[t]$ we proceed by induction on \mathbf{d} . If $\mathbf{d} = 1$ the statement is trivial. Now let us suppose that it is true up to $\mathbf{d} = N$ and prove it for $N + 1$. We consider the polynomial

$$P_{N+1}(t, \lambda) = \prod_{i=1}^{N+1} (t + \lambda_i) - 2 \sum_{i=1}^{N+1} \lambda_i \prod_{k \neq i} (t + \lambda_k)$$

and specify to $\lambda_{N+1} = 0$. We obtain

$$P_{N+1}(t, \lambda_1, \dots, \lambda_N, 0) = t \left(\prod_{i=1}^N (t + \lambda_i) - 2 \sum_{i=1}^N \lambda_i \prod_{k \neq i} (t + \lambda_k) \right) = t P_N(t, \lambda_1, \dots, \lambda_N)$$

and by the inductive hypothesis the second factor is irreducible. Assume by contradiction that P_{N+1} is not irreducible: then it must factorize a linear term of the form $(t - c\lambda_{N+1})$. Note that c is a number, this is due to the fact that P_{N+1} is homogeneous of degree $N + 1$ in (t, λ) and P_N is homogeneous of degree N . We repeat the same argument specifying to $\lambda_1 = 0$ and obtain a contradiction. Indeed we would have

$$P_{N+1}(t, \lambda) = (t - c_1\lambda_{N+1})\widehat{P}_1(t, \lambda) = (t - c_2\lambda_1)\widehat{P}_2(t, \lambda)$$

with $\widehat{P}_1, \widehat{P}_2$ irreducible. Now this equality can hold only if $(t - c_1\lambda_{N+1})$ divides \widehat{P}_2 which is impossible by the irreducibility.

Now remark that for a polynomial with coefficients in a field with characteristic 0, a sufficient condition to have distinct roots is the polynomial to be irreducible. Indeed if there were a double root then $f(t)$ and $f'(t)$ would have a common divisor, thus contradicting the irreducibility. Hence the eigenvalues of the symmetric matrix M are distinct (and obviously real for positive λ) outside a finite number of algebraic surfaces. Since M is homogeneous of degree one then so are the $\mu_i(\lambda)$.

(ii) In order to conclude our proof we restrict to a connected component which does not cross any surface where two eigenvalues coincide. Then we apply the theory of quadratic Hamiltonians as described above.

(iii) The irreducibility can be verified immediately by computing the roots of $Q(t, \lambda_i, \lambda_k)$, which are given by

$$\mu_{i,k}^+(\lambda) = \frac{\lambda_i - \lambda_k - \sqrt{\lambda_i^2 + \lambda_k^2 - 14\lambda_i\lambda_k}}{2}, \quad \mu_{i,k}^-(\lambda) = \frac{\lambda_i - \lambda_k + \sqrt{\lambda_i^2 + \lambda_k^2 - 14\lambda_i\lambda_k}}{2} \quad (5.60)$$

(iv) We restrict λ_i, λ_k to a region where we have 2 distinct, real eigenvalues. To this purpose we impose the condition $\text{Tr}^2 N(\lambda_i, \lambda_k) > 4 \det N(\lambda_i, \lambda_k)$, which is equivalent to

$$(\lambda_k - c_+\lambda_i)(\lambda_k - c_-\lambda_i) > 0, \quad c_{\pm} = (5 \pm \sqrt{21})/2.$$

This selects two conic regions of parameters λ_i, λ_k . In each such region we may apply the theory of quadratic Hamiltonians in order to obtain our result. \square

Proof of Lemma 5.11. We first notice that $\widehat{\mathcal{D}}, \widehat{\mathcal{Z}}_1$ depend on different variables and hence do not interact; in the same way, due to our genericity condition, the first line in $\widehat{\mathcal{Z}}_1$, see (5.48), does not interact with the second and third one, see (5.49)-(5.50). Finally (5.48) is the infinite sum over $n \neq 0$ of the finite dimensional Hamiltonian:

$$\sum_{1 \leq i \leq d} (\mathfrak{m}_i^2 - \omega_i + n^2) |b_{(\mathfrak{m}_i, n)}|^2 + 2\varepsilon \sum_{\substack{1 \leq i, k \leq d, i \neq k \\ n \in \mathbb{Z} \setminus \{0\}}} \sqrt{\lambda_i \lambda_k} b_{(\mathfrak{m}_i, n)} \bar{b}_{(\mathfrak{m}_k, n)}$$

supported on the Fourier indices

$$\mathcal{S}_n := \bigcup_{\pm n} \mathcal{S}_{0n}.$$

Since $\mathfrak{m}_i^2 - \omega_i + n^2 = \varepsilon\lambda_i + n^2$ the Hamiltonian above is of type I, see (5.56), we just identify $z = (b_{(\mathfrak{m}_1, n)}, b_{(\mathfrak{m}_2, n)}, \dots, b_{(\mathfrak{m}_k, n)})$ for all $n \neq 0$ and fix $K = n^2$. Regarding (5.49)-(5.50) we have Hamiltonians of type II, see (5.58). Indeed for each $\vec{j} = (m, n) \in \mathcal{C}_{i,k}^+$ we identify $z_1 = z_{\vec{j}}, z_2 = z_{\vec{j}}$ where $\vec{j} = (\mathfrak{m}_i + \mathfrak{m}_k - m, -n)$. In this case the coefficient $K = m^2 - \mathfrak{m}_i^2 + n^2$. In conclusion $\widehat{\mathcal{D}} + \widehat{\mathcal{Z}}_1$ is block diagonal with respect to the blocks

$$\{\mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C} \cup \mathcal{S}_0)\} \cup_{n>0} \mathcal{S}_n \cup_{i<k} \mathcal{C}_{i,k}.$$

We consider an open connected region in \mathcal{O}_0 where the eigenvalues $\mu_i(\lambda)$ are distinct and for each $i \neq k$, the eigenvalues $\mu_{i,k}^{\pm}(\lambda)$ are real and distinct. Such a region exists and is the intersection between \mathcal{O}_0 and a cone due to Lemma 5.12. Now we may choose a compact domain \mathcal{O}_1 strictly contained in a connected component of the open cone and define

$$\tilde{\gamma} \leq \text{distance between } \mathcal{O}_1 \text{ and the border of the cone,} \quad (5.61)$$

so that

$$\min_{\lambda \in \mathcal{O}_1} \left(|\mu_{i,k}^\sigma(\lambda) - \mu_{i',k'}^{\sigma'}(\lambda)|, |\mu_i(\lambda) - \mu_{i',k'}^{\sigma'}(\lambda)|, |\mu_i(\lambda) - \mu_{i'}(\lambda)| \right) > \tilde{\gamma} \quad (5.62)$$

for any choice of the distinct eigenvalues.

In \mathcal{O}_1 the changes of variables $\mathcal{U}_1(\lambda)$ and $(\mathcal{U}_2(\lambda_i, \lambda_k))_{i < k}$ of Lemma 5.12 are well defined, analytic and we may estimate $\mathcal{U}_i, \partial_\lambda \mathcal{U}_i$ by Cauchy estimates (recall that \mathcal{U}_i and $\tilde{\gamma}$ are ε independent). We are ready to define U , which is a block diagonal matrix with respect to the blocks

$$(\mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{C} \cup \mathcal{S}_0)) \cup_{n \neq 0} \mathcal{S}_n \cup_{i < k} \mathcal{C}_{i,k}.$$

On the first block U is the identity, on each block \mathcal{S}_n it is the matrix U_1 of Lemma 5.12, on each 4×4 block $\vec{j} = (m, n) \in \mathcal{C}_{i,k}^+$, $\vec{j} = (\mathfrak{m}_i + \mathfrak{m}_k - m, -n)$, the matrix U coincides with the matrix $U_2(\lambda_i, \lambda_k)$. We have proved items (i) and (ii). Regarding item (iii) we only need to bound the norm of \mathcal{R} and \mathcal{U} together with their λ derivatives. The bound on \mathcal{R} is trivial, as for the one on \mathcal{U} it follows by our definition of \mathcal{O}_1 .

Finally we prove item (iv). Clearly $\tilde{\mathcal{M}} \circ \mathcal{U} = \tilde{\mathcal{M}}$ since this functions *does not* depend on the variables $(b_{\vec{j}})_{\vec{j} \in \mathcal{S} \cup \mathcal{C}}$, as one sees inspecting formulas (2.31). Regarding $\tilde{\mathcal{P}}_x$ and $\tilde{\mathcal{P}}_y$ we notice that they are decomposed in the same blocks as $\hat{\mathcal{Z}}_1$:

$$\begin{aligned} \tilde{\mathcal{P}}_x &= \sum_i \mathfrak{m}_i \mathcal{Y}_i + \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{S}_0 \cup \mathcal{C})} m |b_{(m,n)}|^2 + \sum_{i < k, (m,n) \in \mathcal{C}_{i,k}^+} (m - \mathfrak{m}_i) (|b_{(m,n)}|^2 - |b_{(\mathfrak{m}_i + \mathfrak{m}_k - m, -n)}|^2), \\ \tilde{\mathcal{P}}_y &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus (\mathcal{S} \cup \mathcal{S}_0 \cup \mathcal{C})} n |b_{(m,n)}|^2 + \sum_{n \neq 0} n \sum_{(m,n) \in \mathcal{S}_n} |b_{(m,n)}|^2 + \sum_{i < k, (m,n) \in \mathcal{C}_{i,k}^+} n (|b_{(m,n)}|^2 - |b_{(\mathfrak{m}_i + \mathfrak{m}_k - m, -n)}|^2). \end{aligned}$$

On the first block \mathcal{U} acts as the identity. On each block \mathcal{S}_n , \mathcal{U} acts as \mathcal{U}_1 and moreover $\tilde{\mathcal{P}}_x$ is null on this block while $\tilde{\mathcal{P}}_y$ is proportional to the invariant Hamiltonian $\mathcal{Q}_0 = \sum_{i=1}^d |b_{(m_i, n)}|^2$. Finally on the last blocks \mathcal{U} acts as \mathcal{U}_2 and $\tilde{\mathcal{P}}_x, \tilde{\mathcal{P}}_y$ are *both* proportional to the invariant Hamiltonian \mathcal{Q}_0 (recall that these blocks are of type II and hence $\mathcal{Q}_0 = |b_{(m,n)}|^2 - |b_{(\mathfrak{m}_i + \mathfrak{m}_j - m, -n)}|^2$, see formula (5.58)). Finally we rename the canonical variables $(\mathcal{Y}, \theta, \mathbf{a})$. \square

Remark 5.13. *The fact that the eigenvalues of $N(\lambda_i, \lambda_k)$ might in principle be imaginary shows that the tori containing the family of the finite gap solutions might be linearly hyperbolic, hence linearly unstable. Here we want to rule out such behavior.*

Proof of Theorems 5.1 and 5.3. We just apply Lemmata 5.4, 5.8, 5.10 and 5.11. \square

6 Non-degeneracy of the Hamiltonian $\tilde{\mathcal{D}}(\lambda, \varepsilon)$

By Theorem 5.1 the quadratic Hamiltonian $\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{H}^{(0)}$ is now reduced to a $\mathcal{O}(\varepsilon^2)$ perturbation of the diagonal Hamiltonian $\omega \cdot \mathcal{Y} + \tilde{\mathcal{D}}(\lambda, \varepsilon)$, where $\tilde{\mathcal{D}}(\lambda, \varepsilon)$ is defined in (5.53).

In this section we prove that we can impose second and third order Melnikov conditions on the frequencies $\tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon)$ of the operator $\tilde{\mathcal{D}}(\lambda, \varepsilon)$. Now we have the following lemma:

Lemma 6.1. *Let $\tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon)$ be defined as in (2.37). For a generic choice of \mathcal{S}_0 the following holds: for each admissible $(\mathbf{j}, \ell, \boldsymbol{\sigma}) \neq ((\vec{j}, \vec{j}), 0, (\sigma_1, -\sigma_1))$ in the sense of Definition 2.7 one has*

$$\omega \cdot \ell + \sigma_1 \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) \neq 0, \quad (6.1)$$

in the set \mathcal{O}_1 of Theorem 5.1.

Proof. Without loss of generality we may assume that $\sigma_1 = 1$ and set $\sigma_2 = \sigma$ and $\vec{j}_i = (m_i, n_i)$ for $i = 1, 2$. In the expression (6.1) we separate the terms of order ε^0 and ε :

$$\omega \cdot \ell + \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) \equiv K_{\mathbf{j}, \ell}^\sigma + \varepsilon F_{\mathbf{j}, \ell}^\sigma(\lambda), \quad (6.2)$$

where we denote by $K_{\mathbf{j}, \ell}^\sigma$ the part of (6.2) which is of order ε^0 and by $F_{\mathbf{j}, \ell}^\sigma(\lambda)$ the part of order ε . Explicitly

$$K_{\mathbf{j}, \ell}^\sigma := \omega^{(0)} \cdot \ell + \tilde{\Omega}_{\vec{j}_1}(\lambda, 0) + \sigma \tilde{\Omega}_{\vec{j}_2}(\lambda, 0) \quad (6.3)$$

$$F_{\mathbf{j}, \ell}^\sigma(\lambda) := \partial_\varepsilon \left(\omega(\lambda) \cdot \ell + \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) \right) \Big|_{\varepsilon=0} = -\lambda \cdot \ell + \vartheta_{\vec{j}_1}(\lambda) + \sigma \vartheta_{\vec{j}_2}(\lambda) \quad (6.4)$$

where $K_{\vec{j},\ell}^\sigma$ is an integer while the functions $\vartheta_{\vec{j}}(\lambda)$ belong to the finite list of functions

$$\vartheta_{\vec{j}}(\lambda) \in \left\{ (\mu_i(\lambda))_{1 \leq i \leq d}, (\mu_{i,k}^\pm(\lambda))_{1 \leq i < k \leq d} \right\}.$$

Clearly in order for the resonance (6.1) to hold identically, we must have $K_{\vec{j},\ell}^\sigma = F_{\vec{j},\ell}^\sigma(\lambda) \equiv 0$.

Remark that \vec{j}_1 and \vec{j}_2 can belong to either $\mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$, or \mathcal{S} , or \mathcal{C} and all the possible combinations are possible. Thus we perform a case analysis and show that, however one chooses admissible $((\vec{j}_1, \vec{j}_2), \ell, (\sigma_1, \sigma_2)) \neq ((\vec{j}, \vec{j}), 0, (\sigma_1, -\sigma_1))$, the function (6.1) cannot be identically 0.

1. $\vec{j}_1, \vec{j}_2 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. In this case $F_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell$, which is identically zero iff $\ell = 0$. The conservation of mass and momentum reads

$$\eta(\ell) + 1 + \sigma = 0, \quad \pi(\ell) + m_1 + \sigma m_2 = 0, \quad n_1 + \sigma n_2 = 0. \quad (6.5)$$

Since $\ell = 0$, the first equation above fixes $\sigma = -1$ and the other two imply $\vec{j}_1 = \vec{j}_2$, which is the trivial case we excluded.

2. $\vec{j}_1 \in \mathcal{S}, \vec{j}_2 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. We have

$$F_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell + \mu_i(\lambda) \quad \text{for some } 1 \leq i \leq d.$$

This expression may be identically zero only if for some $1 \leq i \leq d$ and some $\ell \in \mathbb{Z}^d$ one has $\mu_i(\lambda) \equiv \lambda \cdot \ell$. Since by definition $\mu_i(\lambda)$ is a root of $P(t, \lambda)$ (defined in (2.38)), we would have that $(t - \lambda \cdot \ell) \in \mathbb{Z}[t, \lambda_1, \dots, \lambda_d]$ would be a divisor of $P(t, \lambda)$. However by Lemma 5.12 $P(t, \lambda)$ is irreducible in $\mathbb{Z}[t, \lambda_1, \dots, \lambda_d]$, and we have obtained a contradiction.

3. $\vec{j}_1 \in \mathcal{C}_{i,k}^\pm$ for some $i < k$, $\vec{j}_2 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. We have

$$F_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell + \mu_{i,k}^\pm(\lambda) \quad \text{for some } 1 \leq i < k \leq d.$$

This expression may be zero only if for some $1 \leq i < k \leq d$, $s = \pm$ and some $\ell \in \mathbb{Z}^d$ one has $\mu_{i,k}^s(\lambda) \equiv \lambda \cdot \ell$. But in this case the polynomial $(t - \lambda \cdot \ell) \in \mathbb{Z}[t, \lambda_1, \dots, \lambda_d]$ would be a divisor of $Q(t, \lambda_i, \lambda_k)$ defined in (2.39), which is irreducible by Lemma 5.12. We have obtained a contradiction.

4. $\vec{j}_1, \vec{j}_2 \in \mathcal{S}$. W.l.o.g. we may assume that $m_1 = \mathbf{m}_i$ and $m_2 = \mathbf{m}_k$. By conservation of mass and momentum of Remark 5.2

$$\eta(\ell) = 0, \quad \pi(\ell) = 0, \quad n_1 + \sigma n_2 = 0. \quad (6.6)$$

Then we have $K_{\vec{j},\ell}^\sigma = \omega^{(0)} \cdot \ell + (1 + \sigma)n_1^2$ and finally

$$F_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell + \mu_i(\lambda) + \sigma \mu_k(\lambda).$$

As usual assume that (6.2) is identically 0. If $\ell = 0$ then $K_{\vec{j},\ell}^\sigma = 0$ implies $\sigma = -1$. Then $F_{\vec{j},\ell}^\sigma(\lambda) \equiv 0$ iff $\mu_i(\lambda) = \mu_k(\lambda)$. By the irreducibility of $P(t, \lambda)$ this may hold only if $i = k$ and hence $m_1 = m_2 = \mathbf{m}_i$. Since $n_1 = n_2$ this implies $\vec{j}_1 = \vec{j}_2$, which contradicts the assumptions.

Now suppose $\ell \neq 0$. Then $F_{\vec{j},\ell}^\sigma(\lambda) \equiv 0$ iff $\mu_i(\lambda) \equiv -\sigma \mu_k(\lambda) + \lambda \cdot \ell$. This means that $\mu_k(\lambda)$ is a root both of $P(t, \lambda)$ and $P(-\sigma t + \lambda \cdot \ell, \lambda)$, so the two polynomials have a common divisor. We claim that $P(-\sigma t + \lambda \cdot \ell, \lambda)$ is irreducible over $\mathbb{Z}[t, \lambda_1, \dots, \lambda_d]$. Indeed, suppose that

$$P(-\sigma t + \lambda \cdot \ell, \lambda) = P_1(t, \lambda)P_2(t, \lambda),$$

then setting $\tau = -\sigma t + \lambda \cdot \ell$ we would have

$$P(\tau, \lambda) = P_1(-\sigma \tau + \sigma \lambda \cdot \ell, \lambda)P_2(-\sigma \tau + \sigma \lambda \cdot \ell, \lambda),$$

but this contradicts the irreducibility of P since $P_1(-\sigma \tau + \sigma \lambda \cdot \ell, \lambda) \in \mathbb{Z}[t, \lambda_1, \dots, \lambda_d]$.

Now in order for $P(t, \lambda)$ and $P(-\sigma t + \lambda \cdot \ell, \lambda)$ to have a common divisor they have to be equal (or opposite). Since $\ell \neq 0$ and $\eta(\ell) = 0$ then $\lambda \cdot \ell$ depends on at least two variables, say λ_1, λ_2 . We specialize the equality $P(t, \lambda) = sP(-\sigma t + \lambda \cdot \ell, \lambda)$, $s = \pm$, to $\lambda_i = 0$ for all $i \neq 1, 2$ and get

$$\begin{aligned} (-\sigma t + \lambda \cdot \ell)^{d-2} [(-\sigma t + \lambda \cdot \ell + \lambda_1)(-\sigma t + \lambda \cdot \ell + \lambda_2) - 2\lambda_1(-\sigma t + \lambda \cdot \ell + \lambda_2) - 2\lambda_2(-\sigma t + \lambda \cdot \ell + \lambda_1)] = \\ st^{d-2} [(t + \lambda_1)(t + \lambda_2) - 2\lambda_1(t + \lambda_2) - 2\lambda_2(t + \lambda_1)]. \end{aligned}$$

Equating the leading terms we get $(-\sigma)^d = s$, then from the subsequent degree

$$(-\sigma)^{d-1}(2\lambda \cdot \ell - (\lambda_1 + \lambda_2)) = -(-\sigma)^d(\lambda_1 + \lambda_2) \quad \Rightarrow \quad 2\sigma \lambda \cdot \ell = (1 + \sigma)(\lambda_1 + \lambda_2),$$

which is not compatible with $\eta(\ell) = 0, \ell \neq 0$.

5. $\vec{j}_1 \in \mathcal{C}$, $\vec{j}_2 \in \mathcal{S}$. W.l.o.g. assume that $\vec{j}_1 \in \mathcal{C}_{i,k}^s$ for some $1 \leq i < k \leq d$, $s = \pm$, and $m_2 = \mathfrak{m}_h$ for some $1 \leq h \leq d$. In this case

$$\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell + \mu_{i,k}^s(\lambda) + \sigma\mu_h(\lambda).$$

Assume that $\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) \equiv 0$. Then we would have

$$\mu_{i,k}^s(\lambda) = -\sigma\mu_h(\lambda) + \lambda \cdot \ell,$$

which means that $P(t, \lambda)$ would have $\mu_h(\lambda)$ as a common root with $Q(-\sigma t + \lambda \cdot \ell, \lambda_i, \lambda_k)$. Since both polynomials are irreducible (one can prove the irreducibility of $Q(-\sigma t + \lambda \cdot \ell, \lambda_i, \lambda_k)$ as we did in 4.), then they must coincide up to a scale factor. This is absurd unless $d = 2$. In this last case we can verify directly that the two polynomials never coincide.

6. $\vec{j}_1, \vec{j}_2 \in \mathcal{C}$. W.l.o.g. assume that $\vec{j}_r \in \mathcal{C}_{i_r, k_r}^{s_r}$ for $r = 1, 2$ and $1 \leq i_r < k_r \leq d$, $s_r = \pm$. We have

$$\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) = -\lambda \cdot \ell + \mu_{i_1, k_1}^{s_1}(\lambda) + \sigma\mu_{i_2, k_2}^{s_2}(\lambda)$$

so that $\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) \equiv 0$ would require

$$\mu_{i_1, k_1}^{s_1}(\lambda) \equiv \lambda \cdot \ell - \sigma\mu_{i_2, k_2}^{s_2}(\lambda).$$

This is trivially false if $(i_1, k_1) \neq (i_2, k_2)$ (one just remarks that the square roots in formula (5.60) cannot cancel out). If $(i_1, k_1) = (i_2, k_2)$ we divide two cases. If $\ell = 0$, $\mu_{i_1, k_1}^{s_1}(\lambda) \equiv -\sigma\mu_{i_2, k_2}^{s_2}(\lambda)$ can happen only if $\sigma = -1$ and $s_2 = s_1$. By conservation of $\tilde{\mathcal{P}}_y$ we have $n_1 = n_2$ and by conservation of $\tilde{\mathcal{P}}_x$ we get $(m_1 - \mathfrak{m}_i) - (m_2 - \mathfrak{m}_i) = 0$, which implies $m_1 = m_2$. Hence $\vec{j}_1 = \vec{j}_2$, which is a contradiction.

If $\ell \neq 0$, for the square root to cancel identically we must have $\sigma = 1$, $s_1 = -s_2$ and $\lambda \cdot \ell = -(\lambda_{i_1} - \lambda_{k_1})$. Then \vec{j}_1, \vec{j}_2 are on the same circle \mathcal{C}_{i_1, k_1} and by conservation of $\tilde{\mathcal{P}}_y$ we have $n_1 = -n_2$. Since both \vec{j}_1 and \vec{j}_2 are on a circle, this implies that either $m_1 = m_2$ or they are on the opposite sides of a diameter, i.e. $m_1 + m_2 = \mathfrak{m}_{i_1} + \mathfrak{m}_{k_1}$. But by conservation of $\tilde{\mathcal{P}}_x$ we get

$$0 = (-\mathfrak{m}_{i_1} + \mathfrak{m}_{k_1}) + (m_1 - \mathfrak{m}_{i_1}) + (m_2 - \mathfrak{m}_{k_1}) = m_1 + m_2 - 2\mathfrak{m}_{i_1}$$

and in both cases we get contradictions. □

We pass to third order Melnikov conditions.

Lemma 6.2. *There exists $L \in \mathbb{N}$ such that for a L -generic choice of \mathcal{S}_0 the following holds: for any $(\mathbf{j}, \ell, \sigma) \in \mathfrak{A}_3 \setminus \mathfrak{K}_3$, i.e. admissible and non action preserving in the sense of Definitions 2.7, 2.8 we have*

$$\omega(\lambda) \cdot \ell + \sigma_1 \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) + \sigma_3 \tilde{\Omega}_{\vec{j}_3}(\lambda, \varepsilon) \neq 0 \quad (6.7)$$

in the set \mathcal{O}_1 of Lemma 5.11.

Proof. As in the previous proof we set

$$\omega(\lambda) \cdot \ell + \sigma_1 \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) + \sigma_3 \tilde{\Omega}_{\vec{j}_3}(\lambda, \varepsilon) \equiv \mathbf{K}_{\vec{j},\ell}^\sigma + \varepsilon \mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) \quad (6.8)$$

where $\mathbf{K}_{\vec{j},\ell}^\sigma$ is the term of order 1 and $\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda)$ is the term of order ε . Explicitly

$$\mathbf{K}_{\vec{j},\ell}^\sigma := \omega^{(0)} \cdot \ell + \sigma_1 \tilde{\Omega}_{\vec{j}_1}(\lambda, 0) + \sigma_2 \tilde{\Omega}_{\vec{j}_2}(\lambda, 0) + \sigma_3 \tilde{\Omega}_{\vec{j}_3}(\lambda, 0) \quad (6.9)$$

$$\mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) := \partial_\varepsilon \left(\omega(\lambda) \cdot \ell + \sigma_1 \tilde{\Omega}_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \tilde{\Omega}_{\vec{j}_2}(\lambda, \varepsilon) + \sigma_3 \tilde{\Omega}_{\vec{j}_3}(\lambda, \varepsilon) \right) \Big|_{\varepsilon=0} \quad (6.10)$$

$$= -\lambda \cdot \ell + \sigma_1 \vartheta_{\vec{j}_1}(\lambda) + \sigma_2 \vartheta_{\vec{j}_2}(\lambda) + \sigma_3 \vartheta_{\vec{j}_3}(\lambda) \quad (6.11)$$

As before $\mathbf{K}_{\vec{j},\ell}^\sigma$ is an integer while the functions $\vartheta_{\vec{j}}(\lambda)$ belong to the finite list of functions

$$\vartheta_{\vec{j}}(\lambda) \in \left\{ (\mu_i(\lambda))_{1 \leq i \leq d}, (\mu_{i,k}^\pm(\lambda))_{1 \leq i < k \leq d} \right\}.$$

Clearly in order for the resonance (6.7) to not hold identically, we need to ensure that

$$\mathbf{K}_{\vec{j},\ell}^\sigma = 0 \quad \text{and} \quad \mathbf{F}_{\vec{j},\ell}^\sigma(\lambda) \equiv 0 \quad (6.12)$$

cannot occur for $(\mathbf{j}, \ell, \sigma)$ admissible. As before we study all the possible combinations, each time we assume that (6.12) holds and we deduce a contradiction.

1. $\vec{j}_1, \vec{j}_2, \vec{j}_3 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. Then $F_{\vec{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell$ is identically 0 iff $\ell = 0$. However by mass conservation $\eta(\ell)$ is odd hence $\ell \neq 0$.
2. $\vec{j}_1, \vec{j}_2 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$, $\vec{j}_3 \in \mathcal{S}$. Then $F_{\vec{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell + \sigma_3 \mu_i(\lambda)$ for some $1 \leq i \leq \mathbf{d}$. If $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$ then $\mu_i(\lambda) = \sigma_3 \lambda \cdot \ell$ is a root in $\mathbb{Z}[\lambda]$ of the polynomial $P(t, \lambda)$ defined in (2.38), which is irreducible over $\mathbb{Z}[t, \lambda_1, \dots, \lambda_{\mathbf{d}}]$ (see Lemma 5.12), thus leading to a contradiction.
3. $\vec{j}_1, \vec{j}_2 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$, $\vec{j}_3 \in \mathcal{C}$. W.l.o.g. assume that $\vec{j}_3 \in \mathcal{C}_{i,k}^s$ for some $1 \leq i < k \leq \mathbf{d}$, $s = \pm$. Then

$$F_{\vec{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell + \sigma_3 \mu_{i,k}^s(\lambda) .$$

If $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$, $\mu_{i,k}^s(\lambda) = \sigma_3 \lambda \cdot \ell$ is a root in $\mathbb{Z}[t, \lambda_i, \lambda_k]$ of the polynomial $Q(t, \lambda_i, \lambda_k)$ defined in (2.39), which is irreducible over $\mathbb{Z}[t, \lambda_i, \lambda_k]$ (see Lemma 5.12), thus leading to a contradiction.

4. $\vec{j}_1, \vec{j}_2 \in \mathcal{S}$, $\vec{j}_3 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. W.l.o.g. let $\vec{j}_1 = (\mathbf{m}_i, n_1)$, $\vec{j}_2 = (\mathbf{m}_k, n_2)$ for some $1 \leq i, k \leq \mathbf{d}$. Then

$$F_{\vec{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell + \sigma_1 \mu_i(\lambda) + \sigma_2 \mu_k(\lambda) .$$

By conservation of mass $\eta(\ell) = \pm 1$, thus $\ell \neq 0$. Assume $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$. Then $\mu_k(\lambda) = -\sigma_1 \sigma_2 \mu_i(\lambda) + \sigma_2 \lambda \cdot \ell$. This means that $P(t, \lambda)$ has $\mu_{i_1}(\lambda)$ as a common divisor with $P(-\sigma_1 \sigma_2 t + \sigma_2 \lambda \cdot \ell)$. Proceeding as in case 4. of the previous proof one gets a contradiction.

5. $\vec{j}_1, \vec{j}_2 \in \mathcal{C}$, $\vec{j}_3 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$. W.l.o.g. let $\vec{j}_r \in \mathcal{C}_{i_r, k_r}^{s_r}$ for some $1 \leq i_r < k_r \leq \mathbf{d}$, $s_r = \pm$, $r = 1, 2$. Then

$$F_{\vec{j}, \ell}^\sigma(\lambda) \equiv -\lambda \cdot \ell + \sigma_1 \mu_{i_1, k_1}^{s_1}(\lambda) + \sigma_2 \mu_{i_2, k_2}^{s_2}(\lambda) .$$

By mass conservation $\eta(\ell) = \pm 1$, hence $\ell \neq 0$. Then $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$ would require

$$\mu_{i_1, k_1}^{s_1}(\lambda) \equiv \sigma_1 \lambda \cdot \ell - \sigma_1 \sigma_2 \mu_{i_2, k_2}^{s_2}(\lambda) .$$

This is trivially false if $(i_1, k_1) \neq (i_2, k_2)$ (one just remarks that the square roots in formula (5.60) cannot cancel out). If $(i_1, k_1) = (i_2, k_2)$ then by the explicit formula (5.60) we obtain

$$0 \equiv -\lambda \cdot \ell + (\sigma_1 + \sigma_2) \frac{\lambda_{i_1} - \lambda_{k_1}}{2} - (\sigma_1 s_1 + \sigma_2 s_2) \frac{\sqrt{\lambda_{i_1}^2 + \lambda_{k_1}^2 - 14 \lambda_{i_1} \lambda_{k_1}}}{2} .$$

If $\sigma_1 s_1 + \sigma_2 s_2 \neq 0$ then we get a contradiction since the root cannot be canceled identically by a linear function. If $\sigma_1 s_1 + \sigma_2 s_2 = 0$ and $\sigma_1 + \sigma_2 = 0$, we also get a contradiction as $\ell \neq 0$. Thus we are left with the case $\sigma_1 + \sigma_2 = \pm 2$, which implies $0 \equiv -\lambda \cdot (\ell \pm (\mathbf{e}_{i_1} - \mathbf{e}_{i_k}))$. This can hold only if $\ell = \mp (\mathbf{e}_{i_1} - \mathbf{e}_{i_k})$. But for such ℓ one has $\eta(\ell) = 0$, contradicting the mass conservation.

6. $\vec{j}_1 \in \mathbb{Z}^2 \setminus (\mathcal{S}_0 \cup \mathcal{S} \cup \mathcal{C})$, $\vec{j}_2 \in \mathcal{S}$, $\vec{j}_3 \in \mathcal{C}$. W.l.o.g. let $\vec{j}_2 = (\mathbf{m}_{i_2}, n_2)$ for some $1 \leq i_2 \leq \mathbf{d}$, $\vec{j}_3 = (\mathbf{m}_3, n_3) \in \mathcal{C}_{i_3, k_3}^{s_3}$ for some $1 \leq i_3 < k_3 \leq \mathbf{d}$, $s_3 = \pm$. Then

$$F_{\vec{j}, \ell}^\sigma(\lambda) \equiv -\lambda \cdot \ell + \sigma_2 \mu_{i_2}(\lambda) + \sigma_3 \mu_{i_3, k_3}^{s_3}(\lambda) .$$

Assume that $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$. Then $\mu_{i_2}(\lambda) = -\sigma_2 \sigma_3 \mu_{i_3, k_3}^{s_3}(\lambda) + \sigma_2 \lambda \cdot \ell$. Thus $\mu_{i_3, k_3}^{s_3}(\lambda)$ is a root of $Q(t, \lambda_{i_3}, \lambda_{k_3})$ and of $P(-\sigma_2 \sigma_3 t + \sigma_2 \lambda \cdot \ell, \lambda) \in \mathbb{Z}[t, \lambda_1, \dots, \lambda_{\mathbf{d}}]$. But both polynomials are irreducible, thus they must coincide up to a sign. This is absurd unless $\mathbf{d} \neq 2$. In the last case we can verify directly that the two polynomials never coincide.

7. $\vec{j}_1, \vec{j}_2, \vec{j}_3 \in \mathcal{S}$. W.l.o.g. let $\vec{j}_1 = (\mathbf{m}_{i_1}, n_1)$, $\vec{j}_2 = (\mathbf{m}_{i_2}, n_2)$, $\vec{j}_3 = (\mathbf{m}_{i_3}, n_3)$ for some $1 \leq i_1, i_2, i_3 \leq \mathbf{d}$ and $n_1, n_2, n_3 \neq 0$. Then

$$F_{\vec{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell + \sigma_1 \mu_{i_1}(\lambda) + \sigma_2 \mu_{i_2}(\lambda) + \sigma_3 \mu_{i_3}(\lambda) .$$

Assume $F_{\vec{j}, \ell}^\sigma(\lambda) \equiv 0$. This fix $\ell^{(\mathbf{i}, \sigma)} \in \mathbb{Z}^{\mathbf{d}}$ uniquely, $\mathbf{i} := (i_1, i_2, i_3)$. By conservation of $\tilde{\mathcal{P}}_x$ we have $\sum_k \mathbf{m}_k \ell_k^{(\mathbf{i}, \sigma)} = 0$. If $\ell^{(\mathbf{i}, \sigma)} \neq 0$, such expression defines a hyperplane $\mathbf{v}^{(\mathbf{i}, \sigma)} \subset \mathbb{C}^{\mathbf{d}}$, which depends just on the functions $\mu_i(\lambda)$. By taking all the possible (\mathbf{i}, σ) (there are only a finite number of possibilities!) we get a finite number of hyperplanes. Then it is enough to choose the sites $(\mathbf{m}_k)_{1 \leq k \leq \mathbf{d}}$ outside the set $\bigcup_{\mathbf{i}, \sigma} \mathbf{v}^{(\mathbf{i}, \sigma)}$ (remark that the $\mu_i(\lambda)$ **do not** depend on the sites $(\mathbf{m}_k)_{1 \leq k \leq \mathbf{d}}$,

see Remark 2.11). In order to impose such condition we only have to choose in the L-genericity condition 2.28:

$$L > \max_{k,i,\sigma} |\ell_k^{(i,\sigma)}|.$$

Now assume $\ell^{(i,\sigma)} = 0$. In this case we are not able to prove that $F_{\mathbf{j},\ell}^\sigma(\lambda) \neq 0$, however we show that in such case $K_{\mathbf{j},0}^\sigma \neq 0$. Assume the contrary. By conservation of $\tilde{\mathcal{P}}_y$ we have $\sum_{k=1}^3 \sigma_k n_k = 0$ and furthermore $0 = K_{\mathbf{j},0}^\sigma = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$. Exploring all the possible choices of $\sigma_1, \sigma_2, \sigma_3$, one deduces that at least one among n_1, n_2, n_3 must equal 0. But this is impossible.

8. $\vec{j}_1, \vec{j}_2, \vec{j}_3 \in \mathcal{C}$. W.l.o.g. let $\vec{j}_r \in \mathcal{C}_{i_r, k_r}^{s_r}$ for some $1 \leq i_r < k_r \leq \mathbf{d}$, $s_r = \pm$, $r = 1, 2, 3$. Then

$$F_{\mathbf{j},\ell}^\sigma(\lambda) \equiv -\lambda \cdot \ell + \sigma_1 \frac{\lambda_{i_1} - \lambda_{k_1} - s_1 \sqrt{\lambda_{i_1}^2 + \lambda_{k_1}^2 - 14\lambda_{i_1}\lambda_{k_1}}}{2} + \sigma_2 \frac{\lambda_{i_2} - \lambda_{k_2} - s_2 \sqrt{\lambda_{i_2}^2 + \lambda_{k_2}^2 - 14\lambda_{i_2}\lambda_{k_2}}}{2} + \sigma_3 \frac{\lambda_{i_3} - \lambda_{k_3} - s_3 \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 - 14\lambda_{i_3}\lambda_{k_3}}}{2}$$

Assume $F_{\mathbf{j},\ell}^\sigma(\lambda) \equiv 0$. Assume first that $(i_1, k_1) = (i_2, k_2) = (i_3, k_3)$. Then there is an odd number of roots in the expression for $F_{\mathbf{j},\ell}^\sigma(\lambda)$ and thus they cannot cancel identically.

Otherwise there is a couple of indexes which appears just once. Assume it is (i_1, k_1) . Then specify to $\lambda_{i_1} = \lambda_{k_1} = 1$ and all the rest at 0. Then the first root, namely $\sqrt{\lambda_{i_1}^2 + \lambda_{k_1}^2 - 14\lambda_{i_1}\lambda_{k_1}}$ is complex, while all the others terms are real. Hence $F_{\mathbf{j},\ell}^\sigma(\lambda)$ cannot vanish identically.

9. $\vec{j}_1, \vec{j}_2 \in \mathcal{S}$, $\vec{j}_3 \in \mathcal{C}$. W.l.o.g. let $\vec{j}_1 = (\mathbf{m}_{i_1}, n_1)$, $\vec{j}_2 = (\mathbf{m}_{i_2}, n_2)$ for some $1 \leq i_1, i_2 \leq \mathbf{d}$ and $\vec{j}_3 \in \mathcal{C}_{i_3, k_3}^{s_3}$ for some $1 \leq i_3 < k_3 \leq \mathbf{d}$ and $s_3 = \pm$. Then

$$F_{\mathbf{j},\ell}^\sigma(\lambda) \equiv -\lambda \cdot \ell + \sigma_1 \mu_{i_1}(\lambda) + \sigma_2 \mu_{i_2}(\lambda) + \sigma_3 \mu_{i_3 k_3}^{s_3}(\lambda).$$

Assume $F_{\mathbf{j},\ell}^\sigma(\lambda) \equiv 0$. Specify all the λ -depending functions to $E := \{\lambda \in \mathbb{C}^{\mathbf{d}} : \lambda_i = 0 \text{ for any } i \neq i_3, k_3\}$. Then $\mu_{i_1}(\lambda)|_E, \mu_{i_2}(\lambda)|_E$ are roots of

$$P(t, \lambda)|_E = P(t, 0, \dots, \lambda_{i_3}, 0, \dots, \lambda_{k_3}, \dots, 0) = t^{k-2} (t^2 - (\lambda_{i_3} + \lambda_{k_3})t - 3\lambda_{i_3}\lambda_{k_3})$$

and thus belong to the set

$$\left\{ 0, \frac{\lambda_{i_3} + \lambda_{k_3} \pm \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 + 14\lambda_{i_3}\lambda_{k_3}}}{2} \right\}.$$

Thus we get three cases:

- $\mu_{i_1}(\lambda)|_E = \mu_{i_2}(\lambda)|_E = 0$, thus

$$F_{\mathbf{j},\ell}^\sigma(\lambda)|_E = -\lambda_{i_3} \ell_{i_3} - \lambda_{k_3} \ell_{k_3} + \sigma_3 \mu_{i_3 k_3}^{s_3}(\lambda).$$

Then one proceeds as in 3. to get a contradiction.

- $\mu_{i_1}(\lambda)|_E = 0, \mu_{i_2}(\lambda)|_E = \frac{1}{2} (\lambda_{i_3} + \lambda_{k_3} \pm \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 + 14\lambda_{i_3}\lambda_{k_3}})$. In such a case

$$F_{\mathbf{j},\ell}^\sigma(\lambda)|_E = -\lambda_{i_3} \ell_{i_3} - \lambda_{k_3} \ell_{k_3} + \sigma_2 \frac{\lambda_{i_3} + \lambda_{k_3} + \tilde{\sigma}_2 \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 + 14\lambda_{i_3}\lambda_{k_3}}}{2} + \sigma_3 \frac{\lambda_{i_3} - \lambda_{k_3} - s_3 \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 - 14\lambda_{i_3}\lambda_{k_3}}}{2}$$

But the roots cannot vanish identically, since they are different, hence we exclude such a case.

- $\mu_{i_1}(\lambda)|_E = \mu_{i_2}(\lambda)|_E = \frac{1}{2} (\lambda_{i_3} + \lambda_{k_3} \pm \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 + 14\lambda_{i_3}\lambda_{k_3}})$. Thus

$$F_{\mathbf{j},\ell}^\sigma(\lambda)|_E = -\lambda_{i_3} \ell_{i_3} - \lambda_{k_3} \ell_{k_3} + (\sigma_1 + \sigma_2) \frac{\lambda_{i_3} + \lambda_{k_3}}{2} + (\sigma_1 s_1 + \sigma_2 s_2) \frac{\sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 + 14\lambda_{i_3}\lambda_{k_3}}}{2} + \sigma_3 \frac{\lambda_{i_3} - \lambda_{k_3} - s_3 \sqrt{\lambda_{i_3}^2 + \lambda_{k_3}^2 - 14\lambda_{i_3}\lambda_{k_3}}}{2}$$

One checks directly that however one chooses $\sigma_1, \sigma_2, \sigma_3, s_1, s_2, s_3$ the roots cannot vanish identically.

10. $\vec{j}_1, \vec{j}_2 \in \mathcal{C}$, $\vec{j}_3 \in \mathcal{S}$. W.l.o.g. let $\vec{j}_r \in \mathcal{C}_{i_r, k_r}^{s_r}$ for some $1 \leq i_r < k_r \leq d$, $s_r = \pm$, $r = 1, 2$, and $\vec{j}_3 = (\mathbf{m}_{i_3}, n_3)$ for some $1 \leq i_3 \leq d$. Then

$$\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) \equiv -\lambda \cdot \ell + \sigma_1 \mu_{i_1 k_1}^{s_1}(\lambda) + \sigma_2 \mu_{i_2 k_2}^{s_2}(\lambda) + \sigma_3 \mu_{i_3}(\lambda).$$

Assume $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) \equiv 0$. We study different cases separately:

- if $(i_1, k_1) = (i_2, k_2)$, then

$$\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) = -\lambda \cdot \ell + (\sigma_1 + \sigma_2) \frac{\lambda_{i_1} + \lambda_{k_1}}{2} - (\sigma_1 s_1 + \sigma_2 s_2) \frac{\sqrt{\lambda_{i_1}^2 + \lambda_{k_1}^2 - 14\lambda_{i_1} \lambda_{k_1}}}{2} + \sigma_3 \mu_{i_3}(\lambda)$$

Now if $\sigma_1 s_1 + \sigma_2 s_2 = 0$, we are reduced to case 3., thus we have a contradiction.

If $\sigma_1 s_1 + \sigma_2 s_2 \neq 0$, we specify to $\lambda_{i_1} = \lambda_{k_1} = 1$, $\lambda_i = 0$ if $i \neq i_1, k_1$. We obtain that $\text{Im } \mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) \neq 0$, hence it does not vanish identically. Indeed $\mu_{i_3}(\lambda)$ for $\lambda \geq 0$ is an eigenvalue of a self-adjoint matrix, and thus it is real.

- if $i_2 = i_1, k_2 \neq k_1$, then as before we put $\lambda_{i_1} = \lambda_{k_1} = 1$, all the other $\lambda_i = 0$. We obtain that $\text{Im } \mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) \neq 0$, hence $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda)$ does not vanish identically.
- if i_2, k_2 are both different from i_1, k_1 , then once again we we put $\lambda_{i_1} = \lambda_{k_1} = 1$, all the other $\lambda_i = 0$ and proceed as before.

□

Remark 6.3. For any \vec{j}_1, \vec{j}_2 with $|m_1| > |m_2|$, $\forall \lambda \in \mathcal{O}_1$, $\forall \varepsilon \leq \varepsilon_1$

$$|\tilde{\Omega}_{(m_1, n)}(\lambda, \varepsilon) - \tilde{\Omega}_{(m_2, n)}(\lambda, \varepsilon)| \geq \begin{cases} |m_1|, & \text{if } |m_1| \geq \max_{1 \leq k \leq d} |\mathbf{m}_k| \\ \varepsilon \tilde{\gamma}, & \text{if } |m_1| \leq \max_{1 \leq k \leq d} |\mathbf{m}_k| \end{cases} \quad (6.13)$$

(see formulas (5.61), (5.62)) while

$$|\tilde{\Omega}_{(m_1, n)}(\lambda, \varepsilon) + \tilde{\Omega}_{(m_2, -n)}(\lambda, \varepsilon)| \geq \begin{cases} m_1^2 + n^2 & \text{if } |m_1| \geq \max_{1 \leq k \leq d} |\mathbf{m}_k| \\ n^2 & \text{if } |m_1| \leq \max_{1 \leq k \leq d} |\mathbf{m}_k| \end{cases} \quad (6.14)$$

We conclude the section with the following lemma; first let us fix $\mathbf{M}_0 = \mathbf{M}_0(\mathcal{O}_1, \varepsilon_0)$ as

$$\begin{aligned} \mathbf{M}_0 := & \sum_{1 \leq i \leq d} |\mu_i(\cdot)|_{\mathbb{C}}^{\mathcal{O}_1} + \sum_{\substack{1 \leq i < k \leq d \\ \pm}} |\mu_{i_k}^\pm(\cdot)|_{\mathbb{C}}^{\mathcal{O}_1} + \\ & + \sup_{\varepsilon \leq \varepsilon_0} \frac{1}{\varepsilon^2} \left(\sup_{m \in \mathbb{Z}} |\varpi_m(\cdot, \varepsilon)|_{\mathcal{O}_1}^{\mathcal{O}_1} + 2[\tilde{\mathcal{H}}^{(0, \text{hor})}]_{s, -2}^{\mathcal{O}_1} + 2[\tilde{\mathcal{H}}^{(0, \text{mix})}]_{s, -\bar{2}}^{\mathcal{O}_1} \right). \end{aligned} \quad (6.15)$$

We have the following

Lemma 6.4. There exists $\gamma_1 > 0$ (independent of ε) and a compact domain $\mathcal{C}_c \subset \mathcal{O}_1$ s.t.

$$\min_{|\ell| \leq 4\mathbf{M}_0, \mathbf{j}, \sigma} \inf_{\lambda \in \mathcal{C}_c \cap \mathcal{O}_1} |\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda)| \geq \gamma_1 > 0. \quad (6.16)$$

Proof. For each \mathbf{j}, ℓ, σ the function $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda)$ is a \mathbb{Z} -linear combination of algebraic functions, thus it is algebraic. Furthermore $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda)$ is homogeneous of degree 1 (see Lemma 5.12), its zeroes set is a union of a finite numbers of algebraic surfaces of codimension 1. By homogeneity, such surfaces are in fact cones. Note that outside these surfaces, the eigenvalues $\mu_i(\lambda)$ and $\mu_{i_k}^\pm(\lambda)$ are distinct. As in the construction of \mathcal{O}_1 , each equation $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) = 0$ divides \mathcal{O}_0 and \mathcal{O}_1 in a finite number of distinct open cones in which $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda) \neq 0$.

Since we consider just a finite number of distinct, non zero, analytic functions $\mathbf{F}_{\mathbf{j}, \ell}^\sigma(\lambda)$, any compact domain \mathcal{C}_c strictly contained in \mathcal{O}_1 satisfies an estimate of type (6.16) with some $\tilde{\gamma}_0 > 0$. We just need to fix $\gamma_1 \leq \tilde{\gamma}_0, \tilde{\gamma}$ (see formula (5.61)). □

7 KAM algorithm for the reducibility

In the previous section we have reduced the original Hamiltonian to the normal form

$$\tilde{\mathcal{H}} := \omega \cdot \mathcal{Y} + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \tilde{\Omega}_{\bar{j}} |a_{\bar{j}}|^2 + \tilde{\mathcal{H}}^{(0)} + \tilde{\mathcal{H}}^{(1)} + \tilde{\mathcal{H}}^{(\geq 2)}, \quad (7.1)$$

as described in Theorem 5.1. The aim of this section is to put the Hamiltonian $\tilde{\mathcal{H}}^{(0)}$ to diagonal form with a KAM algorithm. We prove the following result:

Theorem 7.1. *Assume that \mathcal{S}_0 is L-generic (in the sense of Definition 2.4) with L fixed in Lemma 6.2. There exists a $\gamma_* > 0$ such that For all $0 < \gamma_0 < \gamma_*$, there exist $0 < \varepsilon_1(\gamma_0) \ll \varepsilon_0$ s.t that the following holds true for all $0 < \varepsilon \leq \varepsilon_1(\gamma_0)$. There exist functions $(\Omega_{\bar{j}}(\lambda, \varepsilon))_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$ defined and Lipschitz in λ on the set \mathcal{O}_1 of Theorem 5.1, such that:*

(i) *The functions $\Omega_{\bar{j}}(\lambda, \varepsilon)$ satisfy (2.35), (2.36), (2.40) with M_0 as in (6.15). For $\gamma_0/2 \leq \gamma \leq \gamma_0$ and τ sufficiently large, the set*

$$\mathcal{C} := \left\{ \lambda \in \mathcal{O}_1 : |\omega \cdot \ell + \sigma_1 \Omega_{\bar{j}_1}(\lambda, \varepsilon) + \sigma_2 \Omega_{\bar{j}_2}(\lambda, \varepsilon) + \sigma_3 \Omega_{\bar{j}_3}(\lambda, \varepsilon)| \geq \varepsilon \frac{\gamma}{\langle \ell \rangle^\tau}, \forall (\mathbf{j}, \ell, \boldsymbol{\sigma}) \in \mathfrak{A}_3 \setminus \mathfrak{K}_3 \right\} \quad (7.2)$$

has positive measure.

(ii) *For each $\lambda \in \mathcal{C}$ there exists a symplectic change of variables $\hat{\mathcal{T}}$ well defined and majorant analytic $D(s/16, \varrho r/2) \rightarrow D(s/8, \varrho r)$ for all $0 < r \leq r_0$, $s_0/64 \leq s \leq s_0$ (s_0, r_0 and ϱ are defined in Theorem (5.1)) and such that*

$$(\tilde{\mathcal{H}} \circ \hat{\mathcal{T}})(\mathcal{Y}, \theta, a, \bar{a}) =: \mathcal{K} = \omega \cdot \mathcal{Y} + \sum_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\bar{j}} |a_{\bar{j}}|^2 + \mathcal{K}^{(1)} + \mathcal{K}^{(\geq 2)}, \quad (7.3)$$

where $\mathcal{K}^{(1)}$ contains just monomials of scaling degree 1, while $\mathcal{K}^{(\geq 2)}$ of scaling degree at least 2 and

$$\left| \mathcal{K}^{(1)} \right|_{s/16, \varrho r/2}^{\mathcal{C}} \leq \sqrt{\varepsilon} r, \quad \left| \mathcal{K}^{(\geq 2)} \right|_{s/16, \varrho r/2}^{\mathcal{C}} \leq r^2, \quad \forall 0 < r \leq r_0. \quad (7.4)$$

(iii) *Finally one has $\tilde{\mathcal{M}} \circ \hat{\mathcal{T}} = \tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}} \circ \hat{\mathcal{T}} = \tilde{\mathcal{P}}$.*

In the course of the proof, for notational convenience we shall rename $s/8 \rightsquigarrow s$, $\varrho r \rightsquigarrow r$. We first describe the *KAM step*, namely a standard change of variables which we shall apply recursively in order to diagonalize the Hamiltonian (5.2). **The smallness constant $\varepsilon_1(\gamma_0)$ is fixed so that $\varepsilon_1 M_0 / \gamma_0^2 \ll 1$.**

KAM step. Fix an $0 < \varepsilon_1 \ll \varepsilon_0$ and let γ_1 be defined in Lemma 6.16. For all $\varepsilon \leq \varepsilon_1$ consider a Hamiltonian of the form

$$\mathcal{H}(\lambda, \varepsilon) := \omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Q}, \quad \mathcal{Q} := \sum_{\ell, |\alpha|+|\beta|=2} Q_{\alpha, \beta, \ell}(\lambda, \varepsilon) e^{i\ell \cdot \theta} a^\alpha \bar{a}^\beta \quad (7.5)$$

where $\mathcal{D}(\lambda, \varepsilon) := \text{diag}(\mathcal{D}_{\bar{j}}(\lambda, \varepsilon))_{\bar{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0}$. Assume that the functions

$$\mathcal{D}_{\bar{j}}(\lambda, \varepsilon) = \tilde{\Omega}_{\bar{j}}(\lambda, \varepsilon) + \mathcal{D}_{\bar{j}}^{\text{hor}}(\lambda, \varepsilon) + \mathcal{D}_{\bar{j}}^{\text{mix}}(\lambda, \varepsilon)$$

are defined and Lipschitz for $\lambda \in \mathcal{O}_1$ and fulfill the estimates

$$\sum_{1 \leq i \leq d} |\mu_i(\cdot)|_{\mathbb{C}}^{\mathcal{O}_1} + \sum_{\substack{1 \leq i < k \leq d \\ \pm}} |\mu_{ik}^{\pm}(\cdot)|_{\mathbb{C}}^{\mathcal{O}_1} + \sup_{\varepsilon \leq \varepsilon_1} \frac{1}{\varepsilon^2} \left([\mathcal{D}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{O}_1} + [\mathcal{D}^{\text{hor}}]_{s, -2}^{\mathcal{O}_1} \right) \leq M, \quad (7.6)$$

with $\varepsilon_1 M \ll 1$. Note that for a diagonal matrix

$$[\mathcal{D}^{\text{hor}}]_{s, -2}^{\mathcal{O}_1} = \sup_{m \in \mathbb{Z}} \langle m \rangle^2 |\mathcal{D}_m^{\text{hor}}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}, \quad (7.7)$$

same for the mix part.

Assume moreover that \mathcal{Q} is a quadratic Hamiltonian, defined for λ in a compact set $\mathcal{C} \subseteq \mathcal{O}_1$, which commutes with $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}}$, and admits the decomposition

$$\mathcal{Q} = \mathcal{Q}^{\text{hor}} + \mathcal{Q}^{\text{mix}}, \quad \mathcal{Q}^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\mathcal{C}, \text{hor}}, \quad \mathcal{Q}^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\mathcal{C}}, \quad \text{for some } s > 0.$$

Fix $\tau > \mathbf{d} + 1$, $0 < \gamma < \gamma_1$, $\mathbf{N} \gg \max(1, \mathbf{M})$ and assume that for $\varepsilon \leq \varepsilon_1$

$$\mathbf{N}^{3\tau+2} (\gamma^2 \varepsilon)^{-1} \left([\mathcal{Q}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}} \right) \leq 1. \quad (7.8)$$

For $\varepsilon \leq \varepsilon_1$ define $\mathcal{C}^{\text{New}} \equiv \mathcal{C}^{\text{New}}(\gamma, \tau, \varepsilon, \mathcal{D}, \mathbf{N}) \subset \mathcal{C}$ as the set of $\lambda \in \mathcal{C}$ fulfilling

$$|\omega \cdot \ell + \mathcal{D}_{\vec{i}}(\lambda, \varepsilon) + \sigma \mathcal{D}_{\vec{j}}(\lambda, \varepsilon)| \geq \varepsilon \frac{\gamma}{2} \mathbf{N}^{-\tau}, \quad \forall ((\vec{i}, \vec{j}), \ell, \sigma) \in \mathfrak{A}_2 \setminus \mathfrak{B}_2, \quad |\ell| \leq \mathbf{N}. \quad (7.9)$$

Note that at this point the set \mathcal{C}^{New} might be empty.

In the next lemma we will write $a < b$ meaning $a \leq Cb$ with a constant C independent of $\gamma, \varepsilon, \mathbf{N}$.

Proposition 7.2 (Homological equation). *The following holds true:*

(i) For $\lambda \in \mathcal{C}^{\text{New}}$, there exists $\chi = \chi^{\text{hor}} + \chi^{\text{mix}}$, commuting with $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{P}}$, with $\chi^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\mathcal{C}^{\text{New}}, \text{hor}}$ and $\chi^{\text{mix}} \in \mathcal{Q}_{s, -\bar{2}}^{\mathcal{C}^{\text{New}}, \text{mix}}$ which satisfies the bounds

$$[\chi^{\text{hor}}]_{s, -2}^{\mathcal{C}^{\text{New}}} \leq \mathbf{N}^{2\tau+1} (\gamma^2 \varepsilon)^{-1} [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}, \quad [\chi^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}^{\text{New}}} \leq \mathbf{N}^{3\tau+1} (\gamma^2 \varepsilon)^{-1} ([\mathcal{Q}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}) \quad (7.10)$$

and such that

$$\{\omega \cdot \mathcal{Y} + \mathcal{D}, \chi\} = \Pi_{\leq \mathbf{N}}([\mathcal{Q}] - \mathcal{Q}), \quad [\mathcal{Q}](\lambda, \varepsilon; \mathbf{a}) := \sum_{\vec{j} \in \mathbb{Z}^2} Q_{e_{\vec{j}}, e_{\vec{j}}, 0}(\lambda, \varepsilon) |a_{\vec{j}}|^2 \quad (7.11)$$

(ii) For $\lambda \in \mathcal{C}^{\text{New}}$, the time 1-flow \mathcal{T}_χ of χ defines a symplectic analytic change of variables which transforms the Hamiltonian $\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Q}$ into the new Hamiltonian

$$(\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Q}) \circ \mathcal{T}_\chi = \omega \cdot \mathcal{Y} + \mathcal{D}^{\text{New}} + \mathcal{Q}^{\text{New}} \quad (7.12)$$

with

$$\mathcal{D}^{\text{New}} := \sum_{\vec{j} \in \mathbb{Z}^2} \mathcal{D}_{\vec{j}}^{\text{New}}(\lambda, \varepsilon) |a_{\vec{j}}|^2, \quad \mathcal{Q}^{\text{New}} = \mathcal{Q}_{\text{New}}^{\text{hor}} + \mathcal{Q}_{\text{New}}^{\text{mix}}, \quad \mathcal{Q}_{\text{New}}^{\text{hor}} \in \mathcal{Q}_{s', -2}^{\mathcal{C}^{\text{New}}, \text{hor}}, \quad \mathcal{Q}_{\text{New}}^{\text{mix}} \in \mathcal{Q}_{s', -\bar{2}}^{\mathcal{C}^{\text{New}}}$$

for all $0 < s' < s$. The new frequencies $\mathcal{D}_{\vec{j}}^{\text{New}}(\lambda, \varepsilon)$ are defined for all $\lambda \in \mathcal{O}_1$ and are given by

$$\mathcal{D}_{\vec{j}}^{\text{New}}(\lambda, \varepsilon) = \tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon) + \mathcal{D}_{\vec{j}}^{\text{hor}, \mathbf{N}}(\lambda, \varepsilon) + \mathcal{D}_{\vec{j}}^{\text{mix}, \mathbf{N}}(\lambda, \varepsilon), \quad (7.13)$$

and satisfies the estimates

$$[\mathcal{D}^{\text{mix}, \mathbf{N}} - \mathcal{D}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{O}_1} + [\mathcal{D}^{\text{hor}, \mathbf{N}} - \mathcal{D}^{\text{hor}}]_{s, -2}^{\mathcal{O}_1} \leq [\mathcal{Q}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}} \quad (7.14)$$

Finally we have

$$\begin{aligned} ([\mathcal{Q}_{\text{New}}^{\text{mix}}]_{s', -\bar{2}}^{\mathcal{C}^{\text{New}}} + [\mathcal{Q}_{\text{New}}^{\text{hor}}]_{s', -2}^{\mathcal{C}^{\text{New}}}) &\leq e^{-(s-s')\mathbf{N}} ([\mathcal{Q}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}) \\ &+ \mathbf{N}^{3\tau+1} (\gamma^2 \varepsilon)^{-1} ([\mathcal{Q}^{\text{mix}}]_{s, -\bar{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}})^2 \end{aligned} \quad (7.15)$$

In order to prove Proposition 7.2 we need a preliminary result regarding the asymptotics of the frequencies $\mathcal{D}_{\vec{j}}$. For $\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0$ we define $\tilde{\Omega}_{\vec{j}}^*(\lambda, \varepsilon)$ as

$$\tilde{\Omega}_{\vec{j}}^*(\lambda, \varepsilon) := \begin{cases} m^2 + n^2, & \vec{j} = (m, n) \notin \mathcal{S}, \\ \varepsilon \mu_i(\lambda) + n^2, & \vec{j} = (\mathbf{m}_i, n), n \neq 0 \end{cases} \quad (7.16)$$

and the frequencies $\mathcal{D}_{\vec{j}}^*$ as

$$\mathcal{D}_{\vec{j}}^* := \tilde{\Omega}_{\vec{j}}^* + \mathcal{D}_{\vec{j}}^{\text{hor}}.$$

Lemma 7.3. *For any $\lambda \in \mathcal{C}^{\text{New}}$ and $\forall \varepsilon \leq \varepsilon_1$, we have*

$$|\omega \cdot \ell + \mathcal{D}_{\vec{i}}^*(\lambda, \varepsilon) - \mathcal{D}_{\vec{j}}^*(\lambda, \varepsilon)| \geq \varepsilon \frac{\gamma}{4} \mathbf{N}^{-\tau}, \quad \forall ((\vec{i}, \vec{j}), \ell, (1, -1)) \in \mathfrak{A}_2 \setminus \mathfrak{B}_2, \quad |\ell| \leq \mathbf{N}. \quad (7.17)$$

Proof. We deduce (7.17) from the Melnikov conditions (7.9). To do so, recall that for $|\vec{j}| \geq C \max_{1 \leq k \leq d} |\mathbf{m}_k|$, one has $\tilde{\Omega}_{\vec{j}}^* = \tilde{\Omega}_{\vec{j}}$. Moreover it is easily seen that $\mathcal{D}_{\vec{i}}^* - \mathcal{D}_{\vec{j}}^*$ is independent of n for all compatible $((\vec{i}, \vec{j}), \ell, (1, -1)) \in \mathfrak{A}_2$. Thus defining $\vec{i}_1 = \vec{i} + (0, K)$, $\vec{j}_1 = \vec{j} + (0, K)$ with a sufficiently large K , we have

$$\begin{aligned} \mathcal{D}_{\vec{i}}^* - \mathcal{D}_{\vec{j}}^* &= \mathcal{D}_{\vec{i}_1}^* - \mathcal{D}_{\vec{j}_1}^* = \tilde{\Omega}_{\vec{i}_1}^* - \tilde{\Omega}_{\vec{j}_1}^* + \mathcal{D}_{\vec{i}_1}^{\text{hor}} - \mathcal{D}_{\vec{j}_1}^{\text{hor}} \\ &= \tilde{\Omega}_{\vec{i}_1} - \tilde{\Omega}_{\vec{j}_1} + \mathcal{D}_{\vec{i}_1}^{\text{hor}} - \mathcal{D}_{\vec{j}_1}^{\text{hor}} = \mathcal{D}_{\vec{i}_1} - \mathcal{D}_{\vec{j}_1} - \mathcal{D}_{\vec{i}_1}^{\text{mix}} + \mathcal{D}_{\vec{j}_1}^{\text{mix}} \end{aligned}$$

If we take $K \geq N^{\tau/2}$ we bound (recall that $\varepsilon M \ll 1$)

$$|\mathcal{D}_{\vec{i}_1}^{\text{mix}}| + |\mathcal{D}_{\vec{j}_1}^{\text{mix}}| \stackrel{(7.7)}{\leq} [\mathcal{D}^{\text{mix}}]_{s, -\vec{z}} \left(\frac{1}{|\vec{i}_1|^2} + \frac{1}{|\vec{j}_1|^2} \right) \stackrel{(7.6)}{\leq} \varepsilon \frac{\gamma}{4} N^{-\tau}.$$

Then (7.17) follows easily from (7.9). \square

Proof of Proposition 7.2. (i) As usual we represent the quadratic Hamiltonians as

$$F = \sum_{\ell, |\alpha|+|\beta|=2} F_{\alpha, \beta, \ell} e^{i\ell \cdot \theta} a^\alpha \bar{a}^\beta.$$

In Taylor-Fourier components, the equation (7.11) reads

$$i(\omega \cdot \ell + \mathcal{D}(\lambda, \varepsilon) \cdot (\alpha - \beta)) \chi_{\alpha, \beta, \ell} = Q_{\alpha, \beta, \ell}, \quad (\alpha, \beta, \ell) \neq (\alpha, \alpha, 0), \quad |\ell| \leq N. \quad (7.18)$$

The solution of this equation is given by

$$\chi_{\alpha, \beta, \ell} = \frac{1}{i(\omega \cdot \ell + \mathcal{D}(\lambda, \varepsilon) \cdot (\alpha - \beta))} Q_{\alpha, \beta, \ell}, \quad \alpha \neq \beta, \quad |\ell| \leq N \quad (7.19)$$

provided $\lambda \in \mathcal{C}^{\text{New}}$, thanks to the estimate (7.9). Indeed we have that for all admissible choices of ℓ, α, β (recall $N \gg M$):

$$\left| \frac{1}{i(\omega \cdot \ell + \mathcal{D}(\lambda, \varepsilon) \cdot (\alpha - \beta))} \right|_{\mathcal{C}}^{\mathcal{C}^{\text{New}}} \leq N^{2\tau+1} (\varepsilon \gamma^2)^{-1}. \quad (7.20)$$

We show now that such χ can be written as $\chi = \chi^{\text{hor}} + \chi^{\text{mix}}$, and we bound the two components. To do so, consider first the out-diagonal part χ^{out} . In this case we set $\vec{i} = (m_1, n)$, $\vec{j} = (m_2, -n)$, so that (7.19) reads

$$\chi_{m_1, m_2, n}^+ = \frac{1}{i(\omega \cdot \ell + \mathcal{D}_{\vec{i}}(\lambda, \varepsilon) + \mathcal{D}_{\vec{j}}(\lambda, \varepsilon))} Q_{m_1, m_2, n}^+, \quad \alpha \neq \beta, \quad |\ell| \leq N. \quad (7.21)$$

Such term is clearly mixing, and we define $\chi^{\text{mix}, 0}$ the Hamiltonian with coefficients (7.21). To estimate it we need first to control the small divisor. By Remark 6.3

$$\min(|\vec{i}|^2, |\vec{j}|^2) = \min(m_1^2, m_2^2) + n^2 \geq 4N \max_k |\mathbf{m}_k|^2 \implies |\omega \cdot \ell + \mathcal{D}_{\vec{i}} + \mathcal{D}_{\vec{j}}| \geq (m_1^2 + n^2)/2,$$

thus for $\lambda \in \mathcal{C}^{\text{New}}$ there exists $\mathbf{C} = \mathbf{C}(d, \max_k |\mathbf{m}_k|)$ such that

$$\left| \frac{\langle m_1 \rangle^2 + \langle n \rangle^2}{\omega \cdot \ell + \mathcal{D}_{\vec{i}} + \mathcal{D}_{\vec{j}}} \right| \leq 2 + \max_{\substack{\min(|\vec{i}|^2, |\vec{j}|^2) \leq 4N \max_k |\mathbf{m}_k|^2 \\ ((\vec{i}, \vec{j}), \ell, (1, 1)) \in \mathfrak{A}_2 \\ |\ell| \leq N}} \left| \frac{\langle m_1 \rangle^2 + \langle n \rangle^2}{\omega \cdot \ell + \mathcal{D}_{\vec{i}} + \mathcal{D}_{\vec{j}}} \right| \leq \mathbf{C} \frac{N^{\tau+1}}{\gamma \varepsilon}.$$

Its Lipschitz norm is then

$$\left| \frac{\langle m_1 \rangle^2 + \langle n \rangle^2}{\omega \cdot \ell + \mathcal{D}_{\vec{i}} + \mathcal{D}_{\vec{j}}} \right|_{\mathcal{C}}^{\mathcal{C}^{\text{New}}} \leq \mathbf{C} \frac{N^{2\tau+2}}{\varepsilon \gamma^2}, \quad (7.22)$$

which implies that

$$[\chi^{\text{mix}, 0}]_{s, -\vec{z}}^{\mathcal{C}^{\text{New}}} \leq N^{2\tau+2} (\varepsilon \gamma^2)^{-1} |\mathcal{Q}|_s^{\mathcal{C}}. \quad (7.23)$$

We pass to χ^{diag} . In this case we set $\vec{i} = (m_1, n)$, $\vec{j} = (m_2, n)$, so (7.19) reads

$$\chi_{m_1, m_2, n}^- = \frac{1}{i(\omega \cdot \ell + \mathcal{D}_{\vec{i}}(\lambda, \varepsilon) - \mathcal{D}_{\vec{j}}(\lambda, \varepsilon))} Q_{m_1, m_2, n}^-, \quad \alpha \neq \beta, \quad |\ell| \leq N. \quad (7.24)$$

To analyze it we divide the small divisor and Q^{diag} in their horizontal and mixing parts. Recalling $D_{\vec{j}} = \tilde{\Omega}_{\vec{j}} + D_{\vec{j}}^{\text{hor}} + D_{\vec{j}}^{\text{mix}}$ and $D_{\vec{j}}^* = \tilde{\Omega}_{\vec{j}}^* + D_{\vec{j}}^{\text{hor}}$, write

$$\begin{aligned} \frac{1}{i(\omega \cdot \ell + D_{\vec{i}} - D_{\vec{j}})} &= \frac{1}{i(\omega \cdot \ell + D_{\vec{i}}^* - D_{\vec{j}}^*)} + \frac{\tilde{\Omega}_{\vec{i}}^* - \tilde{\Omega}_{\vec{i}} + \tilde{\Omega}_{\vec{j}} - \tilde{\Omega}_{\vec{j}}^* + D_{\vec{i}}^{\text{mix}} - D_{\vec{j}}^{\text{mix}}}{i(\omega \cdot \ell + D_{\vec{i}} - D_{\vec{j}}) (\omega \cdot \ell + D_{\vec{i}}^* - D_{\vec{j}}^*)} \\ &=: A_{m_1, m_2, \ell}(\lambda, \varepsilon) + B_{\vec{i}, \vec{j}, \ell}(\lambda, \varepsilon). \end{aligned} \quad (7.25)$$

We prove that $A_{m_1, m_2, \ell}(\lambda, \varepsilon)$ is independent of n , while $B_{\vec{i}, \vec{j}, \ell}(\lambda, \varepsilon)$ is decreasing in n . It is easy to see that $D_{\vec{i}}^* - D_{\vec{j}}^*$ is independent of n for all $((\vec{i}, \vec{j}), \ell, (1, -1)) \in \mathfrak{A}_2$ and by (7.17), as in (7.20)

$$\left| \frac{1}{\omega \cdot \ell + D_{\vec{i}}^* - D_{\vec{j}}^*} \right|_{\mathbb{C}}^{\mathcal{C}^{\text{new}}} \leq \mathbf{C} \frac{\mathbf{N}^{2\tau+1}}{\varepsilon \gamma^2}. \quad (7.26)$$

Consider now $B_{\vec{i}, \vec{j}, \ell}(\lambda, \varepsilon)$. First we study its numerator. Note that $\tilde{\Omega}_{\vec{j}}^* = \tilde{\Omega}_{\vec{j}}$ if $\vec{j} \notin \mathcal{C}$, so it is convenient to separate the case $|n| > \max_k |\mathbf{m}_k|$ and $|n| \leq \max_k |\mathbf{m}_k|$. One has in the former case

$$\left| \tilde{\Omega}_{\vec{i}}^* - \tilde{\Omega}_{\vec{i}} + \tilde{\Omega}_{\vec{j}} - \tilde{\Omega}_{\vec{j}}^* + D_{\vec{i}}^{\text{mix}} - D_{\vec{j}}^{\text{mix}} \right|_{\mathbb{C}}^{\mathcal{O}_1} \leq \frac{4M\varepsilon^2}{\min(\langle m_1 \rangle^2, \langle m_2 \rangle^2) + \langle n \rangle^2}, \quad \text{for } |n| > \max_k |\mathbf{m}_k|. \quad (7.27)$$

Then (7.20), (7.26) and (7.27) give

$$|B_{\vec{i}, \vec{j}, \ell}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1} \leq \mathbf{C} \frac{\mathbf{N}^{3\tau+1}}{\gamma^2 \langle n \rangle^2}, \quad \text{for } |n| > \max_k |\mathbf{m}_k|, \quad (7.28)$$

while using the definition $B_{\vec{i}, \vec{j}, \ell} := (\omega \cdot \ell + D_{\vec{i}} - D_{\vec{j}})^{-1} - A_{m_1, m_2, \ell}$ and the estimates (7.20), (7.26) one has

$$|B_{\vec{i}, \vec{j}, \ell}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{C}^{\text{new}}} \leq |(\omega \cdot \ell + D_{\vec{i}} - D_{\vec{j}})^{-1}| + |A_{m_1, m_2, \ell}| \leq \mathbf{C} \frac{\mathbf{N}^{2\tau+1}}{\gamma^2 \varepsilon}, \quad \forall n. \quad (7.29)$$

We are ready to estimate χ^{diag} defined in (7.24). Divide $\mathcal{Q}^{\text{diag}}$ into its horizontal and mixing components, $\mathcal{Q}^{\text{diag}} = \mathcal{Q}^{\text{diag, hor}} + \mathcal{Q}^{\text{diag, mix}}$ so that

$$\chi_{m_1, m_2, n}^- = (A_{m_1, m_2, \ell} + B_{\vec{i}, \vec{j}, \ell}) \left(Q_{m_1, m_2, \ell}^{-, \text{hor}} + Q_{\vec{i}, \vec{j}, \ell}^{-, \text{mix}} \right) \quad (7.30)$$

$$= A_{m_1, m_2, \ell} Q_{m_1, m_2, \ell}^{-, \text{hor}} + B_{\vec{i}, \vec{j}, \ell} Q_{m_1, m_2, \ell}^{-, \text{hor}} + \frac{1}{i(\omega \cdot \ell + D_{\vec{i}} - D_{\vec{j}})} Q_{\vec{i}, \vec{j}, \ell}^{-, \text{mix}}. \quad (7.31)$$

We estimate the three terms separately. Note that the first one is horizontal, while the last two are mixing, so we define

$$\chi_{m_1, m_2, \ell}^{\text{hor}} := A_{m_1, m_2, \ell} Q_{m_1, m_2, \ell}^{-, \text{hor}}.$$

To estimate it, use the assumption $\mathcal{Q}^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\mathcal{C}, \text{hor}}$ and (7.26); one obtains immediately the first of (7.10). Define now the Hamiltonian $\chi^{\text{mix}, 1}$ with coefficients

$$\chi_{m_1, m_2, \ell}^{\text{mix}, 1} := B_{\vec{i}, \vec{j}, \ell} Q_{m_1, m_2, \ell}^{-, \text{hor}};$$

using $\mathcal{Q}^{\text{hor}} \in \mathcal{Q}_{s, -2}^{\mathcal{C}, \text{hor}}$ and (7.29) one has

$$[\chi^{\text{mix}, 1}]_{s, -2}^{\mathcal{C}^{\text{new}}} \leq \frac{\mathbf{N}^{2\tau+1}}{\gamma^2 \varepsilon} [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}, \quad (7.32)$$

while by (7.28) one has

$$[\chi^{\text{mix}, 1}]_{s, (0, -2)}^{\mathcal{C}^{\text{new}}} \leq \frac{\mathbf{N}^{3\tau+1}}{\gamma^2 \varepsilon} [\mathcal{Q}^{\text{hor}}]_s^{\mathcal{C}}. \quad (7.33)$$

Then (3.11) implies that

$$[\chi^{\text{mix}, 1}]_{s, -\bar{2}}^{\mathcal{C}^{\text{new}}} \leq \frac{\mathbf{N}^{3\tau+1}}{\gamma^2 \varepsilon} [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}. \quad (7.34)$$

Finally define $\chi^{\text{mix},2}$ to be the Hamiltonian with coefficients

$$\chi_{m_1, m_2, \ell}^{\text{mix},2} := \frac{1}{i(\omega \cdot \ell + \mathcal{D}_{\vec{i}} - \mathcal{D}_{\vec{j}})} Q_{\vec{i}, \vec{j}, \ell}^{-, \text{mix}}.$$

Using that $\mathcal{Q}^{\text{mix}} \in \mathcal{Q}_{s, -\vec{2}}^{\mathcal{C}, \text{mix}}$ and (7.20) one has

$$[\chi^{\text{mix},2}]_{s, -\vec{2}}^{\mathcal{C}^{\text{new}}} \ll \frac{N^{2\tau+1}}{\gamma^2 \varepsilon} [\mathcal{Q}^{\text{mix}}]_{s, -\vec{2}}^{\mathcal{C}}. \quad (7.35)$$

Define now $\chi^{\text{mix}} := \chi^{\text{mix},0} + \chi^{\text{mix},1} + \chi^{\text{mix},2}$. Estimates (7.23), (7.34) and (7.35) imply the second of (7.10). This concludes the proof of item (i).

(ii) Now by Lemma 3.18 and estimate (7.8), the change of variables \mathcal{T}_χ is well defined. We have

$$\begin{aligned} (\omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Q}) \circ \mathcal{T}_\chi &= \omega \cdot \mathcal{Y} + \mathcal{D} + \mathcal{Q} + \{\omega \cdot \mathcal{Y} + \mathcal{D}, \chi\} + \widetilde{\mathcal{V}}_2(\chi; \omega \cdot \mathcal{Y} + \mathcal{D}) + \widetilde{\mathcal{V}}_1(\chi; \mathcal{Q}) \\ &= \omega \cdot \mathcal{Y} + \mathcal{D} + [\mathcal{Q}] + \Pi_{>N} \mathcal{Q} + \widetilde{\mathcal{V}}_1(\chi; \mathcal{Q}) + \sum_{j=1}^{\infty} \frac{\text{ad}^j(\chi)}{(j+1)!} ([\mathcal{Q}] - \Pi_{\leq N} \mathcal{Q}) \end{aligned}$$

We recall that $[\mathcal{Q}](\lambda, \varepsilon; \mathbf{a}) = \sum_{\vec{j}} Q_{\mathbf{e}_{\vec{j}}, \mathbf{e}_{\vec{j}}, 0}(\lambda, \varepsilon) |a_{\vec{j}}|^2$ is defined for $\lambda \in \mathcal{C}$ and by construction

$$[\mathcal{Q}] = [\mathcal{Q}^{\text{hor}}] + [\mathcal{Q}^{\text{mix}}], \quad Q_{\mathbf{e}_{\vec{j}}, \mathbf{e}_{\vec{j}}, 0} = Q_m^{\text{hor}} + Q_{\vec{j}}^{\text{mix}}$$

with

$$\sup_{m \in \mathbb{Z}} \langle m^2 \rangle |Q_m^{\text{hor}}|^{\mathcal{C}} + \sup_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \langle \vec{j} \rangle^2 |Q_{\vec{j}}^{\text{mix}}|^{\mathcal{C}} \leq [\mathcal{Q}^{\text{mix}}]_{s, -\vec{2}}^{\mathcal{C}} + [\mathcal{Q}^{\text{hor}}]_{s, -2}^{\mathcal{C}}.$$

By Kirszbraun theorem we may extend the $Q_m^{\text{hor}}, Q_{\vec{j}}^{\text{mix}}$ to functions $\tilde{Q}_m^{\text{hor}}, \tilde{Q}_{\vec{j}}^{\text{mix}}$ defined on the whole \mathcal{O}_1 and with the same Lipschitz norm. This in turn defines a diagonal Hamiltonian $[\tilde{\mathcal{Q}}]$. We set, for $\lambda \in \mathcal{O}_1$, $\mathcal{D}^{\text{New}} := \mathcal{D} + [\tilde{\mathcal{Q}}]$, while for all $\lambda \in \mathcal{C}^{\text{New}}$:

$$\mathcal{Q}^{\text{New}} := \Pi_{>N} \mathcal{Q} + \widetilde{\mathcal{V}}_1(\chi; \mathcal{Q}) + \sum_{j=1}^{\infty} \frac{\text{ad}^j(\chi)}{(j+1)!} ([\mathcal{Q}] - \Pi_{\leq N} \mathcal{Q}).$$

The bounds (7.14) follow. The bounds 7.15 follow by applying Lemma 3.19 and Proposition B.1 (ii). \square

We now apply the KAM step recursively starting from

$$\mathcal{H}_0 := \omega \cdot \mathcal{Y} + \mathcal{D}_0(\lambda, \varepsilon) + \mathcal{Q}_0 := \omega \cdot \mathcal{Y} + \tilde{\mathcal{D}}(\lambda, \varepsilon) + \tilde{\mathcal{H}}^{(0)}$$

which is well defined for all $\lambda \in \mathcal{C}_0 := \mathcal{O}_1$. More precisely we construct a sequence of sets $\mathcal{C}_\nu \subset \mathcal{C}_0$ and on such sets we define a sequence of canonical transformations \mathcal{T}_ν such that we may recursively set for $\nu \geq 0$

$$\mathcal{H}_\nu \circ \mathcal{T}_{\nu+1} := \mathcal{H}_{\nu+1} = \omega \cdot \mathcal{Y} + \mathcal{D}_{\nu+1}(\lambda, \varepsilon) + \mathcal{Q}_{\nu+1}$$

with $\mathcal{D}_{\nu+1}$ diagonal and defined on the whole \mathcal{O}_1 , and $\mathcal{Q}_{\nu+1} \ll \mathcal{Q}_\nu$ (in appropriate norms, see Proposition 7.4 (iv) below).

Given $s, \eta_0 > 0$ we fix for all $\nu \geq 1$ the following sequences of parameters

$$s_\nu := s_{\nu-1} - \frac{c_* s}{\nu^2}, \quad c_*^{-1} = 2 \sum_{\nu=1}^{\infty} \frac{1}{\nu^2}, \quad \eta_\nu := \eta_0^{(3/2)^\nu}, \quad N_\nu := (s_\nu - s_{\nu+1})^{-1} \log \eta_\nu^{-1}.$$

Proposition 7.4 (Iterative scheme). *There exist $\varepsilon_1, \eta_* > 0$, such that $\forall \varepsilon \leq \varepsilon_1$ and $\forall \gamma \leq \gamma_1$ setting for $s_0/16 < s \leq s_0/8$*

$$\eta_0 := \frac{[\tilde{\mathcal{H}}^{(0, \text{mix})}]_{s, -\vec{2}}^{\mathcal{O}_1} + [\tilde{\mathcal{H}}^{(0, \text{hor})}]_{s, -2}^{\mathcal{O}_1}}{\gamma^2 \varepsilon} \leq \eta_*, \quad (7.36)$$

one has that $\forall \nu \geq 0$ we may define a set $\mathcal{C}_{\nu+1}$, a map $\mathcal{T}_{\nu+1}$ and Hamiltonians $\mathcal{D}_{\nu+1} = \text{diag}(\mathcal{D}_{\vec{j}}^{(\nu+1)})$, $\mathcal{Q}_{\nu+1}$ s.t. the following holds true:

(i) Setting $\mathcal{C}_0 = \mathcal{O}_1$, we have recursively:

$$\mathcal{C}_{\nu+1} := \left\{ \omega \in \mathcal{C}_\nu : \left| \omega \cdot \ell + \mathcal{D}_{\vec{i}}^{(\nu)}(\lambda, \varepsilon) + \sigma \mathcal{D}_{\vec{j}}^{(\nu)}(\lambda, \varepsilon) \right| \geq \varepsilon \frac{\gamma}{2} \mathbf{N}_\nu^{-\tau}, \quad \forall ((\vec{i}, \vec{j}), \ell, \sigma) \in \mathfrak{A}_2 \setminus \mathfrak{B}_2, \quad |\ell| \leq \mathbf{N}_\nu \right\}. \quad (7.37)$$

(ii) $\mathcal{T}_{\nu+1}$ is a canonical transformation defined for all $\lambda \in \mathcal{C}_{\nu+1}$ and s.t.

$$(\omega \cdot \mathcal{Y} + \mathcal{D}_\nu + \mathcal{Q}_\nu) \circ \mathcal{T}_{\nu+1} = \omega \cdot \mathcal{Y} + \mathcal{D}_{\nu+1} + \mathcal{Q}_{\nu+1}$$

and

$$[\mathcal{T}_{\nu+1}^{\text{hor}} - \mathbb{I}]_{s_\nu, -\bar{\mathbf{2}}}^{\mathcal{C}_{\nu+1}} + [\mathcal{T}_{\nu+1}^{\text{mix}}]_{s_\nu, -\bar{\mathbf{2}}}^{\mathcal{C}_{\nu+1}} \leq \eta_\nu \mathbf{N}_\nu^{3\tau+2}$$

(iii) $\mathcal{D}_{\nu+1} = \text{diag}(\mathcal{D}_{\vec{j}}^{(\nu+1)})$ is a diagonal Hamiltonian

$$\mathcal{D}_{\vec{j}}^{(\nu+1)}(\lambda, \varepsilon) = \tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon) + \mathcal{D}_{\vec{j}}^{(\nu+1, \text{hor})}(\lambda, \varepsilon) + \mathcal{D}_{\vec{j}}^{(\nu+1, \text{mix})}(\lambda, \varepsilon).$$

The functions $\mathcal{D}_{\vec{j}}^{(\nu+1, \text{hor})}, \mathcal{D}_{\vec{j}}^{(\nu+1, \text{mix})}$ are defined for $\lambda \in \mathcal{O}_1$ and fulfill the estimates

$$[\mathcal{D}^{(\nu+1, \text{mix})} - \mathcal{D}^{(\nu, \text{mix})}]_{s, -\bar{\mathbf{2}}}^{\mathcal{O}_1} + [\mathcal{D}^{(\nu+1, \text{hor})} - \mathcal{D}^{(\nu, \text{hor})}]_{s, -\bar{\mathbf{2}}}^{\mathcal{O}_1} \leq \gamma^2 \varepsilon \eta_\nu \quad (7.38)$$

(iv) $\mathcal{Q}_{\nu+1} = \mathcal{Q}_{\nu+1}^{\text{hor}} + \mathcal{Q}_{\nu+1}^{\text{mix}}$ and

$$[\mathcal{Q}_{\nu+1}^{\text{hor}}]_{s_{\nu+1}, -\bar{\mathbf{2}}}^{\mathcal{C}_{\nu+1}} + [\mathcal{Q}_{\nu+1}^{\text{mix}}]_{s_{\nu+1}, -\bar{\mathbf{2}}}^{\mathcal{C}_{\nu+1}} \leq \gamma^2 \varepsilon \eta_{\nu+1}$$

Proof. Step $\nu = 0$: We fix ε_1 sufficiently small so that $\varepsilon_1 \mathbf{M}_0 \ll 1$. We apply the KAM step with $\mathcal{D} := \mathcal{D}_0$, $\mathcal{Q} := \mathcal{Q}_0$, $\mathcal{C} = \mathcal{C}_0 := \mathcal{O}_1$, $\mathbf{N} = \mathbf{N}_0$. The smallness condition (7.8) follows from (7.36) provided that η_\star is appropriately small depending only on τ, s_0 . Then we apply the KAM step procedure, getting items (i)-(iii). As for item (iv) it follows from the bound (7.15) by setting $\sigma = \sigma_0 = s - s_1 \geq s_0/24$.

Step $\nu \rightsquigarrow \nu + 1$: we just have to show that we may apply the KAM step with $\mathcal{D} := \mathcal{D}_\nu$, $\mathcal{Q} := \mathcal{Q}_\nu$, $\mathcal{C} = \mathcal{C}_\nu$, $\mathbf{N} = \mathbf{N}_\nu$. The smallness condition (7.8) follows by noting that

$$\mathbf{N}_\nu^{3\tau+2} \eta_\nu < \mathbf{N}_0^{3\tau+2} \eta_0 < 1.$$

We may estimate $\mathbf{M} = \mathbf{M}^{(\nu)}$ in (7.6) by using (7.38), obtaining $\mathbf{M}^{(\nu)} \leq \mathbf{M}^{(0)} + \varepsilon \sum_{i \leq \nu-1} \eta_i \leq 2\mathbf{M}^{(0)} \leq \mathbf{M}_0$. Hence again one has $\varepsilon_1 \mathbf{M}_0 \ll 1$. Then we note that $\mathcal{C}^{\text{New}} \equiv \mathcal{C}_{\nu+1}$, we apply the KAM step procedure and define $\mathcal{D}_{\nu+1} := \mathcal{D}^{\text{New}}$, $\mathcal{Q}_{\nu+1} := \mathcal{Q}^{\text{New}}$. We get items (i)-(iii) directly. As for item (iv) it follows from the bound (7.15) by setting $\sigma = \sigma_\nu = s_\nu - s_{\nu+1}$, provided η_\star is appropriately small depending only on τ, s_0 . \square

The iterative KAM scheme above can be applied to diagonalize $\tilde{\mathcal{H}}^{(0)}$ provided that

$$\frac{[\tilde{\mathcal{H}}^{(0, \text{mix})}]_{s, -\bar{\mathbf{2}}}^{\mathcal{O}_1} + [\tilde{\mathcal{H}}^{(0, \text{hor})}]_{s, -\bar{\mathbf{2}}}^{\mathcal{O}_1}}{\gamma^2 \varepsilon} \sim \varepsilon \gamma^{-2} \leq \varepsilon_1 \gamma^{-2} \leq \eta_\star, \quad \varepsilon_1 \mathbf{M}_0 \leq 1. \quad (7.39)$$

More precisely the following holds:

Corollary 7.5. *Assume that*

$$\frac{\varepsilon_1 \mathbf{M}_0}{\gamma^2} \ll 1, \quad (7.40)$$

with \mathbf{M}_0 as in (6.15). For all $\lambda \in \mathcal{O}_1$ and for any $\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0$ we have that $\mathcal{D}_{\vec{j}}^{(\nu)}$ is a Cauchy sequence. We denote by $\Omega_{\vec{j}}$ its limit. We have that

$$\Omega_{\vec{j}}(\lambda, \varepsilon) = \begin{cases} \tilde{\Omega}_{\vec{j}}(\lambda, \varepsilon) + \Omega_{\vec{j}}^{\text{hor}}(\lambda, \varepsilon) + \Omega_{\vec{j}}^{\text{mix}}(\lambda, \varepsilon), & \vec{j} \neq (m, 0) \\ m^2 + \frac{\varpi_m(\lambda, \varepsilon)}{\langle m \rangle}, & \vec{j} = (m, 0) \end{cases}$$

where

$$\Omega_{\vec{j}}^{\text{hor}}(\lambda, \varepsilon) := \frac{\Theta_m(\lambda, \varepsilon)}{\langle m \rangle^2}, \quad \Omega_{\vec{j}}^{\text{mix}}(\lambda, \varepsilon) := \frac{\Theta_{m, n}(\lambda, \varepsilon)}{\langle m \rangle^2 + \langle n \rangle^2} \quad (7.41)$$

are defined for $\lambda \in \mathcal{O}_1$ and fulfill the estimates

$$\begin{aligned} & \sum_{1 \leq i \leq d} |\mu_i(\cdot)|^{\mathcal{O}_1} + \sum_{\substack{1 \leq i < k \leq d \\ \pm}} |\mu_{ik}^\pm(\cdot)|^{\mathcal{O}_1} \\ & + \sup_{\varepsilon \leq \varepsilon_1} \frac{1}{\varepsilon^2} \left(\sup_{m \in \mathbb{Z}} |\varpi_m(\cdot, \varepsilon)|^{\mathcal{O}_1} + \sup_{m \in \mathbb{Z}} |\Theta_m(\cdot, \varepsilon)|^{\mathcal{O}_1} + \sup_{(m,n) \in \mathbb{Z}^2 \setminus \mathbb{Z}} |\Theta_{m,n}(\cdot, \varepsilon)|^{\mathcal{O}_1} \right) \leq M_0 \end{aligned} \quad (7.42)$$

For all $\lambda \in \cap_\nu \mathcal{C}_\nu$ the sequence

$$\hat{\mathcal{T}}_\nu := \mathcal{T}_1 \circ \mathcal{T}_2 \cdots \circ \mathcal{T}_\nu$$

is a Cauchy sequence of changes of variables, converging to a $\hat{\mathcal{T}}$, which satisfies the bounds:

$$[\hat{\mathcal{T}}^{\text{hor}} - \mathbb{I}]_{s_\nu, -2}^{\mathcal{C}_{\nu+1}} + [\hat{\mathcal{T}}^{\text{mix}}]_{s_\nu, -2}^{\mathcal{C}_{\nu+1}} \leq \varepsilon, \quad \forall \nu \geq 0.$$

Proof. This is a standard KAM convergence argument, see for instance [Pös96]. \square

It turns out that the set on which we can diagonalize can be described in terms of the final frequencies. Indeed we have the following

Lemma 7.6. *Consider the set*

$$\mathcal{C}^{\text{fin}} := \left\{ \lambda \in \mathcal{O}_1 : |\omega \cdot \ell + \Omega_{\vec{i}}(\lambda, \varepsilon) + \sigma \Omega_{\vec{j}}(\lambda, \varepsilon)| \geq \varepsilon \gamma \langle \ell \rangle^{-\tau}, \quad \forall ((\vec{i}, \vec{j}), \ell, \sigma) \in \mathfrak{A}_2 \setminus \mathfrak{R}_2 \right\}. \quad (7.43)$$

We have that $\mathcal{C}^{\text{fin}} \subseteq \cap_\nu \mathcal{C}_\nu$.

Proof. By definition $\mathcal{C}^{\text{fin}} \subseteq \mathcal{O}_1$. By induction, we assume it is contained in $\cap_{\nu \leq n} \mathcal{C}_\nu = \mathcal{C}_n$. Recall that by definition

$$\mathcal{C}_{n+1} = \left\{ \lambda \in \mathcal{C}_n : \left| \omega \cdot \ell + \mathcal{D}_{\vec{i}}^{(n)}(\lambda, \varepsilon) + \sigma \mathcal{D}_{\vec{j}}^{(n)}(\lambda, \varepsilon) \right| \geq \varepsilon \frac{\gamma}{2} \mathbb{N}_n^{-\tau}, \quad \forall ((\vec{i}, \vec{j}), \ell, \sigma) \in \mathfrak{A}_2 \setminus \mathfrak{R}_2, |\ell| \leq \mathbb{N}_n \right\}.$$

First recall that

$$\left| \Omega_{\vec{i}}(\lambda, \varepsilon) - \mathcal{D}_{\vec{i}}^{(n)}(\lambda, \varepsilon) \right| \leq \sum_{k \geq n} \left| \mathcal{D}_{\vec{i}}^{(k+1)}(\lambda, \varepsilon) - \mathcal{D}_{\vec{i}}^{(k)}(\lambda, \varepsilon) \right| \leq \varepsilon \gamma^2 \sum_{\nu \geq n} \eta_\nu \leq 2\varepsilon \gamma^2 \eta_n \leq \frac{\gamma}{4} \varepsilon \mathbb{N}_n^{-\tau}.$$

Thus for $|\ell| \leq \mathbb{N}_n$

$$\begin{aligned} \left| \omega \cdot \ell + \mathcal{D}_{\vec{i}}^{(n)}(\lambda, \varepsilon) + \sigma \mathcal{D}_{\vec{j}}^{(n)}(\lambda, \varepsilon) \right| & \geq \left| \omega \cdot \ell + \Omega_{\vec{i}}(\lambda, \varepsilon) + \sigma \Omega_{\vec{j}}(\lambda, \varepsilon) \right| - 2 \sup_{\vec{i}} \left| \Omega_{\vec{i}}(\lambda, \varepsilon) - \mathcal{D}_{\vec{i}}^{(n)}(\lambda, \varepsilon) \right| \\ & \geq \varepsilon \gamma \langle \ell \rangle^{-\tau} - \varepsilon \frac{\gamma}{2} \mathbb{N}_n^{-\tau} \geq \varepsilon \frac{\gamma}{2} \mathbb{N}_n^{-\tau}. \end{aligned}$$

\square

Finally we prove that the set of λ in which our diagonalization scheme holds has positive measure:

Proposition 7.7. *There exists $\gamma_0 > 0$ s.t. for all $0 < \gamma < \gamma_0$, $\varepsilon < \varepsilon_1$ fulfilling (7.40), the set \mathcal{C} defined in (7.2) has measure $\text{meas}(\mathcal{C}) \geq \text{meas}(\mathcal{C}_c)/2$ (where \mathcal{C}_c is defined in Lemma 6.4).*

The proposition, being quite technical, it is proved in Appendix C.

Proof of Theorem 7.1. Fix γ_0 as in Proposition 7.7 and ε_\star s.t. $4\varepsilon_\star M_0 \gamma_0^{-2} \ll 1$, so that (7.40) holds for $\varepsilon_1 \leq \varepsilon_\star$ and $\gamma_0/2 \leq \gamma \leq \gamma_0$. This ensures that (7.39) holds, so we apply Corollary 7.5. The change of variables $\hat{\mathcal{T}}$ and the final frequencies $\Omega_{\vec{j}}$ are the ones defined in Corollary 7.5, so all the desired bounds follow. The measure of the set \mathcal{C} is studied in Proposition 7.7. \square

8 Birkhoff normal form for the cubic terms

After Theorem 7.1 we have the Hamiltonian

$$\mathcal{K} := \omega \cdot \mathcal{Y} + \sum_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\vec{j}}(\lambda, \varepsilon) |a_{\vec{j}}|^2 + \mathcal{K}^{(1)} + \mathcal{K}^{(\geq 2)}. \quad (8.1)$$

We now perform a Birkhoff change of variables which cancels out $\mathcal{K}^{(1)}$. In order to define such a change of variables we use third order Melnikov conditions, which hold true in the set \mathcal{C} of Proposition 7.7. The main result of this section is the following theorem:

Theorem 8.1. *For all $\varepsilon \leq \varepsilon_*$, there exists $r_0 \ll \sqrt{\varepsilon}$ such that, for all $\lambda \in \mathcal{C}$ there exists a symplectic change of variables $\mathcal{T}^{(1)}$ well defined and majorant analytic: $D(s/32, \varrho r/4) \rightarrow D(s/16, \varrho r/2)$ for all $r \leq r_0$, $s_0/64 \leq s \leq s_0$ and such that*

$$\mathcal{K} \circ \mathcal{T}^{(1)} := \omega \cdot \mathcal{Y} + \sum_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\vec{j}}(\lambda, \varepsilon) |a_{\vec{j}}|^2 + \mathcal{R}^{(\geq 2)},$$

where

(i) the map $\mathcal{T}^{(1)}$ is the time-1 flow of a cubic Hamiltonian $\chi^{(1)}$ such that $|\chi^{(1)}|_{\frac{s}{24}, \frac{\varrho r}{2}}^{\mathcal{C}} < \frac{r}{\sqrt{\varepsilon}}$.

(ii) $\mathcal{R}^{(\geq 2)}$ contains just monomials of scaling degree at least 2 and

$$\left| \mathcal{R}^{(\geq 2)} \right|_{\frac{s}{32}, \frac{\varrho r}{4}} < r^2.$$

(iii) One has $\widetilde{\mathcal{M}} \circ \mathcal{T}^{(1)} = \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{P}} \circ \mathcal{T}^{(1)} = \widetilde{\mathcal{P}}$.

Proof. As usual we look for $\mathcal{T}^{(1)}$ as the time one flow of an Hamiltonian $\chi^{(1)}$. With $\mathcal{N} := \omega \cdot \mathcal{Y} + \sum_{\vec{j} \in \mathbb{Z}^2 \setminus \mathcal{S}_0} \Omega_{\vec{j}}(\lambda, \varepsilon) |a_{\vec{j}}|^2$, we have

$$\mathcal{K} \circ \mathcal{T}^{(1)} = \mathcal{N} + \{\mathcal{N}, \chi^{(1)}\} + \mathcal{K}^{(1)} \quad (8.2)$$

$$+ \widetilde{\mathcal{B}}_2(\chi^{(1)}; \mathcal{N}) + \widetilde{\mathcal{B}}_1(\chi^{(1)}; \mathcal{K}^{(1)}) \quad (8.3)$$

$$+ \mathcal{K}^{(\geq 2)} \circ \mathcal{T}^{(1)} \quad (8.4)$$

We choose $\chi^{(1)}$ to solve the homological equation $\{\mathcal{N}, \chi^{(1)}\} + \mathcal{K}^{(1)} = 0$. Thus

$$\text{setting } \mathcal{K}^{(1)} = \sum_{\substack{(\mathbf{j}, \ell, \sigma) \in \mathfrak{A}_3 \\ \sigma_i = \pm 1}} K_{\ell, \mathbf{j}}^{\sigma}(\lambda, \varepsilon) e^{i\theta \cdot \ell} a_{\vec{j}_1}^{\sigma_1} a_{\vec{j}_2}^{\sigma_2} a_{\vec{j}_3}^{\sigma_3}, \quad \Rightarrow \quad \chi^{(1)} = \sum_{\substack{(\mathbf{j}, \ell, \sigma) \in \mathfrak{A}_3 \\ \sigma_i = \pm 1}} \chi_{\ell, \mathbf{j}}^{\sigma}(\lambda, \varepsilon) e^{i\theta \cdot \ell} a_{\vec{j}_1}^{\sigma_1} a_{\vec{j}_2}^{\sigma_2} a_{\vec{j}_3}^{\sigma_3}$$

with

$$\chi_{\ell, \mathbf{j}}^{\sigma}(\lambda, \varepsilon) := \frac{-i K_{\ell, \mathbf{j}}^{\sigma}(\lambda, \varepsilon)}{\omega \cdot \ell + \sigma_1 \Omega_{\vec{j}_1}(\lambda, \varepsilon) + \sigma_2 \Omega_{\vec{j}_2}(\lambda, \varepsilon) + \sigma_3 \Omega_{\vec{j}_3}(\lambda, \varepsilon)}.$$

Since $\lambda \in \mathcal{C}$, the third order Melnikov conditions of Theorem 7.1 hold, thus we have

$$|\chi^{(1)}|_{\frac{s}{24}, \frac{\varrho r}{2}}^{\mathcal{C}} \leq C(s, \tau) \varepsilon^{-1} |\mathcal{K}^{(1)}|_{\frac{s}{16}, \frac{\varrho r}{2}} < \frac{r}{\sqrt{\varepsilon}}.$$

Thus choosing

$$0 < r \leq r_0 \leq c \sqrt{\varepsilon} \quad (8.5)$$

with c sufficiently small, Proposition 3.3 (ii) guarantees that $\chi^{(1)}$ generates a flow. We come to the terms of line (8.3). First we use the homological equation $\{\mathcal{N}, \chi^{(1)}\} + \mathcal{K}^{(1)} = 0$ to get that

$$\widetilde{\mathcal{B}}_2(\chi^{(1)}; \mathcal{N}) = \sum_{k \geq 2} \frac{\text{ad}(\chi^{(1)})^{k-1}[\{\chi^{(1)}, \mathcal{N}\}]}{k!} = \sum_{k \geq 1} \frac{\text{ad}(\chi^{(1)})^k[\mathcal{K}^{(1)}]}{(k+1)!},$$

and Proposition 3.3(iii) implies that $\widetilde{\mathcal{B}}_2(\chi^{(1)}; \mathcal{N})$, $\widetilde{\mathcal{B}}_1(\chi^{(1)}; \widetilde{\mathcal{H}}^{(1)})$ and $\mathcal{K}^{(2)} \circ \mathcal{T}^{(1)}$ have scaling degree at least 2 and fulfill the quantitative estimates

$$\left| \widetilde{\mathcal{B}}_2(\chi^{(1)}; \mathcal{N}) \right|_{\frac{s}{32}, \frac{\varrho r}{4}}^{\mathcal{C}} \leq r^2, \quad \left| \widetilde{\mathcal{B}}_1(\chi^{(1)}; \widetilde{\mathcal{H}}^{(1)}) \right|_{\frac{s}{32}, \frac{\varrho r}{4}}^{\mathcal{C}} \leq r^2, \quad \left| \mathcal{K}^{(2)} \circ \mathcal{T}^{(1)} \right|_{\frac{s}{32}, \frac{\varrho r}{4}}^{\mathcal{C}} \leq r^2.$$

To conclude the proof we show that $\{\widetilde{\mathcal{M}}, \chi^{(1)}\} = \{\widetilde{\mathcal{P}}, \chi^{(1)}\} = 0$. This follows since $\mathcal{K}^{(1)}$ commutes with $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{P}}$, hence its monomials fulfill the selection rules of Remark 5.2. By the explicit formula for $\chi^{(1)}$ it follows that the same selection rules hold for $\chi^{(1)}$. This concludes the proof. \square

Proof of Theorems 2.9 and 2.10. Given $s_0 > 0$, we fix L, ε_* as in Theorem (7.1), and r_0 to fulfill (4.23) and (8.5). Then Theorem 2.9 follows by applying the sequence of transformations $\mathcal{T}^{(1)}, \widehat{\mathcal{T}}, \mathcal{T}^{(0)}$ defined in Theorems 8.1, 7.1, 5.1 and setting $\varrho_0 = \varrho/4$.

The asymptotics of the frequencies claimed in Theorem 2.10 is proved in Theorem 7.1, more precisely in Corollary 7.5. \square

9 Proof of Theorem 1.3

In this last section we show how Theorem 2.9 implies Theorem 1.3. We start by fixing $s_0 > 0, p > 1$, a L -generic support set $\mathcal{S}_0, 0 < \varepsilon < \varepsilon_*$ (ε_* is given by Theorem 2.9) and $\lambda \in \mathcal{C}$ of (2.32). This fixes the torus $\mathbb{T}^d(\mathcal{S}_0, \lambda) \equiv \mathbb{T}^d(\mathcal{S}_0, I_m(\lambda))$. Note that the actions I_m are fixed by (2.12).

First we study the dynamics of the Hamiltonian (4.15) showing that $\mathcal{Y} = 0, a = 0$ is orbitally stable. More precisely we prove that there exist $K_0, T_0 > 0$, independent of r, ε s.t.

$$(\mathcal{Y}(0), \theta(0), a(0)) \in D\left(\frac{s_0}{64 \cdot 32}, \varrho_0 K_0 r\right) \implies (\mathcal{Y}(t), \theta(t), a(t)) \in D(s_0, r) \quad \forall |t| \leq T_0/r^2. \quad (9.1)$$

To prove 9.1 we apply the change of coordinates \mathcal{T} of Theorem 2.9. Recall that both \mathcal{T} and its inverse \mathcal{T}^{-1} map $D(s/32, \varrho_0 r) \rightarrow D(s, r)$ for any $0 < r \leq r_0$ and $\frac{s_0}{64} \leq s \leq s_0$. Denoting $(\mathcal{Y}, \theta, \mathbf{a}) = \mathcal{T}(\mathcal{Y}', \theta, \mathbf{a}')$, the Hamiltonian in the variables $(\mathcal{Y}', \theta, \mathbf{a}')$ is given by (2.33), and its equations of motion are

$$\begin{cases} \dot{\mathcal{Y}}' = -\partial_{\theta} \mathcal{R}^{(\geq 2)}(\mathcal{Y}', \theta, a', \bar{a}') \\ \dot{\theta} = \omega + \partial_{\mathcal{Y}} \mathcal{R}^{(\geq 2)}(\mathcal{Y}', \theta, a', \bar{a}') \\ i\dot{a}'_{\bar{j}} = \Omega_{\bar{j}} a'_{\bar{j}} + \partial_{\bar{a}'} \mathcal{R}^{(\geq 2)}(\mathcal{Y}', \theta, a', \bar{a}') \end{cases} \quad (9.2)$$

Then we prove the following bootstrap lemma:

Lemma 9.1. *Let $K_0 < \varrho_0$ and consider the system (9.2) with initial datum $(\mathcal{Y}'(0), \theta(0), \mathbf{a}'(0)) \in D(s_0/64, K_0 r)$. Assume that exists $T_0 > 0$ s.t. the quantities $\mathcal{J}(t) := \|a'(t)\|^2 + |\mathcal{Y}'(t)|_1$ and $\Theta(t) := \text{Im}|\theta(t)|$ fulfill*

$$\sup_{|t| \leq T_0 r^{-2}} \mathcal{J}(t) \leq \varrho_0^2 r^2, \quad \sup_{|t| \leq T_0 r^{-2}} \Theta(t) \leq \frac{s_0}{32}. \quad (9.3)$$

Then, provided K_0, T_0 are small enough (independently from r) one has

$$\sup_{|t| \leq T_0 r^{-2}} \mathcal{J}(t) \leq \frac{\varrho_0^2 r^2}{2}, \quad \sup_{|t| \leq T_0 r^{-2}} \Theta(t) \leq \frac{s_0}{40}. \quad (9.4)$$

Proof. In the course of the proof we drop the superscript $'$ from the variables. Consider first the dynamics of \mathcal{Y} . By the very definition of $|\cdot|_{s,r}$ (see (2.21) and (2.18)) we have

$$\sup_{(\mathcal{Y}, \theta, \mathbf{a}) \in D(s_0/32, \varrho_0 r)} \left| \partial_{\theta} \mathcal{R}^{(\geq 2)}(\mathcal{Y}, \theta, \mathbf{a}) \right|_1 \leq \varrho_0^2 r^2 |\mathcal{R}|_{\frac{s_0}{32}, \varrho_0 r} \stackrel{(2.34)}{\leq} C_{\mathcal{R}} \varrho_0^2 r^4.$$

Thus for any $|t| \leq T_0 r^{-2}$ one estimates

$$\begin{aligned} |\mathcal{Y}(t)|_1 &\leq |\mathcal{Y}(0)|_1 + |t| \sup_{|t| \leq T_0/r^2} \left| \partial_{\theta} \mathcal{R}^{(\geq 2)}(\mathcal{Y}(t), \theta(t), a(t), \bar{a}(t)) \right|_1 \\ &\leq K_0^2 r^2 + T_0 C_{\mathcal{R}} \varrho_0^2 r^2 \leq \frac{\varrho_0^2 r^2}{4} \end{aligned} \quad (9.5)$$

provided $K_0^2 \leq \varrho_0^2/8$ and $T_0 < 1/8C_{\mathcal{R}}$. Similarly, using the equations of motion for a and the estimate

$$\sup_{(\mathcal{Y}, \theta, \mathbf{a}) \in D(s_0/32, \varrho_0 r)} \left| \partial_{\bar{a}} \mathcal{R}^{(\geq 2)}(\mathcal{Y}, \theta, \mathbf{a}) \right|_1 \leq |\mathcal{R}|_{\frac{s_0}{32}, \varrho_0 r} \varrho_0 r \stackrel{(5.20)}{\leq} C_{\mathcal{R}} \varrho_0 r^3,$$

for any $|t| \leq T_0 r^{-2}$ one gets

$$\|a(t)\| \leq \|a(0)\| + |t| \sup_{|t| \leq T_0/r^2} \left| \partial_{\bar{a}} \mathcal{R}^{(\geq 2)}(\mathcal{Y}(t), \theta(t), a(t), \bar{a}(t)) \right| \leq K_0 r + T_0 C_{\mathcal{R}} \varrho_0 r \leq \frac{\varrho_0 r}{2} \quad (9.6)$$

provided $K_0 \leq \varrho_0/4$ and T_0 as above.

Finally the equation for θ and the estimate $\sup_{(\mathcal{Y}, \theta, a) \in D(s_0/32, \varrho_0 r)} \left| \partial_{\mathcal{Y}} \mathcal{R}^{(\geq 2)}(\mathcal{Y}, \theta, a) \right|_{\infty} \leq C_{\mathcal{R}} r^2$ give

$$|\operatorname{Im} \theta(t)| \leq |\operatorname{Im} \theta(0)| + T_0 C_{\mathcal{R}} \leq \frac{s_0}{40} \quad (9.7)$$

provided $T_0 < s_0/C_{\mathcal{R}} 10^3$ as above. Estimates (9.5), (9.6) and (9.7) imply (9.4). \square

This bootstrap lemma and the properties of \mathcal{T} and \mathcal{T}^{-1} imply that, taking $K_0 \leq \min\left(\frac{\varrho_0}{4}, \frac{\varrho_0^2}{8}\right)$ and $T_0 \leq \min\left(\frac{s_0}{C_{\mathcal{R}} 10^3}, \frac{1}{8C_{\mathcal{R}}}\right)$, one has

$$\begin{aligned} (\mathcal{Y}(0), \theta(0), a(0)) \in D\left(\frac{s_0}{64 \cdot 32}, \varrho_0 K_0 r\right) &\Rightarrow (\mathcal{Y}'(0), \theta(0), a'(0)) \in D\left(\frac{s_0}{64}, K_0 r\right) \\ &\Rightarrow (\mathcal{Y}'(t), \theta(t), a'(t)) \in D(s_0/32, \varrho_0 r) \Rightarrow (\mathcal{Y}(t), \theta(t), a(t)) \in D(s_0, r) \end{aligned}$$

which is valid for all $|t| \leq T_0/r^2$; this proves (9.1).

Finally we show how the stability result (9.1) implies Theorem 1.3. Thus fix $\delta_0 > 0$ so that

$$c_{\star}^{-1} \delta_0 < K_0 \varrho_0 r_0 \quad (9.8)$$

where c_{\star} is defined in Proposition 4.2. Take $0 < \delta < \delta_0$ and $u \in \mathcal{V}_{\delta}$. Then by Proposition 4.2(ii), $u(0) \in \mathcal{V}_{\delta}$ implies that $(\mathcal{Y}(0), \theta(0), \mathbf{a}(0)) \in D\left(\frac{s_0}{64 \cdot 32}, c_{\star}^{-1} \delta\right)$. Choose now $r = c_{\star}^{-1} (K_0 \varrho_0)^{-1} \delta$ (note that (9.8) guarantees that $r \leq r_0$ is fulfilled) and apply (9.1) to get that $(\mathcal{Y}(t), \theta(t), \mathbf{a}(t)) \in D(s_0, r)$ for $|t| \leq T_0/r^2$. Applying again Proposition 4.2 one gets $u(t) \in \mathcal{V}_{c_{\star}(s_0)r} \equiv \mathcal{V}_{C\delta}$.

A Proof of Lemma 2.5

First we consider (2.28). Let us consider \mathcal{S}_0 as a point $\vec{\mathbf{m}} = (\mathbf{m}_1, \dots, \mathbf{m}_d) \in \mathbb{C}^d$. With this identification B_R is the set of integer vectors in the cube $[-R, R]^d$ with ordered components. An intersection $\mathcal{S} \cap \mathcal{C}$ is a point

$$v_{i,j,l}(\vec{\mathbf{m}}) = (x, y) \in \mathbb{C}^2 : \quad x = \mathbf{m}_l, \quad y = \sqrt{(\mathbf{m}_l - \mathbf{m}_i)(\mathbf{m}_j - \mathbf{m}_l)}$$

for some triple of indexes $i, j, l \in \{1, \dots, d\}$. So fix such indexes and consider the map

$$\Gamma : \mathbb{C}^d \rightarrow \mathbb{C}^{d+2}, \quad \vec{\mathbf{m}} \rightarrow (\vec{\mathbf{m}}, v_{i,j,l}(\vec{\mathbf{m}})).$$

Now we restrict our attention to $\vec{\mathbf{m}} \in \mathbb{Z}^d$ and we wish to avoid to choose $\vec{\mathbf{m}}$ so that $v_{i,j,l}(\vec{\mathbf{m}}) \in \mathbb{Z}^2$. Our first remark is that if $\vec{\mathbf{m}} \in B_R$ then we have that $|v_{i,j,l}(\vec{\mathbf{m}})| \leq 3R$. In the square $\sup_i |\mathbf{m}_i| \leq R$ there are $(2R)^d$ integer points and we claim that once we remove all points with $v_{i,j,l}(\vec{\mathbf{m}}) \in \mathbb{Z}^2$, we still have many integer points left. So in \mathbb{C}^{d+2} let us count the integer valued points $(\vec{\mathbf{m}}, v)$ such that $\sup_i |\mathbf{m}_i| \leq R$, $|v| \leq 3R$ and $v = v_{i,j,l}(\vec{\mathbf{m}})$. We can fix all the \mathbf{m}_s with $s \neq l$ in any way we want (we get $(2R)^{d-1}$ points) then for each such choice, \mathbf{m}_l is the x -coordinate of one of the integer points on the circle with diameter $(\mathbf{m}_i, 0), (\mathbf{m}_j, 0)$. But on a circumference of radius R the number of integer points are of the order R^{δ} with δ arbitrarily small, so the bad points in \mathbb{C}^{d+2} are of order $R^{d-1+\delta}$. Then the pre-image through Γ of these points is composed at most of $R^{d-1+\delta}$ points. Since the total amount of points is of order R^d we have lots of good points, i.e. points $\vec{\mathbf{m}} \in \mathbb{Z}^d$ such that the intersection $v_{i,j,l}(\vec{\mathbf{m}})$ is not an integer. When we play this game for all the triples of indexes (i, j, l) we still get many points. Regarding the intersection $\mathcal{C}_{i,j} \cap \mathcal{C}_{l,s}$ the reasoning is the same. An intersection is a point $v = v_{i,j,l,s}(\vec{\mathbf{m}})$ which solves the equation

$$(x - \mathbf{m}_i)(x - \mathbf{m}_j) + y^2 = 0, \quad (x - \mathbf{m}_l)(x - \mathbf{m}_s) + y^2 = 0.$$

So we consider the map

$$\tilde{\Gamma} : \mathbb{C}^d \rightarrow \mathbb{C}^{d+2}, \quad \vec{\mathbf{m}} \rightarrow (\vec{\mathbf{m}}, v_{i,j,l,s}(\vec{\mathbf{m}})).$$

As before we make the ansatz that $\sup_i |\mathbf{m}_i| \leq R$. Then we have that $|v_{i,j,l,s}(\vec{\mathbf{m}})| \leq 3R$. Now we can fix all the points \mathbf{m}_q for $q \neq s$ (R^{d-1} points) and for each such choice, fix v as one of the R^{δ} integer points on the circle with diameter $(\mathbf{m}_i, 0), (\mathbf{m}_j, 0)$. Then \mathbf{m}_s is fixed by the last equation (which is linear for \mathbf{m}_s). In conclusion we have again $R^{d-1+\delta}$ bad points. Here we have to play such game for all the quadruples (i, j, l, s) .

Finally we consider condition (2.29). Fix ℓ with $|\ell| \leq L$. Then the equation $\sum \ell_i x_i = 0$ defines a hyperplane in \mathbb{C}^d , and in the hypercube of size R there are $\sim R^{d-1}$ integer bad points to exclude. Taking the union over all the possible hyperplanes gives $\sim L^d R^{d-1}$ bad points to exclude. In conclusion the number of bad points in B_R is $\leq C(d)R^{d-1+\delta}$, so

$$\lim_{R \rightarrow \infty} \frac{|G_R|}{|B_R|} \geq 1 - \lim_{R \rightarrow \infty} \frac{C(d)R^{d-1+\delta}}{|B_R|} = 1.$$

B Proof from Section 3

We begin by defining projections of Hamiltonians. More precisely we define projections on *any subspace* defined as the closure in $\mathcal{A}_{s,r}$ of the monomials $e^{i\ell \cdot \theta} \mathcal{Y}^\ell a^\alpha \bar{a}^\beta$ satisfying some rule $\ell, l, \alpha, \beta \in \mathbb{I}$ with $\mathbb{I} \subset \mathbb{Z}^d \times \mathbb{N}^d \times \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0} \times \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0}$:

$$\Pi_{\mathbb{I}} h := \sum_{\substack{\ell \in \mathbb{Z}^d, l \in \mathbb{N}^d, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}^2 \setminus \mathcal{S}_0} \\ \ell, l, \alpha, \beta \in \mathbb{I}}} h_{\alpha, \beta, l, \ell} e^{i\ell \cdot \theta} \mathcal{Y}^\ell a^\alpha \bar{a}^\beta. \quad (\text{B.1})$$

In particular we denote by Π_N the projection on trigonometric polynomials of degree $\leq N$, i.e.

$$\Pi_N h := \sum_{\substack{\ell \in \mathbb{Z}^d, |\ell| \leq N \\ l, \alpha, \beta}} h_{\alpha, \beta, l, \ell} e^{i\ell \cdot \theta} \mathcal{Y}^\ell a^\alpha \bar{a}^\beta. \quad (\text{B.2})$$

Proposition B.1. *For every $s, r > 0$ the following holds true:*

(i) Continuity: *all projections $\Pi_{\mathbb{I}}$ are continuous, namely*

$$|\Pi_{\mathbb{I}} h|_{s,r}^{\mathcal{O}} \leq |h|_{s,r}^{\mathcal{O}}.$$

(ii) Smoothing: *for any $0 < s' < s$ one has*

$$|\Pi_N h|_{s,r}^{\mathcal{O}} \leq e^{s'N} |h|_{s-s',r}^{\mathcal{O}}, \quad |(\mathbb{I} - \Pi_N)h|_{s,r}^{\mathcal{O}} \leq e^{-s'N} |h|_{s+s',r}^{\mathcal{O}}$$

(iii) Partial ordering: *if we have*

$$|f_{\alpha, \beta, l, \ell}|_{\mathbb{C}}^{\mathcal{O}} \leq |h_{\alpha, \beta, l, \ell}|_{\mathbb{C}}^{\mathcal{O}}, \quad \forall \alpha, \beta, l, \ell$$

and $\{h_{\alpha, \beta, l, \ell}\}$ are the coefficients of the Taylor-Fourier expansion of a function $h \in \mathcal{A}_{s,r}^{\mathcal{O}}$, then there exists a unique function $f \in \mathcal{A}_{s,r}^{\mathcal{O}}$ whose Fourier expansion has coefficients $\{f_{\alpha, \beta, l, \ell}\}$ and such that

$$|f|_{s,r}^{\mathcal{O}} \leq |h|_{s,r}^{\mathcal{O}}$$

(iv) Graded Poisson algebra: *Given $f, g \in \mathcal{A}_{s,r}^{\mathcal{O}}$, for any $0 < s' < s$ and $0 < r' < r$ one has*

$$|\{f, g\}|_{s',r'}^{\mathcal{O}} \leq \delta^{-1} C(s) |f|_{s,r}^{\mathcal{O}} |g|_{s,r}^{\mathcal{O}},$$

where $\delta := \min\left(1 - \frac{r'}{r}, 1 - \frac{s'}{s}\right)$. Moreover if f and g are monomials of the form $e^{i\ell \cdot \theta} \mathcal{Y}^\ell a^\alpha \bar{a}^\beta$ one has that $\deg(\{f, g\}) = \deg(f) + \deg(g)$ and $\pi(\{f, g\}) = \pi(f) + \pi(g)$.

The proposition can be proved by adapting the methods of [BBP13].

B.1 Proof of Proposition 3.13

In order to prove (3.13) we have to control

$$\begin{aligned} & |\{(\langle m_1 \rangle^{N_1+k} + \langle n \rangle^{N_2}) H_{m_1, m_2, n}^{\pm, \ell}\}_{|\ell| > c \langle m_1 \rangle}\}_{s'}^{\mathcal{O}} \\ & \stackrel{(3.11)}{\leq} |\{ \langle m_1 \rangle^{N_1+k} |H_{m_1, m_2, n}^{\pm, \ell}\}_{|\ell| > c \langle m_1 \rangle}\}_{s'}^{\mathcal{O}} + |\{ \langle n \rangle^{N_2} |H_{m_1, m_2, n}^{\pm, \ell}\}_{s'}^{\mathcal{O}}|. \end{aligned} \quad (\text{B.3})$$

In the first summand since $|\ell| > c \langle m_1 \rangle$ we have

$$|\{ \langle m_1 \rangle^{N_1+k} |H_{m_1, m_2, n}^{\pm, \ell}\}_{|\ell| > c \langle m_1 \rangle}\}_{s'}^{\mathcal{O}} \leq c^{-k} |\{ \langle m_1 \rangle^{N_1} |\ell|^k |H_{m_1, m_2, n}^{\pm, \ell}\}_{s'}^{\mathcal{O}}|.$$

Now we note that

$$|\ell|^k e^{-(s-s')|\ell|} \leq \frac{k!}{(s-s')^k} = k! \delta^{-k}$$

so that

$$|\{\langle m_1 \rangle^{N_1+k} H_{m_1, m_2, n}^{\pm, \ell}\}_{|\ell| > c \langle m_1 \rangle}|_{s'}^{\mathcal{O}} \leq k! c^{-k} \delta^{-k} |\{\langle m_1 \rangle^{N_1} |H_{m_1, m_2, n}^{\pm, \ell}|\}_s^{\mathcal{O}} \stackrel{(3.11)}{\leq} k! c^{-k} \delta^{-k} [H]_{s, -(N_1, N_2)}^{\mathcal{O}}.$$

In the second summand of (B.3) we just have

$$|\{\langle n \rangle^{N_2} |H_{m_1, m_2, n}^{\pm, \ell}|\}_{s'}^{\mathcal{O}} \leq [H]_{s', -(N_1, N_2)}^{\mathcal{O}} \leq [H]_{s, -(N_1, N_2)}^{\mathcal{O}}.$$

B.2 Proof of Lemma 3.14

Proof of Lemma 3.14 (i). By 3.11 we have

$$[\{F, G\}]_{s', -\mathbf{N}-\mathbf{M}}^{\mathcal{O}} \leq [\{F, G\}]_{s', -(N_1+M_1, 0)}^{\mathcal{O}} + [\{F, G\}]_{s', -(0, N_2+M_2)}^{\mathcal{O}}. \quad (\text{B.4})$$

We start by discussing the first term. Following Lemma 3.13 we divide $F = F^B + F^R$ and $G = G^B + G^R$, so that

$$\{F, G\} = \{F^B, G^B\} + \{F, G^R\} + \{F^R, G^B\}.$$

We claim that

$$[\{F^B, G^B\}]_{s', -(N_1+M_1, 0)}^{\mathcal{O}} \leq 4^{\max(M_1, N_1)} [F^B]_{s', -\mathbf{N}}^{\mathcal{O}} [G^B]_{s', -\mathbf{M}}^{\mathcal{O}} \quad (\text{B.5})$$

$$[\{F, G^R\}]_{s', -(N_1+M_1, 0)}^{\mathcal{O}} + [\{F^R, G^B\}]_{s', -(N_1+M_1, 0)}^{\mathcal{O}} \leq C_{N_1, M_1} \delta^{-N_1-M_1} [F]_s^{\mathcal{O}} [G]_s^{\mathcal{O}} \quad (\text{B.6})$$

Let us start by proving (B.5): we divide

$$\begin{aligned} \{F^B, G^B\} &= \{F^B, G^B\}^{\text{diag}} + \{F^B, G^B\}^{\text{out}} + \{F^B, G^B\}^{\text{line}} \\ &= \{F_B^{\text{diag}}, G_B^{\text{diag}}\} + \{F_B^{\text{out}}, G_B^{\text{out}}\} + \{F_B^{\text{diag}}, G_B^{\text{out}}\} + \{F_B^{\text{out}}, G_B^{\text{diag}}\} + \{F_B^{\text{line}}, G_B^{\text{line}}\} \end{aligned}$$

where we put

$$\begin{aligned} F_B^{\text{diag}}(\lambda; \theta, a, \bar{a}) &= \sum_{\substack{m_1, m_2, \ell, n \neq 0 \\ |\ell| \leq c \langle m_1 \rangle}} F_{m_1, m_2, n}^{-, \ell}(\lambda) e^{i\theta \cdot \ell} a_{(m_1, n)} \bar{a}_{(m_2, n)}, \\ F_B^{\text{out}}(\lambda; \theta, a, \bar{a}) &= \sum_{\substack{m_1, m_2, \ell, n > 0 \\ |\ell| \leq c \langle m_1 \rangle}} F_{m_1, m_2, n}^{+, \ell}(\lambda) e^{i\theta \cdot \ell} a_{(m_1, n)} a_{(m_2, -n)} + c.c., \end{aligned}$$

and similarly for G . Then

$$\begin{aligned} \{F_B^{\text{diag}}, G_B^{\text{diag}}\} &= i \sum_{m_1, m_2, \ell, n \neq 0} A_{m_1, m_2, n}^{\ell}(\lambda) e^{i\theta \cdot \ell} a_{(m_1, n)} \bar{a}_{(m_2, n)}, \\ A_{m_1, m_2, n}^{\ell}(\lambda) &:= \left(\sum_{\substack{m_3, \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c \langle m_1 \rangle, |\ell_2| \leq c \langle m_3 \rangle}} F_{m_1, m_3, n}^{-, \ell_1}(\lambda) G_{m_3, m_2, n}^{-, \ell_2}(\lambda) - G_{m_1, m_3, n}^{-, \ell_1}(\lambda) F_{m_3, m_2, n}^{-, \ell_2}(\lambda) \right) \end{aligned}$$

and by momentum conservation $m_1 - m_3 + \pi(\ell_1) = 0$, $m_3 - m_2 + \pi(\ell_2) = 0$, thus $m_1 - m_2 + \pi(\ell) = 0$. By definition of $[\cdot]_{s, -\mathbf{N}}^{\mathcal{O}}$, we need to compute

$$\left| \{\langle m_1 \rangle^{N_1+M_1} A_{m_1, m_2, n}^{\ell}(\lambda)\}_{s'}^{\mathcal{O}} \right| \quad (\text{B.7})$$

We note that

$$\begin{aligned} \langle m_1 \rangle^{N_1+M_1} |A_{m_1, m_2, n}^{\ell}(\lambda)| &\leq \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c \langle m_1 \rangle}} \langle m_1 \rangle^{N_1} \left(\frac{\langle m_1 \rangle}{\langle m_3 \rangle} \right)^{M_1} |F_{m_1, m_3, n}^{-, \ell_1}(\lambda)| \langle m_3 \rangle^{M_1} |G_{m_3, m_2, n}^{-, \ell_2}(\lambda)| \quad (\text{B.8}) \\ &+ \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c \langle m_1 \rangle}} \langle m_1 \rangle^{M_1} |G_{m_1, m_3, n}^{-, \ell_1}(\lambda)| \left(\frac{\langle m_1 \rangle}{\langle m_3 \rangle} \right)^{N_1} \langle m_3 \rangle^{N_1} |F_{m_3, m_2, n}^{-, \ell_2}(\lambda)| \end{aligned}$$

Now we remark that since $m_1 - m_3 = -\pi(\ell_1)$, one has

$$\langle m_1 \rangle - |\pi(\ell_1)| \leq \langle m_3 \rangle \leq \langle m_1 \rangle + |\pi(\ell_1)|. \quad (\text{B.9})$$

Moreover since $|\ell_1| \leq c\langle m_1 \rangle$ with $c^{-1} = 2 \max(|m_i|)$ this implies $2|\pi(\ell_1)| \leq \langle m_1 \rangle$ so that

$$\frac{1}{2} \leq \frac{\langle m_1 \rangle}{\langle m_3 \rangle} \leq 2. \quad (\text{B.10})$$

In conclusion we may bound

$$\begin{aligned} (\text{B.7}) &\leq 2^{M_1} \left| \left\{ \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle}} \langle m_1 \rangle^{N_1} |F_{m_1, m_3, n}^{-, \ell_1}(\lambda)| \langle m_3 \rangle^{M_1} |G_{m_3, m_2, n}^{-, \ell_2}(\lambda)| \right\}_{s'} \right. \\ &\quad \left. + 2^{N_1} \left| \left\{ \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle}} \langle m_1 \rangle^{M_1} |G_{m_1, m_3, n}^{-, \ell_1}(\lambda)| \langle m_3 \rangle^{N_1} |F_{m_3, m_2, n}^{-, \ell_2}(\lambda)| \right\}_{s'} \right| \right. \\ &\leq C_{M_1, N_1} [F^B]_{s', -(N_1, 0)}^{\mathcal{O}} [G^B]_{s', -(M_1, 0)}^{\mathcal{O}} \leq C_{M_1, N_1} [F^B]_{s', -\mathbf{N}}^{\mathcal{O}} [G^B]_{s', -\mathbf{M}}^{\mathcal{O}}, \end{aligned}$$

by using the algebra property (3.9) and Remark 3.11. The term $\{F_B^{\text{out}}, G_B^{\text{out}}\}$ is treated exactly in the same way. Now we analyze $\{F_B^{\text{diag}}, G_B^{\text{out}}\}$. An explicit computation shows that

$$\{F_B^{\text{diag}}, G_B^{\text{out}}\} = -i \sum_{m_1, m_2, \ell, n > 0} \left(\sum_{\substack{m_3, \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle, |\ell_2| \leq c\langle m_2 \rangle}} F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{+, \ell_2} + G_{m_1, -m_3, n}^{+, \ell_1} F_{m_2, -m_3, -n}^{-, \ell_2} \right) e^{i\theta \cdot \ell} a_{(m_1, n)} a_{(m_2, -n)}$$

Let us consider the equivalent of (B.8), namely

$$\begin{aligned} &\langle m_1 \rangle^{N_1 + M_1} \left| \sum_{\substack{m_3, \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle, |\ell_2| \leq c\langle m_2 \rangle}} F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{+, \ell_2} + G_{m_1, -m_3, n}^{+, \ell_1} F_{m_2, -m_3, -n}^{-, \ell_2} \right| \quad (\text{B.11}) \\ &\leq \sum_{\substack{m_3, \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle, |\ell_2| \leq c\langle m_2 \rangle}} \left(\frac{\langle m_1 \rangle}{\langle m_3 \rangle} \right)^{M_1} \langle m_1 \rangle^{N_1} |F_{m_1, m_3, n}^{-, \ell_1}(\lambda)| \langle m_3 \rangle^{M_1} |G_{m_3, m_2, n}^{+, \ell_2}(\lambda)| \\ &\quad + \sum_{\substack{m_3, \ell_1 + \ell_2 = \ell \\ |\ell_1| \leq c\langle m_1 \rangle, |\ell_2| \leq c\langle m_2 \rangle}} \left(\frac{\langle m_1 \rangle}{\langle m_2 \rangle} \right)^{N_1} \langle m_1 \rangle^{M_1} |G_{m_1, -m_3, n}^{+, \ell_1}(\lambda)| \langle m_2 \rangle^{N_1} |F_{m_2, -m_3, -n}^{-, \ell_2}(\lambda)| \end{aligned}$$

The first line is treated as in (B.8). For the second we note that $\langle m_2 \rangle - |\pi(\ell_2)| \leq \langle m_3 \rangle \leq \langle m_2 \rangle + |\pi(\ell_2)|$ so that since

$$\frac{\langle m_1 \rangle}{\langle m_2 \rangle} = \frac{\langle m_1 \rangle \langle m_3 \rangle}{\langle m_3 \rangle \langle m_2 \rangle}$$

we have

$$\frac{1}{4} \leq \frac{\langle m_1 \rangle}{\langle m_2 \rangle} \leq 4.$$

The term $\{F_B^{\text{out}}, G_B^{\text{diag}}\}$ can be estimated in an analogous way, we skip the details. This proves (B.5).

Regarding (B.6) let us suppose first that in (B.8) we have $|\ell_1| > c\langle m_1 \rangle$, then

$$\begin{aligned} &\langle m_1 \rangle^{N_1 + M_1} \left| \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell \\ |\ell_1| > c\langle m_1 \rangle}} F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{-, \ell_2} - G_{m_1, m_3, n}^{-, \ell_1} F_{m_3, m_2, n}^{-, \ell_2} \right| \quad (\text{B.12}) \\ &\leq c^{-N_1 - M_1} \sum_{\substack{m_3 \\ \ell_1 + \ell_2 = \ell}} |\ell_1|^{M_1 + N_1} (|F_{m_1, m_3, n}^{-, \ell_1}(\lambda)| |G_{m_3, m_2, n}^{-, \ell_2}(\lambda)| + |G_{m_1, m_3, n}^{-, \ell_1}(\lambda)| |F_{m_3, m_2, n}^{-, \ell_2}(\lambda)|) \end{aligned}$$

We have proved that

$$\begin{aligned}
& \left| \{ \langle m_1 \rangle^{N_1+M_1} \sum_{\substack{m_3, \ell_1+\ell_2=\ell \\ |\ell_1| > c\langle m_1 \rangle}} (F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{-, \ell_2}(\lambda) - G_{m_1, m_3, n}^{-, \ell_1}(\lambda) F_{m_3, m_2, n}^{-, \ell_2}(\lambda)) \} \right|_{s'}^{\mathcal{O}} \\
& \leq \frac{1}{c^{N_1+M_1}} \left| \{ |\ell_1|^{N_1+M_1} F_{m_1, m_3, n}^{-, \ell_1}(\lambda) \} \right|_{s'}^{\mathcal{O}} \left| \{ G_{m_3, m_2, n}^{-, \ell_2}(\lambda) \} \right|_{s'}^{\mathcal{O}} \\
& \quad + \frac{1}{c^{N_1+M_1}} \left| \{ |\ell_1|^{N_1+M_1} G_{m_1, m_3, n}^{-, \ell_1}(\lambda) \} \right|_{s'}^{\mathcal{O}} \left| \{ F_{m_3, m_2, n}^{-, \ell_2}(\lambda) \} \right|_{s'}^{\mathcal{O}}
\end{aligned} \tag{B.13}$$

Now we remark that

$$\sup_{\ell} |\ell|^{M_1+N_1} e^{-(s-s')|\ell|} \leq C_{M_1, N_1} \delta^{-M_1-N_1}$$

so that

$$\left| \{ |\ell_1|^{N_1+M_1} F_{m_1, m_3, n}^{-, \ell_1}(\lambda) \} \right|_{s'}^{\mathcal{O}} \leq C \delta^{-M_1-N_1} \left| \{ F_{m_1, m_3, n}^{-, \ell_1}(\lambda) \} \right|_s^{\mathcal{O}},$$

and (B.5) follows.

Consider finally the case $|\ell_1| \leq c\langle m_1 \rangle$, $|\ell_2| \geq c\langle m_3 \rangle$. Then (B.10) holds and

$$\begin{aligned}
& \left| \langle m_1 \rangle^{N_1+M_1} \sum_{\substack{m_3 \\ \ell_1+\ell_2=\ell \\ |\ell_2| > c\langle m_3 \rangle}} F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{-, \ell_2}(\lambda) - G_{m_1, m_3, n}^{-, \ell_1}(\lambda) F_{m_3, m_2, n}^{-, \ell_2}(\lambda) \right| \\
& \leq 2^{N_1+M_1} c^{-N_1-M_1} \sum_{\substack{m_3 \\ \ell_1+\ell_2=\ell}} |\ell_2|^{M_1+N_1} (|F_{m_1, m_3, n}^{-, \ell_1}(\lambda)| |G_{m_3, m_2, n}^{-, \ell_2}(\lambda)| + |G_{m_1, m_3, n}^{-, \ell_1}(\lambda)| |F_{m_3, m_2, n}^{-, \ell_2}(\lambda)|)
\end{aligned} \tag{B.14}$$

and we proceed as in the previous case. We are left with the second summand in (B.4). We claim

$$[\{F, G\}]_{s', -(0, N_2+M_2)}^{\mathcal{O}} \leq 2[F]_{s', -(0, N_2)}^{\mathcal{O}} [G]_{s', -(0, M_2)}^{\mathcal{O}} \tag{B.15}$$

indeed

$$\begin{aligned}
& \left| \{ \langle n \rangle^{N_2+M_2} (\sum_{\substack{m_3, \ell_1+\ell_2=\ell \\ |\ell_1| \leq c\langle m_1 \rangle}} F_{m_1, m_3, n}^{-, \ell_1} G_{m_3, m_2, n}^{-, \ell_2}(\lambda) - G_{m_1, m_3, n}^{-, \ell_1}(\lambda) F_{m_3, m_2, n}^{-, \ell_2}(\lambda)) \} \right|_{s'}^{\mathcal{O}} \\
& \leq 2 \left| \{ \langle n \rangle^{N_2} F_{m_1, m_3, n}^{-, \ell_1}(\lambda) \} \right|_{s'}^{\mathcal{O}} \left| \{ \langle n \rangle^{M_2} G_{m_3, m_2, n}^{-, \ell_2}(\lambda) \} \right|_{s'}^{\mathcal{O}},
\end{aligned} \tag{B.16}$$

and the result follows.

Proof of Lemma 3.14 (ii). The proof is standard, we repeat it here for completeness. We need to estimate

$$\mathcal{G}_1(F; G) = \sum_{k \geq 1} \frac{\text{ad}(F)^k(G)}{k!}. \tag{B.17}$$

By the algebra property (3.9) we have

$$|\text{ad}(F)^k(G)|_{s'}^{\mathcal{O}} \leq (C_0 |F|_{s'}^{\mathcal{O}})^k |G|_{s'}^{\mathcal{O}}.$$

Then by using (3.14) we have

$$\begin{aligned}
[\text{ad}(F)^k(G)]_{s', -\mathbf{N}}^{\mathcal{O}} &= [\{F, \text{ad}(F)^{k-1}(G)\}]_{s', -\mathbf{N}}^{\mathcal{O}} \leq C_{\mathbf{N}} ([F]_{s', -\mathbf{N}}^{\mathcal{O}} |\text{ad}(F)^{k-1}(G)|_{s'}^{\mathcal{O}} + \delta^{-N_1} |F|_s^{\mathcal{O}} |\text{ad}(F)^{k-1}(G)|_s^{\mathcal{O}}) \\
&\leq C_{\mathbf{N}} (C_0 |F|_s^{\mathcal{O}})^{k-1} ([F]_{s', -\mathbf{N}}^{\mathcal{O}} + \delta^{-N_1} |F|_s^{\mathcal{O}}) |G|_s^{\mathcal{O}}
\end{aligned} \tag{B.18}$$

Proof of Lemma 3.14 (iii). Let us prove inductively that

$$[\text{ad}(F)^k(G)]_{s', -k\mathbf{N}}^{\mathcal{O}} \leq \hat{C}_{k, \mathbf{N}} (\delta^{-N_1} [F]_{s', -\mathbf{N}}^{\mathcal{O}})^k |G|_s^{\mathcal{O}} \tag{B.19}$$

We use (3.14); writing $\text{ad}(F)^k(G) = \{\text{ad}(F)^{k-1}(G), F\}$ we get

$$\begin{aligned}
[\text{ad}(F)^k(G)]_{s', -k\mathbf{N}}^{\mathcal{O}} &\leq C_{k\mathbf{N}} \left([\text{ad}(F)^{k-1}(G)]_{s', -(k-1)\mathbf{N}}^{\mathcal{O}} |F|_{s', -\mathbf{N}}^{\mathcal{O}} + \delta^{-kN_1} |\text{ad}(F)^{k-1}(G)|_s |F|_s \right) \\
&\leq C_{k\mathbf{N}} \left(|F|_{s', -\mathbf{N}}^{\mathcal{O}} \hat{C}_{k-1, \mathbf{N}} (\delta^{-N_1} [F]_{s', -\mathbf{N}}^{\mathcal{O}})^{k-1} (k-1) |G|_s^{\mathcal{O}} + \delta^{-kN_1} C_0^k |G|_s |F|_s^k \right).
\end{aligned} \tag{B.20}$$

The result follows provided

$$C_{k\mathbf{N}} \left(\hat{C}_{k-1, \mathbf{N}} (k-1) \delta^{-(k-1)N_1} + \delta^{-kN_1} C_0^k \right) \leq \hat{C}_{k, \mathbf{N}} k \delta^{-kN_1}$$

which in turn follows by setting

$$C_{k\mathbf{N}} \left(\hat{C}_{k-1, \mathbf{N}} (k-1) + C_0^k \right) \leq k \hat{C}_{k, \mathbf{N}}.$$

Now setting $s_1 = s - (s - s')/2$ we have

$$[\widetilde{\mathcal{V}}_i(F; G)]_{s', -i\mathbf{N}}^{\mathcal{O}} \leq [(\text{ad}F)^i G \circ \mathcal{T}_F]_{s', -i\mathbf{N}}^{\mathcal{O}} \leq \mathbf{i}(\hat{C}_{i, \mathbf{N}} [F]_{s_1, -\mathbf{N}}^{\mathcal{O}})^i (2\delta)^{-iN_1} |G \circ \mathcal{T}_F|_{s_1}^{\mathcal{O}}$$

and the result follows.

B.3 Proof of Lemma 3.18

First note that $\widetilde{\mathcal{V}}_i(F; G)$ is linear in the second argument. Thus

$$\widetilde{\mathcal{V}}_i(F; G) = \widetilde{\mathcal{V}}_i(F; G^{\text{hor}}) + \widetilde{\mathcal{V}}_i(F; G^{\text{mix}}) = \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}}) + \widetilde{\mathcal{V}}_i(F; G^{\text{mix}}) + (\widetilde{\mathcal{V}}_i(F; G^{\text{hor}}) - \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}})).$$

By Remark 3.17, $\widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}})$ is horizontal. We define $\widetilde{\mathcal{V}}_i(F; G)^{\text{hor}} := \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}})$, which by Lemma 3.14(iii) belongs to $\mathcal{Q}_{s, -i\alpha}^{\text{hor}}$ and fulfills the claimed estimate.

Now consider $\widetilde{\mathcal{V}}_i(F; G^{\text{mix}})$. We prove that $\widetilde{\mathcal{V}}_i(F; G^{\text{mix}}) \in \mathcal{Q}_{s', -\bar{\mathbf{z}}}^{\mathcal{O}}, \forall 0 < s' < s$. We proceed as in the proof of Lemma 3.14(iii). We prove inductively

$$[\text{ad}(F)^k (G^{\text{mix}})]_{s', -\bar{\mathbf{z}}}^{\mathcal{O}} \leq (\hat{C}_{\bar{\mathbf{z}}} |F|_s^{\mathcal{O}})^k \left([G^{\text{mix}}]_{s, -\bar{\mathbf{z}}}^{\mathcal{O}} + k\delta^{-2} |G|_s^{\mathcal{O}} \right). \quad (\text{B.21})$$

Indeed:

$$\begin{aligned} [\text{ad}(F)^k (G^{\text{mix}})]_{s', -\bar{\mathbf{z}}}^{\mathcal{O}} &\leq [\{F, \text{ad}(F)^{k-1} (G^{\text{mix}})\}]_{s', -\bar{\mathbf{z}}}^{\mathcal{O}} \\ &\stackrel{(3.14)}{\leq} C_{\bar{\mathbf{z}}} (|F|_{s'}^{\mathcal{O}} [\text{ad}(F)^{k-1} (G^{\text{mix}})]_{s', -\bar{\mathbf{z}}}^{\mathcal{O}} + \delta^{-2} |F|_s^{\mathcal{O}} |\text{ad}(F)^{k-1} (G)|_s^{\mathcal{O}}) \\ &\stackrel{(\text{B.21})}{\leq} C_{\bar{\mathbf{z}}} |F|_s^{\mathcal{O}} \left(\max(C_0, \hat{C}_{\bar{\mathbf{z}}}) |F|_s^{\mathcal{O}} \right)^{k-1} \left([G^{\text{mix}}]_{s, -\bar{\mathbf{z}}}^{\mathcal{O}} + (k-1)\delta^{-2} |G|_s^{\mathcal{O}} + \delta^{-2} |G|_s^{\mathcal{O}} \right) \end{aligned}$$

and the desired bound follows by taking $\hat{C}_{\bar{\mathbf{z}}} = \eta^{-1} = \max(C_0, C_{\bar{\mathbf{z}}})$, where C_0 is the constant in (3.9). Then substituting we get

$$[\widetilde{\mathcal{V}}_i(F; G^{\text{mix}})]_{s', -\bar{\mathbf{z}}}^{\mathcal{O}} \leq C_{\bar{\mathbf{z}}} \delta^{-2} \frac{[G^{\text{mix}}]_{s, -\bar{\mathbf{z}}}^{\mathcal{O}} \left(\eta^{-1} |F|_s^{\mathcal{O}} \right)^i}{1 - \eta^{-1} |F|_s^{\mathcal{O}}}. \quad (\text{B.22})$$

Finally consider $\widetilde{\mathcal{V}}_i(F; G^{\text{hor}}) - \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}})$. Since the series are summable

$$\widetilde{\mathcal{V}}_i(F; G^{\text{hor}}) - \widetilde{\mathcal{V}}_i(F^{\text{hor}}; G^{\text{hor}}) = \sum_{k \geq \mathbf{i}} \frac{\text{ad}(F)^k (G^{\text{hor}}) - \text{ad}(F^{\text{hor}})^k (G^{\text{hor}})}{k!}.$$

Now remark that $F = F^{\text{hor}} + F^{\text{mix}}$, so $\text{ad}(F^{\text{hor}} + F^{\text{mix}})^k (G^{\text{hor}}) - \text{ad}(F^{\text{hor}})^k (G^{\text{hor}})$ equals

$$\sum_{h=0}^{k-1} \text{ad}(F)^h \text{ad}(F^{\text{mix}}) \text{ad}(F^{\text{hor}})^{k-h-1} (G^{\text{hor}}) \quad (\text{B.23})$$

Now by (3.14)

$$\begin{aligned} [\text{ad}(F^{\text{mix}}) \text{ad}(F^{\text{hor}})^{k-h-1} (G^{\text{hor}})]_{s', -\bar{\mathbf{z}}} &\leq C_{\bar{\mathbf{z}}} \left([F^{\text{mix}}]_{s', -\bar{\mathbf{z}}} \text{ad}(F^{\text{hor}})^{k-h-1} (G^{\text{hor}})|_{s'} + \delta^{-2} |F|_s |\text{ad}(F^{\text{hor}})^{k-h-1} (G^{\text{hor}})|_s \right) \\ &\leq C_{\bar{\mathbf{z}}} ([F^{\text{mix}}]_{s', -\bar{\mathbf{z}}} + \delta^{-2} |F|_s) (C_0 |F|_s)^{k-h-1} |G|_s. \end{aligned} \quad (\text{B.24})$$

Now we use (B.21) with $k \rightsquigarrow h$ and $G^{\text{mix}} \rightsquigarrow \tilde{G}^{\text{mix}} := \text{ad}(F^{\text{mix}})\text{ad}(F^{\text{hor}})^{k-h-1}(G^{\text{hor}})$. We get

$$\begin{aligned} [\text{ad}(F)^h \tilde{G}^{\text{mix}}]_{s', -\bar{\mathbf{z}}} &\leq (\eta^{-1}|F|_s)^h \left([\tilde{G}^{\text{mix}}]_{s, -\bar{\mathbf{z}}} + k\delta^{-2}|\tilde{G}^{\text{mix}}|_s \right) \\ &\stackrel{\text{(B.24)}}{\leq} (\eta^{-1}|F|_s)^h \left(C_{\bar{\mathbf{z}}}[F^{\text{mix}}]_{s', -\bar{\mathbf{z}}} + \delta^{-2}|F|_s(C_0|F|_s)^{k-h-1}|G^{\text{hor}}|_s \right. \\ &\quad \left. + k\delta^{-2}(C_0|F|_s)^{k-h}|G^{\text{hor}}|_s \right) \\ &\leq (\eta^{-1}|F|_s)^{k-1} \left(C_{\bar{\mathbf{z}}}[F^{\text{mix}}]_{s', -\bar{\mathbf{z}}} + (k+1)\delta^{-2}|F|_s \right) |G^{\text{hor}}|_s \end{aligned}$$

Then it follows easily that

$$[\tilde{\mathcal{U}}_1(F; G^{\text{hor}}) - \tilde{\mathcal{U}}_1(F^{\text{hor}}; G^{\text{hor}})]_{s', -\bar{\mathbf{z}}} \leq 2 \left(C_{\bar{\mathbf{z}}}[F^{\text{mix}}]_{s', -\bar{\mathbf{z}}} + \delta^{-2}|F|_s \right) |G^{\text{hor}}|_s \frac{(\eta^{-1}|F|_s)^{i-1}}{1 - \eta^{-1}|F|_s}. \quad (\text{B.25})$$

We define $\tilde{\mathcal{U}}_1(F; G)^{\text{mix}} := \tilde{\mathcal{U}}_1(F; G^{\text{mix}}) + (\tilde{\mathcal{U}}_1(F; G^{\text{hor}}) - \tilde{\mathcal{U}}_1(F^{\text{hor}}; G^{\text{hor}}))$. Estimates (B.22), (B.25) show that $\tilde{\mathcal{U}}_1(F; G)^{\text{mix}} \in \mathcal{Q}_{s, -\bar{\mathbf{z}}}^{\mathcal{O}}$.

B.4 Proof of Lemma 3.19

The lemma follows using the same strategy of the proof of Lemma 3.18, but replacing the estimate of the Poisson bracket $\{F, G\}$ by the following one: let $\mathbf{N} = (N_1, N_2), \mathbf{M} = (M_1, M_2) \in \mathbb{N}^2$, $F \in \mathcal{Q}_{s, -\mathbf{N}}^{\mathcal{O}}$, $G \in \mathcal{Q}_{s, -\mathbf{M}}^{\mathcal{O}}$. Then $\{F, G\} \in \mathcal{Q}_{s, -\mathbf{O}}^{\mathcal{O}}$, where $\mathbf{O} = (\min(N_1, M_1), \min(N_2, M_2))$ with the quantitative estimate

$$[\{F, G\}]_{s, -\mathbf{O}}^{\mathcal{O}} \leq C_0 [F]_{s, -\mathbf{N}}^{\mathcal{O}} [G]_{s, -\mathbf{M}}^{\mathcal{O}}. \quad (\text{B.26})$$

Such an estimate follows easily by exploiting the algebra property of the norm, see also Remark 3.8 and Remark 3.10.

C Proof of Proposition 7.7

Recall that, by Theorem 7.1, the frequencies $\Omega_{\bar{\mathbf{j}}}(\lambda, \varepsilon)$ of Hamiltonian (8.1) have the form (2.35), (2.36). Expanding $\Omega_{\bar{\mathbf{j}}}(\lambda, \varepsilon)$ in Taylor series in powers of ε we get that

$$\omega(\lambda) \cdot \ell + \sigma_1 \Omega_{\bar{\mathbf{j}}_1}(\lambda, \varepsilon) + \sigma_2 \Omega_{\bar{\mathbf{j}}_2}(\lambda, \varepsilon) + \sigma_3 \Omega_{\bar{\mathbf{j}}_3}(\lambda, \varepsilon) = \mathbf{K}_{\bar{\mathbf{j}}, \ell}^{\sigma} + \varepsilon \mathbf{F}_{\bar{\mathbf{j}}, \ell}^{\sigma}(\lambda) + \varepsilon^2 \mathbf{G}_{\bar{\mathbf{j}}, \ell}^{\sigma}(\lambda, \varepsilon), \quad (\text{C.1})$$

where $\mathbf{K}_{\bar{\mathbf{j}}, \ell}^{\sigma}$ is defined in (6.9) and $\mathbf{F}_{\bar{\mathbf{j}}, \ell}^{\sigma}(\lambda)$ is defined in (6.10). Recall that given any $0 < \gamma < \gamma_1$, the functions above are well defined provided that (7.39) holds. From now on we assume (7.40) which clearly implies (7.39).

We wish to prove that the set of $\lambda \in \mathcal{O}_1$ such that

$$|\omega \cdot \ell + \sigma_1 \Omega_{\bar{\mathbf{j}}_1}(\lambda, \varepsilon) + \sigma_2 \Omega_{\bar{\mathbf{j}}_2}(\lambda, \varepsilon) + \sigma_3 \Omega_{\bar{\mathbf{j}}_3}(\lambda, \varepsilon)| \geq \varepsilon \frac{\gamma}{\langle \ell \rangle^{\tau}}, \quad \forall (\bar{\mathbf{j}}, \ell, \sigma) \in \mathfrak{A}_3 \setminus \mathfrak{R}_3 \quad (\text{C.2})$$

has positive measure. To do this, we will show that the set of $\lambda \in \mathcal{C}_c$ such that (C.2) holds has positive measure; this clearly implies that also \mathcal{C} has positive measure.

We discuss two cases separately, recall that \mathbf{M}_0 is defined in (6.15).

Case $|\ell| \leq 4\mathbf{M}_0$. Then $\sup_{\lambda} |\mathbf{F}_{\bar{\mathbf{j}}, \ell}^{\sigma}(\lambda)| \leq 8\mathbf{M}_0$. Assume first that $\mathbf{K}_{\bar{\mathbf{j}}, \ell}^{\sigma} \in \mathbb{Z} \setminus \{0\}$, then for ε sufficiently small one has

$$|(\text{C.1})| \geq |\mathbf{K}_{\bar{\mathbf{j}}, \ell}^{\sigma}| - \varepsilon 8\mathbf{M}_0 - \varepsilon^2 3\mathbf{M}_0 \geq \frac{1}{2}$$

hence, for such ℓ 's, (C.2) is trivially fulfilled $\forall \lambda \in \mathcal{O}_1$.

If instead $\mathbf{K}_{\bar{\mathbf{j}}, \ell}^{\sigma} = 0$, we use Lemma 6.4 and (7.40) to estimate for any $\lambda \in \mathcal{C}_c$

$$|(\text{C.1})| \geq \varepsilon \gamma_1 - \varepsilon^2 3\mathbf{M}_0 \geq \frac{\varepsilon \gamma_1}{2}.$$

Thus so far we have proved that for all $|\ell| \leq 4\mathbf{M}_0$ and for any $\lambda \in \mathcal{C}_c$, (C.2) hold.

Case $|\ell| > 4\mathbf{M}_0$. For $1 \leq i \leq \mathbf{d}$, $0 \leq k \leq \mathbf{d}$ define the functions

$$\widehat{\mathbf{F}}_{i,k}(\lambda) = \begin{cases} \varepsilon\mu_i(\lambda) & \text{if } k = 0 \\ \varepsilon\mu_{i,k}^+(\lambda) & \text{if } 1 \leq i < k \leq \mathbf{d} \\ \varepsilon\mu_{i,k}^-(\lambda) & \text{if } 1 \leq k < i \leq \mathbf{d} \\ 0 & \text{if } 1 \leq i = k \leq \mathbf{d} \end{cases} \quad (\text{C.3})$$

Consider an expression of the form

$$\begin{aligned} & \omega(\lambda) \cdot \ell + K \\ & + \eta_1 \widehat{\mathbf{F}}_{i_1, k_1}(\lambda) + \eta_2 \widehat{\mathbf{F}}_{i_2, k_2}(\lambda) + \eta_3 \widehat{\mathbf{F}}_{i_3, k_3}(\lambda) \\ & + \eta_{11} \frac{\Theta_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2} + \eta_{12} \frac{\Theta_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2} + \eta_{13} \frac{\Theta_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2} \\ & + \eta_{21} \frac{\Theta_{m_1, n_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2 + \langle n_1 \rangle^2} + \eta_{22} \frac{\Theta_{m_2, n_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2 + \langle n_2 \rangle^2} + \eta_{23} \frac{\Theta_{m_3, n_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2 + \langle n_3 \rangle^2} \\ & + \eta_{31} \frac{\varpi_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle} + \eta_{32} \frac{\varpi_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle} + \eta_{33} \frac{\varpi_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle} \end{aligned} \quad (\text{C.4})$$

where $K \in \mathbb{Z}$, $m_i \in \mathbb{Z}$, $n_i \in \mathbb{Z} \setminus \{0\}$ while $\eta_r, \eta_{r_1 r_2} \in \{-1, 0, 1\}$. First we have the following

Lemma C.1. *If $|K| \geq 4|\ell| \max_{1 \leq i \leq \mathbf{d}} (m_i^2)$ then for any $m_i \in \mathbb{Z}$, $n_i \in \mathbb{Z} \setminus \{0\}$, $\eta_r, \eta_{r_1 r_2} \in \{-1, 0, 1\}$, one has $|(C.4)| \geq \mathbf{M}_0$.*

Proof. We have

$$\begin{aligned} |(C.4)| & \geq |K| - |\omega(\lambda)| |\ell| - \sum_{r=1}^3 |\widehat{\mathbf{F}}_{i_r, k_r}|_{\mathbb{C}}^{\mathcal{O}_1} - \sum_{r=1}^3 \frac{|\Theta_{m_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle^2} - \sum_{r=1}^3 \frac{|\Theta_{m_r, n_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle^2 + \langle n_r \rangle^2} \\ & - \sum_{r=1}^3 \frac{|\varpi_{m_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle} \\ & \geq 4 \max_{1 \leq i \leq \mathbf{d}} (m_i^2) |\ell| - \max_{1 \leq i \leq \mathbf{d}} (m_i^2) |\ell| - \varepsilon |\ell| - 3\varepsilon \mathbf{M}_0 - 3\varepsilon^2 \mathbf{M}_0 \geq \mathbf{M}_0. \end{aligned} \quad (\text{C.5})$$

□

Next we have the following result:

Lemma C.2. *Fix arbitrary $K \in \mathbb{Z}$, $m_i \in \mathbb{Z}$, $n_i \in \mathbb{Z} \setminus \{0\}$, $\eta_r, \eta_{r_1 r_2} \in \{-1, 0, 1\}$. For any $\alpha > 0$ then*

$$\text{meas}(\{\lambda \in \mathcal{O}_1 : |(C.4)| < \varepsilon \alpha\}) < 16\alpha |\ell|^{-1}. \quad (\text{C.6})$$

Proof. Let $\hat{\ell} := \ell/|\ell|$ and let us denote the expression in (C.4) as $f(\lambda)$. We have

$$\begin{aligned} \inf_{\substack{\lambda \neq \mu \in \mathcal{O}_1 \\ \lambda - \mu = |\lambda - \mu| \hat{\ell}}} \frac{|f(\lambda) - f(\mu)|}{|\lambda - \mu|} & \geq \varepsilon |\ell| - \sum_{r=1}^3 |\widehat{\mathbf{F}}_{i_r, k_r}|_{\mathbb{C}}^{\mathcal{O}_1} - \sum_{r=1}^3 \frac{|\Theta_{m_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle^2} - \sum_{r=1}^3 \frac{|\Theta_{m_r, n_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle^2 + \langle n_r \rangle^2} \\ & - \sum_{r=1}^3 \frac{|\varpi_{m_r}(\cdot, \varepsilon)|_{\mathbb{C}}^{\mathcal{O}_1}}{\langle m_r \rangle} \\ & \geq \varepsilon |\ell| - 3\varepsilon \mathbf{M}_0 - 3\varepsilon^2 \mathbf{M}_0 \geq \frac{\varepsilon}{8} |\ell|. \end{aligned} \quad (\text{C.7})$$

In order to prove our claim we first perform an orthogonal change of variables so that $\hat{\ell}$ becomes the first basis vector. Formula (C.7) amounts to

$$|f(x, \lambda_2, \dots, \lambda_n) - f(y, \lambda_2, \dots, \lambda_n)| \geq \frac{\varepsilon}{8} |x - y|$$

for all $x \neq y$ such that $(x, \lambda_2, \dots, \lambda_n), (y, \lambda_2, \dots, \lambda_n) \in \mathcal{O}_1$.

We consider the map $F : \lambda \mapsto \lambda' = (f(\lambda), \lambda_2, \dots, \lambda_n)$ which maps \mathcal{O}_1 bijectively to some set B . F is a lipeomorphism and its inverse has Lipschitz constant $< 8\varepsilon^{-1} |\ell|^{-1}$. In the λ' variables the volume of the set of $\mu \in B$ such that $|\mu_1| < \varepsilon \alpha$ can be estimated by $2\varepsilon \alpha$ hence on \mathcal{O}_1 it can be estimated by $16\alpha |\ell|^{-1}$. □

We will employ such lemma to prove the following result:

Lemma C.3. *There exist $\gamma_2, \tau > 0$ such that for $0 < \gamma < \gamma_2$, the set*

$$\mathcal{C}_\star = \mathcal{C}_\star(\gamma, \varepsilon) := \left\{ \lambda \in \mathcal{O}_1 : |(C.4)| \geq \varepsilon \frac{\gamma}{\langle \ell \rangle^\tau}, \quad \forall |\ell| \geq 4M_0 \right\} \quad (C.8)$$

has positive measure. More precisely

$$\text{meas}(\mathcal{O}_1 \setminus \mathcal{C}_\star) \leq C_\star \gamma$$

for some positive C_\star independent of ε .

Proof. We prove such a claim by finite induction on the number of η_{r_1, r_2} different from 0. More precisely for every $0 \leq n \leq 9$ we shall show that for γ small enough, there exist a positive increasing sequence τ_n and a sequence of nested sets $\mathcal{C}^n = \mathcal{C}^n(\gamma, \tau_n)$ such that provided

$$|\eta_{1,1}| + \dots + |\eta_{3,3}| = n,$$

then

$$\text{meas}(\mathcal{O}_1 \setminus \mathcal{C}^0) \leq C\gamma, \quad \text{meas}(\mathcal{C}^n \setminus \mathcal{C}^{n+1}) \leq C\gamma \quad (C.9)$$

with some $C > 0$ independent of ε . Moreover for $\lambda \in \mathcal{C}^n$, $|\ell| \geq 4M_0$, one has

$$|(C.4)| \geq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_n}}. \quad (C.10)$$

Then Lemma C.3 follows by taking $\mathcal{C}_\star := \mathcal{C}^9$, $\tau = \tau_9$ and $C_\star = 10C$. $\gamma_2 \leq \gamma_1$ is fixed in order to ensure that $10C\gamma < \text{meas}(\mathcal{O}_1)$, so that the measure of \mathcal{C}_\star is positive.

Case $n = 0$. Given $K \in \mathbb{Z}$, $\mathbf{i} = (i_1, i_2, i_3) \in \{1, \dots, \mathbf{d}\}^3$, $\mathbf{k} = (k_1, k_2, k_3) \in \{0, \dots, \mathbf{d}\}^3$, $\ell \in \mathbb{Z}^{\mathbf{d}}$ with $|\ell| \geq 4M_0$, $\eta = (\eta_1, \eta_2, \eta_3) \in \{-1, 0, 1\}^3$, we define the set

$$G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell}^0(\gamma, \tau_0) := \left\{ \lambda \in \mathcal{O}_1 : |(C.4)| \leq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_0}} \text{ and } \eta_{r_1 r_2} = 0 \quad \forall r_1, r_2 \right\}.$$

If $|K| \geq 4 \max_{1 \leq i \leq \mathbf{d}} (m_i^2) |\ell|$ then by Lemma C.1 we have $G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell}^0(\gamma, \tau_0) = \emptyset$, provided that $M_0 \geq \varepsilon\gamma/2$.

If $|K| \leq 4 \max_{1 \leq i \leq \mathbf{d}} (m_i^2) |\ell|$, then by Lemma C.2 with $\alpha = \gamma \langle \ell \rangle^{-\tau_0}$ we have

$$\text{meas}(G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell}^0(\gamma, \tau_0)) \leq \frac{16\gamma}{\langle \ell \rangle^{\tau_0+1}}.$$

Taking the union over all the possible values of $K, \mathbf{i}, \mathbf{k}, \eta, \ell$ one gets that

$$\text{meas} \left(\bigcup_{\substack{|\ell| \geq 4M_0, \mathbf{i}, \mathbf{k}, \eta \\ |K| \leq 4 \max_i (m_i^2) |\ell|}} G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell}^0(\gamma, \tau_0) \right) \leq C(\mathbf{d}) \gamma \sum_{|\ell| \geq 4M_0} \frac{1}{\langle \ell \rangle^{\tau_0}} \leq C\gamma,$$

which is finite provided $\tau_0 \geq \mathbf{d} + 1$. Letting

$$\mathcal{C}^0 := \mathcal{O}_1 \setminus \bigcup_{\substack{|\ell| \geq 4M_0, \mathbf{i}, \mathbf{k}, \eta \\ |K| \leq 4 \max_i (m_i^2) |\ell|}} G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell}^0(\gamma, \tau_0)$$

one has clearly that $\text{meas}(\mathcal{O}_1 \setminus \mathcal{C}^0) \leq C\gamma$ and for $\lambda \in \mathcal{C}^0$ we have

$$|\omega(\lambda) \cdot \ell + K + \eta_1 \widehat{\mathbf{F}}_{i_1, k_1}(\lambda) + \eta_2 \widehat{\mathbf{F}}_{i_2, k_2}(\lambda) + \eta_3 \widehat{\mathbf{F}}_{i_3, k_3}(\lambda)| \geq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_0}} \quad (C.11)$$

for any choice of $\ell, K, \mathbf{i}, \mathbf{k}, \eta$. This proves the inductive step for $n = 0$.

Case $n \rightsquigarrow n + 1$. Assume that (C.10) holds for any possible choice of $\eta_{11}, \dots, \eta_{33}$ s.t. $|\eta_{11}| + \dots + |\eta_{33}| \leq n$ for some $(\tau_r)_{r=1}^n$. We prove now the step $n + 1$. Suppose first that

$$\exists m_i \text{ s.t. } |m_i| \geq \langle \ell \rangle^{\tau_n} \text{ and } |\eta_{1i}| + |\eta_{2i}| + |\eta_{3i}| \neq 0. \quad (C.12)$$

W.l.o.g. assume it is m_3 . Then

$$\left| \frac{\Theta_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2} \right| + \left| \frac{\Theta_{m_3, n_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2 + \langle n_3 \rangle^2} \right| + \left| \frac{\varpi_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle} \right| \leq \frac{\mathbf{M}_0 \varepsilon^2}{\langle \ell \rangle^{\tau_n}}$$

and by the inductive assumption and (7.40), for any $\lambda \in \mathcal{C}^n$

$$\begin{aligned} |(C.4)| &\geq \left| \omega(\lambda) \cdot \ell + K + \eta_1 \widehat{\mathbf{F}}_{i_1, k_1}(\lambda) + \eta_2 \widehat{\mathbf{F}}_{i_2, k_2}(\lambda) + \eta_3 \widehat{\mathbf{F}}_{i_3, k_3}(\lambda) \right. \\ &\quad + \eta_{11} \frac{\Theta_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2} + \eta_{12} \frac{\Theta_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2} + \eta_{21} \frac{\Theta_{m_1, n_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2 + \langle n_1 \rangle^2} + \eta_{22} \frac{\Theta_{m_2, n_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2 + \langle n_2 \rangle^2} \\ &\quad \left. + \eta_{31} \frac{\varpi_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle} + \eta_{32} \frac{\varpi_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle} \right| - \frac{\mathbf{M}_0 \varepsilon^2}{\langle \ell \rangle^{\tau_n}} \\ &\geq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_n}} - \frac{\mathbf{M}_0 \varepsilon^2}{\langle \ell \rangle^{\tau_n}} \geq \frac{\varepsilon \gamma}{2 \langle \ell \rangle^{\tau_n}} \geq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_{n+1}}} \end{aligned}$$

provided $\tau_{n+1} \geq \tau_n + 1$.

If (C.12) is not fulfilled, assume that

$$\exists n_i \text{ s.t. } |n_i|^2 \geq \langle \ell \rangle^{\tau_n} \text{ and } |\eta_{2i}| \neq 0. \quad (C.13)$$

W.l.o.g. assume it is n_3 . Then

$$\left| \frac{\Theta_{m_3, n_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2 + \langle n_3 \rangle^2} \right| \leq \frac{\mathbf{M}_0 \varepsilon^2}{\langle \ell \rangle^{\tau_n}}$$

and again by the inductive assumption and (7.40)

$$\begin{aligned} |(C.4)| &\geq \left| \omega(\lambda) \cdot \ell + K + \eta_1 \widehat{\mathbf{F}}_{i_1, k_1}(\lambda) + \eta_2 \widehat{\mathbf{F}}_{i_2, k_2}(\lambda) + \eta_3 \widehat{\mathbf{F}}_{i_3, k_3}(\lambda) \right. \\ &\quad + \eta_{11} \frac{\Theta_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2} + \eta_{12} \frac{\Theta_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2} + \eta_{13} \frac{\Theta_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle^2} \\ &\quad + \eta_{21} \frac{\Theta_{m_1, n_1}(\lambda, \varepsilon)}{\langle m_1 \rangle^2 + \langle n_1 \rangle^2} + \eta_{22} \frac{\Theta_{m_2, n_2}(\lambda, \varepsilon)}{\langle m_2 \rangle^2 + \langle n_2 \rangle^2} \\ &\quad \left. + \eta_{31} \frac{\varpi_{m_1}(\lambda, \varepsilon)}{\langle m_1 \rangle} + \eta_{32} \frac{\varpi_{m_2}(\lambda, \varepsilon)}{\langle m_2 \rangle} + \eta_{33} \frac{\varpi_{m_3}(\lambda, \varepsilon)}{\langle m_3 \rangle} \right| - \frac{\mathbf{M}_0 \varepsilon^2}{\langle \ell \rangle^{\tau_n}} \\ &\geq \frac{\varepsilon \gamma}{2 \langle \ell \rangle^{\tau_n}} \geq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_{n+1}}} \end{aligned}$$

provided $\tau_{n+1} \geq \tau_n + 1$. If (C.12) and (C.13) are not fulfilled, then one has $|m_i|, |n_i|^2 \leq \langle \ell \rangle^{\tau_n}$ for all the m_i, n_i that appear non-trivially in (C.4). Furthermore, we can safely assume that $|K| \leq 4 \max_i m_i^2 |\ell|$, otherwise Lemma C.1 ensures that (C.10) is met for any $\lambda \in \mathcal{O}_1$. Thus we are left with a finite number of cases and we can impose a finite number of Melnikov conditions.

For $K \in \mathbb{Z}$, $\mathbf{i} = (i_1, i_2, i_3) \in \{1, \dots, \mathbf{d}\}^3$, $\mathbf{k} = (k_1, k_2, k_3) \in \{0, \dots, \mathbf{d}\}^3$, $\ell \in \mathbb{Z}^{\mathbf{d}}$ with $|\ell| \geq 4\mathbf{M}_0$, $\eta = (\eta_1, \eta_2, \eta_3) \in \{-1, 0, 1\}^3$, $\mathbf{m} = (m_1, m_2, m_3)$, $\mathbf{n} = (n_1, n_2, n_3)$ define the set

$$G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell, \mathbf{m}, \mathbf{n}}^{m+1}(\gamma, \tau_{n+1}) := \left\{ \lambda \in \mathcal{C}^n : |(C.4)| \leq \frac{\varepsilon \gamma}{\langle \ell \rangle^{\tau_{n+1}}}, \quad |\eta_{11}| + \dots + |\eta_{33}| = n + 1 \right\}.$$

By Lemma C.2 with $\alpha = \gamma / \langle \ell \rangle^{\tau_{n+1}}$ we have

$$\text{meas} \left(G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell, \mathbf{m}, \mathbf{n}}^{m+1}(\gamma, \tau_{n+1}) \right) \leq \frac{16\gamma}{\langle \ell \rangle^{\tau_{n+1} + 1}},$$

and taking the union over the possible values of $K, \mathbf{i}, \mathbf{k}, \eta, \ell, \mathbf{m}, \mathbf{n}$ one gets that

$$\text{meas} \left(\bigcup_{\substack{|\ell| \geq 4\mathbf{M}_0 \\ \mathbf{i}, \mathbf{k}, \eta}} \bigcup_{\substack{|m_i|, |n_i|^2 \leq \langle \ell \rangle^{\tau_n} \\ |K| \leq 4 \max_i m_i^2 |\ell|}} G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell, \mathbf{m}, \mathbf{n}}(\gamma, \tau_{n+1}) \right) \leq C(\mathbf{d}) \gamma \sum_{|\ell| \geq 4\mathbf{M}_0} \frac{\langle \ell \rangle^{1+9\tau_n/2}}{\langle \ell \rangle^{\tau_{n+1} + 1}},$$

which converges provided $\tau_{n+1} \geq d + 1 + 9\tau_n/2$. Thus we define the set

$$\mathcal{C}^{n+1} := \mathcal{C}^n \setminus \bigcup_{\substack{|\ell| \geq 4\mathfrak{M}_0 \\ \mathbf{i}, \mathbf{k}, \eta}} \bigcup_{\substack{|m_i|, |n_i|^2 \leq \langle \ell \rangle \tau_n \\ |K| \leq 4 \max_i (m_i^2) |\ell|}} G_{K, \mathbf{i}, \mathbf{k}, \eta, \ell, \mathbf{m}, \mathbf{n}}^{n+1}(\gamma, \tau_{n+1})$$

which fulfills (C.9) and (C.10). \square

We conclude the section with the proof of Proposition 7.7.

Proof of Proposition 7.7. Fix τ as in Lemma C.3. By definition for $\gamma < \gamma_2$, any $\lambda \in \mathcal{C}_c \cap \mathcal{C}_*$ fulfills (C.2) for any admissible $(\mathbf{j}, \ell, \boldsymbol{\sigma})$. By taking γ sufficiently small, we can ensure that $\text{meas}(\mathcal{C}_c \cap \mathcal{C}_*) > \text{meas}(\mathcal{C}_c)/2$. This fixes γ_0 . \square

D A crash course on polynomial rings and algebraic extensions

The aim of this section is to recall the reader⁸ some basic algebraic properties of polynomial rings and algebraic extensions which are used repeatedly in the paper.

Basic properties of polynomial rings. Let R be a commutative ring with unit element. We denote by $R[\lambda]$ the polynomial ring in λ over R , which is the set of formal polynomials in λ with coefficients in R . Its elements are of the form $\sum_j a_j \lambda^j$. $R[\lambda]$ is a commutative ring with unit element. Similarly one defines the ring of polynomials in the n -variables $\lambda_1, \dots, \lambda_n$ over R , denoted by $R[\lambda_1, \dots, \lambda_n]$, whose elements are of the form $\sum a_{i_1 i_2 \dots i_n} \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n}$.

The following result is very well known:

Lemma D.1. *If R is an integral domain⁹, then so is $R[\lambda_1, \dots, \lambda_n]$.*

Let R be an integral domain. The characteristic of R is the smallest positive integer such that $n \cdot 1 = 0$, where $n \cdot 1$ stands for $1 + 1 + \dots + 1$ n - times. In case $n \cdot 1$ is never 0, then R has characteristic 0.

If R is an integral domain it is possible to construct its field of quotients F and R has characteristic 0 if and only if F contains the field \mathbb{Q} of rational numbers.

In an integral domain we can speak about *irreducible elements*. More precisely an element $a \in R$, $a \neq 1$ will be called *irreducible* if $a = bc$ with $b, c \in R$ implies that one of b or c must be invertible in R , that is a *unit*. Two elements b, c are *associated* if $b = uc$ with u invertible.

Definition D.2. *An integral domain R is a unique factorization domain (UFD) if*

- (a) *Every nonzero element in R is either a unit or can be written as the product of a finite number of irreducible elements of R .*
- (b) *The decomposition in part (a) is unique up to the order and associates of the irreducible elements.*

The ring \mathbb{Z} is the simplest UFD. A main theorem is that if R is a UFD so is $R[\lambda]$ thus:

Example D.3. *The ring $\mathbb{Z}[\lambda_1, \dots, \lambda_d]$ of polynomials in the variables $\lambda_1, \dots, \lambda_d$ with coefficients in \mathbb{Z} is a UFD (its units are ± 1).*

Given a UFD R let F be its field of quotients. We can then consider $R[\lambda]$ to be a sub-ring of $F[\lambda]$. In particular given any polynomial $f(\lambda) \in F[\lambda]$, then $f(\lambda) = (f_0(\lambda)/a)$, where $f_0(\lambda) \in R[\lambda]$ and $a \in R$. It is natural to ask if a polynomial irreducible in $R[\lambda]$ is still irreducible when it is considered as a polynomial in the larger ring $F[\lambda]$.

Lemma D.4. *If $f(\lambda) \in R[\lambda]$ is irreducible as an element of $R[\lambda]$, then it is irreducible as an element of $F[\lambda]$. Conversely, if $f(\lambda) \in R[\lambda]$ is both primitive¹⁰ and irreducible as an element of $F[\lambda]$, it is also irreducible as an element of $R[\lambda]$.*

⁸and the authors!

⁹An integral domain is a commutative ring with an identity $1 \neq 0$ with no zero-divisors. That is $ab = 0 \Rightarrow a = 0$ or $b = 0$. The ring \mathbb{Z} is an integral domain.

¹⁰ $f(\lambda) \in R[\lambda]$ is said to be primitive if the GCD of its coefficients is 1. Every polynomial $f(\lambda) \in R[\lambda]$ can be decomposed as the product of a primitive polynomial and the GCD of its coefficients, such decomposition being unique.

We want to apply this to $R = \mathbb{Z}[\lambda_1, \dots, \lambda_d]$ with field of quotients $\mathbb{Q}(\lambda_1, \dots, \lambda_d)$ the rational functions with rational coefficients.

Remark D.5. We identify $\mathbb{Z}[\lambda_1, \dots, \lambda_d]$ with the ring $\mathbb{Z}[\lambda_1, \dots, \lambda_d][t]$, namely with the ring of polynomial in the variable t with coefficients in the ring $\mathbb{Z}[\lambda_1, \dots, \lambda_d]$.

Next we recall some basic facts about extension fields that is $F \subset E$ two fields. We begin with the following definition:

Definition D.6. If E is a field extension of F , then an element a of E is called an algebraic element over F if there exists some non-zero polynomial $f(t) \in F[t]$ such that $f(a) = 0$.

The set of elements of E which are algebraic over F form also a field that is are closed under sum difference multiplication and inverse. There is also an extension \bar{F} algebraic over F and algebraically closed that is the only irreducible polynomials are the linear ones, thus every polynomial $f(t) \in F[t]$ of degree n factors in \bar{F} through its n roots (with possible multiplicities). So when we speak of the roots of a polynomial $f(t) \in F[t]$ we think of elements of \bar{F} .

Given $a \in E$ an algebraic element over F , we define the set

$$I_a := \{f(t) \in F[t] : f(a) = 0\} .$$

This is an ideal of $F[t]$.

Lemma D.7. $F[t]$ is a principal ideal domain, namely every ideal of $F[t]$ is generated by a unique monic polynomial $p_a \in F[t]$, which is called the minimal polynomial of a :

$$I_a \equiv \{f(t) p_a(t) : f(t) \in F[t]\} .$$

Clearly the minimal polynomial is the monic polynomial of least degree in I_a , and it is irreducible over F .

Let $f(t) \in F[t]$ be irreducible, then the lemma above implies that f is the minimal polynomial of every root of f in E .

Lemma D.8. Let F be a field with characteristic 0. An irreducible polynomial $f(t) \in F[t]$ has no multiple roots.

Proof. Recall that to any polynomial $f(t) = \sum_{j \geq 0} a_j t^j$, we can associate its formal derivative $f'(t) = \sum_{j \geq 0} j a_j t^{j-1}$. If $\deg f = 1$, the statement is trivial, thus let $\deg f > 1$. Assume that $f(t)$ has multiple roots. Then f and f' have a common root $a \in \bar{F}$. Since f is the minimal polynomial of a , then f/f' . But since $\deg f' < \deg f$, then $f' = 0$, hence $\deg f = 1$, contradiction. \square

Lemma D.9. Let $f(t) \in F[t]$ be irreducible. For any $a \in F$, $f(t+a)$ is irreducible.

Proof. Assume that $f(t+a) = g_1(t)g_2(t)$ with g_1, g_2 not trivial polynomials. Then $f(t) = g_1(t-a)g_2(t-a)$, and $g_1(t-a), g_2(t-a) \in F[t]$, getting a contradiction. \square

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