KAM for gravity water waves in finite depth

Pietro Baldi, Massimiliano Berti, Emanuele Haus, Riccardo Montalto

Abstract: We present the recent result in [3] concerning the existence of Cantor families of small amplitude, linearly stable, time *quasi-periodic* standing water wave solutions – i.e. periodic and even in the space variable x – of a bi-dimensional ocean with finite depth under the action of pure gravity. Such a result holds for all the values of the depth parameter in a Borel set of asymptotically full measure.

Keywords: Water waves, KAM for PDEs, quasi-periodic solutions, standing waves.
MSC 2010: 76B15, 37K55 (37C55, 35S05).

1 Introduction

We consider the Euler equations of hydrodynamics for a 2-dimensional perfect, incompressible, inviscid, irrotational fluid under the action of gravity, filling an ocean with finite depth $\mathtt{h}$ and with space periodic boundary conditions, namely the fluid occupies the region

$$\mathcal{D}_\eta := \big\{ (x,y) \in \mathbb{T} \times \mathbb{R} \, : \, -\mathtt{h} < y < \eta(t,x) \big\} \, , \quad \mathbb{T} := \mathbb{T}_x := \mathbb{R}/2\pi\mathbb{Z} \, .$$

In this note we present the result and the main ideas of [3] concerning the existence and the linear stability of small amplitude quasi-periodic in time solutions of the *pure gravity* water waves system

$$\begin{cases} \partial_t \Phi + \frac12 |\nabla \Phi|^2 + g\eta = 0 & \text{at } y = \eta(x) \\ \Delta \Phi = 0 & \text{in } \mathcal{D}_\eta \\ \partial_y \Phi = 0 & \text{at } y = -\mathtt{h} \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } y = \eta(x) \end{cases} \tag{1.1}$$

where $g > 0$ is the acceleration of gravity. The unknowns of the problem are the free surface $y = \eta(x)$ and the velocity potential $\Phi : \mathcal{D}_\eta \to \mathbb{R}$, i.e. the irrotational velocity field $v = \nabla_{x,y}\Phi$ of the fluid. The first equation in (1.1) is the Bernoulli condition stating the continuity of the pressure at the free surface. The last equation in (1.1) expresses the fact that the fluid particles on the free surface always remain part of it. With no loss of generality we can suppose that the gravity $g = 1$.

Following Zakharov [16] and Craig-Sulem [10], the evolution problem (1.1) may be written as an infinite-dimensional Hamiltonian system in the unknowns $(\eta(x), \psi(x))$ where $\psi(t,x) = \Phi(t,x,\eta(t,x))$ is, at each instant t, the trace at the free boundary of the velocity potential. Given the shape $\eta(t,x)$ of the domain top boundary and the Dirichlet value $\psi(t,x)$ of the velocity potential at the top boundary, there is a unique solution $\Phi(t,x,y;\mathtt{h})$ of the elliptic problem

$$\begin{cases} \Delta \Phi = 0 & \text{in } \{-\mathtt{h} < y < \eta(t,x)\} \\ \partial_y \Phi = 0 & \text{on } y = -\mathtt{h} \\ \Phi = \psi & \text{on } \{y = \eta(t,x)\} \, . \end{cases} \tag{1.2}$$

As proved in [10], system (1.1) is then equivalent to the Craig-Sulem-Zakharov system

$$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi = -\eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2(1+\eta_x^2)} \big(G(\eta)\psi + \eta_x \psi_x \big)^2 \end{cases} \tag{1.3}$$

where $G(\eta)$ is the Dirichlet-Neumann operator defined as

$$G(\eta)\psi := \big\{\Phi_y - \eta_x \Phi_x\big\}_{|y=\eta(t,x)}$$

(we denote by η_x the space derivative $\partial_x \eta$). The operator $G(\eta)$ is linear in ψ, self-adjoint with respect to the L^2 scalar product and positive-semidefinite, and its kernel contains only the constant functions. Moreover the Dirichlet-Neumann operator is a *pseudo-differential* operator with principal symbol $D\tanh(\mathtt{h}D)$, with the property

$$G(\eta) - D\tanh(\mathtt{h}D) \in OPS^{-\infty}$$

when $\eta(x) \in \mathcal{C}^\infty$.

Equations (1.3) are the Hamiltonian system (see [16], [10])

$$\partial_t u = J\nabla_u H(u)\,, \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad J := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \tag{1.4}$$

where ∇_u denotes the L^2-gradient, and the Hamiltonian

$$H(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx + \frac{1}{2}\int_{\mathbb{T}} \eta^2\, dx \tag{1.5}$$

is the sum of the kinetic and potential energies expressed in terms of the variables (η,ψ). The Dirichlet-Neumann operator $G(\eta)$ and the Hamiltonian $H(\eta,\psi)$ depend on $\mathtt{h}$, but, for simplicity, we omit to denote such a dependence.

The phase space of (1.3) is

$$(\eta,\psi) \in H_0^1(\mathbb{T}) \times \dot{H}^1(\mathbb{T}) \qquad \text{where} \qquad \dot{H}^1(\mathbb{T}) := H^1(\mathbb{T})/_{\sim}$$

is the homogenous space obtained by the equivalence relation $\psi_1(x) \sim \psi_2(x)$ if and only if $\psi_1(x) - \psi_2(x) = c$ is a constant. For simplicity of notation we denote the equivalence class $[\psi] = \psi$. Note that the second equation in (1.3) is in $\dot{H}^1(\mathbb{T})$, as it is natural because only the gradient of the velocity potential has a physical meaning. Since the quotient map induces an isometry from $\dot{H}^1(\mathbb{T})$ and $H_0^1(\mathbb{T})$ we shall often identify ψ as a function with zero average.

The water waves system (1.3) exhibits several symmetries. First of all, the mass $\int_{\mathbb{T}} \eta\, dx$ is a first integral of (1.3). In addition, the subspace of functions that are even in x,

$$\eta(x) = \eta(-x)\,, \quad \psi(x) = \psi(-x)\,, \tag{1.6}$$

is invariant under (1.3). In this case also the velocity potential $\Phi(x,y)$ is even and 2π-periodic in x and so the x-component of the velocity field $v = (\Phi_x, \Phi_y)$ vanishes at $x = k\pi$, for all $k \in \mathbb{Z}$. Hence there is no flow of fluid through the lines $x = k\pi$, $k \in \mathbb{Z}$, and a solution of (1.3) satisfying (1.6) describes the motion of a liquid confined between two walls.

Another important symmetry of the water waves system is reversibility, i.e.

$$H \circ \rho = H\,, \quad H(\eta,\psi) = H(\eta,-\psi)\,, \quad \rho : (\eta,\psi) \mapsto (\eta,-\psi)\,. \tag{1.7}$$

As a consequence it is natural to look for solutions of (1.3) satisfying

$$u(-t) = \rho u(t)\,, \quad i.e. \quad \eta(-t,x) = \eta(t,x)\,,\ \psi(-t,x) = -\psi(t,x)\,,\ \forall t,x \in \mathbb{R}\,, \tag{1.8}$$

namely η is even in time and ψ is odd in time. Solutions of the water waves equations (1.3) satisfying (1.6) and (1.8) are called gravity *standing water waves.*

The existence of standing water waves is a small divisor problem, which is particularly difficult because (1.3) is a quasi-linear system of PDEs. Existence of small amplitude time-periodic gravity standing wave solutions for bi-dimensional fluids has been first proved by Plotinkov-Toland [14] in finite depth and by Iooss-Plotnikov-Toland [12] in infinite depth. More recently the existence of time periodic gravity-capillary standing wave solutions has been proved by Alazard-Baldi [1]. Next, both the existence and the linear stability of time quasi-periodic gravity-capillary standing wave solutions have been proved by Berti-Montalto [9], see also the expository paper [8].

The goal of this Note is to present the new result in [3] concerning the existence of time *quasi-periodic*, linearly stable, standing wave solutions of (1.3), i.e. of a space periodic bi-dimensional fluid with finite depth under the action of *pure gravity* (with zero surface tension).

The dynamics of the pure gravity and gravity-capillary water waves equations is very different, since in the first case the linear frequencies grow at infinity as $\sim \sqrt{j}$, see (1.12), while in the presence of surface tension they grow as $\sim j^{3/2}$. The sub/super linear growth of the dispersion relation at high frequencies induces quite a relevant difference for the development of KAM theory. As is well known, the abstract infinite-dimensional KAM theorems available in literature, e.g. [13], require that the eigenvalues of the linear constant coefficient differential operator grow as j^d, $d \geq 1$. The reason is that, in presence of a sublinear growth of the linear frequencies, one may impose only very weak Melnikov non-resonance conditions, see e.g. (2.10), which produce strong losses of derivatives along the iterative KAM scheme. Such a difficulty is overcome in [3] by a regularization procedure performed on the linearized PDE at each approximate quasi-periodic solution. This a very general idea, which can be applied in a broad class of situations, and which we shall explain in Section 2.3.

1.1 Main result

We look for small amplitude solutions of (1.3). Of main importance is therefore the dynamics of the system obtained linearizing (1.3) at the equilibrium $(\eta, \psi) = (0, 0)$, namely

$$\begin{cases} \partial_t \eta = G(0)\psi \\ \partial_t \psi = -\eta \end{cases} \tag{1.9}$$

where $G(0) = D \tanh(\mathtt{h}D)$ is the Dirichlet-Neumann operator at the flat surface $\eta = 0$. In the compact Hamiltonian form as in (1.4), system (1.9) reads

$$\partial_t u = J\Omega u\,, \quad \Omega := \begin{pmatrix} 1 & 0 \\ 0 & G(0) \end{pmatrix}. \tag{1.10}$$

The standing waves solutions of the linear system (1.9) are

$$\eta(t, x) = \sum_{j \geq 1} a_j \cos(\omega_j t) \cos(jx), \quad \psi(t, x) = -\sum_{j \geq 1} a_j \omega_j^{-1} \sin(\omega_j t) \cos(jx)\,, \tag{1.11}$$

with linear frequencies of oscillation

$$\omega_j := \omega_j(\mathtt{h}) := \sqrt{j \tanh(\mathtt{h}j)}\,, \quad j \geq 1\,. \tag{1.12}$$

Note that, since $j \mapsto j \tanh(\mathtt{h}j)$ is monotone increasing, all the linear frequencies are simple.

Fix an arbitrary finite subset $\mathbb{S}^+ \subset \mathbb{N}^+ := \{1, 2, \ldots\}$ (tangential sites) and consider the solutions of the linear system (1.9)

$$\eta(t, x) = \sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \cos(\omega_j t) \cos(jx), \quad \psi(t, x) = -\sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \omega_j^{-1} \sin(\omega_j t) \cos(jx)\,, \quad \xi_j > 0\,, \tag{1.13}$$

which are Fourier supported on $\mathbb{S}^+$. Denote by $\nu := |\mathbb{S}^+|$ the cardinality of $\mathbb{S}^+$. We look for quasi-periodic solutions $u(\tilde{\omega} t) = (\eta, \psi)(\tilde{\omega} t)$ of (1.3), with frequency $\tilde{\omega} \in \mathbb{R}^\nu$ (to be determined), close to the solutions (1.13) of (1.9), in the Sobolev spaces of functions $H^s(\mathbb{T}^{\nu+1}, \mathbb{R}^2) := \big\{u = (\eta, \psi) : \eta, \psi \in H^s\big\}$ where

$$H^s := H^s(\mathbb{T}^{\nu+1}, \mathbb{R}) = \Big\{ f = \sum_{(\ell, j) \in \mathbb{Z}^{\nu+1}} f_{\ell j}\, e^{\mathrm{i}(\ell \cdot \varphi + jx)} : \ \|f\|_s^2 := \sum_{(\ell, j) \in \mathbb{Z}^{\nu+1}} |f_{\ell j}|^2 \langle \ell, j \rangle^{2s} < \infty \Big\}$$

and $\langle \ell, j \rangle := \max\{1, |\ell|, |j|\}$. For $s \geq s_0 := [\frac{\nu+1}{2}] + 1 \in \mathbb{N}$, one has $H^s(\mathbb{T}^{\nu+1}, \mathbb{R}) \subset L^\infty(\mathbb{T}^{\nu+1}, \mathbb{R})$, and $H^s(\mathbb{T}^{\nu+1}, \mathbb{R})$ is an algebra.

Theorem 1.1. (KAM for gravity water waves in finite depth, [3]) *For every choice of the tangential sites* $\mathbb{S}^+ \subset \mathbb{N} \setminus \{0\}$, $|\mathbb{S}^+| < \infty$, *there exist* $\bar{s} > \frac{|\mathbb{S}^+|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that for every* $|\xi| \leq \varepsilon_0^2$, $\xi := (\xi_j)_{j \in \mathbb{S}^+}$, $\xi_j > 0$ *for all* $j \in \mathbb{S}^+$, *there exists a Cantor-like set* $\mathcal{G} \subset [\mathtt{h}_1, \mathtt{h}_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.*

$$\lim_{\xi \to 0} |\mathcal{G}| = \mathtt{h}_2 - \mathtt{h}_1 \,,$$

such that, for any $\mathtt{h} \in \mathcal{G}$, *the gravity water waves system* (1.3) *has a time quasi-periodic solution* $u(\tilde{\omega} t, x) = (\eta(\tilde{\omega} t, x), \psi(\tilde{\omega} t, x))$, *with Sobolev regularity* $(\eta, \psi) \in H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R}^2)$, *of the form*

$$\begin{aligned} \eta(t,x) &= \sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + r_1(\tilde{\omega} t, x), \\ \psi(t,x) &= - \sum_{j \in \mathbb{S}^+} \sqrt{\xi_j} \omega_j^{-1} \sin(\tilde{\omega}_j t) \cos(jx) + r_2(\tilde{\omega} t, x) \end{aligned} \tag{1.14}$$

with a Diophantine frequency vector $\tilde{\omega} := (\tilde{\omega}_j)_{j \in \mathbb{S}^+} \in \mathbb{R}^\nu$ *satisfying* $\tilde{\omega}_j \to \omega_j(\mathtt{h})$, $j \in \mathbb{S}^+$, *as* $\xi \to 0$, *and the functions* $r_1(\varphi, x), r_2(\varphi, x)$ *are* $o(\sqrt{|\xi|})$*-small in* $H^{\bar{s}}(\mathbb{T}^\nu \times \mathbb{T}, \mathbb{R})$, *i.e.* $\|r_i\|_{\bar{s}} / \sqrt{|\xi|} \to 0$ *as* $|\xi| \to 0$ *for* $i = 1, 2$. *The solution* (η, ψ) *is even in* x, η *is even in* t *and* ψ *is odd in* t. *In addition these quasi-periodic solutions are linearly stable.*

This is the first result concerning time quasi-periodic solutions for the pure-gravity water waves equations. We remark that no global in time existence results concerning the initial value problem of the water waves equations (1.3) under *periodic* boundary conditions are known so far. For the local existence theory we refer to Alazard-Burq-Zuily [2].

The Nash-Moser-KAM iterative procedure implemented to prove Theorem 1.1 selects many values of the parameter $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$ which give rise to the quasi-periodic solutions (1.14), which are defined for all times. By a Fubini-type argument it also results that, for most values of $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$, there exist quasi-periodic solutions of (1.3) for most values of the amplitudes $|\xi| \leq \varepsilon_0^2$. The fact that we find quasi-periodic solutions restricting to a proper subset of parameters is not a technical issue, because the gravity water waves equations (1.3) are expected to be not integrable, see for example [11] in the case of infinite depth.

Let us make some further comments on Theorem 1.1.

1. The parameter $\mathtt{h}$ varies in the finite interval $[\mathtt{h}_1, \mathtt{h}_2]$ with $0 < \mathtt{h}_1 < \mathtt{h}_2 < +\infty$. The result does not pass to the limit of zero depth ($\mathtt{h}_1 \to 0^+$), nor of infinite depth ($\mathtt{h}_2 \to +\infty$). Different phenomena arise.

2. From a physical point of view, it is also natural to consider the depth $\mathtt{h}$ of the ocean as a fixed physical quantity and to look for quasi-periodic solutions for most values of the space wavelength $2\pi\lambda$. This can be achieved by rescaling properly time, space and the amplitude of (η, ψ).

3. The linear frequencies (1.12) admit the following asymptotic expansion

$$\sqrt{j \tanh(\mathtt{h} j)} = \sqrt{j} + r(j, \mathtt{h}) \quad \text{where} \quad \left| \partial_{\mathtt{h}}^k r(j, \mathtt{h}) \right| \leq C_k e^{-\mathtt{h} j} \quad \forall k \in \mathbb{N}, \ \forall j \geq 1,$$

uniformly in $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$, where the constant C_k depends only on k and $\mathtt{h}_1$. Even though $\mathtt{h}$ moves the frequencies of exponentially small quantities, we shall be able to use the finite depth parameter $\mathtt{h}$ to impose the required non-resonance conditions.

4. The quasi-periodic solutions (1.14) are mainly supported in Fourier space on the tangential sites $\mathbb{S}^+$. The dynamics of the water waves equations (1.3) on the symplectic subspaces

$$H_{\mathbb{S}^+} := \Big\{ v = \sum_{j \in \mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \Big\}, \quad H_{\mathbb{S}^+}^{\perp} := \Big\{ z = \sum_{j \in \mathbb{N} \setminus \mathbb{S}^+} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \in H_0^1(\mathbb{T}_x) \Big\}, \tag{1.15}$$

is quite different. We shall call $v \in H_{\mathbb{S}^+}$ the *tangential* variable and $z \in H_{\mathbb{S}^+}^{\perp}$ the *normal* one. On the finite dimensional subspace $H_{\mathbb{S}^+}$ we shall describe the dynamics by introducing the action-angle variables $(\theta, I) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$ in Section 2.1.

Linear stability. The quasi-periodic solutions $u(\tilde\omega t) = (\eta(\tilde\omega t), \psi(\tilde\omega t))$ found in Theorem 1.1 are linearly stable. This is not only a dynamically relevant information but also an essential ingredient of the existence proof (it is not necessary for time periodic solutions as in [1], [12]). Let us state precisely the result. By Theorem 1 in [7], around each invariant torus there exist symplectic coordinates

$$(\phi, y, w) = (\phi, y, \eta, \psi) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}$$

in which the water waves Hamiltonian reads

$$\omega \cdot y + \frac{1}{2} K_{20}(\phi) y \cdot y + \big(K_{11}(\phi) y, w \big)_{L^2(\mathbb{T}_x)} + \frac{1}{2} \big(K_{02}(\phi) w, w \big)_{L^2(\mathbb{T}_x)} + K_{\geq 3}(\phi, y, w)$$

where $K_{\geq 3}$ collects the terms at least cubic in the variables (y, w). In these coordinates the quasi-periodic solution reads $t \mapsto (\omega t, 0, 0)$ (for simplicity we denote the frequency $\tilde\omega$ of the quasi-periodic solution by ω) and the corresponding linearized water waves equations are

$$\begin{cases} \dot\phi = K_{20}(\omega t)[y] + K_{11}^T(\omega t)[w] \\ \dot y = 0 \\ \dot w = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y] \, . \end{cases}$$

Thus the actions $y(t) = y(0)$ do not evolve in time and the third equation reduces to the linear PDE

$$\dot w = J K_{02}(\omega t)[w] + J K_{11}(\omega t)[y(0)] \, . \tag{1.16}$$

The self-adjoint operator $K_{02}(\omega t)$ turns out to be the restriction to $H^\perp_{\mathbb{S}^+}$ of the linearized water waves operator $\partial_u \nabla H(u(\omega t))$, explicitly written in (2.21), up to a finite dimensional remainder.

In [3] we prove the existence of a bounded and invertible "symmetrizer" map such that, for all $\varphi \in \mathbb{T}^\nu$,

$$\mathbf{W}_\infty(\varphi) : \big(H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C}) \big) \cap H^\perp_{\mathbb{S}^+} \to \big(H^{s-\frac14}(\mathbb{T}_x, \mathbb{R}) \times H^{s+\frac14}(\mathbb{T}_x, \mathbb{R}) \big) \cap H^\perp_{\mathbb{S}^+} \, , \tag{1.17}$$

$$\mathbf{W}_\infty^{-1}(\varphi) : \big(H^{s-\frac14}(\mathbb{T}_x, \mathbb{R}) \times H^{s+\frac14}(\mathbb{T}_x, \mathbb{R}) \big) \cap H^\perp_{\mathbb{S}^+} \to \big(H^s(\mathbb{T}_x, \mathbb{C}) \times H^s(\mathbb{T}_x, \mathbb{C}) \big) \cap H^\perp_{\mathbb{S}^+} \, , \tag{1.18}$$

and, under the change of variables

$$w = (\eta, \psi) = \mathbf{W}_\infty(\omega t) w_\infty \, , \quad w_\infty = (\mathtt{w}_\infty, \overline{\mathtt{w}}_\infty) \, ,$$

equation (1.16) transforms into the diagonal system

$$\partial_t w_\infty = -\mathrm{i} \mathbf{D}_\infty w_\infty + f_\infty(\omega t) \, , \quad f_\infty(\omega t) := \mathbf{W}_\infty^{-1}(\omega t) J K_{11}(\omega t)[y(0)] = \begin{pmatrix} \mathtt{f}_\infty(\omega t) \\ \overline{\mathtt{f}}_\infty(\omega t) \end{pmatrix} \tag{1.19}$$

where i is the imaginary unit and, denoting $\mathbb{S}_0 := \mathbb{S}^+ \cup (-\mathbb{S}^+) \cup \{0\} \subseteq \mathbb{Z}$ and $\mathbb{S}_0^c := \mathbb{Z} \setminus \mathbb{S}_0$, the operator

$$\mathbf{D}_\infty := \begin{pmatrix} D_\infty & 0 \\ 0 & -D_\infty \end{pmatrix} \, , \quad D_\infty := \mathrm{diag}_{j \in \mathbb{S}_0^c} \{ \mu_j^\infty \} \, , \quad \mu_j^\infty \in \mathbb{R} \, , \tag{1.20}$$

is a Fourier multiplier operator of the form

$$\mu_j^\infty := \mathtt{m}_{\frac12}^\infty |j|^{\frac12} \tanh^{\frac12}(\mathtt{h}|j|) + r_j^\infty \, , \ j \in \mathbb{S}_0^c \, , \quad r_j^\infty = r_{-j}^\infty \, ,$$

where, for some $\mathtt{a} > 0$,

$$\mathtt{m}_{\frac12}^\infty = 1 + O(|\xi|^{\mathtt{a}}) \, , \quad \sup_{j \in \mathbb{S}_0^c} |j|^{\frac12} |r_j^\infty| = O(|\xi|^{\mathtt{a}}) \, .$$

Actually by (2.8)-(2.9) and (2.12) we also have a control of the derivatives of $\mathtt{m}_{\frac12}^\infty$ and r_j^∞ with respect to $(\omega, \mathtt{h})$. The numbers $\mathrm{i}\mu_j^\infty$ are the *Floquet exponents* of the quasi-periodic solution. The second equation of system (1.19) is actually the complex conjugate of the first one, and (1.19) reduces to the infinitely many decoupled scalar equations

$$\partial_t \mathtt{w}_{\infty,j} = -\mathrm{i} \mu_j^\infty \mathtt{w}_{\infty,j} + \mathtt{f}_{\infty,j}(\omega t) \, , \quad \forall j \in \mathbb{S}_0^c \, .$$

By variation of constants the solutions are

$$\mathtt{w}_{\infty,j}(t) = c_j e^{-\mathrm{i}\mu_j^\infty t} + \mathtt{v}_{\infty,j}(t) \qquad \text{where} \qquad \mathtt{v}_{\infty,j}(t) := \sum_{\ell\in\mathbb{Z}^\nu} \frac{\mathtt{f}_{\infty,j,\ell}\, e^{\mathrm{i}\omega\cdot\ell t}}{\mathrm{i}(\omega\cdot\ell + \mu_j^\infty)}\,, \quad \forall j \in \mathbb{S}_0^c\,. \tag{1.21}$$

Note that the first Melnikov conditions (2.10) hold at a solution so that $\mathtt{v}_{\infty,j}(t)$ in (1.21) is well defined. Moreover (1.17) implies $\|f_\infty(\omega t)\|_{H^s_x\times H^s_x} \le C|y(0)|$. As a consequence the Sobolev norm of the solution of (1.19) with initial condition $w_\infty(0) \in H^{\mathfrak{s}_0}(\mathbb{T}_x)\times H^{\mathfrak{s}_0}(\mathbb{T}_x)$, for some $\mathfrak{s}_0 \in (s_0, s)$ (in a suitable range of values), satisfies

$$\|w_\infty(t)\|_{H^{\mathfrak{s}_0}_x\times H^{\mathfrak{s}_0}_x} \le C(s)(|y(0)| + \|w_\infty(0)\|_{H^{\mathfrak{s}_0}_x\times H^{\mathfrak{s}_0}_x})\,,$$

and, for all $t\in\mathbb{R}$, using (1.17), (1.18), we get

$$\|(\eta,\psi)(t)\|_{H_x^{\mathfrak{s}_0-\frac14}\times H_x^{\mathfrak{s}_0+\frac14}} \le C\|(\eta(0),\psi(0))\|_{H_x^{\mathfrak{s}_0-\frac14}\times H_x^{\mathfrak{s}_0+\frac14}}\,,$$

which proves the linear stability of the torus.

Clearly a crucial point is the diagonalization of (1.16) into (1.20). With respect to the pioneering works of Plotnikov-Toland [14] and Iooss-Plotnikov-Toland [12] dealing with time periodic solutions, this requires to analyze more in detail the linearized operator in two respects:

1. We have to perform a reduction of the linearized operator into a constant coefficient pseudo-differential operator, up to smoothing remainders, via changes of variables that are quasi-periodic transformations of the phase space, so that the dynamical system nature of the transformed systems is preserved. We shall perform such reductions by changes of variables generated by pseudo-differential operators, diffeomorphisms of the torus, and "semi-Fourier integral operators" (namely pseudo-differential operators of type $(\frac12,\frac12)$ in the notation of Hörmander), inspired by [1], [9].

2. Once the above regularization has been performed, we implement a KAM iterative scheme which completes the diagonalization of the linearized operator. This scheme uses very weak second order Melnikov non-resonance conditions – which lose derivatives –, which are compensated by the smoothing nature of the variable coefficients remainders.

   Such a diagonalization is not required for the search of time-periodic solutions, as in [1], [12], [14]. In such a case a Neumann series argument is sufficient to invert the linearized operator. The key difference is that, for the search of periodic solutions, a sufficiently regularizing operator in the space variable is also regularizing in the time variable, on the Fourier indices $(\ell,j)\in\mathbb{Z}\times\mathbb{Z}$ that correspond to the small divisors, thus $\omega\ell \sim \sqrt{|j|}$. This is not true for quasi-periodic solutions as we explain in Remark 2.5.

Acknowledgements. This research was supported by PRIN 2015 "Variational methods, with applications to problems in mathematical physics and geometry", by the European Research Council under FP7, project no. 306414 "Hamiltonian PDEs and small divisor problem: a dynamical systems approach" (HamPDEs), partially by the Swiss National Science Foundation, and partially by the Programme STAR, funded by Compagnia di San Paolo and UniNA.

2 Ideas of the proof

There are three major difficulties for proving the existence of time quasi-periodic solutions of the gravity water waves equations (1.3):

1. The water waves equations (1.3) are a quasi-linear system.

2. The dispersion relation (1.12) of the linear water waves equations is sublinear, i.e. $\omega_j \sim \sqrt{j}$ for $j\to+\infty$. This is a relevant difference with respect to the capillary-gravity case studied in [9] where the linear frequencies are $\sqrt{(1+\kappa j^2)j\tanh(\mathtt{h}j)} \sim \sqrt{\kappa}j^{3/2}$.

3. We have to verify all the Melnikov non-resonance conditions required on the frequencies by the KAM scheme. Notice that the parameter $\mathtt{h}$ moves the frequencies $\sqrt{j\tanh(\mathtt{h}j)}$ of exponentially small quantities of order $O(e^{-\mathtt{h}j})$ (on the contrary, the surface tension parameter κ moves in the case with capillarity the frequencies of polynomial quantities $O(j^{3/2})$).

We present below the key ideas of the paper to solve these three major problems.

2.1 Nash-Moser theorem of hypothetical conjugation

Rescaling the variable $u \mapsto \varepsilon u$, we write (1.3) as

$$\partial_t u = J\Omega u + \varepsilon X_{P_\varepsilon}(u) \tag{2.1}$$

where $J\Omega$ is the linearized Hamiltonian vector field in (1.10) and $X_{P_\varepsilon}(u)$ is the Hamiltonian vector field generated by the Hamiltonian

$$P_\varepsilon(u) := \frac{\varepsilon^{-1}}{2}\int_{\mathbb{T}} \psi\big(G(\varepsilon\eta) - G(0)\big)\psi\, dx\,.$$

Then we decompose the phase space

$$H^1_{0,\mathrm{even}} := \Big\{u := (\eta,\psi) \in H^1_0(\mathbb{T}_x) \times \dot H^1(\mathbb{T}_x)\,, \quad u(x) = u(-x)\Big\} = H_{\mathbb{S}^+} \oplus H^\perp_{\mathbb{S}^+}$$

as the direct sum of the symplectic subspaces $H_{\mathbb{S}^+}$ and $H^\perp_{\mathbb{S}^+}$ in (1.15), and we introduce action-angle variables on the tangential sites by setting

$$\eta_j := \sqrt{\frac{2}{\pi}}\,\omega_j^{1/2}\sqrt{\xi_j + I_j}\cos(\theta_j), \quad \psi_j := \sqrt{\frac{2}{\pi}}\,\omega_j^{-1/2}\sqrt{\xi_j + I_j}\,\sin(\theta_j)\,, \quad j \in \mathbb{S}^+\,,$$

where $\xi_j > 0$, $j \in \mathbb{S}^+$, and $|I_j| < \xi_j$. We leave unchanged the normal component z. Hence the Hamiltonian system (2.1) transforms into the new Hamiltonian system

$$\dot\theta = \partial_I H_\varepsilon(\theta, I, z)\,, \ \dot I = -\partial_\theta H_\varepsilon(\theta, I, z)\,, \quad z_t = J\nabla_z H_\varepsilon(\theta, I, z)$$

generated by the Hamiltonian

$$H_\varepsilon := \varepsilon^{-2} H \circ \varepsilon A \tag{2.2}$$

where

$$A(\theta, I, z) := v(\theta, I) + z := \sum_{j\in\mathbb{S}^+}\sqrt{\frac{2}{\pi}}\begin{pmatrix} \omega_j^{1/2}\sqrt{\xi_j + I_j}\,\cos(\theta_j) \\ -\omega_j^{-1/2}\sqrt{\xi_j + I_j}\,\sin(\theta_j)\end{pmatrix}\cos(jx) + z\,.$$

We denote by $X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\theta H_\varepsilon, J\nabla_z H_\varepsilon)$ the Hamiltonian vector field in the variables $(\theta, I, z) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}$. The involution ρ in (1.7) becomes

$$\tilde\rho : (\theta, I, z) \mapsto (-\theta, I, \rho z)\,. \tag{2.3}$$

In these new coordinates, by (1.5) and (2.2) the Hamiltonian H_ε reads (up to a constant)

$$H_\varepsilon = \mathcal{N} + \varepsilon P\,, \quad \mathcal{N} := \vec\omega(\mathtt{h})\cdot I + \frac{1}{2}(z, \Omega z)_{L^2}\,, \quad P := P_\varepsilon \circ A\,, \tag{2.4}$$

where $\vec\omega(\mathtt{h}) := (\omega_j(\mathtt{h}))_{j\in\mathbb{S}^+}$ and Ω is defined in (1.10). We look for an embedded invariant torus

$$i : \mathbb{T}^\nu \to \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^\perp_{\mathbb{S}^+}, \quad \varphi \mapsto i(\varphi) := (\theta(\varphi), I(\varphi), z(\varphi))$$

of the Hamiltonian vector field X_{H_ε} filled by quasi-periodic solutions with Diophantine frequency $\omega \in \mathbb{R}^\nu$ (and which will satisfy also first and second order Melnikov-non-resonance conditions as in (2.10)).

Since the expected quasi-periodic solutions of the autonomous Hamiltonian system (2.4) will have shifted frequencies $\tilde{\omega}_j$ – to be found – close to the linear frequencies $\vec{\omega}(\mathtt{h})$, it is convenient to introduce $\alpha \in \mathbb{R}^\nu$ as a free parameter, considering the modified Hamiltonian

$$H_\alpha := \mathcal{N}_\alpha + \varepsilon P \,, \quad \mathcal{N}_\alpha := \alpha \cdot I + \frac{1}{2}(z, \Omega z)_{L^2} \,. \tag{2.5}$$

We first look for zeros of the nonlinear operator

$$\begin{aligned} \mathcal{F}(i,\alpha) &:= \mathcal{F}(i,\alpha,\omega,\mathtt{h},\varepsilon) := \omega \cdot \partial_\varphi i(\varphi) - X_{H_\alpha}(i(\varphi)) = \omega \cdot \partial_\varphi i(\varphi) - (X_{\mathcal{N}_\alpha} + \varepsilon X_P)(i(\varphi)) \\ &:= \begin{pmatrix} \omega \cdot \partial_\varphi \theta(\varphi) - \alpha - \varepsilon \partial_I P(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \partial_\theta P(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - J(\Omega z(\varphi) + \varepsilon \nabla_z P(i(\varphi))) \end{pmatrix} \end{aligned}$$

where $\Theta(\varphi) := \theta(\varphi) - \varphi$ is $(2\pi)^\nu$-periodic. Thus $\varphi \mapsto i(\varphi)$ is an embedded torus, invariant for the Hamiltonian vector field X_{H_α} and filled by quasi-periodic solutions with frequency ω.

Each Hamiltonian H_α in (2.5) is reversible, i.e. $H_\alpha \circ \tilde{\rho} = H_\alpha$ where the involution $\tilde{\rho}$ is defined in (2.3), and we look for reversible solutions of $\mathcal{F}(i,\alpha) = 0$, namely satisfying

$$\theta(-\varphi) = -\theta(\varphi) \,, \quad I(-\varphi) = I(\varphi) \,, \quad z(-\varphi) = (\rho z)(\varphi) \,. \tag{2.6}$$

The norm of the periodic component of the embedded torus $\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0, 0) := (\Theta(\varphi), I(\varphi), z(\varphi))$, $\Theta(\varphi) := \theta(\varphi) - \varphi$, is

$$\|\mathfrak{I}\|_s^{k_0,\gamma} := \|\Theta\|_{H^s_\varphi}^{k_0,\gamma} + \|I\|_{H^s_\varphi}^{k_0,\gamma} + \|z\|_s^{k_0,\gamma} \,,$$

where $\|z\|_s^{k_0,\gamma} = \|\eta\|_s^{k_0,\gamma} + \|\psi\|_s^{k_0,\gamma}$ and $\|u\|_s^{k_0,\gamma} := \sum_{|k| \le k_0} \gamma^{|k|} \|\partial^k_{\omega,\mathtt{h}} u\|_s$. We define

$$k_0 := k_0^* + 2,$$

where k_0^* is the index of non-degeneracy provided by Proposition 2.3, which only depends on the linear unperturbed frequencies. Thus k_0 is considered as an absolute constant, and we will often omit to explicitly write the dependence of the various constants with respect to k_0. We look for quasi-periodic solutions with frequency ω belonging to a δ-neighborhood (independent of ε)

$$\Omega := \Big\{ \omega \in \mathbb{R}^\nu \,:\, \mathrm{dist}\big(\omega, \vec{\omega}[\mathtt{h}_1, \mathtt{h}_2]\big) < \delta \Big\}, \quad \delta > 0 \,,$$

of the unperturbed linear frequencies $\vec{\omega}[\mathtt{h}_1, \mathtt{h}_2]$ where $\vec{\omega}(\mathtt{h}) := (\omega_j(\mathtt{h}))_{j \in \mathbb{S}^+}$. For a function $r(\omega, \mathtt{h})$ vith values in $\mathbb{R}^\nu$ we set $|r|^{k_0,\gamma} := \sum_{|k| \le k_0} \gamma^{|k|} |\partial^k_{\omega,\mathtt{h}} r|$.

Theorem 2.1. (Nash-Moser theorem of hypothetical conjugation) *Fix finitely many tangential sites $\mathbb{S}^+ \subset \mathbb{N}^+$ and let $\nu := |\mathbb{S}^+|$. Let $\tau \ge 1$. There exist positive constants $a_0, \varepsilon_0, \kappa_1, C$ depending on $\mathbb{S}^+, \nu, k_0, \tau$ such that, for all $\gamma = \varepsilon^a$, $0 < a < a_0$, for all $\varepsilon \in (0, \varepsilon_0)$, there exist a k_0 times differentiable function*

$$\alpha_\infty : \mathbb{R}^\nu \times [\mathtt{h}_1, \mathtt{h}_2] \mapsto \mathbb{R}^\nu \,, \quad \alpha_\infty(\omega, \mathtt{h}) = \omega + r_\varepsilon(\omega, \mathtt{h}) \,, \quad \textit{with} \quad |r_\varepsilon|^{k_0,\gamma} \le C\varepsilon\gamma^{-1} \,, \tag{2.7}$$

a family of embedded tori i_∞ defined for all $\omega \in \mathbb{R}^\nu$ and $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$ satisfying the reversibility property (2.6) *and*

$$\|i_\infty(\varphi) - (\varphi, 0, 0)\|_{s_0}^{k_0,\gamma} \le C\varepsilon\gamma^{-1} \,,$$

a sequence of k_0 times differentiable functions $\mu_j^\infty : \mathbb{R}^\nu \times [\mathtt{h}_1, \mathtt{h}_2] \to \mathbb{R}$, $j \in \mathbb{N}^+ \setminus \mathbb{S}^+$, of the form

$$\mu_j^\infty(\omega, \mathtt{h}) = \mathtt{m}_{\frac{1}{2}}^\infty(\omega, \mathtt{h})(j \tanh(\mathtt{h}j))^{\frac{1}{2}} + r_j^\infty(\omega, \mathtt{h}) \tag{2.8}$$

satisfying

$$|\mathtt{m}_{\frac{1}{2}}^\infty - 1|^{k_0,\gamma} \le C\varepsilon \,, \qquad \sup_{j \in \mathbb{N}^+ \setminus \mathbb{S}^+} j^{\frac{1}{2}} |r_j^\infty|^{k_0,\gamma} \le C\varepsilon\gamma^{-\kappa_1} \tag{2.9}$$

such that for all $(\omega, \mathtt{h})$ in the Cantor like set

$$
\begin{aligned}
\mathcal{C}_\infty^\gamma := \Big\{ & (\omega, \mathtt{h}) \in \Omega \times [\mathtt{h}_1, \mathtt{h}_2] \,:\, |\omega \cdot \ell| \geq 8\gamma \langle \ell \rangle^{-\tau}, \, \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}, \\
& |\omega \cdot \ell + \mu_j^\infty(\omega, \mathtt{h})| \geq 4\gamma j^{\frac12} \langle \ell \rangle^{-\tau}, \, \forall \ell \in \mathbb{Z}^\nu, \, j \in \mathbb{N}^+ \setminus \mathbb{S}^+, \\
& |\omega \cdot \ell + \mu_j^\infty(\omega, \mathtt{h}) + \mu_{j'}^\infty(\omega, \mathtt{h})| \geq 4\gamma (j^{\frac12} + j'^{\frac12}) \langle \ell \rangle^{-\tau}, \ \forall \ell \in \mathbb{Z}^\nu, \ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+, \\
& |\omega \cdot \ell + \mu_j^\infty(\omega, \mathtt{h}) - \mu_{j'}^\infty(\omega, \mathtt{h})| \geq 4\gamma j^{-\mathtt{d}} j'^{-\mathtt{d}} \langle \ell \rangle^{-\tau}, \ \forall \ell \in \mathbb{Z}^\nu, \ j, j' \in \mathbb{N}^+ \setminus \mathbb{S}^+, \, (\ell, j, j') \neq (0, j, j) \Big\}
\end{aligned} \tag{2.10}
$$

the function $i_\infty(\varphi) := i_\infty(\omega, \mathtt{h}, \varepsilon)(\varphi)$ is a solution of $\mathcal{F}(i_\infty, \alpha_\infty(\omega, \mathtt{h}), \omega, \mathtt{h}, \varepsilon) = 0$. As a consequence the embedded torus $\varphi \mapsto i_\infty(\varphi)$ is invariant for the Hamiltonian vector field $X_{H_{\alpha_\infty(\omega,\mathtt{h})}}$ and it is filled by quasi-periodic solutions with frequency ω.

The very weak second Melnikov non-resonance conditions in the last line of (2.10) can be verified for most parameters if $\mathtt{d}$ is large enough, i.e. $\mathtt{d} > \frac34 k_0^*$, see Theorem 2.2 below. The loss of derivatives produced by such small divisors will be compensated in the reducibility scheme by the fact that we will reduce the linearized operator to constant coefficients up to very regularizing terms $O(|D_x|^{-M})$ for some $M := M(\mathtt{d}, \tau)$ large enough with respect to $\mathtt{d}$ and τ. We shall explain in detail this procedure below.

Theorem 2.1 is proved by means of an iterative Nash-Moser scheme whose main step is the analysis of the linearized operator at a non trivial solution that we describe in Section 2.3. We first discuss how to deduce Theorem 1.1 from Theorem 2.1.

2.2 Measure estimates

In order to prove the existence of quasi-periodic solutions of the water waves equations (1.3), and not only of the system with modified Hamiltonian H_α with $\alpha := \alpha_\infty(\omega, \mathtt{h}, \varepsilon)$, we have to prove that the curve of the unperturbed linear frequencies

$$
[\mathtt{h}_1, \mathtt{h}_2] \ni \mathtt{h} \mapsto \vec{\omega}(\mathtt{h}) := \big(\sqrt{j \tanh(\mathtt{h} j)}\big)_{j \in \mathbb{S}^+} \in \mathbb{R}^\nu
$$

intersects the image $\alpha_\infty(\mathcal{C}_\infty^\gamma)$, under the map α_∞ of the set $\mathcal{C}_\infty^\gamma$, for "most" values of $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$. By (2.7) the function $\alpha_\infty(\cdot, \mathtt{h})$ from Ω into the image $\alpha_\infty(\Omega, \mathtt{h})$ is invertible:

$$
\beta = \alpha_\infty(\omega, \mathtt{h}) = \omega + r_\varepsilon(\omega, \mathtt{h}) \quad \Longleftrightarrow \quad \omega = \alpha_\infty^{-1}(\beta, \mathtt{h}) = \beta + \breve{r}_\varepsilon(\beta, \mathtt{h}) \quad \text{with} \quad |\breve{r}_\varepsilon|^{k_0, \gamma} \leq C \varepsilon \gamma^{-1}. \tag{2.11}
$$

Theorem 2.2 below states that for "most" values of $\mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2]$ the vector $(\alpha_\infty^{-1}(\vec{\omega}(\mathtt{h}), \mathtt{h}), \mathtt{h})$ is in $\mathcal{C}_\infty^\gamma$. Hence, for such values of $\mathtt{h}$ we have found an embedded invariant torus for the Hamiltonian H_ε in (2.4), filled by quasi-periodic solutions with Diophantine frequency $\omega = \alpha_\infty^{-1}(\vec{\omega}(\mathtt{h}), \mathtt{h})$.

Theorem 2.2. (Measure estimates, [3]) *Let*

$$
\gamma = \varepsilon^a, \qquad 0 < a < \min\{a_0, 1/(k_0 + \kappa_1)\}, \qquad \tau > k_0^*(\nu + 4), \qquad \mathtt{d} > \frac{3k_0^*}{4}, \tag{2.12}
$$

where k_0^* is the index of non-degeneracy given by Proposition 2.3 and $k_0 = k_0^* + 2$. Then the measure of the set

$$
\mathcal{G}_\varepsilon = \big\{ \mathtt{h} \in [\mathtt{h}_1, \mathtt{h}_2] : \big(\alpha_\infty^{-1}(\vec{\omega}(\mathtt{h}), \mathtt{h}), \mathtt{h}\big) \in \mathcal{C}_\infty^\gamma \big\} \tag{2.13}
$$

satisfies $|\mathcal{G}_\varepsilon| \to \mathtt{h}_2 - \mathtt{h}_1$ as $\varepsilon \to 0$.

Let us give an idea of the proof. By (2.11) the vector

$$
\omega_\varepsilon(\mathtt{h}) := \alpha_\infty^{-1}(\vec{\omega}(\mathtt{h}), \mathtt{h}) = \vec{\omega}(\mathtt{h}) + \mathtt{r}_\varepsilon(\mathtt{h}), \quad \mathtt{r}_\varepsilon(\mathtt{h}) := \breve{r}_\varepsilon(\vec{\omega}(\mathtt{h}), \mathtt{h}), \tag{2.14}
$$

satisfies $|\partial_{\mathtt{h}}^k \mathtt{r}_\varepsilon(\mathtt{h})| \leq C \varepsilon \gamma^{-k-1}$ for all $0 \leq k \leq k_0$. We also denote, with a small abuse of notation, for all $j \in \mathbb{N}^+ \setminus \mathbb{S}^+$,

$$
\mu_j^\infty(\mathtt{h}) := \mu_j^\infty(\omega_\varepsilon(\mathtt{h}), \mathtt{h}) := \mathtt{m}_{\frac12}^\infty(\mathtt{h}) (j \tanh(\mathtt{h} j))^{\frac12} + r_j^\infty(\mathtt{h}), \tag{2.15}
$$

where $\mathtt{m}_{\frac12}^\infty(\mathtt{h}) := \mathtt{m}_{\frac12}^\infty(\omega_\varepsilon(\mathtt{h}),\mathtt{h})$ and $r_j^\infty(\mathtt{h}) := r_j^\infty(\omega_\varepsilon(\mathtt{h}),\mathtt{h})$. By (2.10), (2.14), (2.15), the Cantor set $\mathcal{G}_\varepsilon$ in (2.13) becomes

$$\begin{aligned}\mathcal{G}_\varepsilon := \Big\{ & \mathtt{h}\in[\mathtt{h}_1,\mathtt{h}_2] : |\omega_\varepsilon(\mathtt{h})\cdot\ell| \geq 8\gamma\langle\ell\rangle^{-\tau},\ \forall \ell\in\mathbb{Z}^\nu\setminus\{0\},\\ & |\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})| \geq 4\gamma j^{\frac12}\langle\ell\rangle^{-\tau},\ \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+,\\ & |\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})+\mu_{j'}^\infty(\mathtt{h})| \geq 4\gamma(j^{\frac12}+j'^{\frac12})\langle\ell\rangle^{-\tau},\ \forall\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+\,,\\ & |\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})-\mu_{j'}^\infty(\mathtt{h})| \geq \frac{4\gamma\langle\ell\rangle^{-\tau}}{j^{\mathtt{d}}j'^{\mathtt{d}}}, \forall\ell\in\mathbb{Z}^\nu, j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+,(\ell,j,j')\neq(0,j,j)\Big\}\,. \end{aligned} \tag{2.16}$$

To prove the measure estimate for $\mathcal{G}_\varepsilon$ in (2.13) the key point is to prove the following transversality property.

Proposition 2.3. (Transversality) *There exist $k_0^*\in\mathbb{N}$, $\rho_0>0$ such that, for any $\mathtt{h}\in[\mathtt{h}_1,\mathtt{h}_2]$,*

$$\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\vec\omega(\mathtt{h})\cdot\ell\}| \geq \rho_0\langle\ell\rangle\,,\quad \forall\ell\in\mathbb{Z}^\nu\setminus\{0\}, \tag{2.17}$$

$$\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\vec\omega(\mathtt{h})\cdot\ell+\Omega_j(\mathtt{h})\}| \geq \rho_0\langle\ell\rangle\,,\quad \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+, \tag{2.18}$$

$$\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\vec\omega(\mathtt{h})\cdot\ell+\Omega_j(\mathtt{h})+\Omega_{j'}(\mathtt{h})\}| \geq \rho_0\langle\ell\rangle\,,\quad \forall\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+ \tag{2.19}$$

$$\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\vec\omega(\mathtt{h})\cdot\ell+\Omega_j(\mathtt{h})-\Omega_{j'}(\mathtt{h})\}| \geq \rho_0\langle\ell\rangle\,,\quad \forall\ell\in\mathbb{Z}^\nu\setminus\{0\},\ j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+\,, \tag{2.20}$$

where $\vec\omega(\mathtt{h})=(\omega_j(\mathtt{h}))_{j\in\mathbb{S}^+}$, $\omega_j(\mathtt{h})=\sqrt{j\tanh(\mathtt{h}j)}$, and $\Omega_j(\mathtt{h})=\sqrt{j\tanh(\mathtt{h}j)}$, $j\in\mathbb{N}^+\setminus\mathbb{S}^+$. We recall the notation $\langle\ell\rangle:=\max\{1,|\ell|\}$. We call (following [15]) ρ_0 the "amount of non-degeneracy" *and k_0^* the* "index of non-degeneracy"*.*

Note that in (2.20) we exclude the index $\ell=0$. In this case we directly have that, for all $\mathtt{h}\in[\mathtt{h}_1,\mathtt{h}_2]$

$$|\Omega_j(\mathtt{h})-\Omega_{j'}(\mathtt{h})| \geq c_1|\sqrt{j}-\sqrt{j'}| = c_1\frac{|j-j'|}{\sqrt{j}+\sqrt{j'}}\quad \forall j,j'\in\mathbb{N}^+, \qquad \text{where } c_1:=\sqrt{\tanh(\mathtt{h}_1)}\,.$$

The above conditions are stable under perturbations which are small in $\mathcal{C}^{k_0^*+2}$ norm and therefore we get:

Lemma 2.4. (Perturbed transversality) *For ε small enough, for all $\mathtt{h}\in[\mathtt{h}_1,\mathtt{h}_2]$,*

$$\begin{aligned} &\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\omega_\varepsilon(\mathtt{h})\cdot\ell\}| \geq \frac{\rho_0}{2}\langle\ell\rangle \quad \forall\ell\in\mathbb{Z}^\nu\setminus\{0\},\\ &\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})\}| \geq \frac{\rho_0}{2}\langle\ell\rangle \quad \forall\ell\in\mathbb{Z}^\nu,\ j\in\mathbb{N}^+\setminus\mathbb{S}^+ : j^{\frac12}\leq C\langle\ell\rangle,\\ &\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})-\mu_{j'}^\infty(\mathtt{h})\}| \geq \frac{\rho_0}{2}\langle\ell\rangle \quad \forall\ell\in\mathbb{Z}^\nu\setminus\{0\},\ j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+ : |j^{\frac12}-j'^{\frac12}|\leq C\langle\ell\rangle,\\ &\max_{k\leq k_0^*}|\partial_{\mathtt{h}}^k\{\omega_\varepsilon(\mathtt{h})\cdot\ell+\mu_j^\infty(\mathtt{h})+\mu_{j'}^\infty(\mathtt{h})\}| \geq \frac{\rho_0}{2}\langle\ell\rangle \quad \forall\ell\in\mathbb{Z}^\nu,\ j,j'\in\mathbb{N}^+\setminus\mathbb{S}^+ : j^{\frac12}+j'^{\frac12}\leq C\langle\ell\rangle, \end{aligned}$$

where k_0^* is the index of non-degeneracy given by Proposition 2.3.

Therefore using Rüssmann Theorem 17.1 in [15] we deduce a measure estimate for the sublevels of the functions in (2.16) and finally (2.13).

The key transversality Proposition 2.3 is proved by extending the arguments of degenerate KAM theory of [6], [9], using in an essential way the fact that the unperturbed frequencies $\mathtt{h}\mapsto\omega_j(\mathtt{h})$ are *analytic* and simple (on the subspace of the even functions), they grow asymptotically as $j^{1/2}$ and they satisfy a suitable *non-degeneracy* condition (to be essentially non-planar) in the sense of [6]. This is verified by analyticity and a generalized Van der Monde determinant.

2.3 Analysis of the linearized operators

The other crucial point is to prove that the linearized operators obtained at any approximate solution along the Nash-Moser iterative scheme are, for most values of the parameters, invertible, and that their inverse satisfies *tame* estimates in Sobolev spaces (with, of course, loss of derivatives). This is the key assumption to implement a convergent differentiable Nash-Moser iterative scheme in scales of Sobolev functions.

Linearizing the water waves equations (1.3) at a time-quasi-periodic approximate solution $(\eta, \psi)(\omega t, x)$, and changing ∂_t into the directional derivative $\omega \cdot \partial_\varphi$, we obtain the operator

$$\mathcal{L} = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V + G(\eta) B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B & V\partial_x - BG(\eta) \end{pmatrix} \tag{2.21}$$

where the functions B, V are

$$B := B(\eta, \psi) := \frac{\eta_x \psi_x + G(\eta)\psi}{1 + \eta_x^2}, \qquad V := V(\eta, \psi) := \psi_x - B\eta_x .$$

It turns out that $(V, B) = \nabla_{x,y}\Phi$ is the velocity field evaluated at the free surface $(x, \eta(x))$.

By the symplectic procedure developed in Berti-Bolle [7] for autonomous PDEs, and implemented in [4], [9], it is sufficient to prove the invertibility of (a finite rank perturbation of) the operator $\mathcal{L}$ restricted to the normal subspace $H_{\mathbb{S}^+}^\perp$ introduced in (1.15).

The main goal is to conjugate the operator $\mathcal{L}$ in (2.21) to a diagonal system of infinitely many decoupled, constant coefficients, scalar linear equations, see (2.23) below. Our approach involves two well separated procedures that we shall describe in detail:

1. *Symmetrization and diagonalization of $\mathcal{L}$ up to smoothing operators.* The first task is to conjugate $\mathcal{L}$ to an operator of the form

$$\omega \cdot \partial_\varphi + \mathrm{i}\mathtt{m}_{\frac12} |D|^{\frac12} \tanh^{\frac12}(\mathtt{h}|D|) + \mathrm{i} r(D) + \mathcal{R}_0(\varphi) \tag{2.22}$$

   where $\mathtt{m}_{\frac12} \approx 1$ is a real constant, independent of (φ, x), the symbol $r(\xi)$ is real and independent of (φ, x), of order $S^{-1/2}$, and the remainder $\mathcal{R}_0(\varphi)$, as well as $\partial_\varphi^\beta \mathcal{R}_0$ for all $|\beta| \leq \beta_0$ large enough, is a small, still variable coefficient operator, which is regularizing at a sufficiently high order, and satisfies tame estimates in Sobolev spaces.

2. *KAM reducibility.* The second task is to implement an iterative diagonalization scheme to reduce quadratically the size of the perturbation $\mathcal{R}_0(\varphi)$ in (2.22), completing the conjugation of $\mathcal{L}$ to a diagonal, constant coefficient system of the form

$$\omega \cdot \partial_\varphi + \mathrm{i}\mathrm{Op}(\mu_j) \tag{2.23}$$

   where $\mu_j = \mathtt{m}_{\frac12} |j|^{\frac12} \tanh^{\frac12}(\mathtt{h}|j|) + r(j) + \tilde r(j)$ are real and $\tilde r(j)$ are small. The numbers $\mathrm{i}\mu_j$ are the perturbed Floquet exponents of the quasi-periodic solution.

We underline that all the transformations used to achieve these tasks are quasi-periodically time-dependent changes of variables acting in phase spaces of functions of x (quasi-periodic Floquet operators). Therefore, they preserve the dynamical system structure of the conjugated linear operators.

Moreover all these changes of variables are bounded and satisfy tame estimates between Sobolev spaces. As a consequence, the estimates that we shall obtain on the final system (2.23) directly provide good tame estimates for the inverse of the operator (2.21) in the original physical coordinates.

We also note that the original system $\mathcal{L}$ is reversible and even and that all the transformations that we perform are reversibility preserving and even. The preservation of these properties ensures that in the final system (2.23) the numbers μ_j are real. Under this respect, the linear stability of the quasi-periodic standing wave solutions proved in Theorem 1.1 is obtained as a consequence of the reversible nature of the water waves equations.

Remark 2.5. *The above procedure – which we explain in detail below – is quite different with respect to the approach developed in the pioneering works of Plotnikov-Toland [14] and Iooss-Plotnikov-Toland [12] for time periodic gravity waves, as mentioned at the end of Section 1.1. In particular the item 2 of KAM reducibility is not required for the search of periodic solutions. Let us roughly explain why. The diagonal operator* $\omega \cdot \partial_\varphi + \mathcal{D}$*, where* $\mathcal{D} := \mathrm{i}\mathtt{m}_{\frac12}|D|^{\frac12}\tanh^{\frac12}(\mathtt{h}|D|) + \mathrm{i}r(D)$*, can be inverted assuming a diophantine condition as* $|\omega \cdot \ell + (\mathtt{m}_{\frac12}|j|^{\frac12}\tanh^{\frac12}(\mathtt{h}|j|) + r(j))| \geq \gamma\langle\ell\rangle^{-\tau}$*. In such a case its inverse* $(\omega \cdot \partial_\varphi + \mathcal{D})^{-1}$ *is an unbounded operator which loses* τ *derivatives. Actually it acts as a Fourier multiplier* $O(\gamma^{-1}\langle\partial_\varphi\rangle^\tau)$ *on the subspace of functions which are Fourier supported on the indices* $(\ell, j) \in \mathbb{Z}^\nu \times \mathbb{Z}$ *such that* $\omega \cdot \ell \sim \sqrt{|j|}$*. In the periodic case* $\nu = 1$ *and* $\omega \cdot \ell = \omega\ell \sim \sqrt{|j|}$*, so that* $\langle\partial_\varphi\rangle^\tau \sim \langle\partial_x\rangle^{\frac{\tau}{2}}$*. If the remainder* $\mathcal{R}_0$ *in (2.22) is a sufficiently regularizing operator, i.e.* $\mathcal{R}_0 = O(\partial_x^{-d})$ *for* $d \geq \tau/2$*, then the operator* $(\omega\partial_\varphi + \mathcal{D})^{-1}\mathcal{R}_0 \sim \langle\partial_x\rangle^{\frac{\tau}{2}}O(\partial_x^{-d})$ *is bounded and* $\omega\partial_\varphi + \mathcal{D} + \mathcal{R}_0$ *can be inverted by a Neumann series argument. In the quasi-periodic case, i.e.* $\nu \geq 2$*, the modulus* $|\ell|$ *is not equivalent to* $\sqrt{|j|}$ *and the previous argument fails.*

We now explain in detail the steps for the conjugation of the quasi-periodic linear operator (2.21) to an operator of the form (2.23).

1. *Linearized good unknown of Alinhac.* The first step is to introduce the linearized good unknown of Alinhac, as in [1], [9]. The outcome is the more symmetric system

$$\mathcal{L}_0 = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V & -G(\eta) \\ a & V\partial_x \end{pmatrix} = \omega \cdot \partial_\varphi + \begin{pmatrix} V\partial_x & 0 \\ 0 & V\partial_x \end{pmatrix} + \begin{pmatrix} V_x & -G(\eta) \\ a & 0 \end{pmatrix}, \tag{2.24}$$

where the Dirichlet-Neumann operator admits the expansion

$$G(\eta) = |D|\tanh(\mathtt{h}|D|) + \mathcal{R}_G$$

and $\mathcal{R}_G$ is an $OPS^{-\infty}$ pseudo-differential smoothing operator. Such a representation can be obtained for example by transforming the elliptic problem (1.2), which is defined in the variable fluid domain $\{-\mathtt{h} \leq y \leq \eta(x)\}$, into another elliptic problem, defined on a straight strip $\{-\mathtt{h} - c \leq Y \leq 0\}$, which can be solved by an explicit integration.

2. *Straightening the first order vector field* $\omega \cdot \partial_\varphi + V(\varphi, x)\partial_x$*.* The next step is to conjugate the variable coefficients vector field (we regard equivalently a vector field as a differential operator)

$$\omega \cdot \partial_\varphi + V(\varphi, x)\partial_x$$

to the constant coefficient vector field $\omega \cdot \partial_\varphi$ on the torus $\mathbb{T}^\nu_\varphi \times \mathbb{T}_x$ for $V(\varphi, x)$ small. This a perturbative problem of rectification of a close to constant vector field on a torus, which is a classical small divisor problem. For perturbation of a Diophantine vector field this problem was solved at the beginning of KAM theory. Notice that, despite the fact that $\omega \in \mathbb{R}^\nu$ is Diophantine, the constant vector field $\omega \cdot \partial_\varphi$ is resonant on the higher dimensional torus $\mathbb{T}^\nu_\varphi \times \mathbb{T}_x$. We exploit in a crucial way the reversibility property of $V(\varphi, x)$, i.e. $V(\varphi, x)$ is odd in φ, to prove that it is possible to conjugate $\omega \cdot \partial_\varphi + V(\varphi, x)\partial_x$ to the constant vector field $\omega \cdot \partial_\varphi$ without changing the frequency ω.

From a functional point of view we have to solve the linear transport equation

$$\omega \cdot \partial_\varphi \beta(\varphi, x) + V(\varphi, x)(1 + \beta_x(\varphi, x)) = 0\,,$$

which depends on time in a quasi-periodic way. Actually we solve the equation

$$\omega \cdot \partial_\varphi \breve{\beta}(\varphi, y) = V(\varphi, y + \breve{\beta}(\varphi, y)) \tag{2.25}$$

for the inverse diffeomorphism

$$x + \beta(\varphi, x) = y \quad \Longleftrightarrow \quad x = y + \breve{\beta}(\varphi, y)\,, \quad \forall x, y \in \mathbb{R},\ \varphi \in \mathbb{T}^\nu\,.$$

This problem amounts to proving that all the solutions of the quasi-periodically time-dependent scalar characteristic equation $\dot{x} = V(\omega t, x)$ are quasi-periodic in time with frequency ω.

Remark 2.6. *A geometric interpretation of equation* (2.25) *is the following: under a diffeomorphism of* $\mathbb{T}^\nu \times \mathbb{T}$ *as*

$$\begin{pmatrix} \varphi \\ x \end{pmatrix} = \begin{pmatrix} \psi \\ y + \breve\beta(\psi, y) \end{pmatrix}, \quad \textit{the system} \quad \frac{d}{dt}\begin{pmatrix} \varphi \\ x \end{pmatrix} = \begin{pmatrix} \omega \\ V(\varphi, x) \end{pmatrix}$$

transforms into

$$\frac{d}{dt}\begin{pmatrix} \psi \\ y \end{pmatrix} = \begin{pmatrix} \omega \\ \big\{ -\omega \cdot \partial_\varphi \breve\beta(\psi, y) + V(\varphi, y + \breve\beta(\psi, y))\big\}\big(1 + \breve\beta_y(\psi, y)\big)^{-1} \end{pmatrix}.$$

The vector field in the new coordinates reduces to $(\omega, 0)$ *if and only if* (2.25) *holds. In the new variables the solutions are simply given by* $y(t) = c$, $c \in \mathbb{R}$, *and all the solutions of the scalar quasi-periodically forced differential equation* $\dot x = V(\omega t, x)$ *are time quasi-periodic of the form* $x(t) = c + \breve\beta(\omega t, c)$.

We solve (2.25) using a Nash-Moser implicit function theorem for V small. After inverting the corresponding linearized operator at an approximate solution we apply the Nash-Moser-Hörmander Theorem proved in Baldi-Haus [5]. The main advantage of this approach is to provide the optimal higher order regularity estimates of the solution in terms of V.

We remark that, when searching for time periodic solutions as in [12], [14], the corresponding transport equation is not a small-divisor problem and has been solved in [14] by a direct ODE analysis.

Applying this change of variable to the whole operator $\mathcal{L}_0$ in (2.24), the new conjugated system has the form

$$\mathcal{L}_1 = \omega \cdot \partial_\varphi + \begin{pmatrix} a_1 & -a_2|D|\tanh(\mathtt{h}|D|) + \mathcal{R}_1 \\ a_3 & 0 \end{pmatrix}$$

where $a_i = a_i(\varphi, x)$ are functions (i.e. multiplication operators) and the remainder $\mathcal{R}_1$ is in $OPS^{-\infty}$.

3. *Change of the space variable.* We introduce a change of variable induced by a diffeomorphism of $\mathbb{T}_x$ (independent of φ) of the form

$$y = x + \alpha(x) \qquad \Leftrightarrow \qquad x = y + \breve\alpha(y). \tag{2.26}$$

Conjugating $\mathcal{L}_1$ by the change of variable $u(x) \mapsto u(x + \alpha(x))$, we obtain an operator of the same form

$$\mathcal{L}_2 = \omega \cdot \partial_\varphi + \begin{pmatrix} a_4 & -a_5|D|T_{\mathtt{h}} + \mathcal{R}_2 \\ a_6 & 0 \end{pmatrix}, \qquad T_{\mathtt{h}} := \tanh(\mathtt{h}|D|),$$

where $\mathcal{R}_2$ is in $OPS^{-\infty}$, and the functions a_5, a_6 are given by

$$a_5 = \big[a_2(\varphi, x)(1 + \alpha_x(x))\big]_{|x = y + \breve\alpha(y)}, \quad a_6 = a_3(\varphi, y + \breve\alpha(y)).$$

We shall choose later the function $\alpha(x)$ in order to eliminate the space dependence from the highest order coefficients. The advantage to introduce at this step the diffeomorphism (2.26) is that it is easy to study the conjugation under this change of variable of differentiation and multiplication operators, Hilbert transform, and integral operators in $OPS^{-\infty}$.

4. *Symmetrization of the highest order.* We apply two simple conjugations (with a Fourier multiplier and a multiplication operator) whose goal is to obtain a new operator of the form

$$\mathcal{L}_3 = \omega \cdot \partial_\varphi + \begin{pmatrix} \breve a_4 & -a_7|D|^{\frac12}T_{\mathtt{h}}^{\frac12} \\ a_7|D|^{\frac12}T_{\mathtt{h}}^{\frac12} & 0 \end{pmatrix} + \ldots$$

up to lower order operators. The function a_7 is close to 1 and $\breve a_4$ is small in ε. In the complex unknown $h = \eta + \mathrm{i}\psi$ such an operator reads

$$(h, \bar h) \mapsto \omega \cdot \partial_\varphi h + \mathrm{i} a_7|D|^{\frac12}T_{\mathtt{h}}^{\frac12} h + a_8 h + P_5 h + Q_5 \bar h,$$

(here we neglect a projector π_0 on the constants) where $P_5(\varphi)$ is a φ-dependent families of pseudo-differential operators of order $-1/2$, and $Q_5(\varphi)$ of order 0. We shall call the former operator "diagonal", and the latter "off-diagonal", with respect to the variables $(h, \bar h)$.

5. *Symmetrization of the lower orders.* We reduce the off-diagonal term Q_5 to a pseudo-differential operator with very negative order, i.e. we conjugate the above operator to another one whose first component has the form

$$(h,\bar{h}) \mapsto \omega \cdot \partial_\varphi h + \mathrm{i} a_7(\varphi,x)|D|^{\frac{1}{2}} T_{\mathtt{h}}^{\frac{1}{2}} h + a_8 h + P_6 h + Q_6 \bar{h}\,, \tag{2.27}$$

where P_6 is in $OPS^{-\frac{1}{2}}$ and $Q_6 \in OPS^{-M}$ for some M large enough, in view of the reducibility scheme.

6. *Time and space reduction at the highest order.* We now eliminate the φ- and the x-dependence from the coefficient of the leading operator $\mathrm{i} a_7(\varphi,x)|D|^{\frac{1}{2}} T_{\mathtt{h}}^{\frac{1}{2}}$. We conjugate the operator (2.27) by the time-1 flow of the pseudo-PDE

$$\partial_\tau u = \mathrm{i}\beta(\varphi,x)|D|^{\frac{1}{2}} u$$

where $\beta(\varphi,x)$ is a small function to be chosen. This kind of transformations – which could be called "semi-Fourier integral operators" and correspond to pseudo-differential operators of type $(\frac{1}{2},\frac{1}{2})$ in the notation of Hörmander – have been introduced in [1] and studied as flows in [9].

Choosing appropriately the functions $\beta(\varphi,x)$ and $\alpha(x)$ (introduced in (2.26)), the final outcome is a linear operator of the form

$$\mathcal{L}_7(h,\bar{h}) = \omega \cdot \partial_\varphi h + \mathrm{i}\mathtt{m}_{\frac{1}{2}} T_{\mathtt{h}}^{\frac{1}{2}} |D|^{\frac{1}{2}} h + (a_8 + a_9 \mathcal{H}) h + P_7 h + \mathcal{T}_7(h,\bar{h})\,,$$

which has the constant coefficient $\mathtt{m}_{\frac{1}{2}} \approx 1$ at the highest order, while $\mathcal{H}$ is the Hilbert transform, P_7 is in $OPS^{-1/2}$ and the operator $\mathcal{T}_7$ is small, smoothing and satisfies tame estimates in Sobolev spaces. The constant $\mathtt{m}_{\frac{1}{2}}$ collects the quasi-linear effects of the non-linearity at the highest order.

7. *Reduction of the lower orders.* We further diagonalize the operator $\mathcal{L}_7$, reducing it to constant coefficients up to regularizing smoothing operators of very negative order $|D|^{-M}$. This is realized by applying an iterative sequence of pseudo-differential transformations that eliminate the φ- and the x-dependence of the diagonal symbols. The final system has the form

$$(h,\bar{h}) \mapsto \omega \cdot \partial_\varphi h + \mathrm{i}\mathtt{m}_{\frac{1}{2}} |D|^{\frac{1}{2}} \tanh^{\frac{1}{2}}(\mathtt{h}|D|) h + \mathrm{i} r_j(D) h + \mathcal{R}_0(\varphi)(h,\bar{h}) \tag{2.28}$$

where the constant Fourier multiplier $r(\xi)$ is real, even $r(\xi) = r(-\xi)$, satisfies

$$\sup_{j \in \mathbb{Z}} |j|^{\frac{1}{2}} |r_j|^{k_0,\gamma} \lesssim_M \varepsilon \gamma^{-(2M+1)}\,,$$

and the variable coefficient operator $\mathcal{R}_0(\varphi)$ is regularizing and satisfies tame estimates, see more precisely properties (2.29). We also remark that this final operator (2.28) is reversible and even, since all the previous transformations that we performed are reversibility preserving and even.

Our next goal is to diagonalize the operator (2.28); actually, it is sufficient to "almost-diagonalize" it by a KAM iterative scheme. The expression "almost-diagonalize" refers to the fact that we allow to add in (2.23) some non diagonal remainders which are small as $O(\varepsilon \gamma^{-2(M+1)} N_{n-1}^{-\mathtt{a}})$.

8. *KAM-reducibility scheme.* In order to decrease quadratically the size of the perturbation $\mathcal{R}_0$ we apply a KAM diagonalization iterative scheme. Such a scheme converges because the operators

$$\langle D \rangle^{\mathtt{m}+\mathtt{b}} \mathcal{R}_0 \langle D \rangle^{\mathtt{m}+\mathtt{b}+1}, \quad \partial_{\varphi_i}^{s_0+\mathtt{b}} \langle D \rangle^{\mathtt{m}+\mathtt{b}} \mathcal{R}_0 \langle D \rangle^{\mathtt{m}+\mathtt{b}+1}\,,\ i = 1,\dots,\nu\,, \tag{2.29}$$

satisfy tame estimates for some $\mathtt{b} := \mathtt{b}(\tau,k_0) \in \mathbb{N}$ and $\mathtt{m} := \mathtt{m}(k_0)$ which are large enough, independently of s. Such conditions are verified to hold assuming that M (the order of regularization of the remainder) is large enough (essentially $M = O(\mathtt{m}+\mathtt{b})$). This is the property that compensates, along the KAM iteration, the loss of derivatives in φ and x produced by the small divisors in the second order Melnikov non-resonance conditions.

The big difference of the KAM reducibility scheme implemented in [3] with respect to the one developed in [9] is that the second order Melnikov non-resonance conditions imposed in [3] are very weak – in particular, they lose regularity not only in the φ-variable, but also in the space variable x. For this reason we apply at each iterative step a smoothing procedure also in the space variable.

After the diagonalization into (2.23) of the linearized operator, we invert it, by imposing the first Melnikov non-resonance conditions. Since all the changes of variables that we performed in the diagonalization process satisfy tame estimates in Sobolev spaces, we finally conclude the existence of an approximate inverse of the linearized operator which satisfies tame estimates.

Finally we implement a differentiable Nash-Moser iterative scheme to prove Theorem 2.1.

References

[1] Alazard T., Baldi P., *Gravity capillary standing water waves*, Arch. Rat. Mech. Anal, 217, 3, 741-830, 2015.

[2] Alazard T., Burq N., Zuily C., *Cauchy theory for the gravity water waves system with non-localized initial data*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 33, 337-395, 2016.

[3] Baldi P., Berti M., Haus E., Montalto R., *Time quasi-periodic gravity water waves in finite depth*, preprint, arXiv:1708.01517.

[4] Baldi P., Berti M., Montalto R., *KAM for autonomous quasi-linear perturbations of KdV*. Ann. I. H. Poincaré (C) Anal. Non Linéaire, AN 33, 1589-1638, 2016.

[5] Baldi P., Haus E., *A Nash-Moser-Hörmander implicit function theorem with applications to control and Cauchy problems for PDEs*. Preprint 2016 (arxiv:1609.00213).

[6] Bambusi D., Berti M., Magistrelli E., *Degenerate KAM theory for partial differential equations*, Journal Diff. Equations, 250, 8, 3379-3397, 2011.

[7] Berti M., Bolle Ph., *A Nash-Moser approach to KAM theory*, Fields Institute Communications, special volume "Hamiltonian PDEs and Applications", 255-284, 2015.

[8] Berti M., Montalto R., *Quasi-periodic water waves*, J. Fixed Point Th. Appl. 19, no. 1, 129-156, 2017.

[9] Berti M., Montalto R., *KAM for gravity capillary water waves*, Memoires of AMS, Memo 891, to appear, preprint arXiv:1602.02411.

[10] Craig W., Sulem C., *Numerical simulation of gravity waves*. J. Comput. Phys., 108(1):73–83, 1993.

[11] Craig W., Worfolk P., *An integrable normal form for water waves in infinite depth*. Phys. D, 84 (1995), no. 3-4, 513-531.

[12] Iooss G., Plotnikov P., Toland J., *Standing waves on an infinitely deep perfect fluid under gravity*. Arch. Ration. Mech. Anal., 177(3):367–478, 2005.

[13] Kuksin S., *Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum*, Funktsional Anal. i Prilozhen. 2, 22-37, 95, 1987.

[14] Plotnikov P., Toland J., *Nash-Moser theory for standing water waves*. Arch. Ration. Mech. Anal., 159(1):1–83, 2001.

[15] Rüssmann H., *Invariant tori in non-degenerate nearly integrable Hamiltonian systems*, Regul. Chaotic Dyn. 6 (2) (2001) 119-204.

[16] Zakharov V. E., *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys., 9, 190-194, 1968.

Pietro Baldi, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126, Napoli, Italy. *E-mail:* `pietro.baldi@unina.it`

Massimiliano Berti, SISSA, Via Bonomea 265, 34136, Trieste, Italy. *E-mail:* `berti@sissa.it`

Emanuele Haus, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, Monte S. Angelo, 80126, Napoli, Italy. *E-mail:* `emanuele.haus@unina.it`

Riccardo Montalto, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland. *E-mail:* `riccardo.montalto@math.uzh.ch`