



On complete intersections containing a linear subspace

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Abstract

Consider the Fano scheme $F_k(Y)$ parameterizing k -dimensional linear subspaces contained in a complete intersection $Y \subset \mathbb{P}^m$ of multi-degree $\underline{d} = (d_1, \dots, d_s)$. It is known that, if $t := \sum_{i=1}^s \binom{d_i+k}{k} - (k+1)(m-k) \leq 0$ and $\prod_{i=1}^s d_i > 2$, for Y a general complete intersection as above, then $F_k(Y)$ has dimension $-t$. In this paper we consider the case $t > 0$. Then the locus $W_{\underline{d},k}$ of all complete intersections as above containing a k -dimensional linear subspace is irreducible and turns out to have codimension t in the parameter space of all complete intersections with the given multi-degree. Moreover, we prove that for general $[Y] \in W_{\underline{d},k}$ the scheme $F_k(Y)$ is zero-dimensional of length one. This implies that $W_{\underline{d},k}$ is rational.

Keywords Linear subspaces · Complete intersections · Fano schemes · Enumerative results · Rationality results

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1 Introduction

In this paper we will be concerned with the *Fano scheme* $F_k(Y)$, parameterizing k -dimensional linear subspaces contained in a subvariety $Y \subset \mathbb{P}^m$, when Y is a complete intersection of multi-degree $\underline{d} = (d_1, \dots, d_s)$, with $1 \leq s \leq m - 2$. We will assume that Y is neither a linear subspace nor a quadric, cases to be considered as trivial. Thus we will constantly assume that $\prod_{i=1}^s d_i > 2$.

Let $S_{\underline{d}} := \bigoplus_{i=1}^s H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i))$, and consider its Zariski open subset $S_{\underline{d}}^* := \bigoplus_{i=1}^s (H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d_i)) \setminus \{0\})$. For any $u := (g_1, \dots, g_s) \in S_{\underline{d}}^*$, let $Y_u := V(g_1, \dots, g_s) \subset \mathbb{P}^m$ denote the closed subscheme defined by the vanishing of the polynomials g_1, \dots, g_s . When $u \in S_{\underline{d}}^*$ is general, Y_u is a smooth, irreducible variety of dimension $m - s \geq 2$, so that $S_{\underline{d}}^*$ contains an open dense subset parametrizing s -tuples u such that Y_u is a smooth complete intersection. For any integer $k \geq 1$, we define the locus

$$W_{\underline{d},k} := \left\{ u \in S_{\underline{d}}^* \mid F_k(Y_u) \neq \emptyset \right\} \subseteq S_{\underline{d}}^*$$

and set

$$t(m, k, \underline{d}) := \sum_{i=1}^s \binom{d_i + k}{k} - (k + 1)(m - k).$$

If no confusion arises, we will simply denote $t(m, k, \underline{d})$ by t .

First of all, consider the case $t \leq 0$. This is the most studied case in the literature, and it is now well understood (cf. e.g. [2,3,6,7]). In particular, the following holds.

Result 1 *Let m, k, s and $\underline{d} = (d_1, \dots, d_s)$ be such that $\prod_{i=1}^s d_i > 2$ and $t \leq 0$. Then:*

- (a) $W_{\underline{d},k} = S_{\underline{d}}^*$;
- (b) for general $u \in S_{\underline{d}}^*$, $F_k(Y_u)$ is smooth, of dimension $\dim(F_k(Y_u)) = -t$ and it is irreducible when $\dim(F_k(Y_u)) \geq 1$.

The proof of this result can be found e.g. in [2, Proposition 2.1, Corollary 2.2, Theorem 4.1], for the complex case, and in [3, Theorem 2.1, (b) & (c)], for any algebraically closed field. In addition, in [3, Theorem 4.3] the authors compute $\deg(F_k(Y_u))$ under the Plücker embedding $F_k(Y_u) \subset \mathbb{G}(k, m) \hookrightarrow \mathbb{P}^N$, with $N = \binom{m+1}{k+1} - 1$. Their formulas extend to any $k \geq 1$ enumerative formulas by Libgober in [4], who computed $\deg(F_1(Y_u))$ when $t(m, 1, \underline{d}) = 0$.

On the other hand, we are interested in the case $t > 0$, where the known results can be summarized as follows.

Result 2 *Let m, k, s and $\underline{d} = (d_1, \dots, d_s)$ be such that $\prod_{i=1}^s d_i > 2$ and $t > 0$. Then:*

- (a) $W_{\underline{d},k} \subsetneq S_{\underline{d}}^*$;
- (b) $W_{\underline{d},k}$ contains points u for which $Y_u \subset \mathbb{P}^m$ is a smooth complete intersection of dimension $m - s$ if and only if $s \leq m - 2k$.
- (c) For $s \leq m - 2k$, set $H_{\underline{d},k} := \{ u \in W_{\underline{d},k} \mid Y_u \subset \mathbb{P}^m \text{ is smooth, of dimension } m - s \}$. If $d_i \geq 2$ for any $1 \leq i \leq s$, then $H_{\underline{d},k}$ is irreducible, unirational and $\text{codim}_{S_{\underline{d}}^*}(H_{\underline{d},k}) = t$. Moreover, for general $u \in H_{\underline{d},k}$, $F_k(Y_u)$ is a zero-dimensional scheme.

The proof of Result 2 (a) is contained in [3, Theorem 2.1 (a)], whereas that of assertions (b) and (c) is contained in [5, Corollary 1.2, Remark 3.4]; both proofs therein hold for any algebraically closed field.

The main result of this paper, which improves on Result 2, is the following.

Theorem 1.1 *Let m, k, s and $\underline{d} = (d_1, \dots, d_s)$ be such that $\prod_{i=1}^s d_i > 2$ and $t > 0$. Then $W_{\underline{d},k} \subsetneq S_{\underline{d}}^*$ is non-empty, irreducible and rational, with $\text{codim}_{S_{\underline{d}}^*}(W_{\underline{d},k}) = t$. Furthermore, for a general point $u \in W_{\underline{d},k}$, the variety $Y_u \subset \mathbb{P}^m$ is a complete intersection of dimension $m - s$ whose Fano scheme $F_k(Y_u)$ is a zero-dimensional scheme of length one. Moreover, Y_u has singular locus of dimension $\max\{-1, 2k + s - m - 1\}$ along its unique k -dimensional linear subspace (in particular Y_u is smooth if and only if $m - s \geq 2k$).*

The proof of this theorem is contained in Sect. 2 and it extends [1, Proposition 2.3] to arbitrary $k \geq 1$. Theorem 1.1 improves, via different and easier methods, Miyazaki’s results in [5, Corollary 1.2], showing that for general $u \in W_{\underline{d},k}$ one has $\text{deg}(F_k(Y_u)) = 1$, which implies the rationality of $W_{\underline{d},k}$. Moreover we also get rid of Miyazaki’s hypothesis $m - s \geq 2k$.

2 The proof

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $\mathbb{G} := \mathbb{G}(k, m)$ be the Grassmannian of k -linear subspaces in \mathbb{P}^m and consider the incidence correspondence

$$J := \left\{ ([\Pi], u) \in \mathbb{G} \times S_{\underline{d}}^* \mid \Pi \subset Y_u \right\} \subset \mathbb{G} \times S_{\underline{d}}^*$$

with the two projections

$$\mathbb{G} \xleftarrow{\pi_1} J \xrightarrow{\pi_2} S_{\underline{d}}^*.$$

The map $\pi_1 : J \rightarrow \mathbb{G}$ is surjective and, for any $[\Pi] \in \mathbb{G}$, one has $\pi_1^{-1}([\Pi]) = \bigoplus_{i=1}^s (H^0(\mathcal{I}_{\Pi/\mathbb{P}^m}(d_i)) \setminus \{0\})$, where $\mathcal{I}_{\Pi/\mathbb{P}^m}$ denotes the ideal sheaf of Π in \mathbb{P}^m .

Thus J is irreducible with $\dim(J) = \dim(\mathbb{G}) + \dim(\pi_1^{-1}([\Pi])) = (k + 1)(m - k) + \sum_{i=1}^s h^0(\mathcal{I}_{\Pi/\mathbb{P}^m}(d_i))$. From the exact sequence

$$0 \rightarrow \bigoplus_{i=1}^s \mathcal{I}_{\Pi/\mathbb{P}^m}(d_i) \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^m}(d_i) \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{\Pi}(d_i) \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^k}(d_i) \rightarrow 0, \tag{2.1}$$

one gets

$$\dim(J) = (k + 1)(m - k) + \sum_{i=1}^s \binom{d_i + m}{m} - \sum_{i=1}^s \binom{d_i + k}{k} = \dim(S_{\underline{d}}^*) - t. \tag{2.2}$$

The next step recovers [5, Corollary 1.2] via different and easier methods, and we also get rid of the hypothesis $m - s \geq 2k$ present there. We essentially adapt the argument in [2, Proof of Proposition 2.1], used for the case $t \leq 0$.

Step 1 *The map $\pi_2 : J \rightarrow S_{\underline{d}}^*$ is generically finite onto its image $W_{\underline{d},k}$, which is therefore irreducible and unirational. Moreover $\text{codim}_{S_{\underline{d}}^*}(W_{\underline{d},k}) = t$.*

For general $u \in W_{\underline{d},k}$, $F_k(Y_u)$ is a zero-dimensional scheme and Y_u has singular locus of dimension $\max\{-1, 2k + s - m - 1\}$ along any of the k -dimensional linear subspaces in $F_k(Y_u)$.

Proof of Step 1 One has $W_{\underline{d},k} = \pi_2(J)$, hence $W_{\underline{d},k} \subsetneq S_{\underline{d}}^*$ is irreducible and unirational, because J is rational, being an open dense subset of a vector bundle over \mathbb{G} . Once one shows

that $\pi_2 : J \rightarrow W_{d,k}$ is generically finite, one deduces that $\text{codim}_{S_d^*}(W_{d,k}) = t$ from (2.2). Therefore, we focus on proving that π_2 is generically finite, i.e. that if $u \in W_{d,k}$ is a general point, then $\dim(\pi_2^{-1}(u)) = 0$.

Let $[\Pi] \in \mathbb{G}$ and choose $[y_0, y_1, \dots, y_m]$ homogeneous coordinates in \mathbb{P}^m such that the ideal of Π is $I_\Pi := (y_{k+1}, \dots, y_m)$. For general $([\Pi], u) \in \pi_1^{-1}([\Pi]) \subset J$, with $u = (g_1, \dots, g_s) \in W_{d,k}$, we can write

$$g_i = \sum_{h=k+1}^m y_h p_i^{(h)} + r_i, \quad 1 \leq i \leq s,$$

with

$$r_i \in (I_\Pi^2)_{d_i} \text{ whereas } p_i^{(h)} = \sum_{|\underline{\mu}|=d_i-1} c_{i,\underline{\mu}}^{(h)} y_{\underline{\mu}} \in \mathbb{C}[y_0, y_1, \dots, y_k]_{d_i-1}, \quad 1 \leq i \leq s, \quad k+1 \leq h \leq m, \quad (2.3)$$

where $(I_\Pi^2)_{d_i}$ is the homogenous component of degree d_i of the ideal I_Π^2 , $\underline{\mu} := (\mu_0, \dots, \mu_k) \in \mathbb{Z}_{\geq 0}^{k+1}$, $|\underline{\mu}| := \sum_{r=0}^k \mu_r$, and $y_{\underline{\mu}} := y_0^{\mu_0} y_1^{\mu_1} \dots y_k^{\mu_k}$. By the generality assumption on u , the polynomials $p_i^{(h)}$ and r_i are general.

The Jacobian matrix $\left(\frac{\partial g_i}{\partial y_j}\right)_{1 \leq i \leq s; 0 \leq j \leq m}$ computed along Π takes the block form

$$M = (\mathbf{0} \ \mathbf{P}) \text{ where } \mathbf{P} := \left(p_i^{(h)}\right)_{1 \leq i \leq s; k+1 \leq h \leq m}$$

where the $\mathbf{0}$ -block has size $s \times (k+1)$ and \mathbf{P} has size $s \times (m-k)$, where $m-k \geq s$ because of course $\dim(Y_u) = m-s \geq k$. By the generality of the polynomials $p_i^{(h)}$, the locus of Π where $\text{rk}(M) < s$, which coincides with the singular locus of Y_u along Π , has dimension $\max\{-1, 2k+s-m-1\}$ and, by Bertini’s theorem, it coincides with the singular locus of Y_u .

Next we consider the following exact sequence of normal sheaves

$$0 \rightarrow N_{\Pi/Y_u} \rightarrow N_{\Pi/\mathbb{P}^m} \cong \mathcal{O}_{\mathbb{P}^k}(1)^{\oplus(m-k)} \rightarrow N_{Y_u/\mathbb{P}^m}|_{\Pi} \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^k}(d_i) \quad (2.4)$$

(see [8, Lemma 70.5.6]; when $2k \leq m-s$, namely when Y_u is smooth along Π , the map on the right-side in (2.4) is more precisely surjective). Any $\xi \in H^0(\Pi, N_{\Pi/\mathbb{P}^m})$ can be identified with a collection of $m-k$ linear forms on $\Pi \cong \mathbb{P}^k$

$$\phi_h^\xi(y) := a_{h,0}y_0 + a_{h,1}y_1 + \dots + a_{h,k}y_k, \quad k+1 \leq h \leq m,$$

whose coefficients fill up the $(m-k) \times (k+1)$ matrix

$$A_\xi := (a_{h,j}), \quad k+1 \leq h \leq m, \quad 0 \leq j \leq k;$$

by abusing notation, one may identify ξ with A_ξ .

Thus the map $H^0(\Pi, N_{\Pi/\mathbb{P}^m}) \xrightarrow{\sigma} H^0(\Pi, N_{Y_u/\mathbb{P}^m}|_{\Pi})$, arising from (2.4) is given by (cf. e.g. [2, formula (4)])

$$A_\xi \xrightarrow{\sigma} \left(\sum_{0 \leq j \leq k < h \leq m} a_{h,j} y_j p_i^{(h)} \right)_{1 \leq i \leq s}. \quad (2.5)$$

Notice that the assumption $t > 0$ reads as

$$(k + 1)(m - k) = h^0(\Pi, N_{\Pi/\mathbb{P}^m}) < h^0(\Pi, N_{Y_u/\mathbb{P}^m}|_{\Pi}) = \sum_{i=1}^s \binom{d_i + k}{k}.$$

Claim 2.1 *The map $H^0(\Pi, N_{\Pi/\mathbb{P}^m}) \xrightarrow{\sigma} H^0(\Pi, N_{Y_u/\mathbb{P}^m}|_{\Pi})$ is injective, equivalently $h^0(N_{\Pi/Y_u}) = 0$. In particular, for a general point $u \in W_{\underline{d},k}$, the Fano scheme $F_k(Y_u)$ contains $\{[\Pi]\}$ as a zero-dimensional integral component.*

Proof of Claim 2.1 Using (2.3), the polynomials on the right-hand-side of (2.5) read as

$$\sum_{h=k+1}^m \sum_{j=0}^k a_{h,j} y_j \left(\sum_{|\underline{\mu}|=d_i-1} c_{i,\underline{\mu}}^{(h)} y^{\underline{\mu}} \right), \quad 1 \leq i \leq s.$$

Ordering the previous polynomial expressions via the standard lexicographical monomial order on the canonical basis $\{y^{\underline{\mu}}\}$ of $\mathbb{C}[y_0, y_1, \dots, y_k]_{d_i} = H^0(\mathcal{O}_{\mathbb{P}^k}(d_i))$, $1 \leq i \leq s$, the injectivity of the map σ is equivalent for the homogeneous linear system

$$\sum_{0 \leq j \leq k < h \leq m} c_{i,\underline{\nu}-\underline{e}_j}^{(h)} a_{h,j} = 0, \quad 1 \leq i \leq s, \tag{2.6}$$

to have only the trivial solution, where $\underline{\nu} := (\nu_0, \nu_1, \dots, \nu_k) \in \mathbb{Z}_{\geq 0}^{k+1}$ is such that $|\underline{\nu}| = d_i$, \underline{e}_j is the $(j + 1)$ -th vertex of the standard $(k + 1)$ -simplex in $\mathbb{Z}_{\geq 0}^{k+1} \setminus \{0\}$, and $c_{i,\underline{\nu}-\underline{e}_j}^{(h)} = 0$ when $\underline{\nu} - \underline{e}_j \notin \mathbb{Z}_{\geq 0}^{k+1}$ (this last condition stands for “ $\underline{\nu} - \underline{e}_j$ improper” as formulated in [2, p. 29]). The linear system (2.6) consists of $\sum_{i=1}^s \binom{d_i+k}{k}$ equations in the $(k + 1)(m - k)$ indeterminates $a_{h,j}$, with coefficients $c_{i,\underline{\mu}}^{(h)}$, $0 \leq j \leq k < h \leq m$.

Let $C := (c_{i,\underline{\nu}-\underline{e}_j}^{(h)})$ be the coefficient matrix of (2.6); one is reduced to show that, for general choices of the entries $c_{i,\underline{\nu}-\underline{e}_j}^{(h)}$, the matrix C has maximal rank $(k + 1)(m - k)$. This can be done arguing as in [2, p. 29]. Namely, row-indices of C are determined by the standard lexicographical monomial order on the canonical basis of $\bigoplus_{i=1}^s \mathbb{C}[y_0, y_1, \dots, y_k]_{d_i}$, whereas column-indices of C are determined by the standard lexicographic order on the set of indices (h, j) . If one considers the square sub-matrix \widehat{C} of C formed by the first $(k + 1)(m - k)$ rows and by all the columns of C , then $\det(\widehat{C})$ is a non-zero polynomial in the indeterminates $c_{i,\underline{\mu}}^{(h)}$. Indeed, take the lexicographic order on the set of indices

$$(h, i, \underline{\mu}), \quad \text{where } k + 1 \leq h \leq m, |\underline{\mu}| = d_i - 1, 1 \leq i \leq s,$$

and order the monomials appearing in the expression of $\det(\widehat{C})$ according to the following rule: the monomials m_1 and m_2 are such that $m_1 > m_2$ if, considering the smallest index $(h, i, \underline{\mu})$ for which $c_{i,\underline{\mu}}^{(h)}$ occurs in the monomial m_1 with exponent p_1 and in the monomial m_2 with exponent $p_2 \neq p_1$, one has $p_1 > p_2$. The greatest monomial (in the monomial ordering described above) appearing in $\det(\widehat{C})$ has coefficient ± 1 , since in each column the choice of the $c_{i,\underline{\mu}}^{(h)}$ entering in this monomial is uniquely determined. By maximality of such monomial, it follows that $\det(\widehat{C}) \neq 0$, which shows that C has maximal rank $(k + 1)(m - k)$, i.e. the map σ is injective.

The injectivity of σ and (2.4) yield $h^0(N_{\Pi/Y_u}) = 0$. Since $H^0(N_{\Pi/Y_u})$ is the tangent space to $F_k(Y_u)$ at its point $[\Pi]$, one deduces that $\{[\Pi]\}$ is a zero-dimensional, reduced component of $F_k(Y_u)$, as claimed. \square

Finally, by monodromy arguments, the irreducibility of J and Claim 2.1 ensure that for general $u \in W_{\underline{d},k}$, the Fano scheme $F_k(Y_u)$ is zero-dimensional and reduced, i.e. $\pi_2: J \rightarrow W_{\underline{d},k}$ is generically finite, and that Y_u has a singular locus of dimension $\max\{-1, 2k + s - m - 1\}$ along any of the k -dimensional linear subspaces in $F_k(Y_u)$. This completes the proof of Step 1. \square

To conclude the proof of Theorem 1.1, we need the following numerical result.

Step 2 For $0 \leq h \leq k - 1$ integers, consider the integer

$$\delta_h(m, k, \underline{d}) := \sum_{i=1}^s \binom{d_i + k}{k} - \sum_{i=1}^s \binom{d_i + h}{h} - (k - h)(m + h + 1 - k).$$

If $\delta_h(m, k, \underline{d}) \leq 0$, then

$$t(m, k, \underline{d}) \leq 0.$$

Proof of Step 2 In order to ease notation, we set $\delta_h := \delta_h(m, k, \underline{d})$. Therefore, the condition $\delta_h \leq 0$ implies $m \geq \frac{1}{k-h} \left[\sum_{i=1}^s \binom{d_i+k}{k} - \binom{d_i+h}{h} \right] - (h + 1 - k)$. Plugging the previous inequality in the expression of t , one has

$$t \leq - \sum_{i=1}^s \left[\frac{h+1}{k-h} \binom{d_i+k}{k} - \frac{k+1}{k-h} \binom{d_i+h}{h} \right] + (k+1)(h+1). \tag{2.7}$$

Set $D(x) := \frac{h+1}{k-h} \binom{x+k}{k} - \frac{k+1}{k-h} \binom{x+h}{h}$. Thus, (2.7) reads

$$t \leq - \sum_{i=1}^s D(d_i) + (k+1)(h+1). \tag{2.8}$$

The assumption $0 \leq h \leq k - 1$ gives

$$D(d_i) = \frac{(h+1)(d_i+1) \cdots (d_i+h)}{k!(k-h)} \left((d_i+h+1) \cdots (d_i+k) - (k+1)k \cdots (h+2) \right),$$

$$1 \leq i \leq s.$$

The polynomial $D(x)$ vanishes for $x = 1$, which is its only positive root. Notice that

$$D(2) = \frac{h+1}{k-h} \binom{k+2}{k} - \frac{k+1}{k-h} \binom{h+2}{h} = \frac{(h+1)(k+1)}{2} > 0.$$

In particular, $D(x)$ is increasing and positive for $x > 1$, so from (2.8) it follows that

$$\begin{aligned} t &\leq - \sum_{i=1}^s D(d_i) + (k+1)(h+1) \\ &\leq -s D(2) + (k+1)(h+1) \\ &= (k+1)(h+1) \left(1 - \frac{s}{2} \right). \end{aligned}$$

Therefore, when $s \geq 2$, we have $t \leq 0$ and we are done in this case.

If $s = 1$, set $d := d_1$. In this case (2.8) is $t \leq -D(d) + (k + 1)(h + 1)$, where again $D(d)$ is increasing and positive for $d > 1$. When $s = 1$, we have $d \geq 3$ by assumption. Thus, one computes

$$D(3) = (k + 1)(h + 1) \frac{k + h + 5}{6}$$

and so, for any $d \geq 3$, one has

$$t \leq -D(d) + (k + 1)(h + 1) \leq -D(3) + (k + 1)(h + 1) = (k + 1)(h + 1) \frac{1 - k - h}{6}.$$

Being $0 \leq h \leq k - 1$, one deduces that $t \leq 0$, completing the proof of Step 2. □

The final step of the proof of Theorem 1.1 is the following.

Step 3 For general $u \in W_{d,k}$, the zero-dimensional Fano scheme $F_k(Y_u)$ has length one. In particular, the map $\pi_2: J \rightarrow W_{d,k}$ is birational and $W_{d,k}$ is rational.

Proof of Step 3 Let us consider the (locally closed) incidence correspondence

$$I := \left\{ ([\Pi_1], [\Pi_2], u) \in \mathbb{G} \times \mathbb{G} \times S_d^* \mid \Pi_1 \neq \Pi_2, \Pi_i \subset Y_u, 1 \leq i \leq 2 \right\} \subset \mathbb{G} \times \mathbb{G} \times S_d^*.$$

If I is not empty, let $\varphi: I \rightarrow J$ be the map defined by

$$\varphi([\Pi_1], [\Pi_2], u) = ([\Pi_1], u).$$

We need to prove that φ is not dominant. To do this, consider the (locally closed) subset

$$I_h := \left\{ ([\Pi_1], [\Pi_2], u) \in I \mid \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\}, \quad \text{where } -1 \leq h \leq k - 1$$

(we set $\mathbb{P}^{-1} = \emptyset$, i.e. the case $h = -1$ occurs when Π_1 and Π_2 are skew). Clearly, one has $I = \bigsqcup_{h=-1}^{k-1} I_h$. Setting $\varphi_h := \varphi|_{I_h}$, it is sufficient to prove that φ_h is not dominant, for any $-1 \leq h \leq k - 1$.

So, let h be such that I_h is not empty, and let T_h be an irreducible component of I_h . Of course, if $\dim(T_h) < \dim(J)$, the restriction $\varphi_{h|T_h}: T_h \rightarrow J$ is not dominant. On the other hand, suppose that $\dim(T_h) > \dim(J)$. For any such a component, the map $\varphi_{h|T_h}$ cannot be dominant, otherwise the composition $T_h \xrightarrow{\varphi_{h|T_h}} J \xrightarrow{\pi_2} W_{d,k}$ would be dominant, as π_2 is, which would imply that the general fiber of π_2 is positive dimensional, contradicting Step 1.

Therefore, it remains to investigate the case $\dim(T_h) = \dim(J)$. We estimate the dimension of T_h as follows. Consider

$$\mathbb{G}_h^2 := \left\{ ([\Pi_1], [\Pi_2]) \in \mathbb{G} \times \mathbb{G} \mid \Pi_1 \cap \Pi_2 \cong \mathbb{P}^h \right\} \subset \mathbb{G} \times \mathbb{G},$$

which is locally closed in $\mathbb{G} \times \mathbb{G}$. The projection

$$\widehat{\pi}_1: \mathbb{G}_h^2 \rightarrow \mathbb{G}, \quad ([\Pi_1], [\Pi_2]) \mapsto [\Pi_1]$$

is surjective onto \mathbb{G} and any $\widehat{\pi}_1$ -fiber is irreducible, of dimension equal to $\dim(\mathbb{G}(h, k) \times \mathbb{G}(k - h - 1, m - h - 1)) = (h + 1)(k - h) + (k - h)(m - k)$. Thus

$$\dim \mathbb{G}_h^2 = (k + 1)(m - k) + (h + 1)(k - h) + (k - h)(m - k).$$

One has the projection

$$\psi_h: T_h \rightarrow \mathbb{G}_h^2, \quad ([\Pi_1], [\Pi_2], u) \mapsto ([\Pi_1], [\Pi_2]),$$

which is surjective, because the projective group acts transitively on \mathbb{G}_h^2 . Hence $\dim(T_h) = \dim(\mathbb{G}_h^2) + \dim(\mathfrak{F}_h)$, where $\mathfrak{F}_h := \bigoplus_{i=1}^s \left(H^0(\mathcal{I}_{\Pi_1 \cup \Pi_2 / \mathbb{P}^m}(d_i)) \setminus \{0\} \right)$ is the general fiber of $\psi_h|_{T_h}$ and where $\mathcal{I}_{\Pi_1 \cup \Pi_2 / \mathbb{P}^m}$ denotes the ideal sheaf of $\Pi_1 \cup \Pi_2$ in \mathbb{P}^m .

Claim 2.2 *For every positive integer d one has*

$$h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2 / \mathbb{P}^m}(d)) = \dim(S_d) - 2 \binom{d+k}{k} + \binom{d+h}{h}.$$

Proof of Claim 2.2 We have

$$h^0(\mathcal{I}_{\Pi_1 / \mathbb{P}^m}(d)) = \dim(S_d) - \binom{d+k}{k}. \tag{2.9}$$

Consider the linear system Σ cut out on Π_2 by $|\mathcal{I}_{\Pi_1 / \mathbb{P}^m}(d)|$. We claim that Σ is the complete linear system of hypersurfaces of degree d of Π_2 containing $\Pi := \Pi_1 \cap \Pi_2$. Indeed Σ contains all hypersurfaces consisting of a hyperplane through Π plus a hypersurface of degree $d - 1$ of Π_2 , which proves our claim. In the light of this fact, and arguing as in (2.1) and (2.2), we deduce that

$$\begin{aligned} h^0(\mathcal{I}_{\Pi_1 \cup \Pi_2 / \mathbb{P}^m}(d)) &= h^0(\mathcal{I}_{\Pi_1 / \mathbb{P}^m}(d)) - (\dim(\Sigma) + 1) = h^0(\mathcal{I}_{\Pi_1 / \mathbb{P}^m}(d)) \\ &\quad - \left(\binom{d+k}{k} - \binom{d+h}{h} \right), \end{aligned}$$

which, by (2.9), yields the assertion. □

By Claim 2.2 we have

$$\dim(\mathfrak{F}_h) = \dim(S_d^*) - 2 \sum_{i=1}^s \binom{d_i+k}{k} + \sum_{i=1}^s \binom{d_i+h}{h}.$$

Hence

$$\begin{aligned} \dim(T_h) &= \dim(\mathfrak{F}_h) + \dim(\mathbb{G}_h^2) \\ &= \dim(S_d^*) - 2 \sum_{i=1}^s \binom{d_i+k}{k} + \sum_{i=1}^s \binom{d_i+h}{h} \\ &\quad + (k+1)(m-k) + (h+1)(k-h) + (k-h)(m-k) \tag{2.10} \\ &= \dim(J) - \sum_{i=1}^s \binom{d_i+k}{k} + \sum_{i=1}^s \binom{d_i+h}{h} + (k-h)(m+h+1-k) \\ &= \dim(J) - \delta_h. \end{aligned}$$

Since $\dim(T_h) = \dim(J)$, (2.10) implies $\delta_h = 0$. When $0 \leq h \leq k - 1$, Step 2 gives $t \leq 0$, against our assumptions. When $h = -1$, one has $0 = \delta_{-1} = t$, again contrary to our assumptions.

Since no component $T_h \subset I_h$ can dominate J , the map $\varphi: I \rightarrow J$ is not dominant. We conclude therefore that the map $\pi_2: J \rightarrow W_{d,k}$ is birational, completing the proof of Step 3. □

Steps 1–3 prove Theorem 1.1. □

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