

Asymptotic behavior of an elastic satellite with internal friction

Emanuele Haus, Dario Bambusi

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Abstract

We study the dynamics of an elastic body whose shape and position evolve due to the gravitational forces exerted by a pointlike planet. The main result is that, if all the deformations of the satellite dissipate some energy, then under a suitable nondegeneracy condition there are only three possible outcomes for the dynamics: (i) the orbit of the satellite is unbounded, (ii) the satellite falls on the planet, (iii) the satellite is captured in synchronous resonance i.e. its orbit is asymptotic to a motion in which the barycenter moves on a circular orbit, and the satellite moves rigidly, always showing the same face to the planet. The result is obtained by making use of LaSalle's invariance principle and by a careful kinematic analysis showing that energy stops dissipating only on synchronous orbits. We also use in quite an extensive way the fact that conservative elastodynamics is a Hamiltonian system invariant under the action of the rotation group.

1 Introduction

In this paper, we study the dynamics of an elastic satellite interacting with a pointlike planet. Precisely, we study the dynamics of an elastic body, moving in the gravitational field generated by a pointlike mass. We consider the equations of motion of continuum mechanics, with body forces due to the gravitational fields and internal traction arising from the body deformation, without introducing any further approximation. We prove that, if the internal structure of the satellite is such that any deformation dissipates some energy and if a suitable nondegeneracy condition is satisfied, then the dynamics of the system has only three possible final behaviors:

- (i) the orbit of the satellite is unbounded;
- (ii) the satellite falls on the planet;
- (iii) the satellite is captured in synchronous resonance.

By item (iii) we mean that the shape of the body reaches a final configuration, that its center of mass moves on a circular orbit and that it always shows the same face to the planet, i.e. the planet is at rest in a frame comoving with the satellite.

Concerning the inner structure of the body we make as few assumptions as possible. Precisely we assume that the stress tensor is the sum of two terms, the first one being Hamiltonian, i.e. it is the L^2 gradient of a “stored energy functional”, and a second one being nonconservative. On the second term we only assume that, as a consequence of its presence, there is dissipation of energy at any time at which the time derivative of the Cauchy Green stress tensor does not vanish.

The idea of the proof is to use LaSalle’s principle (see [15]), which is a generalization of Lyapunov theorem. LaSalle’s principle ensures that any precompact orbit approaches an invariant set which is contained in the manifold where the Lie derivative of the energy vanishes. The core of the paper consists in characterizing such an invariant set. Since in such an invariant set the dynamics is conservative, it turns out that a convenient framework for our study is that of Hamiltonian systems with symmetry as developed for example in [18] or, in a form directly useful for our problem, in [20].

So we start by writing down the Lagrangian and the Hamiltonian of the conservative part of the system and then we add to the equations of motions the nonconservative forces.

Then we start analyzing the nondissipating manifold \mathcal{ND} . We first prove that \mathcal{ND} consists of rigid motions, and then we show that the motions laying on \mathcal{ND} are actually circular orbits. Finally we show that they are relative equilibria of the reduced Hamiltonian system obtained by exploiting the rotational invariance of the original Hamiltonian. At this point the application of LaSalle’s principle would allow to conclude that the orbit is asymptotic to a manifold obtained by taking the union of all the synchronous orbits. In order to prove that the system is actually asymptotic to a single synchronous orbit we exploit the conservation of angular momentum and we assume a nondegeneracy property stating that the relative equilibria are isolated. This property is discussed in detail in Section 4 and we show that it is typically fulfilled.

The present result still has some quite strong limitations. The main one is that we do not discuss existence and uniqueness for the Cauchy problem of the equations we study. As is well known this is one of the main open problems of elasticity, so we do not enter its discussion. Here we limit ourselves to assuming that the system we study is well posed. Our assumptions are true for example for any Galerkin cutoff of the system or if one can show that the system behaves as a parabolic system (as is expected in the dissipative case under consideration).

The second limitation of our result rests in the fact that we assume that the system is described by differential equations. This means that we do not consider the case where the system is described by an integrodifferential equation with delay, a case which can occur in elasticity. The case we have in mind is the one in which the dissipation is of the kind of that appearing in Navier Stokes equation. We expect that our theory can be extended to the case with delay, but for sure the methods should be adapted.

The study of the gravitational interaction between a deformable body and a pointlike mass traces its origin back to the pioneering work by Darwin [7, 8]. His work shows that, in some approximation, the effect of the internal dynamics of the satellite is just that of producing an effective dissipating effective force

on the orbital and spin degrees of freedom of the satellite. Darwin’s work was subsequently generalized by Kaula [13] and many other authors (for instance, [2, 12, 17, 19]). Critical reviews of the work by Darwin, Kaula and followers can be found in [9–11]. However, the Darwin-Kaula procedure is heuristic and, from a mathematical point of view, its range of validity is far from being clear. For this reason, in the present paper (as in [6]), the point of view is that of starting from first principles in order to obtain a rigorous mathematical proof of the phenomena under consideration.

We remark that the result of the present paper rules out the possibility that periodic orbits different from synchronous resonance exist. This is quite surprising, since some celestial bodies are known to be in a spin-orbit resonances different from the synchronous one (for example Mercury). Our interpretation is that orbits like the 3:2 resonance of Mercury are very likely to be either metastable or orbits eventually impacting the planet. We plan to investigating further in this direction.

In Section 2 we state the main result of the present paper: to this end, we recall the Lagrangian formalism for elastodynamics, we write down the related Cauchy problem and we formulate the nondegeneracy assumption. Section 3 is devoted to the proof of the main result: we recall the statement of LaSalle’s invariance principle, prove that the only solutions which dissipate no energy are synchronous orbits and apply La Salle principle to our system. Finally, in Section 4 we discuss the nondegeneracy assumption and we prove that typically it is fulfilled.

2 Statement of the main result

2.1 The setting

We study the dynamical system consisting of:

- (i) a pointlike mass M (which we will sometimes call “planet”), which is at rest and which is chosen as the origin of a system of coordinates;
- (ii) an elastic body, free to move in space; we will call this extended body “satellite”.

To deal with elastodynamics we use the framework of [18] and [20] from which we take some notations and formalism, that we now recall.

We denote by $\mathcal{B} \subset \mathfrak{R}^3$ the reference configuration of an elastic body and assume that \mathcal{B} is open and bounded with a smooth boundary $\partial\mathcal{B}$. We define the configuration space \mathcal{Q} to be a Banach space of maps¹ $\zeta : \mathcal{B} \rightarrow \mathfrak{R}^3$. Typically in elastodynamics one assumes $\mathcal{Q} \subset H^s(\mathcal{B})$ with s large enough; we will come back later to this point, for the moment we simply assume that ζ admits as many derivatives as needed.

In the conservative case, classical three dimensional elasticity is a Lagrangian system, the Lagrangian $\mathcal{L} : T\mathcal{Q} \rightarrow \mathfrak{R}$ is the difference of kinetic and potential

¹Actually we should restrict ourselves to the manifold of the maps s.t. $\det D\zeta > 0$, however we will consider this as a condition on the domain of definition of the system.

energy. In our case there are also some dissipative forces that will be added to the Lagrange equations. As usual $T\mathcal{Q} \simeq \mathcal{Q} \oplus \mathcal{Q}$ is the tangent bundle to \mathcal{Q} .

We start by writing down the conservative part of the system. The Lagrangian \mathcal{L} of the system is defined by

$$\mathcal{L} = K - U_g - U_{sg} - U_e , \quad (2.1)$$

where

$$K(\dot{\zeta}) := \frac{1}{2} \int_{\mathcal{B}} \rho_0(\mathbf{x}) \left| \dot{\zeta}(\mathbf{x}) \right|^2 d^3 \mathbf{x} \quad (2.2)$$

$$U_g(\zeta) := \int_{\mathcal{B}} \rho_0(\mathbf{x}) V_g(\zeta(\mathbf{x})) d^3 \mathbf{x} , \quad (2.3)$$

$$U_{sg}(\zeta) := \int_{\mathcal{B}} \rho_0(\mathbf{x}) V_{sg}^\zeta(\zeta(\mathbf{x})) d^3 \mathbf{x} , \quad (2.4)$$

$$U_e(D\zeta) := \int_{\mathcal{B}} W(\mathbf{x}, D\zeta(\mathbf{x})) d^3 \mathbf{x} , \quad (2.5)$$

the functions V_g, V_{sg}^ζ are defined by

$$V_g(\chi) := -\frac{kM}{|\chi|} , \quad (2.6)$$

$$V_{sg}^\zeta(\chi) := -\int_{\mathcal{B}} \frac{k\rho_0(\mathbf{x})}{|\zeta(\mathbf{x}) - \chi|} d^3 \mathbf{x} , \quad (2.7)$$

and W is the stored energy function; k is the universal gravitational constant, and $\rho_0 \in C^\infty(\mathcal{B})$ the density of the body in the reference configuration². The stored energy function is assumed to depend on ζ only through the deformation gradient $\mathbf{F} := D\zeta \equiv \{\partial\zeta^i/\partial x^a\}$. We assume that W is frame independent in the sense that

$$W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, R\mathbf{F}) \text{ for all } R \in SO(3) . \quad (2.8)$$

As shown in [20] this implies that the Kirchoff stress tensor, namely

$$\tau_j^i = \frac{\partial\zeta^i}{\partial x^a} \frac{\partial W}{\partial(\partial\zeta^j/\partial x^a)} \quad (\text{sum over } a \text{ understood}) , \quad (2.9)$$

is symmetric.

Assuming the stress free boundary condition, namely

$$\frac{\partial W}{\partial(\partial\zeta^j/\partial x^a)} n_a \Big|_{\partial\mathcal{B}} = 0 , \quad (2.10)$$

where $\mathbf{n} \equiv (n_1, n_2, n_3)$ is the external normal to $\partial\mathcal{B}$, one deduces the standard Lagrange equations:

$$\rho_0 \ddot{\zeta} = -\nabla_\zeta \mathcal{L} \equiv -\rho_0 \frac{\partial V_g}{\partial \chi}(\zeta) - \rho_0 \frac{\partial V_{sg}^\zeta}{\partial \chi}(\zeta) + \frac{\partial}{\partial x^a} \frac{\partial W}{\partial(\partial\zeta/\partial x^a)} , \quad (2.11)$$

where $\nabla_\zeta \mathcal{L} \equiv (\nabla_{\zeta^1} \mathcal{L}, \nabla_{\zeta^2} \mathcal{L}, \nabla_{\zeta^3} \mathcal{L})$ is as usual the gradient with respect to the L^2 scalar product³ and coincides with the expression at r.h.s. of (2.11).

²Of course we assume that $\rho_0(\mathbf{x}) \neq 0 \forall \mathbf{x} \in \mathcal{B}$.

³i.e. it is defined by $d\mathcal{L}(\zeta)h = \langle \nabla \mathcal{L}(\zeta); h \rangle_{L^2}$ for all $h \in \mathcal{Q}$.

Remark 2.1. *It is easy to check that, if a function $U(\zeta)$ is rotation invariant, i.e. $U(R\zeta) = U(\zeta)$, $\forall R \in SO(3)$, then*

$$[\nabla U](R\zeta) = R\nabla U(\zeta) . \quad (2.12)$$

All the terms of the Lagrangian have this property.

Since the Lagrangian is independent of time, the energy

$$\mathcal{H} = K + U_g + U_{sg} + U_e , \quad (2.13)$$

is formally conserved for the system (2.11) with the boundary conditions (2.10).

Furthermore, since the Lagrangian is invariant under the group action

$$\mathcal{Q} \times SO(3) \ni (\zeta, R) \mapsto R\zeta , \quad (2.14)$$

by Nöther's theorem the quantity

$$\mathbf{L} := \int_B \zeta(\mathbf{x}) \times \rho_0(\mathbf{x}) \dot{\zeta}(\mathbf{x}) d^3 \mathbf{x} , \quad (2.15)$$

is conserved for the system. Of course \mathbf{L} coincides with the total angular momentum.

In order to get the equations governing the non conservative dynamics one has simply to add the nonconservative forces⁴, $G = G(\zeta, \zeta_t)$ i.e. to substitute equation (2.11) with the equation

$$\rho_0 \ddot{\zeta} = -\nabla_{\zeta} \mathcal{L} - G . \quad (2.16)$$

In order to write down the precise assumptions on G (which will be given in the next subsection) we have also to introduce the (right) Cauchy Green deformation tensor $C := (D\zeta)^T D\zeta$ or, componentwise

$$C^{ij} = \sum_a \frac{\partial \zeta^i}{\partial x^a} \frac{\partial \zeta^j}{\partial x^a} . \quad (2.17)$$

As it is well known C is symmetric, positive definite and allows to write $D\zeta$ in the polar decomposition form $D\zeta(\mathbf{x}) = R(\mathbf{x}) \sqrt{C(\mathbf{x})}$, where $R(\mathbf{x}) \in SO(3)$ is a rotation matrix.

2.2 The Cauchy problem

In the following we will always denote by $y \equiv (\zeta, \dot{\zeta}) \in T\mathcal{Q} \simeq \mathcal{Q} \oplus \mathcal{Q}$ a point in the space of initial data for our system (which we keep distinct from the phase space, in which the velocities will be substituted by the momenta).

The problem of existence of solutions for the system (2.11), (2.16) with the boundary conditions (2.10) is a major problem of elastodynamics (see e.g. [5]). Here, we do not want to enter such problem, so *we will limit ourselves to assuming the needed well-posedness properties*. Furthermore we will only consider classical solutions, so we give the following definition:

⁴by this notation we mean that G is a function of the functions ζ, ζ_t , not of their value $\zeta(\mathbf{x}), \zeta_t(\mathbf{x})$, so it can also depend on an arbitrary number of derivatives of such functions.

Definition 2.2. Given $y_0 \in T\mathcal{Q}$, a positive T and a function $y \in C^2((0, T); T\mathcal{Q})$ we say that it is a solution of the system (2.16), (2.10) with initial datum y_0 , if it fulfills the equations and the boundary conditions for all $t \in (0, T)$ and one has

$$\lim_{t \rightarrow 0^+} y(t) = y_0 .$$

Then we also need the following definitions:

Definition 2.3. A solution is said to be impacting (in the future) if $\inf_{t>0} \text{dist}(\zeta(\mathcal{B}, t), 0) = 0$.

Definition 2.4. A configuration is said to be non singular if $\det[D\zeta] > 0$.

Definition 2.5. A solution $y(t)$, $t \in (0, T)$ is said to be regular if it is non impacting and the corresponding configuration is non singular for all times. An initial datum $y_0 \equiv (\zeta_0, \dot{\zeta}_0)$ is regular if $\text{dist}(\zeta_0(\mathcal{B}), 0) > 0$ and $\det[D\zeta_0] > 0$.

Definition 2.6. A regular solution $y(t)$, $t \in (0, \infty)$ is said to be precompact if, for any increasing sequence $\{t_n\} \subset (0, \infty)$ there exists a subsequence $\{t_{n_k}\}$ s.t. the limit $\lim_{k \rightarrow +\infty} y(t_{n_k})$ exists and the limit is regular.

Assumption 1. We assume that

- (i) for all regular initial data $y_0 \in T\mathcal{Q}$, the Cauchy problem for the system (2.16) with the boundary conditions (2.10) is locally well-posed;
- (ii) let $y(t)$ be a non-impacting solution. Then its time of existence is infinite and it is forever non singular.
- (iii) Any regular solution fulfilling $\sup_{\mathbf{x} \in \mathcal{B}, t>0} |\zeta(\mathbf{x}, t)| < \infty$ is precompact;
- (iv) the angular momentum \mathbf{L} is conserved along the solutions; the Lie derivative of the energy (2.13) is nonpositive and vanishes if and only if $\dot{C} = 0$, where C is the Cauchy Green tensor (cf. eq. (2.17)).

Remark 2.7. If one thinks that the nonconservative equations we are studying behave like parabolic equations, then typically the situation of Assumption 1 is fulfilled. Indeed, for positive times the solution of a parabolic equation is contained in all Sobolev spaces. Existence of global solutions has been proved for some equations of viscoelasticity (see [14] and [16]). Moreover, the property of having a compact attractor is known for many classes of evolution equations involving dissipation (see [23]).

Remark 2.8. A situation in which Assumption 1 is fulfilled is that in which the space \mathcal{Q} is finite dimensional. A typical situation we have in mind is that in which the space \mathcal{Q} is composed by maps obtained by Galerkin cutoff from some of the maps belonging to the original infinite dimensional configuration space. For example one could decide to keep only a finite (arbitrarily large) number of spherical harmonics of the maps describing the configuration.

2.3 The nondegeneracy assumption

In the following (see subsect. 3.2) we will prove that the nondissipating orbits are relative equilibria of the Hamiltonian system obtained by Legendre transforming the Lagrangian (2.1). We are now going to recall the notion of relative equilibrium and to state the nondegeneracy condition we need.

Define the momentum

$$\pi := \rho_0 \dot{\zeta} , \quad (2.18)$$

then the Hamiltonian of the system coincides with the function \mathcal{H} given by (2.13), where however

$$K := \int_{\mathcal{B}} \frac{|\pi(\mathbf{x})|^2}{2\rho_0(\mathbf{x})} d^3\mathbf{x} ; \quad (2.19)$$

and the Lagrange equations (2.11) are equivalent to the Hamilton equations of (2.13).

The Hamiltonian is invariant under the action of the symmetry group $SO(3)$ defined by

$$Rz \equiv R(\pi, \zeta) := (R\pi, R\zeta) , \quad (2.20)$$

and the total angular momentum \mathbf{L} (written in terms of positions and momenta) is the corresponding conserved quantity. Then one can use Marsden-Weinstein reduction procedure, that can be summarized as follows.

- (1) Fix a value \mathbf{L}_0 of \mathbf{L} and consider the manifold

$$\mathcal{M}_{\mathbf{L}_0} := \{z \equiv (\pi, \zeta) : \mathbf{L}(z) = \mathbf{L}_0\} ;$$

- (2) Consider the subgroup $\mathcal{G}_{\mathbf{L}_0} \subset SO(3)$ leaving invariant $\mathcal{M}_{\mathbf{L}_0}$, namely the group of the rotations around the axis \mathbf{L}_0 . Consider the quotient manifold $\mathcal{M}_{\mathbf{L}_0}/\mathcal{G}_{\mathbf{L}_0}$. Such a manifold has a natural symplectic structure and furthermore the Hamiltonian \mathcal{H} , as well as its Hamilton equations, pass to the quotient and define a Hamiltonian system on $\mathcal{M}_{\mathbf{L}_0}/\mathcal{G}_{\mathbf{L}_0}$.

We denote by $\mathcal{H}_{\mathbf{L}_0}$ the Hamiltonian of such a reduced system.

Definition 2.9. *The critical points of $\mathcal{H}_{\mathbf{L}_0}$ are called relative equilibria of the Hamiltonian system \mathcal{H} , at angular momentum \mathbf{L}_0 .*

Definition 2.10. *A relative equilibrium is said to be topologically nondegenerate if it is not an accumulation point of relative equilibria with the same angular momentum.*

By abuse of notation a representative z_e of the equivalence class of a relative equilibrium is also called a relative equilibrium of \mathcal{H} .

Remark 2.11. *It is well known (see e.g. [1]) that z_e is a relative equilibrium if and only if the Hamiltonian vector field of \mathcal{H} at z_e is tangent to the orbit of $SO(3)$ through z_e .*

Remark 2.12. *If z_e is a relative equilibrium then the corresponding orbit $z(t)$ (under the flow of the Hamiltonian system \mathcal{H}) is formed by relative equilibria. In the nondegenerate case there are no other relative equilibria with the same angular momentum in a neighborhood of the orbit $\cup_t z(t)$.*

Of course a relative equilibrium z_e corresponding to a value \mathbf{L}_0 of \mathbf{L} is topologically nondegenerate if and only if the same is true for the relative equilibrium Rz_e , where $R \in SO(3)$ is arbitrary.

Definition 2.13. *A value $\ell \in \mathfrak{R}$ of the modulus of the angular momentum is said to be nondegenerate if all the relative equilibria with angular momentum \mathbf{L} satisfying $|\mathbf{L}| = \ell$ are topologically nondegenerate relative equilibria of \mathcal{H} .*

Remark 2.14. *In Section 4 we will comment on this condition and show that it is in general fulfilled.*

Theorem 2.15. *Let $y(t)$ be a solution of eq. (2.16) with the boundary condition (2.10), and let ℓ be the corresponding value of the modulus of the angular momentum. Assume that ℓ is nondegenerate, then, one of the following three (future) scenarios is possible:*

- (i) *the trajectory of \mathcal{B} is unbounded;*
- (ii) *the solution impacts the planet;*
- (iii) *the solution is asymptotic to a synchronous non-dissipating orbit, which is a relative equilibrium with angular momentum ℓ .*

3 Proof of theorem 2.15

3.1 LaSalle's invariance principle

In order to study the dynamics of the system, we make use of the LaSalle's principle which is a refinement of the classical Lyapunov's theorem. We now recall its statement and proof.

Let \mathcal{Y} be a Banach space and let $\mathcal{U} \subset \mathcal{Y}$ be open. Consider a system of differential equations

$$\dot{y} = f(y) \quad y \in \mathcal{U}, \quad (3.1)$$

We denote by φ the flow of (3.1), which we assume to be locally well defined.

Definition 3.1. *Let γ be the orbit of (3.1) with initial condition y_0 . A point η is said to be an ω -limit point of γ if there exists a sequence of times $t_n \rightarrow +\infty$ such that*

$$\lim_{n \rightarrow +\infty} \varphi^{t_n}(y_0) = \eta. \quad (3.2)$$

Definition 3.2. *The ω -limit set of an orbit γ is defined as the union of all the ω -limit points of γ .*

Definition 3.3. *A solution $y(t) \subset \mathcal{U}$, $t \in (0, \infty)$ is said to be precompact if, for any increasing sequence $\{t_n\} \subset (0, \infty)$ there exists a subsequence $\{t_{n_k}\}$ s.t. the limit $\lim_{k \rightarrow +\infty} y(t_{n_k})$ exists and belongs to \mathcal{U} .*

Remark 3.4. *It is well known that the ω -limit of a precompact orbit is a connected set (see e.g. [1]).*

The classical version of LaSalle's principle may be stated as follows:

Theorem 3.5. *Suppose that $\mathcal{H} : \mathcal{U} \rightarrow \mathfrak{R}$ is a real-valued smooth function, such that $\mathcal{L}_f \mathcal{H}(y) \leq 0$, $\forall y \in \mathcal{U}$, where \mathcal{L}_f is the Lie derivative. Let \mathcal{I} be the largest invariant set contained in $\mathcal{ND} := \{y \in \mathcal{U} | \mathcal{L}_f \mathcal{H}(y) = 0\}$, then the ω -limit of every precompact orbit is a non-empty subset of \mathcal{I} .*

Proof. Let $\gamma := \{\varphi^t(y_0) | t > 0\}$ be a precompact orbit, and let $\Gamma \subset \mathcal{U}$ be the ω -limit of γ . We prove now that Γ is invariant. Indeed, let $\eta \in \Gamma$, then there exists a sequence $t_n \rightarrow +\infty$ such that $\varphi^{t_n}(y_0) \rightarrow \eta$. But we have

$$\varphi^t(\eta) = \varphi^t\left(\lim_{n \rightarrow +\infty} \varphi^{t_n}(y_0)\right) = \lim_{n \rightarrow +\infty} \varphi^{t+t_n}(y_0) \in \Gamma .$$

We prove now that the ω -limit is contained in \mathcal{ND} . Let $\eta_0 \in \Gamma$. Then there exists a sequence $t_n \rightarrow +\infty$ such that $\varphi^{t_n}(y_0) \rightarrow \eta_0$. Now, let

$$c := \mathcal{H}(\eta_0) = \lim_{n \rightarrow +\infty} \mathcal{H}[\varphi^{t_n}(y_0)] .$$

Since $\mathcal{H}[\varphi^t(y_0)]$ is a time-nonincreasing function, $\lim_{n \rightarrow +\infty} \mathcal{H}[\varphi^{t_n}(y_0)] = c$ is independent of the subsequence t_n , and thus $\mathcal{H} = c$ on the whole Γ . By the invariance of Γ it follows that $\mathcal{L}_f \mathcal{H} = 0$ on Γ , and therefore $\gamma \subset \mathcal{I}$. \square

3.2 Non-dissipating orbits

Consider the non dissipating manifold defined by

$$\mathcal{ND} := \left\{ y \in \mathcal{Q} \oplus \mathcal{Q} : \dot{\mathcal{H}}(y) = 0 \right\} , \quad (3.3)$$

where, for short we denoted by $\dot{\mathcal{H}}(y)$ the Lie derivative of \mathcal{H} along the vector field corresponding to the equations (2.16). In this section we prove that the subset $\mathcal{I} \subset \mathcal{ND}$ invariant under the dynamics is formed by relative equilibria of the Hamiltonian system (2.13).

Remark 3.6. *On \mathcal{ND} the Lagrange equations (2.11) coincide with the non conservative equations (2.16).*

First we prove that the body is rigid along any orbit in \mathcal{I} (we think that this should be well known, but we were not able to find a reference).

Lemma 3.7. *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND} \forall t \in (0, +\infty)$, then, for all $\mathbf{x}, \mathbf{y} \in \mathcal{B}$, one has*

$$\frac{d}{dt} |\zeta(\mathbf{x}) - \zeta(\mathbf{y})| = 0 . \quad (3.4)$$

Proof. Fix two arbitrary points $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ and consider a path $\gamma \subset \mathcal{B}$ connecting \mathbf{x} to \mathbf{y} . Let s be the arclength parameter, so that $|\frac{d\gamma(s)}{ds}| = 1$ and in the reference configuration the path γ has length

$$\ell(\gamma) = \int_{\gamma} ds .$$

The length of the deformed path $\zeta(\gamma)$ is expressed in terms of the Cauchy Green tensor C by (see e.g. [21])

$$\ell(\zeta(\gamma)) = \int_{\gamma} \left[\left(\frac{d\gamma(s)}{ds} \right)^T C \left(\frac{d\gamma(s)}{ds} \right) \right]^{\frac{1}{2}} ds . \quad (3.5)$$

Therefore, since $\dot{C} \equiv 0$, we have $\frac{d}{dt}[\ell(\zeta(\gamma))] \equiv 0$. Now, take two arbitrary times t_0, t_1 and let ζ_0, ζ_1 be the corresponding body configurations. We have that $\zeta_1 \circ (\zeta_0)^{-1} : \zeta_0(\mathcal{B}) \rightarrow \zeta_1(\mathcal{B})$ is a length-preserving map between the $\zeta_0(\mathcal{B}) \subset \mathfrak{R}^3$ and $\zeta_1(\mathcal{B}) \subset \mathfrak{R}^3$, both equipped with the restriction of the Euclidean metric on \mathfrak{R}^3 . Moreover, $\zeta_1 \circ (\zeta_0)^{-1}$ is a diffeomorphism.

Using the fact that the segments minimize the distance it is easy to conclude the proof of the lemma (some care is needed in order to take care of the fact that $\zeta_1(\mathcal{B})$ could fail to be convex). \square

Corollary 3.8. *Let $y = (\zeta, \dot{\zeta})$ be as in the statement of Lemma 3.7 then there exist $\xi \in \mathcal{Q}$, $R \in C^2((0, +\infty), SO(3))$ and $\mathbf{Y} \in C^2((0, +\infty), \mathfrak{R}^3)$ s.t.*

$$\zeta(\mathbf{x}, t) = R(t)(\xi(\mathbf{x}) + \mathbf{Y}(t)) . \quad (3.6)$$

The reference frame with origin $\mathbf{Y}(t)$ and coordinate axes $R(t)\mathbf{e}_i$ is usually called *comoving frame*. In this frame all the points of the satellite are at rest along the orbit $y(t)$. In particular $-\mathbf{Y}(t)$ is the position of the planet M in the comoving frame.

Lemma 3.9. *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND} \forall t \in (0, +\infty)$, then the quantity $\mathbf{Y}(t)$ in (3.6) evolves in such a way that*

$$\forall i, j, k, \quad \frac{\partial^3 V_g}{\partial \chi^i \partial \chi^j \partial \chi^k}(\mathbf{Y}(t)) = 0 \quad (3.7)$$

is independent of time.

Proof. Inserting the expression (3.6) in the Lagrange equations (2.11) and exploiting the rotational invariance of the r.h.s. (cf. Remark 2.1) one gets the following equation for ξ , R and \mathbf{Y} :

$$\begin{aligned} & \left[\ddot{\mathbf{Y}} + (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega})\mathbf{Y} + 2\hat{\omega}\dot{\mathbf{Y}} + (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega})\xi \right] \\ & = -\frac{\partial V_g}{\partial \chi}(\xi + \mathbf{Y}) - \frac{\partial V_{sg}^\zeta}{\partial \chi}(\xi) + \frac{1}{\rho_0} \frac{\partial}{\partial x^a} \frac{\partial W}{\partial (\partial \zeta / \partial x^a)}(D\xi) , \end{aligned} \quad (3.8)$$

where as usual $\hat{\omega} := R^T \dot{R}$. Denote for short

$$L := (\dot{\hat{\omega}} + \hat{\omega}\hat{\omega}) \quad (3.9)$$

and take the time derivative of (3.8). Taking into account that ξ does not depend on time one gets

$$\left\{ \frac{d}{dt} \left[\ddot{\mathbf{Y}} + L\mathbf{Y} + 2\hat{\omega}\dot{\mathbf{Y}} \right] + \dot{L}\xi \right\} = \frac{d}{dt} \left[-\frac{\partial V_g}{\partial \chi}(\xi + \mathbf{Y}) \right] . \quad (3.10)$$

Take now the derivative of such a quantity with respect to x^a , one gets the componentwise equation

$$\left(\sum_k \dot{L}_k^i \frac{\partial \xi^k}{\partial x^a} \right) = \sum_k \frac{d}{dt} \left[-\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k}(\xi + \mathbf{Y}) \frac{\partial \xi^k}{\partial x^a} \right] , \quad (3.11)$$

or using the invertibility of the matrix $\frac{\partial \xi^k}{\partial x^a}$

$$\dot{L}_k^i = -\frac{d}{dt} \left[\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k} (\xi(\mathbf{x}) + \mathbf{Y}(t)) \right]. \quad (3.12)$$

This equation shows in particular that the r.h.s. is independent of \mathbf{x} . Due to the invertibility of ξ and to the analyticity of V_g this means that the function of χ

$$-\frac{d}{dt} \left[\frac{\partial^2 V_g}{\partial \chi^i \partial \chi^k} (\chi + \mathbf{Y}(t)) \right] \quad (3.13)$$

is actually independent of χ . Thus taking the derivative with respect to χ^j and evaluating at $\chi = 0$ one gets the thesis. \square

Lemma 3.10. *Let $y \in C^2((0, +\infty), \mathcal{Q} \oplus \mathcal{Q})$ be a solution of (2.16) s.t. $y(t) \equiv (\zeta(t), \dot{\zeta}(t)) \in \mathcal{ND} \forall t \in (0, +\infty)$, then the quantity $\mathbf{Y}(t)$ in (3.6) is actually independent of time.*

Proof. We write down explicitly (3.7). We denote $(Y^1(t), Y^2(t), Y^3(t)) = \mathbf{Y}(t)$ and $(Y^1)^2 + (Y^2)^2 + (Y^3)^2 = r^2$. One has

$$\frac{\partial^3 V_g}{\partial (\chi^1)^3}(\mathbf{Y}) = \frac{3kMY^1(5(Y^1)^2 - 3r^2)}{r^7} \quad (3.14)$$

$$\frac{\partial^3 V_g}{\partial (\chi^1)^2 \partial \chi^2}(\mathbf{Y}) = \frac{3kMY^2(5(Y^1)^2 - r^2)}{r^7} \quad (3.15)$$

$$\frac{\partial^3 V_g}{\partial (\chi^1)^2 \partial \chi^3}(\mathbf{Y}) = \frac{3kMY^3(5(Y^1)^2 - r^2)}{r^7}. \quad (3.16)$$

Choose now the comoving frame in such a way that $Y^2 = Y^3 = 0$ and $Y^1 = r$ at $t = 0$ (which is possible up to redefinition of R and ξ). By (3.15) and (3.16), for any t one must have

$$Y^2(5(Y^1)^2 - r^2) = 0$$

$$Y^3(5((Y^1)^2 - r^2)) = 0$$

which by continuity implies $Y^2(t) \equiv Y^3(t) \equiv 0$ and $Y^1(t) \equiv r(t)$. Finally, substituting in (3.14), one has

$$\frac{1}{(Y^1(t))^4} = \frac{1}{(Y_0^1)^4},$$

whose only solution is $Y^1(t) \equiv Y_0^1$. \square

Remark now that, given a non dissipating solution, one can associate to it a shape of the body described by the function $\xi(\mathbf{x}) + \mathbf{Y}$, and the shape evolves by a rigid motion about the fixed point M . Introduce the angular velocity which is defined as usual as the vector ω s.t. the two operators

$$\omega \times \cdot = \hat{\omega}$$

coincide. Then the velocity of the motion, in the comoving frame, is given by $\omega \times (\xi + \mathbf{Y})$.

We have now that, for a non dissipating solution, ω does not depend on time.

Lemma 3.11. *The angular velocity ω of a nondissipating solution y is independent of time.*

Proof. Consider again equation (3.12). Since we now know that \mathbf{Y} is independent of time it follows that the operator L (cf. eq. (3.9)) fulfills $\dot{L} = 0$. This means that, for any vector χ one has

$$\frac{d}{dt} [\dot{\omega} \times \chi + \omega \times (\omega \times \chi)] = 0 . \quad (3.17)$$

To exploit such an equation take $\chi = \mathbf{e}_i$ and project the square bracket on \mathbf{e}_i . Using standard vector identities this implies

$$\frac{d}{dt} |\omega \times \mathbf{e}_i|^2 = 0 \quad \forall i .$$

A trivial computation shows that this implies $\dot{\omega} = 0$. □

Corollary 3.12. *Let $y(t)$ be a nondissipating solution as above, then it is the orbit of a relative equilibrium of the system (2.13).*

Proof. We have proved that along a non dissipating solution $\zeta(t) = R(t)\zeta_0$ with a suitable configuration ζ_0 and a rotation matrix $R(t)$ that we can choose in such a way that $R(0) = I$. It follows $\dot{\zeta}(t) = R(t)[\omega \times \xi] = R\dot{\zeta}(0)$. Passing to the phase space one gets that along such an orbit $\pi(t) = \rho_0\dot{\zeta}(t) = R(t)\pi(0)$. This shows that the solution is actually an orbit of the symmetry group, and this is a characterization of being a relative equilibrium. □

Thus we have that the manifold \mathcal{I} is the union of the trajectories of all the possible relative equilibria of the system.

3.3 End of the proof

Applying La Salle principle to our system, with \mathcal{U} defined to be the set of regular configurations, we get the following Lemma.

Lemma 3.13. *For any solution to (2.16) with the boundary condition (2.10), one of the following three (future) scenarios occur:*

- (i) *the trajectory is unbounded;*
- (ii) *the solution impacts the planet;*
- (iii) *the solution is asymptotic to the non-dissipating invariant manifold \mathcal{I} .*

End of the proof of Theorem 2.15. Let \mathbf{L}_0 be the initial value of the angular momentum, then a regular bounded solution is asymptotic to

$$\mathcal{I} \cap \left\{ (\zeta, \dot{\zeta}) \in T\mathcal{Q} : \mathbf{L}(\zeta, \dot{\zeta}) = \mathbf{L}_0 \right\} , \quad (3.18)$$

but, by the nondegeneracy assumption the set (3.18) is formed by orbits which are isolated in the invariant manifold $\left\{ (\zeta, \dot{\zeta}) \in T\mathcal{Q} : \mathbf{L}(\zeta, \dot{\zeta}) = \mathbf{L}_0 \right\}$. Thus, by Remark 3.4 the ω -limit set of an orbit is a single orbit in the set (3.18), i.e. a synchronous orbit. □

4 On the nondegeneracy assumption

In this section we are going to prove that if the restoring elastic forces described by the potential (2.5) are strong enough, then the nondegeneracy of a relative equilibrium for the system (2.1) is equivalent to the nondegeneracy of the relative equilibrium for a rigid body having the shape given by the asymptotic configuration of the satellite.

For simplicity, in this section we limit the discussion at the formal level, namely we forget all the difficulties related to the existence of unbounded operators. All what follows is rigorous if \mathcal{Q} is finite dimensional. It can also be made rigorous in the case of PDEs by detailing most of the assumptions, following the ideas of [3] and exploiting the ellipticity properties of the elasticity tensor (see [18]), however this is outside the aims of the present paper.

First of all, having fixed a configuration $\bar{\zeta}$ we introduce the dynamical system describing the evolution of a rigid satellite with shape $\bar{\zeta}$. The configuration space $SO(3) \times \mathbb{R}^3 \ni (R, \chi)$ and the dynamics is obtained from the Lagrangian obtained by restricting the Lagrangian (2.1) to the set of motions of the form

$$\zeta(t) = R(t)[\bar{\zeta} + \chi(t)] . \quad (4.1)$$

Denote by $\mathcal{L}_{\bar{\zeta}}$ such a Lagrangian. One also has a corresponding Hamiltonian system, which is deduced from (2.1) in the standard way and whose Hamiltonian coincides with the restriction of the Hamiltonian (2.13) to the phase space of the rigid body. Denote by $H_{\bar{\zeta}}$ the Hamiltonian of the rigid body. Such a Hamiltonian is invariant under rotations so one can pass to the reduced system and to introduce again the relative equilibria and define the nondegenerate relative equilibria for such a system and a nondegenerate value of the modulus of the angular momentum.

In order to make a connection between the nondegeneracy of the relative equilibria for rigid motions and the nondegeneracy of the elastic motions we need to specify an assumption on the elastic potential energy U_e . Essentially we are going to assume that the elastic potential has a very steep, isolated (up to the symmetries) minimum at some shape.

First remark that U_e is invariant under the action

$$(SO(3) \times \mathbb{R}^3) \times \mathcal{Q} \ni ((R, \chi), \zeta) \mapsto R[\zeta + \chi] \in \mathcal{Q} , \quad (4.2)$$

and, given a point ζ , consider the group orbit $\mathcal{G}_{\zeta} := (SO(3) \times \mathbb{R}^3)\zeta \subset \mathcal{Q}$, so, if $\bar{\zeta}$ is a critical point of U_e , then all the orbit $\mathcal{G}_{\bar{\zeta}}$ is critical for U_e .

Assumption 2. *One has $U_e = \frac{1}{\epsilon} \tilde{U}_e$, and \tilde{U}_e is a smooth function invariant under the group action (4.2) with the further property that the set of its critical points is formed by finitely many orbits $\mathcal{G}_{\zeta^{(i)}}$ and each critical point is nondegenerate in the direction transversal to the group orbit⁵.*

Under this assumption we have the following

Proposition 4.1. *Fix a value L_0 of the angular momentum, and assume ϵ is small enough; let ζ_e^ϵ be a relative equilibrium of the Hamiltonian system H*

⁵Transversal nondegeneracy means that the restriction of \tilde{U}_e to any hyperplane transversal to the group orbit has a differential which is an isomorphism.

with angular momentum \mathbf{L}_0 . If $(R, 0) \in SO(3) \times \mathfrak{R}^3$ is a nondegenerate relative equilibrium for the rigid system with Hamiltonian H_{ζ_e} then ζ_e^ϵ is a nondegenerate relative equilibrium for the elastic Hamiltonian system with Hamiltonian H .

Proof. The proof is based on ideas from Lyapunov-Schmidt decomposition (see e.g. [3]). First of all it is useful to introduce suitable coordinates in \mathcal{Q} in a neighborhood of ζ_e (following the ideas of [6]). They are constructed as follows: let $\Sigma \subset \mathcal{Q}$ be a codimension 6 affine subspace transversal to the group orbit \mathcal{G}_{ζ_e} , then a suitable set of coordinates about ζ_e is locally obtained by the map

$$\Sigma \times SO(3) \times \mathfrak{R}^3 \ni (\xi, R, \chi) \mapsto R(\xi + \chi) . \quad (4.3)$$

We now recall (and adapt to the present situation) some results of [20]. Using the method of Lagrange multipliers one immediately sees that the relative equilibria can be obtained by finding the critical points of $H - \omega \cdot (\mathbf{L} - \mathbf{L}_0)$ under the condition

$$\mathbf{L} = \mathbf{L}_0 . \quad (4.4)$$

Here ω is the Lagrange multiplier. In [20] it was shown that this is equivalent to finding the critical points of the ‘‘augmented Hamiltonian’’ $H_{\mathbf{L}_0}$ defined by

$$H_{\mathbf{L}_0} := K_{\mathbf{L}_0} + U_{\mathbf{L}_0} , \quad (4.5)$$

$$K_{\mathbf{L}_0} := \frac{1}{2} \int_{\mathcal{B}} \frac{|\pi - \rho_0(\omega \times \zeta)|^2}{\rho_0} d^3 \mathbf{x} , \quad (4.6)$$

$$U_{\mathbf{L}_0} := U - \frac{1}{2} \int_{\mathcal{B}} \rho_0 |\omega \times \zeta|^2 d^3 \mathbf{x} \quad (4.7)$$

again under the condition (4.4) (actually in the present case this is a straightforward computation). As pointed out in [20] the interest of this formulation is that the equations for the critical points of $H_{\mathbf{L}_0}$ take the form

$$\pi_e = \rho_0 \omega \times \zeta_e , \quad (4.8)$$

$$\nabla U(\zeta_e) + \rho_0 \omega \times (\omega \times \zeta_e) = \mathbf{0} . \quad (4.9)$$

In particular the second equation is independent of π . This allows to study separately (4.9). To this end we use the set of coordinates (4.3). However one has to pay attention to the fact that in general the section Σ is not orthogonal to the group orbit, so first we rewrite (4.9) in the (original) dual form:

$$dU(\zeta_e)h + \langle \rho_0 \omega \times (\omega \times \zeta_e); h \rangle_{L^2} = 0 , \quad \forall h \in \mathcal{Q} , \quad (4.10)$$

which in terms of the coordinates (4.3) takes the form (at $R = I$)

$$d_\chi U_g(\xi_e + \chi_e)h_\chi + \langle \rho_0 \omega \times (\omega \times (\xi_e + \chi_e)); h_\chi \rangle_{L^2} = 0 , \quad \forall h_\chi \in \mathfrak{R}^3 , \quad (4.11)$$

$$d_\xi [U_g(\xi_e + \chi_e) + U_{sg}(\xi_e + \chi_e) + \frac{1}{\epsilon} \tilde{U}_e(\xi_e + \chi_e)]h_\xi + \langle \rho_0 \omega \times (\omega \times (\xi_e + \chi_e)); h_\xi \rangle_{L^2} = 0 , \quad \forall h_\xi \in T_{\zeta_e} \Sigma . \quad (4.12)$$

which have to be solved together with

$$\pi_e = \rho_0 \omega \times (\xi_e + \chi_e) , \quad (4.13)$$

and the condition (4.4). In particular the system (4.11), (4.13), (4.4) is identical to the system for the reduced equilibrium of the rigid body with shape ζ_e , so by Assumption 2 it determines uniquely (up to a finite choice) χ_e and ω (and π_e). We analyze now (4.12). Of course it is a perturbation of $d_\xi \tilde{U}_e = 0$, whose critical points are all nondegenerate (as functions of ξ they are nondegenerate in the standard sense), so, by the implicit function theorem a solution ξ_e of (4.12) must be close to a critical point of \tilde{U}_e and is also nondegenerate. This concludes the proof of the proposition. \square

So we have reduced the problem of checking nondegeneracy to the problem of checking nondegeneracy of the relative equilibria for the motions of rigid bodies. This problem has been studied for example in [22] who obtained a complete characterization of the relative equilibria of a triaxial rigid body, provided the gravitational potential is approximated by its quadrupole expansion. In [22] the author obtained that, provided the distance χ_e of the center of mass from the planet is large enough, there are exactly 24 families of stationary points of the reduced system. These stationary points are such that the principal axes of inertia are one pointing to M and a second one in the plane orthogonal to the plane of motion. The number 24 appears as the number of possible choices of the orientations of the body with prescribed principal axes of inertia.

Remark that in particular it turns out that such critical points are nondegenerate. Furthermore we expect that, if χ_e is large enough, then it should be possible to use the implicit function theorem to prove that the critical points are nondegenerate also for the system in which the gravitational potential is not subjected to any approximation.

Remark 4.2. *In [6] a problem is studied in which Assumption 2 is violated due to the fact that the satellite is assumed to have a spherically invariant reference equilibrium configuration. In that case it was shown that actually the orbit is not asymptotic to a single synchronous orbit; nevertheless, one can still say that also in that case the orbit is asymptotic to a synchronous resonance, but the principal axes of inertia could rotate in the satellite.*

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