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Algebraic and combinatorial rank of divisors on finite graphs

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ABSTRACT

We study the algebraic rank of a divisor on a graph, an invariant defined using divisors on algebraic curves dual to the graph. We prove it satisfies the Riemann–Roch formula, a specialization property, and the Clifford inequality. We prove that it is at most equal to the (usual) combinatorial rank, and that equality holds in many cases, though not in general.

RÉSUMÉ

On étudie le rang algébrique pour les diviseurs sur un graphe, un invariant défini par les diviseurs sur les courbes algébriques duales au graphe. On démontre le Théorème de Riemann–Roch, la proprieté de « spécialisation », et l'inégalité de Clifford. On démontre qu'il est inférieur ou égal au rang combinatoriel, avec égalité dans certains cas mais pas en général.

1. Introduction

Since recent years, a lively trend of research is studying the interplay between the combinatorial and algebro-geometric aspects of the theory of algebraic curves; this has led to interesting progress both in algebraic geometry and graph theory. The goal of this paper is to contribute to this progress by investigating the connection between the notion of combinatorial rank of divisors on graphs, and the notion of rank of Cartier divisors on an algebraic curve.

Loosely speaking, our main result is that the combinatorial rank of a divisor on a graph (a computer-computable quantity bounded above by the degree) is a fitting uniform upper bound on the dimension

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of linear series on curves (a hard to compute quantity, unbounded regardless of the degree). To be more precise, we need some context.

In the theory of algebraic curves, combinatorial aspects naturally appear when dealing with all curves simultaneously, as points of an algebraic variety. Indeed, a typical phenomenon in algebraic geometry is that the set of equivalence classes of varieties with given discrete invariants is itself an algebraic variety, whose geometric properties reflect those of the varieties it parametrizes. The case we should here keep in mind is the space $\overline{M_g}$, of all connected nodal curves of genus g up to stable equivalence; it is a complete variety containing, as a dense open subset, the space of isomorphism classes of smooth curves. $\overline{M_g}$ has been a central object of study for a long time, and it has been successfully used to study the geometry of algebraic curves. Several topics in this field, among which many open problems, concern projective realizations of abstract curves, i.e. the theory of line bundles (or Cartier divisors) and linear series.

A systematic study of these matters requires combinatorial methods to handle singular curves. Moreover, several questions are successfully answered by degeneration techniques (specializing a smooth curve to a singular one), where combinatorial aspects are essential; examples of this are the Griffiths–Harris proof of the Brill–Noether theorem, in [13], or the Kontsevich, and others, recursive formulas enumerating curves on surfaces, see [17] or [7].

In fact, since the first appearances of $\overline{M_g}$, as in the seminal paper [12], one sees associated to every nodal curve its *dual graph*, having as vertices the irreducible components of the curve, and as edges the nodes of the curve; moreover, every vertex of the dual graph is given a *weight*, equal to the geometric genus of the component it represents. For any (weighted) graph G we denote by $M^{\text{alg}}(G)$ the set of isomorphism classes of curves dual to G (i.e. having G as dual graph). Then we have

$$\overline{M_g} = (\sqcup M^{\text{alg}}(G))/\sim$$

where the union is over all connected graphs of genus g, and "$\sim$" denotes stable equivalence (which we don't define here, see [14]).

The dual graph is a key tool to deal with the combinatorial aspects mentioned above, especially in the theory of divisors and line bundles, when studying Néron models of Jacobians, Picard functors and compactified Jacobians, or degenerations of linear series.

More recently, and independently of the picture we just described, a purely combinatorial theory of divisors and linear series on graphs was being developed in a different framework; see [4] and [6]. The discovery that this graph-theoretic theory fits in well with the algebro-geometric set-up came somewhat as a surprise. To begin with, the group, $\text{Div}(G)$, of divisors on a graph G is the free abelian group on its vertices. The connection to line bundles on curves is simple: given a curve X dual to G, the multidegree of a line bundle on X is naturally a divisor on G, so we have a map $\text{Pic}(X) \to \text{Div}(G)$.

Such developments in graph-theory provide a fertile ground to extract and study the combinatorial aspects of the theory of algebraic curves; a remarkable example of this is the recent proof of the above mentioned Brill–Noether theorem, given in [11].

In this spirit, as mentioned at the outset, the goal of this paper is to interpret the combinatorial rank, $r_G(\underline{d})$, of a divisor, $\underline{d}$, on a graph G (as defined in [6] and in [2]) by the theory of algebraic curves. We do that by studying another invariant, the *algebraic rank*

$$r^{\text{alg}}(G, \underline{d})$$

of the divisor $\underline{d}$, defined in a completely different fashion, using algebro-geometric notions.

In algebraic geometry, the notion corresponding to the combinatorial rank is the rank of a line bundle, i.e. the dimension of the space of its global sections diminished by one. Now, the algebraic rank of a divisor on a graph G should be thought of as a uniform "sensible" upper bound on the rank of any line bundle on any curve in $M^{\mathrm{alg}}(G)$ having multidegree equal to the given divisor on G. The word “sensible” signals the fact that we need the algebraic rank to be constant in equivalence classes, hence the precise definition needs some care (see Section 2.2 for details). However, the following simple case is clear and gives the right idea: let G be just one vertex of weight g (no edges), then $M^{\mathrm{alg}}(G)$ parametrizes smooth curves of genus g, and a divisor on G is an integer, d; if d is non-negative and at most $2g-2$, the theorem of Clifford yields that the algebraic rank of d is equal to $\lfloor d/2 \rfloor$, and the bound is achieved exactly by certain line bundles on hyperelliptic curves.

The algebraic rank was introduced in [9], where it was conjectured to coincide with the combinatorial rank. Although, as we will here prove, this conjecture is true in a large number of cases, it fails in general. For convenience, we will now assemble together our results concerning the relation between combinatorial and algebraic rank.

Comparison of combinatorial and algebraic rank: results. *Let G be a (finite, connected, weighted) graph of genus g, and let $\underline{d}$ be a divisor on G. Then we have*

$$r^{\mathrm{alg}}(G, \underline{d}) \leq r_G(\underline{d}). \tag{1}$$

Moreover, equality holds in the following cases.

(a) $r_G(\underline{d}) \leq 0$;
(b) G is a binary graph (see Example 5.3);
(c) G is loopless, weightless and $\underline{d}$ is rank-explicit (see Definition 5.8).

Some recent results of [16] establish the inequality opposite to (1) in some cases; hence, combining with our results, we have that equality holds in (1) for non-hyperelliptic graphs if $g = 3$, and for all hyperelliptic graphs in characteristic other than 2; see Corollary 4.5. For completeness, we mention that equality in (1) holds when $g \leq 2$ (assuming G stable for $g = 2$), and if $d \geq 2g-2$ or $d \leq 0$; see [9].

As we said, we have some counterexamples showing that equality can fail in (1); namely Example 5.15 shows that the hypothesis that G is weightless cannot be removed from (b), and Example 5.16 shows that equality can fail for weightless graphs with three vertices. We do not know how to characterize cases, where we have a strict inequality in (1), but we believe that it would be quite an interesting problem.

Here is an outline of the paper. We start with a study of the algebraic rank, establishing some basic properties: in Section 2 we prove that it satisfies the Riemann–Roch formula, along with the appropriate version of Baker’s specialization lemma; both results are proved using algebraic geometry, with no relation to the analogous facts for the combinatorial rank.

In Section 3 we concentrate on the combinatorial rank and establish ways to compute it, or to bound it from above. We use the theory of reduced divisors from [6], implementing it by constructing what we call the *Dhar decomposition* of the vertex set of a graph, a particularly useful tool for our goals.

Using the material of the previous parts, in Section 4 we prove that the algebraic rank is always at most equal to the combinatorial rank (Theorem 4.2). As a consequence, the algebraic rank satisfies the Clifford inequality; this fact could not be proved using algebro-geometric methods (like the Riemann–Roch formula above), as the Clifford inequality fails for reducible curves.

The last section focuses on the opposite inequality: when is the algebraic rank at least equal to the combinatorial rank? We now must bound the algebraic rank from below, hence the new issue is to find singular curves which play for us the role of hyperelliptic curves (as in the above example), i.e. curves on which line bundles of a fixed multidegree tend to have the highest possible rank. We find such *special curves* in every set $M^{\mathrm{alg}}(G)$ when G is a weightless graph, and we use them to prove that, for so-called *rank explicit* divisors on weightless graphs, the combinatorial rank and the algebraic rank are equal (Theorem 5.13).

Index of non-standard notations

$\underline{d}$	divisor on a graph G.
$\underline{e}$	effective divisor on G.
δ	divisor class on G, usually the class of $\underline{d}$.
$\underline{t}_Z$	principal divisor corresponding to a set of vertices Z; see (2).
$\lvert\underline{d}\rvert$	degree of $\underline{d}$.
$r_G(*)$	combinatorial rank of $*$.
$r^{\mathrm{alg}}(G,*)$	algebraic rank of $*$; see Definition 2.3.
$\widehat{G}$	loopless weightless graph associated to G; see Section 3.1.
$g(v)$	sum of weight and number of loops at v; see Section 3.1.
R_v	subgraph of $\widehat{G}$ consisting of v and the $g(v)$ attached cycles; see (6).
e^g	$e+\min\{e,g\}$, where e and g are non-negative integers.
$\underline{e}^{\deg}$	divisor on G satisfying $\underline{e}^{\deg}(v)=\underline{e}(v)^{g(v)}$; see Definition 3.2.
d_g	$\max\{d-g,\lfloor\frac{d}{2}\rfloor\}$, where d and g are non-negative integers.
$\underline{d}_{\mathrm{rk}}$	divisor on G satisfying $\underline{d}_{\mathrm{rk}}(v)=\underline{d}(v)_{g(v)}$; see Definition 3.13.
$\ell_G(\underline{d})$	see (9) for G weightless, and (11) in general.

2. Algebraic rank of divisors on graphs

2.1. Preliminaries

Throughout the paper G will denote a (vertex weighted) finite graph, $V(G)$ the set of its vertices, $E(G)$ the set of its edges and $\mathrm{Div}(G)=\mathbb{Z}^{V(G)}$ the group of its divisors; when no ambiguity is likely to occur we write $V=V(G)$ and $E=E(G)$. The weight function of G is written $\omega: V\to\mathbb{Z}_{\geq 0}$; if $\omega(v)=0$ for every $v\in V$ we say that G is weightless.

The *genus* of a connected graph G is

$$g(G):=\sum_{v\in V}\omega(v)+|E|-|V|+1;$$

thus, if G is weightless, $g(G)$ is its first Betti number. If G has c connected components, $G_1,\dots,G_c$, we set $g(G)=\sum_1^c g(G_i)+1-c$. Our graphs will always be connected, unless otherwise specified.

Elements in $\mathrm{Div}(G)$ are usually denoted by underlined lower-case letters, for example we write $\underline{d}\in\mathrm{Div}(G)$ and $\underline{d}=\{\underline{d}(v),\ \forall v\in V\}$ with $\underline{d}(v)\in\mathbb{Z}$. We write $|\underline{d}|:=\sum_{v\in V}\underline{d}(v)$ for the "degree" of a divisor $\underline{d}$.

We write $\underline{d}\geq 0$ for an effective divisor (i.e. such that $\underline{d}(v)\geq 0$ for every $v\in V$); if $\underline{d}$ and $\underline{e}$ are effective, and $\underline{d}-\underline{e}$ is also effective, we say that "$\underline{d}$ contains $\underline{e}$".

We will usually abuse notation so that for a set $Z\subset V$ of vertices, we also denote by Z the divisor $\sum_{v\in Z}v\in\mathrm{Div}(G)$ (or, with the above notation, the divisor $\underline{d}$ such that $\underline{d}(v)=1$ for all $v\in Z$, and $\underline{d}(v)=0$ otherwise).

The group $\mathrm{Div}(G)$ is endowed with an intersection product associating to $\underline{d}_1,\underline{d}_2\in\mathrm{Div}(G)$ an integer, written $\underline{d}_1\cdot\underline{d}_2$. If $\underline{d}_i=v_i$ with $v_i\in V$ and $v_1\neq v_2$ we set $v_1\cdot v_2$ equal to the number of edges joining v_1 with v_2, whereas $v_1\cdot v_1=-\sum_{v\in V\smallsetminus\{v_1\}}v\cdot v_1$. The linear extension to the entire $\mathrm{Div}(G)$ gives our intersection product.

The subgroup $\mathrm{Prin}(G)\subset\mathrm{Div}(G)$ of *principal* divisors is generated by divisors of the form $\underline{t}_Z$, for all $Z\subset V$; these principal divisors are defined so that for any $v\in V$ we have

$$\underline{t}_Z(v)=\begin{cases} v\cdot Z & \text{if } v\notin Z,\\ -v\cdot Z^c & \text{if } v\in Z,\end{cases} \tag{2}$$

where $Z^c:=V\smallsetminus Z$.

Remark 2.1. Let $\underline{t} \in \operatorname{Prin}(G)$ be non-trivial. Then there exists a partition $V = Z_0 \sqcup \ldots \sqcup Z_m$, with Z_0 and Z_m non-empty, such that $\underline{t} = \sum_{i=0}^{m} i\underline{t}_{Z_i}$. Indeed, by definition, $\underline{t} = \underline{t}_{Y_1} + \ldots + \underline{t}_{Y_k}$, where each Y_j is a set of vertices. For each $a \geq 0$, let Y'_a be the set of vertices that are contained in a different such sets (Y'_a may be empty). Then the sets Y'_a are a disjoint cover of V, and $\underline{t} = 0 \cdot \underline{t}_{Y'_0} + \ldots + k \cdot \underline{t}_{Y'_k}$. Let b be the first integer so that Y'_b is non-empty. Since $\sum \underline{t}_{Y'_i} = \underline{t}_V = 0$, by subtracting b copies of it from $\underline{t}$, and defining $Z_i = Y'_{i+b}$, we are done.

This implies that we have

$$\underline{t}_{|Z_m} \leq (\underline{t}_{Z_m})_{|Z_m}.$$

Indeed, pick $v \in Z_m$, we have $\underline{t}_{Z_m}(v) = -Z_m^c \cdot v$; on the other hand

$$\underline{t}(v) = \sum_{i=0}^{m-1} iZ_i \cdot v - mZ_m^c \cdot v \leq (m-1)\sum_{i=0}^{m-1} Z_i \cdot v - mZ_m^c \cdot v = -Z_m^c \cdot v.$$

We set $\operatorname{Pic}(G) = \operatorname{Div}(G)/\operatorname{Prin}(G)$; we say that two divisors are *equivalent* if their difference is in $\operatorname{Prin}(G)$, and we refer to $\operatorname{Pic}(G)$ as the group of equivalence classes of divisors; for $\underline{d} \in \operatorname{Div}(G)$ we denote its class in $\operatorname{Pic}(G)$ by $\delta = [\underline{d}]$. Since equivalent divisors have the same degree we write $|\delta| := |\underline{d}|$. For an integer k we write $\operatorname{Div}^k(G)$, and $\operatorname{Pic}^k(G)$, for the set of divisors, or divisor classes, of degree k.

For any divisor $\underline{d} \in \operatorname{Div}(G)$ we have its *combinatorial rank*, denoted by $r_G(\underline{d})$ and defined as in [6] and [2] (see Subsection 3.1 for details). For now, we just recall that $r_G(\underline{d})$ is an integer such that

$$-1 \leq r_G(\underline{d}) \leq \max\{-1, |\underline{d}|\}.$$

Equivalent divisors have the same combinatorial rank, hence we set

$$r_G(\delta) := r_G(\underline{d}).$$

2.2. The algebraic rank

In this paper, unless otherwise specified, the word *curve* stands for one-dimensional scheme, reduced and projective over an algebraically closed field k, and having only nodal singularities. Let X be a curve; its *dual graph*, G, is defined as follows. $V(G)$ is the set of irreducible components of X; $E(G)$ is the set of nodes of X, with one edge joining two vertices if the corresponding node is at the intersection of the two corresponding components (a loop based at a vertex v corresponds to a node of the irreducible component corresponding to v); the weight of $v \in V$ is the genus of the desingularization of the corresponding component. One easily checks that the (arithmetic) genus of X is equal to the genus of its dual graph. We recall that it is well known that the theorem of Riemann–Roch holds for any such curve X, but the Clifford inequality only holds when X is irreducible; see [8] for more details.

The irreducible component decomposition of X is denoted as follows:

$$X = \cup_{v \in V} C_v.$$

$\operatorname{Pic}(X)$ is the group of isomorphism classes of line bundles on X; for $L \in \operatorname{Pic}(X)$ we denote by $\underline{\deg}\, L = \{\deg_{C_v} L,\ \forall v \in V\}$ its *multidegree*. Now, $\underline{\deg}\, L$ can be viewed as a divisor on G, by setting $\underline{\deg}\, L(v) = \deg_{C_v} L$; hence we have a surjective group homomorphism

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Div}(G); \qquad L \mapsto \underline{\deg}\, L.$$

For $\underline{d} \in \operatorname{Div}(G)$ we write $\operatorname{Pic}^{\underline{d}}(X) = \{L \in \operatorname{Pic}(X) :\ \underline{\deg}\, L = \underline{d}\}$.

As already mentioned, $M^{\mathrm{alg}}(G)$ denotes the set of isomorphism classes of curves dual to G. Observe that $M^{\mathrm{alg}}(G)$ is never empty.

For any $X \in M^{\mathrm{alg}}(G)$ and any $\underline{d} \in \mathrm{Div}(G)$ we define

$$r^{\max}(X, \underline{d}) := \max\{r(X, L), \quad \forall L \in \mathrm{Pic}^{\underline{d}}(X)\}$$

where $r(X, L) = \dim H^0(X, L) - 1$. Note the following simple fact:

Remark 2.2. Let $\underline{d}' \in \mathrm{Div}(G)$; if $\underline{d}' \geq \underline{d}$ then $r^{\max}(X, \underline{d}') \geq r^{\max}(X, \underline{d})$.

Varying $\underline{d}$ in its equivalence class $\delta \in \mathrm{Pic}(G)$ we can define

$$r(X, \delta) := \min\{r^{\max}(X, \underline{d}), \quad \forall \underline{d} \in \delta\}.$$

(By contrast, $\max\{r^{\max}(X, \underline{d}),\ \forall \underline{d} \in \delta\} = +\infty$, for every reducible curve X.)

Definition 2.3. For any divisor class $\delta \in \mathrm{Pic}(G)$ of a graph G we set

$$r^{\mathrm{alg}}(G, \delta) := \max\{r(X, \delta), \quad \forall X \in M^{\mathrm{alg}}(G)\},$$

and for every representative $\underline{d} \in \mathrm{Div}(G)$ for δ

$$r^{\mathrm{alg}}(G, \underline{d}) := r^{\mathrm{alg}}(G, \delta).$$

We refer to $r^{\mathrm{alg}}(G, \delta)$ and $r^{\mathrm{alg}}(G, \underline{d})$ as the *algebraic rank* of δ and $\underline{d}$.

Example 2.4. If G has only one vertex, any curve $X \in M^{\mathrm{alg}}(G)$ is irreducible (singular if G has loops); then every class in $\mathrm{Pic}(G)$ contains exactly one element, and we naturally identify $\mathrm{Pic}(G) = \mathrm{Div}(G) = \mathbb{Z}$. Then, for any $d \in \mathrm{Div}(G)$ by the theorems of Riemann–Roch and Clifford,

$$r^{\mathrm{alg}}(G, d) = \begin{cases} -1 & \text{if } d < 0, \\ \lfloor d/2 \rfloor & \text{if } 0 \leq d \leq 2g-2, \\ d - g & \text{if } d \geq 2g-1. \end{cases}$$

The following natural problem arises:

Problem 1. Let G be a graph and $\delta \in \mathrm{Pic}(G)$. Is

$$r^{\mathrm{alg}}(G, \delta) = r_G(\delta)?$$

In [9] the answer to this problem is shown to be positive in a series of cases, and it is conjecture that it be always the case. As already mentioned, we shall prove that we have $r^{\mathrm{alg}}(G, \delta) \leq r_G(\delta)$, but equality does fail in some cases.

Remark 2.5. In this paper we do not consider metric graphs. Nonetheless, we believe it would be very interesting to have a generalization of the algebraic rank to divisors on metric graphs in a way that reflects the algebro-geometric nature of the graph, which we can loosely describe as follows. To a metric graph $\Gamma = (G, \ell)$ there corresponds a set of (equivalence classes of) nodal curves, $\mathcal{X} \to \mathrm{Spec}\, R$, where R is a valuation ring with algebraically closed residue field; the dual graph of the closed fiber is G, and the metric, ℓ, represents the geometry of $\mathcal{X}$ locally at the closed fiber.

We refer to [1] for a treatment of the correspondence between algebraic and combinatorial aspects of the theory using metric graphs (and more generally, "metrized complexes") instead of finite graphs.

2.3. Riemann–Roch for the algebraic rank

We shall now prove that the algebraic rank, exactly as the combinatorial rank, satisfies a Riemann–Roch formula; the proof will be a consequence of Riemann–Roch for curves.

As we said, the Clifford inequality also holds for the algebraic rank, but its proof requires more work and it is quite different as it follows from the Clifford inequality for graphs (indeed, the Clifford inequality fails for reducible curves!); see Proposition 4.6.

We denote by $\underline{k}_G$ the canonical divisor of a graph G, and by K_X the dualizing line bundle of a curve X. Recall that $\underline{k}_G$ is defined as follows:

$$\underline{k}_G(v) = \mathrm{val}(v) + 2\omega(v) - 2,$$

where $\mathrm{val}(v)$ denotes the valency of the vertex v. We have $\underline{\deg}\, K_X = \underline{k}_G$ for every $X \in M^{\mathrm{alg}}(G)$.

Proposition 2.6 *(Riemann–Roch). Let G be a graph of genus g, $\underline{d}$ a divisor of degree d on G, and $X \in M^{\mathrm{alg}}(G)$. Then, setting $\delta = [\underline{d}]$, the following identities hold.*

(a) $r^{\max}(X, \underline{d}) - r^{\max}(X, \underline{k}_G - \underline{d}) = d - g + 1$;
(b) $r(X, \delta) - r(X, \underline{k}_G - \delta) = d - g + 1$;
(c) $r^{\mathrm{alg}}(G, [\underline{d}]) - r^{\mathrm{alg}}(G, [\underline{k}_G - \underline{d}]) = d - g + 1$.

Proof. We begin by introducing some notation. For $L \in \mathrm{Pic}^{\underline{d}}(X)$, set $L^* = K_X L^{-1}$, so that $L^{**} = L$. Similarly, we set $\underline{d}^* = \underline{\deg}\, L^* = \underline{k}_G - \underline{d}$ and $d^* = \deg \underline{d}^* = 2g - 2 - d$, next $\delta^* := [\underline{d}^*]$ (this is well defined, as $\underline{d} \sim \underline{e}$ implies $\underline{d}^* \sim \underline{e}^*$). We have $\delta^* \in \mathrm{Pic}^{\underline{d}^*}(G)$ and $\delta^{**} = \delta$ (as $\underline{d}^{**} = \underline{d}$).

Notice that the correspondence $L \mapsto L^*$ is a bijection between $\mathrm{Pic}^{\underline{d}}(X)$ and $\mathrm{Pic}^{\underline{d}^*}(X)$. Similarly $\underline{d} \mapsto \underline{d}^*$ is a bijection between the representatives of δ and those of δ^*, and $\delta \mapsto \delta^*$ is a bijection between $\mathrm{Pic}^d(G)$ and $\mathrm{Pic}^{d^*}(G)$.

Let $X \in M^{\mathrm{alg}}(G)$ and $L \in \mathrm{Pic}^{\underline{d}}(X)$. We claim the following:

$$r^{\max}(X, \underline{d}) = r(X, L) \quad \Leftrightarrow \quad r^{\max}(X, \underline{d}^*) = r(X, L^*). \tag{3}$$

In other words, whenever L is a line bundle realizing $r^{\max}(X, \underline{d})$, its dual L^* will realize $r^{\max}(X, \underline{d}^*)$. By the algebro-geometric Riemann–Roch applied to L on X it is clear that (3) implies (a).

By the bijection described above, it suffices to prove only one implication of (3). So assume $r^{\max}(X, \underline{d}) = r(X, L)$. By contradiction, suppose $r(X, L^*) < r^{\max}(X, \underline{d}^*)$, and let $M^* \in \mathrm{Pic}^{\underline{d}^*}(X)$ be such that $r(X, M^*) = r^{\max}(X, \underline{d}^*)$. Now by Riemann–Roch for X we have

$$r(X, L) = r(X, L^*) + d - g + 1 < r(X, M^*) + d - g + 1 = r(X, M)$$

hence $r^{\max}(X, \underline{d}) = r(X, L) < r(X, M)$, which is impossible as $M \in \mathrm{Pic}^{\underline{d}}(X)$. (3) is thus proved, and (a) with it.

From (a), to prove (b) it suffices to show the following:

$$r^{\max}(X, \underline{d}) = r(X, \delta) \quad \Leftrightarrow \quad r^{\max}(X, \underline{d}^*) = r(X, \delta^*). \tag{4}$$

As before, we need only prove one implication, so assume $r^{\max}(X, \underline{d}) = r(X, \delta)$ and let $L \in \mathrm{Pic}^{\underline{d}}(X)$ be such that $r(X, L) = r^{\max}(X, \underline{d})$. By (3) we have $r(X, L^*) = r^{\max}(X, \underline{d}^*)$, so it suffices to prove that $r(X, L^*) = r(X, \delta^*)$. By contradiction, suppose this is not the case. Then there exist $\underline{e}^* \in \delta^*$ and $N^* \in \mathrm{Pic}^{\underline{e}^*}(X)$ such that

$$r(X, L^*) > r(X, N^*) = r^{\max}(X, \underline{e}^*).$$

By Riemann–Roch on X we have

$$r(X, L) = r(X, L^*) + d - g + 1 > r(X, N^*) + d - g + 1 = r(X, N).$$

By (3) we have

$$r(X, N) = r^{\max}(X, \underline{e}) \geq r^{\max}(X, \underline{d}) = r(X, L),$$

a contradiction with the previous inequality; (4) and (b) are proved.

Finally, let $L \in \mathrm{Pic}^{\underline{d}}(X)$ be such that $r(X, L) = r^{\mathrm{alg}}(G, \delta)$, and let us prove that $r(X, L^*) = r^{\mathrm{alg}}(G, \delta^*)$. By Riemann–Roch on X this will imply (c).

As $r(X, L) = r^{\max}(X, \underline{d}) = r(X, \delta)$ by (3) and (4) we have $r(X, L^*) = r^{\max}(X, \underline{d}^*) = r(X, \delta^*)$. By contradiction, suppose there exists a curve $Y \in M^{\mathrm{alg}}(G)$ and a line bundle $P^* \in \mathrm{Pic}^{\underline{e}^*}(Y)$ with $\underline{e}^* \in \delta^*$ such that

$$r(X, L^*) < r^{\mathrm{alg}}(G, \delta^*) = r(Y, P^*) = r^{\max}(Y, \underline{e}^*).$$

Arguing as before we get

$$r(X, L) = r(X, L^*) + d - g + 1 < r(Y, P^*) + d - g + 1 = r(Y, P).$$

Now claims (3) and (4) yield, as $\underline{e} \in \delta$,

$$r(Y, P) = r(Y, \delta) \leq r^{\mathrm{alg}}(G, \delta) = r(X, L),$$

contradicting the preceding inequality. The proof is complete. □

2.4. Specialization for the algebraic rank

We shall now prove a result analogous to Baker Specialization Lemma, established in [5], stating that the algebraic rank of divisors varying in a family of curves cannot decrease when specializing to the dual graph of the special fiber.

We need some preliminaries. Let X be a connected curve, and let $\phi : \mathcal{X} \to (B, b_0)$ be a regular one-parameter smoothing of X, i.e. $\mathcal{X}$ is a regular 2-dimensional variety, B is a regular 1-dimensional variety with a marked point $b_0 \in B$, ϕ is a fibration in curves such that the fiber over b_0 is X and the fibers over the other points of B are smooth projective curves. The relative Picard scheme of ϕ is written $\mathrm{Pic}_\phi \to B$, so that the fiber of Pic_ϕ over a point $b \in B$ is the Picard scheme of the curve $X_b := \phi^{-1}(b)$. The set of sections of $\mathrm{Pic}_\phi \to B$ is denoted by $\mathrm{Pic}_\phi(B)$; so, an element $\mathcal{L} \in \mathrm{Pic}_\phi(B)$ gives a line bundle $\mathcal{L}(b) \in \mathrm{Pic}(X_b)$ for every $b \in B$.

Let L_0 and L_0' be two line bundles on X; we say that L_0 and L_0' are ϕ-equivalent, and write $L_0' \sim_\phi L_0$, if for some divisor D on $\mathcal{X}$ entirely supported on X we have

$$L_0^{-1} \otimes L_0' = \mathcal{O}_{\mathcal{X}}(D)_{|X}.$$

For example, for $\mathcal{L} \in \mathrm{Pic}_\phi(B)$ and $D \in \mathrm{Div}(X)$ as above, we can define $\mathcal{L}' \in \mathrm{Pic}_\phi(B)$ that assigns $\mathcal{L}(b) \otimes \mathcal{O}_\mathcal{X}(D)_{|X_b}$ to every $b \in B$. As $\mathrm{Supp}\, D \subset X$ we have $\mathcal{L}(b) = \mathcal{L}'(b)$ for $b \neq b_0$, and $\mathcal{L}(b_0) \sim_\phi \mathcal{L}'(b_0)$.

Finally, let G be the dual graph of X; for any $\phi : \mathcal{X} \to B$ as above we have a natural map (cf. Subsection 2.2)

$$\tau : \mathrm{Pic}_\phi(B) \longrightarrow \mathrm{Div}(G); \qquad \mathcal{L} \mapsto \underline{\deg}\, \mathcal{L}(b_0).$$

Lemma 2.7 *(Specialization). Let $\phi : \mathcal{X} \to B$ be a regular one-parameter smoothing of a connected curve X. Let G be the dual graph of X. Then for every $\mathcal{L} \in \mathrm{Pic}_\phi(B)$ there exists an open neighborhood $U \subset B$ of b_0 such that for every $b \in U \smallsetminus \{b_0\}$ we have*

$$r(X_b, \mathcal{L}(b)) \leq r^{\mathrm{alg}}(G, \underline{\deg}\, \mathcal{L}(b_0)). \tag{5}$$

Proof. We write $L_b := \mathcal{L}(b)$, $L_0 := \mathcal{L}(b_0)$ and $\underline{d} := \tau(\mathcal{L}) = \underline{\deg}\, L_0$; we set $\delta \in \mathrm{Pic}(G)$ to be the class of $\underline{d}$. By uppersemicontinuity of h^0 we have, for every $L'_0 \in \mathrm{Pic}(X)$ such that $L'_0 \sim_\phi L_0$

$$r(X_b, L_b) \leq r(X, L'_0)$$

for every b in some neighborhood $U \subset B$ of b_0. Hence, by definition of $r^{\max}$,

$$r(X_b, L_b) \leq r^{\max}(X, \underline{\deg}\, L'_0).$$

As L'_0 varies in its ϕ-class the values of $\underline{\deg} L'_0$ cover all the representatives of δ, therefore we obtain

$$r(X_b, L_b) \leq r(X, \delta).$$

Since by definition $r(X, \delta) \leq r^{\mathrm{alg}}(G, \delta)$, we are done. $\square$

2.5. Algebraic smoothability of divisors on graphs

Let us now show how Problem 1 is related to the “smoothability” problem for line bundles on curves, i.e. the following general natural problem.

Problem 2. Let L be a line bundle on a curve X such that $r(X, L) = r$. Do there exist a regular one-parameter smoothing $\phi : \mathcal{X} \to B$ of X, and a section $\mathcal{L} \in \mathrm{Pic}_\phi(B)$, such that $\mathcal{L}(b_0) = L$ and $r(X_b, \mathcal{L}(b)) = r$ for every $b \in B$?

Easy cases when the answer is always positive are $r < 0$ (by upper-semicontinuity) and $\deg L \geq 2g - 1$ (by Riemann–Roch). But in other cases this problem is well known to get very hard, even without the regularity assumption on ϕ. We shall now try to simplify it by a combinatorial version.

Definition 2.8. Let G be a graph and $\underline{d} \in \mathrm{Div}(G)$; set $r_G(\underline{d}) = r$. We say that $\underline{d}$ is *(algebraically) smoothable* if there exist $X \in M^{\mathrm{alg}}(G)$, a regular one-parameter smoothing $\phi : \mathcal{X} \to B$ of X, and a section $\mathcal{L} \in \mathrm{Pic}_\phi(B)$, such that $\underline{\deg}\mathcal{L}(b_0) = \underline{d}$ and $r(X_b, \mathcal{L}(b)) = r$ for every $b \in B \smallsetminus b_0$ (hence $r(X, \mathcal{L}(b_0)) \geq r$ by upper-semicontinuity).

The following is the combinatorial counterpart to Problem 2.

Problem 3. Let $\underline{d} \in \mathrm{Div}(G)$. Is $\underline{d}$ smoothable?

As for Problem 2, if $r_G(\underline{d}) = -1$, or if $|\underline{d}| \geq 2g(G) - 1$, then $\underline{d}$ is smoothable.

Remark 2.9. If $\underline{d}$ is smoothable and $\underline{d}' \sim \underline{d}$, then $\underline{d}'$ is also smoothable (by the regularity assumption on ϕ). We shall say that a divisor class $\delta \in \mathrm{Pic}(G)$ is smoothable if so are its representatives.

The next simple result connects Problem 3 to Problem 1.

Proposition 2.10. *If $r_G(\delta) > r^{\mathrm{alg}}(G, \delta)$ then δ is not smoothable (i.e. no representative for δ is smoothable).*

Proof. Set $r = r_G(\delta)$. By contradiction, assume that δ is smoothable. Pick any representative $\underline{d}$ for δ; let $\phi : \mathcal{X} \to B$ and $\mathcal{L}$ as in the above definition, with $\underline{d} = \underline{\deg}\, \mathcal{L}(b_0)$. Then

$$r \leq r(X, \mathcal{L}(b_0)) \leq r^{\max}(X, \underline{\deg}\, \mathcal{L}(b_0)) = r^{\max}(X, \underline{d}).$$

Since the above holds for every $\underline{d}$ in δ, we get $r(X, \delta) \geq r$. Therefore

$$r \leq r(X, \delta) \leq r^{\mathrm{alg}}(G, \delta),$$

a contradiction. $\square$

As we shall see in the examples at the end of Section 5, non-smoothable classes do exist. We refer to the recent preprint [10] for a study of closely related issues and further examples.

We conclude this section asking about the converse of the above proposition, namely, what can be said about the smoothing problem when the algebraic and combinatorial rank coincide (keeping in mind that by Theorem 4.2 below, the algebraic rank is never greater than the combinatorial rank).

Question 2.11. Assume $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$; is δ smoothable?

A negative answer to this question is given in [19, Example 3.2], where a non-smoothable divisor whose algebraic and combinatorial ranks are both 1 is found.

3. Computing combinatorial ranks via reduced divisors

3.1. Combinatorial rank of divisors on graphs

Let G be a graph. For each $v \in V(G)$ we denote by $l(v)$ the number of loops adjacent to v; we set

$$g(v) := \omega(v) + l(v).$$

Let $\widehat{G}$ be the weightless, loopless graph obtained from G by gluing to each vertex $v \in V(G)$ a number of loops equal to $\omega(v)$, and then by inserting a vertex in every loop. Denote by $z_v^1, \ldots, z_v^{g(v)}$ the vertices in $V(\widehat{G}) \smallsetminus V(G)$ adjacent to v, and by R_v the complete subgraph of $\widehat{G}$ whose vertices are $\{v, z_v^1, \ldots, z_v^{g(v)}\}$:

$$R_v := \left[v, z_v^1, \ldots, z_v^{g(v)}\right] \subset \widehat{G}; \tag{6}$$

note that R_v has genus $g(v)$.

There is a natural injective homomorphism

$$\mathrm{Div}(G) \hookrightarrow \mathrm{Div}(\widehat{G}); \qquad \underline{d} \mapsto \widehat{\underline{d}} \tag{7}$$

such that $\widehat{\underline{d}}$ is defined to be zero on each new vertex of $\widehat{G}$, and equal to $\underline{d}$ on the vertices of G. The above homomorphism maps $\mathrm{Prin}(G)$ to $\mathrm{Prin}(\widehat{G})$, hence it descends to an injective homomorphism

$$\mathrm{Pic}(G) \hookrightarrow \mathrm{Pic}(\widehat{G}); \qquad \delta \mapsto \widehat{\delta}$$

such that if $\delta = [\underline{d}]$ then $\widehat{\delta} = [\widehat{\underline{d}}]$. The map (7) is used to define the combinatorial rank of a divisor on a general graph G via the combinatorial rank of a divisor on the weightless, loopless graph $\widehat{G}$; indeed, as in [2], the combinatorial rank of $\underline{d}$ is

$$r_G(\underline{d}) := r_{\widehat{G}}(\widehat{\underline{d}}),$$

where $r_{\widehat{G}}(\widehat{\underline{d}})$ is defined, as in [6] (for weightless and loopless graphs), as follows. If $\widehat{\underline{d}}$ is not equivalent to any effective divisor, we set $r_{\widehat{G}}(\widehat{\underline{d}}) = -1$; otherwise $r_{\widehat{G}}(\widehat{\underline{d}})$ is the maximum integer $k \geq 0$ such that for every effective $\underline{e} \in \mathrm{Div}(\widehat{G})$ of degree k the divisor $\widehat{\underline{d}} - \underline{e}$ is equivalent to an effective divisor. In particular, if $|\widehat{\underline{d}}| < 0$ then $r_{\widehat{G}}(\widehat{\underline{d}}) = -1$.

Now, let g and e be two non-negative integers; we define

$$e^g = e + \min\{e, g\}.$$

Remark 3.1. For every irreducible curve C of genus g, by the theorems of Riemann–Roch and Clifford, e^g is the minimum degree of a line bundle of rank e; more precisely for a line bundle L on C we have:

$$r(C, L) \geq e \implies \deg L \geq e^g.$$

The previous remark together with the subsequent Lemma 3.3 serve as motivation for the next definition.

Definition 3.2. Let G be a graph and $\underline{e}$ an effective divisor of G. We define the (effective) divisor $\underline{e}^{\deg}$ on G so that for every $v \in V$

$$\underline{e}^{\deg}(v) = \underline{e}(v)^{g(v)} = \underline{e}(v) + \min\{\underline{e}(v), g(v)\}.$$

The superscript "deg" indicates that the entries of $\underline{e}^{\deg}$ are "minimum degrees" associated to the entries of $\underline{e}$, as explained in Remark 3.1.

Notice that if G is weightless and loopless then $\underline{e}^{\deg} = \underline{e}$.

Lemma 3.3. *Let G be a graph and let $\underline{d} \in \mathrm{Div}(G)$. If for every effective divisor $\underline{e}$ of degree r the divisor $\underline{d} - \underline{e}^{\deg}$ is equivalent to an effective divisor, then $r_G(\underline{d}) \geq r$.*

Proof. Let $\widehat{G}$ be the weightless loopless graph defined at the beginning of the subsection; we need to show that $r_{\widehat{G}}(\widehat{\underline{d}}) \geq r$, where $\widehat{\underline{d}}$ is the divisor induced by $\underline{d}$ on $\widehat{G}$; see (7).

So, let $\epsilon \in \mathrm{Div}(\widehat{G})$ be any effective divisor of degree r, and let us prove that $\widehat{\underline{d}} - \epsilon$ is equivalent to an effective divisor. We define an effective divisor $\underline{e} \in \mathrm{Div}(G)$ by setting, for every vertex v of G,

$$\underline{e}(v) := \epsilon(v) + \epsilon(z_v^1) + \ldots + \epsilon(z_v^{g(v)}) \tag{8}$$

where $z_v^1, \ldots, z_v^{g(v)} \in V(\widehat{G})$ as in (6). Note that $|\underline{e}| = |\epsilon| = r$ hence, by hypothesis, $\underline{d} - \underline{e}^{\deg}$ is equivalent to an effective divisor. Therefore, it suffices to show that $\widehat{\underline{e}^{\deg}}$ is equivalent to a divisor containing ϵ. For every vertex $v \in \mathrm{Div}(G)$ consider the principal divisor $\underline{t}^v \in \mathrm{Prin}(\widehat{G})$

$$\underline{t}^v := -\sum_{k=1}^{g(v)} a_v^k \underline{t}_{z_v^k},$$

where $\underline{t}_{z_v^k} \in \mathrm{Prin}(G)$ are defined in (2) and, for every $1 \le k \le g(v)$,

$$a_v^k := \left\lceil \frac{\epsilon(z_v^k)}{2} \right\rceil.$$

To prove the lemma we will show that $\widehat{\underline{e}^{\deg}} + \sum_{v \in V(G)} \underline{t}^v$ is effective and contains ϵ. We have, for any $u \in V(G)$,

$$\begin{aligned}\Big(\widehat{\underline{e}^{\deg}} + \sum_{v \in V(G)} \underline{t}^v\Big)(u) &= \underline{e}(u) + \min\{\underline{e}(u), g(u)\} - 2\sum_{k=1}^{g(u)} a_u^k \\ &= \underline{e}(u) + \min\{\underline{e}(u), g(u)\} - \sum_{k=1}^{g(u)} \epsilon(z_u^k) - o(u),\end{aligned}$$

where $o(u)$ denotes the number of indices k with $1 \le k \le g(u)$ such that $\epsilon(z_u^k)$ is odd. We have $o(u) \le \min\{\underline{e}(u), g(u)\}$, hence, using (8),

$$\Big(\widehat{\underline{e}^{\deg}} + \sum_{v \in V(G)} \underline{t}^v\Big)(u) = \epsilon(u) + \min\{\underline{e}(u), g(u)\} - o(u) \ge \epsilon(u),$$

as required. Next,

$$\Big(\widehat{\underline{e}^{\deg}} + \sum_{v \in V(G)} \underline{t}^v\Big)(z_u^k) = 2\left\lceil \frac{\epsilon(z_u^k)}{2} \right\rceil \ge \epsilon(z_u^k).$$

The lemma is proved. $\square$

Remark 3.4. Our proof of the previous lemma has the advantage of being elementary and self-contained. But an alternative proof could be obtained using metric graphs, as follows. By [5, Lemma 1.5], when computing the rank of a divisor, we may regard G as a metric graph. By [18, Proposition 2.5], the vertices of G are a weighted rank determining set, which, by definition, means that a divisor $\underline{d}$ has rank at least r exactly when $\underline{d} - \underline{e}^{\deg}$ is equivalent to an effective divisor for every effective divisor $\underline{e}$ of degree r supported on the vertices. But this is true by assumption.

3.2. Weightless and loopless graphs

An important tool for computing combinatorial ranks is the notion of reduced divisors with respect to a vertex as introduced by Baker and Norine in [6] (see also [20]). In this subsection we consider weightless graphs with no loops, as in [6]; let us recall the definition and a few basic facts.

Definition 3.5. Let $\underline{d}$ be a divisor on a weightless and loopless graph G and fix a vertex $u \in V$. A divisor $\underline{d}$ is said to be *reduced* with respect to u, or *u-reduced*, if

(1) $\underline{d}(v) \ge 0$ for all $v \in V \smallsetminus \{u\}$;
(2) for every non-empty set $A \subset V \smallsetminus \{u\}$, there exists a vertex $v \in A$ such that $\underline{d}(v) < v \cdot A^c$.

Conditions (1) and (2) are equivalent to the following:

(1′) $\underline{d}_{|V\smallsetminus\{u\}} \geq 0$;
(2′) for every non-empty $A \subset V \smallsetminus \{u\}$ we have $(\underline{d} + \underline{t}_A)_{|V\smallsetminus\{u\}} \ngeq 0$.

Remark 3.6. Let $\underline{d}$ be a divisor on G and u be a vertex of G.

(a) If $\underline{d}$ is u-reduced, then for any integer n the divisor $\underline{d} - nu$ is also u-reduced.
(b) If $\underline{d}$ is u-reduced on the graph G, and G is a spanning subgraph of G' (i.e. $V(G) = V(G')$), then $\underline{d}$ is u-reduced on G'.

Fact 3.7. *(See [6, Proposition 3.1].) Let G be a weightless and loopless graph. Fix a vertex $u \in V(G)$. Then for every $\underline{d} \in \mathrm{Div}(G)$ there exists a unique u-reduced divisor $\underline{d}' \in \mathrm{Div}(G)$ such that $\underline{d}' \sim \underline{d}$.*

The next result is probably well known to the experts, but we could not find a proof in the literature.

Lemma 3.8. *Let $\delta \in \mathrm{Pic}(G)$.*

(a) $r_G(\delta) = -1$ if and only if there exists a vertex u whose u-reduced representative $\underline{d}$ in δ has $\underline{d}(u) < 0$.
(b) $r_G(\delta) = 0$ if and only if there exists a vertex u whose u-reduced representative $\underline{d}$ in δ has $\underline{d}(u) = 0$.

Proof. One implication of part (a) is clear. To prove the other, let $\underline{d}$ be u-reduced and suppose $\underline{d}(u) < 0$. By contradiction, suppose $r_G(\underline{d}) \geq 0$; let $\underline{t} \in \mathrm{Prin}(G)$ be such that $\underline{d} + \underline{t} \geq 0$, notice that $\underline{t}$ is not trivial. By Remark 2.1, there exists a non-empty $Z \subsetneq V(G)$ such that $\underline{t}_{|Z} \leq (\underline{t}_Z)_{|Z}$, hence

$$0 \leq (\underline{d} + \underline{t})_{|Z} \leq (\underline{d} + \underline{t}_Z)_{|Z}.$$

In particular, $u \notin Z$ (if $u \in Z$ then $\underline{t}_Z(u) \leq 0$, hence the above inequality yields $0 \leq \underline{d}(u)$, contradicting the initial assumption). For any $v \neq u$ with $v \notin Z$ we have $\underline{t}_Z(v) \geq 0$, hence

$$(\underline{d} + \underline{t}_Z)(v) \geq \underline{d}(v) \geq 0,$$

because $\underline{d}$ is u-reduced. We have thus proved that $(\underline{d}+\underline{t}_Z)_{|V\smallsetminus\{u\}} \geq 0$, which is impossible as $\underline{d}$ is u-reduced.

For (b), suppose $\underline{d}$ is u-reduced with $\underline{d}(u) = 0$; then $r_G(\underline{d}) \geq 0$. As the divisor $u \in \mathrm{Div}(G)$ has degree 1, it suffices to prove that $r_G(\underline{d} - u) = -1$. Remark 3.6 yields that $\underline{d} - u$ is u-reduced, moreover $(\underline{d} - u)(u) = -1$. By the previous part we get $r_G(\underline{d} - u) = -1$.

Conversely, assume $r_G(\delta) = 0$. By the previous part, if $\underline{d} \in \delta$ is u-reduced, then $\underline{d}(u) \geq 0$. It suffices to show that if, for all $u \in V$, the u-reduced representative, $\underline{d}$, satisfies $\underline{d}(u) > 0$, then $r_G(\delta) \geq 1$. Indeed, let $\underline{e}$ be an effective divisor of degree 1; hence $\underline{e} = v$ for some $v \in V$. Let $\underline{d}' \in \delta$ be v-reduced, then $\underline{d}' - v$ is effective, and we are done. □

Let G be a loopless, weightless graph and pick $\underline{d} \in \mathrm{Div}\, G$. Set

$$\ell_G(\underline{d}) = \begin{cases} \min\{\underline{d}(v), \quad \forall v \in V(G)\} & \text{if } \underline{d} \geq 0, \\ -1 & \text{otherwise.} \end{cases} \tag{9}$$

Remark 3.9. It is clear that $r_G(\underline{d}) \geq \ell_G(\underline{d})$ (in fact, "ℓ" stands for "lower bound"). We now look for conditions under which equality occurs.

The following result generalizes to arbitrary combinatorial rank the implications "if" of Lemma 3.8 (the implication "only if" does not generalize; see Example 5.11).

Proposition 3.10. *Let G be a weightless, loopless graph. Let $\underline{d} \in \mathrm{Div}(G)$ be such that for some $u \in V(G)$ with $\underline{d}(u) = \ell_G(\underline{d})$ we have that $\underline{d}$ is reduced with respect to u. Then $r_G(\underline{d}) = \ell_G(\underline{d})$.*

Proof. For simplicity, we write $\ell = \ell_G(\underline{d})$. Let us first suppose $\underline{d} \ngeq 0$, that is $\ell = -1$. Then the statement is a consequence of Lemma 3.8.

Assume now $\ell \geq 0$. Since $r_G(\underline{d}) \geq \ell$ it suffices to prove

$$r_G(\underline{d}) < \ell + 1. \tag{10}$$

Let $\underline{e} := (\ell + 1)u \in \mathrm{Div}(G)$, with u as in the statement; so $\underline{e}$ is an effective divisor of degree $\ell + 1$. Set $\underline{c} = \underline{d} - \underline{e}$; to prove (10) it is enough to show that $r_G(\underline{c}) = -1$. Now, $\underline{c}$ is reduced with respect to u (see Remark 3.6), hence the previous case yields $r_G(\underline{c}) = -1$. □

3.3. Computing the rank for general graphs

We now generalize the previous set-up to weighted graphs. Let g and d be two non-negative integers; set

$$d_g := \max\left\{d - g, \left\lfloor \frac{d}{2} \right\rfloor\right\}.$$

Remark 3.11. For every irreducible curve C of genus g, by the theorems of Riemann–Roch and Clifford, d_g is the maximum rank of a line bundle of degree d; more precisely for a line bundle L on C we have:

$$\deg L \leq d \implies r(C, L) \leq d_g.$$

Similarly, d_g is the maximum combinatorial rank of a divisor of degree d on a graph G of genus g; that is, for every $\underline{d} \in \mathrm{Div}(G)$ we have

$$|\underline{d}| \leq d \implies r_G(\underline{d}) \leq d_g.$$

This follows from the Riemann–Roch formula, [2, Thm. 3.8], and the Clifford inequality, due to Baker and Norine [6, Corollary 3.5] (the extension to weighted graphs is trivial, see [9, Proposition 1.7(4)]).

The notation d_g introduced here is related to the notation e^g introduced before Remark 3.1 by the following trivial lemma, of which we omit the proof.

Lemma 3.12. *Let g, e, d be non-negative integers. Then $(e^g)_g = e$ and*

$$(d_g)^g = \begin{cases} d - 1 & \text{if } d \leq 2g - 1 \text{ and } d \text{ is odd}, \\ d & \text{otherwise}. \end{cases}$$

Now we define:

Definition 3.13. Let G be a graph and let $\underline{d} \in \mathrm{Div}\, G$ be an effective divisor. We define the (effective) divisor $\underline{d}_{\mathrm{rk}}$ such that for every $v \in V$

$$\underline{d}_{\mathrm{rk}}(v) = \underline{d}(v)_{g(v)} = \max\left\{\underline{d}(v) - g(v), \left\lfloor \frac{\underline{d}(v)}{2} \right\rfloor\right\}.$$

Remark 3.14. The subscript "rk" indicates that the entries of $\underline{d}_{\mathrm{rk}}$ are thought of as "maximum ranks", in the spirit of Remark 3.11.

Example 3.15. Suppose G is a graph with no edges, made of a single vertex v having weight $g(v)$. Then for any $\underline{d} \in \mathrm{Div}(G)$ we have $r_G(\underline{d}) = \underline{d}_{\mathrm{rk}}(v)$. More exactly, we have $\widehat{G} = R_v$ (where R_v is the graph introduced in (6)) and

$$r_G(\underline{d}) = r_{R_v}(\widehat{\underline{d}}) = \underline{d}(v)_{g(v)},$$

see [2, Lemma 3.7 and Thm. 3.8].

For any divisor $\underline{d}$ on G we define

$$\ell_G(\underline{d}) = \begin{cases} \min\{\underline{d}_{\mathrm{rk}}(v), \quad \forall v \in V(G)\} & \text{if } \underline{d} \geq 0, \\ -1 & \text{otherwise.} \end{cases} \tag{11}$$

This definition of $\ell_G(\underline{d})$ coincides with the one in (9) if G has no loops and no weights.

Lemma 3.16. *Let G be any graph and let $\underline{d} \in \mathrm{Div}\, G$ be a divisor. Then $r_G(\underline{d}) \geq \ell_G(\underline{d})$.*

Proof. Set $\ell = \ell_G(\underline{d})$; we can clearly assume $\ell \geq 0$, hence $\underline{d} \geq 0$. Note that we have

$$\underline{d}(v) \geq \underline{d}_{\mathrm{rk}}(v) + \min\{\underline{d}_{\mathrm{rk}}(v), g(v)\} = \underline{d}_{\mathrm{rk}}(v)^{g(v)}$$

for every $v \in V(G)$. Now, let $\underline{e}$ be any effective divisor of degree ℓ. In particular, $\underline{e}(v) \leq \ell$ for every $v \in V(G)$, so by the definition of ℓ, we have

$$\underline{d}_{\mathrm{rk}}(v) \geq \ell \geq \underline{e}(v)$$

hence

$$\underline{d}_{\mathrm{rk}}(v)^{g(v)} \geq \underline{e}(v)^{g(v)}.$$

Combining the last inequality with the first we have

$$\underline{d}(v) \geq \underline{d}_{\mathrm{rk}}(v)^{g(v)} \geq \underline{e}^{\deg}(v).$$

We conclude that for every effective divisor $\underline{e}$ with $|\underline{e}| = \ell$ we have $\underline{d} - \underline{e}^{\deg} \geq 0$. By Lemma 3.3 we are done. $\square$

Let G be any graph. As in [2], we denote by G_0 the weightless graph obtained by removing from G every loop, and disregarding all weights. We have natural identifications $V(G) = V(G_0)$ and $\mathrm{Div}(G) = \mathrm{Div}(G_0)$. Now, this identification does not preserve the combinatorial rank, but we have

$$r_{G_0}(\underline{d}) \geq r_G(\underline{d}) \tag{12}$$

for every $\underline{d} \in \mathrm{Div}(G)$ (see [2, Remark 3.3]). Also, we have $\mathrm{Prin}(G_0) = \mathrm{Prin}(G)$ and hence an identification $\mathrm{Pic}(G) = \mathrm{Pic}(G_0)$. Finally, the above facts are easily seen to imply

$$r_{G_0}(\underline{d}) = -1 \Leftrightarrow r_G(\underline{d}) = -1. \tag{13}$$

The next result generalizes Proposition 3.10.

Proposition 3.17. *Let G be any graph. Let $\underline{d} \in \mathrm{Div}(G)$ be such that for some $u \in V(G)$ with $\underline{d}_{\mathrm{rk}}(u) = \ell_G(\underline{d})$ we have that $\underline{d}$ is reduced with respect to u. Then $r_G(\underline{d}) = \ell_G(\underline{d})$.*

Proof. Let us write $\ell = \ell_G(\underline{d})$ and let G_0 be the weightless, loopless graph defined above. We first suppose $\underline{d} \not\geq 0$, i.e., $\ell = -1$. Then the fact that $r_G(\underline{d}) = -1$ follows immediately by combining (13) with Lemma 3.8.

Assume now that $\ell \geq 0$. By Lemma 3.16 it is enough to show that

$$r_G(\underline{d}) < \ell + 1.$$

Denote by G^u the graph obtained from G by adding $\omega(u)$ loops based at u and then by inserting a vertex in each loop based at u. Notice that, using the notation at the beginning of Subsection 3.1, we have a connected subgraph $R_u \subset G^u$ of genus $g(u)$. Denote by $\underline{d}^u$ the divisor obtained by extending $\underline{d}$ to G^u with degree 0 on the new vertices of R_u.

As we saw in Example 3.15, we have

$$\ell = \underline{d}_{\mathrm{rk}}(u) = r_{R_u}(\underline{d}^u_{|R_u}). \tag{14}$$

It is clear that $r_G(\underline{d}) = r_{G^u}(\underline{d}^u)$ (as $\widehat{G^u} = \widehat{G}$ and $\widehat{\underline{d}^u} = \widehat{\underline{d}}$). It suffices to show $r_{G^u}(\underline{d}^u) < \ell + 1$.

By (14), there is on R_u an effective divisor, $\underline{e}_u$, such that $|\underline{e}_u| = \ell + 1$ and

$$r_{R_u}(\underline{d}^u_{|R_u} - \underline{e}_u) = -1.$$

Define $\underline{e} \in \mathrm{Div}(G^u)$ as the extension of $\underline{e}_u$ to G^u with degree 0 on all vertices outside R_u and set $\underline{c} := \underline{d}^u - \underline{e}$. Since $|\underline{e}| = \ell + 1$, to conclude it is enough to check that $r_{G^u}(\underline{c}) = -1$.

Let $\underline{c}'_u \in \mathrm{Div}(R_u)$ be linearly equivalent to $\underline{d}^u_{|R_u} - \underline{e}_u$ and reduced with respect to u; since $r_{R_u}(\underline{c}'_u) = -1$ we have $\underline{c}'_u(u) < 0$. Let $\underline{c}'$ be the extension of $\underline{c}'_u$ to $\mathrm{Div}(G^u)$ such that $\underline{c}'(v) := \underline{c}(v) = \underline{d}(v)$ for all $v \in V(G^u) \smallsetminus V(R_u)$; then $\underline{c}'$ is linearly equivalent to $\underline{c}$ (since we have an inclusion $\mathrm{Prin}(R_u) \hookrightarrow \mathrm{Prin}(G^u)$), hence it suffices to prove that

$$r_{G^u}(\underline{c}') = -1.$$

Now, the restrictions of $\underline{c}'$ to R_u and to G are both u-reduced (the latter because $\underline{c}'$ coincides with $\underline{d}$ on $V(G) \smallsetminus \{u\}$, which is u-reduced by hypothesis). Hence $\underline{c}'$ is u-reduced, and hence $r_{G^u}(\underline{c}') = -1$. □

3.4. Dhar decomposition

Fix any graph G; we are going to define a decomposition, which we call the "Dhar decomposition", that will come very useful with inductive arguments, and which is independent of the weights or of the loops of G.

Fix a vertex u of G and let $\underline{d}$ be a divisor on G whose restriction to $V \smallsetminus \{u\}$ is effective; the Dhar decomposition associated to $\underline{d}$ with respect to u is a decomposition of V that will be denoted as follows:

$$V = Y_0 \sqcup Y_1 \ldots \sqcup Y_l \sqcup W. \tag{15}$$

For the reader familiar with Dhar Burning Algorithm, Y_j will be the set of vertices burned at the j-th day when starting a fire from u, and W is the set which remains unburned at the end. Hence, we shall refer to it as the Dhar decomposition of V associated to u.

Denote $Y_0 = \{u\}$ and set $W_0 = V \smallsetminus \{u\}$. If $\underline{d} + \underline{t}_{W_0}$ is effective (which implies that $\underline{d}$ is not reduced at u), then set $W = W_0$, and the decomposition is just $V = Y_0 \sqcup W$. Otherwise, define Y_1 to be the set of vertices in W_0 where $\underline{d} + \underline{t}_{W_0}$ is negative.

Now, repeat the process: suppose that $Y_0, Y_1, \ldots, Y_{j-1}$ have been defined. Denote $W_{j-1} := V \setminus Y_0 \sqcup \ldots \sqcup Y_{j-1}$, and consider $\underline{d} + \underline{t}_{W_{j-1}}$. If it is effective, then W will be equal to W_{j-1}, and we are done. Otherwise, define Y_j to be the set of vertices in W_{j-1} where $\underline{d} + \underline{t}_{W_{j-1}}$ is negative. If the process eventually exhausts all the vertices of the graph, then W will be the empty set (which occurs exactly when $\underline{d}$ is u-reduced; see [20, Lemma 2.6]).

Remark 3.18. For every $\underline{d}$ and u, vertices in Y_j or W can be characterized as follows. For $j = 1, \ldots, l-1$, we have

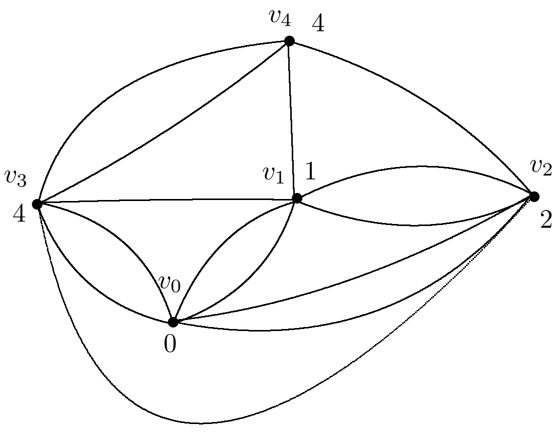
$$v \in Y_j \Leftrightarrow v \notin Y_{j-1} \text{ and } \underline{d}(v) < v \cdot (Y_0 \sqcup Y_1 \ldots \sqcup Y_{j-1}). \tag{16}$$

Example 3.19. In the picture below we illustrate an example of a graph G with vertex set $\{v_0, v_1, v_2, v_3, v_4\}$ and a divisor $\underline{d} = (0, 1, 2, 4, 4) \in \mathrm{Div}(G)$. Then Dhar decomposition of $\underline{d}$ with respect to $u = v_0$ is as follows

$$Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup W = \{v_0\} \sqcup \{v_1\} \sqcup \{v_2\} \sqcup \{v_3, v_4\}.$$

On the other hand, the decomposition with respect to $u = v_4$ is simply:

$$Y_0 \sqcup W = \{v_4\} \sqcup \{v_0, v_1, v_2, v_3\}.$$



Let $\underline{d} \in \mathrm{Div}(G)$ be an effective divisor which is non-reduced with respect to a fixed vertex u of G. Then, by adding a suitable set of edges to G, one can construct a graph on which $\underline{d}$ is u reduced.

Definition 3.20. Let u be a vertex of a graph G. Let $\underline{d} \in \mathrm{Div}(G)$ be such that $\underline{d}(v) \geq 0$ for every $v \in V(G) \setminus \{u\}$. A *saturation* of G (or a G-*saturation*) with respect to u and $\underline{d}$ is a graph G' satisfying the following requirements:

(a) G is a spanning subgraph of G';
(b) $\underline{d}$ is u-reduced as a divisor on G';
(c) every edge in $E(G') \setminus E(G)$ is adjacent to u.

For instance, in Example 3.19, a G-saturation with respect to $\underline{d}$ and v_0 can be obtained by adding one edge between v_0 and v_3 and one edge between v_0 and v_4. And a G-saturation with respect to $\underline{d}$ and v_4 is given adding one edge between v_0 and v_4, or also one edge between v_1 and v_4.

Remark 3.21. It is trivial to check that such saturations always exist.

More precisely, consider the Dhar decomposition (15) with respect to $\underline{d}$ and u. Then, as G is connected, by adding $\underline{d}(w)$ edges between w and u for every $w \in W$, we get a G-saturation with respect to $\underline{d}$ and u.

Proposition 3.22. *Let $\underline{d}$ be an effective divisor on a loopless, weightless graph G, and let $u \in V(G)$ be such that $\underline{d}(u) = \ell_G(\underline{d})$. Let G' be a G-saturation with respect to u and $\underline{d}$. Set $m = |E(G') \smallsetminus E(G)|$; then*

$$r_G(\underline{d}) \le \ell_G(\underline{d}) + m$$

(equivalently: $r_G(\underline{d}) \le r_{G'}(\underline{d}) + m$).

Proof. Write $\ell = \ell_G(\underline{d})$. Recall that the edges in $E(G') \smallsetminus E(G)$ are all adjacent to u; denote by $\{w_1, \dots, w_k\} \subset V(G) \smallsetminus \{u\}$ the vertices adjacent to the different edges in $E(G') \smallsetminus E(G)$. Set

$$m_i = (u \cdot w_i)_{G'} - (u \cdot w_i)_G \tag{17}$$

(the subscripts G and G' indicate the graph where the intersection product is computed). We have $\sum_{i=1}^k m_i = m$.

Write

$$\underline{c} = \underline{d} - (\ell + 1)u - \sum_{i=1}^{k} m_i w_i;$$

we have $\underline{c}(u) = -1$. It suffices to prove that the rank of $\underline{c}$ is -1. By contradiction, suppose this is not the case; then there exists $\underline{t} \in \mathrm{Prin}(G)$ such that $\underline{c} + \underline{t} \ge 0$. According to Remark 2.1 we have $\underline{t} = \sum_{i=1}^k i \cdot \underline{t}_{Z_i}$, where $Z_1, \dots Z_k$ are disjoint sets of vertices with $Z_k \ne \emptyset$; by the same remark we have $\underline{t}_{|Z_k} \le (\underline{t}_{Z_k})_{|Z_k}$. Summarizing:

$$0 \le (\underline{c} + \underline{t})_{|Z_k} = \underline{c}_{|Z_k} + \underline{t}_{|Z_k} \le \underline{c}_{|Z_k} + (\underline{t}_{Z_k})_{|Z_k} \le \underline{c}_{|Z_k} \tag{18}$$

(as $(\underline{t}_{Z_k})_{|Z_k} \le 0$). Hence $u \notin Z_k$. Write $\underline{t}'_{Z_k}$ for the principal divisor corresponding to Z_k in G'. Pick $v \in Z_k$; if $v = w_i$ for some i, then $\underline{t}'_{Z_k}(v) = \underline{t}_{Z_k}(v) - m_i$ by (17); otherwise we have $\underline{t}'_{Z_k}(v) = \underline{t}_{Z_k}(v)$. In either case for every $v \in Z_k$ we have

$$(\underline{d} + \underline{t}'_{Z_k})(v) = (\underline{c} + \underline{t}_{Z_k})(v) \ge 0$$

by (18); a contradiction to the reducedness of $\underline{d}$ on G'. □

Example 3.19 *(continued).* Let us show that in Example 3.19 we have $r_G(\underline{d}) = 2$. Of course, $\ell_G(\underline{d}) = \underline{d}(v_0) = 0$ and, as we already observed, a G-saturation with respect to v_0 can be obtained by adding two edges on G. Hence Proposition 3.22 applies with $m = 2$, giving $r_G(\underline{d}) \le 2$. Let us now check that $r_G(\underline{d}) \ge 2$. We have

$$\underline{d} + \underline{t}_{\{v_3,v_4\}} = (0,1,2,4,4) + (2,2,2,-4,-2) = (2,3,4,0,2) =: \underline{d}'.$$

Hence it remains to check that $\underline{d} - (v_0 + v_3)$ is equivalent to an effective divisor, which we do as follows:

$$\underline{d}' + \underline{t}_{\{v_1,v_2,v_4\}} = (2,3,4,0,2) + (4,-3,-3,4,-2) = (6,0,1,4,0).$$

4. The inequality $r_G(\delta) \geq r^{\mathrm{alg}}(G, \delta)$

4.1. Proof of the inequality

Suppose L is a line bundle on a curve X such that its multidegree corresponds to a u-reduced divisor for some vertex u of the dual graph of X. Then a non-zero section of L cannot vanish identically on the component of X corresponding to u. This is a special case of the next lemma (namely the case when $\underline{d}$ is u-reduced), which is similar to Lemma 4.9 in [2]. Recall that for an effective divisor $\underline{e}$, we introduced the effective divisor $\underline{e}^{\deg}$ in Definition 3.2.

Lemma 4.1. *Let X be a nodal curve whose dual graph is G. Let L be a line bundle on X, and denote $\underline{d} = \underline{\deg} L$. Suppose that for some $u \in V(G)$, and effective divisor $\underline{e} \in \mathrm{Div}(G)$, the divisor $\underline{d} - \underline{e}^{\deg}$ is u-reduced. Then the space of global sections of L vanishing identically on C_u has dimension at most $|\underline{e}| - \underline{e}(u)$.*

Proof. Consider the Dhar decomposition associated to $\underline{d} - \underline{e}^{\deg}$ with respect to u as in 3.4; since $\underline{d} - \underline{e}^{\deg}$ is u-reduced, W is empty and $V = Y_0 \sqcup \ldots \sqcup Y_l$.

For each $0 \leq j \leq l$, denote by $\Lambda_j \subset H^0(X, L)$ the space of sections of L vanishing on the components of X corresponding to the vertices of $Y_0 \sqcup \ldots \sqcup Y_j$. We must prove that $\dim \Lambda_0 \leq |\underline{e}| - \underline{e}(u)$. We will proceed by descending induction on $0 \leq j \leq l$, showing that

$$\dim \Lambda_j \leq \sum_{i=j+1}^{l} |\underline{e}_{|Y_i}|. \tag{19}$$

For $j = l$, the claim is obvious, since Λ_l is the space of sections vanishing on the entire curve, and its dimension is 0. Now, assume that (19) holds and let us consider Λ_{j-1}.

Let v be any vertex of Y_j, and let D_v be the divisor on C_v consisting exactly of the intersection points of C_v with the components of X corresponding to $Y_0 \sqcup \ldots \sqcup Y_{j-1}$. By Remark 3.18, we have

$$(\underline{d} - \underline{e}^{\deg})(v) - \deg(D_v) < 0.$$

Hence

$$\deg_{C_v} L(-D_v) < \underline{e}^{\deg}(v),$$

and therefore, by Remark 3.1, we get

$$h^0(C_v, L(-D_v)) \leq \underline{e}(v).$$

We obtain

$$\dim\Big(\bigoplus_{v \in Y_j} H^0(C_v, L(-D_v))\Big) \leq \sum_{v \in Y_j} \underline{e}(v) = |\underline{e}_{|Y_j}|. \tag{20}$$

Now, consider the exact sequence

$$0 \to \Lambda_j \to \Lambda_{j-1} \xrightarrow{\alpha} \bigoplus_{v \in Y_j} H^0(C_v, L(-D_v)),$$

where α is the map restricting a section to each component. Then

$$\dim \Lambda_{j-1} \leq \dim \Lambda_j + \dim\Big(\bigoplus_{v\in Y_j} H^0(C_v, L(-D_v))\Big) \leq \sum_{i=j}^{l} |\underline{e}_{|Y_i}|,$$

where the last inequality follows from the induction hypothesis and (20). The proof is complete. $\square$

Theorem 4.2. *Let δ be a divisor class on a graph G. Then*

$$r^{\mathrm{alg}}(G,\delta) \leq r_G(\delta).$$

Proof. Denote $s = r^{\mathrm{alg}}(G,\delta)$. If $s = -1$ then the claim is obvious, since the combinatorial rank is always bounded below by -1. Hence we may assume that $s \geq 0$.

In order to prove that $r_G(\delta) \geq s$, by Lemma 3.3 it suffices to show that for any effective divisor $\underline{e}$ with $|\underline{e}| = s$, the divisor class δ admits a representative $\underline{d}$ such that $\underline{d} - \underline{e}^{\deg} \geq 0$.

Let $\underline{e}$ be such a divisor, and let $u \in V$ be a fixed vertex. Using Fact 3.7 we have that there exists a representative $\underline{d}$ for δ such that $\underline{d} - \underline{e}^{\deg}$ is u-reduced. Now, by definition, $\underline{d} - \underline{e}^{\deg}$ is effective on $V \smallsetminus \{u\}$, so it remains to show that

$$\underline{d}(u) - \underline{e}^{\deg}(u) \geq 0. \tag{21}$$

Since $r^{\mathrm{alg}}(G,\delta) = s$, there exist a curve $X \in M^{\mathrm{alg}}(G)$ and a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$ such that $h^0(X,L) \geq s+1$. Consider the exact sequence

$$0 \to \ker(\pi) \to H^0(X,L) \xrightarrow{\pi} H^0(C_u, L_{C_u}),$$

where π is the restriction of sections to C_u. The space $\ker(\pi)$ is exactly the set of global sections of L vanishing on C_u, so by Lemma 4.1,

$$\dim\ker(\pi) \leq s - \underline{e}(u).$$

From the above exact sequence we obtain

$$h^0(C_u, L_{C_u}) \geq h^0(X,L) - \dim(\ker\pi) \geq s + 1 - s + \underline{e}(u) = \underline{e}(u) + 1.$$

Now Remark 3.1 yields

$$\deg_{C_u} L \geq \underline{e}(u)^{g(C_u)},$$

in other words $\underline{d}(u) \geq \underline{e}^{\deg}(u)$, which proves (21) and the theorem. $\square$

Corollary 4.3. *Let δ be a divisor class on G such that $r_G(\delta) \leq 0$. Then for every $X \in M^{\mathrm{alg}}(G)$ we have $r(X,\delta) = r_G(\delta)$. In particular, $r^{\mathrm{alg}}(G,\delta) = r_G(\delta)$.*

Proof. The case $r_G(\delta) = -1$ is obvious. Suppose $r_G(\delta) = 0$; by the previous theorem it suffices to show that for any $X \in M^{\mathrm{alg}}(G)$ we have $r(X,\delta) \geq 0$. Let $\underline{d} \in \delta$ be such that $\underline{d} \geq 0$, now let $L = \mathcal{O}_X(D)$ where D is an effective Cartier divisor of multidegree $\underline{d}$; then $r(X,L) \geq 0$ and hence $r^{\max}(X,\underline{d}) \geq 0$.

Now, let $\underline{c} \in \delta$ be a different representative for δ and write $\underline{c} + \underline{t} = \underline{d}$, for some non-trivial $\underline{t} \in \mathrm{Prin}(G)$. By Remark 2.1, we have

$$\underline{t}_{|Z} \leq (\underline{t}_Z)_{|Z}, \tag{22}$$

for some non-empty $Z \subsetneq V$. We shall abuse notation and denote by Z and Z^c the subcurves of X whose components correspond to the vertices in Z and Z^c respectively. We have $r(Z, L_Z) \geq 0$, of course.

Let $Z \cdot Z^c \in \mathrm{Div}(Z)$ be the (effective, Cartier) divisor cut on Z by Z^c; its multidegree, $\underline{\deg}_Z Z \cdot Z^c$, satisfies

$$\underline{\deg}_Z Z \cdot Z^c = (\underline{t}_{Z^c})_{|Z} = -(\underline{t}_Z)_{|Z}. \tag{23}$$

Now, for every $M \in \mathrm{Pic}^{\underline{c}}(X)$ we have

$$r(X, M) \geq r(Z, M_Z(-Z \cdot Z^c)) \tag{24}$$

(any section of $M_Z(-Z \cdot Z^c)$ vanishes on $Z \cap Z^c$ and hence can be glued to the zero section on Z^c). Moreover, using (23) and (22)

$$\underline{\deg}_Z M(-Z \cdot Z^c) = \underline{c}_{|Z} - \underline{\deg}_Z Z^c = \underline{c}_{|Z} + (\underline{t}_Z)_{|Z} \geq \underline{c}_{|Z} + \underline{t}_{|Z} = \underline{d}_{|Z}.$$

The above inequality implies that we can pick $M \in \mathrm{Pic}^{\underline{c}}(X)$ such that its restriction to Z satisfies $M_Z = L_Z(Z \cdot Z^c + E)$ where E is some effective Cartier divisor on Z. By (24) we have

$$r(X, M) \geq r(Z, L_Z(Z \cdot Z^c + E - Z \cdot Z^c)) \geq r(Z, L_Z) \geq 0.$$

Hence $r^{\max}(X, \underline{c}) \geq 0$, and hence $r(X, \delta) \geq 0$, as required. □

Corollary 4.4. *Let G be a graph of genus g and $\delta \in \mathrm{Pic}^d(G)$ with $d \geq 0$. If $r_G(\delta) = \min\{0, d - g\}$ then $r^{\mathrm{alg}}(G, \delta) = r_G(\delta)$.*

Proof. By Riemann–Roch, for all $X \in M^{\mathrm{alg}}(G)$ and all $L \in \mathrm{Pic}^d(X)$ we have $r(X, L) \geq \min\{0, d - g\}$. By Theorem 4.2 we are done. □

In [15] and [16, Thm. 1.1 and Thm. 1.2] the authors prove the inequality $r^{\mathrm{alg}}(G, \delta) \geq r_G(\delta)$ in certain cases; combining with Theorem 4.2, we have the following partial answer to Problem 1.

Corollary 4.5. *Let G be a graph. We have $r^{\mathrm{alg}}(G, \delta) = r_G(\delta)$ for every $\delta \in \mathrm{Pic}^d(G)$ in the following cases:*

(a) $\mathrm{char}(k) \neq 2$ *and G is hyperelliptic.*
(b) G has genus 3 and it is not hyperelliptic.

4.2. Clifford inequality for the algebraic rank

It is well known that Clifford inequality fails trivially for reducible curves; in fact for any reducible curve X of genus g and any integer d with $0 \leq d \leq 2g - 2$, there exist infinitely many $L \in \mathrm{Pic}^d(X)$ such that $r(X, L) > \lfloor d/2 \rfloor$ (see [9, Proposition 1.7(4)]). But consider the following question:

Pick a graph G of genus g and $\delta \in \mathrm{Pic}^d(G)$ with $0 \leq d \leq 2g - 2$.
Does there exist a multidegree $\underline{d} \in \delta$ such that for every $X \in M^{\mathrm{alg}}(G)$ and every $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have $r(X, L) \leq \lfloor d/2 \rfloor$?

Apart from some special cases (see [8]), the answer to this question was not known; we can now answer it affirmatively in full generality.

Proposition 4.6 *(Clifford inequality). Let G be a graph of genus g and $\delta \in \mathrm{Pic}^d(G)$ with $0 \leq d \leq 2g-2$. Then*

$$r^{\mathrm{alg}}(G, \delta) \leq \lfloor d/2 \rfloor$$

(that is $r(X, \delta) \leq \lfloor d/2 \rfloor$ for every $X \in M^{\mathrm{alg}}(G)$).

Proof. Immediate consequence of Theorem 4.2 and Clifford inequality for graphs [6, Corollary 3.5] (which, as we already mentioned, extends trivially to graphs with loops and weights). □

In other words, for every X there exists a multidegree $\underline{d} \in \delta$ such that every $L \in \mathrm{Pic}^{\underline{d}}(X)$ satisfies Clifford inequality. The following problem naturally arises.

Question 4.7. For which multidegrees does Clifford inequality hold? Do these multidegrees depend on the curve X, or can they be combinatorially characterized (i.e. depend only on G and d)?

A few special cases of this question are answered in [8], namely $|\underline{d}| \leq 4$, or $|V(G)| = 2$. A general answer is not known.

4.3. Reduction to loopless graphs

Let G be a graph with a loop e based at the vertex v. Denote by $G^\bullet$ the graph obtained from G by inserting a weight-zero vertex, u, in e. There is a natural map

$$\mathrm{Div}(G) \longrightarrow \mathrm{Div}(G^\bullet); \qquad \underline{d} \mapsto \underline{d}^\bullet$$

such that $\underline{d}^\bullet(u) = 0$ and $\underline{d}^\bullet$ is equal to $\underline{d}$ on the remaining vertices of $\underline{d}$. The above map is a group homomorphism and sends $\mathrm{Prin}(G)$ into $\mathrm{Prin}(G^\bullet)$, hence we also have a map

$$\mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(G^\bullet); \qquad \delta \mapsto \delta^\bullet.$$

By the definition of combinatorial rank, every $\underline{d} \in \mathrm{Div}(G)$ satisfies

$$r_G(\underline{d}) = r_{G^\bullet}(\underline{d}^\bullet).$$

Let now $X \in M^{\mathrm{alg}}(G)$ and let $X^\bullet \in M^{\mathrm{alg}}(G^\bullet)$ be the curve obtained by "blowing-up" X at the node N_e corresponding to the loop e, i.e.

$$X^\bullet = Y \cup E$$

where Y is the desingularization of X at N_e and $E \cong \mathbb{P}^1$ is attached to Y at the branches over N_e. This process is invertible, i.e. given $X^\bullet$ one reconstructs X by contracting the component E to a node. In conclusion, we have a bijection

$$M^{\mathrm{alg}}(G) \leftrightarrow M^{\mathrm{alg}}(G^\bullet); \qquad X \leftrightarrow X^\bullet.$$

Proposition 4.8. *With the above notation, for any graph G, any divisor $\underline{d} \in \mathrm{Div}(G)$, any class $\delta \in \mathrm{Pic}(G)$, and any curve $X \in M^{\mathrm{alg}}(G)$ we have*

(a) $r^{\max}(X, \underline{d}) = r^{\max}(X^\bullet, \underline{d}^\bullet)$;
(b) $r(X, \delta) \geq r(X^\bullet, \delta^\bullet)$;
(c) $r^{\mathrm{alg}}(G, \delta) \geq r^{\mathrm{alg}}(G^\bullet, \delta^\bullet)$.

Proof. We begin with (a). Let $\sigma : X^\bullet \to X$ be the morphism contracting E, so that its restriction to Y is birational onto X. Then σ induces an isomorphism

$$\mathrm{Pic}^{\underline{d}}(X) \xrightarrow{\cong} \mathrm{Pic}^{\underline{d}^\bullet}(X^\bullet); \qquad L \mapsto \sigma^* L.$$

For any $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have an injection $H^0(X, L) \hookrightarrow H^0(X^\bullet, \sigma^* L)$, therefore $r^{\max}(X, \underline{d}) \leq r^{\max}(X^\bullet, \underline{d}^\bullet)$. For the opposite inequality, pick $L^\bullet = \sigma^* L \in \mathrm{Pic}^{\underline{d}^\bullet}(X^\bullet)$ and notice that the sections of $L^\bullet$ are constant along E (as $\deg_E L^\bullet = 0$), hence they descend to sections of L. (a) is proved.

For (b) it is enough to show that for every $\underline{d} \in \mathrm{Div}(G)$ with $[\underline{d}] = \delta$ we have

$$r^{\max}(X, \underline{d}) \geq r(X^\bullet, \delta^\bullet). \tag{25}$$

Fix such a $\underline{d}$; consider $\underline{d}^\bullet \in \mathrm{Div}(G^\bullet)$. Recall that $r(X^\bullet, \delta^\bullet)$ is the minimum of all $r^{\max}(X^\bullet, \underline{d}')$ as $\underline{d}'$ varies in $\delta^\bullet$. Hence (25) follows from (a). We have thus proved (b), and, since (c) follows trivially from it, we are done. □

Let G be a graph admitting some loops and, by abusing notation, let $G^\bullet$ be the loopless graph obtained by inserting a vertex in the interior of every loop. From the previous result we derive the following:

Proposition 4.9. *If $r_{G^\bullet}(\delta^\bullet) = r^{\mathrm{alg}}(G^\bullet, \delta^\bullet)$, then $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$.*

Proof. By iterating the construction described at the beginning of the subsection, we have a natural map

$$\mathrm{Pic}(G) \to \mathrm{Pic}(G^\bullet); \qquad \delta \mapsto \delta^\bullet$$

such that $r_G(\delta) = r_{G^\bullet}(\delta^\bullet)$. By hypothesis we have

$$r_G(\delta) = r_{G^\bullet}(\delta^\bullet) = r^{\mathrm{alg}}(G^\bullet, \delta^\bullet) \leq r^{\mathrm{alg}}(G, \delta),$$

where the last inequality follows from Proposition 4.8. By Theorem 4.2 the statement follows. □

5. When is $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$?

The purpose of this section is to find cases the answer to Problem 1 is affirmative. Throughout the section we shall restrict our attention to weightless, loopless graphs, unless we specify otherwise.

5.1. *Special algebraic curves*

Let G be a weightless, loopless graph; we now look for curves $X \in M^{\mathrm{alg}}(G)$ which are likely to realize the inequality $r^{\max}(X, \underline{d}) \geq r_G(\underline{d})$ for $\underline{d} \in \mathrm{Div}(G)$. We shall explicitly describe some such curves, after recalling some notation (see [3]).

Let V, E and H be, respectively, the set of vertices, edges and half-edges of G. We have the following structure maps: the *endpoint* map:

$$\epsilon : H \to V;$$

the *gluing* map, which is surjective and two-to-one

$$\gamma : H \to E.$$

The gluing map γ induces a fixed-point-free involution on H, denoted by ι, whose orbits, written $[h, \overline{h}]$, are identified with the edges of G.

For any $v \in V$, we denote by $H_v = \epsilon^{-1}(v)$ and $E_v = \gamma(\epsilon^{-1}(v))$ the sets of half-edges and edges adjacent to v. We denote by

$$H_{v,w} = H_v \cap \gamma^{-1}(E_w)$$

(the set of half-edges adjacent to v and glued to a half-edge adjacent to w). We obviously have $\iota(H_{v,w}) = H_{w,v}$.

Let X be a curve dual to G; we write $X = \cup_{v\in V} C_v$ as usual. We have a set $P_v \subset C_v$ of labeled distinct points of C_v mapping to smooth points of C_v:

$$P_v := \{p_h,\ \forall h \in H_v\} = \sqcup_{w\in V} P_{v,w},$$

where

$$P_{v,w} := \{p_h,\ \forall h \in H_{v,w}\} \subset C_v.$$

We will use the following explicit description of X:

$$X = \frac{\sqcup_{v\in V} C_v}{\{p_h = p_{\bar{h}},\ \forall h \in H\}}. \tag{26}$$

Definition 5.1. Let G be a weightless, loopless graph and $X \in M^{\mathrm{alg}}(G)$. X is *special* if there exists a collection

$$\{\phi_{v,w} : (C_v; P_{v,w}) \longrightarrow (C_w; P_{w,v}),\quad \forall v, w \in V\},$$

where $\phi_{v,w}$ is an isomorphism of pointed curves such that for every $u, v, w \in V$ and $h \in H_{v,w}$ the following properties hold:

(a) $\phi_{v,w}(p_h) = p_{\bar{h}}$;
(b) $\phi_{v,w}^{-1} = \phi_{w,v}$;
(c) $\phi_{v,u} = \phi_{w,u} \circ \phi_{v,w}$.

If G is not connected, $X \in M^{\mathrm{alg}}(G)$ is defined to be special if so is every connected component.

Example 5.2. If G has only vertices of valency at most 3 then every curve $X \in M^{\mathrm{alg}}(G)$ is special.

Example 5.3. We say that G is a *binary graph* of genus g if G consists two vertices joined by $g + 1$ edges. If G is a binary graph of genus $g \geq 2$, then $\dim M^{\mathrm{alg}}(G) = 2(g - 2)$, and the locus of special curves in it has dimension $g - 2$.

Remark 5.4. Let X be a special curve. Then every subcurve of X, and every partial normalization of X, is special. Moreover, let $x \in X$ be a nonsingular point of X lying in the irreducible component C_u; then for every component C_v of X the curve

$$X' := \frac{X}{x = \phi_{u,v}(x)}$$

is also special. The quotient map $\pi : X \to X'$ describes X as a partial normalization of X'. We say that X dominates X'.

Lemma 5.5. *For every weightless, loopless graph G, the set $M^{\mathrm{alg}}(G)$ contains a special curve.*

Proof. The proof is by induction on the number of vertices of G; if $|V(G)| = 1$ there is nothing to prove.

Suppose $|V(G)| \geq 2$. Let $u \in V(G)$ and let $G' = G - u$ be the graph obtained by removing u and all the edges adjacent to it; we choose u so that G' is connected (it is well known that such a vertex u exists for any connected graph G). Let C_u be a copy of $\mathbb{P}^1$. By induction there exists a special curve X' having G' as dual graph; for every $w, v \in V(G')$ let $\phi'_{w,v} : C_w \to C_v$ be the isomorphisms associated to X'.

We now pick $v \in V(G')$ and fix an isomorphism $\phi_{v,u} : C_v \to C_u$. Now, for any other vertex $w \in V(G')$ we set

$$\phi_{w,u} := \phi_{v,u} \circ \phi'_{w,v};$$

we also set $\phi_{w,v} = \phi'_{w,v}$. If $H_{u,w}(G)$ is not empty we pick a set of distinct points $P_{w,u} \subset C_w \subset X'$ labeled by $H_{w,u}(G)$, such that $P_{w,u}$ does not intersect any $P_{w,w'}$ with $w' \neq u$, and such that $\phi_{w,u}(P_{w,u})$ does not intersect any $\phi_{w',u}(P_{w',u})$ with $w' \neq w$; we set $P_{u,w} := \phi_{w,u}(P_{w,u})$. Now let X be obtained by gluing C_u to X' by identifying $p \in P_{u,w}$ with $\phi_{u,w}(p)$ for every $p \in P_{u,w}$ and every $w \in V(G')$. It is clear that X is a special curve. □

5.2. Binary curves

A *binary curve* is a curve whose dual graph is binary, as defined in Example 5.3. For such a curve we write $V(G) = \{v_1, v_2\}$ and $X = C_1 \cup C_2$, so that $C_i = C_{v_i}$ and $C_i \cong \mathbb{P}^1$; recall that $v_1 \cdot v_2 = g + 1$, where g is the genus of X.

Let us show that for binary curves the answer to Problem 1 is "yes".

Proposition 5.6. *Let G be a binary graph of genus g. Then $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$ for every $\delta \in \mathrm{Pic}(G)$.*

Proof. By Theorem 4.2 it suffices to prove $r^{\mathrm{alg}}(G, \delta) \geq r_G(\delta)$. In other words, it suffices to prove that there exists $X \in M^{\mathrm{alg}}(G)$ such that for every $\underline{d} \in \delta$ there exists $L \in \mathrm{Pic}^{\underline{d}}(X)$ for which $r(X, L) \geq r_G(\delta)$.

We can assume $r_G(\delta) \geq 0$ and $0 \leq |\delta| \leq 2g - 2$, see [9, Theorem 2.9 and Lemma 2.4].

Since $r_G(\delta) \geq 0$ we can choose $\underline{d} = (a, b) \in \delta$ such that $0 \leq a \leq b$. There are two cases:

• Case 1: $b \leq g$.

Then $r_G(\underline{d}) = a$, by Proposition 3.10. Let $X = C_1 \cup C_2$ be a special binary curve. Then there clearly exists $L \in \mathrm{Pic}^{(a,a)}(X)$ such that $r(X, L) = a$, hence $r^{\max}(X, (a, a)) \geq a$; Remark 2.2 yields $r^{\max}(X, \underline{d}) \geq r^{\max}(X, (a, a)) \geq a$.

Now let $\underline{d}' \in \delta$ be a different representative, so that $\underline{d}' = (a - n(g+1), b + n(g+1))$ for $n \in \mathbb{Z}$ with $n \neq 0$. Then for any $L' \in \mathrm{Pic}^{\underline{d}'}(X)$, using the simple estimate

$$r(X, L') \geq h^0(C_1, L'_{|C_1}) + h^0(C_2, L'_{|C_2}) - 1 - (g + 1)$$

and recalling that $a \leq b \leq g$, we easily get

$$r(X, L') \geq a + |n|(g + 1) - (g + 1) \geq a.$$

So we are done.

• Case 2: $b \geq g + 1$. We claim that

$$r_G(\underline{d}) = a + b - g. \tag{27}$$

In fact from Proposition 3.10 and the fact that $b \geq g + 1$ we obtain

$$r_G(K_G - (a, b)) = r_G(g - 1 - a, g - 1 - b) = -1.$$

Therefore (27) follows from Riemann–Roch.

On the other hand pick any $X \in M^{\mathrm{alg}}(G)$ and any representative $\underline{d}' = (d_1, d_2) \in \delta$, so that $d_1 + d_2 = a + b$. Let $L \in \mathrm{Pic}^{\underline{d}'}(X)$.

Denote by X^ν the normalization of X and by L^ν the pull back of L to it. If $d_i \geq 0$ for $i = 1, 2$ we have

$$r(X, L) \geq r(X^\nu, L^\nu) - (g + 1) = d_1 + d_2 + 1 - g - 1 = a + b - g = r_G(\underline{d}).$$

If $d_1 \leq -1$ then $d_2 \geq 1 + a + b$ and we have

$$r(X, L) \geq r(X^\nu, L^\nu) - (g + 1) = d_2 - g - 1 \geq a + b - g = r_G(\underline{d}),$$

so we are done. □

Remark 5.7. The proof gives a slightly stronger statement. Indeed in case 1, we proved that for every special curve $X \in M^{\mathrm{alg}}(G)$ we have $r(X, \delta) = r_G(\delta)$. In case 2 we have $r(X, L) \geq r_G(\delta)$ for every $X \in M^{\mathrm{alg}}(G)$, every $\underline{d} \in \delta$ and every $L \in \mathrm{Pic}^{\underline{d}}(X)$.

5.3. The rank-explicit case

For a weightless loopless graph G, in Proposition 3.10 we proved that if a divisor $\underline{d}$ is reduced with respect to a vertex u of minimal degree, then $r_G(\underline{d}) = \underline{d}(u)$, unless $\underline{d}(u) < -1$ in which case $r_G(\underline{d}) = -1$. We think of such a divisor as “explicitly exhibiting” its rank (equal to its minimal entry); this motivates the terminology below.

Definition 5.8. Let G be a weightless loopless graph. We say that a divisor $\underline{d} \in \mathrm{Div}(G)$ is *rank-explicit* if $\underline{d}$ is u-reduced for some vertex u such that $\underline{d}(u) = \ell_G(\underline{d})$.

More generally, we say that a divisor $\underline{d}$ on any graph G is *rank-explicit* if $\underline{d}$ is u-reduced for some vertex u such that $\underline{d}_{\mathrm{rk}}(u) = \ell_G(\underline{d})$.

We say that a divisor class $\delta \in \mathrm{Pic}(G)$ is *rank-explicit* if it admits a rank-explicit representative.

Remark 5.9. If $\underline{d}$ is rank-explicit, then $r_G(\underline{d}) = \ell_G(\underline{d})$, by Proposition 3.17.

Remark 5.10. By Lemma 3.8, if $r_G(\delta) \leq 0$ then δ is rank-explicit.

Example 5.11. Not all divisor classes are rank-explicit. For example, on a binary graph of genus 1 the divisor class $\delta = [(0, 2)]$ has rank 1 and is not rank-explicit.

By Lemma 4.1, if $\underline{d}$ is a u-reduced divisor on a graph G, then for every $X \in M^{\mathrm{alg}}(G)$ and every $L \in \mathrm{Pic}^{\underline{d}}(X)$, the restriction map $H^0(X, L) \to H^0(C_u, L_{C_u})$ is injective (where $C_u \subset X$ is the component corresponding to u). The following lemma tells us under which conditions the restriction map is an isomorphism, using rank-explicit divisors and special curves.

Lemma 5.12. *Let $\underline{d}$ be a rank-explicit divisor on a weightless loopless graph G. Then, for every special curve $X \in M^{\mathrm{alg}}(G)$, there exists a line bundle $L \in \mathrm{Pic}^{\underline{d}}(X)$, such that $H^0(X, L) \cong H^0(\mathbb{P}^1, \mathcal{O}(\ell_G(\underline{d})))$.*

Proof. Set $\ell = \ell_G(\underline{d})$; let $X \in M^{\mathrm{alg}}(G)$ be a special curve. By Remark 2.2, we may assume that $\underline{d} = (\ell, \ell, \dots, \ell)$. If $\ell = -1$ there is nothing to prove, so we will assume $\ell \geq 0$.

We write $X = \cup C_i$, and fix isomorphisms of pointed curves $\phi_{i,j} : (C_i, P_{i,j}) \to (C_j, P_{j,i})$ as in Definition 5.1. Recall that if $x \in X$ is a node whose branches are $x^i \in C_i$ and $x^j \in C_j$, then $\phi_{i,j}(x^i) = x^j$.

For $L \in \mathrm{Pic}^{\underline{d}}(X)$, we write $L_i := L_{C_i}$ for every i, and fix an isomorphism $L_i \cong \mathcal{O}_{\mathbb{P}^1}(\ell)$; now, each $\phi_{i,j}$ induces an isomorphism

$$\chi_{i,j} : H^0(C_i, L_i) \longrightarrow H^0(C_j, L_j); \qquad s \mapsto s \circ \phi_{j,i}. \tag{28}$$

By hypothesis, $\underline{d}$ is u-reduced for some $u \in V$; consider the corresponding Dhar decomposition

$$V = Y_0 \sqcup Y_1 \ldots \sqcup Y_l,$$

(see Subsection 3.4). We can fix, with no loss of generality, an ordering $V = \{v_0 = u, v_1, \ldots, v_t\}$ compatible with the decomposition (i.e. if $v_i \in Y_h$ and $v_{i'} \in Y_{h'}$ with $h < h'$, then $i < i'$), and such that it induces a filtered sequence of connected subcurves

$$C_0 = Z_0 \subset \ldots \subset Z_m \subset \ldots \subset Z_t = X$$

with $Z_m = \cup_{i=0}^m C_i$ and $C_i = C_{v_i}$.

We pick $L \in \mathrm{Pic}^{\underline{d}}(X)$ such that the gluing constants over the nodes are all equal to 1; we will prove that for every $m = 0, \ldots, t$, there is an isomorphism

$$H^0(Z_m, L_{Z_m}) \cong H^0(C_0, L_{C_0}), \tag{29}$$

which, as $H^0(C_0, L_{C_0}) \cong H^0(\mathbb{P}^1, \mathcal{O}(\ell))$, implies the lemma.

The proof of (29) will go by induction on m. The case $m = 0$ is obvious. Assume the statement holds for $m-1$. Write $Z_m = Z_{m-1} \cup C_m$; for every node $x_\alpha \in Z_{m-1} \cap C_m$, we denote its branches by $x_\alpha^m \in C_m$ and $x_\alpha^{i(\alpha)} \in C_{i(\alpha)} \subset Z_{m-1}$. Consider the exact sequence

$$0 \to H^0(Z_m, L_{Z_m}) \xrightarrow{\rho} H^0(Z_{m-1}, L_{Z_{m-1}}) \oplus H^0(C_m, L_m) \xrightarrow{\pi} \bigoplus_\alpha k(x_\alpha),$$

where ρ is the restriction map (i.e. $\rho(s) = s_{|Z_{m-1}} \oplus s_{|C_m}$), $k(x_\alpha)$ is the skyscraper sheaf supported on x_α (equal to k on x_α), the point x_α ranges in $Z_{m-1} \cap C_m$, and

$$\pi(s \oplus t) := \oplus_{x_\alpha \in C_m \cap Z_{m-1}} (s(x_\alpha^{i(\alpha)}) - t(x_\alpha^m)).$$

By our ordering of the vertices of G, and since $\underline{d}$ is u-reduced, we have $|C_m \cap Z_{m-1}| > \deg_{C_m} L$, so the restriction of π to $H^0(C_m, L_m)$ is injective (a section of L_m cannot have more zeroes than its degree).

We now claim that π induces an isomorphism between $H^0(C_m, L_m)$ and $\mathrm{Im}\,\pi$. It suffices to prove that for any $s \in H^0(Z_{m-1}, L_{Z_{m-1}})$, we have $\pi(s) \in \pi(H^0(C_m, L_m))$. Recall that $x_\alpha^{i(\alpha)} = \phi_{m,i(\alpha)}(x_\alpha^m)$, hence

$$\pi(s) = \sum_\alpha s(x_\alpha^{i(\alpha)}) = \sum_\alpha s(\phi_{m,i(\alpha)}(x_\alpha^m)) = \sum_\alpha \chi_{i(\alpha),m}(s_{|C_{i(\alpha)}})(x_\alpha^m)$$

by (28). Since $\chi_{i(\alpha),m}(s_{|C_{i(\alpha)}}) \in H^0(C_m, L_m)$ the proof of the claim is complete. Therefore $\mathrm{Im}\,\pi \cong H^0(C_m, L_m)$, hence

$$H^0(Z_m, L_{Z_m}) \cong H^0(Z_{m-1}, L_{Z_{m-1}}) \cong H^0(C_u, L_u)$$

by induction; (29) is proved, and the lemma with it. $\square$

Theorem 5.13. *Let G be weightless and loopless; let $\delta \in \mathrm{Pic}(G)$ be rank-explicit. Then $r^{\mathrm{alg}}(G,\delta) = r_G(\delta)$.*

Set $r = r_G(\delta)$; by Theorem 4.2 we can assume $r \geq 0$. Let $\underline{d}$ be a rank-explicit representative for δ; then $r_G(\delta) = \ell_G(\underline{d})$ by Remark 5.9. Therefore Theorem 5.13 is a special case of the following more precise result.

Proposition 5.14. *Let G be weightless and loopless; pick $\underline{d} \in \mathrm{Div}(G)$ and set $\delta = [\underline{d}]$. Let $X \in M^{\mathrm{alg}}(G)$ be a special curve. Then*

(a) $r^{\max}(X, \underline{d}) \geq \ell_G(\underline{d})$.
(b) $r(X, \delta) \geq \ell_G(\underline{d})$.
(c) If $r_G(\underline{d}) = \ell_G(\underline{d})$ *then* $r^{\mathrm{alg}}(G,\delta) = r_G(\delta)$.

Proof. Set $\ell = \ell_G(\underline{d})$. Let $X \in M^{\mathrm{alg}}(G)$ be a special curve. If $\underline{d}$ is rank explicit, then part (a) is an immediate consequence of Lemma 5.12.

To treat the general case, fix a vertex u such that $\underline{d}(u) = \ell$, and suppose $\underline{d}$ is not reduced with respect to u. We choose a G-saturation, G', with respect to $\underline{d}$ and u (see Definition 3.20); notice that $\underline{d}$ is rank-explicit on G'. Now, we construct a special curve, X', having G' as dual graph, by gluing together some points in C_u to points on other components of X; hence X dominates X' by a birational map

$$\pi : X \longrightarrow X'.$$

By Lemma 5.12, there exists $L' \in \mathrm{Pic}^{\underline{d}}(X')$ satisfying $r(X', L') = \ell$. Consider $L = \pi^* L'$; then $L \in \mathrm{Pic}^{\underline{d}}(X)$, and, of course,

$$r(X, L) \geq r(X', L') \geq \ell,$$

as required.

For part (b) we must prove that for every $\underline{c} \sim \underline{d}$ with $\underline{d} \neq \underline{c}$ there exists $M \in \mathrm{Pic}^{\underline{c}}(X)$ such that $r(X, M) \geq \ell$. We have $\underline{c} + \underline{t} = \underline{d}$, for some $\underline{t} \in \mathrm{Prin}(G)$. We argue as for Corollary 4.3, with same notation (repeating some things for convenience). By Remark 2.1 we have

$$\underline{t}_{|Z} \leq (\underline{t}_Z)_{|Z}, \tag{30}$$

with $Z \subsetneq V(G)$. We denote by Z and Z^c the subcurves of X corresponding to the vertices in Z and Z^c. By Lemma 5.12 applied to the special curve Z, or to a connected component of Z if Z is not connected, there exists a line bundle $L_Z \in \mathrm{Pic}^{\underline{d}_{|Z}}(Z)$ satisfying

$$r(Z, L_Z) \geq \ell. \tag{31}$$

Let $Z \cdot Z^c \in \mathrm{Div}(Z)$ be the divisor cut on Z by Z^c; we have

$$\underline{\deg}_Z Z \cdot Z^c = (\underline{t}_{Z^c})_{|Z} = -(\underline{t}_Z)_{|Z}. \tag{32}$$

Now, for every $M \in \mathrm{Pic}^{\underline{c}}(X)$ we have

$$r(X, M) \geq r(Z, M_Z(-Z \cdot Z^c)). \tag{33}$$

Moreover, using (32) and (30)

$$\underline{\deg}_Z M(-Z \cdot Z^c) = \underline{c}_{|Z} - \underline{\deg}_Z Z \cdot Z^c = \underline{c}_{|Z} + (\underline{t}_Z)_{|Z} \geq \underline{c}_{|Z} + \underline{t}_{|Z} = \underline{d}_{|Z}.$$

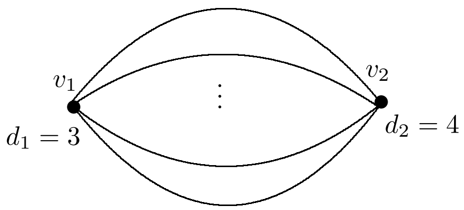
We can therefore pick $M \in \mathrm{Pic}^{\underline{c}}(X)$ such that its restriction to Z satisfies $M_Z = L_Z(Z \cdot Z^c + E)$ for some effective divisor E on Z. By (33) and (31) we have

$$r(X, M) \geq r(Z, L_Z(Z \cdot Z^c + E - Z \cdot Z^c)) \geq r(Z, L_Z) \geq \ell,$$

and part (b) is proved. Part (c) follows from (b) and Theorem 4.2. □

By the following Example 5.15 we have that neither Theorem 5.13, nor Proposition 5.6, extend to weighted graphs.

Example 5.15. Let G be a graph with two vertices v_1 and v_2, whose weights are $\omega(v_1) = 1$, $\omega(v_2) = 2$, and such that $v_1 \cdot v_2 > 12$. Consider the divisor $\underline{d} = (3, 4)$.



$v_1 \cdot v_2 > 12$

We claim that every curve $X \in M^{\mathrm{alg}}(G)$ satisfies $r^{\max}(X, \underline{d}) < r_G(\underline{d})$. Notice that $\underline{d}_{\mathrm{rk}} = (2, 2)$ and $\underline{d}$ is reduced with respect to both vertices. So by Proposition 3.17 we have $r_G(\underline{d}) = 2$, and $\underline{d}$ is rank-explicit.

Now let L be a line bundle on X with $\underline{\deg}\ L = \underline{d}$, where we write $X = C_1 \cup C_2$ and $L_i = L_{|C_i}$, with C_i corresponding to v_i. Let us see that $r(X, L) < 2$. Since this is independent of the choice of X and L, the claim will follow.

Assume by contradiction that $r(X, L) = 2$. Since $\underline{\deg}\ L = (3, 4)$, we have that $r(C_1, L_1) = r(C_2, L_2) = 2$. Note that if a section of L_1 or L_2 can be extended to all of X, then the extension is unique, since the number $|C_1 \cap C_2|$ is large. Hence, the map $\phi_L : X \to \mathbb{P}^2$ determined by L restricts to non-degenerate maps $\phi_1 : C_1 \to \mathbb{P}^2$ and $\phi_2 : C_2 \to \mathbb{P}^2$.

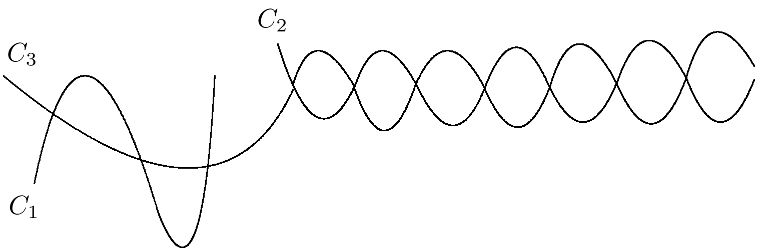
The image of ϕ_1 is an irreducible curve of degree 3, so it is either a cubic or a line with multiplicity 3; since ϕ_1 is non-degenerate, the image is a cubic. Similarly, the image of ϕ_2 is a non-degenerate irreducible curve of degree 4, so it is either a (singular) quartic, or a conic of multiplicity 2.

Hence, $\phi_L(X)$ consists of two distinct irreducible curves of degrees 3, 4. By Bezout Theorem, they intersect in at most 12 points, which is a contradiction.

The next example shows that the hypothesis that δ be rank-explicit is really needed in Theorem 5.13.

Example 5.16. Let $X = C_1 \cup C_2 \cup C_3$ be a curve with 3 rational components meeting as follows (see the picture below).

$$C_1 \cdot C_3 = 3 \text{ and } C_2 \cdot C_3 > 6.$$



Let G be the dual graph of X and let $\underline{d} = (1, 2, 3)$ be the divisor in $\mathrm{Div}(G)$ such that the degree of $\underline{d}$ on C_i is i. We claim that

$$2 = r_G(\underline{d}) > r^{\mathrm{alg}}(G, \underline{d}).$$

It is easy to see that $r_G(\underline{d}) = 2$. To prove the claim we will prove that for every $L \in \mathrm{Pic}^{\underline{d}}(X)$ we have $r(X, L) < 2$.

Assume, by contradiction, that $r(X, L) = 2$. Then L defines a non-degenerate map $\phi : X \to \mathbb{P}^2$. We will treat all possible cases, getting a contradiction in each of them.

Case 0: $\phi(C_3)$ is point. But then $\phi(X)$ is a point (for otherwise $\phi(C_1)$ or $\phi(C_2)$ would have a singular point of too high multiplicity), which is not possible as ϕ is non-degenerate.

Case 1: $\phi(C_3)$ is a line. Then one sees easily that $\phi(C_2) = \phi(C_3)$. Hence $\phi(C_1)$ must be a different line (for the map ϕ is non-degenerate). But then the restriction of ϕ to C_1 is an isomorphism, so ϕ maps the three points of $C_1 \cap C_3$ in different points, which is impossible as $\phi(C_3)$ is a line other than $\phi(C_1)$.

Case 2: $\phi(C_3)$ is a conic. Hence ϕ maps C_3 isomorphically to its image and L has a base point $p \in C_3$. We claim that $\phi(C_2) = \phi(C_3)$. Indeed, $\phi(C_2)$ cannot be a point (for it would be a singular point of $\phi(C_3)$); hence $\phi(C_2)$ is a curve of degree at most 2, which, if different from $\phi(C_3)$, would meet $\phi(C_3)$ in degree greater than 4, contradicting Bezout Theorem.

Let us now consider C_1; if the base point p is one of the three points where C_3 meets C_1, then L has a base point on C_1 and hence $\phi(C_1)$ is a point. This is impossible, since C_1 meets C_3 in two more points which, as we said, have distinct images via ϕ. If $p \notin C_1$, arguing as before we get that $\phi(C_1)$ cannot be a point, and hence it is a line, which intersects $\phi(C_3)$ in three points. By Bezout Theorem this is impossible.

Case 3: $\phi(C_3)$ is a cubic. Then ϕ maps C_3 birationally onto its image, and $\phi(C_2)$ cannot be a point (for it would be a point of too high multiplicity on a cubic). Hence $\phi(C_2)$ is a curve of degree at most 2, which meets $\phi(C_3)$ in degree greater than 6. This contradicts Bezout Theorem.

Remark 5.17. By slightly modifying the example above, we have an example where strict inequality between the algebraic and combinatorial rank holds for a *simple graph* (a graph with at most one edge between any two vertices). Indeed, keeping the notation of the example, consider the graph G' obtained from G by adding a vertex in the interior of each edge, and let $\underline{d}' \in \mathrm{Div}(G')$ be the divisor obtained by extending $\underline{d}$ by zero. Then by [5, Theorem 1.4], $r_{G'}(\underline{d}') = r_G(\underline{d}) = 2$.

Let X' be any curve dual to G', let L' be a line bundle on X' such that $\underline{\deg} L' = \underline{d}'$, and let $u \in V(G') \smallsetminus V(G)$. Since $\underline{d}'(u) = 0$, the map $\phi' : X' \to \mathbb{P}^2$ corresponding to L' maps the component C_u to a point. Therefore the number of intersection points of $\phi'(C_1)$, $\phi'(C_2)$, and $\phi'(C_3)$ is the same as for ϕ, and the exact argument as in the example above yields $r^{\mathrm{alg}}(G', \underline{d}') < 2 = r_{G'}(\underline{d}')$.

We know that if the combinatorial rank is -1 or 0, then the combinatorial and algebraic rank are equal. Notice that in all our examples, where we do not have equality, the combinatorial rank is equal to 2. So the following case is open:

Question 5.18. Suppose $r_G(\delta) = 1$. Is $r_G(\delta) = r^{\mathrm{alg}}(G, \delta)$?

Finally, we may consider the following variant of Problem 1:

Question 5.19. Let G be a graph and $\delta \in \mathrm{Pic}(G)$. Does there exist a representative $\underline{d}_0 \in \delta$ and $X \in M^{\mathrm{alg}}(G)$ such that

$$r^{\max}(X, \underline{d}_0) = r_G(\delta)?$$

Notice that the answer to Question 5.19 is positive in both our counterexamples. For Example 5.15 we take $\underline{d}_0 = (3, 4) + \underline{t}_{v_1} = (3 - k, 4 + k)$, where $k := v_1 \cdot v_2 > 12$; for Example 5.16 we can take $\underline{d}_0 = (1, 2, 3) + \underline{t}_{v_1} = (-2, 6, 2)$.

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