Asymptotic behavior of minimal solutions of $-\Delta u = \lambda f(u)$ as $\lambda \to -\infty$.

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Abstract

We consider the following Dirichlet problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \qquad (\mathcal{P}_f^\lambda)$$

with $\lambda < 0$ and f non-negative and non-decreasing.
We show existence and uniqueness of solutions u_λ for any λ and discuss their asymptotic behavior as $\lambda \to -\infty$. In the expansion of u_λ *large solutions* naturally appear.

1 Introduction and main results

In this paper we consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \qquad (\mathcal{P}_f^\lambda)$$

where λ is a real parameter, Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N \geq 2$ and f is a real function satisfying the following assumption,

$$f \text{ non-decreasing}, \qquad f(0) > 0, \qquad f|_{0<f<f(0)} \text{ is } C^1. \qquad (1.1)$$

In this setting Crandall and Rabinowitz ([10], Section 4, see also [21]) prove for $0 < \lambda < \lambda^*$ and f convex the existence of a branch u_λ of *stable* positive solutions, i.e. satisfying

$$\lambda_1\big(-\Delta - \lambda f'(u_\lambda)I\big) > 0,$$

(here λ_1 demotes the first eigenvalue with zero Dirichlet boundary conditions). This branch is *minimal*, in the sense that any other solution u of $(\mathcal{P}_f^\lambda)$ verifies $u \geq u_\lambda$. There is a huge literature about minimal, non-minimal and stable solutions to $(\mathcal{P}_f^\lambda)$, see [13] as an example. We just recall some results about two nonlinearities which play an important role in this paper:

- $f(t) = \big((t-t_0)^+\big)^p$ with $p \geq 1$ and $t_0 < 0$ (*Problem of confined plasma*). A lot of authors studied this problem ([1, 5, 7, 25]) where the set $\{u > t_0\}$ is the *plasma* and the set $\{u < t_0\}$ is the *vacuum*.
- $f(t) = e^t$ (*the Liouville equation*). There is a very large literature mainly when $\Omega \subset \mathbb{R}^2$, see for instance [4, 14, 16, 24]. Much less is known in higher dimensions ([18]). Observe that in this case the function $v = -u$ solves $-\Delta v = -\lambda e^{-v}$, an equation which has been derived in [17] in the study of the stationary states for a model of evolution of the electronic density in the plasma (see also [8, 9]).

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The aim of this paper is to complete the study of the branch of stable solutions by considering the case $\lambda < 0$. Of course in this event by the maximum principle we get that $u < 0$ in Ω.
Quite surprisingly, this case was not considered in the literature and we will see that some new and interesting phenomena occur. It is easy to show that for any $\lambda < 0$ there exists a unique solution u_λ to $(\mathcal{P}_f^\lambda)$. So the interesting problem is to study the asymptotic behavior of u_λ as $\lambda \to -\infty$.
In order to state our first result let us introduce the following number:

$$t_0 := \inf\{t \in (-\infty, 0) : f(t) > 0\} \in [-\infty, 0). \tag{1.2}$$

We observe that t_0 is the same number which appears in the *plasma problem* and $t_0 = -\infty$ in the Liouville equation. Next theorem gives a description of the solution to $(\mathcal{P}_f^\lambda)$ for any f verifying (1.1).

Theorem 1.1.
Assume f satisfies (1.1). Then, for any $\lambda < 0$, $(\mathcal{P}_f^\lambda)$ has a unique stable solution u_λ. Moreover, $t_0 < u_\lambda(x) < 0$ for any $x \in \Omega$, where t_0 is defined by (1.2), and

$$u_\lambda(x) \underset{\lambda \to -\infty}{\to} t_0 \text{ in } L^\infty_{\text{loc}}(\Omega). \tag{1.3}$$

By the definition of f we have that if $f > 0$ then $t_0 = -\infty$ and so there is a *full blow-up* of the solution u_λ in Ω (these is the case of $f(t) = e^t$). On the other hand, for solutions of the confined plasma problem described before, we get that $u_\lambda(x) \underset{\lambda \to -\infty}{\to} t_0$ in Ω.
This means that when λ is *negative*, i.e. we have negative pressure, there is no *vacuum* in Ω and so no *free boundary* appears (see [7]).

An interesting property of the solution u_λ for $t_0 \in \mathbb{R}$ is the following (see Proposition 2.3)

$$\lambda f(u_\lambda) \to 0 \quad \text{as } \lambda \to -\infty.$$

However note that this is not true if $t_0 = -\infty$ (see Example 4.1).
The result in (1.3) is not surprising if one looks at the functional

$$J_\lambda(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega F(v) dx \tag{1.4}$$

associated with $(\mathcal{P}_f^\lambda)$ for $F(s) = \displaystyle\int_{t_0}^s f(t)dt$. It is easy to see that J_λ is coercive and u_λ is the minimum. The presence of the positive constant $-\lambda$ in front of the potential term suggests that it is convenient for u_λ to minimize it, i.e. to reach the value t_0 even if this increases the kinetic term which becomes infinite near the boundary. Indeed in the examples 3.2 (case *ii*) and 4.1 (case *i*) we find that $J_\lambda(u_\lambda) = (C + o(1))\sqrt{-\lambda}$ as $\lambda \to -\infty$ and both the kinetic and the potential term gives a contribution of order $\sqrt{-\lambda}$.
This phenomenon has some similarities with the Ginzburg-Landau problem

$$J_\varepsilon(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \frac{1}{\varepsilon^2} \int_\Omega F(v) dx$$

where f is a double well potential and the minimizers u_ε are characterized by a phase transition among the two zeroes of the potential term, say ± 1.
In our case, obviously, there is not phase transition, since f has the geometry of a single well and indeed Theorem 1.1 says that $u_\lambda \to t_0 \chi_\Omega$ in L^∞_{loc}.
Nevertheless, since in our case $u_\lambda = 0$ on the boundary, we think that some of the characteristics of the double well potential should appear near $\partial\Omega$.
Since $t_0 < v < 0$ a simple observation and the coarea formula gives

$$\begin{aligned} J_\lambda(v) &\geq \sqrt{-\lambda} \int_\Omega \sqrt{2F(v)} |\nabla v| dx = \sqrt{-\lambda} \int_{t_0}^0 \left(\int_{\Omega \cap \{v = s\}} \sqrt{2F(s)} d\mathcal{H}^{n-1}(y) \right) ds \\ &= \sqrt{-\lambda} \int_{t_0}^0 \sqrt{2F(s)} \mathcal{H}^{n-1}(\{v = s\}) ds \end{aligned}$$

where $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure. Hence if u_λ minimizes J_λ it is natural to expect that, far from the boundary $u_\lambda \to t_0$, which is the unique zero of the potential $F(v)$. It is likewise natural to expect that near the boundary the solution u_λ should minimize $\mathcal{H}^{n-1}(\{v = s\})$ i.e. the level sets are of minimal perimeter among the ones that converges to $\partial\Omega$.
What it should be natural to expect is that $u_\lambda(x) = u_\lambda(d(x, \partial\Omega))$ where $d(x, \partial\Omega)$ is the distance of the point x from the boundary which is what happen for the double well potential.
Next aim is to improve Theorem 1.1 computing a more detailed asymptotic behavior of the expansion (1.3).
Even if our analysis in this paper does not allow to obtain information near $\partial\Omega$ what we will see is that all solutions v of the limit problems which arise in the refined study of u_λ as $\lambda \to \infty$ have this nice geometrical property that near the boundary $v(x) = v(d(x, \partial\Omega))$.
As expected the value of t_0 and the shape of f will play a crucial role. For this reason we will state our results by separating the case in which t_0 is finite from that where $t_0 = -\infty$.

1.1 The case $t_0 \in \mathbb{R}$

In this case from Theorem 1.1 we have that the solution $u_\lambda \to t_0$ in Ω. The aim of this section is to improve (1.3) computing the additional terms of the expansion.
The model nonlinearity is $f(t) = \left((t - t_0)^+\right)^p$ with $p \geq 0$ and t_0 negative.
As remarked before this problem was studied by many people as $\lambda > 0$ and $p \geq 1$. For $p > 1$ we just recall [1] and the references therein and if $p = 1$ we mention [25, 5, 3, 7]. In this last paper it was also studied the asymptotic behavior of the solution as $\lambda \to +\infty$. In this case the region occupied by the plasma, namely $\{x \in \Omega$ such that $u_\lambda > -t_0\}$ has diameter converging to 0. We will see that as λ is negative the opposite phenomenon occurs. On the other hand, as $\lambda \to -\infty$, $p = 1$ is a threshold for our problem where the behavior changes dramatically. In particular, if $p > 1$ *large solutions* v appear in the expansion of the solution. Let us recall that v is a large solution in Ω if it satisfies

$$\begin{cases} \Delta v = g(v) & \text{in } \Omega \\ v(x) \underset{x \to \partial\Omega}{\to} +\infty \end{cases} \tag{1.5}$$

There is a massive literature about existence, uniqueness and asymptotic analysis of solutions v to (1.5), so it is impossible to give en exhaustive list of references. We just recall the seminal papers by Keller [19] and Osserman [23] where it was proved that if g is a positive, continuous, non-decreasing function then (1.5) admits solutions if and only if the following *Keller-Osserman* condition holds:

$$\int_{t_1}^{+\infty} \frac{\mathrm{d}t}{\sqrt{\int_{t_1}^{t} g(s)\mathrm{d}s}} < +\infty, \tag{1.6}$$

for some $t_1 > 0$. The uniqueness of large solutions has been established under some additional assumptions on f and the regularity of the domain Ω (see [12] for references and new results). Here we quote the result in [2] where the authors proved the uniqueness of the large solution to (1.5) when $g(t) = t^p$ and $p > 1$ and [20] in the case when $g(t) = e^t$.
Now we are in position to state our result.

Theorem 1.2.
Let u_λ be the unique solution to $(\mathcal{P}_f^\lambda)$. Assume there exists some $\gamma(\alpha) \underset{\alpha \to 0^+}{\to} 0$ such that

$$g_0(t) \leq \frac{f(\alpha t + t_0)}{\gamma(\alpha)} \underset{\alpha \to 0^+}{\to} t^p, \quad \text{with } p \geq 0 \text{ locally uniformly in } t > 0, \tag{1.7}$$

for some g_0 satisfying (1.6) Then the following alternative holds:

(i) If $\dfrac{\gamma(\alpha)}{\alpha} \underset{\alpha \to 0^+}{\to} 0$, then

$$u_\lambda = t_0 + \alpha_\lambda\big(v + o(1)\big) \quad \text{as } \lambda \to -\infty \text{ in } C^1_{\text{loc}}(\Omega),$$

where v is the unique large solution *to (1.5) with $g(t) = t^p$, for some $\alpha_\lambda \underset{\lambda \to -\infty}{\to} 0^+$.*

(ii) If $\dfrac{\gamma(\alpha)}{\alpha} \underset{\alpha\to 0^+}{\not\to} 0$ *and in addition:*

- *either* Ω *is a ball,*
- *or* $\Omega \subset \mathbb{R}^2$ *and/or* Ω *is strictly convex and*

$$\frac{f(\alpha t + t_0)}{\gamma(\alpha)} \le Ct^q \text{ for some } C > 0, 0 \le q \le 1, t > 0; \tag{1.8}$$

then,

$$u_\lambda(x) = t_0 + \alpha_\lambda \left(v\left(\frac{x - x_\lambda}{\varepsilon_\lambda}\right) + o(1)\right) \quad \text{as } \lambda \to -\infty \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^N),$$

where v *is an entire solution to*

$$\begin{cases} \Delta v = v^p & \text{in } \mathbb{R}^N \\ v \ge v(0) = 1. \end{cases}, \tag{1.9}$$

for some $\alpha_\lambda, \varepsilon_\lambda \underset{\lambda\to-\infty}{\to} 0$ *and* x_λ *being a minimum point of* u_λ*.*

Even if the convergence in Theorem 1.2 does not allow to obtain information on the behavior of u_λ near the boundary of Ω, we observe here that the large solution to (1.5) satisfies

$$\lim_{x\to x_0} \frac{\psi(u(x))}{d(x, \partial\Omega)} = 1,$$

where $x_0 \in \partial\Omega$. Here ψ is a function which depends only on the nonlinear term g in (1.5), see [2].

Remark 1.3.
The assumption that Ω *is planar or strictly convex is essential to have* $x_\lambda \underset{\lambda\to-\infty}{\to} x_0 \in \Omega$*, as in in the paper [15] by Gidas, Ni and Nirenberg (see Corollary 3 and the Problem stated just below it). In the case of a ball, one does not even need to assume (1.8), essentially because all solutions are radial.*
Notice that if Ω *is a ball, then the solution of (1.9) is also radially symmetric. So it is uniquely determined as the solution to the O.D.E.*

$$\begin{cases} v''(r) + \dfrac{N-1}{r} v'(r) = v(r)^p & \text{in } \mathbb{R} \\ v'(0) = 0 \\ v(0) = 1. \end{cases}$$

Remark 1.4.
The assumption $\dfrac{f(\alpha t + t_0)}{\gamma(\alpha)} \underset{\alpha\to 0^+}{\to} t^p$ *in (1.7) is rather general and it is satisfied when the nonlinearity* f *decay at zero at* t_0 *as a power or slowlier. Indeed it is equivalent to ask that*

$$\frac{f(\alpha t + t_0)}{\gamma(\alpha)} \underset{\alpha\to 0}{\to} g(t) \quad \text{for some } g.$$

See Lemma A.1 for details. Observe that the condition that $\dfrac{f(\alpha t + t_0)}{\gamma(\alpha)}$ *is bounded from below by* g_0 *in (1.7) is needed only in case (i).*
When, instead, the nonlinearity f *decay at zero faster, as in the case of*

$$f(t) = \begin{cases} 0 & \text{for } t < t_0 \\ e^{-\frac{1}{t-t_0}} & \text{for } t_0 < t < 0 \end{cases}$$

we still have that a large solution *appears in the expansion of* u_λ*. However we have to modify (1.7) assuming there exists* $\alpha(\beta) \ge 0$ *such that*

$$\begin{aligned} &\alpha(\beta) \underset{\beta\searrow t_0}{\to} 0, \qquad \frac{f(\beta)}{\alpha(\beta)} \underset{\beta\searrow t_0}{\to} 0 \\ &\text{and} \\ &g_0(t) \le \frac{f(\alpha(\beta)t + \beta)}{f(\beta)} \underset{\beta\searrow t_0}{\to} g(t) \text{ locally uniformly for } t > -\sup_\beta \frac{\beta - t_0}{\alpha(\beta)}, \end{aligned} \tag{1.10}$$

for some g_0 satisfying (1.6)*, and we get*

$$u_\lambda(x) = \beta_\lambda + \alpha_\lambda\big(v + o(1)\big) \quad \text{as } \lambda \to -\infty \text{ in } C^1_{\rm loc}(\Omega),$$

where v is the large solution *to* (1.5)*, corresponding to $g(t)$. In this case an exponential function $g(t)$ can appear in the limit problem.*

Due to the important role played by the nonlinearity $f(t) = \left((t-t_0)^+\right)^p$ we would like to state Theorem 1.2 expressly for this case. Note that $p > 1$ corresponds to the case (i) in Theorem 1.2 and $p \le 1$ to (ii).

Corollary 1.5.
Let $\lambda < 0$ and u_λ be the unique negative solution to

$$\begin{cases} -\Delta u = \lambda\left((u-t_0)^+\right)^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the following alternative holds:

(i) If $p > 1$, then

$$u_\lambda = t_0 + \frac{v + o(1)}{(-\lambda)^{\frac{1}{p-1}}} \quad \text{as } \lambda \to -\infty \text{ in } C^\infty_{\rm loc}(\Omega),$$

where v is the unique positive solution to

$$\begin{cases} \Delta v = v^p & \text{in } \Omega \\ v(x) \underset{x\to\partial\Omega}{\to} +\infty \end{cases};$$

(ii) If $0 \le p \le 1$ and Ω is either planar or strictly convex, then setting $\alpha_\lambda = u_\lambda(x_\lambda) - t_0$ and $\varepsilon_\lambda = \sqrt{\dfrac{\alpha_\lambda^{1-p}}{-\lambda}}$ we have that

$$u_\lambda\left(\varepsilon_\lambda x + x_\lambda\right) = t_0 + \alpha_\lambda\big(v + o(1)\big) \quad \text{in } C^\infty_{\rm loc}\left(\mathbb{R}^N\right),$$

where v is a solution to (1.9)*. When Ω is the unit ball instead*

$$u_\lambda(r) = t_0 + \alpha_\lambda v\left(\frac{r}{\varepsilon_\lambda}\right)$$

is the explicit solution to ($\mathcal{P}_f^\lambda$) *if α_λ is such that $\alpha_\lambda v\left(\dfrac{1}{\varepsilon_\lambda}\right) = -t_0$.*

Remark 1.6.
Our result applies also to suitable perturbation of $\left((t-t_0)^+\right)^p$, namely $f(t) = \left((t-t_0)^+\right)^p + \left((t-t_0)^+\right)^q$ with $q > p > 0$ or when f is given by $(t-t_0)^p \log^2(t-t_0)$ for $t > t_0$. The expansion of u_λ is the same as in (i) or (ii) of Corollary 1.5 and $g_0(t) = \left(t^+\right)^p$.

It will be interesting to remove the monotonicity assumption on f at least in the case of an asymptotic linear problem as in the paper [22].

1.2 The case $t_0 = -\infty$

In this case Theorem 1.2 only says that $u_\lambda \to -\infty$ in Ω. Our aim is to give a more precise expansion of u_λ and we will see that a crucial role is played by the limit of $f(t)$ as $t \to -\infty$. Let us recall that f is positive and increasing, so the only options are:

- $\lim\limits_{t\to-\infty} f(t) = c_0 > 0$

- $\lim_{t\to-\infty} f(t) = 0$

Let us consider the first alternative. We have the following

Theorem 1.7.
Let u_λ be the unique negative solution to $(\mathcal{P}_f^\lambda)$ with f verifying (1.1) and

$$\lim_{t\to-\infty} f(t) = c_0 > 0.$$

Then we have that

$$u_\lambda = \lambda\big(c_0 + o(1)\big)\phi \qquad \text{as } \lambda \to -\infty \text{ in } C^1_{\mathrm{loc}}(\Omega)$$

where ϕ is the solution of the torsion problem

$$\begin{cases} -\Delta\phi = 1 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

The proof of the previous result is not difficult and it follows by the standard regularity theory. In the other case $\lim_{t\to-\infty} f(t) = 0$ interesting new phenomena appear.

Theorem 1.8.
Let u_λ be the unique solution to $(\mathcal{P}_f^\lambda)$ with f verifying (1.1) and

$$\lim_{t\to-\infty} f(t) = 0.$$

Assume there exists some $\alpha(\beta), \gamma(\beta) \geq 0$ such that $\gamma(\beta) \underset{\beta\to-\infty}{\to} 0$ and

$$\frac{f(\alpha(\beta)t + \beta)}{\gamma(\beta)} \underset{\beta\to-\infty}{\to} g(t) \text{ locally uniformly in } t \in \mathbb{R}$$

Then, the following alternative holds:

(i) If $\frac{\beta}{\alpha(\beta)} \underset{\beta\to-\infty}{\to} -\infty$ and in addition $\frac{\gamma(\beta)}{\alpha(\beta)} \underset{\beta\to-\infty}{\to} 0$ and $\frac{f(\alpha(\beta)t+\beta)}{\gamma(\beta)} \geq g_0(t)$ for some g_0 satisfying (1.6), then

$$u_\lambda = \beta_\lambda + \alpha_\lambda\big(v + o(1)\big) \qquad \text{as } \lambda \to -\infty \text{ in } C^1_{\mathrm{loc}}(\Omega),$$

where v is the large solution *to (1.5), for some $\beta_\lambda \underset{\lambda\to-\infty}{\to} -\infty, \alpha_\lambda \underset{\lambda\to-\infty}{\to} 0$.*

(ii) If $\frac{\beta}{\alpha(\beta)} \underset{\beta\to-\infty}{\to} A < 0$, then

$$u_\lambda = \alpha_\lambda\big(v + o(1)\big) \qquad \text{as } \lambda \to -\infty \text{ in } C^1_{\mathrm{loc}}(\Omega),$$

where v is the unique (negative) solution to

$$\begin{cases} \Delta v = g(v - A) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.12}$$

for some $\alpha_\lambda \underset{\lambda\to-\infty}{\to} +\infty$

Here we observe that also in the case (ii) the solution v to (1.12) satisfies

$$\lim_{x\to x_0} |v(x) - \psi(d(x, \partial\Omega))| \to 0$$

if ψ is a function which depends only on the nonlinear term g, see [6] as an example.

Remark 1.9.
With respect to Theorem 1.2, the statement of Theorem 1.8 has some differences, also because in this case β cannot be fixed to t_0, as the latter equals $+\infty$. Anyway, some simplifications still occur in case (ii).
In fact, the limit function $g(t)$ is always a negative power of the type $\frac{1}{(-t)^p}$ for some $p \geq 0$ (see Lemma A.1 for details), and in the case $p = 0$ we recover the case of Theorem 1.7. On the other hand, in case (i) other function such as exponentials appear, as explained later on.

Moreover, it is not hard to see that one can take $\gamma(\beta) = f(\beta)$ (see again Lemma A.1).
Finally, local uniform convergence for $\frac{f(\alpha(\beta)t+\beta)}{\gamma(\beta)}$ can actually be assumed only for t for which $\alpha(\beta)t + \beta$ is negative (as we evaluate f on u_λ which attains negative values), namely $t < \sup_\beta \frac{-\beta}{\alpha(\beta)}$.

The model nonlinearity of the case (i) in the previous theorem is $f(t) = e^t$. Due to its importance, it seems useful to state explicitly the result.

Corollary 1.10.
Let $\lambda < 0$ and u_λ a family of negative solutions to

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$u_\lambda = -\log(-\lambda) + v + o(1) \text{ in } C^\infty_{\text{loc}}(\Omega) \text{ as } \lambda \to -\infty$$

and v is the unique positive solution to

$$\begin{cases} \Delta v = e^v & \text{in } \Omega \\ v(x) \underset{x\to\partial\Omega}{\to} +\infty. \end{cases} \tag{1.13}$$

Remark 1.11.
If $\Omega \subset \mathbb{R}^2$ is a simply connected domain then the solution v to (1.13) satisfies

$$v(x) = 2R(x) + \log 8$$

where R is the Robin function associated to Ω, namely the regular part of the Green function computed on the diagonal. Our result, jointly with Suzuki's one (see [24]), gives a complete description for any $\lambda \in \mathbb{R}$ of the bifurcation diagram containing the minimal branch of $(\mathcal{P}^\lambda_f)$ with $f(t) = e^t$. Note that our results does not depend on the dimension of the space, differently from the case where $\lambda > 0$ (see [18] for example).

Remark 1.12.
Some example of nonlinearity f where the previous theorem applies are the following:

- $f(t) = e^{t|t|^{p-1}}$ *with* $\alpha_\lambda = \frac{1}{p(-\beta_\lambda)^{p-1}}$ *and* β_λ *verifying* $(-\beta_\lambda)^{p-1} e^{\beta_\lambda(-\beta_\lambda)^{p-1}} = -\frac{1}{p\lambda}$.
  So we get $u_\lambda = \frac{v_\lambda}{(-\beta_\lambda)^{p-1}} + \beta_\lambda$.

- $f(t) = (1 + |t|)^p e^t$ *with* $\alpha_\lambda = 1$ *and* β_λ *verifying* $(-\beta_\lambda)^p e^{\beta_\lambda} = -\frac{1}{\lambda}$.
  So we get $u_\lambda = v_\lambda + \beta_\lambda$.

The paper is organized as follows: in Section 2 we prove Theorem 1.1. In Section 3 we discuss the case $t_0 \in \mathbb{R}$ and prove Theorem 1.2 and in Section 4 we prove Theorem 1.7 and 1.8. At the end of the both sections we give some examples where explicity solutions are provided. Finally in the Appendix we show that our assumptions on the nonlinearity f are quite general.

2 General properties of the solution u_λ

In this section we prove Theorem 1.1. We start showing some properties of the solution u_λ.

Lemma 2.1.
Assume f satisfies (1.1). Then for any $\lambda < 0$, $(\mathcal{P}_f^\lambda)$ has a unique classical solution which is strictly negative in Ω.

Proof.
As in [10] existence for small λ can be proved applying the implicit function Theorem to $F(\lambda, u) : (-\infty, 0] \times C^{2,\alpha}(\Omega) \cap C_0(\Omega) \to C^{0,\alpha}(\Omega)$ defined as

$$F(\lambda, u) = \Delta u + \lambda f(u)$$

at its trivial zero $(\lambda_0, u_0) = (0, 0)$, as the linearized operator $F'(0, 0) : v \mapsto \Delta v$ is invertible. Here $C^{2,\alpha}(\Omega), C^{0,\alpha}(\Omega)$ are the usual Holder spaces and $C_0(\Omega)$ is the subspace of the continuous functions on Ω that satisfy $u = 0$ on $\partial\Omega$. As $\lambda < 0$ then $u_\lambda < 0$ by the weak and strong maximum principle. More generally, the branch of solutions can be extended at any (λ_0, u_0) with $\lambda_0 < 0$; in fact, the linearized operator is

$$F'(\lambda_0, u_0) : v \mapsto \Delta v + \lambda_0 f'(u_0)v;$$

therefore, being $\lambda_0 < 0$ and $f'(u_0) \geq 0$, for any $v \not\equiv 0$ one has

$$\int_\Omega (F'(\lambda_0, u_0)v)v = \int_\Omega (\Delta v + \lambda_0 f'(u_0)v)v = \int_\Omega \left(-|\nabla v|^2 + \lambda_0 f'(u_0)v^2\right) < 0.$$

This proves the injectivity of $F'(\lambda_0, u_0)$ and hence that the branch of solutions is a regular curve in a neighborhood of any of its point.

Finally, let us show that such a branch exists for every $\lambda \leq 0$. Set $I_\lambda := \{\lambda < 0$ such that $F(\lambda, u_\lambda) = 0\}$ and $\lambda^* = \inf I_\lambda$.
We want to show that if $\lambda^* \in \mathbb{R}$ then λ^* is a minimum for I_λ which contradicts the definition of λ^* since we have already proved that the branch of solutions can be extended from any of its point.
By definition there exists a sequence (λ_n, u_n) with $\lambda_n \to \lambda^*$ such that $F(\lambda_n, u_n) = 0$. By the maximum principle, any solution u to $(\mathcal{P}_f^\lambda)$ is not positive, therefore $f(u_n) \leq f(0)$ and $(\lambda^* - \delta)f(0) \leq -\Delta u_n \leq 0$ for some $\delta > 0$ when n is large enough; hence, by standard elliptic estimates, u_n converges in $C^{2,\alpha}(\Omega)$ to some $u^* \in C_0(\Omega)$ satisfying $F(\lambda^*, u^*) = 0$. This proves existence for any λ.

To show the uniqueness of the solution for $\lambda < 0$, take two solutions u, v to $(\mathcal{P}_f^\lambda)$ and consider their difference $u - v$: it solves

$$\begin{cases} -\Delta(u - v) = \lambda(f(u) - f(v)) & \text{in } \Omega \\ u - v = 0 & \text{on } \partial\Omega \end{cases}.$$

By testing this equation versus $u - v$, we get

$$\int_\Omega |\nabla(u - v)|^2 = \int_\Omega (-\Delta(u - v))(u - v) = \lambda \int_\Omega (f(u) - f(v))(u - v).$$

Since we are taking a non-decreasing f, we have $(f(u) - f(v))(u - v) \geq 0$, therefore we get $\int_\Omega |\nabla(u - v)|^2 \leq 0$, which is possible only if $u \equiv v$. This proves the uniqueness and concludes the proof. □

Next lemma is a comparison principle which is well known as $\lambda \geq 0$. On the other hand the same proof holds for $\lambda < 0$ as well.

Lemma 2.2.
Let $\Omega_1 \subset \Omega_2$ and u_1, u_2 be solutions to

$$-\Delta u_i = g_i(u_i) \quad \text{in } \Omega_i, \qquad\qquad i = 1, 2,$$

such that $u_2(x) \le u_1(x)$ on $\partial\Omega_1$, where g_1, g_2 are locally Lipschitz functions, g_1 nonincreasing, $g_2(t) \le g_1(t)$ for any t. Then,

$$u_2(x) \le u_1(x), \qquad\qquad \forall x \in \Omega_1.$$

Moreover, either one has $\Omega_1 = \Omega_2$, $g_1 \equiv g_2$, $u_2(x) = u_1(x)$ on $\partial\Omega_1$, or $u_2(x) < u_1(x)$ for any $x \in \Omega_1$.

Proof.
The difference $u_1 - u_2$ solves

$$\begin{cases} -\Delta(u_1 - u_2) = g_1(u_1) - g_2(u_2) & \text{in } \Omega_1 \\ u_1 - u_2 \ge 0 & \text{on } \partial\Omega_1 \end{cases}.$$

By writing

$$g_1(u_1) - g_2(u_2) = \frac{g_1(u_1) - g_1(u_2)}{u_1 - u_2}(u_1 - u_2) + g_1(u_2) - g_2(u_2),$$

since $g_1(u_2) \ge g_2(u_2)$, then $u_1 - u_2$ also satisfies $-\Delta(u_1 - u_2) + c(x)(u_1 - u_2) \ge 0$ with $c(x) = -\dfrac{g_1(u_1) - g_1(u_2)}{u_1 - u_2} \ge 0$ by the monotonicity of g_1. Therefore the weak and strong maximum principle gives $u_1 - u_2 \ge 0$, with the strict inequality unless $\Omega_i, g_i, u_{i|\partial\Omega_i}$ all coincide. □

Proof of Theorem 1.1.
Existence and uniqueness of a negative solution follows from Lemma 2.1.
The stability of the solution u_λ is an easy consequence of the fact that λ is negative and f nondecreasing. Moreover, applying Lemma 2.2 with $\Omega_1 = \Omega_2 = \Omega$ and $g_i = \lambda_i f$ we get $u_{\lambda_2}(x) < u_{\lambda_1}(x)$ for any $x \in \Omega$ and $\lambda_2 < \lambda_1$ and therefore the monotonicity in λ gives the existence of a pointwise limit $u_0(x) = \lim\limits_{\lambda \to -\infty} u_\lambda(x)$.
We are left with showing that such a limit equals t_0 for all x; since the monotonicity of u_λ is strict, this would give the inequality $u_\lambda > t_0$.

Let us start with the case when $\Omega = B_R$ is any ball, whose center is omitted for simplicity. As u has constant sign, the Gidas-Ni-Nirenberg Theorem [15] gives that u is radial and radially increasing.
We first show that $u_0 \ge t_0$ in the case $t_0 \in \mathbb{R}$, whereas if $t_0 = -\infty$ it is trivial. By contradiction, we assume $u_\lambda(x) < t_0$ for $x \in B_{R_\lambda}$ and $u_\lambda(x) = t_0$ for $x \in \partial B_{R_\lambda}$, for some $\lambda < 0$ and $R_\lambda \in (0, R)$. Therefore, u_λ solves $-\Delta u_\lambda = \lambda f(u_\lambda)$ in B_{R_λ} but, since $f(t) = 0$ for $t \le t_0$, $u \equiv t_0$ also solves the same equation in B_{R_λ} and the solution is unique in view of Lemma 2.1; hence, $u_\lambda \equiv t_0$ on B_{R_λ} and we found a contradiction.
Now we prove $u_0 \le t_0$, which jointly with the previous inequality gives $u_0 \equiv t_0$ in B_R. If not, $u_\lambda \ge t_1 > t_0$ on some $B_{R_1} \subset B_R$ for any $\lambda < 0$; the monotonicity of f yields $-\Delta u_\lambda = \lambda f(u_\lambda) \le \lambda f(t_1)$, and clearly $u_\lambda \le 0$ on ∂B_{R_1}. Therefore we may apply the comparison principle to u_λ and $\lambda f(t_1)\phi$, with ϕ being the unique solution to (1.11) in B_{R_1}, to get $u_\lambda \le \lambda f(t_1)\phi$: since $f(t_1) > 0$, we get $u_\lambda \underset{\lambda \to -\infty}{\to} -\infty$ a.e. on B_{R_1}, contradicting $u_\lambda \ge t_1$.
Finally, the convergence is locally uniform in B_{R_1} because, since u_λ is radially increasing, for any $r < R$ one has

$$\sup_{B_r} |u_\lambda - t_0| = u_\lambda(r) - t_0 \underset{\lambda \to -\infty}{\to} 0,$$

and when $t_0 = -\infty$

$$\sup_{B_r} u_\lambda = u_\lambda(r) \underset{\lambda \to -\infty}{\to} -\infty.$$

Now, let us consider a generic domain Ω. We consider two balls $B_{R_1} \subset \Omega \subset B_{R_2}$ and the solutions $u_{i,\lambda}$ to ($\mathcal{P}_f^\lambda$) on B_{R_i}: by applying twice Lemma 2.2 with $g_1 = g_2 = \lambda f$ we get $u_{2,\lambda} \le u_\lambda \le u_{1,\lambda}$ on B_{R_1}. Since we already proved that $u_{i,\lambda} \underset{\lambda \to -\infty}{\to} t_0$ in $L^\infty_{\mathrm{loc}}(B_{R_i})$ for both i's, we deduce $u_\lambda \underset{\lambda \to -\infty}{\to} t_0$ in $L^\infty_{\mathrm{loc}}(B_{R_1})$ and, since the choice of B_{R_1} is arbitrary, also in $L^\infty_{\mathrm{loc}}(\Omega)$. □

In the case when $t_0 \in \mathbb{R}$ we can improve Theorem 1.1 getting the following result:

Proposition 2.3.
Let u_λ be the unique solution to $(\mathcal{P}_f^\lambda)$ for $\lambda < 0$. Assume f satisfies (1.1) and that $t_0 \in \mathbb{R}$. Then

$$\lambda f(u_\lambda) \underset{\lambda \to -\infty}{\to} 0 \text{ a.e. in } \Omega, \tag{2.1}$$

and $u_\lambda \underset{\lambda \to -\infty}{\to} t_0$ in $C^1_{\text{loc}}(\Omega)$.

Proof. First we prove that for every compact set $K \subset \Omega$ there exists a constant C_K such that

$$\sup_K |\lambda f(u_\lambda)| \leq C_K. \tag{2.2}$$

By contradiction let us assume that (2.2) does not hold. Then there exists points $x_\lambda \in K$ such that $\lambda f(u_\lambda(x_\lambda)) \to -\infty$ and, up to a sub-sequence $x_\lambda \to \overline{x} \in K$.
We take B_r be a ball centered in $\overline{x}$ and such that $B_{2r} \subset \Omega$. We call $u_{1,\lambda}$ the radial solution to $(\mathcal{P}_f^\lambda)$ in B_{2r}. We know by the proof of Theorem 1.1 that $u_{1,\lambda}$ is radially increasing and that $u_\lambda < u_{1,\lambda}$ in B_{2r}. The monotonicity of f then gives

$$\lambda f(u_\lambda(x)) > \lambda f(u_{1,\lambda}(x))$$

in B_{2r} and since $x_\lambda \in B_{2r}$ for λ large enough then $\lambda f(u_{1,\lambda}(x_\lambda)) \underset{\lambda \to -\infty}{\to} -\infty$. By the monotonicity of $u_{1,\lambda}$ we also have that

$$\lambda f(u_{1,\lambda}(x)) < \lambda f(u_{1,\lambda}(x_\lambda)) \to -\infty$$

for every $x \in B_{2r}$ such that $|x - \overline{x}| > |x_\lambda - \overline{x}|$. In particular we have that, denoting by $A_r = B_r \setminus B_{\frac{r}{2}}$, $\lambda f(u_{1,\lambda}(x)) \to -\infty$ in A_r.
Last step is to show that this cannot happen. Let $M > 0$. There exists $\overline{\lambda} < 0$ such that

$$\lambda f(u_{1,\lambda}(x)) < -M \quad \text{in } A_r \quad \text{for every } \lambda < \overline{\lambda}.$$

We let z_M be the solution to $-\Delta z_M = -M$ in A_r with Dirichlet boundary conditions. Then by the weak and strong maximum principle we have $u_{1,\lambda} < z_M$ in A_r and $z_M = -M\phi$ where ϕ is the unique solution to (1.11) in A_r. Since M is arbitrary this gives a contradiction with

$$t_0 < u_{1,\lambda}(x) < -M\phi(x)$$

which proves (2.2). In order to show (2.1) remark that the r.h.s. of the equation satisfied by u_λ is uniformly bounded in every compact set K of Ω. The standard regularity theory then say that $u_\lambda - t_0$ is uniformly bounded in $W^{2,p}(K)$ per every p, and that, up to a sub-sequence $u_\lambda - t_0 \underset{\lambda \to -\infty}{\to} 0$ in $C^1(K)$. By the weak formulation of $(\mathcal{P}_f^\lambda)$ we then get that $\lambda f(u_\lambda) \underset{\lambda \to -\infty}{\to} 0$ a.e. in Ω. □

3 Second order expansion of the solution u_λ: the case $t_0 \in \mathbb{R}$

The aim of this section is to improve estimate (1.3) in Theorem 1.2.

Proof of Theorem 1.2.

(i) By the assumptions on $\alpha, \gamma(\alpha)$ we have $-\dfrac{\alpha}{\gamma(\alpha)} \underset{\alpha \searrow 0}{\to} -\infty$. Therefore, after a re-labeling, the ratio $-\dfrac{\alpha}{\gamma(\alpha)}$ will decrease monotonically and, for any $\lambda \ll 0$, there will be some α_λ such that $\lambda = -\dfrac{\alpha_\lambda}{\gamma(\alpha_\lambda)}$.
By such a choice, the function v_λ defined by

$$v_\lambda = \frac{u_\lambda - t_0}{\alpha_\lambda}$$

will solve

$$\begin{cases} \Delta v_\lambda = \dfrac{f(\alpha_\lambda v_\lambda + t_0)}{\gamma(\alpha_\lambda)} & \text{in } \Omega \\ v_\lambda = -\dfrac{t_0}{\alpha_\lambda} & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

and, by construction, satisfies $v_\lambda > 0$. Next we show that v_λ is bounded from above.
Observe that it is not restrictive to assume that $g_0 \geq 0$ and non-decreasing in such a way (1.6) still holds. In view of the assumption $\dfrac{f(\alpha t + t_0)}{\gamma(\alpha)} \geq g_0(t)$, we can use Lemma 2.2 to get $v_\lambda \leq v_{0,\lambda}$, with the latter solving

$$\begin{cases} \Delta v_{0,\lambda} = g_0(v_{0,\lambda}) & \text{in } \Omega \\ v_{0,\lambda} = -\dfrac{t_0}{\alpha_\lambda} & \text{on } \partial\Omega. \end{cases}$$

Let us introduce the large solution v_0 which satisfies

$$\begin{cases} \Delta v_0 = g_0(v_0) & \text{in } \Omega \\ v_0(x) \underset{x\to\partial\Omega}{\to} +\infty. \end{cases} \tag{3.2}$$

We have $v_{0,\lambda} \leq v_0$ in Ω (and, actually $v_{0,\lambda} \underset{\lambda\to-\infty}{\to} v_0$) and the boundedness of v_0 gives that v_λ is uniformly bounded from above in $L^\infty_{\text{loc}}(\Omega)$. The boundedness of v_0 on compact sets of Ω follows comparing v_0 with the large solution $v_{0,\rho}$ to (3.2) in a small ball B_ρ centered in $x_0 \in \Omega$ and contained in Ω. By Lemma 2.2 $0 < v_0 < v_{0,\rho}$ and $v_{0,\rho}$ is strictly increasing in the radial variable. This implies that v_0 is bounded in the ball $B_{\frac{\rho}{2}}$.
Since v_λ is bounded in $L^\infty_{\text{loc}}(\Omega)$ and Δv_λ is uniformly bounded for bounded v_λ, it will converge in $C^1_{\text{loc}}(\Omega)$ to some function v, and in view of the limit (1.7), v will solve $\Delta v = v^p$. Last step is to prove that $v(x) \underset{x\to\partial\Omega}{\to} +\infty$. Define $\widetilde{g}(t) := \sup\limits_{\lambda<0} \dfrac{f(\alpha_\lambda t + t_0)}{\gamma(\alpha_\lambda)}$ and because the latter converges for any t, we have $\widetilde{g}(t) < +\infty$ for any t. Therefore one may define $\widetilde{v}_\lambda$ as the solution to

$$\begin{cases} \Delta \widetilde{v}_\lambda = \widetilde{g}\,(\widetilde{v}_\lambda) & \text{in } \Omega \\ \widetilde{v}_\lambda = -\dfrac{t_0}{\alpha_\lambda} & \text{on } \partial\Omega \end{cases}, \tag{3.3}$$

and since $\widetilde{g}$ is non-negative and non-decreasing, arguing as in Lemma 2.1 we get $v_\lambda \geq \widetilde{v}_\lambda$, and since $-\dfrac{t_0}{\alpha_\lambda} \underset{\lambda\to-\infty}{\to} +\infty$, we conclude that $v(x) \underset{x\to\partial\Omega}{\to} +\infty$, namely v is indeed a large solution.

(ii) We first consider the case where Ω is any domain of $\mathbb{R}^2$ or a convex domain of $\mathbb{R}^N$ with $N \geq 3$ and (1.8) holds. Let us take an absolute minimum point $x_\lambda \in \Omega$ and set $\alpha_\lambda := \min u_\lambda - t_0 = u_\lambda(x_\lambda) - t_0$; due to Theorem 1.1, we have $\alpha_\lambda \underset{\lambda\to-\infty}{\to} 0$ and $\lim\limits_{\lambda\to-\infty} x_\lambda = x_0 \in \Omega$ by Remark 1.3.
Now, since we assume $\dfrac{\gamma(\alpha)}{\alpha} \underset{\alpha\searrow 0}{\not\to} 0$, then we set $\varepsilon_\lambda := \sqrt{\dfrac{\alpha_\lambda}{-\lambda\gamma(\alpha_\lambda)}} \underset{\lambda\to-\infty}{\to} 0$.
The rescaled function v_λ defined by

$$v_\lambda(x) = \frac{u_\lambda\,(\varepsilon_\lambda x + x_\lambda) - t_0}{\alpha_\lambda}$$

solves

$$\begin{cases} \Delta v_\lambda = \dfrac{f(\alpha_\lambda v_\lambda + t_0)}{\gamma(\alpha_\lambda)} & \text{in } \dfrac{\Omega - x_\lambda}{\varepsilon_\lambda} \\ v_\lambda(x) \geq v_\lambda(0) = 1 \end{cases}.$$

where $\dfrac{\Omega - x_\lambda}{\varepsilon_\lambda} \to \mathbb{R}^N$ by Remark 1.3.
We have that v_λ satisfies

$$-\Delta v_\lambda + c(x) v_\lambda = 0 \tag{3.4}$$

with $c(x) = \dfrac{f(\alpha_\lambda v_\lambda + t_0)}{\gamma(\alpha_\lambda) v_\lambda} \leq C v_\lambda^{q-1}$ with $0 \leq q \leq 1$ by (1.5).
Since $v_\lambda \geq 1$ we get that $|c(x)| \leq C$. The Harnack inequality applied to (3.4) in any ball B_R then gives

$$\sup_{B_R} v_\lambda < C_H \inf_{B_R} v_\lambda = C_H$$

and so v_λ is uniformly bounded on every compact set of $\mathbb{R}^N$. Using (1.7) we can pass to the limit getting that $v_\lambda \to v$ in $C^1_{\text{loc}}\left(\mathbb{R}^N\right)$ where v is a weak solution to (1.9) concluding the proof under the assumption (1.8).
Let us prove the convergence of v_λ in the case $\Omega = B_R$ is a ball; in this case v_λ is radial and solves

$$\begin{cases} v_\lambda''(r) + \dfrac{N-1}{r} v_\lambda'(r) = \dfrac{f(\alpha_\lambda v_\lambda(r) + t_0)}{\gamma(\alpha_\lambda)} & 0 < r < \dfrac{R}{\varepsilon_\lambda} \\ v_\lambda'(0) = 0 \\ v_\lambda(0) = 1 \end{cases}. \tag{3.5}$$

Because of the uniform convergence to g, there exist sequences M_n and λ_n with $M_n \to +\infty$ and $\lambda_n \to -\infty$ such that

$$\sup_{0<t\leq M_n} \left| \frac{f(\alpha_{\lambda_n} t + t_0)}{\gamma(\alpha_{\lambda_n})} - g(t) \right| \underset{n\to\infty}{\to} 0. \tag{3.6}$$

Therefore a comparison argument gives $v_{\lambda_n}(r) \leq v_0(r)$ as long as $v_{\lambda_n}(r) \leq M_n$, with v_0 solving

$$\begin{cases} v_0''(r) + \dfrac{N-1}{r} v_0'(r) = g(v_0(r)) + 1 & r \in \mathbb{R} \\ v_0'(0) = 0 \\ v_0(0) = 1 \end{cases}.$$

We have that v_0 is well-defined for any r because g does not satisfy the condition (1.6) (see Theorem 4 in [19]). Now taking r_n such that $v_{\lambda_n}(r_n) = M_n$, one has $v_0(r_n) \geq M_n \underset{n\to\infty}{\to} +\infty$ and then $r_n \underset{n\to\infty}{\to} +\infty$. Therefore, for any fixed $r > 0$ we have, for large n, $r \leq r_n$ and $v_{\lambda_n}(r) \leq v_0(r) \leq C$. Since v_{λ_n} is bounded in L^∞_{loc} we can pass to the limit in (3.5) and v_{λ_n} will also converge to the solution v to (1.9). Since from every sequence $\lambda_n \to -\infty$ we can extract a subsequence $\tilde{\lambda}_n$ that satisfies (3.6) then v_λ converges to the solution v.

□

Remark 3.1.
In the case of a more general decay of the function f, as when (1.10) holds instead of (1.7), we can argue as in the proof of Theorem 1.2 (i) choosing $\lambda = -\dfrac{\alpha_\lambda}{f(\beta_\lambda)}$ and replacing the function v_λ with

$$v_\lambda := \frac{u_\lambda - \beta_\lambda}{\alpha_\lambda}.$$

As in the previous case one can find that v_λ is bounded from above since $v_\lambda < v_0$. To prove that v_λ is bounded by below we define $\widetilde{g}(t) := \sup\limits_{t_0<\beta<0} \dfrac{f(\alpha(\beta)t + \beta)}{f(\beta)}$ and because the latter converges for any t, we have $\widetilde{g}(t) < +\infty$ for any t. Therefore one may define $\widetilde{v}$ as the solution to

$$\begin{cases} \Delta \widetilde{v} = \widetilde{g}(\widetilde{v}) & \text{in } \Omega \\ \widetilde{v} = 0 & \text{on } \partial\Omega \end{cases}. \tag{3.7}$$

Since $\widetilde{g}$ is non-negative and non-decreasing, arguing as in Lemma 2.1 we have that $\widetilde{v}$ is uniquely defined and belongs to $L^\infty(\Omega)$; therefore, Lemma 2.2 yields $v_\lambda \geq \widetilde{v}$ on Ω, namely v_λ is also uniformly bounded from below. The convergence of v_λ to v then follows as in the previous case. The only difference is that v is a large solution to (1.5).

We end this section with two examples where Theorem 1.2 applies. In this case we exhibit explicitly the solutions.

Example 3.2.

(i) $f(t) = \left((t+1)^+\right)^{\frac{N+2}{N-2}}$, $\Omega = B_1 \subset \mathbb{R}^N$, $N \geq 3$.
We are in the first alternative of Theorem 1.2, with $t_0 = -1, \alpha(\beta) = \beta + 1$, $g_0(t) = g(t) = (t+1)^{\frac{N+2}{N-2}}$. *In this case we have explicit solutions given by*

$$u_\lambda(x) = \left(\frac{\delta_\lambda - 1}{\delta_\lambda - |x|^2}\right)^{\frac{N-2}{2}} - 1 \qquad \delta_\lambda := \frac{N^2 - 2N - 2\lambda + \sqrt{N(N-2)(N(N-2) - 4\lambda)}}{-2\lambda}.$$

and

$$u_\lambda(x) \to -1 \text{ in } L^\infty_{\text{loc}}(B_1).$$

Taking $\beta_\lambda := -1 + (-\lambda)^{\frac{1}{p-1}}, \alpha_\lambda := (-\lambda)^{\frac{1}{p-1}}$, *we have*

$$v_\lambda(x) := \frac{u_\lambda + 1}{(-\lambda)^{\frac{1}{p-1}}} - 1 = \left(\frac{\lambda^2(\delta_\lambda - 1)}{\delta_\lambda - |x|^2}\right)^{\frac{N-2}{2}} - 1 \underset{\lambda \to -\infty}{\to} \left(\frac{N(N-2)}{1 - |x|^2}\right)^{\frac{N-2}{2}} - 1 =: v(x),$$

the latter being the unique large solution to $\Delta v = (v+1)^{\frac{N+2}{N-2}}$ *in* Ω.

(ii) $f(t) = (t+1)^+$, $\Omega = (-1,1) \subset \mathbb{R}$.
We are in the second alternative of Theorem 1.2, with $\gamma(\alpha) = \alpha$ *and* $g(t) = t$. *In fact, explicit solutions are given by*

$$u_\lambda(x) = \frac{\cosh\left(\sqrt{-\lambda}x\right)}{\cosh\sqrt{-\lambda}} - 1$$

and

$$u_\lambda(x) \to -1 \text{ in } L^\infty_{\text{loc}}(-1,1).$$

Taking $\alpha_\lambda = u_\lambda(0) + 1 = \dfrac{1}{\cosh\sqrt{-\lambda}}$, $x_\lambda = 0$ *and* $\varepsilon_\lambda = \dfrac{1}{\sqrt{-\lambda}}$, *we have*

$$v_\lambda(x) = \frac{u_\lambda\left(\frac{x}{\sqrt{-\lambda}}\right) + 1}{u_\lambda(0) + 1} \equiv \cosh x := v(x),$$

the latter being the solution to

$$\begin{cases} v''(x) = v(x) & x \in \mathbb{R} \\ v(0) = 1 \\ v'(0) = 0 \end{cases}.$$

In the case of the same nonlinearity on $\Omega = B_1 \subset \mathbb{R}^N$, $N \geq 2$, *the same argument holds true with* $\cosh x$ *being replaced with the solution to*

$$\begin{cases} v''(r) + \dfrac{N-1}{r}v'(r) = v(r) \\ v(0) = 1 \\ v'(0) = 0. \end{cases} \tag{3.8}$$

Using the last statement in Corollary 1.5 we can compute the explicit solutions to $(\mathcal{P}_f^\lambda)$ *for any* λ, *which are given by*

$$u_\lambda(r) = \frac{1}{v\left(\sqrt{-\lambda}\right)} v\left(\sqrt{-\lambda}r\right) - 1$$

where v *the unique radial solution to* (3.8).
For $N = 3$, *since it is known that* $v(r) = \dfrac{\sinh r}{r}$, *we have that*

$$u_\lambda(r) = \frac{1}{\sinh\sqrt{-\lambda}} \frac{\sinh\left(\sqrt{-\lambda}r\right)}{r} - 1.$$

A simple computation then gives

$$J_\lambda(u_\lambda) = \frac{1}{2}\int_{B_1} |\nabla u_\lambda|^2 - \lambda(u_\lambda + 1)^2 dx = (2\pi + o(1))\sqrt{-\lambda}$$

as $\lambda \to \infty$ *if* J_λ *is as defined in* (1.4).

4 Refined expansions of the solution u_λ: the case $t_0 = -\infty$

In this section we split the proof in two parts, according the limit of $f(t)$ at $-\infty$. Let us start with the case where the limit at $-\infty$ is positive.

4.1 The case $\lim_{t \to -\infty} f(t) = c_0 > 0$

In this case *second order estimates* for the solution u_λ will be provided without additional assumptions on f.

Proof of Theorem 1.7.
Denote by $v_\lambda = \frac{u_\lambda}{\lambda}$. It verifies,

$$\begin{cases} -\Delta v_\lambda = f(u_\lambda) & \text{in } \Omega \\ v_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

By the properties of f we get that

$$c_0 \le f(u_\lambda) \le f(0)$$

and then by the standard regularity theory we get that there exists ϕ such that $v_\lambda \to \phi$ in $C^1(\Omega)$. Moreover, by Theorem 1.1 and $\lim_{t \to -\infty} f(t) = c_0 > 0$ we get that

$$f(u_\lambda) \to c_0 \quad \text{in } L^\infty_{\text{loc}}(\Omega).$$

Passing to the limit in (4.1) the claim follows. □

Next we consider the other case.

4.2 The case $\lim_{t \to -\infty} f(t) = 0$

Here the argument are very similar to that in Theorem 1.2. We will sketch the main points.

Proof of Theorem 1.8.

(i) Since $\frac{\gamma(\beta)}{\alpha(\beta)} \underset{\beta \to -\infty}{\to} 0$, without loss of generality we may assume the ratio to decrease monotonically and, for $\lambda \ll 0$, we take $\beta_\lambda, \alpha_\lambda = \alpha(\beta_\lambda)$ such that $\lambda = -\frac{\alpha(\beta_\lambda)}{\gamma(\beta_\lambda)}$.

We define $v_\lambda := \frac{u_\lambda - \beta_\lambda}{\alpha(\beta_\lambda)}$, which will solve

$$\begin{cases} \Delta v_\lambda = \dfrac{f(\alpha_\lambda v_\lambda + \beta_\lambda)}{\gamma(\beta_\lambda)} & \text{in } \Omega \\ v_\lambda = -\dfrac{\beta_\lambda}{\alpha_\lambda} & \text{on } \partial\Omega \end{cases},$$

and we have the inequalities $\widetilde{v} \le v_\lambda \le v_0$ in Ω, with $v_0, \widetilde{v}$ respectively defined by (3.2), (3.7). From this we deduce v_λ is locally uniformly bounded in Ω and converges to some solution v to $\Delta v = g(v)$; finally, we take $\widetilde{v}_\lambda$ solving

$$\begin{cases} \Delta \widetilde{v}_\lambda = \widetilde{g}(\widetilde{v}_\lambda) & \text{in } \Omega \\ \widetilde{v}_\lambda = -\dfrac{\beta_\lambda}{\alpha_\lambda} & \text{on } \partial\Omega \end{cases},$$

therefore from the inequality $v_\lambda \ge \widetilde{v}_\lambda$ and $\frac{-\beta_\lambda}{\alpha_\lambda} \underset{\lambda \to -\infty}{\to} +\infty$ we deduce $v|_{\partial\Omega} = +\infty$.

(ii) Since $\dfrac{\beta}{\alpha(\beta)} \geq A-1$, we must have $\alpha(\beta) \underset{\beta\to-\infty}{\to} +\infty$, hence again $\dfrac{\gamma(\beta)}{\alpha(\beta)} \underset{\beta\to-\infty}{\to} 0$; we may assume the latter limit to decrease monotonically and take β_λ so that $-\lambda = \dfrac{\alpha(\beta_\lambda)}{\gamma(\beta_\lambda)}$. Therefore, $v_\lambda := \dfrac{u_\lambda}{\alpha(\beta_\lambda)}$ solves

$$\begin{cases} \Delta v_\lambda = \dfrac{f(\alpha(\beta_\lambda)v_\lambda)}{\gamma(\beta_\lambda)} & \text{in } \Omega \\ v_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

By assumption, we have $v_\lambda \leq 0$ and moreover $\dfrac{f(\alpha(\beta_\lambda)t)}{\gamma(\beta_\lambda)} \underset{\lambda\to-\infty}{\to} g(t-A)$ locally uniformly in t, therefore one may define $\widetilde{g}(t) := \sup_\lambda \dfrac{f(\alpha(\beta_\lambda)t)}{\gamma(\beta_\lambda)} < +\infty$. Since $\widetilde{g}(t) \underset{t\to 0}{\to} +\infty$ and it increases monotonically for any $t<0$, there exists a solution $\widetilde{v}$ to

$$\begin{cases} \Delta\widetilde{v} = \widetilde{g}(\widetilde{v}) & \text{in } \Omega \\ \widetilde{v} = 0 & \text{on } \partial\Omega, \end{cases}$$

which is uniformly bounded in $\overline{\Omega}$ (see Theorem 1.1 in [11] for details).
Therefore, $v_\lambda \geq \widetilde{v}$ hence it is uniformly bounded in $C\left(\overline{\Omega}\right)$ and, in view of the convergence of f, it will converge to the solution to (1.12).

□

As in the previous section we end with two examples where explicit solutions are provided:

Example 4.1.

(i) $f(t) = e^t$ *on* $\Omega = B_1 \subset \mathbb{R}^2$.
We are in the first alternative of Theorem 1.8, with $\alpha(\beta) = 1$ *and* $g(t) = g_0(t) = e^t$*. In fact, explicit solutions are given by*

$$u_\lambda(x) = \log\frac{8\delta_\lambda}{-\lambda\left(\delta_\lambda - |x|^2\right)^2}, \qquad \delta_\lambda = 1 + \frac{4+2\sqrt{4-2\lambda}}{-\lambda} = 1 + \frac{2\sqrt{2}+o(1)}{\sqrt{-\lambda}}.$$

Taking $\beta_\lambda = \log(-\lambda), \alpha_\lambda = 1$*, we have*

$$v_\lambda(x) = \log\frac{8\delta_\lambda}{\left(\delta_\lambda - |x|^2\right)^2} \underset{\lambda\to-\infty}{\to} \log\frac{8}{\left(1-|x|^2\right)^2} =: v(x),$$

the latter being the large solution to $\Delta v = e^v$ *on* Ω*. Moreover a straightforward computation gives that*

$$J_\lambda(u_\lambda) := \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 dx - \lambda\int_\Omega \left(e^{u_\lambda}-1\right)dx = \left(2\sqrt{2}\omega_N + o(1)\right)\sqrt{-\lambda} \tag{4.2}$$

where ω_N *is the area of the unit ball in* $\mathbb{R}^N$.

(ii) $f(t) = \dfrac{1}{(1-v)^3}$ *on* $\Omega = (-1,1) \subset \mathbb{R}$.
We are in the second alternative of Theorem 1.8, with $p = 3, \gamma(\alpha) = \alpha^3$*. In fact, explicit solutions are given by*

$$u_\lambda(x) = 1 - \sqrt{1 + \frac{-2\lambda}{1+\sqrt{1-4\lambda}}\left(1-x^2\right)}.$$

Taking $\alpha_\lambda = \dfrac{1}{(-\lambda)^{\frac{1}{4}}}$*, we have*

$$v_\lambda(x) = \frac{u_\lambda(x)}{(-\lambda)^{\frac{1}{4}}} \underset{\lambda\to-\infty}{\to} -\sqrt{1-x^2} =: v(x),$$

the latter being the solution to $\begin{cases} v'' = \dfrac{1}{(-v)^3} & \text{in } (-1,1) \\ v(1) = v(-1) = 0 \end{cases}.$

A Appendix

In this Appendix we show that the assumption on f considered in Theorems 1.2, 1.8 are rather general. In fact, they occur any time one has the following *asymptotic homogeneity* condition:

$$\frac{f(\alpha(\beta)t+\beta)}{\gamma(\beta)} \underset{\beta\searrow t_0}{\to} g(t) \qquad \text{locally uniformly for } -\sup_\beta \frac{\beta - t_0}{\alpha(\beta)} < t < \sup_\beta \frac{-\beta}{\alpha(\beta)}, \text{ for some } g \not\equiv 0, \tag{A.1}$$

for some $\alpha(\beta),\gamma(\beta) > 0$, with f satisfying (1.1) and (1.2) defined by t_0.
On the other hand, condition (A.1) seems to be necessary in order to pass to the limit in the equation $(\mathcal{P}_f^\lambda)$, after taking a rescalement.

Lemma A.1.
Assume f satisfies (1.1), t_0 is defined by (1.2) and there exist and $\alpha(\beta),\gamma(\beta) > 0$ such that (A.1). Then,

1. *In (A.1), one can take without restriction $\gamma(\beta) = f(\beta)$ and g satisfying $g(0) = 1$;*
2. *If $t_0 \in \mathbb{R}$, then (A.1) can hold only of $\alpha(\beta) \underset{\beta\nearrow t_0}{\to} 0$;*
3. *If $t_0 \in \mathbb{R}$ and $\lim\limits_{\beta\searrow t_0} \dfrac{\beta - t_0}{\alpha(\beta)} = \overline{t} \in \mathbb{R}_{>0}$, then $g(t) = \left(\left(t+\overline{t}\right)^+\right)^p$ for some $p > 0$ and $\dfrac{f(\alpha(\beta)t + t_0)}{f(\beta)} \underset{\beta\searrow t_0}{\to} t^p$ locally uniformly in $t > 0$;*
4. *If $t_0 = +\infty$ and $\lim\limits_{b\to-\infty} \dfrac{-\beta}{\alpha(\beta)} = -\overline{t} \in \mathbb{R}_{<0}$, then $g(t) = \dfrac{1}{\left(\left(\overline{t}-t\right)^+\right)^p}$ for some $p \geq 0$ and $\dfrac{f(\alpha(\beta)t)}{f(\beta)} \underset{\beta\searrow t_0}{\to} \dfrac{1}{(-t)^p}$ locally uniformly in $t < 0$.*

Proof.

1. Because of the convergence at $t = 0$, one has $\dfrac{f(\beta)}{\gamma(\beta)} \underset{\beta\searrow t_0}{\to} g(0)$, therefore $\dfrac{f(\alpha(\beta)t+\beta)}{f(\beta)} \underset{\beta\searrow t_0}{\to} \dfrac{g(t)}{g(0)}$.
2. If $t_0 \in \mathbb{R}$ and $\alpha(\beta) \geq \delta_0 > 0$ on a sub-sequence, then we would get:
$$g(1) = \lim_{\beta\searrow t_0} \frac{f(\alpha(\beta)+\beta)}{f(\beta)} \geq \frac{f(\delta_0+\beta)}{f(\beta)}.$$
Since $f(\delta_0) > f(t_0) = 0$, then passing to the limit on the right-hand side we would get $+\infty$, hence a contradiction.
3. Since $\beta = t_0 + \alpha(\beta)\left(\overline{t} + o(1)\right)$, then for any $t, \varepsilon > 0$ we will have, for β close enough to t_0, $\alpha(\beta)\left(t+\overline{t}-\varepsilon\right) + t_0 \leq \alpha(\beta)t + \beta \leq \alpha(\beta)\left(t+\overline{t}+\varepsilon\right) + t_0$; therefore, by the monotonicity of f,
$$\limsup_{\beta\searrow t_0} \frac{f\left(\alpha(\beta)\left(t+\overline{t}-\varepsilon\right)+t_0\right)}{f(\beta)} \leq g(t) \leq \liminf_{\beta\searrow t_0} \frac{f(\alpha(\beta)\left(t+\overline{t}+\varepsilon\right)+t_0)}{f(\beta)}.$$
As ε is arbitrary and g is continuous, we conclude that $\dfrac{f\left(\alpha(\beta)\left(t+\overline{t}\right)+t_0\right)}{f(\beta)} \underset{\beta\searrow t_0}{\to} g(t)$. Now, we take $\widetilde{\beta}$ such that $\alpha\left(\widetilde{\beta}\right) = 2\alpha(\beta)$ and compute the previous limit with $2t+\overline{t}$ in place of t:
$$g\left(2t+\overline{t}\right) = \lim_{\beta\searrow t_0} \frac{f\left(\alpha(\beta)\left(2t+2\overline{t}\right)+t_0\right)}{f(\beta)} = \lim_{\beta\searrow t_0} \frac{f\left(2\alpha(\beta)\left(t+\overline{t}\right)+t_0\right)}{f(\beta)} = \lim_{\beta\searrow t_0} \frac{f\left(\alpha\left(\widetilde{\beta}\right)\left(t+\overline{t}\right)+t_0\right)}{f\left(\widetilde{\beta}\right)} \frac{f\left(\widetilde{\beta}\right)}{f(\beta)}.$$

Since $\alpha\left(\widetilde{\beta}\right) = 2\alpha(\beta) \to 0$, we have $\widetilde{\beta} \to t_0$, hence the first factor of the right-hand side goes to $g(t)$; therefore we get, for any t, we have $g\left(2t+\overline{t}\right) = Lg(t)$, with $L := \lim \dfrac{f\left(\widetilde{\beta}\right)}{f(\beta)}$. Because of the uniform convergence, g is continuous, and it is also non-negative, non-decreasing and satisfies the condition $g\left(2t+\overline{t}\right) = Lg(t)$: it must be of the kind $g(t) = C\left(\left(t+\overline{t}\right)^+\right)^p$ for some $C > 0, p \geq 0$, and $g(0) = 0$ implies $C = 1$. The final limit follows by passing from $t + \overline{t}$ to t.

4. We argue similarly as before. Since $\beta = \alpha(\beta)\left(-\overline{t} + o(1)\right)$, then $\dfrac{f\left(\alpha(\beta)\left(t-\overline{t}\right)+t_0\right)}{f(\beta)} \underset{\beta \searrow t_0}{\to} g(t)$. We take $\widetilde{\beta}$ such that $\alpha\left(\widetilde{\beta}\right) = 2\alpha(\beta)$, and in this case we have $\widetilde{\beta} \to -\infty$ because the latter goes to $+\infty$. Therefore, as before,

$$g\left(2t-\overline{t}\right) = \lim_{\beta\to-\infty} \frac{f\left(2\alpha(\beta)\left(t-\overline{t}\right)\right)}{f(\beta)} = \lim_{\beta\searrow t_0} \frac{f\left(\alpha\left(\widetilde{\beta}\right)\left(t-\overline{t}\right)\right)}{f\left(\widetilde{\beta}\right)} \frac{f\left(\widetilde{\beta}\right)}{f(\beta)} = Lg(t),$$

which implies g is of a power type and the rest of the statement.

□

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