DISSIPATIVE SCALE EFFECTS IN STRAIN-GRADIENT PLASTICITY: THE CASE OF SIMPLE SHEAR*

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Abstract. We analyze dissipative scale effects within a one-dimensional theory, developed in [L. Anand et al. (2005) J. Mech. Phys. Solids 53], which describes plastic flow in a thin strip undergoing simple shear. We give a variational characterization of the *yield (shear) stress* — the threshold for the onset of plastic flow — and we use this characterization, together with results from [M. Amar et al. (2011) J. Math. Anal. Appl. 397], to obtain an explicit relation between the yield stress and the height of the strip. The relation we obtain confirms that thinner specimens are stronger, in the sense that they display higher yield stress.

Key words. strain–gradient plasticity, rate-independent evolution, energetic formulation, dissipative length scale, size effects, size-dependent strengthening

AMS subject classifications. 74C05, 35K86, 49J40

1. Introduction.

A number of experiments have shown that conventional plasticity fails to capture the size-dependent behavior of metallic specimens undergoing plastic flow in the size range below 100 microns, with smaller samples being, in general, stronger (see [25] for a review).

Substantial theoretical work has been carried out to extend conventional plasticity to the micron scale. It is acknowledged that size effects observed in metallic samples are associated to the inhomogeneity of plastic flow [5], a fact that motivates a number of *strain-gradient plasticity* theories, starting with Ref. [11].

In the so-called *non-local* or *high-order theories*, the flow rule that governs the evolution of plastic strain is a partial differential equation which requires the specification of appropriate boundary conditions. The first of such theories was proposed by Aifantis [1]; the vast majority of subsequent high-order theories were derived using the virtual-power principle, by taking into account power expenditure by higher-order stresses that are work-conjugate to the plastic-strain gradient [6, 15, 20, 21, 22].

Apparently, the theories developed by Gurtin and Anand [21, 22] are those that have inspired most mathematical work. One of the distinctive aspects of [21] is that the full plastic distortion (the sum of a symmetric plastic strain and a skew–symmetric plastic spin) is accounted for. In particular, the issues of existence and uniqueness of solutions for strain-gradient plasticity with plastic spin, as considered in Ref. [21], has been addressed in Ref. [9] in the case of two-dimensional setting of anti-plane shear, and in Refs. [12], [32], and [33] in the full three-dimensional setting. The model for plastically-irrotational materials proposed in Ref. [22] was studied in Ref. [36]. Theoretical and numerical analysis of a related model with no plastic spin is available in [35]. More recently, existence of weak solutions for a model with plastic spin was established in [13] using a Korn's type inequality for incompatible tensor fields

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(see [34] and references therein). Of particular importance for the present paper are the existence theorems for strain-gradient plasticity based on the notion of energetic solution, which have been proved both in the small–strain [19] and in the large–strain [29] setting.

The flow rules proposed in Ref. [22] are of particular interest because they incorporate two length scales:

- an energetic scale L, which appears from letting the free-energy density depend on derivatives of the *plastic strain*, $\mathbf{E}^{\rm p}$, through the *Burgers tensor*, $\mathbf{G} = {\rm curl}\mathbf{E}^{\rm p}$;
- a dissipative scale ℓ, which arises from letting the gradient of plastic strain rate, $\nabla\dot{\mathbf{E}}^{\rm p}$, enter the dissipation-rate density.

The form of the free energy density is motivated by dislocation mechanics. In particular, the choice of letting the free energy to depend on plastic strain gradient through the Burgers tensor follows from the presumption that the so–called geometrically-necessary dislocations (whose density is measured by $\mathbf{G}$) play a major role in determining size-dependent response, a presumption that finds its justification in homogenization results from discrete-dislocation models [18, 28].

Because of the complicated nature of the non–local flow rule, it is not easy to understand how its solutions are affected by the material scales. On the other hand, such understanding is crucial in order to identify these scales by comparison with experiments. Thus, parallel with the literature dealing with modeling, researchers have also endeavored to investigate how the various scales may affect the nature of solutions, not only for the Gurtin-Anand theory, but also for other strain-gradient plasticity theories.

This task is usually accomplished by working out a simple analytical problem that mimics some experimental setup. For example, scale dependence for the torsion experiment was investigated in Ref. [27] (by numerical and asymptotic considerations) in the framework of the Fleck & Willis theory [17] and in Ref. [10] (by rigorous arguments) for energetic scale effects within the Gurtin-Anand theory [22]. Moreover, for the distortion–gradient plasticity (which accounts also for plastic spin), specific finite-element schemes for the torsion problem have been recently proposed in Ref. [8]. Problems involving micro-bending have been scrutinized in Ref. [26] and, more recently, in Ref. [16] in the case of non-proportional plastic-strain histories.

With a similar goal in mind, a simplified flow rule, formulated in one spatial dimension, was derived and analyzed in Ref. [4] to investigate the effects of both the energetic and the dissipative scales, in both isotropic plasticity and crystal plasticity under symmetric double slip (see e.g. [6]). Such flow rule, which mimics the traction problem in simple shear symmetry, will be introduced in Section 2. In the same section we will also make a comparison with conventional plasticity. This comparison illustrates two well-known facts: 1) that the length-scale ℓ is expected to be a source of additional strengthening; 2) that the natural way to quantify strengthening is to consider increase of the *Yield stress*, τ_Y, i.e., the value of the (shear) stress that triggers plastic flow in an initially virgin sample.

The aim of this paper is to rigorously confirm these facts. We will show that the onset of plastic flow, whence τ_Y, is determined by the loss of stability (according to the energetic formulation of rate-independent systems) of the virgin state. As a consequence, we will *explicitly* determine the dependence of τ_Y on ℓ, proving in particular that τ_Y is strictly increasing with ℓ, that is to say, smaller samples are stronger.

Results and proof are stated (in renormalized variables) in Section 3, which also contains an outline of the arguments (details are given in Sections 4-5). In summary, using the above-mentioned characterization of τ_Y in terms of stability, we will argue that τ_Y may also be characterized as the smallest value that the *renormalized dissipation*

$$\frac{S_0}{h}\int_{-h}^{+h}\sqrt{\phi^2(y)+\ell^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}y}(y)\right)^2}\,\mathrm{d}y$$

attains among all $\phi \in H_0^1((-h,+h))$ such that $\int_{-h}^{+h}\phi(y)\mathrm{d}y = 1$ (see Section 2 for the definition of S_0 and h, and Theorem 3.4 for the precise statement). This constrained minimization problem had already been introduced in [4] and analyzed in [2], showing that a minimum is attained in BV, which is smooth in the interior and satisfies the corresponding Euler-Lagrange equation. We will then argue that these results permit to explicitly characterize τ_Y in terms of ℓ (see Theorem 3.5 and Figure 3.1).

2. Problem setup.

2.1. The traction problem. The one dimensional theory developed in Ref. [4] describes plastic flow in a body having the shape of an infinite strip of width $2h$, namely,

$$\Omega_h = \left\{\mathbf{x} = (x,y,z) \in \mathbb{R}^3 : -h < y < h\right\}, \tag{2.1}$$

as sketched in Fig. 1. We restrict attention to the so–called *traction problem*, de-

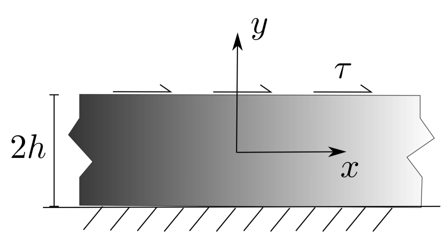

Fig. 2.1. *An infinite strip of height $2h$, clamped on the bottom side and subject to a uniform shear traction τ on the top side.*

scribing an ideal experiment in which the bottom side of the strip is clamped and a uniform *shear traction* τ along the direction x is prescribed on the upper side. We work in the *rate-independent* setting of quasistatic evolution in plasticity and we limit our attention to the case of *proportional loading*, that is to say, we assume that τ is strictly increasing with respect to time. With this assumption, we may label each instant by the corresponding value of the shear stress and adopt τ in place of time as the independent variable.

Because of translational invariance in the x- and z-directions, it is natural to assume that the two kinematic fields of interest, namely the *displacement* $\mathbf{u}$ and the *plastic strain* $\mathbf{E}^{\mathrm{p}}$, are independent of x and z. Moreover, by symmetry considerations (see Appendix A.4), it is natural to assume that $\mathbf{u}$ is parallel to the x-axis and that the only non-vanishing components of $\mathbf{E}^{\mathrm{p}}$ are $(\mathbf{E}^{\mathrm{p}})_{12} = (\mathbf{E}^{\mathrm{p}})_{21}$. Therefore, we make the Ansatz that $\mathbf{u}$ and $\mathbf{E}^{\mathrm{p}}$ have the following representation:

$$\mathbf{u} = u(y,\tau)\mathbf{e}_1, \quad \mathbf{E}^{\mathrm{p}} = \gamma^{\mathrm{p}}(y,\tau)\mathrm{sym}(\mathbf{e}_1 \otimes \mathbf{e}_2), \tag{2.2}$$

with $\{\mathbf{e}_i : i = 1,2,3\}$ the canonical basis of $\mathbb{R}^3$. The stress tensor $\mathbf{T}$, consistent with (2.2) and in view of the balance equation $\mathrm{div}\mathbf{T} = 0$, is taken to be spatially constant and having the representation

$$\mathbf{T}(\tau) = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1).$$

2.2. A local flow rule: strengthening and hardening. If the material is modeled in the framework of *von Mises* plasticity with kinematic hardening, the flow rule governing the evolution of the *shear strain* γ^{p} may be written as

$$\begin{cases} \tau - S_0\kappa\gamma^{\mathrm{p}} = \tau^{\mathrm{dis}}, \\ \frac{\tau^{\mathrm{dis}}}{S_0} \in \mathrm{Sign}(\dot{\gamma}^{\mathrm{p}}), \end{cases} \tag{2.3}$$

where $S_0 > 0$ is the *coarse-grain yield strength*, κ is the *kinematic-hardening coefficient*, a superimposed dot denotes differentiation with respect to the loading parameter τ, and

$$\mathrm{Sign}(x) = \begin{cases} \{+1\} & \text{if } x > 0, \\ [-1,+1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

Note that (2.3) may be equivalently rewritten in its dual form:

$$|\tau - S_0\kappa\gamma^{\mathrm{p}}| \le S_0 \quad \text{and} \quad (\tau - S_0\kappa\gamma^{\mathrm{p}} - \tilde{\tau})\dot{\gamma}^{\mathrm{p}} \ge 0 \quad \text{for all } \tilde{\tau} \in [-S_0, S_0]. \tag{2.4}$$

Note also that $|\tau^{\mathrm{dis}}| \le S_0$ as there is no isotropic hardening. Granted that the body is in its virgin state at the beginning of the experiment, namely,

$$\gamma^{\mathrm{p}}(y,0) = 0, \tag{2.5}$$

the solution of (2.3) is easily worked out and, on introducing the positive-part operator $(\cdot)_+ = \max\{\cdot, 0\}$, can be written as

$$\gamma^{\mathrm{p}}(y,\tau) = \frac{(\tau/S_0 - 1)_+}{\kappa}.$$

This solution displays the typical features of a stress-strain diagram from classical plasticity; in particular:

- the increase of S_0 is associated to *strengthening*, that is, an increase of the threshold for the onset of plastic flow, the Yield shear stress:

$$\tau_Y = S_0; \tag{2.6}$$

- the increase of κ, with S_0 fixed, is associated to *hardening*, that is, an increase of the shear stress required to attain a given amount of plastic shear.

On multiplying (2.3) by $\dot\gamma^{\rm p}$, we obtain the free energy balance

$$\frac{1}{2}S_0\kappa\left(\frac{\rm d}{{\rm d}\tau}(\gamma^{\rm p})^2\right)+S_0|\dot\gamma^{\rm p}|=\tau\dot\gamma^{\rm p}, \tag{2.7}$$

the *free energy density* being given by $\frac{S_0}{2}\kappa(\gamma^{\rm p})^2$. The balance (2.7) can thus be interpreted as a splitting of the internal power $\tau\dot\gamma^{\rm p}$ expended on plastic flow into an *energetic part* and a *dissipative part*, $\tau^{\rm dis}\dot\gamma^{\rm p}=S_0|\dot\gamma^{\rm p}|$. Accordingly, we may say that, in the present context[1],

- strengthening is a *dissipative effect*, whereas
- hardening is an *energetic effect*.

It is worth noticing that the strengthening effect (also referred to as "elastic gap") associated to the dissipative length-scale emerges also in the analysis of non-proportional plastic-straining histories carried out in [16].

2.3. A non-local flow rule: size-dependent strengthening and hardening. Using the strain-gradient plasticity theory of Ref. [4] we derive in Appendix A a non-local, rate-independent flow rule. In particular, we replace the first of (2.3) with:

$$\tau-S_0\left(\kappa\gamma^{\rm p}-L^2\gamma^{\rm p}_{yy}\right)=\tau^{\rm dis}-k^{\rm dis}_y, \tag{2.8a}$$

where the subscript y denotes the partial derivative in the y direction, and the inclusion in (2.3) with:

$$\frac{(\tau^{\rm dis},\ell^{-1}k^{\rm dis})}{S_0}\in{\rm Sign}\left(\dot\gamma^{\rm p},\ell\dot\gamma^{\rm p}_y\right), \tag{2.8b}$$

where the index y denotes partial differentiation with respect to y and

$${\rm Sign}(\mathbf{v})=\begin{cases}\left\{\frac{\mathbf{v}}{|\mathbf{v}|}\right\} & \text{if}\quad |\mathbf{v}|\neq 0,\\ \{\mathbf{v}\in\mathbb{R}^2:\ |\mathbf{v}|\le 1\} & \text{if}\quad |\mathbf{v}|=0\end{cases}$$

(see Remark 3.3 for a discussion of the dual formulation). Problem (2.8) must be complemented by both initial conditions, for which we again choose the virgin-state condition (2.5),

$$\gamma^{\rm p}|_{\tau=0}=0, \tag{2.9a}$$

and boundary conditions, for which we choose *microscopic hard conditions*:

$$\gamma^{\rm p}|_{y=-h}=\gamma^{\rm p}|_{y=+h}=0. \tag{2.9b}$$

As explained in Appendix A, the partial differential equation (2.8a) is a constitutively-augmented microforce balance. The balance is engendered by a version of the principle of virtual powers that accounts for power expenditure on the time derivative of the shear-strain gradient $\gamma^{\rm p}_y$. In particular, taking the formal variation of the *plastic free energy*

$$E^{\rm p}(\gamma^{\rm p})=\frac{S_0}{2}\int_{-h}^{+h}\left(\kappa(\gamma^{\rm p})^2+L^2\left(\gamma^{\rm p}_y\right)^2\right){\rm d}y \tag{2.10}$$

[1]Note however that, as pointed out in [24] (see also [23, § 80]), it is not always possible to discriminate between energetic and dissipative effects.

and defining the *plastic dissipation rate*

$$\Psi^{\mathrm{p}}(\gamma^{\mathrm{p}}) = S_0 \int_{-h}^{+h} \sqrt{(\dot\gamma^{\mathrm{p}})^2 + \ell^2(\dot\gamma^{\mathrm{p}}_y)^2}\mathrm{d}y, \tag{2.11}$$

the following identity is arrived at:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}E^{\mathrm{p}}(\gamma^{\mathrm{p}}) + \Psi^{\mathrm{p}}(\gamma^{\mathrm{p}}) = \int_{-h}^{+h} \tau\dot\gamma^{\mathrm{p}}\mathrm{d}y, \tag{2.12}$$

which is again interpreted as a splitting of work expenditure (the right-hand side of (2.12)) into an energetic part and a dissipative part. Given that L, resp. ℓ, appear in the energetic, resp. dissipative part of the energy balance (2.12), in line with the discussion in §2.2

- one may expect that the extra energy brought into play by L enhances hardening effects, and that the extra dissipation associated to ℓ is a source of additional strengthening.

We have recently scrutinized the role of L in [10], rigorously confirming this expectation in the case of torsion of thin wires. The role of ℓ has been investigated both formally and numerically in [4]. In view of the discussion in §2.2 (cf. in particular (2.6)) a natural way to rigorously quantify the role of ℓ is to determine how the yield shear stress

$$\tau_Y := \sup\Big\{\tau \geq 0 : \ \gamma^{\mathrm{p}} \equiv 0 \ \text{ in } \ (-h,+h)\times[0,\tau]\Big\}, \tag{2.13}$$

i.e. the value attained by τ at the onset of plastic flow, depends on ℓ. Such a relation can not be easily deduced a-priori and is the main point of this paper.

3. Main results.

3.1. Scaling. In order to single out the relevant parameters, we introduce dimensionless independent variables:

$$r := \frac{y}{h}, \quad \theta := \frac{\tau}{S_0}.$$

Consistent with this choice, we introduce the dimensionless parameters:

$$\lambda := \frac{\ell}{h}, \quad \Lambda := \frac{L}{h}. \tag{3.1}$$

The nonlocal flow rule (2.8) can now be reformulated in the domain $I := (-1,+1)$ and takes the form (henceforth, for typographical convenience, we drop the superscript p from γ^{p}):

$$\begin{cases} \theta - \kappa\gamma + \Lambda^2\gamma_{rr} = \bar\tau^{\mathrm{dis}} - \bar k_r^{\mathrm{dis}}, \\ (\bar\tau^{\mathrm{dis}}, \lambda^{-1}\bar k^{\mathrm{dis}}) \in \mathrm{Sign}\,(\dot\gamma, \lambda\dot\gamma_r)\,, \end{cases} \tag{3.2}$$

where the index r denotes partial differentiation with respect to r. Initial and microscopically hard boundary conditions (2.9) now read as

$$\gamma(r,0) = \gamma(-1,\theta) = \gamma(+1,\theta) = 0 \quad (r,\theta) \in I\times[0,+\infty) \tag{3.3}$$

and the *renormalized plastic free energy*, resp. *dissipation-rate*, are given by

$$E(\gamma) := \frac{\kappa}{2}\int_I \left(\gamma^2 + \Lambda^2\gamma_r^2\right) \mathrm{d}r, \quad \Psi(\gamma) := \int_I \sqrt{\gamma^2 + \lambda^2\gamma_r^2}\mathrm{d}r \tag{3.4}$$

(cf. (2.10), resp. (2.11)). In renormalized variables, our aim becomes that of rigorously quantifying the dependence on the *renormalized dissipative scale*, λ, of the *renormalized yield shear stress* (cf. (2.13))

$$\frac{\tau_Y}{S_0} = \theta_Y := \sup\Big\{\theta \geq 0 : \ \gamma \equiv 0 \ \text{ in } \ I \times [0, \theta]\Big\}, \tag{3.5}$$

namely, *the largest value attained by the renormalized shear stress θ prior to the onset of plastic flow.*

3.2. Energetic formulation. We assume hereafter that $\kappa \geq 0$, $\Lambda > 0$, and $\lambda > 0$. Being a *rate-independent* dynamical system, the flow rule (3.2)–(3.3) can be formulated in many equivalent ways. The formulation that best suits our needs is the so–called *energetic formulation* proposed in Ref. [31]. With a view towards formulating (3.2)–(3.3) in the energetic format, we introduce the (renormalized) *energy functional*:

$$\mathscr{E}(\theta, \gamma) := E(\gamma) - \theta \int_I \gamma \mathrm{d}r. \tag{3.6}$$

As usual, we write $\gamma(\theta) := \gamma(\theta, \cdot)$. We can now give the definition of energetic solution.

Definition 3.1 (Energetic solution). *Given $\Theta > 0$, a function $\gamma : [0, \Theta] \to H^1_0(I)$ is an* energetic solution *to* (3.2)–(3.3) *if the function* $[0, \Theta] \ni \theta \mapsto \frac{\partial \mathscr{E}}{\partial \theta}(\theta, \gamma(\theta)) = -\int_I \gamma \mathrm{d}r$ *is in* $L^1((0, \Theta))$ *and if the following conditions are satisfied for all* $\theta \in [0, \Theta]$*:*

$$\mathscr{E}(\theta, \gamma(\theta)) \leq \mathscr{E}(\theta, v) + \Psi(\gamma(\theta) - v) \quad \textit{for all } v \in H^1_0(I), \tag{3.7a}$$

$$\mathscr{E}(\theta, \gamma(\theta)) + \mathrm{dis}_\Psi(\gamma; [0, \theta]) = -\int_0^\theta \int_I \gamma(\vartheta) \mathrm{d}r \, \mathrm{d}\vartheta, \tag{3.7b}$$

where $\mathrm{dis}_\Psi(\gamma; [0, \theta])$ *is the total variation of γ on $[0, \theta]$ with respect to the distance $d(\gamma_1, \gamma_2) = \Psi(\gamma_1 - \gamma_2)$, i.e.,*

$$\mathrm{dis}_\Psi(\gamma; [0, \theta]) := \sup\left\{\sum_{j=1}^N \Psi(\gamma(\theta_j) - \gamma(\theta_{j-1})) : \ N \in \mathbb{N}, \ 0 = \theta_0 < \cdots < \theta_N = \theta\right\}.$$

In the present setting (quadratic energy) the next proposition is established without burden by invoking, for instance, Theorem 2.1 in Ref. [30]:

Proposition 3.2. *There exists a unique solution γ of* (3.2)–(3.3)*. Moreover, $\theta \mapsto \gamma(\theta)$ is Lipschitz continuous as a function from $[0, \Theta]$ to $H^1_0(I)$.*

Remark 3.3. As is well known (see e.g. Section 2.1 in [30]), there are other, equivalent ways to define a solution to (3.2)-(3.3). In particular, the dual formulation (i.e., the strain-gradient counterpart of (2.4)) is given by

$$\theta - \kappa\gamma + \Lambda^2\gamma_{rr} \in \partial\Psi(0) \quad \text{and} \quad \langle \theta - \kappa\gamma + \Lambda^2\gamma_{rr} - \sigma, \dot{\gamma}\rangle \geq 0 \quad \text{for all } \sigma \in \partial\Psi(0),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(I)$ and $H^1_0(I)$ and $\partial\Psi(0) = \{\sigma \in H^{-1}(\Omega):\ \Psi(u) \geq \langle\sigma, u\rangle\ \forall u \in H^1_0(I)\}$. A slight generalization of the arguments in [2, proof of Theorem 6.1] shows that $\partial\Psi(0)$ is characterized as

$$\partial\Psi(0) = \left\{\tilde{\tau} - \tilde{k}_r :\ \|(\tilde{\tau}, \lambda\tilde{k})\|_\infty \leq 1\right\}.$$

3.3. Characterizations of τ_Y. The first main result of this paper is the following characterization of θ_Y:

THEOREM 3.4. *Let γ be the unique energetic solution to* (3.2)-(3.3) *and let*

$$\theta_Y = \frac{\tau_Y}{S_0} = \sup\left\{\theta \geq 0 :\ \gamma \equiv 0 \quad in \quad I \times [0,\theta]\right\}.$$

Then

$$\theta_Y = \inf\left\{\Psi(\phi):\ \phi \in H^1_0(I),\ \int_I \phi \mathrm{d}r = 1\right\}. \tag{3.8}$$

In order to explain the relation between the two quantities, it is convenient to briefly illustrate the main steps in the proof, whose details are given in §4. We begin by observing that the energy–balance condition (3.7b) is identically satisfied for all $\theta \in (0, \theta_Y)$. Thus, what determines the onset of plastic flow is the loss of stability of the trivial state $\gamma \equiv 0$. This leads us to consider the *stability indicator*:

$$m(\theta) := \inf_{\phi \in H^1_0(I)} \Phi_\theta(\phi), \quad \text{where} \quad \Phi_\theta(\phi) := \mathscr{E}(\theta, \phi) + \Psi(\phi). \tag{3.9}$$

We will indeed argue that

$$\theta_Y = \inf\{\theta \geq 0 :\ m(\theta) < 0\}$$

(cf. Proposition 4.3). Next, we note that the plastic dissipation rate Ψ is (positively) homogeneous of degree one in γ, whereas the plastic free energy E is quadratic. Then, a simple scaling argument can be used to show that the *reduced stability indicator*

$$\widetilde{m}(\theta) := \inf_{\phi \in H^1_0(I)} \widetilde{\Phi}_\theta(\phi), \quad \text{where} \quad \widetilde{\Phi}_\theta(\phi) := \Psi(\phi) - \theta \int_I \phi \, \mathrm{d}r \tag{3.10}$$

is equivalent to the *stability indicator*:

$$m(\tau) < 0 \Leftrightarrow \widetilde{m}(\tau) < 0$$

(cf. Proposition 4.4). The last step of our argument consists in observing that, again by homogeneity, for negative values of $\widetilde{m}$ we can restrict our attention to the subspace of tests ϕ satisfying the normalization condition $\int_I \phi \mathrm{d}r = 1$: this leads to Theorem 3.4.

3.4. The formula for τ_Y. The second main result of this paper is the following explicit formula for τ_Y:

THEOREM 3.5. *The renormalized yield shear stress* $\theta_Y = \dfrac{\tau_Y}{S_0}$ *and the* renormalized dissipative scale $\lambda = \dfrac{\ell}{h}$ *are related by*

$$\lambda = \frac{2\sqrt{\theta_Y^2 - 1}}{\pi(\theta_Y - \sqrt{\theta_Y^2 - 1}) + 2\theta_Y \arctan \frac{1}{\sqrt{\theta_Y^2 - 1}}}. \tag{3.11}$$

The proof is provided in Section 5 and relies on results from Ref. [2], guaranteeing that the relaxation in $BV(I)$ of the infimum problem in (3.8) admits a minimizer ϕ_Y which is smooth in I and satisfies the Euler-Lagrange equation

$$\theta_Y = \frac{\phi_Y}{\sqrt{{\phi_Y}^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}} - \lambda^2 \frac{\mathrm{d}}{\mathrm{d}r} \frac{\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}}{\sqrt{{\phi_Y}^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}}. \tag{3.12}$$

By a suitable change of dependent variable, we convert (3.12) into a *first-order* differential equation with *two* side conditions. The extra side condition selects the *eigenvalue* θ_Y of the E-L equation (3.12), yielding (3.11).

The graph of τ_Y/S_0, recovered from (3.11), is plotted in Fig. 3.1 (recall (3.1) and (3.5)). Our result confirms that as the sample becomes smaller, i.e. $\lambda = \ell/h$ increases, the actual yield strength increases: hence *smaller samples are stronger*. Needless to say, the results from our plot agree with the numerical calculations carried out in Ref. [4] and reported in Figure 4 thereof.

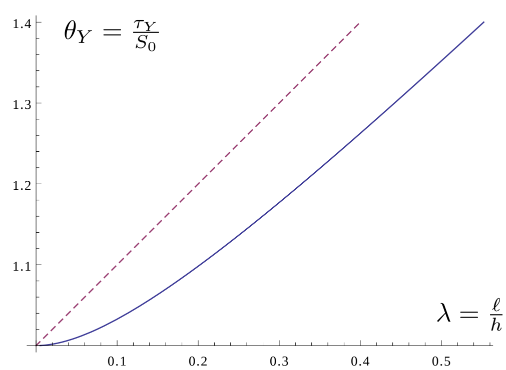

FIG. 3.1. *Solid line: renormalized effective yield strength τ_Y/S_0 versus renormalized dissipative scale ℓ/h, as from formula* (3.11)*. Dashed line: the upper bound $\frac{\tau_Y}{S_0} < 1 + \frac{\ell}{h}$ derived in [4]. This plot agrees with the result computed numerically in [4] and reported in Fig. 4 thereof. When comparing the two figures, the reader should take into account that in the present paper the symbol h denotes half the thickness of the strip, whereas in [4] the same symbol denotes the overall thickness.*

Our explicit formula provides additional insight concerning the asymptotic behavior of the actual yield strength for small and large values of h. In particular, from

(3.11) one finds that, for $0 < \theta_Y - 1 \ll 1$,

$$\lambda \sim \frac{\sqrt{2}}{\pi}\sqrt{\theta_Y - 1},$$

which implies that, for $0 < \lambda \ll 1$, the renormalized actual yield strength has the following asymptotic behavior:

$$\theta_Y - 1 \sim \frac{\pi^2}{2}\lambda^2 \quad \text{for} \ \ 0 < \lambda \ll 1.$$

We also note that, as $\lambda = \ell/h \to \infty$, a linear relation is recovered:

$$\theta_Y - \lambda \sim \frac{\pi}{4} \quad \text{for} \ \ \lambda \gg 1.$$

REMARK 3.6. It would be interesting to see if and how the dependence of τ_Y on ℓ is modified by a generalization of the plastic dissipation-rate density in (2.11) which preserves 1-homogeneity, namely $\left((\dot\gamma^{\rm p})^q + \ell^q(\dot\gamma^{\rm p}_y)^q\right)^{1/q}$ with $q \geq 1$, as there are mechanical arguments supporting it (see e.g. [7, 14]). It would also be interesting to seek for quantitative relations between the onset of plastic flow and the dissipative length-scale under different symmetry assumptions (e.g., torsional symmetry) or, better, in a generic three-dimensional framework.

4. Proof of Theorem 3.4. Existence and uniqueness of the minimum in (3.9) is readily ascertained through the direct method of the calculus of variations, owing to coercivity, lower semicontinuity, and convexity of Φ_θ in $H^1_0(I)$:

LEMMA 4.1. *For any $\lambda > 0$ there exists a unique minimizer ϕ_λ of the infimum problem in* (3.9)*.*

The first step is to show that if the trivial state is not stable at a certain value of the renormalized shear stress θ during the loading process, then it is not stable for whatever higher value:

LEMMA 4.2. *The function $[0,\Theta] \ni \theta \mapsto m(\theta)$ defined in* (3.9) *is non-increasing.*

Proof. Let ϕ_θ be the unique minimizer of Φ_θ. First, we observe that

$$\phi_\theta \geq 0 \qquad \text{a.e. in } I \text{ for all } \theta \geq 0. \tag{4.1}$$

Indeed, obviously $\phi_0 \equiv 0$; for $\theta > 0$, if $\phi_\theta < 0$ in a set J of positive measure, then (by the definitions (3.6) and (3.4) of $\mathcal{E}$, resp. Ψ) we would have $\Phi_\theta(|\phi_\theta|) < \Phi_\theta(\phi_\theta)$, a contradiction. Thus, given $\theta_1 \leq \theta_2$, we have

$$\begin{aligned} m(\theta_2) &\overset{(3.9)}{=} \Phi_{\theta_2}(\phi_{\theta_2}) \\ &\leq \Phi_{\theta_2}(\phi_{\theta_1}) \quad (\text{by def. of } \phi_{\theta_2}) \\ &\overset{(3.6),(4.1)}{\leq} \Phi_{\theta_1}(\phi_{\theta_1}) \overset{(3.9)}{=} m(\theta_1), \end{aligned}$$

as desired. ∎

The previous lemma is expedient to arrive to the following characterization of θ_Y.

PROPOSITION 4.3. *Let γ be the unique energetic solution to* (3.2)-(3.3) *and let θ_Y and m as in* (3.5)*, resp.* (3.9)*. Then*

$$\theta_Y = \inf\left\{\theta \geq 0 : \ m(\theta) < 0\right\}.$$

Proof. Let us set $\widehat{\theta} = \inf\{\theta \geq 0 : \ m(\theta) < 0\}$. We notice that, since $m(\theta)$ is nonincreasing, $m(\theta) = 0$ in $[0, \widehat{\theta})$. Hence, by direct substitution into (3.7), we see that the trivial function $\theta \mapsto 0$ is an energetic solution on the interval $[0, \widehat{\theta})$. By the uniqueness of the energetic solution, and by (3.5), it follows that $\theta_Y \geq \widehat{\theta}$.

The reverse inequality follows from the monotonicity of $\theta \mapsto m(\theta)$: suppose indeed that $\widehat{\theta} < \theta_Y$; then, by Lemma 4.2 there exists $\tilde{\theta} < \theta_Y$ such that $m(\tilde{\theta}) < 0$; however, $\tilde{\theta} < \theta_Y$ implies that $\gamma(\tilde{\theta}) = 0$; thus, by (3.7a) and (3.9), this means that $m(\tilde{\theta}) = 0$, whence a contradiction. ◻

We now show that the reduced stability indicator defined in (3.10) can be used to detect the onset of plastic flow. Indeed, we have the following equivalence:

Proposition 4.4.

$$\theta_Y = \inf\{\theta \geq 0 : \widetilde{m}(\theta) < 0\}. \tag{4.2}$$

Proof. In view of Proposition 4.3, it suffices to show that

$$m(\theta) < 0 \qquad \text{if and only if} \qquad \widetilde{m}(\theta) < 0.$$

Since by definition $\tilde{\Phi}_\theta \leq \Phi_\theta$, $m(\theta) < 0$ obviously implies $\tilde{m}(\theta) < 0$. For the reverse implication, let us assume $\tilde{m}(\theta) < 0$. Then there exists $\widetilde{\phi} \in H^1_0(I)$ such that $\widetilde{\Phi}_\theta(\widetilde{\phi}) < 0$. On the other hand, by the 1-homogeneity of $\widetilde{\Phi}_\theta$,

$$\lim_{\alpha \to 0+} \frac{\Phi_\theta(\alpha\widetilde{\phi})}{\alpha} = \widetilde{\Phi}_\theta(\widetilde{\phi}) < 0.$$

Thus $\Phi_\theta(\alpha\widetilde{\phi}) < 0$ for $\alpha > 0$ sufficiently small, whence $m(\theta) < 0$. ◻

With Proposition 4.4 at hand we are now ready to establish the variational characterization we have been after.

Proof. [Proof of Theorem 3.4] Let

$$\widehat{\theta}_Y(\lambda) := \inf\left\{\Psi(\phi) \colon \ \phi \in H^1_0(I), \ \int_I \phi \mathrm{d}r = 1\right\}. \tag{4.3}$$

On recalling the definitions of Ψ and $\widetilde{\Phi}_\theta$ given in (3.4), respectively (3.10), we see that the inequality

$$\theta_Y \leq \widehat{\theta}_Y(\lambda) \tag{4.4}$$

is implied by the following chain of implications:

$$\begin{aligned}
\widehat{\theta}_Y(\lambda) < \theta \quad &\Rightarrow \quad \Psi(\bar{\phi}) < \theta \text{ for some } \bar{\phi} \in H^1_0(I) \text{ such that } \int_I \bar{\phi}\mathrm{d}r = 1 \\
&\Rightarrow \quad \inf_{\phi \in H^1_0(I)} \left(-\int_I \theta\phi \mathrm{d}r + \Psi(\phi)\right) < 0 \\
&\overset{(3.10)}{\Rightarrow} \quad \widetilde{m}(\theta) < 0 \\
&\overset{(4.2)}{\Rightarrow} \quad \theta_Y \leq \theta.
\end{aligned}$$

Having established (4.4), it remains for us to prove the reverse inequality:

$$\theta_Y \geq \widehat{\theta}_Y(\lambda). \tag{4.5}$$

To this aim, let $\theta \in (0, \widehat{\theta}_Y(\lambda))$. By the definition (4.3) of $\widehat{\theta}_Y(\lambda)$, we have

$$\theta \int_I \phi \mathrm{d}r = \theta < \Psi(\phi) \quad \text{for all } \phi \in H_0^1(I) \text{ such that } \int_I \phi \mathrm{d}r = 1. \tag{4.6}$$

Since both sides of the inequality in (4.6) are positively 1-homogeneous, (4.6) upgrades to

$$\theta \int_I \phi \mathrm{d}r < \Psi(\phi) \quad \text{for all } \phi \in H_0^1(I) \text{ such that } \int_I \phi \mathrm{d}r > 0. \tag{4.7}$$

In turn, since Ψ is non-negative, (4.7) upgrades to

$$0 \leq \Psi(\phi) - \theta \int_I \phi \mathrm{d}r \overset{(3.10)}{=} \widetilde{\Phi}_\theta(\phi) \quad \text{for all } \phi \in H_0^1(I) \tag{4.8}$$

which holds for all $\theta \in (0, \widehat{\theta}_Y(\lambda))$. Summing up, we have the implication:

$$0 \leq \theta < \widehat{\theta}_Y(\lambda) \overset{(4.8)}{\Rightarrow} \widetilde{m}(\theta) \;=\; \inf_{\phi \in H_0^1(I)} \widetilde{\Phi}_\theta(\phi) \geq 0 \overset{(4.2)}{\Rightarrow} \theta \leq \theta_Y,$$

whence (4.5), since $\theta_Y \geq 0$ by definition. □

5. Proof of Theorem 3.5. The infimum problem in (3.8) was addressed in Ref. [2]. Consider the *relaxation* of Ψ,

$$\bar{\Psi}(\phi) := \inf \Big\{ \liminf_{k \to \infty} \Psi(\phi_k) : \ \{\phi_k\} \subseteq W_0^{1,1}(I), \ \phi_k \to \phi \text{ in } L^1(I) \Big\}, \tag{5.1}$$

i.e. the largest lower semicontinous extension of Ψ. It is shown in Ref. [2] that the relaxation $\bar{\Psi}$ has the following representation for $\phi \in BV(I)$:[2]

$$\bar{\Psi}(\phi) = \int_I \sqrt{\phi^2 + \lambda^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}r} \right)^2} \mathrm{d}r + \lambda \| D^s \phi \|(I) + \lambda \left(|\phi(-1)| + |\phi(+1)| \right). \tag{5.2}$$

Notice that, as is customary in the BV setting, homogeneous boundary conditions are now incorporated in the functional through the penalization term $\lambda |\phi|(\partial I) = \lambda |\phi - 0|(\partial I)$, which measures the jump between the trace of ϕ and the prescribed null value.

The following results were established in Ref. [2].

Theorem 5.1 (see Thm. 5.1 in Ref. [2]). *Let* $\bar{\Psi}$ *as in* (5.1). *There exists a unique* $\phi_Y \in SBV(I)$ *such that* $\int_I \phi_Y \mathrm{d}r = 1$ *and*

$$\bar{\Psi}(\phi_Y) = \min \Big\{ \bar{\Psi}(\phi) : \phi \in L^1(I), \int_I \phi \mathrm{d}r = 1 \Big\}.$$

[2]Here $\|\mu\|$ denotes the total variation of a measure μ (see e.g. [3, Def. 1.4]) and $\frac{\mathrm{d}\phi}{\mathrm{d}r}$, resp. $D^s\phi$, denote the absolutely continuous, resp. singular, part of $D\phi$ with respect to the Lebesgue measure (see e.g. [3, Th. 1.28 and §3.9]). We also refer to [3] for definitions and basic properties of the spaces $BV(I)$ and $SBV(I)$.

Moreover, ϕ_Y is even, strictly decreasing in $[0,1)$, *and smooth in* $(-1,1)$*; furthermore, it solves the* Euler-Lagrange equation

$$\bar{\Psi}(\phi_Y) = \frac{\phi_Y}{\sqrt{{\phi_Y}^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}} - \lambda^2 \frac{\mathrm{d}}{\mathrm{d}r}\frac{\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}}{\sqrt{{\phi_Y}^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}} \quad \textit{in } I \tag{5.3}$$

and it satisfies

$$\lim_{r\to 1^-} \frac{\phi_Y(r)}{\phi_Y(0)} = \frac{\theta_Y - 1}{\theta_Y} \quad \textit{and} \quad \lim_{r\to 1^-} \frac{\mathrm{d}\phi_Y}{\mathrm{d}r}(r) = -\infty. \tag{5.4}$$

Remark 5.2. Notably, (5.4) shows that the solution $\phi_Y \in SBV(I)$ of the relaxed minimization problem does *not* satisfy the boundary conditions $\phi(-1) = \phi(1) = 0$; generally speaking, this amounts to saying that, in order to minimize $\bar{\Psi}$ with mass constraint, paying a jump discontinuity at the boundary is cheaper than attaining the boundary value zero.

We are now ready to prove Theorem 3.5.

Proof. [Proof of Theorem 3.5] In view of Theorem 3.4 and since $H^1_0(I)$ is dense in $BV(I)$,

$$\bar{\Psi}(\phi_Y) = \theta_Y. \tag{5.5}$$

We also notice that, since $\mathrm{d}\phi_Y/\mathrm{d}r < 0$ in $[0,1)$ and ϕ_Y is positive with $\int_I \phi_Y(r)\mathrm{d}r = 1$,

$$\theta_Y \overset{(5.5)}{=} \bar{\Psi}(\phi_Y) \overset{(5.2)}{\geq} \int_I \sqrt{{\phi_Y}^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}\mathrm{d}r > \int_I |\phi_Y| dr = 1. \tag{5.6}$$

Now, consider the function

$$\zeta(r) := -\lambda \frac{\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}}{\sqrt{\phi_Y^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}}.$$

Since ϕ_Y is smooth and positive in I, ζ is smooth as well. We note that

$$1 - \zeta^2 = 1 - \lambda^2 \frac{\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}{\phi_Y^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}. = \frac{(\phi_Y)^2}{\phi_Y^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}.$$

Hence, since $\phi_Y > 0$,

$$\frac{\phi_Y}{\sqrt{\phi_Y^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}} = \sqrt{1-\zeta^2}. \tag{5.7}$$

By making also use of the Euler-Lagrange equation, we see that ζ satisfies the following differential equation:

$$\lambda\frac{\mathrm{d}\zeta}{\mathrm{d}r} \overset{(5.3)}{=} \bar{\Psi}(\phi_Y) - \frac{\phi_Y}{\sqrt{\phi_Y^2 + \lambda^2\left(\frac{\mathrm{d}\phi_Y}{\mathrm{d}r}\right)^2}} \overset{(5.7),(5.5)}{=} \theta_Y - \sqrt{1-\zeta^2}. \tag{5.8}$$

It follows from (5.6) and (5.8) that $\frac{\mathrm{d}\zeta}{\mathrm{d}r} > 0$. Hence,

$$\frac{1}{\lambda} \overset{(5.8)}{=} \int_0^1 \frac{\frac{\mathrm{d}\zeta}{\mathrm{d}r}}{\theta_Y - \sqrt{1-\zeta^2}}\,\mathrm{d}r = \int_{\zeta(0)}^{\zeta(1-)} \frac{1}{\theta_Y - \sqrt{1-\zeta^2}}\mathrm{d}\zeta.$$

In addition, since ϕ_Y is even and because of (5.4), we have that

$$\zeta(0) = 0 \quad \text{and} \quad \lim_{r\to 1-} \zeta(r) = 1.$$

Therefore,

$$\frac{1}{\lambda} = \int_0^1 \frac{\mathrm{d}\zeta}{\theta_Y - \sqrt{1-\zeta^2}}. \tag{5.9}$$

The integral on the right-hand side of (5.9) is well defined and can be computed explicitly. As a result, we arrive at formula (3.11) for the renormalized actual yield stress. □

Appendix A. The nonlocal flow rule.

In this section we briefly recapitulate the steps leading to the flow rule (2.8), as devised in Ref. [4], with a few changes from the original path. At variance with the previous sections, we do not assume proportional loading. Accordingly, the independent variables are now $y \in (-h, +h)$ and t, which stands for time, and the index y denotes partial differentiation with respect to y.

A.1. Principle of virtual powers. We start from the decomposition

$$u_y = \gamma^{\mathrm{e}} + \gamma^{\mathrm{p}} \tag{A.1}$$

of the *shear strain* u_y into an *elastic part* γ^{e} and a *plastic part* γ^{p}. This decomposition is accompanied by the prescription that, given any part $P = (a, b) \subset (-h, +h)$, the internal power expended within P has the form:

$$\mathscr{W}_{\mathrm{int}}(P) = \int_P \tau\dot{\gamma}^{\mathrm{e}} + \tau^{\mathrm{p}}\dot{\gamma}^{\mathrm{p}} + k^{\mathrm{p}}\dot{\gamma}^{\mathrm{p}}_y \mathrm{d}y. \tag{A.2}$$

Thus, power expenditure by the *macroscopic shear stress* τ is accompanied by working of the *plastic microstress* τ^{p} and *gradient microstress* k^{p}. If body forces are left out of the picture, the external power expended on $P = (a, b)$ is localized on the boundary $\partial P = \{a, b\}$ and has the form:

$$\mathscr{W}_{\mathrm{ext}}(P) = \widehat{\tau}(b)\dot{u}(b) + \widehat{k}^{\mathrm{p}}(b)\dot{\gamma}^{\mathrm{p}}(b) - \widehat{\tau}(a)\dot{u}(a) - \widehat{k}^{\mathrm{p}}(a)\dot{\gamma}^{\mathrm{p}}(a),$$

where $\widehat{\tau}$ and k^{p} are, respectively, the *macroscopic* and the *microscopic* shear tractions. The application of the principle of virtual powers yields:

1) the identification between stress and traction, namely $\tau = \widehat{\tau}$, along with the *macroscopic-force balance*:

$$\tau_y = 0; \tag{A.3}$$

2) the identification of $\widehat{k}^{\mathrm{p}}$ with k^{p}, along with the *microscopic force-balance*:

$$\tau = \tau^{\mathrm{p}} - k^{\mathrm{p}}_y.$$

A.2. Constitutive prescriptions. Consistent with the choice (A.2) for the internal power expenditure, it is assumed in Ref. [4] that the *free-energy density* φ depends on the triplet $(\gamma^{\mathrm{e}}, \gamma^{\mathrm{p}}, \gamma^{\mathrm{p}}_y)$ through a constitutive equation of the form:

$$\varphi = \widehat{\varphi}(\gamma^{\mathrm{e}}, \gamma^{\mathrm{p}}, \gamma^{\mathrm{p}}_y).$$

It is also assumed that the constitutive mapping delivering the free-energy density is the sum:

$$\widehat{\varphi}(\gamma^{\mathrm{e}}, \gamma^{\mathrm{p}}, \gamma^{\mathrm{p}}_y) = \widehat{\varphi}^{\mathrm{e}}(\gamma^{\mathrm{e}}) + \widehat{\varphi}^{\mathrm{p}}(\gamma^{\mathrm{p}}, \gamma^{\mathrm{p}}_y)$$

of an *elastic-energy mapping* $\widehat{\varphi}^{\mathrm{e}}$, which takes into account the elastic shear, and a *defect-energy mapping* $\hat{\varphi}^{\mathrm{p}}$, which depends on the plastic shear and on its gradient. In particular, the elastic-energy mapping is given the form $\widehat{\varphi}^{\mathrm{e}}(\gamma^{\mathrm{e}}) = \frac{1}{2}G(\gamma^{\mathrm{e}})^2$, with $G > 0$ the *shear modulus*. This assumption is accompanied by the standard constitutive prescription $\tau = \frac{\partial \widehat{\varphi}^{\mathrm{e}}}{\partial \gamma^{\mathrm{e}}}$, whence:

$$\tau = G\gamma^{\mathrm{e}}. \tag{A.4}$$

The microstresses are then split into an energetic part and a dissipative part by setting

$$\tau^{\mathrm{p}} = \tau^{\mathrm{dis}} + \tau^{\mathrm{en}}, \qquad k^{\mathrm{p}} = k^{\mathrm{dis}} + k^{\mathrm{en}},$$

where

$$\tau^{\mathrm{en}} = \frac{\partial \widehat{\varphi}^{\mathrm{p}}}{\partial \gamma^{\mathrm{p}}}, \qquad k^{\mathrm{en}} = \frac{\partial \widehat{\varphi}^{\mathrm{p}}}{\partial \gamma^{\mathrm{p}}_y},$$

so that the following reduced form of the dissipation inequality is arrived at:

$$0 \leq \tau^{\mathrm{dis}} \dot{\gamma}^{\mathrm{p}} + k^{\mathrm{dis}} \dot{\gamma}^{\mathrm{p}}_y.$$

By analogy with the constitutive equations describing viscoplastic behavior in metals, the following constitutive equations have been considered in Ref. [4]:

$$\begin{aligned} &\tau^{\mathrm{dis}} = S\left(\frac{d^{\mathrm{p}}}{d_0}\right)^m \frac{\dot{\gamma}^{\mathrm{p}}}{d^{\mathrm{p}}}, \qquad k^{\mathrm{dis}} = S_0 \ell^2 \left(\frac{d^{\mathrm{p}}}{d_0}\right)^m \frac{\dot{\gamma}^{\mathrm{p}}_y}{d^{\mathrm{p}}}, \\ &d^{\mathrm{p}} = \sqrt{(\dot{\gamma}^{\mathrm{p}})^2 + \ell^2 \left(\dot{\gamma}^{\mathrm{p}}_y\right)^2}, \qquad \dot{S} = H(S) d^{\mathrm{p}}, \qquad S(0) = S_0. \end{aligned} \tag{A.5}$$

Here: S is the *current yield strength*, an internal variable whose value at time $t = 0$ is equal to the *initial yield strength* S_0 and whose time derivative is proportional to the *effective flow rate* d^{p} through a (isotropic) *hardening/softening function* $H(S)$; d_0 is the *reference flow rate*; $m > 0$ is the *rate-sensitivity parameter*.

The constitutive prescription (2.8b) follows by setting $H(S) = 0$ (no isotropic hardening) and by formally letting $m \to 0$ in (A.5) (rate-independent limit). The partial differential equation (2.8a) is recovered by choosing:

$$\widehat{\varphi}^{\mathrm{p}}(\gamma^{\mathrm{p}}, \gamma^{\mathrm{p}}_y) = \frac{1}{2} S_0 \big(\kappa (\gamma^{\mathrm{p}})^2 + L^2 (\gamma^{\mathrm{p}}_y)^2\big).$$

A.3. The traction problem. In the *traction problem*, the bottom side of the strip is clamped, that is,

$$u(-h,t)=0, \tag{A.6}$$

and a time-dependent shear traction $\widehat{\tau}_h(t)$ is prescribed on the upper side, that is,

$$\tau(h,t)=\widehat{\tau}_h(t).$$

On recalling that the shear stress is spatially constant by (A.3), we see that the shear stress $\tau(t)$ appearing in the flow rule (2.8) is a prescribed, spatially–constant field. Thus, the flow rule (A.5) can be solved for the plastic shear γ^{p} without knowing the displacement field. The latter is recovered by integrating (A.1) and (A.4), and by taking (A.6) into account, that is to say,

$$u(y,t)=\int_{-h}^{y}\left(\frac{\tau(t)}{G}+\gamma^{\mathrm{p}}(s,t)\right)\mathrm{d}s. \tag{A.7}$$

A.4. Comparison with the Gurtin-Anand three-dimensional theory. Under constitutive prescriptions analogous to those mentioned above, once augmented with kinematic hardening the three-dimensional theory developed in [22] leads to the following flow rule (see also [23, §90]):

$$\begin{cases} \mathbf{T}_0-\mathbf{T}_{\mathrm{back}}=\mathbf{T}^{\mathrm{p}}_{\mathrm{dis}}-\mathrm{div}\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}, \\ S_0^{-1}\left(\mathbf{T}^{\mathrm{p}}_{\mathrm{dis}},\ell^{-1}\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}\right)\in \mathrm{Sign}(\dot{\mathbf{E}}^{\mathrm{p}},\ell\nabla\dot{\mathbf{E}}^{\mathrm{p}}), \end{cases} \tag{A.8a}$$

together with the standard force balance

$$\mathrm{div}\mathbf{T}=\mathbf{0}, \tag{A.8b}$$

where

$$\mathrm{sym}\nabla\mathbf{u}=\mathbf{E}^{\mathrm{e}}+\mathbf{E}^{\mathrm{p}},\quad \mathbf{T}=\mathbf{2}\mu\mathbf{E}^{\mathrm{e}}+\lambda(\mathrm{tr}\mathbf{E}^{\mathrm{e}})\mathbf{I},$$

$$\mathbf{T}_{\mathrm{back}}=S_0\kappa\mathbf{E}^{\mathrm{p}}-S_0L^2\left(\Delta\mathbf{E}^{\mathrm{p}}-\mathrm{sym}(\nabla\mathrm{div}\mathbf{E}^{\mathrm{p}})+\frac{1}{3}(\mathrm{div}\,\mathrm{div}\mathbf{E}^{\mathrm{p}})\mathbf{I}\right),$$

$$\mathrm{Sign}(\mathbf{V},\mathbb{V})=\begin{cases}\left\{\frac{(\mathbf{V},\mathbb{V})}{\sqrt{|\mathbf{V}|^2+|\mathbb{V}|^2}}\right\} & \text{if } |\mathbf{V}|^2+|\mathbb{V}|^2\neq 0 \\ \left\{(\mathbf{V},\mathbb{V})\in\mathbb{R}^{3\times3}_{\mathrm{Sym},0}\times(\mathbb{R}^{3\times3}_{\mathrm{Sym},0}\times\mathbb{R}^3): |\mathbf{V}|^2+|\mathbb{V}|^2\leq 1\right\} & \text{if } |\mathbf{V}|^2+|\mathbb{V}|^2=0,\end{cases}$$

and $\mathbf{T}_0$ is the deviatoric part of $\mathbf{T}$. Here $\mathbb{R}^{3\times3}_{\mathrm{Sym},0}$ denotes the space of symmetric and traceless 3×3 matrices.

Let Ω_h be as in (2.1) and let $\tau=\tau(t)$ be prescribed. Formally, if $(\mathbf{u},\mathbf{E}^{\mathrm{p}})$ is a solution to (A.8) in $\Omega_h\times(0,\infty)$ with $(\mathbf{T}\mathbf{e}_2)|_{y=+h}=\tau\mathbf{e}_1$ and $\mathbf{u}|_{y=-h}=\mathbf{0}$, one can check that:

(1) by translational invariance, $(\mathbf{u},\mathbf{E}^{\mathrm{p}})$ are independent of x and z;
(2) by odd reflection with respect to $z=0$ and in view of (1), $\mathbf{u}\cdot\mathbf{e}_3=\mathbf{e}_3\cdot\mathbf{E}^{\mathrm{p}}\mathbf{e}_1=\mathbf{e}_3\cdot\mathbf{E}^{\mathrm{p}}\mathbf{e}_2=0$;
(3) since $(-\mathbf{u},-\mathbf{E}^{\mathrm{p}})$ is a solution to (A.8) with τ replaced by $-\tau$, by odd reflection with respect to $x=0$ and in view of (1), $\mathbf{u}\cdot\mathbf{e}_2=\mathbf{e}_i\cdot\mathbf{E}^{\mathrm{p}}\mathbf{e}_i=0$ for $i=1,2,3$.

This motivates Ansatz (2.2) in Section 2.

One can also check that if γ^{p} is a solution to (2.8) and u is defined similarly to (A.7), then $(u\mathbf{e_1}, \gamma^{\mathrm{p}}\mathrm{sym}(\mathbf{e_1} \otimes \mathbf{e_2}))$ is a solution to (A.8). In fact, we could have introduced (2.8) as well in this way rather than through the ad-hoc discussion in Section A.1-A.3. We have opted for the latter in the hope of making the resulting model more transparent.

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