Generalized Connectives for Multiplicative Linear Logic

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Abstract

In this paper we investigate the notion of generalized connective for multiplicative linear logic. We introduce a notion of orthogonality for partitions of a finite set and we study the family of connectives which can be described by two orthogonal sets of partitions.

We prove that there is a special class of connectives that can never be decomposed by means of the multiplicative conjunction $\otimes$ and disjunction $\parr$, providing an infinite family of non-decomposable connectives, called *Girard connectives*. We show that each Girard connective can be naturally described by a type (a set of partitions equal to its double-orthogonal) and its orthogonal type. In addition, one of these two types is the union of the types associated to a family of MLL-formulas in disjunctive normal form, and these formulas only differ for the cyclic permutations of their atoms.

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1 Introduction

In his seminal paper [7], Girard introduced the notion of generalized multiplicative connective for linear logic [6] expressed in terms of permutations over finite sets. This work was then improved by Danos and Regnier in [5] where permutations were replaced by the weaker structure of partitions of finite sets. In particular, the original orthogonality condition for permutations proposed by Girard is replaced by the following:

*two partitions on the same finite domain are orthogonal iff the (bipartite) multigraph with vertices the blocks of the two partitions and edges between blocks sharing an element is connected and acyclic (*ACC *for short).*

This orthogonality relation is extended to sets of partitions: two sets of partitions P and Q are orthogonal (denoted $P \perp Q$) if their elements are pairwise orthogonal (see Figure 1).

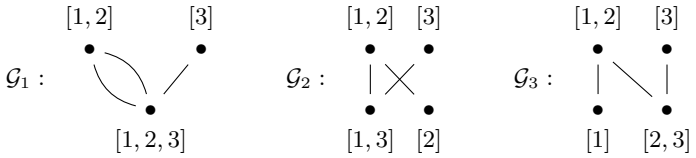

Figure 1 The two partitions $\langle [1,2],[3]\rangle$ and $\langle [1,2,3]\rangle$ are not orthogonal since $\mathcal{G}_1$ contains a cycle. The two sets of partitions $P = \{\langle [1,2],[3]\rangle\}$ and $Q = \{\langle [1,3],[2]\rangle, \langle [1],[2,3]\rangle\}$ are orthogonal.

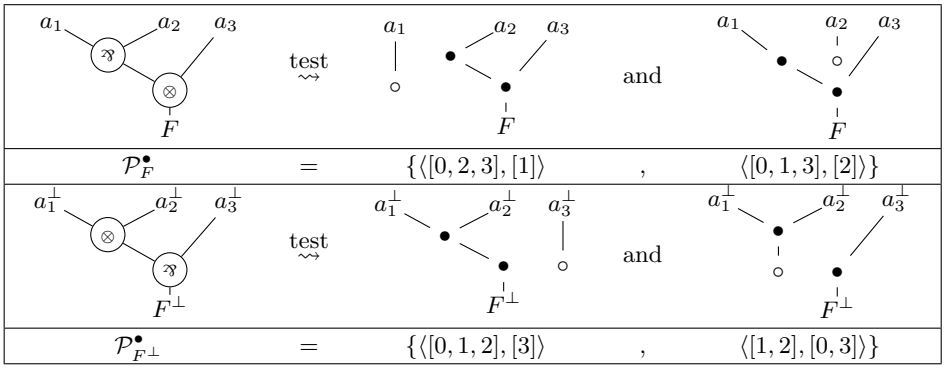

Figure 2 The pretype of the formulas $F = (a_1 \parr a_2) \otimes a_3$ and $F^\perp = (a_1^\perp \otimes a_2^\perp) \parr a_3^\perp$.

Multiplicative linear logic has two well-known proof systems: sequent calculus and proof nets. Thus, we are able to associate sets of partitions to multiplicative formulas $F = F(a_1, \ldots, a_n)$ by means these two syntaxes.

In the *sequential syntax*, a partition keeps the information about how the literals $a_1, \ldots, a_n$ occurring in F are gathered between its m premise sequents. In this way, we can see a (non-logical) derivation of F from $a_1, \ldots, a_n$ as a generalized m-ary rule of the sequent calculus and this rule is completely characterized by the *organization* of its premises – i.e. how premise atoms are split into sequents. This is possible because multiplicative rules are *linear*, that is conservative with respect to literals, and *unconditional*, that is context-free. By means of example, consider the following (non-logical) derivation of $F(a_1, a_2, a_3) = (a_1 \parr a_2) \otimes a_3$ and its associated generalized rule ρ:

$$\dfrac{\dfrac{a_1, a_2}{a_1 \parr a_2}\parr \qquad a_3}{(a_1 \parr a_2) \otimes a_3}\otimes \quad \leftrightsquigarrow \quad \dfrac{a_1, a_2 \qquad a_3}{F(a_1, a_2, a_3)}\,\rho$$

Then, the organization of F is the same of its unique associated generalized rule ρ, that is $\mathcal{O}_F = \{\langle[1,2],[3]\rangle\} = \mathcal{O}_\rho$. However, if we consider its dual formula $F^\perp(a_1^\perp, a_2^\perp, a_3^\perp) = (a_1^\perp \otimes a_2^\perp) \parr a_3^\perp$ we observe two possible derivations associated to two possible generalized rules ρ_1 and ρ_2 and that $\mathcal{O}_{F^\perp} = \{\langle[1,3],[2]\rangle, \langle[2,3],[1]\rangle\} = \mathcal{O}_{\rho_1} \cup \mathcal{O}_{\rho_2}$ since $\mathcal{O}_{\rho_1} = \{\langle[1,3],[2]\rangle\}$ and $\mathcal{O}_{\rho_2} = \{\langle[2,3],[1]\rangle\}$. Moreover $\mathcal{O}_F \perp \mathcal{O}_{F^\perp}$.

$$\dfrac{\dfrac{a_1^\perp, a_3^\perp \qquad a_2^\perp}{a_1^\perp \otimes a_2^\perp, a_3}\otimes}{(a_1^\perp \otimes a_2^\perp) \parr a_3^\perp}\parr \quad \leftrightsquigarrow \quad \dfrac{a_1^\perp, a_3^\perp \qquad a_2^\perp}{F^\perp(a_1^\perp, a_2^\perp, a_3^\perp)}\,\rho_1 \qquad \dfrac{\dfrac{a_2^\perp, a_3^\perp \qquad a_1^\perp}{a_1^\perp \otimes a_2^\perp, a_3}\otimes}{(a_1^\perp \otimes a_2^\perp) \parr a_3^\perp}\parr \quad \leftrightsquigarrow \quad \dfrac{a_2^\perp, a_3^\perp \qquad a_1^\perp}{F^\perp(a_1^\perp, a_2^\perp, a_3^\perp)}\,\rho_2$$

In the *graphical syntax* (i.e. *proof structures*), a partition keeps the information about how the premises are gathered by a *Danos-Regnier switching* [5] in the correction graph of the proof structure with premises $a_1, \ldots, a_n$ and the conclusion of a MLL-formula F (i.e. the formula tree of F). However, as already observed in [12], this construction gives another key information: not all blocks of a partition have the same statute. In fact, only one of its block is *principal*, that is, it is connected with the conclusion. To keep this information, we consider the set of *pointed partitions*, i.e. partitions over $\{0, 1, \ldots, n\}$ where 0 is a marking for principal blocks of a formula $F(a_1, ..., a_n)$: we call this set, the *pretype* of F, denoted by $\mathcal{P}^\bullet_F$ (see Figure 2). Once we define a *forgetting function* $\lfloor - \rfloor$ erasing the occurrence of 0 in a pointed partition, we observe that $\lfloor \mathcal{P}^\bullet_F \rfloor \perp \lfloor \mathcal{P}^\bullet_{F^\perp} \rfloor$.

The sequential and the graphical way to associate a set of partitions to a MLL-formula $F(a_1, \ldots, a_n)$ are in some sense orthogonal since we can show that $\mathcal{O}_F = \lfloor \mathcal{P}^\bullet_F \rfloor^\perp$.

This construction suggests a natural generalization for sequent calculus: given two sets of partitions P and Q over $\{1,\dots,n\}$ such that $P \perp Q$ and $Q = P^{\perp}$ (or $P = Q^{\perp}$), we define a pair of generalized multiplicative connectives $\mathsf{C} = \mathsf{C}_{(P,Q)}$ and $\mathsf{C}^{\perp}$ for which we assume given a set of sequent rules. Each rule introducing C or $\mathsf{C}^{\perp}$ has as organization a partition in P or respectively Q. Moreover, the orthogonality of P and Q assures the existence of a cut-elimination procedure.

Analogously, in proof structures syntax, given two sets of pointed partitions $P^{\bullet}$ and $Q^{\bullet}$ over $\{0,1,\dots,n\}$ such that $\lfloor P^{\bullet}\rfloor \perp \lfloor Q^{\bullet}\rfloor$ and $\lfloor P^{\bullet}\rfloor^{\perp} \perp \lfloor Q^{\bullet}\rfloor^{\perp}$, we define a pair of dual multiplicative connectives satisfying cut-elimination. The information given by the pointed partitions allows us to define the Danos-Regnier switches for these connectives, since it gives us not only the information on how to gather the incoming edges of a node into blocks, but also which one of them is connected with the outgoing edge. This gives an extension of the correctness criterion for proof structures containing such connectives. Furthermore, the orthogonality of $\lfloor P^{\bullet}\rfloor^{\perp}$ and $\lfloor Q^{\bullet}\rfloor^{\perp}$ is mandatory for cut-elimination.

One natural question arises about the *decomposability* by means of $⅋$ and $\otimes$:

given a pair of partitions (P,Q) describing a multiplicative connective $\mathsf{C}_{(P,Q)}$, is it always possible to find a MLL*-formula F such that $\mathcal{O}_F = P$ and $\mathcal{O}_{F^{\perp}} = Q$?*

A preliminary negative answer to this question is given by the non-decompsable connective G_4 defined in [7] in terms of permutations, and here reported as reformulated in [5]:

$$\mathsf{G}_4 = \mathsf{C}_{(P,Q)} \text{ with } P = \{\langle[1,2],[3,4]\rangle, \langle[2,3],[4,1]\rangle\} \text{ and } Q = \{\langle[1,3],[2],[4]\rangle, \langle[2,4],[1],[3]\rangle\}$$

In [11] the second author defines an infinite family of non-decomposable connectives generalizing G_4. Each (sequential) connective of this family is given by a set of two partitions P, called *entangled pair*[1], together with its *orthogonal* set of partitions $P^{\perp} = \{q \mid q \perp p \text{ for all } p \in P\}$.

In this paper we make a step further with respect to [11] by providing an infinite class of sets of partitions $\mathfrak{S}_{\langle u,v\rangle}$ enabling us to define an infinite class of non-decomposable connectives strictly including the entangled ones. We show that this set of partitions can be expressed as the union of the types of a family of formulas obtained by all the possible cyclic permutations of the literals of a formula $F(a_1,\dots,a_n) = (a_{1,1} \otimes \cdots \otimes a_{1,n_1}) ⅋ \cdots ⅋ (a_{k,1} \otimes \cdots \otimes a_{k,n_k})$, i.e. a MLL disjunctive normal form. Besides the combinatorial nature of this property, this allows to prove that if $\mathfrak{S}_{\langle u,v\rangle} \subset P$, then any generalized connective $\mathsf{C}_{(P,Q)}$ cannot be decomposable.

Non-decomposable connectives represent a new challenging research subject in linear logic: a denotational semantics and the geometry of interaction for the extension of MLL with these connectives are still missing. Moreover, we foresee an extension of Andreoli's paradigm of *modular proof construction*, using such connectives as additional modules [3, 10].

Structure of the paper. In Section 2 we give some backgrounds on graphs and partitions of finite sets. In particular, we provide a family of partitions satisfying a property of closure with respect to a notion of *orthogonality*. Furthermore we recall some multiplicative linear logic definitions and results in Section 3. In Section 4 we explain the correspondence between partitions sets and generalized multiplicative connectives and in Section 5 we redefine the notions of decomposable connectives in graphical and sequential syntax. Finally in Section 6 we give the family of non-decomposable connectives called *Girard connectives*.

[1] A pair of partitions, p and q, is *entangled* iff p and q have the same number of blocks and each block contains at most 2 elements of the support $\{1,\dots,n\}$.

2 Graphs and Partitions

A *(direct) multigraph* $\mathcal{G} = (V, E)$ is given by a set of vertices V and a multiset of *edges* $E = \{(u, v) | u, v \in V\}$. We denote $u \frown v$ iff there is a $(u, v) \in E$ and $u \not\frown v$ iff there is no (u, v) in E. A multigraph is *undirected* if the set of edges is reflexive, i.e. $(u, v) \in E$ iff $(v, u) \in E$. Let $u, v \in V$, then a *path form u to v* is a sequence of vertices $v_0, \ldots, v_n \in V$ such that $v_i \frown v_{i+1}$ for all $i \in 0, \ldots, n-1$.A multigraph is *connected* if for all $u, v \in V$ there is a path from u to v. A connected component of a graph is a maximal subset of connected vertices $V' \subset V$. A path is a *cycle* if $v_0 = v_n$. A cycle is *primitive* if $v_i \not\frown v_j$ for all $j \neq i + 1$ with $i \neq 0$ and $j \neq n$. A multigraph is *acyclic* if it contains no cycles. A *graph* is a multigraph such that E is a set of edges, i.e. there is at most one edge (u, v) for each pair of vertices $u, v \in V$.

▶ **Theorem 1** (Euler-Poincaré invariance). *Let $\mathcal{G} = (V, E)$ be a multigraph. If $|\mathsf{Cy}|$ and $|\mathsf{CC}|$ are respectively the number of primitive cycles and the number of connected components of $\mathcal{G}$, then $|V| - |E| + |\mathsf{Cy}| - |\mathsf{CC}| = 0$.*

A *partition* $p = \langle \gamma_1, \ldots, \gamma_u \rangle$ of a finite set $X = \{1, ..., n\}$ is a set of subsets of X (an element of $\mathscr{P}(X)$) such that $X = \bigcup_i \gamma_i$ and if $i \neq j$ then $\gamma_i \cap \gamma_j = \emptyset$. We denote by $\mathbb{P}_X$ the set of partitions of a finite set X and $\mathbb{P}_n = \mathbb{P}_{\{1,\ldots,n\}}$. We call X the *support* of p and γ_i a *block* of p. To simplify reading, we differentiate parenthesis for partitions and blocks as follows $p = \langle [a_{1,1}, \ldots, a_{1,k_1}], \ldots, [a_{u,1}, \ldots, a_{u,k_u}] \rangle$.

▶ **Definition 2** (Orthogonality). *Let $p, q \in \mathbb{P}_n$. The (undirected)* graph of incidence of p and q, *denoted $\mathcal{G}(p, q)$, is the multigraph with vertices the blocks of p and q such that there is an edge $v_{\gamma_1} \frown v_{\gamma_2}$ for each element in $\gamma_1 \cap \gamma_2 \neq \emptyset$. We say that p and q are* orthogonal, *denoted $p \perp q$, iff the induced multigraph $\mathcal{G}(p, q)$ is connected and acyclic (*ACC *for short).*

The notion of orthogonality extends to set of partitions: if $P, Q \subset \mathbb{P}_n$, we say that P and Q are orthogonal ($P \perp Q$) iff they are pointwise orthogonal, that is $p \perp q$ for all $p \in P$ and $q \in Q$. If $P \subset \mathbb{P}_n$, we denote $P^\perp = \{q \in \mathbb{P}_n \mid p \perp q \text{ for all } p \in P\}$ the orthogonal of P. *For an example refer to Figure 1.*

From Theorem 1 we deduce the following

▶ **Corollary 3.** *If $p, q \in P \in \mathbb{P}_n$ and $|p| \neq |q|$ then $P^\perp = \emptyset$.*

▶ **Definition 4** (Type). *A set of partitions $P \subset \mathbb{P}_n$ is a* type *iff $P = P^{\perp\perp}$.*

We here recall some results form [12] which are useful to compute the orthogonal of a set of partitions and to decide whenever a set of partitions is a type.

▶ **Proposition 5** (Partitions and Orthogonality). *Let $A, B \subset \mathbb{P}_n$, then the following facts hold:*

1. *$A^\perp = A^{\perp\perp\perp}$. This means that $A^\perp$ is a type;*
2. *$A \perp B$ iff $A \subseteq B^\perp$ and $A \perp B$ iff $B \subseteq A^\perp$;*
3. *$A \subseteq B$ implies $B^\perp \subseteq A^\perp$;*
4. *if A is a type, then there is B such that $A = B^\perp$;*
5. *$(\bigcup_i A_i)^\perp = \bigcap_i A_i^\perp$;*
6. *$(\bigcap_i A_i)^\perp \supseteq \bigcup_i A_i^\perp$;*
7. *if A admits a set B such that $A \perp B$ then all partitions in A have the same cardinality.*

In particular, the intersection of types is always a type, while the union is not.

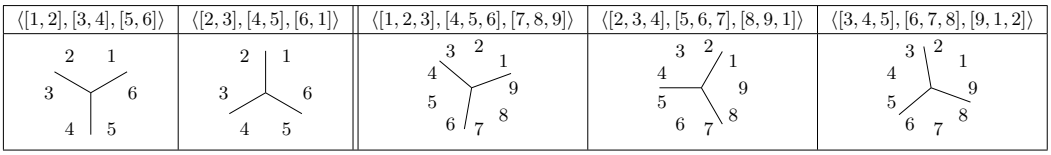

Figure 3 Examples of basic partitions and their corresponding subdivision of the cycle in n parts.

▶ **Example 6.** Let $P, Q \subset \mathbb{P}_4$ be defined as

$$P = \{p_1 = \langle [1,3],[2,4]\rangle, p_2 = \langle [1,4],[2,3]\rangle\} \text{ and } Q = \{q_1 = \langle [1,3,4],[2]\rangle, q_2 = \langle [2,3,4],[1]\rangle$$

Then $P^\perp = \{p_1\}^\perp \cap \{p_2\}^\perp = \{\langle [3,4],[1],[2]\rangle, \langle [1,2],[3],[4]\rangle\}$ and $Q^\perp = \{q_1\}^\perp \cap \{q_2\}^\perp = \{\langle [1,2],[3],[4]\rangle\}$. That is P is a type and Q is not.

▶ **Theorem 7** (No sub-type). *If $T \subset \mathbb{P}_n$ is a type, then there is no type $T' \neq T$ such that $T' \subset \mathbb{P}_n$ and $T' \subset T$.*

Proof. By Proposition 5.3 if $P \subset T$ then $T^\perp \subseteq P^\perp$. In particular, $T \perp P^\perp$. By Proposition 5.2 we have $P \subseteq T^{\perp\perp}$. ◀

▶ **Definition 8** (Entangled pairs of partitions [11]). *A pair of partitions $P = \{p, q\} \subset \mathbb{P}_n$ with $p \neq q$ is an* entangled pair *if $|p| = |q|$ and $1 \leq |\gamma| \leq 2$ for each $\gamma \in p \cup q$.*

By means of example, the set P and $P^\perp$ given in Example 6 are both entangled pairs.

▶ **Theorem 9** (Entangled types [11]). *Every entangled pair of partitions $P \subset \mathbb{P}_n$ is a type.*

2.1 Basic Partitions

For the rest of this paper we assume $n \in \mathbb{N}$ such that $n = uv$ for some $u, v > 1$.

A *basic partition of n* is a partition $p \in \mathbb{P}_n$ with u blocks of v elements such that each block is of the form $[i, i+1, \ldots, i+v-1]$, if $i+v-1 \leq n$, or $[i, \ldots, n, 1, 2, \ldots, i+v-1-n]$ otherwise. Intuitively, if we place the elements in $\{1, \ldots, n\}$ over a circle in an increasing order, a basic partition can be viewed as a subdivision of that circle into u intervals containing v elements as shown in Figure 3.

▶ **Definition 10** (Space of basic partitions). *We call the* space of basic partitions of rank $\langle u, v\rangle$, *denoted $\mathfrak{S}_{\langle u,v\rangle}$, the set of all possible basic partitions of n made of u blocks of v elements. That is, $\mathfrak{S}_{\langle u,v\rangle} \subset \mathbb{P}_n$ is the following set*

$$\left\{\begin{array}{ll} p_1: & \langle [1,\ldots,v],[v+1,\ldots,2v],\ldots,[v(u-1)+1,\ldots,n]\rangle \\ p_2: & \langle [2,\ldots,v+1],[v+2,\ldots,2v+1],\ldots,[v(u-1)+2,\ldots,n,1]\rangle \\ \vdots & \vdots \\ p_i: & \langle [i,\ldots,v+(i-1)],[v+i,\ldots,2v+(i-1)],\ldots,[v(u-1)+i,\ldots,n,1,\ldots,i-1]\rangle \\ \vdots & \vdots \\ p_v: & \langle [v,\ldots,2v-1],[2v,\ldots,3v-1],\ldots,[n,1,\ldots,v-1]\rangle \end{array}\right\}$$

Some examples of spaces with different rank are given in Figure 4.

▶ **Lemma 11** (Cardinality of $\mathfrak{S}_{\langle u,v\rangle}$). *If $\mathfrak{S}_{\langle u,v\rangle}$ is a space of rank $\langle u, v\rangle$, then $|\mathfrak{S}_{\langle u,v\rangle}| = v$.*

Proof. For each $1 \leq i \leq v$ there is a unique $p \in \mathfrak{S}_{\langle u,v\rangle}$ such that $[i, \ldots, i+v-1] \in p$. ◀

$$\mathfrak{S}_{\langle 3,2\rangle} = \{\langle [1,2],[3,4],[5,6]\rangle, \langle [2,3],[4,5],[6,1]\rangle\}$$

$$\mathfrak{S}_{\langle 2,3\rangle} = \{\langle [1,2,3],[4,5,6]\rangle, \langle [2,3,4],[5,6,1]\rangle, \langle [3,4,5],[6,1,2]\rangle\}$$

$$\mathfrak{S}_{\langle 3,3\rangle} = \{\langle [1,2,3],[4,5,6],[7,8,9]\rangle, \langle [2,3,4],[5,6,7],[8,9,1]\rangle, \langle [3,4,5],[6,7,8],[9,1,2]\rangle\}$$

Figure 4 Some examples of spaces of rank $\langle u, v\rangle$.

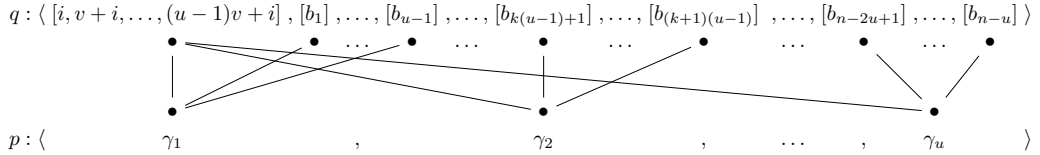

Figure 5 If $p \in \mathfrak{S}_{\langle u,v\rangle}$ and $b_i \in \{1,\dots,n\} \setminus \{i, v+1, \dots, (u-1)v+i\}$ then $p \perp q$ for $q, p \in \mathbb{P}_n$.

▶ **Definition 12** (Distance). *Given $1 \le i, j \le n$, we define the* distance of i and j modulo n

$$\delta_n(i,j) = \begin{cases} min\{j-i, i-j+n\} & \text{if } j \ge i \\ min\{i-j, j-i+n\} & \text{if } j < i \end{cases} \tag{1}$$

E.g the distance of 1 and 9 modulo $n = 9$ is 1 that is, $\delta_9(1,9) = min\{8,1\}$.

▶ **Lemma 13** (Distance). *Let $1 \le i, j \le n = uv$.*

- *$\delta_n(i,j) < v$ iff there is $p \in \mathfrak{S}_{\langle u,v\rangle}$ containing a block γ such that $i, j \in \gamma$;*
- *$\delta_n(i,j) \ge v$ iff for all $p \in \mathfrak{S}_{\langle u,v\rangle}$ there are $\gamma_1 \ne \gamma_2 \in p$ such that $i \in \gamma_1$, $j \in \gamma_2$.*

Proof.
- Since $\delta_n(i,j) = \delta_n(j,i)$, we assume without losing generality that $i < j$. Hence, it suffices to remark that $\mathfrak{S}_{\langle u,v\rangle}$ always contains a partition including block $[i, \dots, i+v-1]$ if $i + v - 1 \le n$, or including block $[i, i+1, \dots, n, 1, 2, \dots, i+v-1-n]$ if $i + v - 1 > n$;
- By similar reasoning. ◀

▶ **Lemma 14.** *If $\mathfrak{S}_{\langle u,v\rangle}$ is a space of basic partitions then its orthogonal $\mathfrak{S}^{\perp}_{\langle u,v\rangle}$ is not empty.*

Proof. Let q be the partition consisting of $n - u + 1$ blocks including a block $[i = a_1, \dots, a_u]$ such that $\delta_n(a_i, a_j) = hv$ with $h \in \mathbb{N}$ for $1 < j \le u$ (called *i^{th}-block of congruence modulo v*), and $n - u$ singleton blocks over $\{1,\dots,n\} \setminus \{a_1, \dots, a_u\}$.

After Lemma 13 the multigraph $\mathcal{G}(p,q)$ is acyclic, that is $|\mathsf{Cy}| = 0$. Hence, by Theorem 1, $p \perp q$ for all $p \in \mathfrak{S}_{\langle u,v\rangle}$ (see Figure 5 for an intuition). ◀

▶ **Corollary 15.** *All partitions of $\mathfrak{S}^{\perp}_{\langle u,v\rangle}$ have size $1 + n - u = 1 + u(v-1)$.*

Proof. It follows Lemma 14. ◀

Moreover, by simple arithmetic argument we have the following results:

▶ **Lemma 16.** *If $p \in \mathfrak{S}_{\langle u,v\rangle}$ and $1 \le i, j \le n = uv$ with $\delta_n(i,j) > v$, then there is $1 \le k \le n$ such that $\delta_n(i,j) = hv$ for a $h \in N$ and $\delta_n(j,k) < v$. That is, for each i, j there is a k at distance a multiple of v from i which belongs to the same block of j in the partition p.*

▶ **Proposition 17.** *If $\mathfrak{S}_{\langle u,v\rangle}$ is a space of rank $\langle u, v\rangle$, then:*

1. *if $1 \le i \le n$, then there exists a partition $q \in \mathfrak{S}^{\perp}_{\langle u,v\rangle}$ such that $[i] \in q$;*

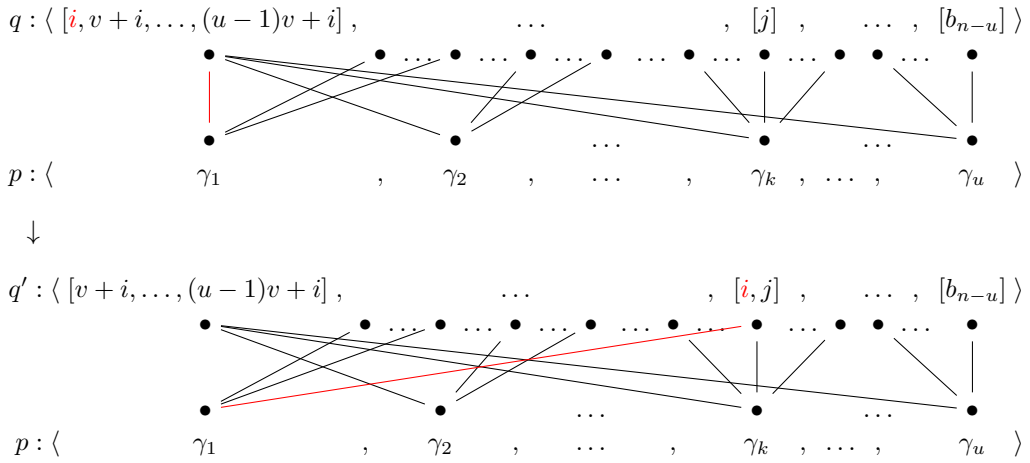

Figure 6 The permutation q including the i^{th}-block of congruence modulo v and q' are both ortogonal to $p \in \mathfrak{S}_{\langle u,v\rangle}$.

2. *if* $1 \leq i, j \leq n$ *such that* $\delta_n(i,j) \geq v$*, then there is a partition* $q \in \mathfrak{S}^{\perp}_{\langle u,v\rangle}$ *containing a block* γ *such that* $i, j \in \gamma$.

Proof. 1. By Lemma 14, given $1 \leq i, j \leq n$ such that $\delta(i,j) > v$ and $\delta(i,j) = hv$ for $h \in \mathbb{N}$, there is a partition q containing the j^{th}-block of congruence modulo v and all singleton blocks is in $\mathfrak{S}^{\perp}_{\langle u,v\rangle}$. Hence, in q there is the singleton block $[i]$.

2. if $\delta_n(i,j) = hv$ for a $h \in \mathbb{N}$, then we consider the partition q made of the i^{th}-block of congruence modulo v and singleton blocks.
If $\delta_n(i,j) > v$ and $\delta_n(i,j) = hv$ for a $h \in \mathbb{N}$ we define a partition q' from q by removing i form the i^{th}-block of congruence modulo v and adding i to the singleton $[j]$ as shown in Figure 6. To prove that $q' \in \mathfrak{S}^{\perp}_{\langle u,v\rangle}$ it suffices to use Lemma 16. In fact, we can assume that i belongs to $\gamma_i \in p \in \mathfrak{S}_{\langle u,v\rangle}$. Then there is k such that $i \in \gamma_k$ for any $p \in \mathfrak{S}_{\langle u,v\rangle}$. Since $j \neq i + hv$ with $h \in \mathbb{N}$, then $\gamma_k \neq \gamma_1$. By Theorem 1, $\mathcal{G}(q', p)$ is acyclic and connected, hence $q' \perp p$. ◀

▶ **Theorem 18.** *Every space of basic partitions* $\mathfrak{S}_{\langle u,v\rangle}$ *is a type.*

Proof. Assume by contradiction that $\mathfrak{S}_{\langle u,v\rangle}$ is not a type, i.e. assume there exists $p' \in (\mathfrak{S}^{\perp}_{\langle u,v\rangle})^{\perp}$ such that $p' \notin \mathfrak{S}_{\langle u,v\rangle}$. By Proposition 17.1, p' cannot contain any singleton block $[i]$. Moreover, by Proposition 17.2, p' cannot contain any block γ such that $i, j \in \gamma$ and $\delta_n(i,j) \geq v$. This means that p' consists only of blocks containing elements at distance strictly smaller than v, hence $|p'| > u$. This contradicts Proposition 5.7. ◀

3 Multiplicative Linear Logic Backgrounds

We consider the class $\mathcal{F}$ of *multiplicative linar logic formulas* (denoted by $A, B, \ldots$) in negation normal form, generated by a countable set $\mathcal{A} = \{a, b, \ldots\}$ of *propositional variables* by the grammar $A, B ::= a \mid A^{\perp} \mid A \parr B \mid A \otimes B$ modulo the involution of $(\cdot)^{\perp}$ and the *de Morgan* laws: $A^{\perp\perp} = A$, $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$ and $(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$. A *sequent* is a set of occurrences of formulas. If $a \in \mathcal{A}$, we say that a and $a^{\perp}$ are *atoms* or *atomic formulas*. The sequent system for MLL is given by the rules in Figure 7. If ρ is a sequent system rule, we call *active* a formula in a premise of a rule which is not in its conclusion and and *principal* the formula introduced by the rule in the conclusion.

$$\frac{}{A, A^{\perp}}\ \mathsf{ax} \qquad \frac{\Gamma, A \qquad \Delta, A^{\perp}}{\Gamma, \Delta}\ \mathsf{cut} \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr$$

Figure 7 Standard MLL Sequent Calculus.

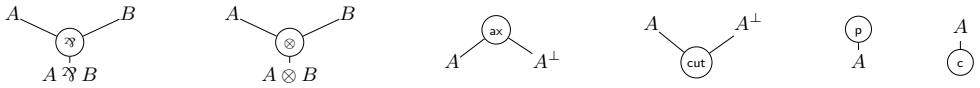

Figure 8 Labels conditions for vertices and edges of a proof structures (also known as *links*).

▶ **Definition 19** (Proof Structure). *A* proof structure $\mathfrak{P}$ *is a direct graph with edges labeled by* MLL*-formulas and vertices labeled by* $\{\mathsf{ax}, \mathsf{cut}, \otimes, \parr, \mathsf{p}, \mathsf{c}\}$ *according to conditions of Figure 8. We call* premises *(*conclusions*) of a proof structure the nodes labeled by* p *(*c*). Moreover, abusing notation, we identify these nodes with the formula labeling the outgoing (respectively incoming) edge of these nodes. Similarly, we call* premises *(*conclusion*) of a node its incoming (outgoing) edges labels.*

To each derivation $\mathfrak{d}$ with conclusion Γ in MLL we associate the proof structure $\mathfrak{P}_{\mathfrak{d}}$ with conclusions Γ defined as follows:

- for all inference rule ρ in $\mathfrak{d}$ there is a corresponding node in $\mathfrak{P}_{\mathfrak{d}}$ labeled by ρ having as premises the active formulas of ρ and as conclusion the principal formula of ρ;
- for each formula in the conclusion of $\mathfrak{d}$ there is a node in $\mathfrak{P}_{\mathfrak{d}}$ labeled by c.

▶ **Definition 20.** *A proof structure* π *is a* proof net *if there is a derivation* $\mathfrak{d}$ *in* MLL *such that* $\pi = \mathfrak{P}_{\mathfrak{d}}$.

We characterize proof nets by means of correctness conditions on proof structures.

▶ **Definition 21** (Switching). *A* switching σ *of a* MLL *proof structure* $\mathfrak{P}$ *is a function associating to each* $\parr$*-node in* $\mathfrak{P}$ *a* switch*, i.e. a block of the partition* $\langle [1], [2] \rangle$*. For each switching, we define* $\sigma(\mathfrak{P})$ *as the undirected* correction graph *(also called* test*) obtained by forgetting the orientation of edges and by removing, for each* $\parr$*-node with conclusion* $A \parr B$*, the edge labeled by* B *if its switch is* $[1]$ *or the edge labeled by* A *if the switch is* $[2]$.

▶ **Theorem 22** (Danos-Regnier sequentialization [5]). *For each switching* σ *of* π*, the graph* $\sigma(\pi)$ *is* ACC *iff there is a derivation* $\mathfrak{d}$ *such that* $\mathfrak{P} = \mathfrak{P}_{\mathfrak{d}}$.

The interest of proof nets lies on the fact that they allow of identify derivations which are equivalent modulo rules permutations. This simplifies the proof of cut-elimination theorem for MLL by eliminating the bureaucracy of rules permutations during cut-elimination procedure. The *rewriting rules* for proof structures cut-elimination are given in Figure 9.

▶ **Theorem 23** (Danos-Regnier cut-elimination [5]). *Cut-elimination procedure for proof structures is convergent and preserves connectedness and acyclicity.*

4 Generalized multiplicative connectives and partitions sets

An *n-ary connective* is a syntactic symbol C we use to construct a new formula $\mathsf{C}(A_1, \ldots, A_n)$ from the formulas $A_1, \ldots, A_n$ in a formal grammar. By means of example, in MLL we have only the (binary) connectives $\parr$ and $\otimes$. We remark that in a complete sequent calculus, each n-ary connective C admits at least one rule ρ with $k \leq n$ premise sequents with active formulas $A_1, \ldots, A_n$ and principal formula $\mathsf{C}(A_1, \ldots, A_n)$.

In [5] the authors define a *generalized multiplicative rule* as a sequent rule which is:

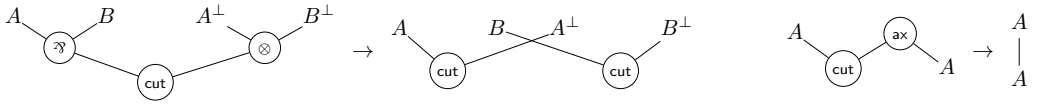

Figure 9 Proof nets cut-elimination rewriting rules.

$$\frac{\vdash \Gamma_1, A_{i_1}, \dots, A_{i_k} \quad \dots \quad \vdash \Gamma_m, A_{i_h}, \dots, A_{i_n}}{\vdash \Gamma_1, \dots, \Gamma_m, \mathsf{C}(A_1, \dots, A_n)}\,\rho\mathsf{C} \qquad \wp_\rho = \langle [i_1, \dots, i_k], \dots, [i_h, \dots i_n] \rangle$$

Figure 10 A sequential rule ρ introducing the connective C and its associate partition $\wp_\rho$.

- *conservative* with respect of the atoms (or *linear*), i.e. the premises of the rule have exactly the same atoms as the conclusion;
- *unconditional*, i.e. the rule does not require information about the contexts.

As remarked in [7] and [5], these conditions allow us to associate set of partitions to multiplicative connectives of linear logic. In fact, in sequent calculus we can associate to each connective C the set of partitions describing how all the sequential rules introducing C gather the principal subformula between its premise sequents.

Similarly, by the Danos-Regnier correctness criterion, each switching of a MLL proof structure determines a partition corresponding to the premises belonging to the same connected component. However, some of the premises can never be connected to the root of a single-conclusion test of a proof net. For this reason, we prove in this paper that a set of partitions is not enough to describe a graphical connective, since each connective has to be given together with its possible switches. This additional information is provided by considering a special symbol to mark the *principal block*, i.e. the unique block selected by the switch to be connected to the conclusion.

4.1 Partitions and generalized sequential connectives

We can associate to a multiplicative rule (i.e. linear and context-free) of the sequent calculus with n active formulas a partition in $\mathbb{P}_n$. That is, a multiplicative m-ary rule $\rho\mathsf{C}$ for a generalized n-ary connective C is completely characterized by the *organization* of its principal subformulas $A_1, \dots, A_n$ (see Figure 10).

▶ **Definition 24** (Organization of a rule). *Let ρ be an m-ary rule (i.e. a rule with m premise sequents) with n active formulas $A_1, \dots, A_n$ and principal formula $\mathsf{C}(A_1, \dots, A_n)$. The partition $\wp_\rho \in \mathbb{P}_n$ associated to ρ is made of m blocks defined as follows: i, j belong to a same block iff the formulas A_1 and A_j belong to the same premise of ρ. We call $\wp_\rho$ the* organization of the rules ρ.

▶ **Example 25.** The organizations of the ⅋-rule and the ⊗-rule are respectively $\{\langle [1,2] \rangle\}$ and $\{\langle [1], [2] \rangle\}$. Moreover, $\{\langle [1,2] \rangle\} \perp \{\langle [1], [2] \rangle\}$.

This allows to describe an n-ary connective by means of a set of partitions.

▶ **Definition 26** (Generalized sequential connective). *We says that a pair (P, Q) of non-empty sets of partitions in $\mathbb{P}_n$ is a* description *of (or it* describes*) a sequential n-ary connective if $P \perp Q$ and if $Q = P^\perp$ or $P = Q^\perp$.*

If (P, Q) is a description of a n-ary sequential connective, we denote by $\mathsf{C}_{(P,Q)}$ a sequential n-ary connective *described by (P, Q) and by $\mathsf{C}^\perp_{(P,Q)} = \mathsf{C}_{(Q,P)}$ its* dual *connective – described by (Q, P). We call $\mathcal{O}(\mathsf{C}_{(P,Q)}) = P$ the* organization of $\mathsf{C}_{(P,Q)}$.

The organization a sequential connective $\mathsf{C} = \mathsf{C}_{(P,Q)}$ can be interpreted as the set of the organizations of the rules introducing C. That is, if C is a sequential connective described by (P, Q), we can think to $\mathcal{O}(\mathsf{C})$ as the organizations of some rules in a two-sided calculus introducing C in the right-hand side and the set $\mathcal{O}(\mathsf{C}^\perp)$ as the organizations of all the rules introducing C on the left-hand side (because $\mathcal{O}_{\mathsf{C}}^{\perp} = \mathcal{O}_{\mathsf{C}^\perp}$ is a type).

▶ Remark 27. If $\mathsf{C}_{(P,Q)}$ is a generalized sequential connective, since $Q \neq \emptyset$, by Corollary 3 all its sequential rules have the same arity $m = |p|$ for any $p \in P$.

Let $\mathcal{C} = \{\mathsf{C}_1, \ldots, \mathsf{C}_n\}$ be a set of multiplicative connectives, we define the generalized *$\mathcal{C}$-multiplicative formulas* $\mathcal{F}_{\mathcal{C}}$ extending $\mathcal{F}$ with the generalized connectives in $\mathcal{C}$, that is, for all $\mathsf{C} = \mathsf{C}_{(P,Q)} \in \mathcal{C}$ with $P, Q \in \mathbb{P}_n$ we extend the grammar of MLL-formulas with $\mathsf{C}(A_1, \ldots A_{n_i})$ and $\mathsf{C}^\perp(A_1, \ldots A_{n_i})$. Thus, for each $p \in P$, we define a sequential rule ρ_{C}^{p} introducing the connective $\mathsf{C}_{(P,Q)}$ such that $o_{\rho_{\mathsf{C}}^{p}} = p$ (see Figure 10). We denote $\mathsf{MLL}(\mathcal{C})$ the extension of MLL with the sequent rules $\bigcup_{\mathsf{C}\in\mathcal{C}} \bigcup_{p\in P} \{\rho_{\mathsf{C}}^{p}\}$.

▶ **Theorem 28.** *The sequent system* $\mathsf{MLL}(\mathcal{C})$ *is cut-free, that is a sequent* Γ *in* $\mathcal{F}_{\mathcal{C}}$ *is derivable in* $\mathsf{MLL}(\mathcal{C}) \cup \{\mathsf{cut}\}$ *iff it is in* $\mathsf{MLL}(\mathcal{C})$.

Proof. The proof is given in [5]. It suffices to remark that the partitions sets describing C and its dual $\mathsf{C}^\perp$ describe the introduction rules for these connectives. Hence a cut-elimination step consist of replacing the cut-rule and the two rules ρ and ρ' introducing the cut-formula by cut-rules between the active formulas of ρ and ρ'. ◀

▶ **Example 29.** If we consider the partitions sets $P = \{\langle [1,2], [3]\rangle\}$ and $Q = \{\langle [1,2,3]\rangle\}$, we have $P \not\perp Q$. If ρ_P and ρ'_Q, are the corresponding sequent rules, we can not define a cut-elimination step as shown below.

$$\frac{\vdash \Gamma, A_1, B_2 \qquad \vdash \Delta, C_3}{\vdash \Gamma, \Delta, \rho(A_1, B_2, C_3)}\,\rho \qquad \frac{\vdash \Gamma, A_1, B_2, C_3}{\vdash \Gamma, \rho'(A_1, B_2, C_3)}\,\rho' \qquad \frac{\dfrac{\vdash \Gamma, A_1, B_2 \qquad \vdash \Delta, C_3}{\vdash \Gamma, \Delta, \rho(A_1, B_2, C_3)}\,\rho \qquad \dfrac{\vdash \Sigma, A_1, B_2, C_3}{\vdash \Sigma, \rho'(A_1, B_2, C_3)}\,\rho'}{\vdash \Gamma, \Delta, \Sigma}\,\mathsf{cut}$$

4.2 Partitions and generalized graphical connectives

We associate to each correction graph of an MLL-proof structure with n premises and one single conclusion a partition in $\mathbb{P}_n$ where each block contains the indices of connected premises. Hence, we are able to associate to each proof net a set of partitions corresponding to all its possible correction graphs where axiom nodes are replaced by pairs of premise nodes.

▶ **Definition 30** (Pointed partition)**.** *A* pointed partition[2] $p^\bullet$ *is defined as a partition of the set* $\{0, k+1, \ldots, n\}$ *with* $k \in \mathbb{N}$ *such that* $[0]$ *is not an allowed block of* $p^\bullet$*. We denote by* $\mathbb{P}_n^\bullet$ *the set of pointed partitions over the set* $\{0, 1, \ldots, n\}$*. We define a* forgetful map

$$\lfloor - \rfloor \colon \mathbb{P}_{\{0,k+1,\ldots,n\}} \to \mathbb{P}_{\{k+1,\ldots,n\}}$$

which associates to each pointed partition $p^\bullet$ *a partition* $\lfloor p^\bullet \rfloor = p$ *called* underlying partition of $p^\bullet$ *given by removing the element* 0 *form its the non-singleton block in which occurs. Similarly if* $p^\bullet$ *is a set of pointed partitions we denote by* $P = \lfloor p^\bullet \rfloor$ *the set* $\{p = \lfloor p^\bullet \rfloor \mid p^\bullet \in p^\bullet\}$.

Intuitively, we use the element 0 to mark the *principal block*, i.e. the block containing the indices of the premises which are connected to the conclusion in a test.

With this definition, we define the analogous of Definition 26 for graphical connectives.

[2] The name "pointed partition" is inspired by *pointed spaces* of topology, which are spaces where a specific point plays a special role.

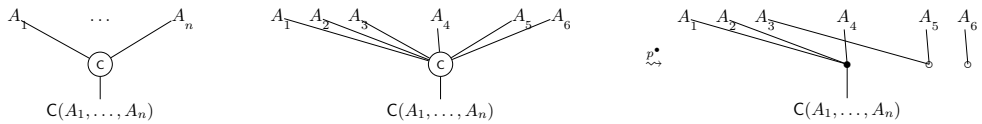

Figure 11 On the left: Labels conditions for generalized connectives. **On the right:** A node labeled by the $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ with $\langle[0,1,2,4],[3,5],[6]\rangle = p^\bullet \in P^\bullet$ and how this node is modified during test computation when $p^\bullet$ is its selected switch.

▶ **Definition 31** (Generalized Graphical Connectives). *We say that the pair $(P^\bullet, Q^\bullet)$ of non-empty sets of pointed partitions in $\mathbb{P}^\bullet_n$ such that for all $0 < i \leq n$ there is a block γ such that $\{0,i\} \subset \gamma \in p^\bullet \in P^\bullet$ (respectively $\{0,i\} \subset \gamma \in q^\bullet \in Q^\bullet$) is a* description *of (or it* describes*) a graphical n-ary connective if $\lfloor P^\bullet \rfloor \perp \lfloor Q^\bullet \rfloor$ and if $\lfloor P^\bullet \rfloor^\perp \perp \lfloor Q^\bullet \rfloor^\perp$.*

We denote by $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ a graphical n-ary connective *described by $(P^\bullet, Q^\bullet)$ and by $\mathsf{C}^\perp_{(P^\bullet,Q^\bullet)} = \mathsf{C}_{(Q^\bullet,P^\bullet)}$ its* dual *connective – described by $(Q^\bullet, P^\bullet)$.*

▶ **Example 32.** If $P^\bullet = \{\langle[1,0],[2]\rangle, \langle[1],[0,2]\rangle\}$ and $Q^\bullet = \{\langle[0,1,2]\rangle\}$, then $⅋$ and $\otimes$ are respectively described by $(P^\bullet, Q^\bullet)$ and $(Q^\bullet, P^\bullet)$.

▶ **Definition 33** (Generalized Proof Structure). *Let $\mathcal{C} = \{\mathsf{C}_{(P^\bullet_1,Q^\bullet_1)}, \dots, \mathsf{C}_{(P^\bullet_k,Q^\bullet_k)}\}$ be a set of graphical n-ary connectives. An* MLL($\mathcal{C}$) proof structure *is a direct graph $\mathfrak{P}$ with edges labeled by* MLL($\mathcal{C}$)*-formulas and vertices labed by $\{\mathsf{ax}, \mathsf{cut}, \otimes, ⅋, \mathsf{p}, \mathsf{c}\} \cup \{\mathsf{C}, \mathsf{C}^\perp\}_{\mathsf{C}\in\mathcal{C}}$ satisfying conditions in Figures 8 and 11.*

As for MLL proof structure, in order to define a correctness criterion, we extend the notion of switching to graphical n-ary connectives.

▶ **Definition 34** (Switching). *Let $\mathcal{C}$ be a set of generalized graphical connectives. A* switching *σ of a* MLL($\mathcal{C}$) *proof structure $\mathfrak{P}$ is a function associating to each $\mathsf{C}_{(P^\bullet,Q^\bullet)}$-node (i.e. a node labeled by $\mathsf{C}_{(P^\bullet,Q^\bullet)} \in \mathcal{C}$) a* switch*, i.e. a pointed partition $p^\bullet \in P^\bullet$.*

Each switching σ defines an undirected graph $\sigma(\mathfrak{P})$ (called correction graph *or* test*) obtained by forgetting edge orientations and modifying each node v labeled by $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ with switch $p^\bullet \in P^\bullet$ as follows: for each block $\gamma \in p^\bullet$ with $0 \notin \gamma$, disconnect the corresponding edges targeting v and we re-link them to a fresh target node v_γ for each γ (see Figure 11).*

▶ **Definition 35** (Pretype). *Let F be a* MLL($\mathcal{C}$) *formula over the atoms $a_1, \dots, a_n$ and $\mathfrak{P}_F$ be the unique proof structure with premise $a_1, \dots, a_n$ and conclusion F, i.e. $\mathfrak{P}_F$ is the formula tree of F. For each switching σ of $\mathfrak{P}_F$ we define a pointed partition $p^\bullet_\sigma \in \mathbb{P}^\bullet_n$ as follows:*

- *i and j belong to the same block in $p^\bullet$ iff a_i and a_j belongs to the same connected component of $\sigma(\mathfrak{P})$;*
- *i belongs in the same block 0 in $p^\bullet$ iff a_i is connected to the conclusion of $\sigma(\mathfrak{P})$.*

The pretype *of F is the set $\mathcal{P}^\bullet_F = \{p^\bullet_\sigma \in \mathbb{P}^\bullet_n \mid \sigma$ is a switching of $\mathfrak{P}_F\}$. We call $\lfloor \mathcal{P}^\bullet_F \rfloor$ the* Danos-Regnier pretype *(or* DR-pretype *for short) of F. The* type *of F is the bi-orthogonal of DR-pretype, i.e. $\mathcal{T}_F = \lfloor \mathcal{P}^\bullet_F \rfloor^{\perp\perp}$.*

▶ **Definition 36** (Generalized Proof Net). *A* MLL($\mathcal{C}$)*-proof structure $\mathfrak{P}$ is a* MLL($\mathcal{C}$)*-proof net iff for each switching σ of $\mathfrak{P}$ the graph $\sigma(\mathfrak{P})$ is* ACC*.*

The computational meaning of generalized connectives is guaranteed by the fact that the elimination of a cut-vertex linking two vertices labeled by C and $\mathsf{C}^\perp$ preserves the correctness criterion [5]. This follows from Definition 31 of a graphical connective: the condition $P \perp Q$ is necessary for ACC of proof structures, while the condition $P^\perp \perp Q^\perp$ is mandatory to ensure the stability of ACC under cut-elimination (see Figure 12).

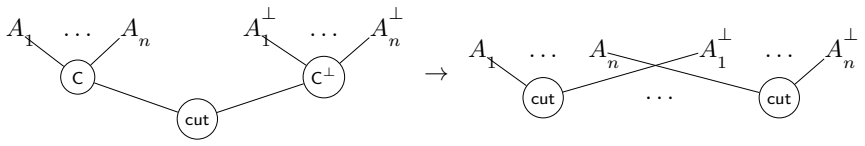

Figure 12 Generalized proof nets cut-elimination rewriting rule.

5 Decomposable connectives

In this section we study a notion of decomposability by means of $\otimes$ and $\parr$ for generalized connectives in both sequential and graphical sense. In particular, we provide a new definition of decomposability for graphical connectives which has to replace the one given in [5].

5.1 Sequential connectives

▶ **Definition 37** (Organization of a formula). *If $F = F(a_1, \dots, a_n)$ is a* MLL*-formula, we define the* organization of F *as the set of all partitions $p \in \mathbb{P}_n$ with $p = \langle \gamma_1, \dots, \gamma_k \rangle$ such that there is a* MLL*-derivation of F form the premise sequents $\{a_i\}_{i\in\gamma_1}, \dots, \{a_i\}_{i\in\gamma_k}$.*

▶ **Definition 38** (Decomposable sequential connectives). *A sequential connective $\mathsf{C}_{(P,Q)}$ is* s-decomposable *if there is a* MLL*-formula F such that $P = \mathcal{O}_F$ (and $Q = \mathcal{O}_{F^\perp}$).*

▶ **Example 39.** Let $P = \{\langle [1,3,4],[2]\rangle, \langle [2,3,4],[1]\rangle, \langle [1,3],[2,4]\rangle, \langle [1,4],[2,3]\rangle\}$ and $Q = \{\langle [1,2],[3],[4]\rangle\}$. Then $\mathsf{C}_{(P,Q)}$ is s-decomposable. In fact $P = \mathcal{O}_F$ for $F = (a_1 \otimes a_2) \parr a_3 \parr a_4$.

We show in Subsection 5.3 (Corollary 48) that if C is a s-decomposable sequential connective, then $\mathcal{O}_\mathsf{C}$ is a type.

5.2 Graphical connectives

As for the sequential case, we define a notion of decomposability for graphical connectives.

▶ **Definition 40** (Decomposable graphical connectives). *A graphical n-ary connective $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ is* g-decomposable *iff there is a* MLL *formula $F(a_1, \dots, a_n)$ such that $P^\bullet = \mathcal{P}^\bullet_F$ and $Q^\bullet = \mathcal{P}^\bullet_{F^\perp}$. It is* DR-decomposable *if $\lfloor P^\bullet \rfloor = \lfloor \mathcal{P}^\bullet_F \rfloor$ and $\lfloor Q^\bullet \rfloor = \lfloor \mathcal{P}^\bullet_{F^\perp} \rfloor$.*

▶ **Lemma 41.** *If a graphical connective is not DR-decomposable then it is not g-decomposable.*

Proof. By absurd, let $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ be a g-decomposable graphical connective which is not DR-decomposable. Thus, there is a MLL formula F such that $P^\bullet = \mathcal{P}^\bullet_F$ and $Q^\bullet = \mathcal{P}^\bullet_{F^\perp}$. Then $\lfloor P^\bullet \rfloor = \lfloor \mathcal{P}^\bullet_F \rfloor$ and $\lfloor Q^\bullet \rfloor = \lfloor \mathcal{P}^\bullet_{F^\perp} \rfloor$. ◀

▶ **Example 42.** Let $F = (((a_1 \parr a_2) \otimes a_3) \otimes a_4) \parr a_5$ and

$$\begin{aligned} \mathcal{P}^\bullet_F \quad &= P^\bullet_1 = \quad \{\langle [0,1,3,4],[2],[5]\rangle, \langle [1,3,4],[2],[0,5]\rangle, \langle [0,2,3,4],[1],[5]\rangle, \langle [2,3,4],[1],[0,5]\rangle\} \\ \mathcal{P}^\bullet_{F^\perp} \quad &= Q^\bullet = \quad \{\langle [0,1,2,5],[3],[4]\rangle, \langle [0,3,5],[1,2],[4]\rangle, \langle [0,4,5],[1,2],[3]\rangle\}. \end{aligned}$$

Let $P^\bullet_2 = P^\bullet_1 \cup \{\langle [1,3,4],[0,2],[5]\rangle\}$ and $\mathsf{C}_1 = \mathsf{C}_{(P^\bullet_1,Q^\bullet)}$ and $\mathsf{C}_2 = \mathsf{C}_{(P^\bullet_2,Q^\bullet)}$. Then C_1 and C_2 are both DR-decomposable, since $\lfloor P^\bullet_1 \rfloor = \lfloor P^\bullet_2 \rfloor = \lfloor \mathcal{P}^\bullet_F \rfloor$ and $\lfloor Q^\bullet \rfloor = \lfloor \mathcal{P}^\bullet_{F^\perp} \rfloor$, while $\mathsf{C}_1 = \mathsf{C}_{(P^\bullet_1,Q^\bullet)}$ is g-decomposable and $\mathsf{C}_2 = \mathsf{C}_{(P^\bullet_2,Q^\bullet)}$ is not g-decomposable.

▶ **Proposition 43** (Switchings composition). *Let $F = F(a_1, \dots, a_n)$ be a* MLL*-formula, then:*

1. *If $F = F_1(a_1, \dots, a_k) \parr F_2(a_{k+1}, \dots a_n)$ and σ is a switching of $\mathfrak{P}_F$, there are $p^\bullet_1 \in \mathbb{P}_{\{0,1,\dots,m\}}$ and a $p^\bullet_2 \in \mathbb{P}_{\{0,m+1,\dots,n\}}$ pointed partitions associated respectively to a test of $\mathfrak{P}_{F_1}$ and a test of $\mathfrak{P}_{F_2}$ such that $p^\bullet_\sigma \in \mathbb{P}^\bullet_n$ is $p^\bullet_\sigma = \lfloor p^\bullet_1 \rfloor \cup p^\bullet_2$ or $p^\bullet_\sigma = p^\bullet_1 \cup \lfloor p^\bullet_2 \rfloor$.*
2. *If $F = F_1(a_1, \dots, a_k) \otimes F_2(a_{k+1}, \dots a_n)$ and σ is a switching of $\mathfrak{P}_F$, then there are $p^\bullet_1 \in \mathbb{P}_{\{0,1,\dots,m\}}$ and a $p^\bullet_2 \in \mathbb{P}_{\{0,m+1,\dots,n\}}$ pointed partitions associated respectively to a test of $\mathfrak{P}_{F_1}$ and a test of $\mathfrak{P}_{F_2}$ such that $p^\bullet_\sigma \in \mathbb{P}^\bullet_n$ is the pointed partition*

$$p^\bullet_\sigma = (p^\bullet_1 \setminus \{\gamma^\bullet_1\}) \cup (p^\bullet_2 \setminus \{\gamma^\bullet_2\}) \cup \{\gamma^\bullet_1 \cup \gamma^\bullet_2\}$$

   with $\gamma^\bullet_1$ and $\gamma^\bullet_2$ respectively the blocks of $p^\bullet_1$ and $p^\bullet_2$ containing 0.
3. *For all $i \in \{1, \dots, n\}$ there is a $p^\bullet \in \mathcal{P}^\bullet_F$ with $\gamma \in p^\bullet$ such that $\{0, i\} \subset \gamma$.*

Proof. 1. Since $F = F_1 \parr F_2$, then every switching σ on $\mathfrak{P}_F$ is given by a switching σ_1 on $\mathfrak{P}_{F_1}$, a switching σ_2 on $\mathfrak{P}_{F_2}$ and a switch for the principal $\parr$ node. If $i, j > 0$, then $i, j \in \gamma \in p^\bullet_\sigma$ iff their corresponding premise are connected in $\sigma(\mathfrak{P}_F)$. Thus i, j belong to a same block iff there is a block in $p^\bullet_{\sigma_1}$ or in $p^\bullet_{\sigma_2}$ which contains both i and j.
For $j = 0$, since only one block may contain 0 accordingly with the switch of the principal $\parr$, i and 0 belong in the same block iff they are either in the same block in $p^\bullet_{\sigma_1}$ or in $p^\bullet_{\sigma_2}$.
2. Similarly to the previous case. It suffices to remark that if $i, j \in \gamma \in p^\bullet_\sigma$ then either i and j belong to the same block in $p^\bullet_{\sigma_1}$ or in $p^\bullet_{\sigma_2}$, or i and 0 belong to the same block in $p^\bullet_{\sigma_1}$ and j and 0 belong to the same block in $p^\bullet_{\sigma_2}$.
3. By induction over F. If $F = a$ is an atomic formula then $\mathcal{P}^\bullet_F = \{\langle [1] \rangle\}$, while if $F = F_1 \otimes F_2$ or $F = F_1 \parr F_2$ then i and 0 belong to the same block of a pointed partition in $\mathcal{P}^\bullet_F$ iff i and 0 belong to a same block of a partition in $\mathcal{P}^\bullet_{F_1} \cup \mathcal{P}^\bullet_{F_2}$. ◀

▶ **Proposition 44** (Pretypes composition). *Let F be a* MLL*-formula.*

1. *If $F = F_1 \parr F_2$, then $p^\bullet \in \mathcal{P}^\bullet_F$ iff $p^\bullet = \lfloor p^\bullet_1 \rfloor \cup p^\bullet_2$ or $p^\bullet = p^\bullet_1 \cup \lfloor p^\bullet_2 \rfloor$ with $p^\bullet_1 \in \mathcal{P}^\bullet_{F_1}$ and $p^\bullet_2 \in \mathcal{P}^\bullet_{F_2}$.*
2. *If $F = F_1 \otimes F_2$, then $p^\bullet \in \mathcal{P}^\bullet_F$ iff $p^\bullet = (p^\bullet_1 \setminus \{\gamma^\bullet_1\}) \cup (p^\bullet_2 \setminus \{\gamma^\bullet_2\}) \cup \{\gamma^\bullet_1 \cup \gamma^\bullet_2\}$ with $p^\bullet_1 \in \mathcal{P}^\bullet_{F_1}$ and $0 \in \gamma^\bullet_1 \in p^\bullet_1$, and $p^\bullet_2 \in \mathcal{P}^\bullet_{F_2}$ and $0 \in \gamma^\bullet_2 \in p^\bullet_2$.*

Proof. It follows the constructions given in the proof of Proposition 44. ◀

▶ **Lemma 45.** *If $F = F_1 \parr F_2$ is a* MLL *formula then $|\lfloor \mathcal{P}^\bullet_F \rfloor| = |\lfloor \mathcal{P}^\bullet_{F_1} \rfloor| \cdot |\lfloor \mathcal{P}^\bullet_{F_2} \rfloor|$.*

Proof. Since $\lfloor p^\bullet_1 \cup \lfloor p^\bullet_2 \rfloor \rfloor = \lfloor \lfloor p^\bullet_1 \rfloor \cup p^\bullet_2 \rfloor = \lfloor p^\bullet_1 \rfloor \cup \lfloor p^\bullet_2 \rfloor$, we conclude by Proposition 44. ◀

5.3 Correspondence between sequential and graphical connectives

There is a strong link between s-decomposable sequential and g-decomposable graphical connectives as exemplified by $\otimes$ and $\parr$:

$$\begin{array}{rcccccc} \lfloor \{\langle [0,1,2] \rangle\} \rfloor^\perp & = & \{\langle [1,2] \rangle\}^\perp & = & \{\langle [1],[2] \rangle\} & = & \mathcal{O}(\otimes) \\ \lfloor \{\langle [1,0],[2] \rangle, \langle [1],[0,2] \rangle\} \rfloor^\perp & = & \{\langle [1],[2] \rangle\}^\perp & = & \{\langle [1,2] \rangle\} & = & \mathcal{O}(\parr) \end{array}$$

In fact, the two syntaxes are orthogonal views of a same decomposable connective:

▶ **Proposition 46** ([5]). *If $\mathsf{C}_{(P^\bullet, Q^\bullet)}$ is a g-decomposable graphical connective, then there is a* MLL *formula F such that $\mathcal{O}(F) = \lfloor \mathcal{P}^\bullet_{\mathsf{C}_{(P^\bullet, Q^\bullet)}} \rfloor^\perp$.*

▶ **Example 47.** If we consider the connective C given in the Example 39, $\lfloor \mathcal{P}^\bullet_{P^\bullet} \rfloor = \lfloor \mathcal{P}^\bullet_F \rfloor = \{\langle [1,2],[3],[4] \rangle, \}$ with $F = ((a_1 \otimes a_2) \parr a_3) \parr a_4$ and $\lfloor \mathcal{P}^\bullet_{P^\bullet} \rfloor^\perp = \mathcal{O}_F = \{\langle [1,3,4],[2] \rangle, \langle [2,3,4],[1] \rangle \langle [1,3],[2,4] \rangle, \langle [1,4],[2,3] \rangle\}$.

▶ **Corollary 48.** *If* C *is a s-decomposable sequential connective,* $\mathcal{O}(\mathsf{C})$ *and* $\mathcal{O}(\mathsf{C}^\perp)$ *are types.*

Proof. It is consequence of Propositions 5.4 and 46. ◀

6 Non-decomposable connectives

In this section we show that not all connectives are decomposable. We start by the following connective given in [7], then reformulated in [5]:

$$\mathsf{G}_4 = \mathsf{C}_{(P,Q)} \text{ with } P = \{\langle[1,2],[3,4]\rangle, \langle[2,3],[4,1]\rangle\} \text{ and } Q = \{\langle[1,3],[2],[4]\rangle, \langle[2,4],[1],[3]\rangle\}$$

Following [11], G_4 belongs to a class of non-decomposable connectives, called *entangled*, given by two sets of partitions P and Q such that one of them is an entangled type (Definition 8).

We now define a more general class of non-decomposable connectives $\mathsf{C}_{(P,Q)}$ where $P = \mathfrak{S}_{\langle u,v\rangle}$ is a basic set of partitions. We then call such connectives *Girard connectives*. We prove that these connectives are not decomposable and that whenever $\mathfrak{S}_{\langle u,v\rangle}$ is contained in a set of partitions P then the connective $\mathsf{C}_{(P,Q)}$ is non-decomposable (for any Q).

Moreover, if G is a Girard connective, the sequent $\mathsf{G}(a_1,\dots,a_n), \mathsf{G}^\perp(a_1^\perp,\dots,a_n^\perp)$ admits no η-exapaded proof in $\mathsf{MLL}(\mathcal{C})$ (this problem is known as "packaging problem"). In fact, since Girard connectives are not decomposable, this sequent is not stepwise derivable in $\mathsf{MLL}(\mathcal{C})$. In other words, for any $\mathcal{C}$ containing at least one non-decomposable connective, any sequent system for $\mathsf{MLL}(\mathcal{C})$ can not be an *initial-coherent system* [13].

▶ **Definition 49** (Girard connectives). *If* $\mathfrak{S}_{\langle u,v\rangle}$ *is a space of basic partitions with* u *and* v *prime numbers, we call the sequential connective* $\mathsf{C}_{\langle u,v\rangle}$ *described by* $(\mathfrak{S}_{\langle u,v\rangle}, \mathfrak{S}^\perp_{\langle u,v\rangle})$ *a* sequential Girard connective. *Moreover, we call the graphical connective* $\mathsf{C}_{\langle u,v\rangle}$ *described by* $(P^\bullet, Q^\bullet)$ *a* graphical Girard connective *iff* $\lfloor P^\bullet \rfloor = \mathfrak{S}_{\langle u,v\rangle}$ *and* $\lfloor Q^\bullet \rfloor = \mathfrak{S}^\perp_{\langle u,v\rangle}$.

▶ **Theorem 50.** *Every Girard graphical connective is not DR-decomposable.*

Proof. Let $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ be a Girard graphical connective. By definition this means that $\lfloor P^\bullet \rfloor = \mathfrak{S}_{\langle u,v\rangle}$ and $\lfloor Q^\bullet \rfloor = \mathfrak{S}^\perp_{\langle u,v\rangle}$. By absurd, if $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ is DR-decomposable, then there is a MLL-formula F such that $\lfloor \mathcal{P}^\bullet_F \rfloor = \mathfrak{S}_{\langle u,v\rangle}$ and $\lfloor \mathcal{P}^\bullet_{F^\perp} \rfloor = \mathfrak{S}^\perp_{\langle u,v\rangle}$. Depending on F, we have three cases:

- if F is an atomic formula, then $\lfloor \mathcal{P}^\bullet_F \rfloor = \{\langle[1]\rangle\} \neq \mathfrak{S}_{\langle u,v\rangle}$ for any $u, v \in \mathbb{N}$;
- if $F = F_1 \parr F_2$, by Lemma 45, $v = |\mathfrak{S}_{\langle u,v\rangle}| = ||\lfloor \mathcal{P}^\bullet_F \rfloor|| = ||\lfloor \mathcal{P}^\bullet_{F_1} \rfloor|| \cdot ||\lfloor \mathcal{P}^\bullet_{F_2} \rfloor||$. Since v is prime, we can assume without loss of generality that $\lfloor \mathcal{P}^\bullet_{F_1} \rfloor = \{p_1\}$, thus there is at least a block $\gamma \in p_1$ such that $\gamma \in p$ for all $p \in \lfloor \mathcal{P}^\bullet_F \rfloor$;
- if $F(a_1,\dots,a_n) = F_1 \otimes F_2$, we can assume without loss of generality that $F_1 = F_1(a_1,\dots,a_k)$ and $F_2 = F_2(a_{k+1,\dots,a_n})$ with $k+1 > v$. Thus, by Proposition 43.3, there is a $\gamma_1 \in p^\bullet_1 \in \mathcal{P}^\bullet_{F_1}$ such that $0, 1 \in \gamma_1$. Since $k+1 > v$ and $n = uv$, then there is $j \geq k+1$ such that $\delta_n(i,j) \leq v$. Moreover, by Proposition 43.3, there is a $\gamma_2 \in p^\bullet_2 \in \mathcal{P}^\bullet_{F_2}$ such that $0, j \in \gamma_2$. By Proposition 43.2 we conclude that there is $\gamma \in p^\bullet \in \mathcal{P}^\bullet_F$ such that $j, i \in \gamma$, which is absurdum after Lemma 13. ◀

▶ **Corollary 51.** *Every graphical Girard connective is not g-decomposable.*

Proof. By Theorem 50 and Lemma 41. ◀

▶ **Corollary 52.** *Every Girard connective is not s-decomposable.*

▶ **Theorem 53** (Danos-Regnier). *Let* $P = \lfloor P^\bullet \rfloor$ *and* $Q = \lfloor Q^\bullet \rfloor$ *s.t.* $P = Q^\perp$ *and* $Q = P^\perp$. *Then a graphical connective* $\mathsf{C}_{(P^\bullet,Q^\bullet)}$ *is DR-decomposable iff* $\mathsf{C}_{(P,Q)}$ *is s-decomposable.*

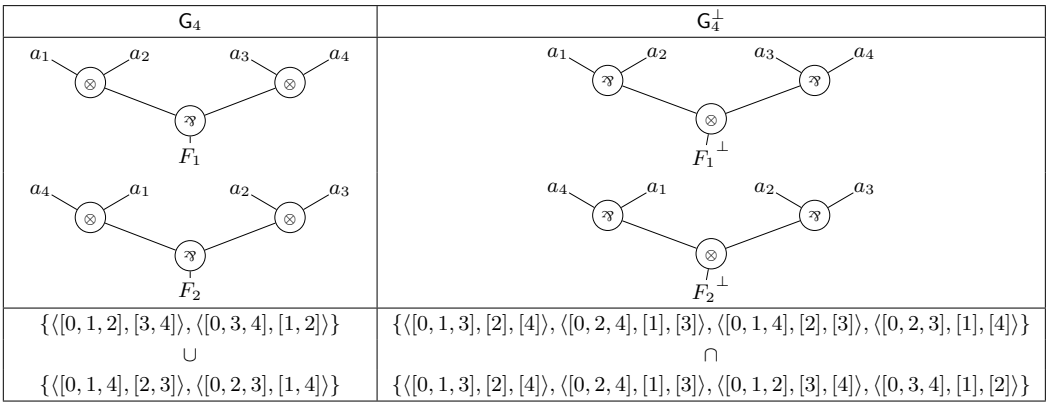

Figure 13 The connectives $\mathsf{G}_4 = \mathsf{C}_{\langle 2,2\rangle}$ and its dual connective $\mathsf{G}_4^\perp$ seen respectively as the union of DNF formulas pretypes and the intersection of CNF formula pretypes.

In [11] it is showed that $P = Q^\perp$ and $Q = P^\perp$ for every sequential connective $\mathsf{C}_{(P,Q)}$.

▶ **Corollary 54** (Completion of a sequential Girard connective). *Let $n = uv$ with u, v prime numbers, and P and Q non empty subsets of $\mathbb{P}_n$. If $\mathsf{C}_{(P,Q)}$ is a s-decomposable sequential, then $\mathsf{C}_{\langle u,v\rangle} \not\subset P$ and $\mathsf{C}^\perp_{\langle u,v\rangle} \not\subset P$.*

Proof. By Theorem 18, both $\mathfrak{S}_{\langle u,v\rangle}$ and $\mathfrak{S}^\perp_{\langle u,v\rangle}$ are types. Moreover, by Proposition 46, if $\mathsf{C}_{(P,Q)}$ is decomposable then P is a type. Then, by Theorem 7, none of $\mathsf{C}_{\langle u,v\rangle}$ and $\mathsf{C}^\perp_{\langle u,v\rangle}$ can be subsets of P. ◀

7 Conclusions and future works

In this paper we studied the generalized multiplicative connectives which can be described by two sets of pairwise orthogonal partitions. The orthogonality condition guarantees the definition of dual connectives for which cut-elimination is satisfied. Thus, multiplicative linear logic can be extended with these connectives preserving a computational interpretation.

We defined a notion of decomposability by means of $\parr$ and $\otimes$ for generalized connectives, with respect to both sequent calculus and proof structures syntax. We then showed the existence of connectives which are not decomposable in both senses. In particular, we exhibited the existence of an infinite family of non-decomposable connectives called Girard connectives. For such non-decomposable generalized connectives, we gave an interpratation as superposition of special decomposable generalized connectives which are connectives associated to a family of MLL disjunctive normal forms.

The class of Girard connectives strictly includes the class of non-decomposable entagled connectives, thus extending the previous work of the second author on the same subject [11]. Although the definition of a Girard connective appears to be highly combinatorial, it admits the following simple geometrical interpretation. Every Girard connective in graphical syntax can be interpreted either as the union of the pretypes of a family of DNF formulas or as the intersection of the pretypes of a family of CNF formulas having the same formula tree but differing for the cyclic permutation of their atoms/leaves (see Figure 13). Observe that cyclic permutations can help to visualize the partition associated to those connectives (see Figure 3). This interpretation is not trivial since, by Proposition 5.6, the union of pretypes is not

necessarily a type. However, these connectives have no relation with the cyclic fragment of multiplicative linear logic [1]: neither the order among blocks nor the order among the elements of each block take role in the definition.

The existence of non-decomposable multiplicative connectives which do not admit any sequentialization via the $\otimes$ and $\parr$, suggests future investigations on their geometry of interaction [8], their connection to syntaxes for *concurrency* such as the *π-calculus* [14] and their denotational semantics [4] expanding the ideas given in [9] for syntectic connectives. Moreover, from the view point of logical programming with proof nets [3], non-decomposable graphical connectives provide additional *modules*. We foresee the use of the Girard connectives which may be interpreted as superposition of DNF for the definition of modules representing the superpositions of *bipoles* [2].

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