SIAM J. MATH. ANAL.
Vol. 49, No. 4, pp. 3161–3207

RESPONSE SOLUTIONS FOR QUASI-PERIODICALLY FORCED, DISSIPATIVE WAVE EQUATIONS*

RENATO C. CALLEJA†, ALESSANDRA CELLETTI‡, LIVIA CORSI§, AND RAFAEL DE LA LLAVE¶

To the memory of Gilberto Flores and Tim Minzoni

Abstract. We consider several models of nonlinear wave equations subject to very strong damping and quasi-periodic external forcing. This is a singular perturbation, since the damping is not the highest order term or it creates multiple time scales. We study the existence of response solutions (i.e., quasi-periodic solutions with the same frequency as the forcing). Under very general nonresonance conditions on the frequency, we show the existence of asymptotic expansions of the response solution; moreover, we prove that the response solution indeed exists and depends analytically on ε (where ε is the inverse of the coefficient multiplying the damping) for ε in a complex domain, which in some cases includes disks tangent to the imaginary axis at the origin. In other models, we prove analyticity in cones of aperture $\pi/2$ and we conjecture it is optimal. These results have consequences for the asymptotic expansions of the response solutions considered in the literature. The proof of our results relies on reformulating the problem as a fixed point problem in appropriate spaces of smooth functions, constructing an approximate solution, and studying the properties of iterations that converge to the solutions of the fixed point problem. In particular we do not use dynamical properties of the models, so the method applies even to ill-posed equations.

Key words. dissipative wave equation, quasi-periodic solution, response solution

AMS subject classifications. 70K43, 70K20, 34D35

DOI. 10.1137/151005397

1. Introduction.

In recent times there has been extensive interest in strongly damped systems, namely, systems in which the term describing the damping contains a factor ε^{-1} (where ε is a small parameter), and subject to external forcing. Since the damping is not the term which corresponds to the time-derivative of highest order (see (1.1) below) or it might be responsible for the onset of multiple time scales (see (1.2) below), the system contains a singular perturbation in ε. We are interested in finding *response solutions*, i.e., solutions which have the same frequency as the forcing term, for two classes of equations, namely,

$$\partial_{tt}u(t,x) + \frac{1}{\varepsilon}\,\text{Friction} - \Delta_x u(t,x) + h(u(t,x),x) = f(\omega t, x) \tag{1.1}$$

*Received by the editors January 26, 2015; accepted for publication (in revised form) March 16, 2017; published electronically August 17, 2017.
http://www.siam.org/journals/sima/49-4/100539.html
Funding: The work of the first author was partially supported by NSF grant DMS-1162544 and CONACYT grant 133036. The work of the second author was partially supported by PRIN-MIUR 2010JJ4KPA_009 and GNFM/INdAM. The work of the third author was partially supported by the ERC project "Hamiltonian PDEs and Small Divisor Problems: A Dynamical Systems Approach." The work of the third and fourth authors was partially supported by NSF grant DMS-1500943.
†Department of Mathematics and Mechanics, IIMAS, National Autonomous University of Mexico (UNAM), Apdo. Postal 20-126, C.P. 01000, Mexico D.F., Mexico (calleja@mym.iimas.unam.mx).
‡Department of Mathematics, University of Rome Tor Vergata, 00133 Rome, Italy (celletti@mat.uniroma2.it).
§School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0161 (lcorsi@math.gatech.edu).
¶School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 (rafael.delallave@math.gatech.edu).

or

$$\varepsilon^2 \partial_{tt} u(t,x) + \text{Friction} - \Delta_x u(t,x) + h(u(t,x),x) = f(\omega t, x)\ , \tag{1.2}$$

where in (1.2) the nonlinearity can be replaced by $\varepsilon h(u(t,x),x)$ (see models A, A′, B, and B′ in section 2.1). In (1.1) we will consider different models of friction, which have appeared in the literature, namely, $\partial_t u(t,x)$, $\partial_t \Delta_x u(t,x)$. In (1.1) and (1.2) we will assume that h, f are real for real arguments, so that the problems make sense for $\varepsilon \in \mathbb{R}$, in such case, we will obtain that u is real for real values but we will later consider $\varepsilon \in \mathbb{C}$ which is natural in order to obtain analyticity results in ε. Of course the equations will be supplemented with the boundary conditions. Both problems considered as evolutions are singular perturbations, since the highest order term has coefficients that are smaller than those of a term of lower order. Of course, one can multiply and rescale the time to make the models look similar. Note that, after a time rescaling in (1.2), the main remaining difference between the two classes of equations (1.1) and (1.2) is that the forcing term in (1.2) takes the form of $f(\varepsilon\omega t)$ which is slowly varying in time, so that there are different time scales.

A first attempt to understand these problems is to use perturbation theory in ε and obtain formal series in powers of ε. Nevertheless, since the perturbation is singular, one does not expect that the resulting formal series is convergent and one needs to use resummation techniques to obtain that there is an analytic solution defined in an open complex domain which does not include $\varepsilon = 0$ but has it on the boundary. This approach has been used for ODEs in [GBD05, GBD06, Gen10a, Gen10b]. Different arguments for other singular perturbation problems can be found in [Bal94, Rou15].

In [CCdlL13] one can find an alternative approach for singular problems in ODEs, which inspired our present treatment for PDEs. One considers the perturbative expansion to low orders and obtains a reasonably good approximate solution in a neighborhood of $\varepsilon = 0$ (i.e., an expression that solves the equation up to a small error). Then, starting from the approximate solution, one switches to another perturbative method (a contraction mapping argument in this paper) to prove the existence of a true solution. Since the problem is analytic in ε for ε ranging in a complex domain, one obtains analytic dependence in ε of the solution for ε in a certain domain. The domains we consider do not include any ball centered at zero. Indeed, we find that there are arbitrarily small values of ε for which the map is not a contraction and the method of proof breaks down. We conjecture that this is a real effect and not just a shortcoming of the method (see section 8).

To motivate the procedure adopted in [CCdlL13] and in this paper, we argue heuristically that since $\varepsilon = 0$ is the most singular value of ε, one attempts to do as little work as possible based on it. One tries to implement a perturbation theory on small but nonzero values of ε; as soon as one gets even a flimsy foothold on nonzero values of ε one switches to another perturbation method that is not affected by singularities (even if it contains some large terms, they can be beaten by pairing them with small ones). This procedure is somewhat reminiscent of some works in Celestial Mechanics, notably Hill's theory of the moon [Hil78], [Poi87b], in which one uses a perturbation theory from an intermediate model which is controlled in turn by another perturbative argument.

As happens often in perturbative expansions, the way one deals with the first order term is different from the subsequent ones (the equation for the leading term is often called the "inner equation"): this is even more evident in situations like the present one, since we are dealing with singular perturbations. In [CCdlL13] the first term of the expansion, corresponding to $\varepsilon = 0$, was obtained by means of an implicit function theorem, but the subsequent steps were all similar and they involved the same hypotheses. In this paper, the difference between the zeroth order term and the higher order ones is even more dramatic. The term in the expansion corresponding to $\varepsilon = 0$ is very different from the others and in principle can be dealt with by a variety of methods, including implicit function theorems (at least for certain cases, as we do for the model described by (2.3) below—see section 7.2) or using variational methods (as we can do for the model described by (2.1) below), depending on the model we are studying. As we will see, when we apply variational methods, we may get even infinitely many solutions of the order 0 equation. Each of them continues to a family of solutions analytic in ε.

Hence, the procedure adopted in the present work has two steps, with the first step having two substeps.

(a) obtaining an approximate solution to high order, and precisely
 (a1) obtaining the order zero solution;
 (a2) obtaining high order approximations.
(b) polishing off the approximate solutions to obtain true solutions.

Each of these steps has its own methodology (indeed, step (a1) will be accomplished by means of several different methodologies depending on the model) and requires different conditions on the frequency as well as different nondegeneracy assumptions. Hence, the conditions required in the main theorem are obtained by joining together the conditions of all the steps.

Nevertheless, the final assumptions are very weak. For example, the nonresonance conditions needed to carry out the whole argument are weaker than the Brjuno condition and they allow exponentially growing small divisors.

The strategy above is widely applicable. In this paper we decided to document its breadth by applying it to four different models in the literature with several variations, such as different boundary conditions. We call these models A, A′, B, and B′ (more details will be given in section 2). On the other hand, we have not optimized the hypotheses: It seems clear that one could obtain slightly sharper domains of analyticity, better regularity conditions, less assumptions on the domain, etc. We conjecture (and present arguments in favor of the conjecture) that the domains obtained are essentially optimal (see section 8).

The main result for step (a) is Theorem, 4.3, the main result for step (b) is Theorem 4.6, and the final result is Theorem 4.2.

Step (a1) is the solution of a nonperturbative functional equation. Step (a2) is a Lindstedt procedure, which entails very mild conditions on the small divisors and requires very weak nonresonance conditions on the frequency. In this way one produces polynomials in ε which solve the equation up to some (high) power of ε. Under a bit stronger conditions on the small divisors, the Lindstedt procedure provides the existence of a formal solution up to all orders in ε (see Theorem 4.3). As we will show, the solutions of the equations for the perturbative expansion are unique once we fix the solution of order 0; however, as already pointed out, this solution to order 0 can be very nonunique.

Step (b) is based on a contraction mapping principle. Hence no small divisors are involved but, on the other hand, we need to consider ε in an appropriate complex domain to carry out the argument. We note that step (b) also works in cases where the spectrum of the operators driving the evolution is not discrete. Unfortunately, we do not know good conditions that ensure that one can perform step (a) when the spectrum is not discrete. If, by any chance, one is dealing with a particular problem having a continuous spectrum and step (a) can be performed, then step (b) can be performed too and one can obtain the result.

The final result is that the response solution is an analytic function of ε defined in a domain (selected in step (b)) which does not include zero, even if it might include circles with real centers and tangent to the imaginary axis. Hence, the method does not guarantee that the Lindstedt series (the formal power expansion) converges because the analyticity domain established does not contain any circle centered at the origin. Indeed, in [CCdlL13] one can find arguments that suggest that the Lindstedt series does not converge in general, even in the case of ODEs. Here, we also present similar arguments in section 8.

Nevertheless, the domain of analyticity established here for models A and A′ (describing dissipative wave equations) is large enough so that the Nevanlinna–Sokal theory [Nev19, Sok80, Har49] on asymptotic expansions applies. As a consequence, the response solutions constructed here have an asymptotic expansion and these functions can be reconstructed from their asymptotic expansions by resummation. Notice that this procedure is very different from establishing the existence of the solution by resumming the series. Of course, since the problem is nonlinear, resumming the series is not enough and one needs other arguments to show that the resummation solves the equation [Har49] (see also [Bal94, BLS02, GBD05, GBD06, Gen10a, Gen10b]). We note that the solutions u are real for real values of ε and x, t. When f is real for real arguments.

In some models such as models B and B′ (describing large stiffness equations), we obtain domains of analyticity which are cones containing the imaginary axis and have an aperture of $\pi/2$. We conjecture that these domains are essentially optimal (see section 8). We will show that the functions we construct have the same asymptotic expansions as the formal power series. On the other hand we note that in domains of this kind it is not clear that the response solution can be obtained by resumming the asymptotic expansions: indeed in these domains there are nontrivial functions whose asymptotic expansion vanishes, so that the expansion is not unique (e.g., the Cauchy example $\exp(-\varepsilon^{-2})$ which has an asymptotic series vanishing in domains of aperture less than $\pi/2$). As a consequence, it could well happen that for these models the response solutions lead to exponentially small phenomena. Notice that model B is an infinite dimensional analogue of fast oscillators for which exponentially small phenomena have been established (see [BFGS12]).

In [CCdlL13] the problem considered is the *varactor* equation, which is a single ODE. Even if step (b) in [CCdlL13] was just an elementary one (based on contraction arguments), the results obtained in [CCdlL13] improved the existence domains and weakened the nonresonance conditions that have been imposed in the previous literature. The analyticity domain was later extended for ODEs in [CFG13] (where a domain of analyticity tangent more than quadratically to the origin was established) and the nondegeneracy assumption on the nonlinearity has been relaxed (for real ε) in [CFG14]. It seems plausible that using a more efficient fixed point argument (e.g., the KAM theory) or higher order perturbations in step (a) would improve the results of the present paper.

As further references, we mention also [Rab67, Rab68, Cra83], where the periodic case with real small damping has been considered. We point out that we only consider the existence of response solutions and we do not discuss their stability. Since we will consider complex values, this is not so straightforward (for some of the values of ε the equation might even become ill-posed).

1.1. Description of the main results. The goal of this paper is to extend the method of [CCdlL13] to some PDEs. The method is very flexible and we will present results for four different models considered in the literature, each with three different types of boundary conditions (see section 2). It is clear that there are many more models that could have been considered by the method. Of course the main difficulty of the extension of the method of [CCdlL13] to PDEs is that the operators appearing in PDEs are unbounded. Hence the reformulation of the problem as a fixed point problem requires some more thought, even to get a viable formulation. For example, we need to ensure that the operator maps some space into itself and that the space satisfies suitable properties (such as Banach algebra properties). So, considerable effort goes into the choice of spaces as it happened in the classical study of elliptic problems (see section 3.5).

In all models, given a domain $\mathcal{D}$ (as specified in section 2.1) and denoting by $\overline{\mathcal{D}}$ its topological closure, $u : \mathbb{R} \times \overline{\mathcal{D}} \to \mathbb{R}$ is the unknown (later we will also consider complex extensions). We will require that the following data of the problem are fixed:

- the boundary conditions;
- $h : \mathbb{R} \times \overline{\mathcal{D}} \to \mathbb{R}$ to which we refer as the nonlinearity;
- $f : \mathbb{T}^d \times \overline{\mathcal{D}} \to \mathbb{R}$ (with $\mathbb{T}^d \equiv (\mathbb{R}/\mathbb{Z})^d$) to which we refer as the forcing;
- $\omega \in \mathbb{R}^d$, which denotes the frequency of the forcing. We assume without loss of generality that ω has rationally independent components, namely, that $\omega \cdot k \neq 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$.

Of course we assume that the forcing and the nonlinearity are such that the boundary conditions are maintained. We will also need

- quantitative estimates on the size of $|\omega \cdot k|^{-1}$ as a function of $|k|$ (which will turn out to be weaker than the Diophantine or Bryuno conditions);
- a nondegeneracy condition on the nonlinearity.

Then we shall prove the following "meta" result.

Meta-Theorem 1.1. *Under the above requirements there exists a response solution for the models of the form* (1.1) *or* (1.2) *above, defined for ε in an appropriate complex domain: the specific form of such domain depends on the model considered as well as on the boundary conditions.*

The precise statement of the result requires the introduction of the spaces, the domains, and a precise formulation of the regularity condition that we will give later on; see sections 3–4. The existence of the solutions of the zeroth order term is discussed in section 5 and Appendix B. The proof for the case of dissipative wave equations is provided in section 6, while the modifications of the proof for the other models are given in section 7. Some arguments supporting that the domains are almost optimal are given in section 8.

2. Models considered and some preliminary assumptions.

In this section we present the models we intend to study and we state the required nondegeneracy assumptions.

In what follows we will assume that Δ_x is a self-adjoint elliptic operator of second order; in the physical applications we have in mind it is the Laplace–Beltrami operator. We will not necessarily assume that Δ_x is a constant coefficient operator.

2.1. PDEs considered. We will consider PDEs for which the space variables range in the topological closure of an ℓ-dimensional domain $\mathcal{D}$ and we will look for solutions quasi-periodic in time. The domain $\mathcal{D}$ can be

D1. a compact manifold without boundary, for example, $\mathbb{T}^\ell$ (we will refer to this as the periodic case), D2. an open, bounded, connected subset of $\mathbb{R}^\ell$ with a C^∞ boundary. In this case, we will supplement the solutions with either Dirichlet or Neumann boundary conditions.

Therefore, we will consider the following (standard) boundary conditions, each one leading to a different functional setting, which we will specify below:

D. homogeneous Dirichlet boundary conditions,

N. homogeneous Neumann boundary conditions,

P. periodic boundary conditions.

Following a usual approach, we interpret the boundary conditions as describing a space of solutions. In our case, the spaces will consist of very smooth functions so that the boundary conditions make sense in the classical sense. Then, the operator Δ_x acts on this space. Of course the spectral properties of Δ_x depend on the space too. To specify the function space we also need to specify a norm. Our treatment will be for spaces of functions which are analytic in t and differentiable to a high order in x.

We will consider four different PDEs: models A, A′, B, and B′. Each of them may have entirely different boundary conditions (periodic, Dirichlet, and Neumann).

A. The dissipative wave model. The first model is a direct analogue of the varactor equation studied, e.g., in [CCdlL13, CGV05, GBD05, GBD06, Gen10a, Gen10b, CFG14, CFG13]; the model is obtained from the wave equation by adding a singular friction proportional to the velocity:

$$\partial_{tt}u(t,x) + \frac{1}{\varepsilon}\partial_t u(t,x) - \Delta_x u(t,x) + h(u(t,x),x) = f(\omega t,x)\ . \tag{2.1}$$

Equations of the form (2.1) are often referred to in the literature as (nonlinear, forced) "telegraph equations" [GGG12], [Ber08], [Maw05], [Ort05], [MORP05].

A′. The frequency overdamped model. We modify the friction of model A, as described by the following equation:

$$\partial_{tt}u(t,x) \pm \frac{1}{\varepsilon}\partial_t \Delta_x u(t,x) - \Delta_x u(t,x) + h(u(t,x),x) = f(\omega t,x)\ ; \tag{2.2}$$

we consider both well-posed (negative sign in the second term of (2.2)) and ill-posed (positive sign in (2.2)) models, since the method of proof encompasses both cases without too much additional effort (see section 7.1). The equation with a positive sign has connections with the backward heat equation, which finds applications in different fields, for example, in problems related to image enhancement. The equation with a negative sign has been studied, for instance, in [PSM09], where the damping is stronger for the spatial modes with larger spatial frequency. Indeed, not only is the damping term $\varepsilon^{-1}\,\partial_t\Delta_x u$ in (2.2) affected by a factor which is the inverse of the small parameter ε, but also it contains the unbounded operator Δ_x.

For simplicity, we have considered the case where the Δ_x appearing in the damping and in the restoring force are the same operator. Some slight generalizations are possible (but we will not consider them), such as taking different operators for the damping and the restoring force. The argument presented here works with small modifications provided that the damping and the restoring force commute. In many physical applications, it is natural that the operators describing the damping and the force commute, since they have to be translation invariant and isotropic.

Note that we will allow the parameter ε appearing in (2.2) to vary in a domain symmetric by the exchange $\varepsilon \to -\varepsilon$ (see (4.6)) so in principle there is no need to distinguish the sign $\pm$ in (2.2). However, we decided to highlight it in order to stress the fact that we allow the equation to be ill-posed.

B. Large stiffness model. This is a generalization of the model introduced in [Flo14, FMPS07], described by the equation

$$\varepsilon^2\partial_{tt}u(t,x)+\partial_t u(t,x)-\Delta_x u(t,x)+h(u(t,x),x)=f(\omega t,x)\ ; \tag{2.3}$$

in [Flo14, FMPS07] one can find the specific case $h(u,x)=\gamma/(1+u)^2$ with $\varepsilon\in\mathbb{R}$ and with $\gamma\geq 0$ a dimensionless parameter which provides the relative strengths of electrostatic and mechanical forces in a MEMS.

Equation (2.3) models an electrostatically actuated microelectromechanical systems (MEMS) device. Precisely, the physical interpretation of the model (2.3) is that the restoring force of the oscillators forming the wave equations is very large. This type of equation is used to model the deflection of an elastic membrane suspended above a rigid ground plate, with a voltage source and a fixed capacitor. The model represents the limit of a small aspect ratio, when the gap size is small compared to the device length. The paper [FMPS07] contains a detailed discussion of the motivation. It is interesting to note that the varactor equation considered in [CCdlL13] is somehow a model for all the physical problems motivating model B.

B$'$. The modified large stiffness model. We modify the nonlinearity of model B by assuming that it is of order ε, as described by the following equation:

$$\varepsilon^2\partial_{tt}u(t,x)+\partial_t u(t,x)-\Delta_x u(t,x)+\varepsilon h(u(t,x),x)=f(\omega t,x)\ ;$$

the above equation appears in the study of MEMS with high aspect ratios and/or when the applied tension is high.

Technically, we shall see that models A$'$ and B$'$ require further assumptions on the operator Δ_x with respect to models A and B; see H1$'$ and H2$'$ below.

2.2. Regularity assumptions and boundary conditions. We will require that f is smooth in x, satisfies the boundary conditions, and is analytic in the variable $\theta\equiv\omega t$. We will formulate this assumption more precisely by saying that f belongs to a Hilbert space which we shall call $\mathcal{A}_{\rho,j,m}$; see the definition in section 3.5.2, where we will impose some restrictions on the parameters (as we will see, ρ, j measure the analyticity properties and m measures the regularity properties in the space variables).

We will assume that h has some regularity properties too. Roughly, we will require that h is analytic in its first argument and differentiable when the second argument ranges over $\mathcal{D}$. Slightly more precisely, we will require that h is such that given a function $u\in\mathcal{A}_{\rho,j,m}$, then $h(u(\theta,x),x)$ is also in $\mathcal{A}_{\rho,j,m}$ and that the map $u\mapsto h(u(\cdot),\cdot)$ is differentiable in the sense of maps in Banach spaces. See later for the definition of the spaces $\mathcal{A}_{\rho,j,m}$. We will also require that h satisfies certain geometric conditions ensuring that the boundary conditions are preserved. We make the following requirements:

BCD. For Dirichlet boundary conditions we require that $h(u,x)=0$ for all $x\in\partial\mathcal{D}$.

BCN. For Neumann boundary conditions we require that

$$n(x)\cdot(D_x h)(u,x)=0\qquad \forall x\in\partial\mathcal{D}\ ,\quad u\in\mathbb{R}\ , \tag{2.4}$$

where $n(x)$ denotes the normal to the domain $\mathcal{D}$ at x. In this way, we obtain that

$$n(x)\cdot D_x[h(u(t,x),x)]=h'(u(t,x),x)\ n(x)\cdot D_x u(t,x)+n(x)\cdot(D_x h)(u(t,x),x)\ ,$$

which equals zero if $u(t,\cdot)$ satisfies the Neumann boundary conditions and (2.4) holds. In what follows, h' denotes the derivative of h with respect to u. Notice that we have used that h' is one-dimensional to rewrite $n(x)\cdot(h'\ D_x u) = h'\cdot n(x) D_x u$.

We anticipate that, besides the above regularity and boundary conditions, we will also require some nondegeneracy conditions on h.

Remark 2.1. In this paper we will construct solutions analytic in time. The proofs work similarly in spaces of functions with Sobolev regularity in time (with a high-enough Sobolev exponent depending on the dimension of the frequency) when developing the theory for finitely differentiable cases (i.e., when the functions f, h are only assumed to be finitely differentiable). Of course in this case one can consider only $\varepsilon \in \mathbb{R}$.

3. Formulation of the problem and overview of the method for model A. In this section we go over the method for model A and reduce it to a fixed point problem. We will present first the formal manipulations, since they are the motivation for the constructions and the precise definitions given later on. Notably, the choice of spaces in section 3.5 will be motivated by the need that the operator appearing in the fixed point equation derived in this section maps the spaces into themselves and it is a contraction.

3.1. Response solutions and formal power series. Our goal is to find response solutions of the form

$$u_\varepsilon(t,x) = c_0(x) + U_\varepsilon(\omega t, x)\ , \tag{3.1}$$

where for each fixed ε, $U_\varepsilon : \mathbb{T}^d \times \overline{\mathcal{D}} \to \mathbb{R}$ is at least $O(\varepsilon)$. We will refer to c_0 as the zeroth order term and omit the index ε whenever this does not lead to confusion.

We will first show that when we write U as a formal power series in ε

$$U_\varepsilon = \sum_{j=1}^{\infty} \varepsilon^j U_j\ , \tag{3.2}$$

the coefficients U_j can be formally defined: the appropriate Banach spaces of functions in which the coefficients actually exist will be specified in section 3.5. Such Banach spaces will include regularity properties as well as the boundary conditions.

Inserting (3.1) in (2.1), we get that the functions c_0, U_ε must satisfy the equation[1]

$$\begin{aligned}(\omega\cdot\nabla_\theta)^2 U_\varepsilon(\theta,x) + \frac{1}{\varepsilon}(\omega\cdot\nabla_\theta)U_\varepsilon(\theta,x) - \Delta_x U_\varepsilon(\theta,x) - \Delta_x c_0(x)\\ + h(c_0(x) + U_\varepsilon(\theta,x), x) = f(\theta,x).\end{aligned} \tag{3.3}$$

The solution of (3.3) will be the centerpiece of our treatment. Later, we will develop analogous procedures for models A′, B, and B′ (see section 7). We remark that the series expansion (3.2) does not contain the term $j=0$; in fact, if we add a term U_0 to the series (3.2), then taking the coefficient of order ε^{-1} in (3.3), the term U_0 would satisfy

$$(\omega\cdot\nabla_\theta)U_0(\theta,x) = 0\ ,$$

showing that the solution U_0, which can be found under the nonresonance assumption on ω, is independent on θ. However, having written the response function as in (3.1) with c_0 being the θ-independent part, we conclude that c_0 plays the same role as U_0.

[1]The search of quasi-periodic solutions with frequency ω having rationally independent components is equivalent to looking for a solution $u = u(\theta,x)$ of the differential equation in which ωt is replaced by θ and ∂_t is replaced by $\omega\cdot\nabla_\theta$; this is why we shall study functions of the form $u = u(\theta,x)$.

3.2. Formal solutions of the equation for response functions. In this section we describe how to obtain a formal power series solution for (3.3). This is step (a) of the strategy discussed in the introduction.

3.2.1. Dividing the problem into zeroth order and higher orders. We introduce the notation

$$N_\varepsilon U(\theta,x) \equiv \left[(\omega\cdot\nabla_\theta)^2 + \frac{1}{\varepsilon}(\omega\cdot\nabla_\theta) + \mathcal{L}\right] U(\theta,x)\ , \tag{3.4}$$

$$\mathcal{L}\eta(x) \equiv -\Delta_x\eta(x) + h'(c_0(x),x)\eta(x)\ , \tag{3.5}$$

$$G(U)(\theta,x) \equiv h(c_0(x)+U(\theta,x),x) - h(c_0(x),x) - h'(c_0(x),x)\,U(\theta,x)\ , \tag{3.6}$$

where h' denotes the derivative of h with respect to its first argument. Note that the operator N_ε depends on ε, whereas G and $\mathcal{L}$ are independent of ε.

With the above notation and denoting by $\langle\cdot\rangle$ the average with respect to θ, we split (3.3) into the pair of equations

$$N_\varepsilon U_\varepsilon(\theta,x) + G(U_\varepsilon)(\theta,x) = f(\theta,x) - \langle f\rangle(x), \tag{3.7}$$

$$-\Delta_x c_0(x) + h(c_0(x),x) = \langle f\rangle(x). \tag{3.8}$$

The reason to divide (3.3) into (3.7) and (3.8) is that (3.8) is the leading order in ε.

Notice that the order ε^0-term in (3.3) is

$$(\omega\cdot\nabla_\theta)U_1(\theta,x) - \Delta_x c_0(x) + h(c_0(x),x) = f(\theta,x)\ . \tag{3.9}$$

Hence for a solution U_1 of (3.9) to exist, it is necessary that the average of (3.9) with respect to θ is zero (hence (3.8)). Of course, if ω satisfies suitable nonresonance conditions and the functions f, h, and c_0 are smooth, it is indeed possible to obtain U_1. The solution of (3.9), which is standard in KAM theory and which can be dealt with by Fourier expansions, will be discussed in section 6.1. In conclusion the system (3.7), (3.8) is equivalent to (3.3), if we look for formal solutions as in (3.2).

Notice that the system (3.7), (3.8) has an upper triangular structure. In particular, (3.8) involves only c_0: once we obtain a solution c_0 of (3.8), we can substitute it in (3.9) and obtain the solution U_1 and then proceed to higher orders.

The existence of solutions of (3.8) has been studied extensively in the literature through a great variety of methods. In Appendix B we will present some of the results available in the literature.

As a result we obtain that under many circumstances there are several (often infinitely many) c_0 solving (3.8). For each of them we will see that (under appropriate nondegeneracy conditions) we can find a unique solution U_ε (first as formal power series and then as analytic function in a domain). Hence the upper triangular system may have many solutions, but the only source of nonuniqueness is (3.8) for c_0.

3.2.2. Preliminary assumptions on the operator $\mathcal{L}$. We now specify the spectral properties assumed of the operator $\mathcal{L}$.

Remark 3.1. It is well known [Agm10] that if the operator $\mathcal{L}$ is elliptic (second order), then the boundary value problem can be formulated in $H^m(\overline{\mathcal{D}})$ with $m \geq 2$ for Dirichlet and $m \geq 3$ for Neumann boundary conditions. Note that we assumed $\mathcal{D}$ to have a smooth boundary so that we can use restriction theorems to make sense of boundary conditions. Furthermore the operator $\mathcal{L}$ has compact resolvent in H^m

(the solution is smoother than the data by elliptic estimates) and therefore the spectrum consists of eigenvalues with finite multiplicity. Again by regularity theory, the eigenfunctions are very smooth: hence the spectrum is the same in all the H^m spaces. Finally we observe that if the operator is symmetric w.r.t. an inner product $\langle , \rangle$, namely, if $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$, then the eigenfunctions corresponding to different eigenvalues are orthogonal w.r.t. the inner product $\langle , \rangle$ and moreover the eigenvalues are real. In the physical applications, the inner product is the usual L^2 product; for discussion about the choice of the norm see section 3.5.1.

The following assumptions H1–H2 will be requested for model A as well as for models A′, B, and B′ for the corresponding operator $\mathcal{L}$ in those cases.

H1. The spectrum of $\mathcal{L}$ is real, discrete and its eigenvalues λ_n satisfy for some $J \geq 0$

$$\lambda_1 \leq \cdots \leq \lambda_J \leq 0 \leq \lambda_{J+1} \leq \cdots \leq \lambda_{J+n} \leq \cdots \quad \forall\ n \geq 1$$

with the convention that for $J = 0$ all eigenvalues are positive; the multiplicity of each eigenvalue is finite, possibly increasing with n.

H2. All eigenvalues are different from zero: $\lambda_n \neq 0$ for any $n \geq 1$.

In the case of models A′ and B′ we will also assume the following hypotheses on the operator $-\Delta_x$:

H1′. The spectrum of $-\Delta_x$ is discrete and its eigenvalues λ_n^Δ satisfy for some $J_\Delta \geq 0$:

$$\lambda_1^\Delta \leq \cdots \leq \lambda_{J_\Delta}^\Delta \leq 0 \leq \lambda_{J_\Delta+1}^\Delta \leq \cdots \leq \lambda_{J_\Delta+n}^\Delta \leq \cdots \quad \forall\ n \geq 1$$

with the convention that for $J_\Delta = 0$ all eigenvalues are positive; the multiplicity of each eigenvalue is finite, possibly increasing with n.

H2′. All eigenvalues are different from zero: $\lambda_n^\Delta \neq 0$ for any $n \geq 1$.

Remark 3.2. Note that a consequence of H1 and H1′ is that there is an orthonormal basis of eigenfunctions Φ_n for the operator $P = \mathcal{L}$ *or* $-\Delta_x$ in $L^2(\mathcal{D})$ such that

$$P\Phi_n = \lambda_n^{(P)}\Phi_n \qquad \text{for} \ \ n = 1, 2, \ldots \qquad \text{with} \ \ \lambda_n^{(P)} \equiv \begin{cases} \lambda_n & \text{if } P = \mathcal{L}\ , \\ \lambda_n^\Delta & \text{if } P = -\Delta_x\ . \end{cases}$$

Remark 3.3. As indicated before, for domains $\mathcal{D}$ as considered here H1 is automatic (because of the compact resolvent of elliptic operators). Of course, H2 is nontrivial, but we point out that it holds *generically* with respect to h.

We also note that, in concrete models, H2 can be verified by a finite numerical computation. Using that the operators are elliptic, there are a posteriori theorems on the spectrum. These theorems give explicit upper bounds between the distance of the spectrum of $\mathcal{L}$ and that of a finite dimensional matrix [NN01, CW15].

Remark 3.4. We note that H1 and H2 are in turn assumptions on h, c_0. In some arguments, we will need to assume only H1, but in order to get the crucial estimates on the "small divisors" (and hence to obtain the final result; see Theorem 4.6), we will need to assume that there is the spectral gap in H2, so that we can invert $\mathcal{L}$.

The above assumptions can be slightly modified; in particular, the previous assumptions H1–H2–H1′–H2′ can be extended to encompass the case of a continuous spectrum.

3.2.3. The nonlinear term and the boundary conditions. We start by noticing that G is the operator analogue of the nonlinear term used in [CCdlL13], provided of course that the operator G is defined from some appropriate space to itself.

In this section we will just check that, if we assume h to satisfy the conditions BCD and BCN in section 2.2, then the operator G preserves the spaces of functions satisfying these conditions.

In the case of periodic boundary conditions, there is nothing to check.

For Dirichlet boundary conditions we observe that if $x \in \partial\mathcal{D}$ and c_0, U satisfy the Dirichlet boundary conditions, then $c_0(x) = 0$, $U(\theta, x) = 0$ and hence $G(U)(\theta, x) = 0$.

For Neumann boundary conditions we observe that if h satisfies (2.4), then we just need to check that

$$n(x) \cdot D_x[h'(c_0(x), x)U(\theta, x)] = 0 \qquad \forall x \in \partial\mathcal{D},\ \theta \in \mathbb{T}^d\ . \tag{3.10}$$

The left-hand side of (3.10) can be written as the sum of three pieces, i.e., for $x \in \partial\mathcal{D}$,

$$\begin{aligned} n(x)\cdot D_x[h'(c_0(x), x)U(\theta, x)] &= h''(c_0(x), x)(n(x) \cdot D_x c_0(x))U(\theta, x) \\ &\quad + n(x) \cdot (D_x h')(c_0(x), x)U(\theta, x) + h'(c_0(x), x)\ n(x) \cdot (D_x U)(\theta, x)\ . \end{aligned} \tag{3.11}$$

The first and second terms in the right-hand side of (3.11) vanish, since we impose that $c_0(\cdot)$ satisfies Neumann boundary conditions and h satisfies (2.4). Therefore, if $U(\theta, \cdot)$ satisfies Neumann boundary conditions, the last term in (3.11) will also be equal to zero.

For model B we further require the nonlinearity to be of order at least one in u, namely,

HQ. $h(0, x) = 0$.

3.3. The higher order equations. We are looking for a formal power series solution of (3.7). Assume that we solved (3.8), insert (3.2) into (3.7), and expand the power series (this requires enough regularity for the function h, which we will make explicit later).

Equating the coefficients of the same power ε^N for $N \geq 0$, we obtain the following recursive equations for $N \geq 0$:

$$\begin{aligned} &(\omega \cdot \nabla_\theta)U_{N+1}(\theta, x) + (\omega \cdot \nabla_\theta)^2 U_N(\theta, x) - \Delta_x U_N(\theta, x) + h'(c_0(x), x)\ U_N(\theta, x) \\ &\qquad = S_N(c_0(\theta, x), U_1(\theta, x), \ldots, U_{N-1}(\theta, x)), \end{aligned} \tag{3.12}$$

where S_N is a polynomial expression in $U_1, \ldots, U_{N-1}$ obtained by applying the Taylor theorem to order N in (3.3) and gathering terms.

We think of (3.12) as an equation for U_{N+1}, given all the previous terms of the expansion. Of course we need to assume that $\omega \cdot k \neq 0$ and indeed that it is not too small as $|k|$ increases. Provided that

$$\langle \Delta_x U_N(\theta, x) - h'(c_0(x), x)\ U_N(\theta, x) + S_N(c_0, U_1, \ldots, U_{N-1})(\theta, x)\rangle = 0\ , \tag{3.13}$$

we can find U_{N+1} which is unique up to the choice of an additive function of x alone.

Hence, as it is standard when dealing with Lindstedt series, proceeding by induction we assume that we have determined $U_1, \ldots, U_N$ and then using (3.12) we can determine U_{N+1} up to an additive function of x: such a function is obtained by solving (3.13) and this can be done because of H2. As is usual in Lindstedt series, these constant functions left undetermined up to (3.13) will be determined in the next step, so that the condition of zero average for the next order satisfies the zero average condition. The fact that we can adjust the average (3.13) by choosing the constant component of U_N follows from a nondegeneracy assumption.

Sufficient conditions for the existence of an *approximate solution* provided by a truncation to order N of the series expansion (3.2) are given in Theorem 6.1; see also section 6.1 for a discussion of the existence of an approximate solution to a finite order by solving (3.12) (compare with (6.2) in section 6.1).

3.4. Formulation of the fixed point problem equivalent to (3.3). As we shall see, the operator N_ε in (3.4) is boundedly invertible in the spaces $\mathcal{A}_{\rho,j,m}$ alluded to above (see section 3.5), if ε ranges in a suitable domain, so that (3.7) can be rewritten as

$$U_\varepsilon(\theta,x) = N_\varepsilon^{-1}\,\left[-G(U_\varepsilon)(\theta,x) + f(\theta,x) - \langle f\rangle(x)\right] \equiv \mathcal{T}_\varepsilon(U_\varepsilon)(\theta,x)\ , \tag{3.14}$$

where we have introduced for convenience the operator $\mathcal{T}_\varepsilon$; we will show that (3.14) can be solved by a contraction mapping argument.

Therefore one of the crucial points of the strategy will be to study the invertibility of N_ε and give quantitative estimates on its inverse, notably the Lipschitz constants, which are valid for ε in a complex domain. In order to do so, we assume a uniform lower bound on the eigenvalues of N_ε (which will depend on ε), using the assumption H2 on the eigenvalues of $\mathcal{L}$. By carefully examining such ε-dependent bounds, we will show that, for ε in a suitable domain, the operator appearing in the right-hand side of (3.14) sends a ball centered at the approximate solution (given by the perturbative expansion) into itself and that it is a contraction inside this ball. Hence, the fixed point can be obtained by iteration, starting from the approximate solutions.

We think of the iterative procedure as taking a function analytic in ε and producing another analytic function of ε. We will show that the convergence is uniform for ε in a suitably chosen complex domain. Then it is a standard argument that the limit is an analytic function of ε in this domain. Note that the iterative procedure maps real valued functions into real valued functions.

The contraction mapping argument is classical; however, it requires we use spaces in which we have sharp estimates, so that we do not lose any regularity and we obtain that the operator in (3.14) sends the spaces into themselves. These requirements are the reason for the choices in section 3.5.

Of course, once we have defined the spaces, we will have to justify the formal manipulations, such as the existence of functional derivatives. This amounts to making regularity assumptions on the term h, which justify the use of the Taylor's theorem up to order N for the composition operator.

3.5. Choice of spaces. In this section we present the spaces we will use. We discuss some of their elementary properties in Appendix A.

The leading principle is that the norms of the functions can be expressed in terms of generalized Fourier coefficients, namely, the coefficients associated to the basis given by the product of the Fourier basis in θ and the eigenfunctions of $\mathcal{L}$ with boundary conditions in x.

This principle allows us to estimate rather easily the inverse of the linear operator N_ε in (3.4) just by estimating its eigenvalues, because we are allowed to use the base in which N_ε is diagonal.

We also need the spaces to have other properties allowing us to control the nonlinear terms, such as Banach algebra properties and properties of the composition operator, so that we can study the operator G with ease. Since we want to obtain analyticity in ε, we will also need spaces of analytic functions and, in order to simplify the analysis, we require that they are Hilbert spaces, so that some of the analysis is easier. Note that we think of the functions in x as "scalars" in analogy to what happens in [CCdlL13]; hence, it is natural to consider Hilbert spaces of analytic functions in θ taking values in another Hilbert space of functions of x.

The choice of the spaces presented here satisfies such properties and leads to simple proofs. Of course we are not claiming that the choices we make are optimal and it is quite plausible that other choices (e.g., analytic functions in both variables) could lead to better regularity. The main problem that prevents us from using spaces of analytic functions in x is that it is not clear to us how to express the analyticity of a function in terms of the coefficients of the expansions in eigenvalues.

We will present several equivalent norms, since some of the properties of the space will be easier to verify in one norm than in another. Henceforth, given two (finite or infinite dimensional) equivalent norms $\|\cdot\|$, $\|\cdot\|'$, we write $\|\cdot\| \cong \|\cdot\|'$.

3.5.1. Sobolev spaces with boundary conditions. In this section we introduce the Sobolev-like spaces which we will use; we will define them only for indices $m \in 2\mathbb{N}$, since this is enough for our purposes. The advantage is that, for these indices, it is possible to give particularly simple characterizations of the norm in terms of the eigenfunction expansions. Using several characterizations of the norms allows one to obtain simple proofs of Lipschitz properties of operators.

For functions $S : \overline{\mathcal{D}} \to \mathbb{C}$ satisfying the corresponding boundary conditions and for $m \in 2\mathbb{N}$, we define the family of norms as

$$\|S\|_{H^m_{\mathcal{L}}} \equiv \|\mathcal{L}^{\frac{m}{2}} S\|_{L^2} \ . \tag{3.15}$$

For notational convenience we shall not write explicitly the dependence on the boundary conditions unless needed. The space $H^m_{\mathcal{L}_{BC}}$ will be the completion in the norm (3.15) of the set of C^∞ functions satisfying the boundary conditions BC. As is well known, this space is the same for all second order elliptic operators $\mathcal{L}$ (Gårding inequalities), and indeed it is the standard Sobolev space.

If $S(x) = \sum_{n=1}^\infty \widehat{S}_n \Phi_n(x)$ with $\widehat{S}_n \in \mathbb{R}$ and Φ_n as in Remark 3.2, then the Sobolev norm (3.15) is given by

$$\|S\|^2_{H^m_{\mathcal{L}}} = \sum_{n=1}^{\infty} |\lambda_n|^m |\widehat{S}_n|^2 \ ,$$

where λ_n are the eigenvalues of $\mathcal{L}$ (recall that the Φ_n's form a basis of eigenfunctions of $\mathcal{L}$). Since $\mathcal{L}$ is elliptic, by Gårding's inequality (see [Tay11a, Theorem 6.1, Chapter 7]), we have

$$\|S\|_{H^m_{\mathcal{L}}} \cong \|S\|_{H^m} \ , \tag{3.16}$$

where $\|\cdot\|_{H^m}$ is the standard Sobolev norm, namely,

$$\|S\|_{H^m} = \left\|(1-\Delta_0)^{m/2} S\right\|_{L^2}$$

with Δ_0 the standard constant coefficient Laplacian and we are considering S satisfying the specified boundary conditions.

The spaces $H^m_{\mathcal{L}}(\mathcal{D})_{BC}$ and $H^m(\mathcal{D})_{BC}$ are the completion of C_0^∞—the set of C^∞ functions with compact support contained in the interior of $\mathcal{D}$—under the above norms.

It is well known that for $m > \ell/2$ the Sobolev spaces satisfy the Banach algebra property [Tay11b] and hence the equivalence of the norms $\|\cdot\|_{H^m_{\mathcal{L}}}$ and $\|\cdot\|_{H^m}$ in (3.16) implies for every $S_1, S_2 \in H^m_{\mathcal{L}}$,

$$\|S_1 S_2\|_{H^m_{\mathcal{L}}} \le C \|S_1\|_{H^m_{\mathcal{L}}} \|S_2\|_{H^m_{\mathcal{L}}} \ , \qquad m > \frac{\ell}{2},$$

for some constant $C > 0$.

When $m > \ell/2$ the Sobolev embedding theorem says that the functions in H^m are continuous, so that the Dirichlet boundary conditions have classical meaning.

Similarly, when $m > \ell/2 + 1$, the gradient of functions in H^m are continuously differentiable. Hence, the Neumann boundary conditions have classical meaning.

3.5.2. Spaces of analytic functions of complex variables taking values into Banach spaces. We introduce domains that consist of a strip around the torus $\mathbb{T}^d$ in the imaginary direction. We will consider analytic functions in these domains.

DEFINITION 3.5. *Given $\rho > 0$, we denote by $\mathbb{T}^d_\rho$ the set*

$$\mathbb{T}^d_\rho = \left\{ \theta \in (\mathbb{C}/\mathbb{Z})^d \, : \, \mathrm{Re}(\theta_j) \in \mathbb{T} \ , \quad |\, \mathrm{Im}(\theta_j)| \le \rho \ , \quad j = 1, \ldots, d \right\} \ .$$

When we consider functions of $\theta \in \mathbb{T}^d_\rho$ and $x \in \overline{\mathcal{D}}$, we can think of them as functions from $\mathbb{T}^d_\rho$ into $H^m_{\mathcal{L}}$ which are analytic.[2] The spaces which we will consider are the standard Bargmann spaces taking values into $H^m_{\mathcal{L}}$.

Given a function $u = u(\theta, x)$ which we expand as

$$u(\theta, x) = \sum_{k \in \mathbb{Z}^d} e^{2\pi i k \cdot \theta} \hat{u}_k(x) = \sum_{k \in \mathbb{Z}^d, n \ge 1} e^{2\pi i k \cdot \theta} \Phi_n(x) \hat{u}_{k,n} \ , \tag{3.17}$$

we will consider the space of analytic functions of θ endowed with the $H^j(\mathbb{T}^d_\rho; H^m_{\mathcal{L}})$ norm defined below. We emphasize that we are considering $\mathbb{T}^d_\rho$ as a $2d$-dimensional real manifold with boundary. Again, for simplicity, we just consider the even Sobolev exponents j.

Precisely, for $\rho > 0$, $j, m \in 2\mathbb{N}$, setting $\nabla^2_\theta \equiv \sum_{n=1}^d \nabla_{\theta_n} \nabla_{\bar{\theta}_n}$ (where the bar denotes complex conjugation), we define the $H^j(\mathbb{T}^d_\rho; H^m_{\mathcal{L}})$ norm as

$$\begin{aligned} \|u\|^2_{\rho,j,m} &= \int_{\mathbb{T}^d_\rho} \left\| (1 - \nabla^2_\theta)^{\frac{j}{2}} u(\theta, \cdot) \right\|^2_{H^m_{\mathcal{L}}} \ d^{2d}\theta \\ &= \int_{\mathbb{T}^d_\rho} \left\| \left(1 - \sum_{n=1}^d \nabla_{\theta_n} \nabla_{\bar{\theta}_n} \right)^{\frac{j}{2}} u(\theta, \cdot) \right\|^2_{H^m_{\mathcal{L}}} \ d^{2d}\theta \ . \end{aligned} \tag{3.18}$$

We denote by $\mathcal{A}_{\rho,j,m}$ the space of functions analytic in θ whose norm $\|\cdot\|_{\rho,j,m}$ is finite. Note that $\mathcal{A}_{\rho,j,m}$ are Hilbert spaces, since the norm (3.18) clearly comes from the inner product

$$\langle u, v \rangle = \int_{\mathbb{T}^d_\rho} \langle u, (1 - \nabla^2_\theta)^j v \rangle_{H^m} \ d^{2d}\theta \ .$$

[2] When we consider complex domains which are closed with a smooth boundary, we refer to analytic functions in a domain $\mathcal{D}$ as functions which are analytic in the interior and that extend continuously to the boundary. For us, using domains which are compact is slightly more convenient in order to quote embedding theorems, etc. Nevertheless, in order to avoid repetitions, we omit that analyticity is meant only for the interior and that we assume the continuous extension to the boundary.

Moreover they are complete, since they are a closed subspace of $H^j(\mathbb{T}^d_\rho; H^m_{\mathcal{L}})$ for $j > d$, so the limit in the $\|\cdot\|_{\rho,j,m}$-norm of analytic functions is an analytic function [RS80] in $\mathcal{D}$ for any j. For j large enough, the continuous extensions are also continuous in the closure since the convergence is uniform.

Remark 3.6. We think of a function $u \in \mathcal{A}_{\rho,j,m}$ as an analytic function from $\mathbb{T}^d_\rho$ into $H^m_{\mathcal{L}}$, say, $\theta \to u(\theta, \cdot)$. In this way, the problems considered here look closer to the formulation of the varactor problem considered in [CCdlL13]. The PDE looks formally like an ODE in $H^m_{\mathcal{L}}$ and the response solutions will be analytic functions from the torus into $H^m_{\mathcal{L}}$. For sufficiently high m, these will be classical functions which are analytic in the t variable and differentiable in the variable x. Hence, they will be classical solutions for the PDE.

4. Precise statement of the results.

4.1. Approximate solutions of the fixed point problem. For all models we can give a definition of "approximate solution" as follows.

DEFINITION 4.1. *Let us consider a family of functional equations*

$$\mathcal{F}_\varepsilon(U) = 0 \ , \tag{4.1}$$

where $\mathcal{F}_\varepsilon : \mathcal{A}_{\rho,j,m} \to \mathcal{A}_{\rho,j,m_2}$ is an operator that maps ε-dependent families into families (of course the operator $\mathcal{F}_\varepsilon$ may have an explicit ε-dependence). We say that

$$U_\varepsilon^{(M)} = \sum_{k=0}^{M} \varepsilon^k U_k \tag{4.2}$$

for some $M \in \mathbb{Z}_+$ is an approximate solution up to order M of (4.1) *if*

$$\|\mathcal{F}_\varepsilon(U_\varepsilon^{(M)})\|_{\rho,j,m_2} = O(\varepsilon^{M+1}) \ .$$

4.2. Main results. Our main results are provided by the following Theorems 4.2, 4.3, and 4.6.

Theorem 4.2 is based on a contraction mapping argument and it states the existence of a solution, provided we assume the existence of an approximate solution.

Theorem 4.3 gives sufficient conditions for the existence of an approximate solution up to any order under some nonresonance assumptions on the frequency: the higher the order of approximation we want to reach, the more restrictive the condition on the frequency will be.

Theorem 4.6 summarizes the results above, i.e., it gives the existence of an analytic solution under the requirements of Theorem 4.3, which provides the approximate solution. The proof of Theorem 4.6 relies on applying Theorem 4.2 to the approximate solutions provided by Theorem 4.3.

In Theorems 4.2, 4.3, and 4.6 we will assume that f belongs to the space of functions $\mathcal{A}_{\rho,j,m}$ with j, m, d as in Proposition A.3, which ensures the validity of the Banach algebra property.

THEOREM 4.2. *Assume that f is in $\mathcal{A}_{\rho,j,m}$ for $\rho > 0$, $j, m \in 2\mathbb{N}$, $j > d$, $m > \ell/2$ ($m > \ell/2 + 1$ in the case of Neumann boundary conditions). Let $h : \mathcal{B} \times \overline{\mathcal{D}} \to \mathbb{C}$ with $\mathcal{B} \subset \mathbb{C}$ open set, and let $\mathcal{D}$ be either of the form $D1$ or $D2$ as in section* 2.1. *We assume that h is analytic in $\mathcal{B}$ and $C^m(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ in x.*

Consider the models A, A′, B, *and* B′ *with D, N, P boundary conditions. Assume that the equation for the zeroth order term c_0 (see* (3.8)*) admits a solution (some* *sufficient conditions are given in Appendix* B *for models* A, *and* A′ *and section* 5.2 *for* B *and* B′*); assume that the hypotheses* H1–H2 *are satisfied (see section* 3.2.2*) and that the nonlinearity* h *satisfies* BCD *or* BCN *(see section* 2.2*) in case of* D *or* N *boundary conditions, respectively. For model* B *assume also* HQ *and for models* A′ *and* B′ *assume also* H1′–H2′.

Assume that for some $M \in \mathbb{N}$, $M \geq 2$, *there exists an approximate solution in* ε *up to order* M *of* (3.3) *for model* A *of*

$$
\begin{aligned}
&(\omega \cdot \nabla_\theta)^2 U_\varepsilon(\theta, x) \pm \frac{1}{\varepsilon}(\omega \cdot \nabla_\theta)\ \Delta_x U_\varepsilon(\theta, x) - \Delta_x U_\varepsilon(\theta, x) - \Delta_x c_0(x) \\
&\qquad + h(c_0(x) + U_\varepsilon(\theta, x), x) = f(\theta, x)
\end{aligned}
\tag{4.3}
$$

for model A′, *of*

$$
\varepsilon^2(\omega \cdot \nabla_\theta)^2 U_\varepsilon(\theta, x) + (\omega \cdot \nabla_\theta) U_\varepsilon(\theta, x) - \Delta_x U_\varepsilon(\theta, x) + h(U_\varepsilon(\theta, x), x) = f(\theta, x) \tag{4.4}
$$

for model B, *and of*

$$
\varepsilon^2(\omega \cdot \nabla_\theta)^2 U_\varepsilon(\theta, x) + (\omega \cdot \nabla_\theta) U_\varepsilon(\theta, x) - \Delta_x U_\varepsilon(\theta, x) + \varepsilon h(U_\varepsilon(\theta, x), x) = f(\theta, x) \tag{4.5}
$$

for model B′.

For model A, let ε *be in the domain* $\Omega_B = \cup_\sigma \Omega_{\sigma,B}$ *with* $B > B_0$ *for some* $B_0 > 0$ *sufficiently large,* $\sigma > 0$ *sufficiently small, where*

$$
\Omega_{\sigma,B} \equiv \{\varepsilon = \xi + i\eta \in \mathbb{C} \, : \, |\xi| > B\,\eta^2\,,\, \sigma < |\varepsilon| < 2\sigma\} \tag{4.6}
$$

and θ *in the strip of size* $\rho > 0$

$$
\mathbb{T}^d_\rho \equiv \{\theta \in (\mathbb{C}/\mathbb{Z})^d \, : \, \mathrm{Re}(\theta_j) \in \mathbb{T}\ , \quad |\operatorname{Im}(\theta_j)| \leq \rho\ , \quad j = 1, \ldots, d\}\ .
$$

Then, there exists a function $U_\varepsilon = U_\varepsilon(\theta, x) \in \mathcal{A}_{\rho,j,m}$, *which provides an exact solution of* (3.3).

For model A′, *let* ε *be in a domain of the form* (4.6). *Then, there exists a function* $u = u(\theta, x) = c_0(x) + U_\varepsilon(\theta, x)$ *as in* (3.1), *belonging to* $\mathcal{A}_{\rho,j,m}$, *which provides an exact solution of* (4.3).

For model B, *assuming that* ε *belongs to the domain*

$$
\Omega_\delta \equiv \{\varepsilon = \xi + i\eta \in \mathbb{C} \, : \, \mathrm{Re}(-\varepsilon^2) \geq \delta\ ,\ |\varepsilon| < \sigma\} \cup \{\varepsilon = \xi \in \mathbb{R} \, : \, \delta < |\xi| < 2\delta\} \tag{4.7}
$$

for some $\delta > 0$ *and for* σ *sufficiently small, then there exists a function* $U_\varepsilon = U_\varepsilon(\theta, x) \in \mathcal{A}_{\rho,j,m}$, *which provides an exact solution of* (4.4).

For model B′, *assuming that* ε *belongs to* Ω_δ *as in* (4.7) *for some* $\delta > 0$, *then there exists a function* $U_\varepsilon = U_\varepsilon(\theta, x) \in \mathcal{A}_{\rho,j,m}$, *which provides an exact solution of* (4.5).

In all the cases above, the solution U_ε *is analytic in the considered domains as a function of* ε *and it is asymptotic to the approximate solution, namely, the truncated Taylor expansion in* ε *of the solution at order* M *equals the approximate solution. We also note that the approximations are real for real values of the arguments.*

Note that Theorem 4.2 involves mainly regularity assumptions and the requirement that there exist approximate solutions at least of order 2. Sufficient conditions for the existence of an approximate solution (given by the expansion to any arbitrary order) are provided by the following result.

THEOREM 4.3. *Assume that f is in $\mathcal{A}_{\rho,j,m}$ for $\rho > 0$, $j, m \in 2\mathbb{N}$, $j > d$, $m > \ell/2$ ($m > \ell/2 + 1$ in the case of Neumann boundary conditions). Let $h : \mathcal{B} \times \overline{\mathcal{D}} \to \mathbb{C}$ with $\mathcal{B} \subset \mathbb{C}$ open set, and let $\mathcal{D}$ be either of the form* D1 *or* D2*. We assume that h is analytic in $\mathcal{B}$ and $C^m(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ in x.*

Consider the models A, A′ *and* B′ *with either D, N, P boundary conditions and assume that h satisfies either* BCD *or* BCN *in case of D, N boundary conditions, respectively (depending on the boundary condition considered for the equation).*

Assume that the zeroth order term admits a solution (see, respectively, Appendix B for models A *and* A′ *and section* 5.2 *for model* B′*) and that this solution satisfies H*1 *and H*2*.*

(i) *If we assume that there exists $M \in \mathbb{N}$, $c \in \mathbb{R}$, such that*

$$\log |\omega \cdot k|^{-1} \le \frac{2\pi\rho}{M} \; |k| + c \qquad \forall k \in \mathbb{Z}^d \setminus \{0\} \ , \tag{4.8}$$

then there exists an approximate solution U_ε in ε of (3.3), (4.3), (4.5) *up to order M. In particular, if*

$$\limsup_{|k|\to\infty} |k|^{-1} \log(|\omega \cdot k|^{-1}) = 0 \ , \tag{4.9}$$

then we can obtain a formal power series in ε (U_ε) solving (3.3), (4.4), (4.5) *up to all orders.*

(ii) *If we assume that f is a trigonometric polynomial in θ, then there is a formal power series in ε which is a solution of* (3.3), (4.3), (4.5) *up to all orders, without requiring any nonresonance condition on the frequency ω.*

Remark 4.4. Recall that we are assuming that $\omega \cdot k = 0$, $k \in \mathbb{Z}^d$, implies $k = 0$. This can be arranged without loss of generality by just considering a basis for the module generated by ω.

Remark 4.5. It seems likely that the condition (4.8) (which is even weaker than the Bryuno condition) is optimal. Each step of the computation of the perturbative expansion involves solving a differential equation with $(\omega \cdot k)^{-1}$ as small divisors. Hence, we expect that the solutions, in general, lose a domain of definition of size ρ/M at each step.

Combining Theorems 4.2 and 4.3 we obtain the following result.

THEOREM 4.6. *Assume that f, h satisfy the assumptions of Theorem* 4.2 *regarding the regularity, the boundary conditions, and the existence of the zeroth order solution. Fix $M \ge 2$ and assume* (4.8)*. Then, we have the following results.*

For model A *there exists a solution of* (3.3)*, which is analytic in ε and θ, and satisfies the D, N, or P boundary conditions. The analytic solution exists for ε in the domain Ω_B as in Theorem* 4.2 *and θ in $\mathbb{T}^d_\rho$.*

For model A′ *assuming* H1′ *and* H2′*, there exists an analytic solution of* (4.3) *in $\theta \in \mathbb{T}^d_\rho$ and ε with ε in a domain of the form Ω_B as in Theorem* 4.2*.*

For model B *with D, N, P boundary conditions, provided the existence of a solution of the order zero equation, there exists an analytic solution of* (4.4) *in $\theta \in \mathbb{T}^d_\rho$ and ε in a domain of the form Ω_δ as in* (4.7) *for some $\delta > 0$.*

For model B′ *with D, N, P boundary conditions, assuming* H1′–H2′ *there exists an analytic solution of* (4.5) *in $\theta \in \mathbb{T}^d_\rho$ and ε in a domain of the form Ω_δ as in* (4.7) *for some $\delta > 0$.*

Remark 4.7. Note that the nonresonance condition which we need to impose on ω to obtain the existence to all orders is more restrictive than the nonresonance condition we need to obtain the existence of an analytic solution defined in the domain $\Omega_{\sigma,B}$ or Ω_δ. Since, as we argued in Remark 4.5, we believe that the conditions are optimal, it seems that given an M_0 and an ω that satisfy (4.8) for $M = M_0$, but not for $M = M_0 + 1$, then we can obtain functions that have an expansion up to order M_0, but not $M_0 + 1$.

It seems, therefore, possible to arrange the existence of models with a solution analytic in a domain of the form $\Omega_{\sigma,B}$ or Ω_δ, but without a Taylor expansion beyond a certain order.

Remark 4.8. By restricting the domain in Theorem 4.6, we can obtain stronger contraction properties for the operator $\mathcal{T}$ defined in (3.14).

For example, for model A one possibility would be to consider only one of the domains $\Omega_{\sigma,B}$ defined in (4.6) with σ small enough. Another possibility is to consider conic domains defined as

$$\Upsilon_{\delta,\sigma} = \{\varepsilon \in \mathbb{C} : |\,\mathrm{Im}(\varepsilon)|/|\varepsilon| < \delta \ , \quad \sigma < |\varepsilon| < 2\sigma\}$$

for some δ, $\sigma > 0$. We refer to [CCdlL13] for further details on the study of the solution of a forced strongly dissipative ODE on conic domains. By restricting to the real line, it seems possible to obtain results for finitely differentiable nonlinearities.

5. Existence of solutions of the zeroth order term. The existence of solutions of the zeroth order equation for models A and A′ has a very extensive literature and can be done by a variety of methods; with reference to classical textbooks like [Str08, AM07, Pre13], we defer the presentation of some of such results to Appendix B, since they are based on very different methods and we just take them from the literature. For models A and A′, we will produce several (even infinitely many) solutions of the zeroth order equation. However, we mention that there are other possibilities which we have not covered, for example, methods based on index theory [FG95, Ber73].

In this section we confine ourselves to discussing models B and B′, using arguments based on an implicit function theorem in Banach spaces, which are similar to the arguments in the rest of the paper. Later in this paper we show that each of these solutions of the zeroth order equation which satisfy H1 and H2 continues into a formal solution to all orders and that, furthermore, it can be modified to be a true solution for ε in a complex domain.

5.1. Solution of the zeroth order term for model B. The zeroth order equation for model B is

$$(\omega \cdot \nabla_\theta) U_0(\theta, x) - \Delta_x U_0(\theta, x) + h(U_0(\theta, x), x) = f(\theta, x) \ . \tag{5.1}$$

We will solve (5.1) by reducing it to a fixed point problem and providing easy-to-check conditions that ensure solvability.

Let us denote by Γ the operator

$$\Gamma \equiv \omega \cdot \nabla_\theta + \mathcal{L} \ , \tag{5.2}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}\eta \equiv -\Delta_x \eta + h'(0, x)\eta \ . \tag{5.3}$$

As said, we assume that

$$h(0,x) = 0\ .$$

Then, (5.1) becomes

$$\Gamma U_0(\theta,x) + G(U_0(\theta,x),x) = f(\theta,x) \tag{5.4}$$

with G given by

$$G(U_0(\theta,x),x) = h(U_0(\theta,x),x) - h'(0,x)U_0(\theta,x)\ . \tag{5.5}$$

We notice that Γ is a diagonal operator in the Fourier basis, since it is separated into the sum of two parts, one of which acts only on θ and the other acting only on x. If we assume that $\mathcal{L}$ satisfies H1 and H2 and we denote its eigenvalues by λ_n, then

$$\Gamma(e^{2\pi i k\cdot\theta}\Phi_n) = (2\pi i\omega\cdot k + \lambda_n)e^{2\pi i k\cdot\theta}\Phi_n\ .$$

Thus, we notice that Γ is boundedly invertible and we can reduce (5.4) to the following fixed point problem:

$$U_0(\theta,x) = -\Gamma^{-1}\ G(U_0(\theta,x),x) + \Gamma^{-1}\ f(\theta,x)\ . \tag{5.6}$$

We define the operator $\mathcal{T}$ by

$$\mathcal{T}(U) \equiv -\Gamma^{-1}\ G(U(\theta,x),x) + \Gamma^{-1}\ f(\theta,x)\ . \tag{5.7}$$

Using properties of composition of functions (see Proposition A.7 in Appendix A), we can show that for $U_0, V_0 \in \mathcal{A}_{\rho,j,m}$, $\mathcal{T}$ satisfies the inequality

$$\begin{aligned}\|\mathcal{T}(U_0) - \mathcal{T}(V_0)\|_{\rho,j,m} &= \|\Gamma^{-1}\ h(U_0,x) - \Gamma^{-1}\ h(V_0,x)\\ &\qquad - \Gamma^{-1}h'(0,x)U_0 + \Gamma^{-1}h'(0,x)V_0\|_{\rho,j,m}\\ &\le C\alpha_0\ \|\Gamma^{-1}\|_{\rho,j,m}\ \|U_0 - V_0\|_{\rho,j,m}\ ,\end{aligned}$$

where the Lipschitz constant of the composition with G is bounded by a constant C times α_0. Finally, if we choose α_0 small enough so that $\mathcal{T}$ is a contraction, we obtain a solution U_0 in a ball of radius α_0. Thus, we have proven the following result.

PROPOSITION 5.1. *Assume that f is in $\mathcal{A}_{\rho,j,m}$ for $\rho > 0$, $j,m \in 2\mathbb{N}$, $j > d$, $m > \ell/2$ ($m > \ell/2 + 1$ for Neumann boundary conditions). Let $h : \mathcal{B}\times\overline{\mathcal{D}} \to \mathbb{C}$ with $\mathcal{B} \subset \mathbb{C}$ open set, and let $\mathcal{D}$ be either of the form* D1 *or* D2. *We assume that h is analytic in $\mathcal{B}$ and $C^m(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ in x and assume that $h(0,x) = 0$ and that Δ_x satisfies* H1′ *and* H2′.

Consider model B *with either D, N, or P boundary conditions and assume that h satisfies, respectively,* BCD, BCN *in case of D, N boundary conditions (depending on the boundary condition considered for the equation). Then, the zeroth order term of model* B *given by* (5.1) *admits a solution U_0 contained in a ball around the origin in $\mathcal{A}_{\rho,j,m}$ of small enough radius α_0.*

5.2. Solution of the zeroth order term for model B′. The zeroth order equation for model B′ is

$$(\omega\cdot\nabla_\theta)U_0(\theta,x) - \Delta_x U_0(\theta,x) = f(\theta,x)\ , \tag{5.8}$$

which can be solved under the assumptions H1′–H2′. In fact, defining the operator Γ acting on U as in (5.2), but with Δ_x instead of $\mathcal{L}$, we can write (5.8) as

$$\Gamma U_0(\theta, x) = f(\theta, x) \ ,$$

which can be solved, because Γ is invertible (using H1′–H2′).

Indeed, the operator Γ is diagonal in the Fourier basis. Let us expand U_0 as

$$U_0(\theta, x) = \sum_{k\in\mathbb{Z}^d} \sum_{n\geq 0} e^{2\pi i k\cdot\theta} \Phi_n(x) \tilde{U}_{0,k,n}$$

for some coefficients $\tilde{U}_{0,k,n}$; in a similar way, let

$$f(\theta, x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \sum_{n\geq 0} e^{2\pi i k\cdot\theta} \Phi_n(x) f_{k,n} \ .$$

Then, we obtain

$$\tilde{U}_{0,k,n} = \frac{f_{k,n}}{(2\pi i\omega \cdot k + \lambda_n^{\Delta})} \tag{5.9}$$

for $k \neq 0$ (with $\lambda_n^{\Delta} \in \mathbb{R}$ denoting the eigenvalues of $-\Delta_x$) and $\tilde{U}_{0,0,n} = 0$. From (5.9) we see that the assumptions H1′–H2′ and the regularity of $f(\theta, x)$ imply the regularity of $\tilde{U}_0$. Because of H1′–H2′, we have that $|2\pi i\omega \cdot k + \lambda_n^{\Delta}| \geq \nu$ for some $\nu > 0$; using the expression of the norms in terms of the Fourier coefficients, we obtain the desired result that $\tilde{U}_0$ exists.

6. Proof of Theorems 4.2, 4.3, and 4.6 for model A.

In this section we present detailed arguments that complete the proof of Theorems 4.2, 4.3, and 4.6 for the case of the dissipative wave equation (2.1) of model A; in section 7 we will present the necessary modifications in order to have the results for models A′, B, and B′.

We start by proving the existence of an approximate solution up to prescribed orders (section 6.1); then, we bound the operator N_ε in (3.4) providing estimates in a parabolic domain (section 6.2) and we conclude by showing the existence of a solution of (3.7) through a fixed point argument (section 6.3).

6.1. Existence of an approximate solution up to prescribed orders.

In this section we describe the construction of the approximate solution up to a prescribed order M, as in the statement of Theorem 4.3.

For model A described by (2.1), the first order term c_0 of the expansion of the response solution (see (3.1)) satisfies the semilinear second order elliptic equation (3.8).

To perform the formal manipulations that lead to the approximate solution up to order M, we find it convenient to write (3.3) as

$$\begin{aligned} &[\varepsilon(\omega\cdot\nabla_\theta)^2 + (\omega\cdot\nabla_\theta) - \varepsilon\Delta_x] U_\varepsilon(\theta, x) - \varepsilon\Delta_x c_0(x) \\ &\qquad + \varepsilon h(c_0(x) + U_\varepsilon(\theta, x), x) = \varepsilon f(\theta, x) \ . \end{aligned} \tag{6.1}$$

We assume that a solution c_0 for (3.8) can be found as described in Appendix B. Next, we write formally $U_\varepsilon \equiv U_\varepsilon(\theta)$ in powers of ε (see (3.2)). We now show that we can define an approximate solution of (6.1) as a finite truncation of (3.2) up to order M. Inserting (3.2) into (6.1), we get

$$\sum_{j=1}^{\infty} \varepsilon^j [\varepsilon(\omega \cdot \nabla_\theta)^2 + (\omega \cdot \nabla_\theta) - \varepsilon \Delta_x] U_j(\theta, x) - \varepsilon \Delta_x c_0(x) + \varepsilon h(c_0(x) + U_\varepsilon(\theta, x), x) - \varepsilon f(\theta, x) = 0 \ . \tag{6.2}$$

Hence the first order in ε in (6.2) is given by (3.9). Since c_0 satisfies (3.8), then the equation for the first order in ε becomes

$$(\omega \cdot \nabla_\theta) U_1(\theta, x) = f(\theta, x) - \langle f \rangle(x) \tag{6.3}$$

and it is easy to see that by the nonresonance condition (4.8), (6.3) has a solution in $\mathcal{A}_{\rho', j, m}$ for some $\rho' < \rho$. Let us define $g(\theta, x) \equiv f(\theta, x) - \langle f \rangle(x)$; let us expand U_1 and g as

$$U_1(\theta, x) = \sum_{k \in \mathbb{Z}^d} \sum_{n \geq 0} e^{2\pi i k \cdot \theta} \Phi_n(x)\ \tilde{U}_{1,k,n} \ , \qquad g(\theta, x) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \sum_{n \geq 0} e^{2\pi i k \cdot \theta} \Phi_n(x)\ \tilde{g}_{k,n}$$

for suitable coefficients $\tilde{U}_{1,k,n}$, $\tilde{g}_{k,n}$. From (6.3) we obtain that

$$\tilde{U}_{1,k,n} = \frac{\tilde{g}_{k,n}}{2\pi i \omega \cdot k} \ ,$$

which is well defined thanks to (4.8). The appearance of the small divisors in the expression for $\tilde{U}_{1,k,n}$ is the origin of the loss of analyticity domain of U_1 with respect to f.

Note that $U_1(\theta, x)$, as a solution of (6.3), has a free parameter, namely, $\langle U_1 \rangle(x)$, which will be determined at the equation for the next order term.

Recalling the definition (3.5) of $\mathcal{L}$, the order ε^2 in (6.2) is

$$(\omega \cdot \nabla_\theta) U_2(\theta, x) = -[(\omega \cdot \nabla_\theta)^2 + \mathcal{L}] U_1(\theta, x) \ ,$$

which admits a solution $U_2(\theta, x)$, provided that the average of the right-hand side is zero. In particular, the average of U_1 must satisfy the equation $\mathcal{L}(\langle U_1 \rangle) = 0$ and since $\mathcal{L}$ satisfies H1–H2, then $\langle U_1 \rangle = 0$.

Now, by using the nonresonance condition (4.8) one can proceed recursively as follows. Suppose that after N many steps the functions $U_1, \ldots, U_{N-1}$ are completely determined while U_N is determined up to its average. Then we pass to (3.12) for the order $N+1$, and we see that we can determine U_{N+1} up to the average; at the same time we need to require that the average of (3.12) is zero and this leads to

$$\mathcal{L}(\langle U_N \rangle) = -\langle S_N(c_0, U_1 \ldots, U_{N-1}) \rangle \ , \tag{6.4}$$

which is a recursive equation for the average of U_N. The existence of $\langle U_N \rangle$ satisfying (6.4) is guaranteed by assumptions H1–H2.

In this way we obtain the approximate solution up to order M under the assumption (4.8). It follows that we obtain a well-defined approximate solution to all orders under the condition (4.9); for instance one can adapt the argument given in Appendix H of [CG12]. Finally, if we assume further that f is a trigonometric polynomial of degree $L > 0$, then U_N is a trigonometric polynomial of degree NL and we can obtain the formal solution up to any order N. This concludes the proof of Theorem 4.3.

6.2. Bounds on the operator N_ε. In this section we will obtain bounds on N_ε in appropriate spaces, when ε is contained in the domain Ω_B defined as the union over σ of the complex domains (4.6). This section is the most delicate section of the paper, as we need to study the domains. We also remark that it is not possible to obtain the same bounds when ε is on the imaginary axis: indeed, we present separate arguments that lead us to conjecture that the bound on the spectrum of N_ε cannot be obtained when ε is imaginary.

Note that, since $\mathcal{L}$ acts on the x variable only and $\omega \cdot \nabla_\theta$ on the θ variable, we can apply separation of variables and obtain that the spectrum of N_ε is

$$\lambda_{n,k} \equiv \lambda_{n,k}(\varepsilon) = -(2\pi\omega \cdot k)^2 + \frac{2\pi i}{\varepsilon}(\omega \cdot k) + \lambda_n \ . \tag{6.5}$$

As already pointed out, the invertibility of N_ε in the spaces we consider follows from the fact that its spectrum is bounded away from zero. For a general operator, the bounds on the inverse would need to estimate not only the spectrum but also the spectral projections, though this is trivial in this case since $\mathcal{L}$ is self-adjoint and $\omega \cdot \nabla_\theta$ is anti-self-adjoint, so that $\mathcal{L} + \omega \cdot \nabla_\theta$ is a normal operator.

We study the spectrum of N_ε, when ε ranges in the domain

$$\Omega_{\sigma,B,\alpha} \equiv \{\varepsilon = \xi + i\eta \in \mathbb{C} \,:\, |\xi| > B\,\eta^\alpha \,,\, \sigma < |\varepsilon| < 2\sigma\} \ . \tag{6.6}$$

Afterward, we will fix α in such a way that we can use the fixed point argument of section 6.3 and it will turn out that the best choice is $\alpha = 2$, thus leading to defining the solution in the domain $\Omega_{\sigma,B,2} = \Omega_{\sigma,B}$ as defined in (4.6). To study the spectrum of N_ε, we will use the maximum principle for $\lambda_{n,k}$ as a function of ε; hence, we will get a lower bound on $|\lambda_{n,k}|$ on the boundary of the domain.

Observing (6.5) we see that $\lambda_{n,k}(\varepsilon) = 0$ can happen only when $\varepsilon \in i\mathbb{R}$. Hence $\varepsilon\,\lambda_{n,k}(\varepsilon) \neq 0$ on $\Omega_{\sigma,B,\alpha}$ and $[\varepsilon\,\lambda_{n,k}(\varepsilon)]^{-1}$ is holomorphic on $\Omega_{\sigma,B,\alpha}$. By the maximum modulus principle,

$$\sup_{\varepsilon \in \Omega_{\sigma,B,\alpha}} |\varepsilon\,\lambda_{n,k}(\varepsilon)|^{-1} = \left(\inf_{\varepsilon \in \Omega_{\sigma,B,\alpha}} |\varepsilon\,\lambda_{n,k}(\varepsilon)|\right)^{-1}$$

is reached for some $\varepsilon \in \partial\Omega_{\sigma,B,\alpha}$. That is, we have

$$\inf_{\varepsilon \in \Omega_{\sigma,B,\alpha}} |\varepsilon\,\lambda_{n,k}(\varepsilon)| = \inf_{\varepsilon \in \partial\Omega_{\sigma,B,\alpha}} |\varepsilon\,\lambda_{n,k}(\varepsilon)| \ .$$

Hence, to establish the desired result it suffices to estimate from below the infimum on the boundary.

Let us suppose that $\lambda_n \to \infty$ whenever $n \to \infty$. We will show later that this assumption can be relaxed to encompass the case that the sequence of the λ_n's has a finite supremum, even if this case does not appear in the applications we have in mind. Thus, we can fix $K \in \mathbb{N}$ large enough so that $\lambda_K - 1 \geq \lambda_{J+1}$; we will first provide a bound for $1 \leq n \leq K$ on the whole region $\Omega_{\sigma,B,\alpha}$.

For every $n \in \mathbb{N}$, we want to estimate $\inf_{k \in \mathbb{Z}^d \setminus \{0\}} |\lambda_{n,k}(\varepsilon)|^2$ for $\varepsilon \in \Omega_{\sigma,B,\alpha}$; therefore, we study the behavior of

$$\begin{aligned}
|\varepsilon\lambda_{n,k}(\varepsilon)|^2 &= |-\varepsilon(2\pi\omega \cdot k)^2 + i(2\pi\omega \cdot k) + \varepsilon\lambda_n|^2 \\
&= |-(\xi + i\eta)(2\pi\omega \cdot k)^2 + i(2\pi\omega \cdot k) + (\xi + i\eta)\lambda_n|^2 \\
&= \xi^2\ [-(2\pi\omega \cdot k)^2 + \lambda_n]^2 + [-\eta(2\pi\omega \cdot k)^2 + (2\pi\omega \cdot k) + \eta\lambda_n]^2 \ .
\end{aligned}$$

For a given n, consider the function

$$\Gamma_n(\tau,\xi,\eta) \equiv \xi^2\ (-\tau^2+\lambda_n)^2 + [\eta(-\tau^2+\lambda_n)+\tau]^2\ , \tag{6.7}$$

where $\tau\in\mathbb{R}$, $\xi+i\eta\in\Omega_{\sigma,B,\alpha}$. Clearly,

$$\Gamma_n(2\pi\omega\cdot k,\xi,\eta) = |(\xi+i\eta)\lambda_{n,k}(\xi+i\eta)|^2\ ;$$

since $\inf_{\tau\in\mathbb{R}}\Gamma_n(\tau,\xi,\eta)\le\inf_{k\in\mathbb{Z}^d\setminus\{0\}}\Gamma_n(2\pi\omega\cdot k,\xi,\eta)$, it suffices to bound from below $\inf_{\tau\in\mathbb{R}}\Gamma_n(\tau,\xi,\eta)$. As argued before, we will get the bounds on the infimum absolute value by bounding from below the absolute value on the boundary.

Let us start by considering the boundary $\{\xi=B\eta^\alpha\}$; namely, we consider ε as

$$\varepsilon = \pm B\eta^\alpha + i\eta \tag{6.8}$$

with $\eta\in\mathbb{R}\setminus\{0\}$, $B>0$ large enough, say, $B>B_0$ for some $B_0\in\mathbb{R}_+$. Clearly, for every n we have that

$$\inf_{\varepsilon=B\eta^\alpha+i\eta}\ \inf_{k\in\mathbb{Z}^d\setminus\{0\}}|\varepsilon\lambda_{n,k}(\varepsilon)|^2 \ge \inf_{\tau\in\mathbb{R}}|\Gamma_n(\tau,B\eta^\alpha,\eta)|\ .$$

We recall that H1 and H2 imply that $\inf_{n\ge1}|\lambda_n|>0$.

We analyze separately the negative eigenvalues $\lambda_1\le\cdots\le\lambda_J<0$ and the positive eigenvalues $0<\lambda_{J+1}\le\lambda_{J+2}\le\dots$.

For $1\le n\le J$ from (6.7) we have

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &= \xi^2(\tau^2+|\lambda_n|)^2+[\eta(-\tau^2-|\lambda_n|)+\tau]^2\\ &\ge \xi^2(\tau^2+|\lambda_n|)^2 \ge \xi^2|\lambda_n|^2\ge\xi^2|\lambda_J|^2=(B\eta^\alpha)^2|\lambda_J|^2\ .\end{aligned}$$

Let us now consider the positive eigenvalues, namely, $J<n\le K$. Obtaining lower bounds of (6.7) for ε of the form (6.8) is very simple: indeed it is the sum of two nonnegative terms, which vanish at very different places. We analyze carefully each of the places where one of the terms vanishes. If none of the terms vanishes, the lower bound is clear.

Let us define the three regions

$$\begin{aligned}I_+ &\equiv \left[\sqrt{\lambda_n}-10^{-3}\sqrt{\lambda_n},\sqrt{\lambda_n}+10^{-3}\sqrt{\lambda_n}\right],\\ I_- &\equiv \left[-\sqrt{\lambda_n}-10^{-3}\sqrt{\lambda_n},-\sqrt{\lambda_n}+10^{-3}\sqrt{\lambda_n}\right]\end{aligned} \tag{6.9}$$

and the complement of $I_+\cup I_-$. When τ is in such complement we have

$$\Gamma_n(\tau,\xi,\eta)\ge(B\eta^\alpha)^2(-\tau^2+\lambda_n)^2\ge C_1^2(B\eta^\alpha)^2\lambda_n^2\ge C_1^2(B\eta^\alpha)^2\ \lambda_{J+1}^2 \tag{6.10}$$

for a suitable constant $C_1>0$. When $\tau\in I_+\cup I_-$ we have

$$|\lambda_n-\tau^2| = \left|\sqrt{\lambda_n}-\tau\right|\ \left|\sqrt{\lambda_n}+\tau\right|\le 10^{-3}\sqrt{\lambda_n}\ (2+10^{-3})\sqrt{\lambda_n}\ ,$$

so that we obtain

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &\ge [\eta(\lambda_n-\tau^2)+\tau]^2\\ &\ge \tau^2-2|\tau|\,|\eta|\,|\lambda_n-\tau^2|\\ &\ge \tau^2-2|\tau|\,|\eta|\,(2+10^{-3})\ 10^{-3}\,\lambda_n\ .\end{aligned}$$

Since we have that $|\tau| \le (1+10^{-3})\sqrt{\lambda_n}$ and $\tau^2 \ge (1-10^{-3})^2\lambda_n$, we obtain for $\tau \in I_+ \cup I_-$

$$\Gamma_n(\tau,\xi,\eta) \ge (1-10^{-3})^2\lambda_n - 2(1+10^{-3})\ (2+10^{-3})\, 10^{-3}\, \lambda_n^{\frac{3}{2}}\ |\eta|\ .$$

Since we are considering $\lambda_{J+1} \le \lambda_n \le \lambda_K$, the following inequality holds for all τ:

$$\Gamma_n(\tau,\xi,\eta) \ge C_2^2(B\eta^\alpha)^2 \tag{6.11}$$

for some constant $C_2 > 0$, provided $|\eta|$ is sufficiently small to satisfy the condition

$$C_2^2(B\eta^\alpha)^2 + 2(1+10^{-3})\ (2+10^{-3})\, 10^{-3}\, \lambda_K^{3/2}\ |\eta| \le (1-10^{-3})^2\, \lambda_{J+1}\ .$$

We now consider the remaining parts of the boundary of the region $\Omega_{\sigma,B,\alpha}$, starting from the circle $|\varepsilon|^2 = \sigma^2$. We first consider the negative eigenvalues, so that for $n \le J$ one has

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &= \sigma^2(\tau^2-\lambda_n)^2 + [\eta(\lambda_n-\tau^2)+\tau]^2\\ &\ge \sigma^2(\tau^2+|\lambda_n|)^2 \ge \sigma^2|\lambda_n|^2 \ge \sigma^2|\lambda_J|^2 > 0\ .\end{aligned}$$

For $J < n \le K$, we begin from the case $|\lambda_n - \tau^2| < \delta$ for some positive δ. Then, for σ sufficiently small and setting $\varepsilon = \xi + i\eta$, we have

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &= \sigma^2(\tau^2-\lambda_n)^2 + (1-2\eta\tau)(\tau^2-\lambda_n) + \lambda_n\\ &\ge \sigma^2(\tau^2-\lambda_n)^2 + \frac{1}{2}(\tau^2-\lambda_n) + \lambda_n \ge \frac{\lambda_{J+1}}{2} \ge C_3\xi^2\end{aligned} \tag{6.12}$$

for $C_3 > 0$ if σ and δ are small enough with $0 < \delta \le \lambda_{J+1}$ and provided

$$|\xi| \le \sqrt{\frac{\lambda_{J+1}}{2C_3}}\ .$$

When $|\lambda_n - \tau^2| \ge \delta$, we have—as before—that the minimum of $\Gamma_n(\tau,\xi,\eta)$ is reached for $|\lambda_n - \tau^2| = \delta$ and we obtain the following bound:

$$\Gamma_n(\tau,\xi,\eta) = |\varepsilon|^2(\lambda_n-\tau^2)^2 + \tau^2 + 2\tau\eta(\lambda_n-\tau^2) = \xi^2\delta^2 + (\eta(\lambda_n-\tau^2)+\tau)^2 \ge \xi^2\delta^2\ . \tag{6.13}$$

Of course on the circle $|\varepsilon|^2 = 4\sigma^2$ we can reason in the same way, possibly changing the constants.

Let us discuss now the case $n > K$; again we distinguish two cases.

For τ such that $|\lambda_n - \tau^2| < 1$ we have $\tau^2 > \lambda_K - 1$; therefore for $|\eta|$ sufficiently small and for some $C_4 > 0$ we obtain the bound

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &= |\varepsilon|^2(\lambda_n-\tau^2)^2 + \tau^2 + 2\tau\eta(\lambda_n-\tau^2) \ge \tau^2 - 2|\tau||\eta| \ge \frac{1}{2}\tau^2\\ &\ge \frac{1}{2}(\lambda_K-1) > \frac{\lambda_{J+1}}{2} \ge C_4\xi^2\ ,\end{aligned} \tag{6.14}$$

provided $|\xi|$ is sufficiently small, namely,

$$|\xi| \le \sqrt{\frac{\lambda_{J+1}}{2C_4}}\ .$$

Finally, for $|\lambda_n - \tau^2| \geq 1$, $n > K$, we have

$$\Gamma_n(\tau, \xi, \eta) \geq \xi^2 |\lambda_n - \tau^2|^2 \geq \xi^2 \ . \tag{6.15}$$

Casting together the bounds (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), there exists a constant $C_5 > 0$, depending on λ_J and λ_{J+1}, such that for every n,

$$\Gamma_n(\tau, \xi, \eta) \geq C_5 \ \max\{(B\eta^\alpha)^2, \xi^2\} = C_5 \xi^2 \ , \tag{6.16}$$

since we are in the domain $\Omega_{\sigma,B,\alpha} \subseteq \{\xi + i\eta \in \mathbb{C} : \ |\xi| \geq B|\eta|^\alpha\}$. This concludes the bounds on the spectrum of N_ε.

Remark 6.1.

(i) The bounds providing the invertibility of N_ε fail when ε is on the imaginary axis.

(ii) The discussion above does not depend explicitly on the boundary conditions assumed for the PDE. However, the boundary conditions enter through the assumption H2.

(iii) The case in which

$$\sup_{n\geq 1} \lambda_n = \Lambda < \infty$$

is even simpler than the previous discussion, since in this case we can reason exactly as we did just for the case $n \leq K$.

To use the fixed point argument formulated in section 6.3, we need the bound

$$(\Gamma_n(\tau, \xi, \eta))^{\frac{1}{2}} \geq C_6 \sigma^2$$

for some constant C_6. This bound will be used in (6.19) below and, in view of (6.16), it amounts to requiring

$$|\xi| \geq \tilde{C}_6 \sigma^2$$

for some constant $\tilde{C}_6 > 0$. This inequality is in turn implied by

$$|B\eta^\alpha| \geq \tilde{C}_6 \sigma^2 \ ,$$

which is possible only if

$$|B\eta^\alpha| \geq \tilde{C}_6 (B^2 \eta^{2\alpha} + \eta^2) \ .$$

Therefore, since $|\eta| < 1$, we must have $\alpha \leq 2$; in conclusion, we take $\alpha = 2$, being the best possible exponent, thus leading to define the domain $\Omega_{\sigma,B}$ as in (4.6).

6.3. Existence of the fixed point. As we have discussed in section 3, we can rewrite (3.7) as a fixed point equation, namely,

$$U(\theta, x) = N_\varepsilon^{-1}(f(\theta, x) - \langle f \rangle(x)) - N_\varepsilon^{-1} G(U)(\theta, x) \ ,$$

where U denotes a function of ε defined by $U_\varepsilon = U_\varepsilon(\theta, x)$. In this way, we define an operator $\mathcal{T}$ acting on functions analytic in ε, taking values in $\mathcal{A}_{\rho,j,m}$, given by

$$\mathcal{T}(U) \equiv N_\varepsilon^{-1}(f - \langle f \rangle) - N_\varepsilon^{-1} G(U) \ . \tag{6.17}$$

For a fixed ε, we find a fixed point of $\mathcal{T}$ by considering a domain $\mathcal{P} \subset \mathcal{A}_{\rho,j,m}$ with $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$ on which $\mathcal{T}$ is a contraction. Since we want to obtain analyticity in ε, we reinterpret (6.17) as an operator acting on a space of analytic functions in ε and we consider the domain $\tilde{\mathcal{P}}$ in the space $\mathcal{A}_{\rho,j,m,\sigma,B}$ consisting of analytic functions of ε taking values in $\mathcal{A}_{\rho,j,m}$ with ε ranging on the domain $\Omega_{\sigma,B}$.

We endow $\mathcal{A}_{\rho,j,m,\sigma,B}$ with the supremum norm

$$\|U\|_{\rho,j,m,\sigma,B} \equiv \sup_{\varepsilon\in\Omega_{\sigma,B}} \|U\|_{\rho,j,m} \ , \tag{6.18}$$

for which $\mathcal{A}_{\rho,j,m,\sigma,B}$ is a Banach space. Moreover, due to Proposition A.3 of Appendix A, if $j > d$ and $m > \ell/2$, then $\mathcal{A}_{\rho,j,m,\sigma,B}$ with the norm (6.18) is a Banach algebra.

Notice that (6.16) and the fact that N_ε is diagonal imply that we can estimate its norm in the domains $\Omega_{\sigma,B}$. We note that the infimum is reached at the boundary of the domains, namely,

$$\|N_\varepsilon^{-1}\|_{\rho,j,m,\sigma,B} \le C_7 B^{-1}\sigma^{-2}\sigma \tag{6.19}$$

for some $C_7 > 0$, provided that B is sufficiently large. Using the Banach algebra property of $\mathcal{A}_{\rho,j,m,\sigma,B}$ and the fact that $D_U G(0)(\theta,x) = 0$, we note that the operator $\mathcal{T}_\varepsilon$ is Lipschitz in a ball $\mathcal{B}_\alpha(0) \subset \mathcal{A}_{\rho,j,m,\sigma,B}$ of radius $\alpha > 0$. Indeed, using Proposition A.7 we have that the Lipschitz constant of the composition with G is bounded by a constant times α. Thus, we have shown that

$$\|\mathcal{T}(U) - \mathcal{T}(V)\|_{\rho,j,m,\sigma,B} \le C_7 B^{-1}\sigma^{-1}\alpha\|U-V\|_{\rho,j,m,\sigma,B} \ .$$

We continue as in [CCdlL13] by showing that $\mathcal{T}$ is a contraction in a ball centered around the approximate solution that gets mapped into itself. First, we notice that the approximate solution $U^{(M)} = U^{(M)}(\theta,x)$ (see Definition 4.1) satisfies

$$\|U^{(M)}\|_{\rho,j,m,\sigma,B} \le C_8\sigma$$

for some $C_8 > 0$.

We fix $\alpha_0 > 0$ which will be the radius of a ball around zero in $\mathcal{A}_{\rho,j,m,\sigma,B}$, so that we will take the constants corresponding to this ball. We will refer to this ball as the *ambient ball*.

Our next goal will be to identify balls around the approximate solutions such that the operator $\mathcal{T}$ maps them into themselves and is a contraction. The following discussion is very similar to what is done in [CCdlL13].

Consider a ball $\mathcal{B}_\beta(U^{(M)})$ of radius β around $U^{(M)}$. We will impose several conditions on β that ensure that the ball is mapped into itself by $\mathcal{T}$ and that $\mathcal{T}$ is a contraction.

The ball $\mathcal{B}_\beta(U^{(M)})$ is contained in the ambient ball, $\mathcal{B}_{\alpha_0}(0) \in \mathcal{A}_{\rho,j,m,\sigma,B}$, provided that

$$C_8\sigma + \beta \le \alpha_0 \ . \tag{6.20}$$

Hence, we will assume (6.20) to ensure we can use the constants of the operator in the ambient ball.

The operator $\mathcal{T}$ is a contraction on $\mathcal{B}_\beta(U^{(M)})$ provided that

$$C_7(C_8\sigma + \beta)B^{-1}\sigma^{-1} < 1 \ . \tag{6.21}$$

Moreover, we have that the approximate solution satisfies the inequality

$$\left\|\mathcal{T}(U^{(M)}) - U^{(M)}\right\|_{\rho,j,m,\sigma,B} \leq C_9\sigma^3 B^{-1}\sigma^{-1}$$

for some constant $C_9 > 0$, since $U^{(M)}$ is a solution at least to $O(\varepsilon^3)$ as in Theorem 4.2 or 4.6. The ball $\mathcal{B}_\beta(U^{(M)})$ is mapped into itself whenever

$$C_7(C_8\sigma + \beta)B^{-1}\sigma^{-1}\beta + C_9 B^{-1}\sigma^2 \leq \beta \ . \tag{6.22}$$

Notice that to fulfill (6.20), (6.21), (6.22), we are allowed to choose β. Namely, we want to show that for some B large enough and for all σ sufficiently small, say, $\sigma \leq \sigma^*(B)$, we can find $\beta > 0$ such that the three conditions (6.20), (6.21), (6.22) are satisfied.

It is natural to choose

$$\beta = 100\ \sigma$$

and then (6.20), (6.21), (6.22) are implied by

$$(C_8 + 100)\sigma \leq \alpha_0, \tag{6.23}$$

$$C_7(C_8 + 100)B^{-1} < 1, \tag{6.24}$$

$$100C_7(C_8 + 100)B^{-1} + C_9 B^{-1}\sigma \leq 100. \tag{6.25}$$

We see that we can choose B large enough so that (6.24) is satisfied and then (6.23), (6.25) are satisfied for σ small enough.

In conclusion, we obtain that $\mathcal{T}$ admits a fixed point in the domain $\tilde{\mathcal{P}}$, provided σ and B are suitably chosen. This fixed point will be a function analytic in $\Omega_{\sigma,B}$.

As a corollary, the solution is locally unique; namely, we have the following result.

COROLLARY 6.2. *For a fixed $\varepsilon \in \Omega_{\sigma,B}$ with σ, B such that* (6.20), (6.21), (6.22) *are satisfied, let $U^{(M)}$ be an approximate solution. Then, for any $\theta \in \mathbb{T}^d$, we have that*

$$\lim_{n\to\infty} \mathcal{T}^n\ U^{(M)}(\theta) = U(\theta)\ . \tag{6.26}$$

In particular, the convergence in (6.26) *is uniform for $\varepsilon \in \Omega_{\sigma,B}$ with $\Omega_{\sigma,B}$ as in* (4.6)*, since $\|U^{(M)} - U\| \leq C\sigma^M$ for a positive constant $C > 0$, which implies that the solution is analytic for $\varepsilon \in \Omega_{\sigma,B}$.*

7. Modifications of the proof in section 6 to models A′, B, and B′.

In this section we consider models A′, B, and B′, providing the necessary modifications to the proof developed for model A. In particular, we concentrate on the extension of Theorem 4.3 to construct an approximate solution and on the bound of the Lipschitz constants of N_ε for ε in a suitable domain $\Omega_{\sigma,B}$, as it was done for model A in section 6.2. The other parts of the proof can be extended to models A′, B, and B′, trivially. The existence of the order zero solution is considered in Appendix B and sections 5.1 and 5.2, respectively.

7.1. Model A′.

For model A′ we look for a response solution of the form (3.1) and define the operators N_ε, $\mathcal{L}$, and G as

$$\begin{aligned}
N_\varepsilon U(\theta, x) &\equiv \left[(\omega\cdot\nabla_\theta)^2 \pm \frac{1}{\varepsilon}(\omega\cdot\nabla_\theta)\Delta_x + \mathcal{L}\right] U(\theta, x),\\
\mathcal{L}U(\theta, x) &\equiv -\Delta_x U(\theta, x) + h'(c_0(x), x)U(\theta, x),\\
G(U)(\theta, x) &\equiv h(c_0(x) + U(\theta, x), x) - h(c_0(x), x) - h'(c_0(x), x)U(\theta, x)\ .
\end{aligned} \tag{7.1}$$

Equation (4.3) is equivalent to the equation

$$N_\varepsilon U_\varepsilon(\theta,x)+G(U_\varepsilon)(\theta,x)=f(\theta,x)-\langle f\rangle(x)\ ,$$

while the function c_0 must satisfy (3.8)

To construct an approximate solution, let us again write formally U_ε as $U_\varepsilon = \sum_{j=1}^{\infty}\varepsilon^j U_j$. Then, after solving (3.8) for c_0 (see Appendix B), at the first order in ε we need to solve

$$\pm(\omega\cdot\nabla_\theta)\Delta_x U_1(\theta,x)=f(\theta,x)-\langle f\rangle(x)\ ,$$

which yields the nonaverage part of U_1 using H2$'$ and the nonresonance condition on ω. At the second order in ε we obtain the equation

$$\pm(\omega\cdot\nabla_\theta)\Delta_x U_2(\theta,x)=-[(\omega\cdot\nabla_\theta)^2+\mathcal{L}]U_1(\theta,x)\ ,$$

from which we first deduce that the average $\langle U_1\rangle$ should be zero by imposing that the right-hand side has zero average and noting that $\mathcal{L}$ is invertible. Then, we determine the nonaverage part of U_2 by solving the remaining equation.

Then we again proceed recursively, by assuming that after $N-1$ steps we completely determined $U_1,\dots,U_{N-2}$ while U_{N-1} is determined up to its average. The equation at step N is

$$\begin{aligned}\pm(\omega\cdot\nabla_\theta)\Delta_x U_N(\theta,x)=&-[(\omega\cdot\nabla_\theta)^2+\mathcal{L}]U_{N-1}(\theta,x)\\ &+S_N(c_0(x),U_1(\theta,x),\dots,U_{N-2}(\theta,x))\end{aligned}\tag{7.2}$$

for a suitable function S_N, depending on c_0 and on the functions U_j with $j<N-1$; by imposing that the average of the right-hand side of (7.2) is zero we obtain the relation

$$\mathcal{L}(\langle U_{N-1}\rangle)=\langle S_N(c_0,U_1\dots,U_{N-2})\rangle\ ,$$

which allows us to fix the average of U_{N-1}, so that we obtain the nonaverage part of U_N by solving (7.2).

To conclude the proof for model A$'$, we proceed to estimate the eigenvalues of the operator N_ε in a way similar to that of model A.

Indeed, let us write $\varepsilon=\xi+i\eta$ with ε belonging to the domain $\Omega_{\sigma,B}$ defined in (4.6); denoting by λ_n^Δ the eigenvalues associated to $-\Delta_x$, we have

$$\begin{aligned}|\varepsilon\lambda_{n,k}|^2&=|-\varepsilon(2\pi\omega\cdot k)^2\pm i(2\pi\omega\cdot k)\lambda_n^\Delta+\varepsilon\lambda_n|^2\\ &=\xi^2((2\pi\omega\cdot k)^2-\lambda_n)^2+[-\eta(2\pi\omega\cdot k)^2\pm(2\pi\omega\cdot k)\lambda_n^\Delta+\eta\lambda_n]^2\ .\end{aligned}$$

As for model A, we introduce an auxiliary function $\Gamma_n(\tau,\xi,\eta)$ to obtain bounds on the eigenvalues of N_ε:

$$\Gamma_n(\tau,\xi,\eta)\equiv\xi^2(\tau^2-\lambda_n)^2+[\eta(-\tau^2+\lambda_n)\pm\tau\lambda_n^\Delta]^2\ .\tag{7.3}$$

Again we fix $K\in\mathbb{Z}$ such that $\lambda_K-1\ge\lambda_{J+1}$ and consider first the case $1\le n\le K$. Let us first consider the case of negative eigenvalues $\lambda_1\le\dots\le\lambda_J<0$. From (7.3) we have

$$\Gamma_n(\tau,\xi,\eta)\ge\xi^2(\tau^2+|\lambda_n|)^2\ge\xi^2|\lambda_n|^2\ge\xi^2|\lambda_J|^2>0\ .$$

Next, we consider the positive eigenvalues $0 < \lambda_{J+1} \le \lambda_{J+2} \le \cdots$. Define the regions I_- and I_+ as in (6.9). In the region $(I_- \cup I_+)^c$ we obtain

$$\Gamma_n(\tau,\xi,\eta) \ge \xi^2(\tau^2-\lambda_n)^2 \ge C_{10}\xi^2\lambda_n^2 \ge C_{10}\xi^2\lambda_{J+1}^2$$

for a suitable constant $C_{10} > 0$.

Within the region $I_- \cup I_+$ we first consider the case of negative eigenvalues of $-\Delta_x$: $\lambda_1^\Delta \le \cdots \le \lambda_{J_\Delta}^\Delta < 0$. From (7.3) we have

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &\ge (-\eta\tau^2\mp\tau|\lambda_n^\Delta| + \eta\lambda_n)^2 = [\mp\tau|\lambda_n^\Delta| + \eta(\lambda_n-\tau^2)]^2\\ &\ge \tau^2|\lambda_n^\Delta|^2\mp 2|\tau|\,|\eta|\,|\lambda_n^\Delta|\,|\lambda_n-\tau^2|\\ &\ge \tau^2|\lambda_n^\Delta|^2\mp 2|\tau|\,|\eta|\,|\lambda_n^\Delta|\,(2+10^{-3})\,10^{-3}\,\lambda_n\\ &\ge (1-10^{-3})^2\lambda_n|\lambda_n^\Delta|^2\mp 2|\eta|\,|\lambda_n^\Delta|\,(2+10^{-3})\,10^{-3}\,(1+10^{-3})\lambda_n^{\frac{3}{2}} \ge C_{11}'\xi^2\end{aligned}$$

for some constant C_{11}' such that

$$C_{11}'\xi^2 + 2|\eta|\,|\lambda_1^\Delta|\,(2+10^{-3})\,(1+10^{-3})\,10^{-3}\,\lambda_K^{\frac{3}{2}} \le (1-10^{-3})^2\lambda_{J+1}\,|\lambda_{J_\Delta}^\Delta|^2$$

when taking the negative sign in model A$'$ or

$$C_{11}'\xi^2 \le (1-10^{-3})^2\ \lambda_{J+1}\ |\lambda_{J_\Delta}^\Delta|^2$$

when taking the positive sign. Then, we consider the case $0 < \lambda_{J_\Delta+1}^\Delta \le \lambda_{J_\Delta+2}^\Delta \le \cdots$; again from (7.3) we have

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &\ge (-\eta\tau^2\pm\tau\lambda_n^\Delta + \eta\lambda_n)^2 \ge \tau^2(\lambda_n^\Delta)^2 - 2|\eta|\ |\tau|\ |\lambda_n-\tau^2|\lambda_n^\Delta\\ &\ge \tau^2(\lambda_n^\Delta)^2 - 2\lambda_n^\Delta|\eta|\ |\tau|(2+10^{-3})\,10^{-3}\,\lambda_n\\ &\ge (1-10^{-3})^2\lambda_n(\lambda_n^\Delta)^2 - 2\lambda_n^\Delta(1+10^{-3})\lambda_n^{3/2}(2+10^{-3})\,10^{-3}\,|\eta| \ge C_{11}\xi^2\end{aligned}$$

for some constant $C_{11} > 0$, if

$$C_{11}\xi^2 + 2\lambda_K^\Delta(1+10^{-3})(2+10^{-3})\,10^{-3}\,\lambda_K^{3/2}|\eta| \le (1-10^{-3})^2(\lambda_1^\Delta)^2\lambda_1\ . \tag{7.4}$$

In the case $n > K$, we recall the expression of $\Gamma_n(\tau,\xi,\eta)$ in (7.3). Then, we consider the subcase $|\lambda_n-\tau^2| < 1$, which provides $\tau^2 > \lambda_K - 1$. We first consider the negative eigenvalues of $-\Delta_x$, namely, $\lambda_1^\Delta \le \cdots \le \lambda_{J_\Delta}^\Delta < 0$. From (7.3) we obtain

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &\ge [-\eta\tau^2\mp\tau|\lambda_n^\Delta| + \eta\lambda_n]^2 = [\tau|\lambda_n^\Delta|\mp\eta(\lambda_n-\tau^2)]^2\\ &\ge \tau^2|\lambda_n^\Delta|^2 - 2|\tau|\,|\eta|\,|\lambda_n^\Delta|\\ &\ge \frac{1}{2}\tau^2|\lambda_n^\Delta|^2 \ge \frac{1}{2}|\lambda_1^\Delta|^2\,(\lambda_K-1) > \frac{1}{2}|\lambda_{J_\Delta}^\Delta|^2\,\lambda_{J+1}\end{aligned}$$

if

$$|\eta| \le \frac{1}{4}|\lambda_{J_\Delta}^\Delta|\,\sqrt{\lambda_K-1}\ .$$

Next, we consider the positive eigenvalues of $-\Delta_x$; recalling (7.3), one obtains for $|\eta|$ sufficiently small

$$\begin{aligned}\Gamma_n(\tau,\xi,\eta) &= |\varepsilon|^2(\lambda_n-\tau^2)^2+(\lambda_n^\Delta)^2\tau^2 \pm 2\tau\ \lambda_n^\Delta\ \eta\ (\lambda_n-\tau^2)\\ &\geq (\lambda_n^\Delta)^2\tau^2-2\lambda_n^\Delta|\eta|\ |\tau|\\ &\geq \frac{1}{2}(\lambda_n^\Delta)^2\tau^2 \geq \frac{1}{2}(\lambda_{J_\Delta+1}^\Delta)^2\ (\lambda_K-1) > \frac{1}{2}(\lambda_{J_\Delta+1}^\Delta)^2\,\lambda_{J+1}\ ,\end{aligned}$$

provided

$$|\eta| \leq \frac{1}{4}\lambda_1^\Delta\sqrt{\lambda_K-1}\ .$$

In the case $|\lambda_n-\tau^2|\geq 1$, recalling (7.3) we obtain

$$\Gamma_n(\tau,\xi,\eta) \geq \xi^2|\lambda_n-\tau^2|^2 \geq \xi^2\ .$$

Concerning the boundaries $|\varepsilon|=\sigma$ and $|\varepsilon|=2\sigma$, again we can reason as for model A.

Note that the assumption H2$'$ is crucial in order to get the bound; if Δ_x is the standard Laplace–Beltrami operator, the invertibility of the operator N_ε is guaranteed only for D boundary conditions, because H2$'$ is violated for N, P boundary conditions.

This concludes the discussion of the invertibility of the operator N_ε for model A$'$. The existence of a fixed point can be done in full analogy to model A; see section 6.3.

7.2. Model B. To construct an approximate solution, let us write formally $U_\varepsilon(\theta,x)=U_0(\theta,x)+\sum_{j=1}^\infty \varepsilon^j U_j(\theta,x)$. Given the zeroth order solution (see (5.1)) as in section 5.1, we proceed to determine the higher order terms, matching powers of the formal series expansion in the equation

$$\varepsilon^2(\omega\cdot\nabla_\theta)^2U_\varepsilon(\theta,x)+(\omega\cdot\nabla_\theta)U_\varepsilon(\theta,x)-\Delta_xU_\varepsilon(\theta,x)+h(U_\varepsilon(\theta,x),x)=f(\theta,x)\ . \tag{7.5}$$

At first order in ε, we get the equation

$$(\omega\cdot\nabla_\theta)U_1(\theta,x)-\Delta_xU_1(\theta,x)+h'(U_0(\theta,x),x)U_1(\theta,x)=0\ , \tag{7.6}$$

which can be used to determine U_1. Indeed, writing (7.6) as

$$[(\omega\cdot\nabla_\theta)-\Delta_x+h'(U_0(\theta,x),x)]\ U_1(\theta,x)=0\ ,$$

we may fix $U_1\equiv 0$.

Remark 7.1. Note that if U_0 is contained in a ball around the origin in $\mathcal{A}_{\rho,j,m}$ with small enough radius as in Proposition 5.1, then the operator

$$\tilde\Gamma=(\omega\cdot\nabla_\theta)-\Delta_x+h'(U_0(\theta,x),x) \tag{7.7}$$

is invertible. Indeed, if we write it as the sum of the invertible operator (5.2), introduced in section 5.1 plus the multiplication operator T defined as

$$T\phi=[h'(U_0(\theta,x),x)-h'(0,x)]\phi\ ,$$

which is small when U_0 is in a small ball, we obtain the inverse of $\tilde\Gamma=\Gamma+T$ by a Neumann series argument.

At the general order $N \geq 2$, we obtain the equation

$$\tilde{\Gamma}[U_N](\theta, x) = -(\omega \cdot \nabla_\theta)^2 U_{N-2}(\theta, x) + S_N(U_0(\theta, x), \ldots, U_{N-1}(\theta, x)) \ , \tag{7.8}$$

where S_N is a known function of the U_j's with $j < N$. If the operator $\tilde{\Gamma}$ is invertible, we can determine U_N uniquely.

Now we assume that it is possible to solve the zeroth order equation (5.1) (some sufficient conditions have been presented is section 5.1) as well as to solve the recursive equations (7.6), (7.8) by taking U_0 in a small ball around the origin in the $\mathcal{A}_{\rho,j,m}$ norm. We proceed to study the conditions under which (7.5) can be solved.

We start by introducing the operator

$$\Lambda_\varepsilon = \varepsilon^2(\omega \cdot \nabla_\theta)^2 + (\omega \cdot \nabla_\theta) - \Delta_x + h'(0, x) \ . \tag{7.9}$$

If this operator is invertible, by the same argument as in Remark 7.1 the operator

$$\tilde{\Lambda}_\varepsilon = \varepsilon^2(\omega \cdot \nabla_\theta)^2 + (\omega \cdot \nabla_\theta) - \Delta_x + h'(U_0(\theta, x), x) \tag{7.10}$$

is invertible whenever $\|U_0\|_{\rho,j,m}$ is sufficiently small. Now, if we write (7.5) as

$$\tilde{\Lambda}_\varepsilon U_\varepsilon(\theta, x) + H(U_\varepsilon)(\theta, x) = f(\theta, x), \tag{7.11}$$

where we write $U_\varepsilon = U_0 + \tilde{U}_\varepsilon$ with $\tilde{U}_\varepsilon = \sum_{j=1}^{\infty} \varepsilon^j U_j(\theta, x)$ and

$$H(U_\varepsilon)(\theta, x) = h(U_\varepsilon(\theta, x), x) - h'(U_0(\theta, x), x) U_\varepsilon(\theta, x) \ ,$$

we are led to solve the equation

$$U_\varepsilon(\theta, x) = -\tilde{\Lambda}_\varepsilon^{-1}[H(U_\varepsilon)(\theta, x) - f(\theta, x)] \ .$$

Let us define the operator $\mathcal{T}$ acting on a function $U = U(\theta, x)$ by

$$\mathcal{T}[U](\theta, x) \equiv -\tilde{\Lambda}_\varepsilon^{-1}[H(U)(\theta, x) - f(\theta, x)] \ . \tag{7.12}$$

Using Proposition A.7 of Appendix A, we can show that for $U, V \in \mathcal{A}_{\rho,j,m}$, $\mathcal{T}$ satisfies the inequality

$$\begin{aligned} \|\mathcal{T}(U) - \mathcal{T}(V)\|_{\rho,j,m} &= \|\tilde{\Lambda}_\varepsilon^{-1} \ (H(U)) - \tilde{\Lambda}_\varepsilon^{-1} \ (H(V))\|_{\rho,j,m} \\ &\leq C\alpha_0 \ \|\tilde{\Lambda}_\varepsilon^{-1}\|_{\rho,j,m} \ \|U - V\|_{\rho,j,m} \ , \end{aligned}$$

since the Lipschitz constant of the composition with H is bounded by a constant times α_0.

As in the case of model A, to check that $\mathcal{T}$ maps a small enough ball around an approximate solution $U^{(M)}(\theta, x)$ into itself and it is a contraction, we need to investigate the domain on which Λ_ε can be inverted with "good bounds."

The multiplier $\lambda_{n,k,\varepsilon}$ associated to $\tilde{\Lambda}_\varepsilon$ is given by

$$\lambda_{n,k,\varepsilon} \equiv \varepsilon^2(2\pi i\omega \cdot k)^2 + 2\pi i\omega \cdot k + \lambda_n \ ,$$

where the eigenvalues λ_n of $\mathcal{L} \equiv -\Delta_x + h'(U_0(\theta, x), x)$ satisfy H1–H2 with the assumption that $\|U_0\|_{\rho,j,m}$ is small. For a given n, we consider the function

$$\tilde{\Gamma}_n(\tau, \varepsilon) \equiv -\varepsilon^2\tau^2 + i\tau + \lambda_n$$

for $\tau \in \mathbb{R}$. We are interested in evaluating the quantity $\inf |\tilde{\Gamma}_n(\tau, \varepsilon)|$. This function can be easily analyzed geometrically, since the part corresponding to $-\varepsilon^2\tau^2 + i\tau$ is a parabola. The infimum is generated by considering the minimum distance of the parabola from the quantity λ_n (which is a real number).

For $\varepsilon = 0$ the parabola coincides with the vertical axis, so that if $\lambda_n \neq 0$, the distance is always positive. Indeed, the parabola $-\varepsilon^2\tau^2 + i\tau$ passes through the origin and it is tangent there to $i\mathbb{R}$. The axis of the parabola coincides with $-\varepsilon^2$.

We assume that

$$\mathrm{Re}(-\varepsilon^2) \geq \delta > 0 \ , \tag{7.13}$$

which amounts to requiring that $\eta^2 - \xi^2 \geq \delta > 0$.

Let us start by considering the case of ε real, say, $\varepsilon = \xi$ with $\xi \in \mathbb{R}$ as in (4.7) with δ small enough. Then, setting

$$|\tilde{\Gamma}_n(\tau, \xi)|^2 \equiv |-\xi^2\tau^2 + i\tau + \lambda_n|^2 = (\lambda_n - \xi^2\tau^2)^2 + \tau^2 \ ,$$

it follows that

$$\frac{d}{d\tau}\Gamma_n(\tau, \xi) = 2\tau(2\xi^4\tau^2 - 2\lambda_n\xi^2 + 1) \ .$$

We have two cases:

Case 1. $2\lambda_n\xi^2 \leq 1$, so that the minimum is attained at $\tau = 0$ and one has $\Gamma_n(0, \xi) = \lambda_n^2 \geq \min\{\lambda_J^2, \lambda_{J+1}^2\}$.

Case 2. $2\lambda_n\xi^2 > 1$ and hence the minimum is attained at $\tau_\pm = \pm\sqrt{(2\lambda_n\xi^2 - 1)/(2\xi^4)}$ ($\Gamma_n(\tau_\pm, \xi)$ are equal for parity reasons) and one has

$$\Gamma_n(\tau_\pm, \xi) = \frac{1}{4\xi^4} + \frac{\lambda_n}{\xi^2} - \frac{1}{2\xi^4} = \frac{\lambda_n}{\xi^2} - \frac{1}{4\xi^4} \geq \frac{1}{4\xi^4} \geq 1$$

for ξ small enough.

Next we consider the case of ε imaginary, say $\varepsilon = \eta$ with $\eta \in \mathbb{R}$. Then, we have

$$|\tilde{\Gamma}_n|^2 = [-\eta^2\tau^2 + |\lambda_n|]^2 + \tau^2 \ .$$

For $\alpha_0 > 0$ let us define the interval

$$I_{\alpha_0} \equiv \{\tau \in \mathbb{R} : \ |\tau| \leq \alpha_0\} \ .$$

Then, in the complement of I_{α_0} we have

$$|\tilde{\Gamma}_n|^2 \geq \alpha_0^2 > 0 \ ,$$

while in I_{α_0} we have

$$|\tilde{\Gamma}_n|^2 \geq [-\eta^2\alpha_0^2 + |\lambda_n|]^2 \geq (1 - 10^{-6}) \ |\lambda_J|^2 \ ,$$

provided

$$|\eta| \, \alpha_0 < 10^{-3}|\lambda_J| \ .$$

Next, we consider the case $\xi \neq 0$, $\eta \neq 0$ and we split the proof by considering first the negative eigenvalues $\lambda_1 \leq \ldots \lambda_J < 0$. Recall that

$$|\tilde{\Gamma}_n|^2 = [-(\eta^2 - \xi^2)\tau^2 + |\lambda_n|]^2 + (\tau - 2\xi\eta \ \tau^2)^2 \ .$$

For $\alpha > 0$ let us define the interval

$$I_\alpha \equiv \{\tau : \ |1 - 2\xi\eta\tau| \le 1 + \alpha\} \ .$$

In the complement of I_α we have that $|1 - 2\xi\eta\tau| > 1 + \alpha$, which implies that $\tau > \frac{2+\alpha}{2\xi\eta}$ and that

$$|\tilde\Gamma_n|^2 \ge \tau^2(1+\alpha)^2 > 0 \ .$$

Within I_α we have that $-\alpha \le 2\xi\eta\tau \le 2+\alpha$ and therefore we obtain that

$$\begin{aligned} |\tilde\Gamma_n|^2 &\ge [-|\eta^2 - \xi^2|\tau^2 + |\lambda_n|]^2 \\ &\ge \left[-|\eta^2 - \xi^2| \, \frac{(2+\alpha)^2}{4|\xi\eta|^2} + |\lambda_n|\right]^2 \\ &\ge (1 - 10^{-6})^2 |\lambda_J|^2 > 0 \ , \end{aligned}$$

provided

$$\frac{(2+\alpha)^2}{4} \frac{|\eta^2 - \xi^2|}{|\xi\eta|} < 10^{-6} \ |\lambda_J| \ . \tag{7.14}$$

From $\eta^2 + \xi^2 < \sigma^2$ we have $\eta^2 - \xi^2 < \sigma^2$; on the other hand, from $\eta^2 + \xi^2 \ge \eta^2 - \xi^2 \ge \delta > 0$ we obtain $\eta^2 \ge \delta + \xi^2 \ge \delta$. Hence, we conclude that $\xi^2 > \delta - \sigma^2$, from which we obtain that $|\xi^2\eta^2| \ge \delta(\delta - \sigma^2)$. Therefore, (7.14) is implied by

$$\frac{(2+\alpha)^2}{4} \ \frac{\sigma^2}{(\delta(\delta - \sigma^2))^{\frac12}} < 10^{-6} \ |\lambda_J| \ ,$$

which is satisfied by a suitable choice of σ, δ.

Finally, we consider the positive eigenvalues and we obtain the estimate

$$|\mathrm{Re}(-\varepsilon^2\tau^2 + i\tau + \lambda_n)| \ge \delta\tau^2 + \lambda_n \ge \lambda_n \ ,$$

which ensures that the spectrum of $\tilde\Lambda_\varepsilon$ is away from zero due to H1–H2. Therefore, we infer that the operator $\Gamma_n(\tau,\varepsilon)$ is invertible and we get uniform bounds within the domain

$$\Omega_\delta \equiv \{\varepsilon = \xi + i\eta : \ \mathrm{Re}(-\varepsilon^2) \ge \delta\}$$

for $\delta > 0$.

Summarizing, for ε real we get

$$\Gamma_n(\tau,\xi) \ge \min\{\lambda_J^2, \lambda_{J+1}^2, 1\} \ . \tag{7.15}$$

7.3. Model B′. We write the solution as $U_\varepsilon(\theta,x) = U_0(\theta,x) + \tilde U_\varepsilon(\theta,x)$, where $\tilde U_\varepsilon \equiv \sum_{j=1}^\infty \varepsilon^j U_j(\theta,x)$. The zeroth order solution has been already discussed in section 5.2. With respect to model B, the only modification is that here we do not need the assumption $h(0) = 0$. To determine the higher order terms U_j, we start by considering the equation

$$\varepsilon^2(\omega\cdot\nabla_\theta)^2 U_\varepsilon(\theta,x) + (\omega\cdot\nabla_\theta)U_\varepsilon(\theta,x) - \Delta_x U_\varepsilon(\theta,x) + \varepsilon h(U_\varepsilon(\theta,x),x) = f(\theta,x) \ . \tag{7.16}$$

Inserting the series expansion for $\tilde U_\varepsilon$ into (7.16) and matching the same powers of ε, we get the equations for the functions U_j, $j \ge 1$.

At the first order in ε we obtain the equation

$$(\omega \cdot \nabla_\theta) U_1(\theta, x) - \Delta_x U_1(\theta, x) = -h(U_0(\theta, x), x) \ . \tag{7.17}$$

Let $\Lambda \equiv (\omega \cdot \nabla_\theta) - \Delta_x$; then (7.17) can be rewritten as

$$\Lambda U_1(\theta, x) = -h(U_0(\theta, x), x) \ , \tag{7.18}$$

and note that the right-hand side is a known function, once we solved the zeroth order equation.

At the order $N \geq 2$ we get a recursive equation of the form

$$\Lambda U_N(\theta, x) = S_N(U_0(\theta, x), U_1(\theta, x), \ldots, U_{N-1}(\theta, x)) \tag{7.19}$$

for a function S_N depending on the terms U_j, $0 \leq j < N$, which are assumed to be determined at the previous steps.

Both (7.18) and (7.19) can be solved, provided the operator Λ is boundedly invertible in the spaces $\mathcal{A}_{\rho,j,m}$. This requirement is satisfied under the conditions H1′–H2′ on the eigenvalues of $-\Delta_x$ appearing in Λ.

After solving the zeroth order equation as well as (7.19) up to a finite order N, we consider the formulation of (7.16) as a fixed point equation and establish the existence of solutions.

Let us define the operator Λ_ε as

$$\tilde{\Lambda}_\varepsilon = \varepsilon^2 (\omega \cdot \nabla_\theta)^2 + (\omega \cdot \nabla_\theta) - \Delta_x \ ; \tag{7.20}$$

then, (7.16) can be written as

$$\tilde{\Lambda}_\varepsilon U_\varepsilon(\theta, x) + H_\varepsilon(U_\varepsilon)(\theta, x) = f(\theta, x) \ ,$$

where H_ε is defined as

$$H_\varepsilon(U_\varepsilon)(\theta, x) = \varepsilon h(U_\varepsilon(\theta, x), x) \ .$$

To obtain the existence of solutions, it suffices to solve the fixed point equation

$$U_\varepsilon(\theta, x) = -\tilde{\Lambda}_\varepsilon^{-1} [H_\varepsilon(U_\varepsilon)(\theta, x) - f(\theta, x)] \ . \tag{7.21}$$

The invertibility of the operator $\tilde{\Lambda}_\varepsilon$ in the space $\mathcal{A}_{\rho,j,m}$ has been already discussed for model B and we conclude that condition (7.13) together with H1′–H2′ ensures that the spectrum of $\tilde{\Lambda}_\varepsilon$ is bounded away from zero, if U_0 is in a sufficiently small ball around the origin, as required in Proposition 5.1. Comparing (7.12) for model B and (7.21), we conclude that we can reason as for model B to apply the contraction mapping argument.

8. Optimality of the results. The domains of analyticity for response solutions established in Theorem 4.2 are not optimal. Clearly, many details of the argument can be optimized and it is quite possible that one can use better fixed point theorems or better arguments.

Nevertheless, we want to argue in this section that the results presented so far cannot be improved very dramatically.

We will present rigorous results (Theorem 8.2) and heuristic arguments (Conjectures 8.6, 8.7, and 8.8) that indicate that the results obtained are qualitatively optimal and quantitatively almost optimal.

In particular, we believe that the domains of analyticity of the response solutions for models B and B′ do not contain sectors with aperture bigger than $\pi/2$ for generic perturbations. Note that $\pi/2$ is precisely the critical aperture of the Phragmén–Lindelof theorem [PL08, SZ65], which makes the function theoretic properties of the perturbative functions considered here very tantalizing. Of course, the aperture of the cone in the domain has deep consequences for the properties of the asymptotic expansions and how to recover the function from the computed asymptotic expansion (note that not even the uniqueness of the asymptotic expansion is clear for functions in these domains).

The argument presented in this section is very general and it applies to many problems that can be reduced to a fixed point problem with parameters and which satisfy some mild conditions on analyticity and compactness. In particular, it applies to the treatment of the varactor problem carried out in [CCdlL13], but we will not formulate here the precise results in this case, even if they have fewer technicalities than those used in this work.

The argument we present here goes by contradiction. We show rigorously (see Theorem 8.2) that if there is a family of solutions u_ε whose domain includes a *resonance* (see Definition 8.1), if we embed the problem in a two-parameter family of problems, then for most families we do not have a two-parameter family of solutions. The second rigorous result (Lemma 8.3) strengthens a bit the previous one by showing that if we had solutions for all the one-parameter families in a neighborhood (in the space of analytic families), we could find an analytic two-parameter family. The conclusion of the two results above is that it will be very unlikely to find a family of problems, so that the domain of the response function includes a resonance. That is, if we find a family whose solutions include a resonance, we can find arbitrarily small perturbations whose response solutions have a domain that does not include the resonance.

By examining the argument carefully, and by proposing alternative points of view, we speculate—but we do not prove it rigorously—that this argument applies to all resonances simultaneously. This leads to Conjecture 8.6.

Of course, since our contradictions are obtained by constructing perturbations which cannot be continued, if we consider a restricted class of models, one has to wonder whether the perturbations can be constructed in this class.

Similar lines of argumentation have appeared in the literature. Notably, we have been inspired by the use of uniform integrability in [Poi87a] to obtain insights on the problem of integrability.

8.1. Statement of rigorous results on optimality. The key to the arguments in this section is the concept of resonance for a parameter family of solutions.

DEFINITION 8.1. *Let $\mathcal{O}_\varepsilon$ be an analytic family of bounded operators from a Banach space to itself. We say that ε_0 is a resonant value for the family $\mathcal{O}_\varepsilon$ whenever the operator $\mathcal{O}_{\varepsilon_0}$ has a zero eigenvalue.*

We say that the resonance is isolated if for all $0 < |\varepsilon - \varepsilon_0| \ll 1$, we have that $\mathcal{O}_\varepsilon$ is invertible. Note that the arguments presented here do not require that the resonance is isolated.

In our applications to nonlinear problems, say $\mathcal{F}_\varepsilon(U_\varepsilon) = 0$, we will take as the linear operators $\mathcal{O}_\varepsilon$, the derivatives at the solution $\mathcal{O}_\varepsilon = D\mathcal{F}_\varepsilon(U_\varepsilon)$.

Of course for a general family of operators there are other alternatives between having an eigenvalue zero and being invertible (having continuous spectrum, having residual spectrum, etc.). In our case, the operators have a spectrum which is the closure of the set of the eigenvalues.

The important fact about resonances is that if ε_0 is a resonant value, the range of the operator $\mathcal{O}_{\varepsilon_0}$ has codimension at least 1.

To prove our results it will be useful to introduce a two-parameter family, say, $\mathcal{F}_{\varepsilon,\mu}$, so that it will be easier to compute obstructions generated by resonances. Precisely, our result is based on the following arguments:

(i) we will show (see part (b) of Theorem 8.2) that given a two-parameter family of problems, $\mathcal{F}_{\varepsilon,\mu}(U_{\varepsilon,\mu}) = 0$, such that $D\mathcal{F}_{\varepsilon,0}(U_{\varepsilon,0})$ is resonant, we cannot expect to obtain solutions analytic in μ near $\mu = 0$ (we call this phenomenon *automatic analyticity*);

(ii) we will show that if there is a perturbative solution for every one-parameter family, there has to be a jointly analytic solution in two parameters (see Lemma 8.3);

(iii) the consequence of these two results is that it is impossible that there is a solution that drives through the resonances for every one-parameter family (see part (a) of Theorem 8.2).

We will work mainly with (3.7), but—as anticipated before—we extend it adding a parameter for the nonlinearity. In particular, we rewrite (3.7) as

$$\mathcal{F}_{\varepsilon,\mu}(U_{\varepsilon,\mu}) = N_\varepsilon U_{\varepsilon,\mu} + A_\mu(U_{\varepsilon,\mu}) = 0 \ , \tag{8.1}$$

where $A_\mu(U_{\varepsilon,\mu}) \equiv G_\mu(U_{\varepsilon,\mu}) - f + \langle f \rangle$ and G_μ is any smooth function of μ such that $G_0 = G$.

We define the family of operators $\mathcal{O}_\varepsilon$ as

$$\mathcal{O}_\varepsilon \equiv D\mathcal{F}_{\varepsilon,0}(U_{\varepsilon,0}) = N_\varepsilon + A_0'(U_{\varepsilon,0}) \ . \tag{8.2}$$

We will argue that if $\mathcal{O}_\varepsilon$ has a resonance at $\varepsilon = \varepsilon_0$, it is very difficult to have a family A_μ that allows us to have $U_{\varepsilon_0,\mu}$ analytic in μ and which solves $\mathcal{F}_{\varepsilon_0,\mu}(U_{\varepsilon_0,\mu}) = 0$.

To make all this precise, we endow the space of analytic families of linear operators with the topology of the supremum of the norm in a complex domain, so that it is a Banach space. We will always consider the domains in μ to be a ball around $\mu = 0$. The domains in ε could be either a ball around $\varepsilon = 0$ for the perturbative expansions or a ball around $\varepsilon = \varepsilon_0$, where ε_0 is a resonance. When dealing with functions of two variables, we consider domains which are the product.

The key to the argument is to show that if there are analytic solutions, the family $\mathcal{F}_{\varepsilon_0,\mu}$ has to satisfy constraints and that generic families violate them. Of course, if one considers specific models in (8.1), it could in principle happen that the family automatically satisfies the constraint. We will, however, show that this does not happen in general and that, even in specific models for (8.1), it is unlikely that one can make deformations satisfying the constraints imposed by the existence of analytic solutions.

THEOREM 8.2. *Let $\mathcal{F}_\varepsilon$ be an analytic family of analytic operators from a Banach space to itself. Assume that there is a family U_ε defined in a domain of analyticity including ε_0, such that $\mathcal{F}_\varepsilon(U_\varepsilon) = 0$ and that ε_0 is a resonant value for the $D\mathcal{F}_\varepsilon(U_\varepsilon)$ family.*

Consider an arbitrary small ball $B \subset \mathbb{C}$ centered at ε_0 and define $\mathcal{A}_B$ the space of analytic families of operators defined in the ball endowed with the supremum topology.

Then, we have the following results:

(a) *In any sufficiently small ball of $\mathcal{A}_B$ centered at $\mathcal{F}_{\varepsilon_0}$, we can find a family of operators $\tilde{\mathcal{F}}_\varepsilon$ such that the domain of analyticity of the solution of $\tilde{\mathcal{F}}_\varepsilon(U_\varepsilon) = 0$ does not include ε_0.*

(b) *For restricted two-parameter families $\mathcal{F}_{\varepsilon,\mu}$ of the form* (8.1)*, we have the same result. Namely, if there is an analytic family $U_{\varepsilon,0}$ satisfying $\mathcal{F}_{\varepsilon,0}(U_{\varepsilon,0}) = 0$ and ε_0 is a resonant value for $D\mathcal{F}_{\varepsilon,0}(U_{\varepsilon,0})$, then for an open and dense set of families A_μ, we can find arbitrary small values μ such that the family $\mathcal{F}_{\varepsilon,\mu}$ does not admit a solution $U_{\varepsilon,\mu}$ which is analytic near $\mu = 0$.*

8.2. Proof of Theorem 8.2. The first element in the proof of Theorem 8.2 is the following elementary lemma showing that if one has analytic solutions for all equations, then they have to be analytic in a second parameter. Afterward, we will identify obstructions for analyticity in two parameters near a resonance; this obstruction is very similar to Poincaré's obstructions to uniform integrability [Poi87a, section 81] (see [dlL96] for a reexamination of [Poi87a] with modern techniques and for a converse of the results of [Poi87a]).

LEMMA 8.3. *Consider a family of equations $\mathcal{F}_\varepsilon(U) = 0$, where $\mathcal{F}_\varepsilon$ is an analytic family of nonlinear operators. Endow the space of analytic operators with the supremum topology.*

Assume that for all $\mathcal{G}_\varepsilon$ in a neighborhood of $\mathcal{F}_\varepsilon$ in the space of analytic functions there is an analytic solution U_ε, which is locally unique. Then, for every two-parameter family $\mathcal{F}_{\varepsilon,\mu}$, such that $\mathcal{F}_{\varepsilon,0} = \mathcal{F}_\varepsilon$, there exists a solution $U_{\varepsilon,\mu}$, which is analytic in the two parameters for arbitrarily small values of μ.

8.2.1. Proof of Lemma 8.3. Given a family of operators depending on two parameters $\mathcal{F}_{\varepsilon,\mu}$, we fix α, β and consider the one-parameter family defined as $\mathcal{G}_\varepsilon = \mathcal{F}_{\varepsilon,\alpha\varepsilon+\beta}$. By the hypothesis, if α, β are small, we can find a solution U_ε, so that $\mathcal{G}_\varepsilon(U_\varepsilon) = 0$.

Geometrically, if we let β vary, then the lines $(\varepsilon, \alpha\varepsilon + \beta)$ form a foliation. For a different value of α, we obtain a transversal foliation. The solution is analytic when we restrict it to the leaves of each of the two transversal foliations. Note that we are using the hypothesis of local uniqueness to conclude that the solutions for two families are the same.

Precisely, if we choose $\alpha_1 \neq \alpha_2$, we can consider a change of coordinates from (ε, μ) to β_1, β_2 given by

$$\alpha_1\varepsilon - \mu = \beta_1 \ , \qquad \alpha_2\varepsilon - \mu = \beta_2 \ , \tag{8.3}$$

which gives

$$\varepsilon = \frac{-\beta_1 + \beta_2}{-\alpha_1 + \alpha_2} \ , \qquad \mu = \frac{-\alpha_2\beta_1 + \alpha_1\beta_2}{-\alpha_1 + \alpha_2} \ .$$

By hypothesis the solution U_{β_1,β_2} of the two-parameter family $\mathcal{F}_{\varepsilon,\mu}$ is analytic in β_1 for β_2 fixed and in β_2 for β_1 fixed. This is the hypothesis of Hartog's theorem [Kra01, Nar71], so that we can conclude that the function U_{β_1,β_2} is jointly analytic in β_1, β_2 and, hence, it is jointly analytic in ε, μ.

If the operators act on infinite dimensional Banach spaces, we can reduce the proof to the classical result for complex valued functions by observing that we can apply Hartog's theorem to $\ell(U_\varepsilon)$, where ℓ is a linear functional from the Banach space to the complex. It is also well known [RS80, HP57] that functions that are analytic in this weak sense are strongly analytic. Alternatively, we could just note that the proof of Hartog's theorem [Kra01, Nar71] works for functions taking values in Banach spaces.

Remark 8.4. It is amusing to note that Lemma 8.3 allows one to improve the results of [Poi87a]. It immediately shows that if all systems in a neighborhood remained analytically integrable, any two-parameter family would be uniformly integrable. Hence, the obstructions to uniform integrability discovered by [Poi87a] show that we can get nonintegrable systems in any neighborhood. Of course, even if Poincaré was one of the creators of the theory of several complex variables, he did not know about Hartog's theorem. Under the extra assumption of uniform boundedness (which is not so unreasonable in the present case), the analogue of Hartog's theorem was presumably known.

Our next result shows that there are obstructions to the existence of solutions analytic in two variables in two-parameter families near resonances. This is an elementary application of power series matching. Notice that the argument works in the generality of mappings into Banach spaces, since it is really a soft argument which applies in many other contexts.

One subtlety is that if we consider the restricted class of families of operators as in (8.1), it can, in principle, happen that the obstructions vanish for the restricted family. So, when we consider restricted families such as (8.1), we will need to verify that the family is general enough to be affected by the obstructions.

LEMMA 8.5. *Consider the two-parameter family $\mathcal{F}_{\varepsilon,\mu}$. For some ε_0, assume that the following equation holds: $\mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0}) = 0$. If the range of $D\mathcal{F}_{\varepsilon_0,0}$ has codimension at least* 1*, then the space of families for which there is a solution is contained in a set of infinite codimension.*

Moreover, for the restricted families of the form (8.1)*, if we can find $U_{\varepsilon,0}$ solving $\mathcal{F}_{\varepsilon,0}(U_{\varepsilon,0}) = 0$, then there exists an arbitrarily small μ, such that the family $\mathcal{F}_{\varepsilon,\mu}$ does not have solutions close to $U_{\varepsilon,0}$, which are analytic near $\mu = 0$.*

Proof of Lemma 8.5. If there is a solution $U_{\varepsilon,\mu}$ of $\mathcal{F}_{\varepsilon,\mu}(U_{\varepsilon,\mu}) = 0$ analytic in μ, we should have

$$D\mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0})\ \partial_\mu U_{\varepsilon_0,0} + \partial_\mu\ \mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0}) = 0\ . \tag{8.4}$$

Clearly, if the perturbation is such that $\partial_\mu \mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0})$ is not in the range of $D\mathcal{F}_{\varepsilon_0,0}$, then there is no possibility of finding a solution of (8.4) and, a fortiori, no possibility of finding an analytic solution.

Of course, the families $\mathcal{F}_{\varepsilon,\mu}$ for which the first jet is in the range is a codimension one set of perturbations. Hence, the derived necessary conditions imply that the perturbation has to be in this set.

Obviously, the necessary condition above is not the only one. Indeed, one can obtain even more obstructions for the existence of another branch by considering higher order terms. Matching terms up to order N, we obtain that

$$D\mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0})(\partial_\mu)^N U_{\varepsilon_0,0} + (\partial_\mu)^N \mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0}) + R_N = 0\ , \tag{8.5}$$

where R_N is an expression involving only derivatives of order up to $N-1$.

Clearly, the fact that $R_N + (\partial_\mu)^N \mathcal{F}_{\varepsilon_0,0}(U_{\varepsilon_0,0})$ is in the range of $D\mathcal{F}_{\varepsilon_0,0}$ gives another obstruction for the perturbations.

If the range of $D\mathcal{F}_{\varepsilon_0,0}$ has codimension k, we claim that the set of families that matches the necessary conditions up to order N is a submanifold of codimension

Nk. In particular, the set of maps that satisfy all the obstructions is contained in a submanifold of infinite codimension, which becomes a very meager set in the sense of Baire category theory.

The proof of the first claim of the lemma is very easy. The key observation is that the obstruction at order N (see (8.5)) involves that a given expression is in the range of $D\mathcal{F}_{\varepsilon_0,0}$. This expression is very complicated in the coefficients of order up to $N-1$, but its dependence on the coefficient of order N is very simple. Hence, we obtain that for each of the functions satisfying the condition at order $N-1$, the obstruction at order N takes the form that the Nth derivative should be an explicit expression over all the previous ones plus the range of $D\mathcal{F}_{\varepsilon_0,0}$. Since the range of $D\mathcal{F}_{\varepsilon_0,0}$ has codimension k, this increments by k the codimension of the solution of (8.5). In the limit we obtain that the solution of (8.4) is contained in a set of infinite codimension.

The second claim of the lemma is obtained observing that for the families of operators as in (8.1), (8.4) gives restrictions to the derivatives $\partial_\mu \mathcal{F}_{\varepsilon_0,0}$. Then, we want to show that the range of $\partial_\mu \mathcal{F}_{\varepsilon_0,0}$ is in the complementary of the range of $D\mathcal{F}_{\varepsilon_0,0}$. Given that in the restricted family one has $\partial_\mu \mathcal{F}_{\varepsilon,\mu} = \partial_\mu A_\mu(U_{\varepsilon,\mu})$ and recalling that the set of A_μ has infinite codimension, we conclude that the range of $\partial_\mu \mathcal{F}_{\varepsilon_0,0}$ is not in the range of $D\mathcal{F}_{\varepsilon_0,0}$ and therefore the family $\mathcal{F}_{\varepsilon,\mu}$ does not admit solutions close to $U_{\varepsilon,0}$, which are analytic near $\mu = 0$. ∎

Now we are in position to finish the proof of Theorem 8.2. Consider the family $\mathcal{F}_\varepsilon$ of analytic operators and let ε_0 be a resonant value. Let $B \subset \mathbb{C}$ be a ball around ε_0. We assume that all the perturbations of the family admit an analytic solution that goes across the ball B. If indeed there were solutions for all perturbations, then using Lemma 8.3 the family should be analytic in two variables. However, near a resonance, which is contained in the ball B, by Lemma 8.5 we obtain that there are many perturbations for which this is impossible. Hence, we conclude that the assumption that there were solutions for all perturbations analytic in the ball B is false. This provides part (a) of Theorem 8.2.

We conclude by mentioning that part (b) of Theorem 8.2 is obtained from a straightforward implementation of the second statement of Lemma 8.5.

Note that the proof of Theorem 8.2 goes by contradiction. We started by assuming that all the systems gave solutions that were analytic in a ball and we concluded that they were not, except in a set of infinite codimension. This, of course, contradicts the hypothesis that for any perturbation, there are analytic solutions extending through a neighborhood of the resonant ε_0 and we conclude that there is one family which does not extend.

Unfortunately, this does not allow us to conclude anything beyond the fact that there are families which do not admit solutions that extend through the resonance. Once we conclude that the hypothesis fails, we cannot obtain any of the conclusions that we obtained from assuming its existence (and which we used to derive a contradiction). In particular, the argument does not allow us to conclude that the set of families which extend is infinite codimension. The infinite codimension statement was predicated on the fact that we had at least a solution.

If, indeed, we could show that the set of functions for which a solution extends through a resonance is infinite codimension, we could use the Baire category theorem to show that there is a residual set of families for which the analyticity domain does not include any resonance. Even if the argument above does not allow us to conclude that rigorously, we formulate the following conjecture.

Conjecture 8.6. *For an open and dense set of families (in the topology indicated above), there is no solution defined in a neighborhood of any of the resonances.*

Notice also that for the equations considered in (8.1), if we have a perturbative solution of the equation, the resonances of the perturbed equation have to be close to the resonances of N_ε. Hence, we also have the following conjecture.

Conjecture 8.7. *Consider the problem in* (8.1). *For an open and dense set of nonlinearities, the response solution has a singularity at a distance less than $C|\varepsilon|$ from the resonances of N_ε.*

We hope that, perhaps, the argument used in Theorem 8.2 can be strengthened to obtain Conjecture 8.7. There could also be other strategies to prove Theorem 8.2, which are direct and not just by contradiction. A more constructive argument could possibly take the form of observing that, near the resonances, one small change in the model leads to a very large change in the response function. Hence, one could hope to pile up perturbations of the model in such a way that the model remains well defined but that the response function breaks down.

To apply the above results to our models, one slightly delicate point is that the linearization depends on the solution U_ε. Arguing again by contradiction and in a nonrigorous way, we observe that we can compute the eigenvalues of $N_\varepsilon U_\varepsilon + \mu G(U_\varepsilon)$ by using a perturbative expansion as in [Kat76]. Even if a full proof will be complicated, one can imagine that the eigenvalues can be continued analytically in μ. Hence, the values of ε for which an eigenvalue vanishes will move continuously (of course, if there are some nondegeneracy assumptions, which is reasonable to conjecture hold generically) and they will move differentiably.

Hence, we are led to the following conjecture.

Conjecture 8.8. *For a generic family, we can find a constant C such that no ball of the form $\{\varepsilon :\ |\varepsilon - \varepsilon_0| \le C\varepsilon_0^2\}$, with ε_0 a resonance for N_ε, is completely contained in the domain of analyticity of the response function. In other words, for each of the balls as above, we can find a point not in the domain of analyticity.*

Appendix A. Some properties of $\mathcal{A}_{\rho,j,m}$.

A.1. Characterization of the norm in terms of the Fourier coefficients. Here we provide a norm equivalent to (3.18) which can be expressed in terms of the Fourier coefficients. In this way, it is easy to study the boundedness of operators which are diagonal in the Fourier basis (products of complex exponentials in θ and eigenfunctions of $\mathcal{L}$ in x). As before for two equivalent norms $\|\cdot\|$, $\|\cdot\|'$, we will write $\|\cdot\| \cong \|\cdot\|'$.

Proposition A.1. *Let $u \in \mathcal{A}_{\rho,j,m}$ have a Fourier expansion as in* (3.17). *We have*

$$\|u\|_{\rho,j,m} \cong \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{e^{4\pi|k|\rho}}{B(k,\rho)} ((2\pi)^d\ |k|^2+1)^j\ \|\hat u_k\|^2_{H^m_{\mathcal{L}}} + \frac{\|\hat u_0\|^2_{H^m_{\mathcal{L}}}}{B(0,\rho)} \right)^{1/2}$$
$$= \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}, n\in\mathbb{N}} \frac{e^{4\pi|k|\rho}}{B(k,\rho)} ((2\pi)^d\ |k|^2+1)^j |\lambda_n|^m |\hat u_{k,n}|^2 + \frac{|\lambda_n|^m}{B(0,\rho)} |\hat u_{0,n}|^2 \right)^{1/2},$$

where for $k \in \mathbb{Z}^d$ we denote $|k| \equiv |k_1| + \cdots + |k_d|$ and

$$B(k, \rho) \equiv \prod_{j=1}^{d} a(k_j, \rho) \ , \qquad a(j, \rho) \equiv \begin{cases} 4\pi|j| & \text{if } j \neq 0, \\ \frac{1}{4\pi\rho} & \text{if } j = 0. \end{cases}$$

Proof. For $d = 1$ we have that for $k \neq 0$

$$\int_{\mathbb{T}_\rho} |e^{2\pi i k\theta}|^2 d^2\theta = \int_{|\operatorname{Im}(\theta)| \leq \rho} e^{-4\pi k \operatorname{Im}\theta} \ d(\operatorname{Im}(\theta)) = \frac{e^{4\pi|k|\rho} - e^{-4\pi|k|\rho}}{a(k, \rho)} \ .$$

Of course, when $k = 0$, the integral is just $4\pi\rho$. For any $k \neq 0$ we have that the integral can be bounded as

$$C_- \ \frac{e^{4\pi|k|\rho}}{a(k, \rho)} \leq \int_{\mathbb{T}_\rho} |e^{2\pi i k\theta}|^2 d^2\theta \leq C_+ \ \frac{e^{4\pi|k|\rho}}{a(k, \rho)} \tag{A.1}$$

with $C_- \equiv 1 - e^{-4\pi\rho}$ and $C_+ \equiv 2$. For $d \geq 2$ we can use Fubini's theorem, applying (A.1) to each factor and obtaining the following inequalities:

$$C_-^d \ \frac{e^{4\pi|k|\rho}}{B(k, \rho)} \leq \int_{\mathbb{T}_\rho^d} |e^{2\pi i k\cdot\theta}|^2 d^{2d}\theta \leq C_+^d \ \frac{e^{4\pi|k|\rho}}{B(k, \rho)} \ .$$

Finally, we note that the Laplacian is diagonal in the exponentials, so that

$$|(1 - \nabla_\theta^2)^{j/2} e^{2\pi i k\cdot\theta}|^2 = ((2\pi)^d \, |k|_2^2 + 1)^j |e^{2\pi i k\cdot\theta}|^2 \ ,$$

where $|k|_2$ denotes the Euclidean norm.

Since the exponentials are orthogonal with respect to the L^2 inner product, we obtain

$$\int_{\mathbb{T}_\rho^d} \|(1 - \nabla_\theta^2)^{\frac{j}{2}} u(\theta, \cdot)\|_{H_\mathcal{L}^m}^2 \ d^{2d}\theta \cong \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{4\pi|k|\rho}}{B(k, \rho)} ((2\pi)^d \, |k|_2^2 + 1)^j \|\hat{u}_k\|_{H_\mathcal{L}^m}^2 + \frac{\|\hat{u}_0\|_{H_\mathcal{L}^m}^2}{B(0, \rho)} \ .$$

This concludes the proof. ∎

Remark A.2. Since the Euclidean norm $|k|_2$ in d dimensions is equivalent to the ℓ^1-norm $|k|$, we can substitute $|k|_2$ for $|k|$ in the polynomial factor and obtain an equivalent norm in the Banach space. On the other hand, if we change the ℓ^1-norm of $|k|$ in the argument of the exponential by the ℓ^2-norm, we obtain a nonequivalent norm in the space of analytic functions.

The space $\mathcal{A}_{\rho,j,m}$ is a closed subspace of the Sobolev space of maps from the $2d$-dimensional manifold $\mathbb{T}_\rho^d$ into the Banach algebra $H_\mathcal{L}^m$ (remember that we are assuming $m > \ell/2$, so that $H_\mathcal{L}^m$ is a Banach algebra under multiplication).

In particular, since $\mathbb{T}_\rho^d$ is a $2d$-dimensional real manifold, the Sobolev embedding theorem implies

$$\|u\|_{L^\infty(\mathbb{T}_\rho^d; H_\mathcal{L}^m)} \leq C\|u\|_{H^j(\mathbb{T}_\rho^d; H_\mathcal{L}^m)} \ ,$$

whenever $j > d$ for some constant $C > 0$. Hence, the convergence in $H^j(\mathbb{T}_\rho^d; H_\mathcal{L}^m)$ implies the uniform convergence.

It is well known that uniform limits of analytic functions are analytic, so that under the above bounds, it is classical that $\mathcal{A}_{\rho,j,m}$ is a closed subspace. The fact that $\mathcal{A}_{\rho,j,m}$ is a closed subspace of $H^j(\mathbb{T}_\rho^d; H_\mathcal{L}^m)$ is also true when $j \leq d$ (see [RS80]).

Summarizing we have the following propositions.

Proposition A.3. *Let $\rho > 0$, $j, m \in 2\mathbb{N}$. The space $\mathcal{A}_{\rho,j,m}$ of functions analytic in θ, endowed with the norm given in* (3.18)*, is a Banach algebra under multiplication, when $j > d$, $m > \ell/2$.*

Proposition A.4. *If we have a linear operator $\mathcal{M}$ which is diagonal in the Fourier basis, say,*

$$\mathcal{M}(e^{2\pi i k\cdot\theta}\Phi_n(x)) = \lambda_{n,k}\ e^{2\pi i k\cdot\theta}\Phi_n(x)\ ,$$

for suitable coefficients $\lambda_{n,k}$, then we have

$$\|\mathcal{M}\|_{\rho,j,m} \le C \sup_{n,k} |\lambda_{n,k}|\ . \tag{A.2}$$

The bound (A.2) is immediate since the operator $\mathcal{M}$ is diagonal in the Fourier basis and the norm squared is just the sum of the Fourier coefficients squared.

A.2. Analytic functions from a Banach space to another. Now we present some analyticity properties of nonlinear functions in Banach spaces (the literature on this subject is very wide; see, for example, [HP57, Muj86]). In analogy with the finite dimensional case, there are results which show that some weak definitions such as differentiability imply stronger ones (convergence of Taylor series around every point). In the infinite dimensional cases the results are more subtle since there are different notions of differentiability and different notions of convergence of power series, but it is true that extremely weak notions turn out to be equivalent to the strongest one [HP57, Chapter III].

For our purposes, we only need to apply an easy implicit function theorem and to study the analyticity properties of the nonlinear operator G defined in (3.6). The Banach spaces in which G acts are the spaces defined in section 3.5.2.

The following definition of analytic functions will be enough for us.

Definition A.5. *Let X and Y be complex Banach spaces. We say that $f : \Omega \subset X \to Y$ is analytic if it is uniformly differentiable at all points of Ω, namely, the derivative is uniformly bounded and there exists a function $\gamma = \gamma(|z|)$, with $\gamma(|z|) \to 0$ as $|z| \to 0$, such that the following uniform bound holds:*

$$\|f(x + z\zeta) - f(x) - Df(x)\ z\zeta\| \le \gamma(|z|) \tag{A.3}$$

for all $x \in \Omega$, $\zeta \in X$ with $|\zeta| = 1$, $z \in \mathbb{C}$.

Remark A.6. It is clear that Definition A.5 is a rather weak notion of differentiability. However, it is a remarkable fact (see [HP57, Theorem 3.17.1]) that Definition A.5 is equivalent to requiring that for every x the function f has a Taylor expansion of the form

$$f(x + \zeta) = \sum_\alpha \frac{1}{\alpha!}\ \partial^\alpha f(x)\zeta^\alpha$$

that converges uniformly for $\|\zeta\| \le M_x$ for some $M_x > 0$ (indeed, $\sum_\alpha \|\frac{1}{\alpha!}\partial^\alpha f(x)\| M_x^{|\alpha|} < \infty$).

It is also true that even weaker notions of differentiability imply (A.3). For our purposes, Definition A.5 will be enough, since it allows us to apply a contraction mapping argument. We refer to [HP57, Chapter III], [Muj86, Her71] for other definitions of analyticity that turn out to be equivalent.

To make sense of the composition operator which given u produces $h(u, \cdot)$, we need an analytic extension of the nonlinearity h to the complex plane with respect to its first argument.

PROPOSITION A.7. *Let $h : \mathcal{B} \times \overline{\mathcal{D}} \to \mathbb{C}$ with $\mathcal{B}$ an open set in $\mathbb{C}$ and assume that h is analytic in $u \in \mathcal{B}$ and it is $C^m(\mathcal{D}) \cap C(\overline{\mathcal{D}})$.*

Assume that $\partial_u^{\alpha_1} \partial_x^{\alpha_2} h(u, x)$ are bounded in $\mathcal{B} \times \mathcal{D}$ for $\alpha_1 + \alpha_2 \leq m$. We denote by ∂_u the complex derivatives and we assume that $m > \ell/2$.

Let $u_0 \in \mathcal{A}_{\rho,j,m}$ be such that $\operatorname{dist}(u_0(\mathbb{T}^d_\rho, \overline{\mathcal{D}}), \mathbb{C} \setminus \mathcal{B}) > a_0$ with $a_0 > 0$.

Then, for all u in a neighborhood $\mathcal{U}$ of u_0 in $\mathcal{A}_{\rho,j,m}$, we can define the operator

$$\mathcal{C}_h[u](\theta, x) = h(u(\theta, x), x)$$

from $\mathcal{A}_{\rho,j,m}$ to itself, which is an analytic operator in the sense of Definition A.5.

Moreover, for $v \in \mathcal{A}_{\rho,j,m}$ the derivative of $\mathcal{C}_h$ is given by

$$(D\mathcal{C}_h[u]v)(\theta, x) = h'(u(\theta, x), x)v(\theta, x) \ ,$$

where by h' we denote the complex derivative of h with respect to its first argument, namely, $h'(u, x) = (\partial_1 h)(u, x)$ (in the proof h'' will denote the second derivative of h with respect to its first argument).

Proof. The proof is rather straightforward, but it requires that we interpret some elementary calculations (the Taylor theorem up to order two with remainder) in different levels of abstraction.

Because of Sobolev's embedding theorem we have that the functions u, v are bounded and continuous, so that, for fixed θ, x and for fixed u, v, we can think of $u(\theta, x), v(\theta, x)$ as numbers and assume that $|v(\theta, x)|$ is so small that $u(\theta, x) + s\, v(\theta, x)$ is in the domain of h for $s \in [0, 1]$.

The fundamental theorem of calculus implies that for all $\theta \in \mathbb{T}^d_\rho, x \in \mathcal{D}$ and for all $u \in \mathcal{U}$ and all $v \in \mathcal{A}_{\rho,j,m}$, we have

$$\begin{aligned} h(u(\theta, x) + v(\theta, x), x) &= h(u(\theta, x), x) + \int_0^1 h'(u(\theta, x) + s\, v(\theta, x), x) v(\theta, x)\; ds \\ &= h(u(\theta, x), x) + h'(u(\theta, x), x)v(\theta, x) \\ &\quad + \int_0^1 \int_0^s h''(u(\theta, x) + s\ t\ v(\theta, x), x)\ v(\theta, x)^2 \ \ dt\ ds \ . \end{aligned} \tag{A.4}$$

Now we interpret the formula (A.4) as an equality in function spaces.

By standard Gagliardo–Nirenberg–Moser composition estimates in Sobolev spaces [Tay11b, Proposition 3.9], we have that for some $C_\alpha > 0$ depending on the norm of u, one has

$$\|D^\alpha h(u, x)\|_{H^j(\mathbb{T}^d_\rho; H^m_{\mathcal{L}})} \leq C_\alpha(\|u\|_{L^\infty(\mathbb{T}^d_\rho; \mathcal{D})})\ (1 + \|u\|_{H^j(\mathbb{T}^d_\rho; H^m_{\mathcal{L}})}) \ .$$

It is also easy to check that if $u(\theta, x), v(\theta, x)$ are complex differentiable in θ and h is also differentiable in θ, we obtain that $h''(u(\theta, x) + s\ t\ v(\theta, x), x)$ is a function in $\mathcal{A}_{\rho,j,m}$ with uniform bounds.

Using the Banach algebra properties, we obtain indeed the desired result, since we can bound the integral by a constant times $\|v\|^2_{\rho,j,m}$ in the last term of (A.4). ∎

Appendix B. Solution of the zeroth order term for models A and A′.

For models A and A′ the zeroth order term c_0 must satisfy (3.8). The literature on the solution of (3.8) is very wide and the results strongly depend on the form of the nonlinearity. To be concrete, we quote—among others—a result on the existence of weak solutions (see Proposition B.1), which are indeed regular solutions as noticed in Remark B.2 below, and a result on the existence of an unbounded sequence of solutions if h is odd (see Proposition B.3). To fix the notation, let $f_0(x) \equiv \langle f(\theta, x)\rangle$.

PROPOSITION B.1 (see [Pre13, Theorem 9.7]). *Let $\mathcal{D} \subset \mathbb{R}^\ell$, $\ell \geq 3$, be a bounded open set, $f \in H^{-1}(\mathcal{D})$; let $\underline{u}$, $\overline{u}$ be, respectively, a lower and an upper*[3] *solution of* (3.8) *with D boundary conditions and with $\underline{u}(x) \leq \overline{u}(x)$ for almost every $x \in \mathcal{D}$. If the function $h : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions*[4] *and is increasing in its first variable, i.e.,*

$$h(u_1, x) \leq h(u_2, x)$$

with $\underline{u}(x) \leq u_1 \leq u_2 \leq \overline{u}(x)$ for almost every $x \in \mathcal{D}$, then (3.8) *with D boundary conditions has at least one weak solution $u \in H_0^1(\mathcal{D})$, satisfying $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for almost every $x \in \mathcal{D}$.*

Remark B.2. For the models we are considering, it can be proved that all the weak solutions are smooth: under regularity conditions on the coefficients defining the elliptic operator Δ_x, a weak solution is regular. We refer the reader to Chapter 4 in [LU68] and to [Agm10, Eva10].

Assuming that h is odd, one obtains an infinite number of solutions as shown by the following result.

PROPOSITION B.3 (see [Str08, Theorem 7.2]). *Let $\mathcal{D} \subset \mathbb{R}^\ell$, $\ell \geq 3$, be a smoothly bounded domain. Assume that*

(i) *$h : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is continuous and odd with primitive $h_p(x, u) = \int_0^u h(x, v)dv$;*
(ii) *there exists $p < 2\ell/(\ell - 2)$ and $C > 0$, such that*

$$|h(x, u)| \leq C(1 + |u|^{p-1})$$

almost everywhere;

(iii) *there exists $q > 2$, $R_0 > 0$ such that*

$$0 < q\ h_p(x, v) \leq h(x, v)\ v$$

for almost every x, $|v| \geq R_0$;

(iv) *the quantities p and q satisfy*

$$\frac{2p}{\ell(p-2)} - 1 > \frac{q}{q-1}\ .$$

Then, for any $f_0 \in L^2(\mathcal{D})$, (3.8) *with $c_0(x) = 0$ on the boundary of $\mathcal{D}$ has an unbounded sequence of solutions $c_{0k} \in H_0^{1,2}(\mathcal{D})$, $k \in \mathbb{N}$.*

Remark B.4. Notice that Proposition B.3 only addresses the existence of solutions of (3.8). Of course, we also need to verify that these solutions satisfy H1, H2. Since c_0 are smooth because of the regularity theory of elliptic equations, we obtain that H1 is automatic. H2 holds in a codimension one family of problems. Since there are infinitely many functions c_0, we expect that for some of them H2 holds *generically*.

[3]By a lower (upper) solution of problem (3.8), we mean a function $\underline{u}\,(\overline{u}) \in H^1(\mathcal{D})$, such that $h(\underline{u}, \cdot)\,(h(\overline{u}, \cdot)) \in L^{\frac{2\ell}{\ell+2}}(\mathcal{D})$ satisfies $-\Delta_x \underline{u}(x) + h(\underline{u}(x), x) \leq \langle f\rangle(x) (-\Delta_x \overline{u}(x) + h(\overline{u}(x), x) \geq \langle f\rangle(x))$ and $\underline{u}(x) \leq 0$ ($\overline{u}(x) \geq 0$) when $x \in \partial\mathcal{D}$.

[4]A function $h : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition if $h(y, \cdot) : \mathcal{D} \to \mathbb{R}$ is measurable for every $y \in \mathbb{R}$ and $h(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \mathcal{D}$.

Remark B.5.

(i) In the above results, Δ_x can typically be any second order uniformly elliptic operator with smooth coefficients (compare with [AM07]).

(ii) The multiplicity of solutions can be studied using the Lusternik–Schnirelman theory, which allows one to find critical points of the variational functional on a given manifold. The number of critical points is related to the Lusternik–Schnirelman category of the manifold, for which a lower bound is provided by the cup length of the manifold [AM07].

(iii) In our discussion we considered D boundary conditions; results are available also for N boundary conditions (see, e.g., [TW05, Tan02, WT06]) or can be extended to P boundary conditions.

Acknowledgments. We thank the referees for a very thorough reading and constructive suggestions that improved the paper. In particular, one of the referees pointed out that an older version of hypothesis H1 was unduly restrictive. Part of this work was carried out while the first author was a visiting assistant professor in the School of Mathematics at the Georgia Institute of Technology. The first author thanks the late G. Flores for useful discussions and for sharing a preprint version of [Flo14]. The second author is grateful to S. Terracini for several discussions. Part of this work was done as the third author was a Canada Research Chair Postdoctoral fellow at McMaster University.

REFERENCES

[Agm10] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, AMS, Providence, RI, 2010.

[AM07] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Stud. Adv. Math., 104, Cambridge University Press, Cambridge, UK, 2007.

[Bal94] W. Balser, *From Divergent Power Series to Analytic Functions*: Theory and Application of Multisummable Power Series, Lecture Notes in Math. 1582, Springer-Verlag, Berlin, 1994.

[Ber73] M. S. Berger, *Applications of global analysis to specific nonlinear eigenvalue problems*, Rocky Mountain J. Math., 3 (1973), pp. 319–354.

[Ber08] C. Bereanu, *An Ambrosetti-Prodi-type result for periodic solutions of the telegraph equation*, Proc. Roy. Soc. Edinburgh Sect. A, 138 (2008), pp. 719–724.

[BFGS12] I. Baldomá, E. Fontich, M. Guardia, and T. M. Seara, *Exponentially small splitting of separatrices beyond Melnikov analysis: Rigorous results*, J. Differential Equations, 253 (2012), pp. 3304–3439.

[BLS02] W. Balser, D. A. Lutz, and R. Schäfke, *On the convergence of Borel approximants*, J. Dyn. Control Syst., 8 (2002), pp, 65–92.

[CCdlL13] R. C. Calleja, A. Celletti, and R. de la Llave, *Construction of response functions in forced strongly dissipative systems*, Discrete Contin. Dyn. Syst., 33 (2013), pp. 4411–4433.

[CFG13] L. Corsi, R. Feola, and G. Gentile, *Domains of analyticity for response solutions in strongly dissipative forced systems*, J. Math. Phys., 54 (2013), 122701.

[CFG14] L. Corsi, R. Feola, and G. Gentile, *Convergent series for quasi-periodically forced strongly dissipative systems*, Commun. Contemp. Math., 16 (2014), 1350022.

[CG12] L. Corsi and G. Gentile, *Oscillator synchronisation under arbitrary quasi-periodic forcing*, Comm. Math. Phys., 316 (2012), pp. 489–529.

[CGV05] V. V. Chepyzhov, A. Yu. Goritsky, and M. I. Vishik, *Integral manifolds and attractors with exponential rate for nonautonomous hyperbolic equations with dissipation*, Russ. J. Math. Phys., 12 (2005), pp. 17–39.

[Cra83] W. Craig, *A bifurcation theory for periodic solutions of nonlinear dissipative hyperbolic equations*, Ann. Sc. Norm. Sup. Pisa Cl. Sci., (4), 10 (1983), pp. 125–167.

[CW15] S. Cai and Y. Watanabe, *A computer-assisted method for excluding eigenvalues of an elliptic operator linearized at a solution of a nonlinear problem*, Jpn. J. Ind. Appl. Math., 32 (2015), pp. 263–294.

[dlL96] R. de la Llave, *On necessary and sufficient conditions for uniform integrability of families of Hamiltonian systems*, in International Conference on Dynamical Systems (Montevideo, 1995), Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996, pp. 76–109.

[Eva10] L. C. Evans, *Partial Differential Equations*, 2nd ed., Grad. Stud. Math., 19, AMS, Providence, RI, 2010.

[FG95] I. Fonseca and W. Gangbo, *Degree Theory in Analysis and Applications*, Oxford Lecture Ser. Math. Appl., 2, Oxford University Press, New York, 1995.

[Flo14] G. Flores, *Dynamics of a damped wave equation arising from MEMS*, SIAM J. Appl. Math., 74 (2014), pp. 1025–1035.

[FMPS07] G. Flores, G. Mercado, J. A. Pelesko, and N. Smyth, *Analysis of the dynamics and touchdown in a model of electrostatic MEMS*, SIAM J. Appl. Math., 67 (2007), pp. 434–446.

[GBD05] G. Gentile, M. V. Bartuccelli, and J. H. B. Deane, *Summation of divergent series and Borel summability for strongly dissipative differential equations with periodic or quasiperiodic forcing terms*, J. Math. Phys., 46 (2005), 062704.

[GBD06] G. Gentile, M. V. Bartuccelli, and J. H. B. Deane, *Quasiperiodic attractors, Borel summability and the Bryuno condition for strongly dissipative systems*, J. Math. Phys., 47 (2006), 072702.

[Gen10a] G. Gentile, *Quasi-periodic motions in strongly dissipative forced systems*, Ergodic Theory Dynam. Systems, 30 (2010), pp. 1457–1469.

[Gen10b] G. Gentile, *Quasiperiodic motions in dynamical systems: Review of a renormalization group approach*, J. Math. Phys., 51 (2010), 015207.

[GGG12] C. G. Gal, S. G. Gal, and J. A. Goldstein, *Wave and telegraph equations with real time variable and complex spatial variables*, Complex Var. Elliptic Equ., 57 (2012), pp. 91–109.

[Har49] G. H. Hardy, *Divergent Series*, AMS, Providence, RI, 1949.

[Her71] M. Hervé, *Analytic and Plurisubharmonic Functions in Finite and Infinite Dimensional Spaces*, Lecture Notes in Math., 198, Springer-Verlag, Berlin, 1971.

[Hil78] G. W. Hill, *Researches in the lunar theory*, Amer. J. Math., 1 (1878), pp. 5–26.

[HP57] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. 31, AMS, Providence, RI, 1957.

[Kat76] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Grundlehren Math. Wiss., 132, Springer Berlin, 1976.

[Kra01] S. G. Krantz, *Function Theory of Several Complex Variables*, AMS, Providence, RI, 2001.

[LU68] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[Maw05] J. Mawhin, *Maximum principle for bounded solutions of the telegraph equation: The case of high dimensions*, in Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl., 63, Birkhäuser, Basel, 2005, pp. 343–351.

[MORP05] J. Mawhin, R. Ortega, and A. M. Robles-Pérez, *Maximum principles for bounded solutions of the telegraph equation in space dimensions two and three and applications*, J. Differential Equations, 208 (2005), pp. 42–63.

[Muj86] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Math. Stud., 120, North-Holland, Amsterdam, 1986.

[Nar71] R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Math., University of Chicago Press, Chicago, IL, 1971.

[Nev19] F. Nevanlinna, Zur theorie der Asymptotischen Potenzreihen, Ann. Acad. Sci. Fenn. Ser. A, 12 (1916), pp. 1–12.

[NN01] K. Nagatou and M. T. Nakao, *An enclosure method of eigenvalues for the elliptic operator linearized at an exact solution of nonlinear problems*, Linear Algebra Appl., 324 (2001), pp. 81–106.

[Ort05] R. Ortega, *Stability and index of periodic solutions of a nonlinear telegraph equation*, Commun. Pure Appl. Anal., 4 (2005), pp. 823–837.

[PL08] L. E. Phragmén and E. Lindelöf, *Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogénes dans le voisinage d'un point singulier*, Acta Math., 31 (1908).

[Poi87a] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste. Tome* I, Librairie Scientifique et Technique Albert Blanchard, Paris, 1987.

[Poi87b] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste. Tome* II, Grands Classiques Gauthier-Villars, Librairie Scientifique et Technique Albert Blanchard, Paris, 1987.

[Pre13] R. Precup, *Linear and Semilinear Partial Differential Equations: An Introduction*, Walter de Gruyter, Berlin, 2013.

[PSM09] M. Pellicer and J. Solà-Morales, *Optimal decay rates and the selfadjoint property in overdamped systems*, J. Differential Equations, 246 (2009), pp. 2813–2828.

[Rab67] P. H. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math., 20 (1967), pp. 145–205.

[Rab68] P. H. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations, Vol.* II, Comm. Pure Appl. Math., 22 (1968), pp. 15–39.

[Rou15] C. Rousseau, *Divergent Series: Past, Present, Future*, preprint, arXiv:1312.5712, 2015.

[RS80] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol.* I, 2nd ed., Academic Press, New York, 1980.

[Sok80] A. D. Sokal, *An improvement of Watson's theorem on Borel summability*, J. Math. Phys., 21 (1980), pp. 261–263.

[Str08] M. Struwe, *Variational Methods*, Ergeb. Math. Grenzgeb (3) 34, Springer-Verlag, Berlin, 2008.

[SZ65] S. Saks and A. Zygmund, *Analytic Functions*, Monogr. Mat. 28., Państwowe Wydawnietwo Naukowe, Warsaw, 1965.

[Tan02] C.-Lei Tang, *Some existence theorem for the sublinear Neumann boundary value problem*, Nonlinear Anal., 48 (2002), pp. 1003–1011.

[Tay11a] M. E. Taylor, *Partial Differential Equations* II. *Qualitative Studies of Linear Equations*, 2nd ed., Appl. Math. Sci., 116, Springer, New York, 2011.

[Tay11b] M. E. Taylor, *Partial Differential Equations* III*, 2nd ed., Nonlinear Equations*, *Appl. Math. Sci.*, 117, Springer, New York, 2011.

[TW05] C.-L. Tang and X.-P. Wu, *Multiple solutions of a class of Neumann problem for semilinear elliptic equations*, Nonlinear Anal., 62 (2005), pp. 455–465.

[WT06] X. Wu and K.-K. Tan, *On existence and multiplicity of solutions of Neumann boundary value problems for quasi-linear elliptic equations*, Nonlinear Anal., 65 (2006), pp. 1334–1347.