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UNIVERSALITY FOR CRITICAL KCM: FINITE NUMBER OF STABLE DIRECTIONS

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In this paper, we consider kinetically constrained models (KCM) on $\mathbb{Z}^2$ with general update families $\mathcal{U}$. For $\mathcal{U}$ belonging to the so-called "critical class," our focus is on the divergence of the infection time of the origin for the equilibrium process as the density of the facilitating sites vanishes. In a recent paper (*Probab. Theory Related Fields* **178** (2020) 289–326), Marêché and two of the present authors proved that if $\mathcal{U}$ has an infinite number of "stable directions," then on a doubly logarithmic scale the above divergence is twice the one in the corresponding $\mathcal{U}$-bootstrap percolation.

Here, we prove instead that, contrary to previous conjectures (*Comm. Math. Phys.* **369** (2019) 761–809), in the complementary case the two divergences are the same. In particular, we establish the full universality partition for critical $\mathcal{U}$. The main novel contribution is the identification of the leading mechanism governing the motion of infected critical droplets. It consists of a peculiar hierarchical combination of mesoscopic East-like motions.

1. Introduction.

1.1. *Kinetically constrained models.* We directly define the models of interest and refer the reader to the companion paper [14] for more background. Let $\mathcal{U}$ be a finite collection of finite subsets of $\mathbb{Z}^2 \setminus \{0\}$ called *update rules* and consider the following interacting particle systems on $\Omega = \{0,1\}^{\mathbb{Z}^2}$ parametrized by $\mathcal{U}$ and $q \in (0,1)$. We say that a site $x \in \mathbb{Z}^2$ is *infected* if $\omega(x) = 0$ and *healthy* otherwise. The dynamics is as follows. On each site $x \in \mathbb{Z}^2$, we are given an independent Poisson clock of parameter one and at each arrival time $t := t_{x,k}$ of the clock the process attempts to update the current state $\omega_x(t)$ according to the following rule. If the configuration $\omega(t)$ is such that there exists $U \in \mathcal{U}$ such that $\forall y \in U$, $\omega_{x+y}(t) = 0$, then $\omega_x(t)$ is resampled from the Bernoulli$(1-q)$-measure $\mu_q(1) = 1-q$, $\mu_q(0) = q$. In this case, we say that a *legal update* occurs at time t at site x. Otherwise, the attempted update is rejected.

Using the fact that all rings $(t_{x,k})_{x\in\mathbb{Z}^2, k\in\mathbb{N}}$ of the Poisson clocks are different a.s. and that all updating rules are finite sets, it is easy to check that the above process is well defined [18]. Moreover, since the update rules do not contain the origin, the process is reversible w.r.t. the product measure $\mu := \mu_q^{\mathbb{Z}^2}$.

We will refer to the above process as the *kinetically constrained spin model* with update family $\mathcal{U}$, for short $\mathcal{U}$-KCM or just KCM if $\mathcal{U}$ is clear from the context. KCM have been introduced several years ago in the physics literature (but only for certain specific choices of $\mathcal{U}$) in order to reproduce in simple and fundamental interacting particle systems some of the main features of the so-called *glassy dynamics*, that is, the dynamics of a supercooled liquid near the *glass transition* [3, 9, 10, 16, 23].

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1.2. *$\mathcal{U}$-Bootstrap percolation.* The $\mathcal{U}$-KCM can also be seen as the nonmonotone stochastic counterpart of the *$\mathcal{U}$-bootstrap percolation*, a monotone cellular automaton on Ω (see, e.g., [5, 22]). In the latter process, one says that $x \in \mathbb{Z}^2$ is *infected* for $\omega \in \Omega$ if $\omega_x = 0$ and *healthy* otherwise and the relevant time evolution concerns the set of infected sites A_t at integer times t. Given A_t, the set A_{t+1} is constructed by adding to A_t any healthy site x for which there exists $U \in \mathcal{U}$ such that $U + x \subset A_t$:

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2, \exists U \in \mathcal{U}, x + U \subset A_t\}.$$

The only randomness in the process occurs at time $t = 0$ by taking the initial infection as the random set $A_\omega = \{x \in \mathbb{Z}^2 : \omega_x = 0\}$ with $\omega \sim \mu$. If $A_{t=0} = A$, then the *closure* of A under the $\mathcal{U}$-bootstrap percolation is the set $[A]_{\mathcal{U}} := \bigcup_{t \geq 0} A_t$.

For both processes the main focus has been on the typical value (e.g., in mean, median, or w.h.p.) of the infection time of the origin defined as

$$\tau_0 = \inf\{t : \omega_0(t) = 0\}.$$

Notice that for $\mathcal{U}$-bootstrap percolation $\tau_0 \in \mathbb{N} \cup \{+\infty\}$ and it only depends on the initial set of infection A_ω. On the other hand, for the KCM $\tau_0 \in [0, +\infty]$ and it depends on the initial state $\omega(0)$, on the occurrences of the Poisson clocks at the vertices of $\mathbb{Z}^2$ and on the resampling values of the legal updates. In order to present our main result on τ_0, we need some further notation (see [6]).

Let $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and scalar product on $\mathbb{R}^2$, respectively. For each unit vector $u \in S^1 = \{x \in \mathbb{R}^2, \|x\| = 1\} \sim \mathbb{R}/2\pi\mathbb{Z}$, let $\mathbb{H}_u := \{x \in \mathbb{R}^2 : \langle x, u \rangle < 0\}$ denote the half-plane whose boundary is perpendicular to u.

DEFINITION 1.1 (Stable directions). The set of *stable directions* of $\mathcal{U}$ is

$$\mathcal{S} = \mathcal{S}(\mathcal{U}) = \{u \in S^1 : [\mathbb{H}_u \cap \mathbb{Z}^2]_{\mathcal{U}} = \mathbb{H}_u \cap \mathbb{Z}^2\}.$$

Using the above definition, the update family $\mathcal{U}$ was classified in [6] as:

- *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$,
- *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has nonempty intersection with $\mathcal{S}$,
- *subcritical* if every semicircle in S^1 has infinite intersection with $\mathcal{S}$.

It is known from [2, 6] that for all $q \in (0, 1)$ the $\mathcal{U}$-bootstrap percolation τ_0 is a.s. finite iff $\mathcal{U}$ is supercritical or critical. The next definition quantifies the difficulty of propagation of infection in a stable direction for bootstrap percolation.

DEFINITION 1.2 (Definition 1.2 of [5]). Let $\mathcal{U}$ be an update family and $u \in S^1$ be a direction. The *difficulty* of u, $\alpha(u)$, is defined as follows:

- If u is unstable, then $\alpha(u) = 0$.
- If u is an isolated (in the topological sense) stable direction, then

$$\alpha(u) = \min\{n \in \mathbb{N} : \exists Z \subset \mathbb{Z}^2, |Z| = n, |[\mathbb{Z}^2 \cap (\mathbb{H}_u \cup Z)] \setminus \mathbb{H}_u| = \infty\},$$

  that is, the minimal number of infections allowing $\mathbb{H}_u$ to grow infinitely.
- Otherwise, $\alpha(u) = \infty$.

The *difficulty* of $\mathcal{U}$ is

$$\alpha(\mathcal{U}) = \inf_{C \in \mathcal{C}} \sup_{u \in C} \alpha(u), \tag{1}$$

where $\mathcal{C}$ is the set of open semicircles of S^1.

REMARK 1.3. It was proved in [6], Lemma 5.2 (see also [5], Lemma 2.8) that $1 \leq \alpha(u) < \infty$ if and only if u is an isolated stable direction. Moreover, it is easy to prove [5], Lemma 6.6, that if $\alpha(u) < +\infty$ then there exists a set Z of cardinality $\alpha(u)$ such that $|[\mathbb{Z}^2 \cap (\mathbb{H}_u \cup Z] \cap \ell(u)| = +\infty$, where $\ell_u = \{x \in \mathbb{R}^2 : \langle x, u\rangle = 0\}$.

It follows from the main result of [5] that for any critical $\mathcal{U}$ with difficulty α the $\mathcal{U}$-bootstrap percolation infection time τ_0 w.h.p. satisfies

$$\lim_{q\to 0} \frac{\log\log(\tau_0)}{\log(1/q)} = \alpha. \tag{2}$$

1.3. *Main results*. Our main result is that (2) holds also for the $\mathcal{U}$-KCM if $\mathcal{S}$ is finite. The core of the proof is based on the discovery of a new and efficient relaxation mechanism completely different from the one occurring in bootstrap percolation. While for the latter the dominant mechanism to grow infection is a linear expansion from some rare large groups of infected sites (critical droplets), for KCM we find that these droplets, in order to move around in an efficient way to infect the origin, perform a complex hierarchical motion (see Section 2 for an heuristic detailed description). The above motion is a novel type of relaxation mechanism with respect to all those considered so far in the KCM literature. In particular, it is different from the random walk like motion that captures the dominant behavior for 2-neighbor model [21], and it is also different from the purely East-like motion used to establish the scaling for models with an infinite number of stable directions [14, 19, 20]. Indeed, based on the wrong intuition that the two former mechanisms were essentially the only two possible efficient ways to move critical droplets around, a conjecture was put forward in [20], Conjecture 3, which is disproved by our result, Theorem 1 below.

Write $\mathbb{E}_\mu(\tau_0)$ for the expectation of the infection time for the $\mathcal{U}$-KCM with initial law μ (i.e., for the stationary process).

THEOREM 1. *Let $\mathcal{U}$ be a critical update family with* finite *stable set $\mathcal{S}$ and difficulty α. Then*

$$\mathbb{E}_\mu(\tau_0) = e^{O(\log(1/q)^3/q^\alpha)}. \tag{3}$$

Moreover,

$$\lim_{q\to 0} \frac{\log\log(\mathbb{E}_\mu(\tau_0))}{\log(1/q)} = \alpha. \tag{4}$$

The second statement (4) follows immediately from (3) together with [21], Lemma 4.3, and (2). Theorem 1 together with [14], Theorem 2.8, and [20], Theorem 2(a), corrects [20], Conjecture 3, and gives the following full universality picture for $\mathcal{U}$-KCM with critical $\mathcal{U}$.

THEOREM 2. *Let $\mathcal{U}$ be a critical update family. Then*

$$\lim_{q\to 0} \frac{\log\log(\mathbb{E}_\mu(\tau_0))}{\log(1/q)} = \begin{cases} \alpha & \text{if } |\mathcal{S}| < +\infty, \\ 2\alpha & \text{otherwise.} \end{cases} \tag{5}$$

Some key partial results in the direction of proving Theorem 2 were established in [20, 21]. In particular, in [20] the scaling (4) was proved for any update family $\mathcal{U}$ with $\max_{u\in S^1} \alpha(u) = \alpha$ while [19] proved (5) with a higher degree of precision for a specific model with $|\mathcal{S}| = \infty$, the Duarte model. We refer the interested reader to the introduction of [14] for a detailed account of the history leading to (5). For supercritical update families, the correct scaling was determined in [20], Theorem 1, and [19], Corollary 4.3.

1.4. *Organization of the paper.* We start by providing a heuristic explanation of the relaxation mechanism underlying our main result in Section 2. In Section 3, we fix some notation and gather some preliminary tools from bootstrap percolation that are by now well established in the literature. We will not dwell on the technical aspects of the definitions and invite the reader to refer to Section 4.3 of [20], which we follow closely, for more details. For reader's convenience, we have collected in Section 3.2 three useful technical lemmas on certain one-dimensional kinetically constrained Markov processes. Although the proof of these lemma can be found or derived from the existing literature on KCM, we have added it in the Appendix for completeness. Section 4 contains the main new technical Poincaré inequality, while Theorem 1 is proved in Section 5. Finally, in Section 6 we discuss some natural open problems raised by the present work.

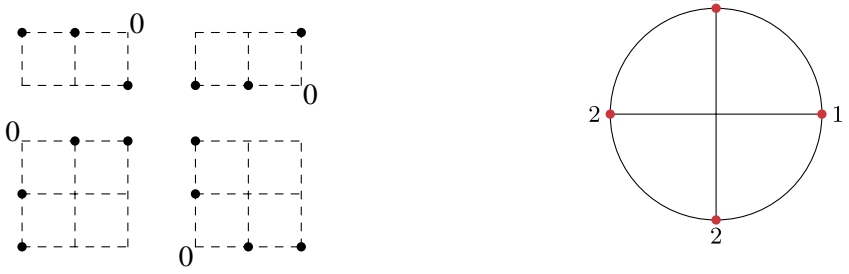
2. Some heuristics behind Theorem 1. For a high-level and accessible introduction to the main general ideas and techniques involved in bounding from above $\mathbb{E}_\mu(\tau_0)$, we refer to [20], Section 2.4. There, in particular, it was stressed that while the necessary intuition is developed using dynamical considerations (e.g., by guessing some efficient mechanism to create/heal infection inside the system), the actual mathematical tools are mostly analytic and based on suitable (and, unfortunately, sometimes very technical) Poincaré inequalities. This paper makes no exception.

In order to go beyond the results of [20] and get the sharp scaling of Theorem 1 in the case of a *finite* set of stable directions, the following new key input is needed.

For simplicity, imagine that $\mathcal{U}$ has only four stable directions coinciding with the four natural directions of $\mathbb{Z}^2$. For a generic model with $|\mathcal{S}| < \infty$, the mechanism is the same, the only difference being that in general "droplets" have a more complex geometry. Assume further that $\alpha(\vec{e}_1) = 1$ and $\alpha(-\vec{e}_1) = \alpha(\pm\vec{e}_2) = 2$ (see Figure 1).

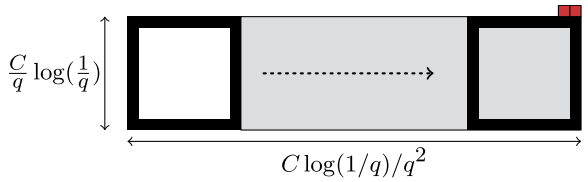
Consider now a *critical droplet*, i.e. a square frame D, centered at the origin, of side length $\approx C\log(1/q)/q$, $C \gg 1$, and $O(1)$-thickness, and suppose that D is infected. Then w.h.p. (w.r.t. μ) there will be extra infected sites next to D in the $\vec{e}_1$-direction allowing D to infect $D + \vec{e}_1$. However, it will be extremely unlikely to find a pair of infected sites near each other and next to the other three sides of D because of the choice of the side length of D. We conclude that w.h.p. it is easy for D to advance forward in the $\vec{e}_1$-direction but not in the other directions. Moreover, as explained in detail in [20], Section 2.4, an efficient way to effectively realize the motion in the $\vec{e}_1$-direction is via a generalized East path. In its essence, the latter can be described by the following game. At every integer time a token is added or removed (if already present) at some integer point according to the following rules:

- each integer can accommodate at most one token;
- a token can be freely added or removed at 1;

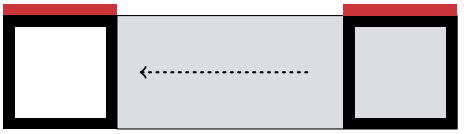


(a) The four update rules belonging to $\mathcal{U}$. (b) The difficulties of the four stable directions of the model.

FIG. 1. *An example of an update family $\mathcal{U}$ with finite number of stable directions examined in Section 2.*



(a) The infected droplet (black frame with width $O(1)$) progressively moves to the right in an East-like way using the extra infected sites present w.h.p. in each column of the gray rectangle. This progression stops when reaching the first infected horizontal pair of sites at the correct height (red pair).



(b) The infected droplet on the right grows into an e_2-extended droplet thanks to the infected pair of sites. The movement is then reverted, progressively retracting the extended droplet in an East way until reaching the original position.

FIG. 2. *The mechanism for the droplet to grow in the $\vec{e}_2$-direction.*

- for any $j \geq 2$ the operation of adding/removing a token at j is allowed iff there is already a token at $j-1$.

Given $n \in \mathbb{N}$, by an efficient path reaching distance n we mean a way of adding tokens to the original empty configuration to finally place one at n which uses a minimal number of tokens. A combinatorial result (see [8]) says that the optimal number grows like $\log_2(n)$.

The main new idea now is that, while w.h.p. the droplet D will not find a pair of infected sites (which are necessary to grow an extra layer of infection in the $\vec{e}_2$-direction) *next* to, for example, its top side, w.h.p. it will find it at the right height within distance $C\log(1/q)/q^2$ in the $\vec{e}_1$-direction (see Figure 2). Hence, a possible efficient way for D to move one step in the $\vec{e}_2$-direction is to:

(a) travel in the $\vec{e}_1$-direction in a East-like way until finding the necessary pair of infected sites within distance $C\log(1/q)/q^2$ from the origin;
(b) grow there an extra layer in the $\vec{e}_2$-direction and retrace back to its original position while keeping the acquired extra layer of infection.

A similar mechanism applies to the $-\vec{e}_2$-direction. Slightly more involved is the way in which D can advance in the $-\vec{e}_1$-direction. In this case the extra infected pair needs to be found within distance $C\log(1/q)/q^2$ from the origin in the *vertical* direction (see Figure 3). In order to reach it, D performs an East-like movement upwards, each of whose steps is itself realized by the back-and-forth East motion in the e_1 direction described above.

Using the result for the typical time scales of the generalised East process (see [20], formula (3.5)) it is easy to see that the typical excursion of D for a distance $\ell \equiv C\log(1/q)/q^2$ in the $\vec{e}_1$-direction requires a time lag

$$\Delta t = q^{-|D|O(\log(\ell))} = e^{O(\log^3(1/q))/q}.$$

This time scale also bounds from above the time scale necessary to advance by one step in the "hard" directions $-\vec{e}_1, \pm\vec{e}_2$.

In conclusion, by making a "quasi-local" (*i.e.*, on a length scale ℓ) East-like motion in the easy direction $\vec{e}_1$, the infected critical droplet D can actually perform a sort of random

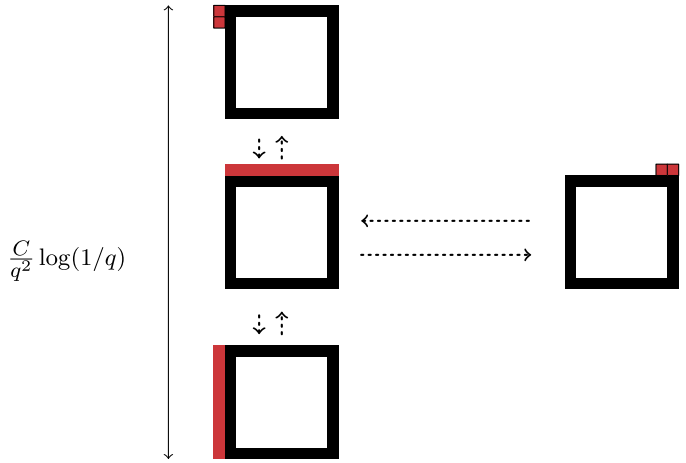


FIG. 3. *The mechanism for the droplet growth in the $-\vec{e}_1$-direction. The droplet moves in an East way in the $\vec{e}_2$-direction by making long excursions in the $\vec{e}_1$-direction as in Figure 2.*

walk in which each step requires a time Δt. The result of Theorem 1 becomes now plausible provided that one proves that anomalous regions of missing helping infected sites do not really constitute a serious obstacle.

The above dynamic heuristics can be turned into a rigorous argument using canonical paths. However, a much neater approach is to prove a Poincaré inequality for the $\mathcal{U}$-KCM restricted to a suitable *finite* domain of $\mathbb{Z}^2$ (see Theorem 4.6). More precisely, in the toy example discussed above the inequality that we establish is as follows.

Let $V = B \cup T_0 \cup T_1^- \cup T_1^+$ where B, T_0, $T_1^\pm$ are as in Figure 4. The set V is an example of a more general geometric construction developed in Section 4 and denoted *snail with base B and trapezoids T_0, $T_1^\pm$*. The ratio of the sides of the rectangle B is $\Theta(q)$ while for the other rectangles it is $\Theta(1)$.

Let Ω_0 consist of all configurations of $\{0,1\}^V$ such that:

- each column of B contains an infected site;
- each row of T_0 contains a pair of adjacent infected sites;
- each column of $T_1^\pm$ contains a pair of adjacent infected sites.

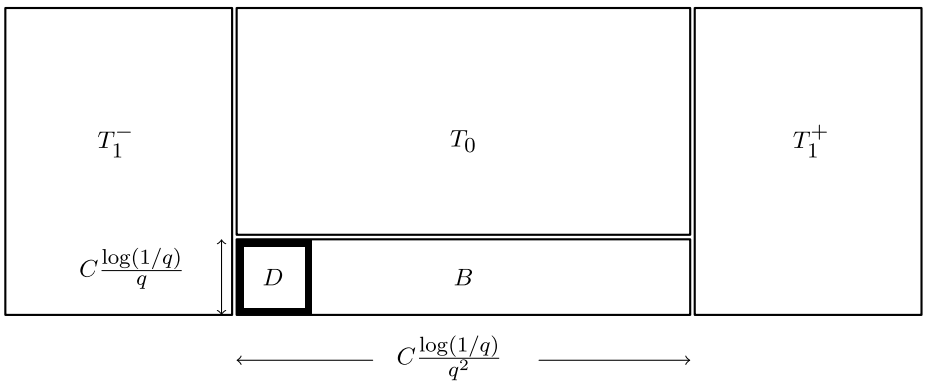


FIG. 4. *The geometric setting for the toy model of Figure 1.*

Notice that by choosing C large enough $\mu(\Omega_0) = 1 - o(1)$ as $q \to 0$. Then, in the key Theorem 4.6, we prove that for any $f : \{0,1\}^V \to \mathbb{R}$,

$$\mathbb{1}_{\{D \text{ is infected}\}} \operatorname{Var}_V(f \mid \Omega_0) \le e^{O(\log(1/q)^3)/q} \times \mathcal{D}(f),$$

where $\mathcal{D}(f)$ is the Dirichlet form of f (see (8)). One can interpret the above inequality as saying that the KCM in V restricted to the good set Ω_0 has a relaxation time at most $e^{O(\log(1/q)^3)/q}$. We prove this by an inductive procedure over T_0, $T_1^{\pm}$ which, in some sense, makes rigorous the dynamic heuristics described above.

3. Notation and preliminaries. In this section, we gather the relevant notation and basic inputs from bootstrap percolation and KCM theories. We shall always denote spatial regions (either in $\mathbb{Z}^2$ or in $\mathbb{R}^2$) with capital letters and events in the various probability spaces with calligraphic capital letters.

3.1. *Bootstrap percolation.*

3.1.1. *Stable and quasi-stable directions.* For every integer n, we write $[n] := \{0, 1, \dots, n-1\}$. We fix a critical update family $\mathcal{U}$ with difficulty $\alpha = \alpha(\mathcal{U})$ and with a *finite* set $\mathcal{S}$ of stable directions.[1] Using the definition of $\alpha(\mathcal{U})$ (see (1)), one can fix an open semicircle C with midpoint u_0, one of whose endpoints is in $\mathcal{S}$ and such that $\max_{u \in C} \alpha(u) = \alpha$. Using [6], Lemma 5.3 (see also [5], Lemma 3.5, and [20], Lemma 4.6), one can choose a set of rational directions[2] $\mathcal{S}' \supset \mathcal{S}$, so that for every two consecutive elements u and v of $\mathcal{S}'$ there exists an update rule $X \in \mathcal{U}$ such that $X \subset (\mathbb{H}_u \cup \ell_u) \cap (\mathbb{H}_v \cup \ell_v)$, where $\ell_{u'} = \{x \in \mathbb{R}^2 : \langle x, u' \rangle = 0\}$ is the boundary of $\mathbb{H}_{u'} = \{x \in \mathbb{R}^2 : \langle x, u' \rangle < 0$ for any $u' \in S^1$. The elements of $\mathcal{S}'$ are usually referred to as *quasi-stable directions*. Then our fundamental set of directions will be

$$\widehat{\mathcal{S}} = \bigcup_{u \in \mathcal{S}'} \big(\{u, u_0 - (u - u_0)\} + \{0, \pi/2, \pi, 3\pi/2\}\big). \tag{6}$$

In other words, we start with the stable directions, add to them the quasi-stable ones, reflect them at u_0 and finally make the set obtained invariant by rotation by $\pi/2$. By construction, the cardinality of $\widehat{\mathcal{S}}$ is a multiple of 4.

REMARK 3.1. Let us note that invariance by rotation and reflection is cosmetic and one could in fact deal directly with the set of quasi-stable directions from [6], though notation would be more laborious and drawings less aesthetic.

We write $u_0, u_1, \dots, u_{4k-1}$ for the elements of $\widehat{\mathcal{S}}$ ordered clockwise starting with u_0 and $\widehat{\mathcal{S}}_0$ for those elements of $\widehat{\mathcal{S}}$ belonging to the semicircle C. For all figures, we shall take $\widehat{\mathcal{S}} = \{i\pi/4, i \in \{0, 1, \dots, 7\}\}$ and $u_0 = \pi$. When referring to u_i, the index i will be considered modulo $4k$. With this convention, $\widehat{\mathcal{S}}_0 = \{u_{-k+1}, \dots, u_{k-1}\}$.

3.1.2. *$\mathcal{U}$-Bootstrap percolation restricted to $\Lambda \subset \mathbb{Z}^2$.* In the sequel, we will sometimes need the following slight variation of the $\mathcal{U}$-bootstrap percolation. Given $\Lambda \subset \mathbb{Z}^2$ and a set $A \subset \Lambda$ of initial infection, we will write $[A]_{\mathcal{U}}^{\Lambda}$ for the closure $\bigcup_{t \ge 0} A_t^{\Lambda}$ of the $\mathcal{U}$-bootstrap percolation restricted to Λ, $(A_t^{\Lambda})_{t \ge 0}$, defined by

$$A_{t+1}^{\Lambda} = A_t^{\Lambda} \cup \{x \in \Lambda, \exists U \in \mathcal{U}, x + U \subset A_t^{\Lambda}\}.$$

[1] By Lemmas 2.6 and 2.8 of [5], this is equivalent to the fact that all (stable) directions have finite difficulty.

[2] A direction $u \in S^1$ is rational if $\tan(u) \in \mathbb{Q}$ or, equivalently, if $su \in \mathbb{Z}^2\}$ for some $s > 0$.

3.1.3. *Geometric setup*. We next turn to defining the various geometric domains we will need to consider. As the notation is a bit cumbersome, the reader is invited to systematically consult the relevant figures. We fix a large integer w and a small positive number δ depending on $\mathcal{U}$ (e.g., w much larger than the diameter of $\mathcal{U}$ and of the largest difficulty of stable directions), but not depending on q. When using asymptotic notation (as $q \to 0$), we will assume that the implicit constants do not depend on w, δ and q. Throughout the entire paper, we shall consider that q is small, as we are interested in the $q \to 0$ limit. In particular, we shall assume that q is so small that any length scale diverging to $+\infty$ as $q \to 0$ will be (much) larger than the constant w.

DEFINITION 3.2. Consider a closed convex polygon P in $\mathbb{R}^2$. Assume that the outward normal vectors to the sides of P belong to $\widehat{\mathcal{S}}$ and that u is one of them. Then we write $\partial_u P$ for the side whose outward normal is u_i.

We can now define the notion of droplet that will be relevant for our setting (see Figure 5). In the sequel for $u \in S^1$, we set $\overline{\mathbb{H}}_u = \mathbb{H}_u \cup \ell_u$ for the closure of $\mathbb{H}_u$. Moreover, given $x \in \mathbb{R}^2$ and $s \in \mathbb{R}$, we set $\mathbb{H}_u(x) = \mathbb{H}_u + x$, $\mathbb{H}_u(s) = \mathbb{H}_u(su)$ and similarly for $\overline{\mathbb{H}}_u$ and ℓ_u. Finally, for any $u_i \in \hat{\mathcal{S}}$ we set $\rho_i = \inf\{\rho > 0, \exists x \in \mathbb{Z}^2, \langle x, u_i\rangle = \rho\}$ for the smallest positive s such that $\ell_{u_i}(s) \neq \ell_{u_i}$ and $\ell_{u_i}(s) \cap \mathbb{Z}^2 \neq \varnothing$.

DEFINITION 3.3 (Quasi-stable annulus and half-annulus). Fix a radius $R = R(q)$ such that $\lim_{q\to 0} R(q) = +\infty$ and let $R_i = \rho_i \lfloor \frac{R}{\rho_i} \rfloor$ for $i \in [4k]$. We call the subset of $\mathbb{R}^2$

$$A = \bigcap_{i\in[4k]} \overline{\mathbb{H}}_{u_i}(R_i) \setminus \bigcap_{i\in[4k]} \mathbb{H}_{u_i}(R_i - w)$$

the *quasi-stable annulus* (or simply the *annulus*) with radius R and width w centered at the origin. We write A^{int} for the region $\bigcap_{i\in[4k]} \mathbb{H}_{u_i}(R_i - w)$ enclosed by A. Clearly, the outer boundary of A is a closed convex polygon P satisfying the assumption of Definition 3.2 and we write $\partial_{u_i} A$ for $\partial_{u_i} P$. We also let

$$\partial_{\widehat{\mathcal{S}}_0} A = \bigcup_{u\in\widehat{\mathcal{S}}_0} \partial_u A, \tag{7}$$

and we call

$$HA = \bigcap_{i=-k}^{k} \overline{\mathbb{H}}_{u_i}(R_i) \setminus \bigcap_{i=-k+1}^{k-1} \mathbb{H}_{u_i}(R_i - w)$$

the *quasi-stable half-annulus* of radius R and width w.

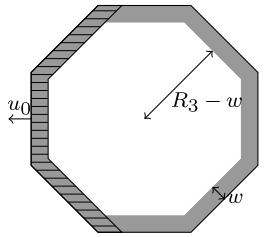


FIG. 5. *The shaded region is the* quasi-stable annulus A, *while the hatched one is the* quasi-stable half-annulus HA. *As anticipated all the radii* R_i *are much larger than the width* w.

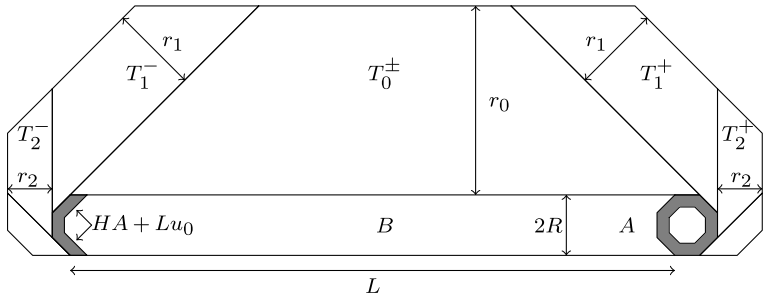


FIG. 6. *A snail $V = V^+ \cup V^-$ with its base B and its trapezoids $T_0^\pm, T_1^\pm, \dots$. The right-snail of V is $V^+ = B \cup \bigcup_i T_i^+$ while the left-snail is $V^- = B \cup \bigcup_i T_i^-$. In Section 4, the shaded quasi-stable annulus A and the half-annulus $HA + Lu_0$ will act as an infected boundary condition.*

Our approach will consist in building progressively larger domains for which we can bound the Poincaré constant of the finite volume KCM process conditionally on the simultaneous occurrence of a certain likely event *and* the presence of an infected annulus. We next define these domains (see Figure 6). Recall that δ is a small constant depending on the update family $\mathcal{U}$.

DEFINITION 3.4 (Snails). Recall R and R_i from Definition 3.3. Let $L = L(q) > 0$ be such that $\lim_{q\to 0} L(q) = +\infty$ and assume that $\frac{L\langle u_0, u_{k-1}\rangle}{\rho_{k-1}} \in \mathbb{N}$ (i.e., $\ell_{u_{k-1}}(Lu_0)$ contains lattice sites). We call a sequence of nonnegative numbers $\underline{r} = (r_0, r_1, \dots, r_{2k})$ *admissible* if

$$0 \le r_0 \le \delta L, \qquad r_i \le \delta r_{i-1}, \qquad r_{2k} = 0.$$

Given an admissible $\underline{r}$, we call the set

$$V_L^{R,+}(\underline{r}) = \bigcap_{i=-k+1}^{k-1} \overline{\mathbb{H}}_{u_i}\big(R_i + L\langle u_0, u_i\rangle\big) \cap \bigcap_{i=k}^{3k} \overline{\mathbb{H}}_{u_i}(R_i + r_{i-k})$$

the *right-snail* with parameters $(R, L, \underline{r})$. Using the symmetric construction of $\widehat{\mathcal{S}}$, the left-snail $V_L^{R,-}(\underline{r})$ with parameters $(R, L, \underline{r})$ is simply defined as the reflection of the right-snail w.r.t. the line orthogonal to u_0 and passing through the point $\frac{1}{2}Lu_0$. Finally, the *snail* with parameters $(R, L, \underline{r})$ is the set

$$V_L^R(\underline{r}) = V_L^{R,+}(\underline{r}) \cup V_L^{R,-}(\underline{r}).$$

We systematically drop the parameters R, L, and $\underline{r}$ from our notation when no ambiguity arises.

DEFINITION 3.5. We observe that any right-snail $V_L^{R,+}(\underline{r})$ can be thought of as the set obtained by stacking together as in Figure 6 its *base* defined as

$$B = V_L^R\big((0, \dots, 0)\big)$$

and its trapezoids defined as

$$\begin{aligned} T_i^+ &= V_L^{R,+}(r_0, \dots, r_i, 0, \dots, 0) \setminus V_L^{R,+}(r_0, \dots, r_{i-1}, 0, \dots, 0) \\ &= \big(\overline{\mathbb{H}}_{u_{k+i}}(R_{k+i} + r_i) \setminus \overline{\mathbb{H}}_{u_{k+i}}(R_{k+i})\big) \cap \overline{\mathbb{H}}_{u_{k+i-1}}(R_{k+i-1} + r_{i-1}) \cap \overline{\mathbb{H}}_{u_{k+i+1}}(R_{k+i+1}) \end{aligned}$$

with the convention $r_{-1} = L\langle u_0, u_{k-1}\rangle$. Notice that the base B is characterized by two parameters R, L called *radius* and *length*, respectively.

With this picture in mind the positive values of $\underline{r}$ coincide with the heights of the corresponding nonempty trapezoids. A similar decomposition holds for the left-snail. In the sequel, it will be convenient to partition the lattice sites in each trapezoid T_i^+ into disjoint *slices* $ST_{i,j}^+$, $j = 1, 2, \dots$, with each slice consisting of all the lattice sites of the trapezoid lying on a *common line* of $\mathbb{R}^2$ orthogonal to the direction u_{k+i}. Similarly, for the lattice sites contained in the *truncated base* $B^\circ := B \setminus (A \cup A^{\text{int}} \cup (HA + Lu_0))$. In this case each slice, denoted SB_j, $j = 1, 2, \dots$, will consist of all the sites belonging to a common suitable translate in the u_0-direction of $\partial_{\widehat{\mathcal{S}}_0} A$ defined in (7). Recall from Definition 3.3 ρ_i and $R_i = \rho_i \lfloor R/\rho_i \rfloor$, where R is the radius of the annulus A.

DEFINITION 3.6. Fix a snail and suppose that its trapezoid T_i^+, $i \in [2k]$ is nonempty. The jth slice of $T_i^+ \cap \mathbb{Z}^2$ is the set

$$ST_{i,j}^+ = T_i^+ \cap \ell_{u_{k+i}}(R_{k+i} + j\rho_{k+i}) \cap \mathbb{Z}^2,$$

so that $T_i^+ \cap \mathbb{Z}^2 = \bigcup_{j>0} ST_{i,j}^+$. Similarly, for the left trapezoid T_i^- if nonempty.

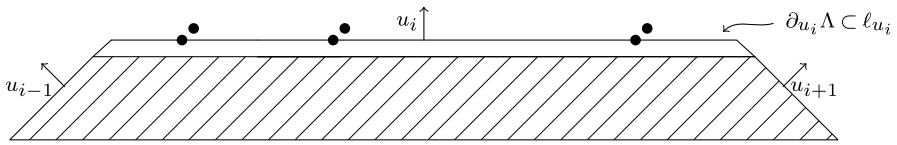
Turning to the truncated base B°, we first set $\lambda_{j+1} = \inf\{\lambda > \lambda_j, (\lambda u_0 + \partial_{\widehat{\mathcal{S}}_0} A) \cap \mathbb{Z}^2 \neq \varnothing\}$ with $\lambda_0 = 0$. Then we define the jth slice of the truncated base B° of the snail, SB_j, and its ith-side, $SB_{i,j}$, as

$$SB_j = (\lambda_j + \partial_{\widehat{\mathcal{S}}_0} A) \cap B^\circ \cap \mathbb{Z}^2,$$

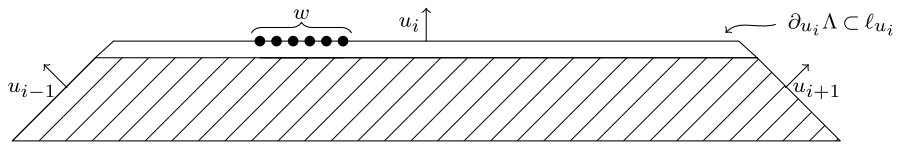
$$SB_{i,j} = (\lambda_j + \partial_{u_i} A) \cap B^\circ \cap \mathbb{Z}^2.$$

Note that for any admissible sequence $\underline{r}$ a nonempty slice of the trapezoid T_i^+ consists of all lattice points of a segment $I \subset \mathbb{R}^2$ orthogonal to u_{k+i} with length $\Omega(r_{i-1})$ and such that $I \cap \mathbb{Z}^2 \neq \varnothing$. Similarly, the number of lattice sites in each slice of B° is $\Theta(R)$. In the sequel, we will only consider nonempty slices without explicitly specifying the range of the index $j > 0$.

3.1.4. *Helping sets.* Recall Definition 1.2 and Remark 1.3. If u is a stable direction, then the infected half-plane $\mathbb{H}_u$ needs *finitely* many (exactly $\alpha(u)$) extra infected sites in $\mathbb{R}^2 \setminus \mathbb{H}_u$ in order to infect *infinitely* many sites on the line ℓ_u. If only a finite portion of $\mathbb{H}_u$ is infected, for example, the dashed region in Figure 7, then the propagation of infection to some portion



(a) An example helping set (the black dots) consisting of three disjoint copies of Z shifted along ℓ_{u_i}. In the figure, $\alpha(u_i) = 2$ with $Z = \{(0, 0), (1, 1)\}$ and $m = 3$.



(b) Illustration of w consecutive sites of $\partial_{u_i}\Lambda$.

FIG. 7. *The setting of Lemma 3.7. In the figure, u_i is the upwards direction and the hatched trapezoid represents the lattice sites in $\Lambda \setminus \partial_{u_i}\Lambda$. The lemma states that if the hatched region and the black sites are infected, $\partial_{u_i}\Lambda$ also becomes infected (in the $\mathcal{U}$ bootstrap percolation process restricted to a suitable region).*

of the line ℓ_u is a delicate problem. A special case which suffices for our purposes is covered in the next lemma (see [5], Lemma 3.4, and [6], Lemma 5.2).

LEMMA 3.7. *Fix $u = u_i$, $i \in [4k]$ and recall that w is a large enough integer (depending on $\mathcal{U}$) and let $r \geq w^2$. Let*

$$\Lambda := \Lambda(u, w, r) = \overline{\mathbb{H}}_{u_{i-1}}(r) \cap \overline{\mathbb{H}}_{u_i} \cap \overline{\mathbb{H}}_{u_{i+1}}(r) \cap \overline{\mathbb{H}}_{u_{i+2k}}(w)$$

be the (closed) trapezoid in Figure 7 of height w and bases orthogonal to u. Note that $\partial_u \Lambda \subset \ell_u$.

(a) *Let $Z \subset \mathbb{Z}^2 \setminus \mathbb{H}_u$ be a set of $\alpha(u)$ sites at distance at most $\sqrt{w}$ from the origin such that $[\mathbb{H}_u \cup Z]_{\mathcal{U}} \cap \ell_u$ is infinite. Then there exist finitely many lattice points $a_1, \dots, a_m, b$, on the line ℓ_u such that the following holds. If $\Lambda \setminus \partial_u \Lambda$ and $\bigcup_{j=1}^m (Z + a_j + k_j b)$ are infected, where $k_1, \dots, k_m \in \mathbb{Z}$ are such that $\{a_1 + k_1 b, a_2 + k_2 b, \dots, a_m + k_m b\}$ form m distinct lattice sites of $\partial_u \Lambda$ at distance at least w from the endpoints of $\partial_u \Lambda$, then the $\mathcal{U}$-bootstrap percolation restricted to the larger trapezoid*

$$\overline{\Lambda} = \overline{\mathbb{H}}_{u_{i-1}}(r) \cap \overline{\mathbb{H}}_{u_i}(w/2) \cap \overline{\mathbb{H}}_{u_{i+1}}(r) \cap \overline{\mathbb{H}}_{u_{i+2k}}(w)$$

is able to infect $\partial_u \Lambda$.

(b) *If $\Lambda \setminus \partial_u \Lambda$ and w consecutive lattice sites in $\partial_u \Lambda$ are infected, then the $\mathcal{U}$-bootstrap percolation restricted to Λ is able to infect $\partial_u \Lambda$.*

DEFINITION 3.8 (u-helping sets). Let $i \in [-k+1, k-1]$. Any collection of lattice sites of the form $\{a_1 + k_1 b, a_2 + k_2 b, \dots, a_m + k_m b\}$ satisfying the assumption in (a) above will be referred to as *u_i-helping set* for $\partial_{u_i} \Lambda$ or simply u_i-helping set.

3.2. *Some KCM tools*. For reader's convenience, we next collect some general tools from KCM theory that will be applied several times throughout the proof of the main result.

Notation. For every statement $\mathcal{P}$, define $\mathbb{1}_{\{\mathcal{P}\}} = 1$ if $\mathcal{P}$ holds and $\mathbb{1}_{\{\mathcal{P}\}} = 0$ otherwise. For any subset Λ of $\mathbb{R}^2$, we write $(\Omega_\Lambda, \mu_\Lambda)$ for the product probability space $(\{0,1\}^{\Lambda \cap \mathbb{Z}^2}, \bigotimes_{x \in \Lambda \cap \mathbb{Z}^2} \mu_q)$. If $\Lambda = \mathbb{R}^2$, we simply write (Ω, μ). Given $f : \Omega_\Lambda \to \mathbb{R}$ we shall write $\mu_\Lambda(f)$ and $\mathrm{Var}_\Lambda(f)$ for the mean and variance of f w.r.t. μ_Λ, respectively, whenever they exist. For any $\omega \in \Omega$ and $\Lambda \subset \mathbb{R}^2$, we write ω_Λ for the collection $(\omega_x)_{x \in \Lambda \cap \mathbb{Z}^2}$. Given a function $f : \Omega \to \mathbb{R}$ depending on finitely many variables, we write

$$\mathcal{D}(f) = \sum_{x \in \mathbb{Z}^2} \mu\big(c_x \, \mathrm{Var}_x(f)\big), \tag{8}$$

for the KCM *Dirichlet form* of f, where $c_x(\omega)$ is the indicator of the event $\{\exists X \in \mathcal{U} : \forall y \in X, \omega_{x+y} = 0\}$ and $\mathrm{Var}_x(f) := \mathrm{Var}_{\{x\}}(f)$ denotes the conditional variance $\mathrm{Var}(f \mid (\omega_z)_{z \neq x})$. Finally, we shall write $\mathbb{P}_\mu(\cdot)$ for the law of the $\mathcal{U}$-KCM process on $\mathbb{Z}^2$ with initial law μ and $\mathbb{E}_\mu(\cdot)$ for the expectation w.r.t. $\mathbb{P}_\mu(\cdot)$.

Poincaré inequalities. We begin with a well-known general fact on product measures which we state here in ready-to-use form.

LEMMA 3.9. *Let Λ_i, $i \in \{1,2,3\}$ be three disjoint finite subsets of $\mathbb{Z}^2$ and ν_i be a probability measures on Ω_{Λ_i}. Let ν be the product measure $\bigotimes_{i=1}^3 \nu_i$ on $\bigotimes_{i=1}^3 \Omega_{\Lambda_i}$. Then for any function f we have*

$$\nu_1\big(\mathrm{Var}_{\nu_2 \otimes \nu_3}(f)\big) \leq \mathrm{Var}_\nu(f) \leq \nu_1\big(\mathrm{Var}_{\nu_2 \otimes \nu_3}(f)\big) + \nu_2\big(\mathrm{Var}_{\nu_1 \otimes \nu_3}(f)\big).$$

PROOF OF LEMMA 3.9. The first inequality follows from the total variance formula

$$\mathrm{Var}_\nu(f) = \nu_1\big(\mathrm{Var}_{\nu_2\otimes\nu_3}(f)\big) + \mathrm{Var}_{\nu_1}\big(\nu_2\otimes\nu_3(f)\big).$$

For the second inequality, we observe that

$$\begin{aligned}\mathrm{Var}_{\nu_1}\big(\nu_2\otimes\nu_3(f)\big) &= \nu_1\big(\nu_2\otimes\nu_3\big(f-\nu(f)\big)^2\big)\\ &\le \nu\big(\big(f-\nu_1\otimes\nu_3(f)\big)^2\big) = \nu_2\big(\mathrm{Var}_{\nu_1\otimes\nu_3}(f)\big)\end{aligned}$$

by Jensen's inequality. □

In order to understand the general framework for the last two results, we begin by recalling a standard Poincaré inequality for n independent random variables $X_1,\dots,X_n$ (for simplicity each one taking finitely many values). For any $f = f(X_1,\dots,X_n)$,

$$\mathrm{Var}(f) \le \sum_i \mathbb{E}\big(\mathrm{Var}_i(f)\big),$$

where $\mathrm{Var}_i(f)$ is the conditional variance computed w.r.t. the variable X_i given all the other variables. The sum in the right-hand side above can be interpreted as the Dirichlet form of the continuous time *Gibbs sampler*, reversible w.r.t. the product law of $(X_i)_i$, which with rate n chooses a random index $i\in[n]$ and resamples X_i w.r.t. its marginal. From this perspective, the above inequality tells us that the relaxation time (see, e.g., [17]) of the Gibbs sampler is bounded from above by 1.

Now consider n events $(\mathcal{H}_i)_{i=1}^n$, in the sequel *facilitating events*, and suppose that each $\mathcal{H}_i$ depends only on the variables $(X_j)_{j\neq i}$. An example of a *constrained* Poincaré inequality with facilitating events $(\mathcal{H}_i)_{i=1}^n$ is the inequality

$$\mathrm{Var}(f\mid\Omega_{\mathcal{H}}) \le C\sum_i \frac{\mathbb{E}(\mathbb{1}_{\mathcal{H}_i}\mathrm{Var}_i(f))}{\mathbb{P}(\Omega_{\mathcal{H}})}, \tag{9}$$

where $\Omega_{\mathcal{H}} = \bigcup_{i=1}^n \mathcal{H}_i$ and $C\in[1,+\infty]$. Notice that the sum in the right-hand side above can be interpreted as the Dirichlet form of the continuous time *constrained* Gibbs sampler on $\Omega_{\mathcal{H}}$, which with rate n chooses a random index $i\in[n]$ and resamples X_i w.r.t. its marginal iff $\mathcal{H}_i$ holds. If the facilitating events are such that the constrained Gibbs sampler on $\Omega_{\mathcal{H}}$ is ergodic, then $C<+\infty$.

Each one of the two results we are about to discuss next is just a special instance of the above general problem.

LEMMA 3.10. *Let X_1, X_2 be two independent random variable taking values in two finite sets $\mathbb{X}_1$, $\mathbb{X}_2$. Let also $\mathcal{H}\subset\mathbb{X}_1$ with $\mathbb{P}(X_1\in\mathcal{H})>0$. Then for any function $f(X_1,X_2)$,*

$$\mathrm{Var}(f) \le 2\mathbb{P}(X_1\in\mathcal{H})^{-1}\mathbb{E}\big(\mathrm{Var}_1(f)+\mathbb{1}_{\{X_1\in\mathcal{H}\}}\mathrm{Var}_2(f)\big).$$

REMARK 3.11. The above inequality coincides with (9) with $\mathcal{H}_1=\mathbb{X}_2$, $\mathcal{H}_2=\{X_1\in\mathcal{H}\}$ and $C=2/\mathbb{P}(X_1\in\mathcal{H})$. Clearly, the constrained Gibbs sampler is irreducible because $\mathbb{P}_1(X_1\in\mathcal{H})>0$.

PROOF OF LEMMA 3.10. It follows from [7], Proof of Proposition 4.4, that

$$\begin{aligned}\mathrm{Var}(f) &\le \frac{1}{1-\sqrt{1-\mathbb{P}(X_1\in\mathcal{H})}}\mathbb{E}\big(\mathrm{Var}_1(f)+\mathbb{1}_{\{X_1\in\mathcal{H}\}}\mathrm{Var}_2(f)\big)\\ &\le 2\mathbb{P}(X_1\in\mathcal{H})^{-1}\mathbb{E}\big(\mathrm{Var}_1(f)+\mathbb{1}_{\{X_1\in\mathcal{H}\}}\mathrm{Var}_2(f)\big).\end{aligned}$$

□

The second result concerns a generalization of the standard (finite volume) constrained Poincaré inequality for the 1-neighbor KCM process, or FA1f KCM, [7].

Let $(\widehat{S}, \widehat{\nu})$ be a finite probability space with $\widehat{\nu}$ a positive probability measure, let $\Omega_n = \widehat{S}^{[n]}$ and $\nu = \bigotimes_{i\in[n]} \nu_i$, where $\nu_i = \widehat{\nu}$ for all $i \in [n]$. Elements of Ω_n are denoted $\omega = (\omega_0, \dots, \omega_{n-1})$ with $\omega_i \in \widehat{S}$. Fix a single site event $\mathcal{H} \subset \widehat{S}$ and a positive integer $\kappa < n$. Then, according to whether we view the set $[n]$ as the n-cycle or not, we define the *facilitating event* $\mathcal{H}_i$ as follows. If $[n]$ is the n-cycle,

$$\mathcal{H}_i = \bigcap_{j=i+1}^{i+\kappa} \{\omega_j \in \mathcal{H}\} \cup \bigcap_{j=i-1}^{i-\kappa} \{\omega_j \in \mathcal{H}\}.$$

If instead $[n]$ is linear,

$$\mathcal{H}_i = \begin{cases} \displaystyle\bigcap_{j=i-1}^{i-\kappa} \{\omega_j \in \mathcal{H}\} & \text{if } i+\kappa \geq n, \\ \displaystyle\bigcap_{j=i+1}^{i+\kappa} \{\omega_j \in \mathcal{H}\} & \text{if } i-\kappa < 0, \\ \displaystyle\bigcap_{j=i+1}^{i+\kappa} \{\omega_j \in \mathcal{H}\} \cup \bigcap_{j=i-1}^{i-\kappa} \{\omega_j \in \mathcal{H}\} & \text{otherwise.} \end{cases}$$

In words, in the periodic case $\mathcal{H}_i$ requires the κ variables immediately after or before i (in the clockwise order) to be in a state belonging to $\mathcal{H}$, while in the linear case the same requirement holds when i is farther than κ from the boundary points of $[n]$. When i is closer than κ to, for example, the the right boundary of $[n]$, then $\mathcal{H}_i$ requires the κ variables immediately before i to be in states belonging to $\mathcal{H}$. The case when $\widehat{S} = \{0, 1\}$, $\widehat{\nu}$ is the Bernoulli$(1-q)$-measure, $\mathcal{H} = \{0\}$ and $\kappa = 1$ is the usual 1-neighbor KCM setting.

LEMMA 3.12. *Assume that* $(1 - \widehat{\nu}(\mathcal{H})^k)^{n/(3\kappa)} < \frac{1}{16}$ *and set* $\Omega_{\mathcal{H}} = \bigcup_{i=0}^{n-1} \mathcal{H}_i$. *Then, for all* $f : \Omega_n \to \mathbb{R}$,

$$\operatorname{Var}_\nu(f \mid \Omega_{\mathcal{H}}) \leq \left(\frac{2}{\widehat{\nu}(\mathcal{H})}\right)^{O(\kappa)} \sum_{i=1}^{n} \nu\big(\mathbb{1}_{\mathcal{H}_i} \operatorname{Var}_{\nu_i}(f)\big). \tag{10}$$

The proof is left to the Appendix.

REMARK 3.13. We will apply the lemma with $(\widehat{S}, \widehat{\nu})$ equal to the probability space given by $\{0,1\}^m$ equipped with the Bernoulli$(1-q)$ product measure conditioned on some event whose probability tends to one as $q \to 0$. The integers $1 \ll m \ll n$ may diverge to infinity as $q \to 0$ while the integer κ will be large but independent of q.

4. The core of the proof. In this section, we prove a Poincaré inequality which will represent the key step in the proof of Theorem 1.

4.1. *Roadmap*. Before we dive into the technical details, let us give a hands-on roadmap of the argument. Although it is underlined by the dynamical intuition explained in Section 2, the latter is not very transparent in the Poincaré language of the formal proof.

The goal of this section is to prove Theorem 4.6. It says that the $\mathcal{U}$-KCM ($\mathcal{U}$ being a fixed critical update family with a finite number of stable directions) on a snail $V = V_L^R(\underline{r})$ (recall Definition 3.4 and Figure 6), conditioned on a well-chosen *super good* event $\mathcal{SG}(V)$ is able to relax in a time $\exp(\log^3(1/q)/q^\alpha)$, which is the dominating contribution leading to (5).

For the purposes of the roadmap, the reader should think of the snail as having dimensions $R = w^2 \log(1/q)/q^\alpha$, $L = q^{-3w}$ and $r_i = \delta^{i-(2k-1)} q^{-2w}$, $i \in [2k]$ for some small positive δ. Let us explain the Definition 4.2 of $\mathcal{SG}(V)$ before outlining the proof of Theorem 4.6.

4.1.1. *Good and super good events.* The super good event $\mathcal{SG}(V)$ will decompose as a product w.r.t. the partition of V into its annulus A, half-annulus $HA + Lu_0$, annulus interior A^{int}, truncated base B° and trapezoids $T_i^\pm$ from Section 3.1.3. On A (HA), we require the event $\mathcal{A}$ ($\mathcal{HA}$) that A (HA) is fully infected. These are the only *unlikely* events involved in $\mathcal{SG}(V)$ and we will denote by $\mathcal{SG}$ only events requiring the occurrence of (spatial translates of) $\mathcal{A}$ and $\mathcal{HA}$. Events of type $\mathcal{SG}$ will all have very small probability $\mu(\mathcal{SG})$ of the order of $\exp(-\log^2(1/q)/q^\alpha)$.

We will use instead write $\mathcal{G}$ to denote good events, which are *likely* and only involve the presence of appropriate helping sets as in Definition 3.8 or sets of w consecutive infections as in Lemma 3.7(b). Recall the decomposition of B° into slices SB_j from Definition 3.6. We say that the event $\mathcal{SB}_j$ occurs if each side of SB_j (which consists of at most one segment in each direction) has a helping set for the corresponding direction. We then define $\mathcal{G}(B^\circ) = \bigcap_j \mathcal{SB}_j$ and it is not hard to see that this way the occurrence of $\mathcal{SG}(B) = \mathcal{A} \cap \mathcal{HA} \cap \mathcal{G}(B^\circ)$ implies that the infections in B are sufficient to fully infect B.

Notice that in general the event $\mathcal{SB}_j$ depends on the values of ω in the set $\bigcup_{i=0}^k SB_{j+i}$ for some $k \geq 0$ depending only on $\mathcal{U}$. In order to avoid this (annoying) technical detail, we will use the following simplifying Assumption 4.1 implying that $k = 0$.

ASSUMPTION 4.1. For every stable direction $u \in \widehat{\mathcal{S}}_0$, there exists a subset Z_u of the line ℓ_u of cardinality α such that $[\mathbb{H}_u \cup Z_u]_{\mathcal{U}} \cap \ell_u$ has infinite cardinality.

This is by no means restrictive, as the proof applies directly without this assumption up to changing $\mathcal{SB}_{i,j}$ in Definition 4.2, following [20], Section 7. We will spare the reader the tedious details, as they already appeared previously in the above mentioned paper. This assumption is only relevant for treating the base B, for which we will import the result from [20], where the assumption was introduced.

Having defined the good event for the base B, we now define the good event for the trapezoids of the snail V. Let $\mathcal{ST}_{i,j}^\pm$ be the event that the slice $ST_{i,j}^\pm$ in the decomposition of $T_i^\pm$ from Definition 3.6 contains a set of w consecutive infected sites. We then define $\mathcal{G}(T_i^\pm) = \bigcap_j \mathcal{ST}_{i,j}^\pm$. Again, by Lemma 3.7 it is not hard to see that if B and $T_{i'}^+$ for $i' < i$ are fully infected and $\mathcal{G}(T_i^+)$ occurs, then the $\mathcal{U}$-bootstrap percolation can also infect T_i^+ (and similarly for T_i^-).

Finally, the super good event $\mathcal{SG}(V)$ is defined as $\mathcal{SG}(B) \cap \bigcap_i (\mathcal{G}(T_i^+) \cap \mathcal{G}(T_i^-))$ and it clearly implies that the entire snail V can be infected from within.

4.1.2. *Structure of the proof.* The fact that V can be fully infected on $\mathcal{SG}(V)$ is reassuring and implies that the relaxation time we are after in Theorem 4.6 is finite, but we need an efficient relaxation mechanism to prove the theorem. It is not hard to see that it suffices to treat the right-snail V^+, so we concentrate on it and drop all $+$ superscripts. In the sequel, whenever we refer to the *relaxation in a given region* Λ mathematically this will translate into proving a Poincaré inequality like the one in (12) with V replaced by Λ.

The proof proceeds by proving an efficient relaxation in progressively larger and larger volumes always conditioned on a corresponding $\mathcal{SG}$ event. In the process, we will often rely on auxiliary constrained block dynamics of several types like those in Section 3.2. These auxiliary dynamics allow us to relate the relaxation in a given region to the relaxation in smaller subregions, each subregion having an additional convenient constraint on the configuration

outside it. The auxiliary dynamics we will use are of FA1f type (like the one in Lemma 3.12) or two-blocks type (like the one in Lemma 3.10). By performing such reductions, we reduce the problem of proving an efficient relaxation on a large region to a similar problem on suitable smaller regions. The base case of the above inductive procedure is then treated directly. We now describe the various steps of the above iterative reduction.

The base case: *The annulus interior* A^{int}. First, in Lemma 4.10 we treat A^{int} on the event $\mathcal{A}$ that the annulus is fully infected, which serves as a boundary condition. This is fairly easy and can be done in various ways. To give a formal argument, we split A^{int} into strips of bounded width (see Figure 8). Fully infected strips perform an FA1f auxiliary dynamics. The boundary condition provides all the sets of w consecutive infections needed for an infected strip to infect its neighbor using Lemma 3.7 (see Figure 7a).

From A^{int} *to the base* B. Up to now, we have a Poincaré inequality on the annulus and its interior. In Proposition 4.9, we extend that to a base B. We will not insist on this step, as it was essentially done already in [20]. Indeed, using an East-like dynamics in direction u_0 the relaxation time of B° (on $\mathcal{G}(B^\circ)$) with infected boundary condition in A was shown to be roughly $\exp(\log^3(1/q)/q^\alpha)$. Combining this with the result for A^{int}, we obtain a Poincaré inequality for B.

Adding the first trapezoid to B. Our next goal is to consider the relaxation in $B \cup T_0$. In turn, this step is split into two distinct parts.

Adding the first slice $ST_{0,1}$ *to* B. This is achieved in Lemma 4.13 (see Figure 9). Relaxation in B has already been established in the previous step, so we focus on the relaxation in $ST_{0,1}$. In doing this, we are allowed our knowledge of the relaxation in B. We use the FA1f-like dynamics of Lemma 3.12, asking for w consecutive infections in $ST_{0,1}$ next to the site to be updated. In other words, we have to understand how to efficiently resample a site $x \in ST_{0,1}$ using the $\mathcal{U}$-KCM dynamics when its neighboring w sites are infected. Using a two-block dynamics (Lemma 3.10), resampling B roughly $q^{-O(1)}$ times, we may further impose the condition that the site we wish to resample has a fully infected neighborhood in B in addition to the next w sites in $ST_{0,1}$, which are already infected. This is exactly the situation in Figure 9 and makes the flip of the site we want to update legal for the original $\mathcal{U}$-KCM. Thus, this step produces terms of the Dirichlet form of the $\mathcal{U}$-KCM in (8), as well as a term $\mathrm{Var}_B(f \mid \mathcal{SG}(B))$, which we already know how to control.

Adding more slices to B. In a sense, this part embodies the East-like motion of droplets in direction u_1 hinted in Section 2. This connection is rather indirect in the sense that the *bisection method* used to analyze the relaxation in the union of B with several slices of the first trapezoid coincides with the bisection method used to efficiently bound from above the relaxation time of the standard East model in [7].

Consider the problem of the relaxation in a snail consisting of B and $2n$ slices of the first trapezoid. Our aim is to reduce it to the same problem on two similar snails, each one with essentially the same base B but with only n slices. This is achieved in Lemma 4.14. We start by introducing an auxiliary constrained two-block dynamics in which B and the first n slices form the first block $\widetilde{V}$, while the second group of n slices form the second block $\overline{T}_0$ (see Figure 10). The constraint of the two-blocks dynamics is that a translated base $\overline{B}$ (corresponding to $\overline{V}_{i-1}$ in Figure 10) is super good. The base $\overline{B}$ is constructed so that together with $\overline{T}_0$ it forms a snail $\overline{V}$ with size similar to that of $\widetilde{V}$. The relaxation to equilibrium on $\widetilde{V}$ is dealt with by induction on the number of slices, so it remains to analyze the relaxation to equilibrium on $\overline{T}_0$ under the above constraint. The relaxation time of the auxiliary model is $1/\mu(\mathcal{SG})$ (the number of times one needs to update the first block until the constraint becomes satisfied). Then in order to relax on $\overline{T}_0$ it suffices to do so on the larger region $\overline{V}$. We are done since $\overline{V}$ and $\widetilde{V}$ are already treated by the induction.

In Corollary 4.16, repeating the above bisection several times, we manage to reproduce the relaxation on a snail with base B and arbitrary number r_0 of slices of the first trapezoid. Indeed, starting from the snail with a single slice in T_0 provided above, we double its height $\log(1/q)$ times to reach the desired $r_0 \simeq \delta^{-2k+1} q^{-2w}$. Thus, the Poincaré constant of B is multiplied by $1/\mu(\mathcal{SG})^{\log(1/q)}$ in this process.

Adding all trapezoids of the original snail V. Finally, repeating the above steps for each trapezoid, we obtain the desired Poincaré constant for the entire snail, concluding Theorem 4.6.

4.2. *Setup.* Given a snail $V = V_L^R(\underline{r})$, we shall work in the associated probability space $\Omega_V = \{0,1\}^{V \cap \mathbb{Z}^2}$ endowed with the probability measure $\mu_V(\cdot \mid \mathcal{SG}(V))$ conditioned to the simultaneous occurrence of the following events on Ω_V.

DEFINITION 4.2 (Good and super good events).

- Recalling Definition 3.3, we define $\mathcal{A}$ as the event that A is infected and $\mathcal{HA}$ as the event that $HA + Lu_0$ is infected.
- Recalling Definitions 3.6 and 3.8, for each SB_j and $u_i \in \widehat{\mathcal{S}}_0$, let $\mathcal{SB}_{i,j}$ denote the event that $SB_{i,j} = \varnothing$ or $SB_{i,j}$ contains an infected u_i-helping set. Then set $\mathcal{SB}_j = \bigcap_{u_i \in \widehat{\mathcal{S}}_0} \mathcal{SB}_{i,j}$.
- Recalling Definition 3.5, for each nonempty $ST_{i,j}^{\pm}$ let $\mathcal{ST}_{i,j}^{\pm}$ be the event that there exist w consecutive infected sites in $ST_{i,j}^{\pm}$.

Using the above events, we then define

$$\mathcal{G}(B^\circ) = \bigcap_{j>0} \mathcal{SB}_j, \qquad \mathcal{SG}(B) = \mathcal{A} \cap \mathcal{HA} \cap \mathcal{G}(B^\circ), \qquad \mathcal{G}(T_i^{\pm}) = \bigcap_{j>0} \mathcal{ST}_{i,j}^{\pm}.$$

Finally, we set $\mathcal{SG}(V^+) = \mathcal{SG}(B) \cap \bigcap_{i \in [2k]} \mathcal{G}(T_i^+)$ and $\mathcal{SG}(V) = \mathcal{SG}(V^+) \cap \mathcal{SG}(V^-)$, with $\mathcal{SG}(V^-)$ the analog of $\mathcal{SG}(V^+)$ for the left-snail.

We note that the event $\mathcal{HA}$ is there only to ensure the easy removal of the simplifying Assumption 4.1.

REMARK 4.3. The events above are defined so as to preserve as much as possible the original product structure of μ in the conditional measure $\mu_V(\cdot \mid \mathcal{SG}(V))$. In fact,

$$\begin{aligned}
\mu_{V^\pm}(\cdot \mid \mathcal{SG}(V^\pm)) &= \mu_B(\cdot \mid \mathcal{SG}(B)) \otimes \Big(\bigotimes_{i \in [2k]} \mu_{T_i^\pm}(\cdot \mid \mathcal{G}(T_i^\pm)) \Big), \\
\mu_{T_i^\pm}(\cdot \mid \mathcal{G}(T_i^\pm)) &= \bigotimes_{j>0} \mu_{ST_{i,j}^\pm}(\cdot \mid \mathcal{ST}_{i,j}^\pm), \\
\mu_B(\cdot \mid \mathcal{SG}(B)) &= \mu_{A^{\mathrm{int}}} \otimes \delta_{\omega_{A \cup (HA+Lu_0)}=0} \otimes \mu_{B^\circ}(\cdot \mid \mathcal{G}(B^\circ)), \\
\mu_{B^\circ}(\cdot \mid \mathcal{G}(B^\circ)) &= \bigotimes_{j>0} \mu_{SB_j}(\cdot \mid \mathcal{SB}_j),
\end{aligned}$$

since trapezoids and the base are pairwise disjoint by construction and likewise for the slices of the trapezoids, the slices of the base, the annulus, its interior and the translated half-annulus.

Taking into account this product structure, in the next observations we establish that, as claimed in Section 4.1, all $\mathcal{G}$ events we will use are likely and all $\mathcal{SG}$ events have roughly the same probability, $q^{\Theta(Rw)}$.

OBSERVATION 4.4. Let $R \geq w^2 \log(1/q)/q^\alpha$ and $L \leq q^{4w}$, let $\underline{r}$ be admissible (see Definition 3.4) and $r_{i-1} \geq q^{-2w}$ for some $i \in [2k]$. Then $\mu(\mathcal{G}(T_i^\pm)) \geq 1 - o(1)$ and $\mu(\mathcal{G}(B^\circ)) \geq 1 - o(1)$.

PROOF. For the first assertion, notice that the condition implies that for all $j > 0$, $ST_{i,j}^\pm$ is either empty or has cardinality at least $\Omega(q^{-2w})$. Then by Remark 4.3,

$$\mu(\mathcal{G}(T_i^\pm)) = \prod_j \mu(\mathcal{ST}_{i,j}^\pm) \geq \big(1 - (1 - q^w)^{\Omega(q^{-2w}/w)}\big)^{O(r_i)} \geq 1 - o(1),$$

since $r_i \leq L \leq q^{4w}$ by Definition 3.4 and by assumption.

The second assertion is proved similarly (see, e.g., [20], Lemma 6.5). □

OBSERVATION 4.5. Let $R \geq w^2 \log(1/q)/q^\alpha$, $L \leq q^{-4w}$, let $\underline{r}$ be admissible such that for some $i \in [2k]$, $r_{i+1} = 0$ and $r_{i-1} \geq q^{-2w}$ with the convention $r_{-1} = L$. Then

$$\mu\big(\mathcal{SG}\big(V_L^R(\underline{r})\big)\big) = q^{\Theta(Rw)}. \tag{11}$$

PROOF. Using Remark 4.3 and Observation 4.4, it suffices to note that

$$\mu(\mathcal{A} \cap \mathcal{HA}) \geq q^{|A|+|HA|} = q^{\Theta(Rw)}.$$ □

4.3. *Key step*. We are ready to state the main result of this section. In the sequel, for any $\Lambda \subset \mathbb{Z}^2$, any $x \in \Lambda$ and any $\omega_\Lambda \in \Omega_\Lambda$ we shall write $c_x^\Lambda(\omega_\Lambda)$ for the constraint $c_x(\omega)$ computed for the configuration ω equal to ω_Λ in Λ and equal to 1 elsewhere. By construction, $c_x^\Lambda(\omega_\Lambda) \leq c_x(\omega')$ for any $\omega' \in \Omega$ such that $\omega'_\Lambda = \omega_\Lambda$ and $c_x^\Lambda \geq c_x^{\Lambda'}$ for any $\Lambda' \subset \Lambda$. Then for any snail V (or base) we write γ_V for the smallest constant $\gamma \in [1, \infty]$ such that the Poincaré inequality

$$\mathrm{Var}_V\big(f \mid \mathcal{SG}(V)\big) \leq \gamma \sum_{x \in V} \mu_V\big(c_x^V \mathrm{Var}_x(f)\big) \tag{12}$$

holds for every function $f : \Omega \to \mathbb{R}$.

THEOREM 4.6. *There exist $w_0, \delta_0 > 0$ not depending on q such that for any $0 < \delta \leq \delta_0$ and $w \geq w_0$ the following holds for any $R = \Theta(w^2 \log(1/q)/q^\alpha)$. Consider the snail $V = V_L^R(\underline{r})$ for admissible $L, \underline{r}$ such that $r_{2k-1} \geq q^{-2w}$ and $L \leq q^{-4w}$. Then*

$$\gamma_V \leq e^{-O(w^4 \log^3(1/q)/q^\alpha)}. \tag{13}$$

CLAIM 4.7. *We have $\gamma_V \leq 3 \max(\gamma_{V^+}, \gamma_{V^-})$.*

PROOF. Set $\Lambda_1 = V^+ \setminus (B \cup T_0^+)$, $\Lambda_2 = V^- \setminus (B \cup T_0^+)$, $\Lambda_3 = B \cup T_0^+ = B \cup T_0^-$, $\nu_1 = \mu_{\Lambda_1}(\cdot \mid \bigcap_{i=1}^{2k-1} \mathcal{G}(T_i^+))$, similarly for ν_2 and $\nu_3 = \mu_{\Lambda_3}(\cdot \mid \mathcal{SG}(B) \cap \mathcal{G}(T_0^+))$. By Remark 4.3, we can apply Lemma 3.9 to obtain

$$\begin{aligned}
\mathrm{Var}_V\big(f \mid \mathcal{SG}(V)\big) &\leq \gamma_{V^-} \sum_{x \in V^-} \nu_1\big(\mu_{V^-}\big(c_x^{V^-} \mathrm{Var}_x(f)\big)\big) \\
&\quad + \gamma_{V^+} \sum_{x \in V^+} \nu_2\big(\mu_{V^+}\big(c_x^{V^+} \mathrm{Var}_x(f)\big)\big) \\
&\leq \big(1 + o(1)\big)(\gamma_{V^-} + \gamma_{V^+}) \sum_{x \in V} \mu\big(c_x^V \mathrm{Var}_x(f)\big),
\end{aligned}$$

where in the last inequality we used Observation 4.4 to remove the conditioning of ν_1 and ν_2. □

Therefore, in order to prove (13) it suffices to prove the analogous statement with V replaced by $V^{\pm}$. In the sequel, we will concentrate on proving (13) for the best constant γ_{V^+} in the Poincaré inequality (12) with V replaced by its right-snail V^+. The proof is based on comparison methods between Markov processes and induction over right-snails with different L and $\underline{r}$ as outlined in Section 4.1. If we exchange right-snails with left-snails the same proof will then apply to the left-snail V^- as well. Since our arguments no longer require a left-snail, for lightness of notation, we drop the superscript "+" from our notation whenever possible.

The proof of the theorem is decomposed into two quite different steps (see Propositions 4.9 and 4.12 below). In the first one, labeled the *base case*, we consider a right-snail V with no trapezoids ($\underline{r} = 0$). In the second step, labeled *reduction step*, roughly speaking we compare the Poincaré constant γ_V of a generic right-snail V with the same constant computed for its base B.

The conclusion of Theorem 4.6 follows at once from (11), Proposition 4.9 and Proposition 4.12. In the sequel fix δ, w, R as in the statement of the theorem and recall that $B = V_L^R(0)$.

REMARK 4.8. For future purposes (see the discussion in Section 6), it is very important to emphasise that it is only in the first step that we use directly the definition of the event $\mathcal{SG}(B)$ entering in the event $\mathcal{SG}(V)$ (cf. Definition 4.2). In the second step, the only property of the event $\mathcal{SG}(B)$ that is needed is that it is a decreasing event in Ω_B w.r.t. the partial order $\omega \prec \omega'$ iff $\omega_x \le \omega'_x$ for all $x \in B$.

4.4. *Base case.*

PROPOSITION 4.9. *For any* $f : \Omega_B \to \mathbb{R}$,

$$\operatorname{Var}_B\big(f \mid \mathcal{SG}(B)\big) \le q^{-O(Rw\log L)} \sum_{x\in B} \mu_B\big(c_x^B \operatorname{Var}_x(f)\big).$$

PROOF OF PROPOSITION 4.9. We first observe that, up to minor modifications, in [20], Proposition 6.6, it was proved that for all $f : \Omega_{B^\circ} \to \mathbb{R}$,

$$\mathbb{1}_{\mathcal{A}\cap\mathcal{H}\mathcal{A}} \operatorname{Var}_{B^\circ}\big(f \mid \mathcal{G}(B^\circ)\big) \le q^{-O(Rw\log L)} \mathbb{1}_{\mathcal{A}} \sum_{x\in B^\circ} \mu_{B^\circ}\big(c_x^B \operatorname{Var}_x(f)\big). \tag{14}$$

The next step in the proof is an analogous result for A^{int}.

LEMMA 4.10. *For any* $f : \Omega_{A^{\mathrm{int}}} \to \mathbb{R}$,

$$\mathbb{1}_{\mathcal{A}} \operatorname{Var}_{A^{\mathrm{int}}}(f) \le q^{-O(Rw)} \mathbb{1}_{\mathcal{A}} \sum_{x\in A^{\mathrm{int}}} \mu_{A^{\mathrm{int}}}\big(c_x^{A\cup A^{\mathrm{int}}} \operatorname{Var}_x(f)\big).$$

PROOF OF LEMMA 4.10. Let us partition A^{int} into disjoint strips K_i of width w perpendicular to u_0 and number them from left to right (see Figure 8).

We can then apply [20], Proposition 3.4, on the generalized FA1f KCM to obtain

$$\operatorname{Var}_{A^{\mathrm{int}}}(f) \le q^{-O(Rw)} \sum_i \mu\big((\mathbb{1}_{\mathcal{H}_i^+} + \mathbb{1}_{\mathcal{H}_i^-}) \operatorname{Var}_{K_i}(f)\big),$$

where $\mathcal{H}_i^{\pm}$ are the events that $K_{i\pm1}$ is fully infected and we use the convention that $\mathcal{H}_i^+$ occurs for the last strip and $\mathcal{H}_i^-$ does for the first one, which corresponds to the boundary condition

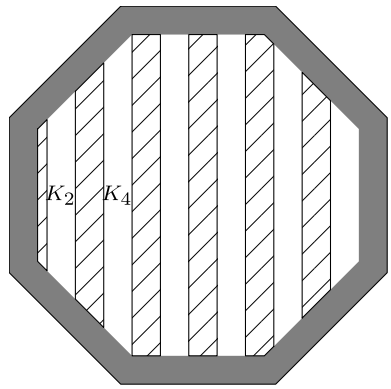


FIG. 8. *Setting of the proof of Lemma 4.10. Every second strip K_i of A^{int} is hatched. The annulus A is shaded.*

provided by A. W.l.o.g. it then suffices to bound the generic term $\mu(\mathbb{1}_{\mathcal{H}_i^+}\operatorname{Var}_{K_i}(f))$. But this can be done using Lemma 5.2 of [20] and Lemma 3.7(b), which guarantees that if A and K_{i+1} are infected, then K_i can also be infected by the $\mathcal{U}$-bootstrap percolation restricted to $K_i \cup K_{i+1} \cup A$. □

Using Lemma 3.9 with $\Lambda_1 = B^\circ$, $\Lambda_2 = A^{\text{int}}$, $\Lambda_3 = \varnothing$, $\nu_1 = \mu_{B^\circ}(\cdot \mid \mathcal{G}(B^\circ))$, and $\nu_2 = \mu_{A^{\text{int}}}$, we obtain

$$\begin{aligned}&\mathbb{1}_{\mathcal{A}\cap\mathcal{H}\mathcal{A}}\operatorname{Var}_{B^\circ\cup A^{\text{int}}}(f \mid \mathcal{G}(B^\circ))\\&\quad\le \mathbb{1}_{\mathcal{A}\cap\mathcal{H}\mathcal{A}}\big(\mu_{A^{\text{int}}}(\operatorname{Var}_{B^\circ}(f \mid \mathcal{G}(B^\circ))) + \mu_{B^\circ}(\operatorname{Var}_{A^{\text{int}}}(f) \mid \mathcal{G}(B^\circ))\big).\end{aligned}$$

The first term in the right-hand side above is bounded by

$$q^{-O(Rw\log L)}\mathbb{1}_{\mathcal{A}\cap\mathcal{H}\mathcal{A}}\sum_{x\in B^\circ}\mu_{B^\circ\cup A^{\text{int}}}(c_x^B\operatorname{Var}_x(f)),$$

using (14), while the second one is bounded by

$$q^{-O(Rw\log L)}\mathbb{1}_{\mathcal{A}\cap\mathcal{H}\mathcal{A}}\sum_{x\in A^{\text{int}}}\mu_{B^\circ\cup A^{\text{int}}}(c_x^{A\cup A^{\text{int}}}\operatorname{Var}_x(f) \mid \mathcal{G}(B^\circ))$$

by Lemma 4.10. By Remark 4.3, we immediately get that

$$\begin{aligned}\operatorname{Var}_B(f \mid \mathcal{S}\mathcal{G}(B)) &= \mu_B(\operatorname{Var}_{B^\circ\cup A^{\text{int}}}(f \mid \mathcal{G}(B^\circ)) \mid \mathcal{A}\cap\mathcal{H}\mathcal{A})\\&\le \mu(\mathcal{S}\mathcal{G}(B))^{-1}q^{-O(Rw\log L)}\sum_{x\in B}\mu_B(c_x^B\operatorname{Var}_x(f))\end{aligned}$$

and the proposition follows from Observation 4.5. □

4.5. *Reduction step.* Before we can state a relationship between γ_V and γ_B, we need the following notion, which will cover all snail shapes that may arise during the reduction. Recall that δ, w, $V = V_L^{R,+}(\underline{r})$ are fixed as in the statement of Theorem 4.6 and that we do not write the $+$ index, though all snails we refer to are right-snails. Also recall that all snails are defined by admissible sequences (see Definition 3.4).

DEFINITION 4.11. Let C be a constant chosen sufficiently large depending on $\widehat{\mathcal{S}}$, but much smaller than $1/\delta_0$ in Theorem 4.6. We say that a snail $\widehat{V} = V_{\widehat{L}}^R(\widehat{\underline{r}})$ is *of type* $i \in [2k]$ if:

(a) $\widehat{r}_{i+1} = 0$,

(b) $\widehat{r}_i \le r_i$,
(c) for all $j < i$ it holds that $0 \le r_j - \widehat{r}_j \le C(r_i - \widehat{r}_i + \sum_{l=i+1}^{2k-1} r_l)$,
(d) $0 \le L - \widehat{L} \le C(r_i - \widehat{r}_i + \sum_{l=i+1}^{2k-1} r_l)$.

We say that $\widehat{V}$ is *relevant* if there exists $i \in [2k]$ such that $\widehat{V}$ is of type i. In particular, a base $\widehat{B} = V_{\widehat{L}}^R(0)$ is relevant iff $0 \le L - \widehat{L} = O(r_0)$.

In words, $\widehat{V}$ is relevant if all trapezoids except the last one are only slightly shorter than the corresponding ones for V and similarly for the base, while the last trapezoid may be as much shorter as needed. Indeed, observe that by admissibility $\sum_{l=i+1}^{2k-1} r_l < 2r_{i+1}$ for any $i \in [2k]$.

Let us mention that the technical second inequalities in conditions (c) and (d) in the definition above are only needed for the inductive procedure below to always yield relevant snails. We invite the reader to ignore those conditions and admit that all smaller snails arising in our argument have sizes which can be treated by induction.

PROPOSITION 4.12. *Let* $\sigma = 1/\min_{\widehat{V}} \mu_{\widehat{V}}(\mathcal{SG}(\widehat{V}))$ *and* $\Gamma = \max_{\widehat{B}} \gamma_{\widehat{B}}$, *where the* min *and* max *run over relevant snails and relevant bases, respectively. Then*

$$\gamma_V \le \left(q^{-w^4}\sigma\right)^{O(\log L)}\Gamma.$$

In the rest of the section, we slowly build the proof of this proposition. The first step of reduction consists in removing a trapezoid consisting of a single slice. This is done using Lemma 3.12 and may be intuitively understood as an FA1f dynamics of w consecutive infected sites in the slice. Recall the definition of ρ_i from Section 3.1.1.

LEMMA 4.13 (Removing a single slice). *Let* $\widehat{V} = V_{\widehat{L}}^R(\underline{\widehat{r}})$ *be a snail of type* i *such that* $\widehat{r}_i = \lambda\rho_{k+i}$ *for* $\lambda \in \mathbb{N}$. *In other words, the last nonempty trapezoid,* $\widehat{T}_i$ *of* $\widehat{V}$ *consists of* λ *segments orthogonal to* u_{i+k}. *Then, setting* $\underline{\widetilde{r}} = (\widehat{r}_0, \dots, \widehat{r}_{i-1}, 0, \dots, 0)$, $\widetilde{V} = V_{\widehat{L}}^R(\underline{\widetilde{r}})$, *we have*

$$\gamma_{\widehat{V}} \le \left(q^{-w^4}\right)^{O(\lambda)} \max\left(\gamma_{\widetilde{V}}, 1/\mu_{\widehat{V}}(\mathcal{SG}(\widehat{V}))\right).$$

PROOF OF LEMMA 4.13. By induction on λ, it suffices to prove the lemma for $\lambda = 1$, in which case the last trapezoid is simply $\widehat{T}_i = \widehat{ST}_{i,1}$.

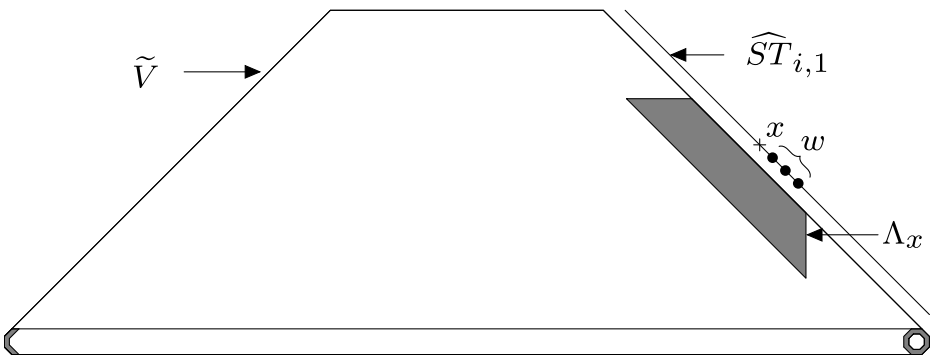


FIG. 9. *The geometric setting of Lemma 4.13. The snail is* $\widetilde{V}$, *while* $\widehat{ST}_{i,1} = \widehat{V} \setminus \widetilde{V}$ *is the remaining slice on the top-right. The site* x *to be updated in* (17) *is marked by a cross. The event* $\widetilde{\mathcal{H}}_x$ *corresponds to the shaded trapezoid* Λ_x *being infected and the event* $\mathcal{H}_x$ *corresponds to the* w *consecutive sites next to* x *on one of its sides being infected.*

We will proceed in two steps. First, we will divide $\widehat{V}$ into $\widetilde{V}$ and $\widehat{ST}_{i,1}$. The $\widetilde{V}$ part is harmless, as it directly relates to $\gamma_{\widetilde{V}}$ appearing in the right-hand side of the statement of the lemma. In order to reproduce a "resampling" of $\widehat{ST}_{i,1}$ we will proceed in two steps. First, using the FA1f-like dynamics, Lemma 3.12, we will reduce the problem to resampling a single site in $\widehat{ST}_{i,1}$ given that next to it there are w consecutive infections. Then we will use $\widetilde{V}$ to provide additional infections to ensure that $c_x^{\widehat{V}}$ is satisfied and this will yield the x term of the Dirichlet form from (8). The lemma is illustrated in Figure 9.

Recalling Remark 4.3, for any $f : \Omega_{\widehat{V}} \to \mathbb{R}$ Lemma 3.9 gives

$$\operatorname{Var}_{\widehat{V}}\big(f \mid \mathcal{SG}(\widehat{V})\big) \le \mu_{\widehat{V}}\big(\operatorname{Var}_{\widetilde{V}}\big(f \mid \mathcal{SG}(\widetilde{V})\big) + \operatorname{Var}_{\widehat{ST}_{i,1}}\big(f \mid \widehat{\mathcal{ST}}_{i,1}\big) \mid \mathcal{SG}(\widehat{V})\big). \tag{15}$$

Since $\mathcal{SG}(\widehat{V}) = \mathcal{SG}(\widetilde{V}) \times \widehat{ST}_{i,1}$, the first term in the right-hand side above is

$$\begin{aligned}\mu_{\widehat{ST}_{i,1}}\big(\operatorname{Var}_{\widetilde{V}}\big(f \mid \mathcal{SG}(\widetilde{V})\big) \mid \widehat{\mathcal{ST}}_{i,1}\big) &\le \frac{\gamma_{\widetilde{V}}}{\mu_{\widehat{ST}_{i,1}}(\widehat{\mathcal{ST}}_{i,1})}\mu_{\widehat{V}}\Big(\sum_{x\in\widetilde{V}} c_x^{\widehat{V}} \operatorname{Var}_x(f)\Big)\\ &= \big(1+o(1)\big)\gamma_{\widetilde{V}} \sum_{x\in\widetilde{V}} \mu_{\widehat{V}}\big(c_x^{\widehat{V}} \operatorname{Var}_x(f)\big)\end{aligned}$$

by the definition (12) of $\gamma_{\widetilde{V}}$, Observation 4.4, and the fact that $c_x^{\widetilde{V}} \le c_x^{\widehat{V}}$. To bound the second term in (15), we use Lemma 3.12 for $\mu_{\widehat{ST}_{i,1}}(\cdot \mid \widehat{\mathcal{ST}}_{i,1})$ with $\kappa = w$ and constraining event $\mathcal{H} = \{0\} \subset \{0,1\} = \widehat{S}$, the hypothesis of the lemma following from Observation 4.4. This gives

$$\operatorname{Var}_{\widehat{ST}_{i,1}}(f \mid \widehat{\mathcal{ST}}_{i,1}) \le q^{-O(w)} \sum_{x\in\widehat{ST}_{i,1}} \mu_{\widehat{ST}_{i,1}}\big(\mathbb{1}_{\mathcal{H}_x} \operatorname{Var}_x(f)\big), \tag{16}$$

where $\mathcal{H}_x$ is the event that w consecutive sites immediately to the left or to the right of x in $\widehat{ST}_{i,1}$ are infected. Plugging this back in (15), we see that we need to bound from above a generic term

$$\mu_{\widehat{V}}\big(\mathbb{1}_{\mathcal{H}_x} \operatorname{Var}_x(f) \mid \mathcal{SG}(\widetilde{V})\big), \quad x \in \widehat{ST}_{i,1}. \tag{17}$$

At this point, we have succeeded in bringing w consecutive infected sites next to the site x, which we want to update. In order to be sure that the constraint $c_x^{\widehat{V}}$ is satisfied, we would like to also bring some infections next to x in $\widetilde{V}$. To do that, we first use Lemma 3.9 to include $\widetilde{V}$ in the variance, so that we are allowed to "resample" it and then use the two-block dynamics, Lemma 3.10, to indeed obtain the desired infections by resampling $\widetilde{V}$ enough times.

Applying Lemma 3.9 to $\Lambda_1 = \widetilde{V}$, $\Lambda_2 = \{x\}$, $\Lambda_3 = \varnothing$, $\nu_1 = \mu_{\widetilde{V}}(\cdot \mid \mathcal{SG}(\widetilde{V}))$ and $\nu_2 = \mu_x$, we bound the generic term (17) from above by

$$\mu_{\widehat{ST}_{i,1}\setminus\{x\}}\big(\mathbb{1}_{\mathcal{H}_x} \operatorname{Var}_{\widetilde{V}\cup\{x\}}\big(f \mid \mathcal{SG}(\widetilde{V})\big)\big).$$

We next apply Lemma 3.10 to the product space

$$\big(\Omega_{\widetilde{V}}, \mu_{\widetilde{V}}(\cdot \mid G_{\widetilde{V}})\big) \otimes (\Omega_{\{x\}}, \mu_x)$$

with constraining event $\widetilde{\mathcal{H}}_x \subset \Omega_{\widetilde{V}}$ that the trapezoid

$$\Lambda_x = x + \big(\overline{\mathbb{H}}_{u_{i+k-1}}(w^2) \cap \mathbb{H}_{u_{i+k}} \cap \overline{\mathbb{H}}_{u_{i+k+1}}(w^2) \cap \mathbb{H}_{u_{i+3k}}(w)\big) \cap \widetilde{V}$$

being infected. It is not hard to check that $\partial_{u_{i+k}}\Lambda_x$ contains x and the w infected sites guaranteed by $\mathcal{H}_x$. In other words, we are in the setting of Figure 9. Using $|\Lambda_x \cap \mathbb{Z}^2| = O(w^4)$ and noticing that by the Harris inequality [11] $\mu_{\widetilde{V}}(\widetilde{\mathcal{H}}_x \mid \mathcal{SG}(\widetilde{V})) \ge \mu_{\widetilde{V}}(\widetilde{\mathcal{H}}_x)$, Lemma 3.10 gives

$$\begin{aligned}\mathbb{1}_{\mathcal{H}_x} \operatorname{Var}_{\widetilde{V}\cup\{x\}}\big(f \mid \mathcal{SG}(\widetilde{V})\big) \le q^{-O(w^4)}\mu_{\widetilde{V}\cup\{x\}}\big(&\operatorname{Var}_{\widetilde{V}}\big(f \mid \mathcal{SG}(\widetilde{V})\big)\\ &+ \mathbb{1}_{\mathcal{H}_x\cap\widetilde{\mathcal{H}}_x} \operatorname{Var}_x(f) \mid \mathcal{SG}(\widetilde{V})\big).\end{aligned}$$

Finally, since u_{i+k} is an isolated (quasi-)stable direction, it is easily seen (see Figure 9 and Lemma 3.7) that $\mathbb{1}_{\mathcal{H}_x \cap \widetilde{\mathcal{H}}_x} \le c_x^{\widehat{V}}$. Recalling the definition of the Poincaré constant $\gamma_{\widehat{V}}$ (see (12)), we conclude that

$$\begin{aligned}&\mu_{\widehat{ST}_{i,1}\setminus\{x\}}\big(\mathbb{1}_{\mathcal{H}_x}\operatorname{Var}_{\widetilde{V}\cup\{x\}}(f \mid \mathcal{SG}(\widetilde{V}))\big)\\&\quad\le q^{-O(w^4)}\big(\gamma_{\widetilde{V}} + 1/\mu\big(\mathcal{SG}(\widetilde{V})\big)\big)\sum_{y\in\widetilde{V}\cup\{x\}}\mu_{\widehat{V}}\big(c_y^{\widehat{V}}\operatorname{Var}_y(f)\big).\end{aligned}$$

Putting all together, we finally get

$$\begin{aligned}\operatorname{Var}_{\widehat{V}}\big(f \mid \mathcal{SG}(\widehat{V})\big) \le |\widehat{ST}_{i,1}|q^{-O(w^4)}\max\big(\gamma_{\widetilde{V}}, 1/\mu_{\widetilde{V}}\big(\mathcal{SG}(\widetilde{V})\big)\big)\\ \times\sum_{x\in\widehat{V}}\mu_{\widehat{V}}\big(c_x^{\widehat{V}}\operatorname{Var}_x(f)\big),\end{aligned}$$

where the factor $|\widehat{ST}_{i,1}| = O(\widehat{L}) = O(q^{-4w})$ comes from the fact that each vertex $x \in \widehat{ST}_{i,1}$ produces a term of the form $\sum_{y\in\widetilde{V}}\mu_{\widehat{V}}(c_y^{\widehat{V}}\operatorname{Var}_y(f))$. □

The remaining induction step allows us to reduce the size of the last nonempty trapezoid $\widehat{T}_i$ twice. The proof is illustrated in Figure 10.

LEMMA 4.14 (Bisection of a trapezoid). *Let $\widehat{V} = V_{\widehat{L}}^R(\underline{\widehat{r}})$ be a snail of type i such that $\widehat{r}_i$ is larger than some sufficiently large constant. Let $\lambda = \min\{\ell > 0, \ell u_{i+1} \in \mathbb{Z}^2\} = O(1)$ and let $x = u_{i+1}\lambda\lfloor \widehat{r}_i/(2\lambda\langle u_{i+k}, u_{i+1}\rangle)\rfloor$. With this choice, $\langle u_{i+k}, x\rangle \simeq \widehat{r}_i/2$. In other words, x is the vector by which the ring should be translated so that half of the last trapezoid, $\widehat{T}_i$, remains above it (see Figure 10). Then we set*:

$$\begin{aligned}\underline{\widetilde{r}} &= \big(\widehat{r}_0, \dots, \widehat{r}_{i-1}, \langle u_{i+k}, x\rangle, 0, \dots, 0\big),\\ \underline{\overline{r}} &= \big(\widehat{r}_0 - \langle u_k, x\rangle, \dots, \widehat{r}_i - \langle u_{k+i}, x\rangle, 0, \dots, 0\big),\\ \overline{L} &= \min\Big(\widehat{L}, \widehat{L} - \frac{\langle u_{k-1}, x\rangle}{\langle u_{k-1}, u_0\rangle}\Big),\\ \widetilde{V} &= V_{\widehat{L}}^R(\underline{\widetilde{r}}),\\ \overline{V} &= x + V_{\overline{L}}^R(\underline{\overline{r}}).\end{aligned}$$

In words, $\widetilde{V}$ is $\widehat{V}$ with half of $\widehat{T}_i$ removed, while $\overline{V}$ is the snail such that its last trapezoid $\overline{T}_i$ is exactly that missing half, but with length eventually shortened, so that $\overline{V}$ fits inside $\widehat{V}$ (see Figure 10). With these notation,

$$\gamma_{\widehat{V}} \le \gamma_{\widetilde{V}}/\mu(G_{\overline{V}}) + \gamma_{\overline{V}}/\mu(G_{\widetilde{V}})$$

and $\widetilde{V}$ and $\overline{V}$ are snails of type i.

PROOF OF LEMMA 4.14. The proof goes as follows. In Claim 4.15, we show that the two polygons $\widetilde{V}$ and $\overline{V}$ are indeed snails (defined by admissible sequences) of type i and that they do correspond to their informal definitions in the statement of the lemma. Though technical, this claim hides no subtlety and we invite the reader to skip it. Then we apply Lemma 3.10 to reduce the problem of relaxation on $\widehat{V}$ to the one on $\widetilde{V}$ and on $\overline{V}$ which yields the desired result. The event $\mathcal{SG}(\widetilde{V})$ is implied by $\mathcal{SG}(\widehat{V})$ by construction, but the second block, $\overline{V}$, of the dynamics corresponding to Lemma 3.10, is updated only when the part of $\mathcal{SG}(\overline{V})$ witnessed in $\widetilde{V}$ occurs.

We begin with some geometric observations following directly from Definitions 3.4 and 3.5.

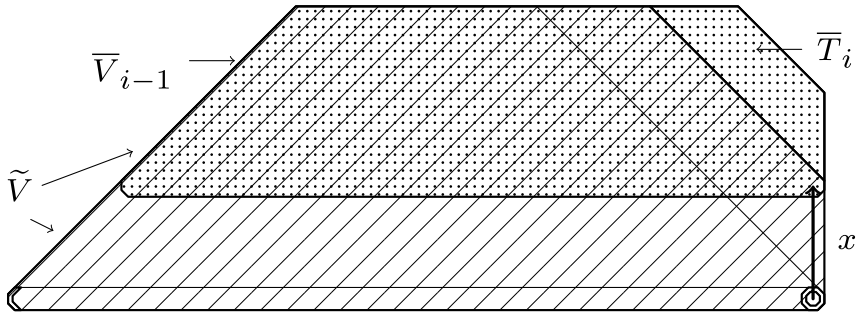


FIG. 10. *The geometric setting of Lemma 4.14. The snail $\widetilde{V}$ is hatched, $\overline{V}$ is dotted and their union is the original snail $\widetilde{V}$. The dotted-hatched region is $\overline{V}_{i-1}$, while the dotted trapezoid is $\overline{T}_i$.*

CLAIM 4.15. *$\widetilde{V}$ and $\overline{V}$ are snails of type i. Furthermore, we have $\widetilde{V} \cup \overline{V} = \widehat{V}$ and $\widehat{V} \setminus \widetilde{V} = \overline{T}_i$ (the last trapezoid of $\overline{V}$).*

PROOF OF THE CLAIM. The statement that $\widetilde{V}$ is a snail of type i follows from the definition of $\underline{\widetilde{r}}$ and the same fact for $\widehat{V}$, since $\widetilde{r}_i \le \widehat{r}_i$.

Turning to $\overline{V}$, notice that $\langle u_j, x\rangle \ge 0$ for all $i \in [2k]$ and $j \in [k, k+i]$ with equality iff $i = 2k-1$ and $j = k$. Thus, for all $j \in [2k]$ we have $\widehat{r}_j \ge \overline{r}_j$, and clearly $\widehat{L} \ge \overline{L}$. Thus, recalling the definition of $\overline{V}$ and that $\widehat{V}$ is of type i, conditions (a) and (b) and the left inequalities in (c) and (d) of Definition 4.11 are satisfied. Moreover, for $0 \le j < i$ we have

$$\widehat{r}_j - \overline{r}_j = \langle u_{k+j}, x\rangle = \frac{\langle u_{i+1}, u_{k+j}\rangle}{\langle u_{i+1}, u_{k+i}\rangle}(\widehat{r}_i - \overline{r}_i) \le C(\widehat{r}_i - \overline{r}_i), \tag{18}$$

so the right inequality of (c) for $\overline{V}$ follows from the one for $\widehat{V}$. Similarly,

$$\widehat{L} - \overline{L} \le \frac{|\langle u_{k-1}, x\rangle|}{\langle u_{k-1}, u_0\rangle} \le C(\widehat{r}_i - \overline{r}_i), \tag{19}$$

gives that (d) of Definition 4.11 holds for $\overline{V}$.

We next prove that $\overline{V}$ is a snail (with admissible $\overline{L}$ and $\underline{\overline{r}}$). Recalling from Definition 3.4 that we need to prove that:

(i) $\overline{r}_j \ge 0$ for all $j \in [2k]$,
(ii) $\overline{r}_j \le \delta \overline{r}_{j-1}$ for all $1 \le j < 2k$,
(iii) $\overline{r}_0 \le \delta \overline{L}$, and
(iv) $(\widehat{L} - \overline{L})\langle u_0, u_{k-1}\rangle / \rho_{k-1} \in \mathbb{N}$.

To check (i), observe that $\overline{r}_j = \widehat{r}_j - O(\widehat{r}_i) > 0$ for $j < i$ by admissibility of $\underline{\widehat{r}}$ and $\overline{r}_i \simeq \widetilde{r}_i/2 > 0$. By admissibility of $\widehat{V}$, $\widehat{r}_j - \overline{r}_j = \langle u_{k+j}, x\rangle = \Theta(\widehat{r}_i)$ and $C < 1/\delta$, we get (ii). For the last two properties, we consider two cases.

First, assume that $i \in \{2k-2, 2k-1\}$ (i.e., x corresponds to a horizontal translation to the right—in direction u_{2k}). It is easy to check from the definition of $\overline{L}$ that in this case $\overline{L} = \widehat{L}$, so that (iv) is trivial and (iii) follows from $\overline{r}_0 \le \widehat{r}_0 \le \delta \widehat{L}$. This concludes the proof that $\overline{V}$ is a snail of type i in this case.

Assume that, on the contrary, $i < 2k-2$, so that the $\widehat{L} - \overline{L} = \frac{\langle u_{k-1}, x\rangle}{\langle u_{k-1}, u_0\rangle} = \Theta(\widehat{r}_i)$. Then (iii) follows from the fact that $\widehat{r}_0 - \overline{r}_0 = \Theta(\widehat{r}_i)$ as above. For (iv), simply observe that

$$(\widehat{L} - \overline{L})\langle u_0, u_{k-1}\rangle = \langle u_{k-1}, x\rangle \in \rho_{k-1}\mathbb{N},$$

since $x \in \mathbb{Z}^2$ by the definition of x and λ. This concludes the proof that $\overline{V}$ is a snail of type i.

By Definition 3.5, it is clear that

$$\widehat{V} \setminus \overline{V} = \big(\overline{\mathbb{H}}_{u_{k+i}}(R_{k+i} + \widehat{r}_i) \setminus \overline{\mathbb{H}}_{u_{k+i}}(R_{k+i} + \langle u_{k+i}, x\rangle)\big) \\ \cap \overline{\mathbb{H}}_{u_{k+i-1}}(R_{k+i-1} + \widehat{r}_{i-1}) \cap \overline{\mathbb{H}}_{u_{k+i+1}}(R_{k+i+1}).$$

It also follows from Definition 3.5 that the above trapezoid $\widehat{V} \setminus \overline{V}$ is also equal to $\overline{T}_i$ as claimed. Finally, we have that $\overline{V} \subset \widehat{V}$ using Definition 3.4, which completes the proof of the claim. □

Let now

$$\overline{V}_{i-1} = V_L^R(\overline{r}_0, \dots, \overline{r}_{i-1}, 0, \dots, 0) = \overline{V} \setminus \overline{T}_i = \overline{V} \cap \widetilde{V}.$$

By Claim 4.15 and Remark 4.3, we have

$$(\Omega_V, \mu_{\widehat{V}}(\cdot \mid G_{\widehat{V}})) = (\Omega_{\widetilde{V}}, \mu_{\widetilde{V}}(\cdot \mid \mathcal{SG}(\widetilde{V}))) \otimes (\Omega_{\overline{T}_i}, \mu_{\overline{T}_i}(\cdot \mid \mathcal{G}(\overline{T}_i))) \tag{20}$$

and we can apply Lemma 3.10 with the facilitating event

$$\mathcal{SG}(\overline{V}_{i-1}) = \mathcal{SG}(\overline{B}) \cap \bigcap_{j<i} \mathcal{G}(\overline{T}_j) \subset \Omega_{\widetilde{V}},$$

where $\overline{B}$ and $\overline{T}_j$ are the base and trapezoids of $\overline{V}$. We get

$$\begin{aligned} \operatorname{Var}_{\widehat{V}}(f \mid \mathcal{SG}(\widehat{V})) &\le \mu(\mathcal{SG}(\overline{V}_{i-1}))^{-1} \mu_{\widehat{V}}\big(\operatorname{Var}_{\widetilde{V}}(f \mid \mathcal{SG}(\widetilde{V})) \\ &\quad + \mathbb{1}_{\mathcal{SG}(\overline{V}_{i-1})} \operatorname{Var}_{\overline{T}_i}(f \mid \mathcal{G}(\overline{T}_i)) \mid \mathcal{SG}(\widehat{V})\big), \end{aligned} \tag{21}$$

where we used that $\mu_{\widetilde{V}}(\mathcal{SG}(\overline{V}_{i-1}) \mid \mathcal{SG}(\widetilde{V})) \ge \mu(\mathcal{SG}(\overline{V}_{i-1}))$ by the Harris inequality. Using the definition of the Poincaré constant $\gamma_{\widetilde{V}}$, the fact that $c_x^{\widetilde{V}} \le c_x^V$ together with $\mu(\mathcal{SG}(\overline{V}_{i-1})) \ge \mu(\mathcal{SG}(\overline{V}))$ the first term is bounded from above by

$$\frac{\gamma_{\widetilde{V}}}{\mu(\mathcal{SG}(\overline{V}))} \mu_{\widehat{V}}\Big(\sum_{x \in \widetilde{V}} c_x^{\widehat{V}} \operatorname{Var}_x(f)\Big). \tag{22}$$

The term $\mu(\mathcal{SG}(\overline{V}_{i-1}))^{-1} \mu_{\widehat{V}}(\mathbb{1}_{\mathcal{SG}(\overline{V}_{i-1})} \operatorname{Var}_{\overline{T}_i}(f \mid \mathcal{G}(\overline{T}_i)) \mid \mathcal{SG}(\widehat{V}))$ from the right-hand side of (21) can be bounded from above by

$$\begin{aligned} &\mu(\mathcal{SG}(\widetilde{V}))^{-1} \mu_{\widetilde{V}}(\mu_{\overline{V}_{i-1}}(\operatorname{Var}_{\overline{T}_i}(f \mid \mathcal{G}(\overline{T}_i)) \mid \mathcal{SG}(\overline{V}_{i-1}))) \\ &\le \mu(\mathcal{SG}(\widetilde{V}))^{-1} \mu_{\widetilde{V}}(\operatorname{Var}_{\overline{V}}(f \mid \mathcal{SG}(\overline{V}))) \\ &\le \frac{\gamma_{\overline{V}}}{\mu(\mathcal{G}(\widehat{V}))} \mu_{\widetilde{V}}\Big(\sum_{x \in \overline{V}} c_x^{\widehat{V}} \operatorname{Var}_x(f)\Big), \end{aligned} \tag{23}$$

using (20), Lemma 3.9, the definition of $\gamma_{\overline{V}}$ and the fact that $c_x^{\overline{V}} \le c_x^{\widehat{V}}$.

If we now combine (21), (22) and (23), we get the statement of the lemma. □

We can now assemble our main induction step from Lemmas 4.13 and 4.14. Namely, we repeatedly use Lemma 4.14 until the last trapezoid is reduced to a bounded number of lines and then apply Lemma 4.13 to remove them as well.

COROLLARY 4.16 (Removing a trapezoid). *Let* $\sigma_i = 1/\min_{V_i} \mu(\mathcal{SG}(V^i))$ *with* min *running over all snails of type* i. *Let* $\Gamma_i = \max_{V_i'} \gamma_{V_i'}$, *where the* max *runs over all snails of type* i *with* $r_i = 0$. *Let* $\widehat{V} = V_L^R(\widehat{r})$ *be a snail of type* i. *Then*

$$\gamma_{\widehat{V}} \le q^{-O(w^4)} \sigma_i^{O(\max(1, \log \widehat{r}_i))} \Gamma_i.$$

PROOF OF COROLLARY 4.16. Let c be a sufficiently large constant. We prove by induction on $\widehat{r}_i$ that

$$\gamma_{\widehat{V}} \le q^{-cw^4} \sigma_i^{c \max(1, \log \widehat{r}_i)} \Gamma_i.$$

The base of the induction, $\widehat{r}_i \le \sqrt{c}$, follows from Lemma 4.13, since $\gamma_{\widetilde{V}} \ge 1$ by definition. Assume that $\widehat{r}_i > \sqrt{c}$. Then Lemma 4.14 and the induction hypothesis applied to both $\widetilde{V}$ and $\overline{V}$ from that lemma give

$$\gamma_{\widehat{V}} \le \sigma_i q^{-cw^4} \sigma_i^{c \log(2\widehat{r}_i/3)} \Gamma_i \le q^{-cw^4} \sigma_i^{c \log \widehat{r}_i} \Gamma_i,$$

since both $\widetilde{r}_i$ and $\overline{r}_i$ in Lemma 4.14 are smaller than $2\widehat{r}_i/3$. This completes the proof of the induction step and the corollary. □

We are now ready to conclude the proof of Proposition 4.12 and of Theorem 4.6.

PROOF OF PROPOSITION 4.12. Applying Corollary 4.16 to each nonzero coordinate of $\underline{r}$, we obtain

$$\gamma_V \le \left(q^{-O(w^4)} \sigma\right)^{O(\log L)} \Gamma$$

with the notation of the statement of Proposition 4.12. □

PROOF OF THEOREM 4.6. Combining Propositions 4.9 and 4.12, we get

$$\gamma_V \le \left(q^{-w^4} \sigma q^{-Rw}\right)^{O(\log L)} \le e^{-O(w^4 \log^3(1/q)/q^\alpha)},$$

where the last equality follows from Observation 4.5. □

5. Proof of Theorem 1. Recall that w is a large constant much bigger than the constants in any $O(\cdot)$ notation. Let $t_* = \frac{1}{w} e^{w^5 \log^3(1/q)/q^\alpha}$ and $T = e^{1/q^{3\alpha}}$. Then we have

$$\begin{aligned}
\mathbb{E}_\mu(\tau_0) &= \int_0^{+\infty} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s) \\
&= \int_0^{t_*} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s) + \int_{t_*}^{T} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s) + \int_T^{+\infty} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s) \\
&\le t_* + T \mathbb{P}_\mu(\tau_0 > t_*) + \int_T^{+\infty} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s).
\end{aligned}$$

The term t_* has exactly the form required in Theorem 1. The last term in the right-hand side above tends to zero as $q \to 0$. Indeed, using [20], Theorem 2, we have that $\forall s > 0$, $\mathbb{P}_\mu(\tau_0 > s) \le e^{-s\lambda_0}$ with $\lambda_0 \ge e^{-\Omega((\log q)^4/q^{2\alpha})}$ and, therefore,

$$\int_T^{+\infty} \mathrm{d}s\, \mathbb{P}_\mu(\tau_0 > s) \le \lambda_0^{-1} e^{-T\lambda_0} \to 0 \quad \text{as } q \to 0.$$

In conclusion, the proof of the upper bound in Theorem 1 boils down to proving

$$\lim_{q \to 0} T \mathbb{P}_\mu(\tau_0 > t_*) = 0. \tag{24}$$

That requires a sequence of simple steps ((a)–(d) below) and a more involved part ((e) below). Before turning to the details of the proof of Theorem 1, let us sketch our approach.

5.1. *Roadmap.*

(a) In order to prove that w.h.p. $\tau_0 \le t_*$, it suffices to prove the result for the (stationary) $\mathcal{U}$-KCM process on to the torus Λ and with side $K = 2e^{w^5 \log^3(1/q)/q^\alpha}$ (see (25)).

(b) Let $L = \Theta(\lambda)/q^{3w}$ for a large positive constant $\lambda = \lambda(\mathcal{U}, \delta)$, let $R = w^2 \log(1/q)/q^\alpha$, and recall the good and super good events described in Section 4.1.1 and Definition 4.2. Given a snail $V = V_L^R(\underline{r}) = B \cup \bigcup_{i\in[2k]} T_i^{\pm} \subset \Lambda$ (recall Definitions 3.4 and 3.5) with base B and trapezoids $T_i^{\pm}$, we will construct a new event $\mathcal{E} \subset \Omega_{\Lambda\cap\mathbb{Z}^2}$ which will guarantee that (in particular) the following occurs.

(i) For any (translate of) $V \subset \Lambda$ as above, the good events $\mathcal{G}(T_i^{\pm})$ occur for all $i \in [2k]$.

(ii) In every strip of Λ parallel to u_0 and of width $2R$, there exists a translate of the base B for which the super good event holds.

(c) We will prove that $\mu(\mathcal{E}) \ge 1 - e^{-1/q^w}$, which will allow us to conclude that it is sufficient to analyze the infection time of the origin of the stationary $\mathcal{U}$-KCM in Λ restricted to $\mathcal{E}$ (see (29)).

(d) For the latter process, we will follow the standard "variational" approach (see [1], Theorem 2, and also [20], Section 2.2) and get that

$$T\mathbb{P}_\mu(\tau_0 \ge t_*) \le T e^{-t_*\lambda_{\mathcal{F}}}\big(1 + o(1)\big).$$

Here, $\lambda_{\mathcal{F}}$ is related to the Dirichlet problem for the $\mathcal{U}$-KCM on the torus and restricted to $\mathcal{E}$ with boundary condition $f|_{\{\omega\in\mathcal{E}:\omega_0=0\}} = 0$. In particular (see (31)),

$$\lambda_{\mathcal{F}} \ge \inf_f q \frac{\mathcal{D}_\Lambda^{\mathrm{per}}(f)}{\mathrm{Var}_\Lambda(f \mid \mathcal{E})},$$

where $\mathcal{D}_\Lambda^{\mathrm{per}}(f)$ is the Dirichlet form of the $\mathcal{U}$-KCM on the torus Λ and the supremum is taken over all $f : \mathcal{E} \to \mathbb{R}$.

(e) The last and most important step will be to prove that

$$\mathrm{Var}_\Lambda(f \mid \mathcal{E}) \le e^{O(w^4 \log^3(1/q))/q^\alpha} \mathcal{D}_\Lambda^{\mathrm{per}}(f),$$

implying that $t_*\lambda_{\mathcal{F}}$ diverges as $q \to 0$ rapidly enough. The high-level intuition behind the above Poincaré inequality is as follows. A super good base (i.e., a base B for which the super good $\mathcal{SG}(B)$ event holds), whose presence is guaranteed by (b.ii), will be able to move in Λ using an FA1f-like dynamics as in Lemma 3.12 with $\widehat{\nu}(\mathcal{H})$ given by

$$e^{\Theta(w^3 \log^2(1/q))/q^\alpha}.$$

Indeed, we will reproduce each step of that dynamics with a resampling of an appropriate super good translate of the snail V, since (b.i) guarantees that the super good base does extend to a super good translate of the snail V. Indeed, the snail (see Figure 6) does extend on both sides of the base for a distance $\Theta(r_{2k-1})$, so taking r_{2k-1} of order L, one can induce a change on both sides of the base by resampling the configuration inside the snail. Thanks to Theorem 4.6, the relaxation time of the super good snail is $e^{O(w^4 \log^3(1/q))/q^\alpha}$. The conclusion of Theorem 1 then follows rather naturally.

5.2. *Proof.* Let $K = 2\exp(w^5 \log^3(1/q)/q^\alpha)$ and let $\Lambda = \mathbb{R}^2/(Ku_0\mathbb{Z} + Ku_k\mathbb{Z})$ be the torus in $\mathbb{R}^2$ of side K directed by u_0, which we think of as centered at 0. Further set

$$R = w^2 \log(1/q)/q^\alpha, \qquad W = 1/q^{3w}, \qquad M = K/(2R_0 + W),$$

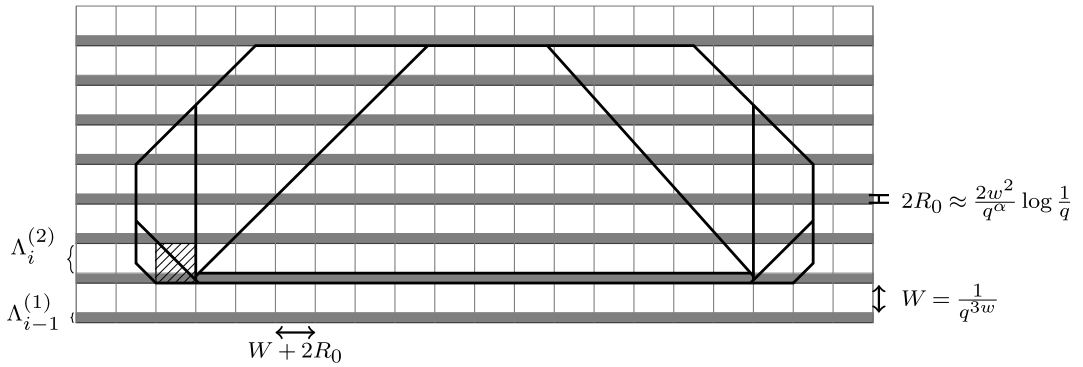


FIG. 11. *The partition of Λ into strips $\Lambda_i = \Lambda_i^{(1)} \cup \Lambda_i^{(2)}$, $i \in [M]$. The hatched region represents a square $Q_{i,j}$, which we would like to resample. The thick polygon is the snail V with its trapezoids. Note that its base B does not intersect $Q_{i,j}$ and (almost) spans the squares $Q_{i,j+1}, \dots, Q_{i,j+\lambda}$.*

recalling the notation $R_0 = \rho_0 \lfloor R/\rho_0 \rfloor$ from Definition 3.3. For simplicity, we assume that $u_0(2R_0 + W) \in \mathbb{Z}^2$ and that M is an even integer (W and K can be modified by $O(1)$ and $O(1/q^{3w})$, resp., so that these both hold).

We partition Λ into alternating strips $\Lambda_i^{(1)}$, $\Lambda_i^{(2)}$, $i \in [M]$, of length K and parallel to u_0 (see Figure 11). The strips $\Lambda_i^{(1)}$ have width $2R_0$ while the strips $\Lambda_i^{(2)}$ have width W. We write $\Lambda_i = \Lambda_i^{(1)} \cup \Lambda_i^{(2)}$ and we think of the thin strip $\Lambda_i^{(1)}$ as being just below the thick one $\Lambda_i^{(2)}$, when u_0 points left. In turn, we partition Λ_i into consecutive squares $Q_{i,j}$, $j \in [M]$, of side length equal to $2R_0 + W$ and sides parallel to u_0 and u_k and we write $Q_{i,j}^{(a)} = Q_{i,j} \cap \Lambda_i^{(a)}$, $a \in \{1, 2\}$.

REMARK 5.1. Recalling Definition 3.3, the width of the thin strips is chosen so that an annulus A of radius R would fit tightly inside.

We are now ready to detail the steps (a)–(e) sketched in the roadmap above.

Step (*a*). Notice that $t_* = K/(2w)$ and let τ_0, τ_0^Λ denote the infection times of the origin for the $\mathcal{U}$-KCM process on $\mathbb{Z}^2$ and for the $\mathcal{U}$-KCM process on the discrete torus $\Lambda \cap \mathbb{Z}^2$, respectively. Using the fact that the jump rates of the KCM are bounded, a standard argument of finite speed of information propagation (see, e.g., [18]) implies that

$$\mathbb{P}_\mu(\tau_0 \geq t_*) \leq \mathbb{P}_{\mu_\Lambda}\big(\tau_0^\Lambda \geq t_*\big) + e^{-\Omega(K)} \quad \text{as } q \to 0. \tag{25}$$

Step (*b*). Given a small positive constant $\varepsilon = \varepsilon(\mathcal{U}) = \Omega(1)$ and a large one $\lambda = \lambda(\mathcal{U}, \delta)$ to be specified later (recall the constant δ from Definition 3.4) let:

(i) $\mathcal{SG}(Q_{i,j}^{(1)})$ be the event that the rightmost and leftmost annuli A in $Q_{i,j}^{(1)}$ are infected and any segment $I \subset Q_{i,j}^{(1)}$ intersecting $\mathbb{Z}^2$, of length εR and orthogonal to some $u_i \in \widehat{\mathcal{S}}_0$ contains an infected u_i-helping set in $Q_{i,j}^{(1)}$;

(ii) $\mathcal{G}(Q_{i,j})$ be the event that any segment $I \subset Q_{i,j}$ intersecting $\mathbb{Z}^2$, of length εW and orthogonal to some $u \in \widehat{\mathcal{S}}$ contains w infected consecutive sites;

(iii) $\mathcal{E}_i$ be the event that for all the squares $Q_{i,j} \subset \Lambda_i$ the event $\mathcal{G}(Q_{i,j})$ holds and moreover there exists $j \in [M]$ such that $\bigcap_{j'=j+1}^{j+\lambda} \mathcal{SG}(Q_{i,j'}^{(1)})$ also holds;

(iv) $\mathcal{E} = \bigcap_{i \in [M]} \mathcal{E}_i$.

REMARK 5.2. Similar to Definition 4.2, (i) needs to be modified slightly if Assumption 4.1 is not satisfied, but we keep working under that assumption.

Step (*c*). With our choice of K, R, W, as in Observation 4.5, it follows that $\mu(\mathcal{SG}(Q_{i,j}^{(1)})) = q^{O(Rw)}$. Moreover, using the Harris inequality

$$\mu\Big(\bigcap_{j\in[\lambda]} \mathcal{SG}(Q_{i,j}^{(1)})\Big) \geq q^{O(\lambda Rw)}. \tag{26}$$

Also,

$$\mu\Big(\bigcap_{j\in[M]} \mathcal{G}(Q_{i,j})\Big) \geq 1 - O(MW^2)e^{-q^w \varepsilon W/w^2} \geq 1 - e^{-q^{-2w+o(1)}}.$$

In conclusion,

$$1 - \mu(\mathcal{E}_i) \leq e^{-q^{-2w+o(1)}} + \big(1 - q^{O(\lambda Rw)}\big)^{\lfloor M/\lambda \rfloor} \leq e^{-q^{-2w+o(1)}} \tag{27}$$

and $\mu(\mathcal{E}) \geq 1 - M(1-\mu(\mathcal{E}_i)) \geq 1 - e^{-q^{-2w+o(1)}}$. Therefore, writing $\tau_{\mathcal{E}^c}^{\Lambda}$ for the hitting time of $\mathcal{E}^c$ for the $\mathcal{U}$-KCM process in Λ and recalling that $t_* = K/(2w)$, we obtain

$$\mathbb{P}_{\mu_\Lambda}\big(\tau_{\mathcal{E}^c}^{\Lambda} \leq t_*\big) \leq O\big(K^2 t_*\big)\mu\big(\mathcal{E}^c\big) + e^{-\Omega(K^2 t_*)} \leq e^{-q^{-2w+o(1)}}. \tag{28}$$

In the second inequality above, we used a simple union bound over the updates for the $\mathcal{U}$-KCM in Λ together with the fact that the law of the $\mathcal{U}$-KCM process in Λ started from μ_Λ is equal to μ_Λ at any given time and a simple large deviations result on the number of updates.

Thus, if $\mathcal{F} = \{\omega : \omega_0 = 0\} \cup \mathcal{E}^c$ then (25) together with (28) imply that

$$\begin{aligned}\mathbb{P}_\mu(\tau_0 \geq t_*) &\leq \mathbb{P}_\mu\big(\tau_0^{\Lambda} \geq t_*\big) + e^{-\Omega(K)} \\ &\leq \mathbb{P}_{\mu_\Lambda}\big(\tau_{\mathcal{F}}^{\Lambda} \geq t_*\big) + \mathbb{P}_{\mu_\Lambda}\big(\tau_{\mathcal{E}^c}^{\Lambda} \leq t_*\big) + e^{-\Omega(K)} \\ &\leq \mathbb{P}_{\mu_\Lambda}\big(\tau_{\mathcal{F}}^{\Lambda} \geq t_*\big) + e^{-q^{-2w+o(1)}}.\end{aligned} \tag{29}$$

Step (*d*). As in [1], Theorem 2,

$$\mathbb{P}_{\mu_\Lambda}\big(\tau_{\mathcal{F}}^{\Lambda} \geq t_*\big) \leq e^{-\lambda_{\mathcal{F}} t_*}, \tag{30}$$

with

$$\lambda_{\mathcal{F}} = \inf\left\{\frac{\mathcal{D}_\Lambda^{\mathrm{per}}(f)}{\mu_\Lambda(f^2)},\, f|_{\mathcal{F}} = 0\right\},$$

where $\mathcal{D}_\Lambda^{\mathrm{per}}(f)$ denotes the Dirichlet form of the $\mathcal{U}$-KCM process on the torus Λ (see (8)). Observe now that for any $f : \Omega_\Lambda \to \mathbb{R}$ such that $f|_{\mathcal{F}} = 0$,

$$\begin{aligned}\mathrm{Var}_\Lambda(f \mid \mathcal{E}) &= \frac{1}{2}\sum_{\omega}\sum_{\omega'} \mu_\Lambda(\omega \mid \mathcal{E})\mu_\Lambda(\omega' \mid \mathcal{E})\big(f(\omega) - f(\omega')\big)^2 \\ &\geq \mu_\Lambda(\omega_0 = 0 \mid \mathcal{E})\mu_\Lambda\big(f^2 \mid \mathcal{E}\big) \geq q\,\mu_\Lambda\big(f^2\big),\end{aligned}$$

where for the last inequality we used the Harris inequality ($\{\omega : \omega_0 = 0\}$ and $\mathcal{E}$ are both decreasing events) and the fact that $f^2 \mathbb{1}_{\mathcal{E}} = f^2$. Hence,

$$\lambda_{\mathcal{F}} \geq q \inf_f \frac{\mathcal{D}_\Lambda^{\mathrm{per}}(f)}{\mathrm{Var}_\Lambda(f \mid \mathcal{E})}. \tag{31}$$

Notice the absence of the conditioning event $\mathcal{E}$ in the Dirichlet form $\mathcal{D}_\Lambda^{\mathrm{per}}(f)$.

Step (*e*). Our main result on the above variational problem is as follows.

THEOREM 5.3. *For all $w > 0$ large enough, all $\varepsilon > 0$ small enough and all $f : \Omega_\Lambda \to \mathbb{R}$*

$$\mathrm{Var}_\Lambda(f \mid \mathcal{E}) \le e^{O(w^4(\log(1/q))^3)/q^\alpha} \mathcal{D}_\Lambda^{\mathrm{per}}(f), \tag{32}$$

i.e.

$$\lambda_{\mathcal{F}} \ge e^{-O(w^4(\log(1/q))^3)/q^\alpha}. \tag{33}$$

Before proving this theorem, let us first complete the proof of (24). Using $t_* = w^{-1}\exp(w^5(\log(1/q))^3/q^\alpha)$ and (33), we get that for any w large enough

$$t_* \lambda_{\mathcal{F}} \ge 1/q^w$$

which, together with (29) and (30) and the choice of $T = e^{1/q^{3\alpha}}$, gives

$$T\mathbb{P}_\mu(\tau_0 \ge t_*) \le T\big(e^{-\lambda_{\mathcal{F}} t_*} + e^{-q^{-2w+o(1)}}\big) \to 0. \tag{34}$$

This proves (24) and, therefore, Theorem 1 modulo Theorem 5.3.

PROOF OF THEOREM 5.3. The two main ingredients of the proof will be Lemma 3.12 and Theorem 4.6. The definition of the event $\mathcal{E} = \bigcap_i \mathcal{E}_i$ and the fact that the strips Λ_i are disjoint imply that $\mu_\Lambda(\cdot \mid \mathcal{E}) = \bigotimes_i \mu_{\Lambda_i}(\cdot \mid \mathcal{E}_i)$. In turn, Lemma 3.9 gives

$$\mathrm{Var}_\Lambda(f \mid \mathcal{E}) \le \sum_i \mu_\Lambda\big(\mathrm{Var}_{\Lambda_i}(f \mid \mathcal{E}_i) \mid \mathcal{E}\big). \tag{35}$$

Hence, it is enough to analyse a generic term $\mu_\Lambda(\mathrm{Var}_{\Lambda_i}(f \mid \mathcal{E}_i) \mid \mathcal{E})$ and for this purpose we plan to apply Lemma 3.12 to bound from above $\mathrm{Var}_{\Lambda_i}(f \mid \mathcal{E}_i)$.

Recall that the strip Λ_i is the disjoint union of M squares $(Q_{i,j})_{j=1}^M$ and recall the definition of the "single square" events $\mathcal{SG}(Q_{i,j}^{(1)})$ and $\mathcal{G}(Q_{i,j})$ given in (i) and (ii) above. Those definitions allow us to write (in what follows the index i of the strip is fixed)

$$\mu_{\Lambda_i}(\cdot \mid \mathcal{E}_i) = \nu_i\left(\cdot \,\Big|\, \bigcup_j \bigcap_{j'=j+1}^{j+\lambda} \mathcal{SG}(Q_{i,j'}^{(1)})\right),$$

where $\nu_{i,j} = \mu_{Q_{i,j}}(\cdot \mid \mathcal{G}(Q_{i,j}))$ and $\nu_i = \bigotimes_j \nu_{i,j}$. We can now apply Lemma 3.12 to the product measure ν_i with $\mathcal{SG}(Q_{i,j}^{(1)})$ as the event $\mathcal{H}$, M as the parameter n, and λ as the parameter κ. The choice of the key parameter κ entering the definition of the associated facilitating events $\mathcal{H}_{i,j}$ in the periodic case,

$$\mathcal{H}_{i,j} = \bigcap_{j'=j+1}^{j+k} \mathcal{SG}(Q_{i,j'}^{(1)}) \cup \bigcap_{j'=j-1}^{j-k} \mathcal{SG}(Q_{i,j'}^{(1)}),$$

will be postponed to Lemma 5.4 below. There κ will be chosen to be large enough but independent of q. The requirement of Lemma 3.12 that $(1 - \widehat{\nu}(\mathcal{H})^\kappa)^{n/(3\kappa)} < 1/16$ is implied by (27).

In the above setting, Lemma 3.12 gives

$$\mathrm{Var}_{\Lambda_i}(f \mid \mathcal{E}_i) \le q^{-O(Rw\lambda)} \sum_j \nu_i\big(\mathbb{1}_{\mathcal{H}_{i,j}} \mathrm{Var}_{Q_{i,j}}(f \mid \mathcal{G}(Q_{i,j}))\big).$$

By combining the above with (35), we finally get

$$\mathrm{Var}_\Lambda(f \mid \mathcal{E}) \le q^{-O(Rw\lambda)} \sum_{i,j} \mu_\Lambda\left(\mathbb{1}_{\mathcal{H}_{i,j}} \mathrm{Var}_{Q_{i,j}}(f \mid \mathcal{G}(Q_{i,j})) \,\Big|\, \bigcap_{i',j' \in [M]} \mathcal{G}(Q_{i',j'})\right).$$

We shall now analyze a generic term in the sum above with the help of Theorem 4.6.

LEMMA 5.4. *There exists an constant $\lambda = \lambda(\mathcal{U}, \delta)$ such that the following holds. If the parameter κ of the facilitating events $\mathcal{H}_{i,j}$ is taken equal to λ then, for any function $f : \Omega_\Lambda \to \mathbb{R}$ and any i, j,*

$$\mu_\Lambda\Big(\mathbb{1}_{\mathcal{H}_{i,j}} \operatorname{Var}_{Q_{i,j}}(f \mid \mathcal{G}(Q_{i,j})) \,\Big|\, \bigcap_{i',j'} \mathcal{G}(Q_{i',j'})\Big) \le q^{-O(w^4 \log^3(1/q)/q^\alpha)} \sum_{\substack{x\in\Lambda \\ d(x,Q_{i,j})\le O(\lambda W)}} \mu_\Lambda(c_x \operatorname{Var}_x(f)).$$

If we assume the lemma, we immediately recover (32), concluding the proof of Theorem 5.3. □

PROOF OF LEMMA 5.4. We assume that $\mathbb{1}_{\mathcal{H}_{i,j}} = 1$ and that w.l.o.g. the event $\mathcal{H}^+_{i,j} = \bigcap_{j'=j+1}^{j+\lambda} \mathcal{SG}(Q^{(1)}_{i,j'})$ occurs. Next, we recall Definition 3.4 of the snail $V^R_L(\underline{r})$ and we choose $r_l = \rho_{k+l}\lfloor \delta r_{l-1}/\rho_{k+l} \rfloor$ for all $l \in [2k]$, setting $r_{-1} = L = \lambda(W + 2R_0) - 2R_0$. We choose λ sufficiently large, depending on δ and $\mathcal{U}$ but not on w and q, in such a way that $Q_{i,j} \subset x + V^R_L(\underline{r})$, where x is the center of the rightmost annulus in $Q^{(1)}_{i,j+\lambda}$. We write $V = x + V^R_L(\underline{r})$ and observe that, by construction, $Q_{i,j} \cap B = \varnothing$, where B is the base V. Finally, we recall Definition 4.2 of the events $\mathcal{SG}(B)$, $\mathcal{G}(T^\pm_l)$ and $\mathcal{SG}(V) = \mathcal{SG}(B) \cap \bigcap_{l\in[2k]}(\mathcal{G}(T^+_l) \cap \mathcal{G}(T^-_l))$ for the snail V. It is easy to verify the following implications (see Figure 11):

$$\mathcal{H}^+_{i,j} \subset \mathcal{SG}(B) \qquad \bigcap_{i',j'} \mathcal{G}(Q_{i',j'}) \subset \bigcap_{l\in[2k]} (\mathcal{G}(T^+_l) \cap \mathcal{G}(T^-_l)). \tag{36}$$

Indeed, for the first inclusion, recalling (i) it is clear that $\mathcal{A}$ and $\mathcal{HA}$ occur (since the leftmost annulus in $Q_{i,j+1}$ contains HA and the rightmost one in $Q_{i,j+\lambda}$ contains A) and that all $\mathcal{SB}_{m,p}$ occur (for $SB_{m,p}$ contained in two consecutive squares $Q_{i,j'}$, $Q_{i,j'+1}$ at least in one of them we are guaranteed to have the helping sets; for $SB_{0,p}$ close to the left boundary of $Q_{i,j'}$ the infected rightmost annulus provides the desired helping sets). To see the second one, observe that for all l, m, $ST^\pm_{l,m}$ intersects at least one of the squares $Q_{i',j'}$ in a segment of length at least εW.

Using (36) and $\mu_\Lambda(\mathcal{E}) = 1 - o(1)$, we have that

$$\begin{aligned}
&\mu_\Lambda\Big(\mathbb{1}_{\mathcal{H}^+_{i,j}} \operatorname{Var}_{Q_{i,j}}(f \mid \mathcal{G}(Q_{i,j})) \,\Big|\, \bigcap_{i',j'} \mathcal{G}(Q_{i',j'})\Big) \\
&\le (1 + o(1))\mu_\Lambda\Big(\mathbb{1}_{\mathcal{SG}(B)}\mathbb{1}_{\bigcap_{i',j'}\mathcal{G}(Q_{i',j'})} \inf_a \mu_{Q_{i,j}}(\mathbb{1}_{\mathcal{G}(Q_{i,j})}(f-a)^2)\Big) \\
&\le (1 + o(1))\mu_\Lambda\Big(\inf_a \mu_{Q_{i,j}}(\mathbb{1}_{\mathcal{SG}(V)}(f-a)^2)\Big) \\
&\le \mu_\Lambda\big(\mathbb{1}_{\mathcal{SG}(V)}(f - \mu_V(f \mid \mathcal{SG}(V)))^2\big)/\mu(\mathcal{SG}(V)) \\
&= \mu_\Lambda(\operatorname{Var}_V(f \mid \mathcal{SG}(V))).
\end{aligned}$$

If we now apply the bound (13) of Theorem 4.6 and use the fact that V is contained in a deterministic $O(\lambda W)$-neighborhood of the square $Q_{i,j}$ we get the conclusion of the lemma, once we observe that $c^V_x \le c_x$, where c_x are the constraints on the torus Λ. □

6. Open problems. With Theorem 1 establishing universality, the next natural goal is to determine the relaxation time up to a constant factor. This would correspond to reaching the refined universality partition in bootstrap percolation proved in [5]. However, for KCM,

we expect that the partition, even restricted to the finite stable set case studied in this work, will be more subtle. In order to state it, we need one more definition adapted from [22], Definition 2.3.

DEFINITION 6.1. A critical update family of difficulty α is called *rooted* if there exist two nonopposite directions of difficulty strictly larger than α and *unrooted* otherwise.

The importance of this distinction, though not the one initially suggested in [20, 22], is the following. In a rooted model, a droplet may not reproduce in certain directions, which forces it to perform an East-like motion, as explained in Section 2. As it was established in the present work, this obstruction can be circumvented using the mechanism explained in Section 2 on length scale $q^{-O(1)}$. However, for example, for the rooted model in Figure 1, typically the first pairs of infections are found at a distance q^{-2} from the droplet and, before reaching them, the droplet can only move right East-like. Therefore, for rooted models with finitely many stable directions we expect that the bottleneck of the dynamics consists in creating $\log(1/q)$ disjoint droplets close to each other. For unrooted models, one may hope to directly move droplets in an FA1f-like way (creating a droplet and immediately erasing the previous one), as in Section 5, since they may locally move in all directions.

On the other hand, as identified in [5], balanced models have droplets whose probability of occurrence is $\exp(-\Theta(1)/q^{\alpha})$, while unbalanced ones only have droplets with probability $\exp(-\Theta(\log^2(1/q))/q^{\alpha})$. Putting these two intuitions together and the mechanism put forward in the present work, this leads us to the following conjecture.

CONJECTURE 6.2. *Let $\mathcal{U}$ be a critical update family with finite number of stable directions and difficulty α. Then*

$$\mathbb{E}_{\mu}[\tau_0] = \exp\left(\frac{\log^{\gamma}(1/q)(\log\log(1/q))^{O(1)}}{q^{\alpha}}\right),$$

where

- $\gamma = 0$ *if there exists at most one direction $u \in S^1$ such that $\alpha(u) > \alpha$ (balanced unrooted models)*,
- $\gamma = 1$ *if there exist at least two directions $u \in S^1$, such that $\alpha(u) > \alpha$, but not two opposite ones (balanced rooted models)*,
- $\gamma = 2$ *if there exists u such that $\min(\alpha(u), \alpha(u+\pi)) > \alpha$, but $\alpha(v) \le \alpha$ for $v \neq u, u+\pi$ (unbalanced unrooted models)*,
- $\gamma = 3$ *otherwise, that is, there exist three different directions with difficulty larger than α, two of which are opposite (unbalanced rooted models)*.

Furthermore, we conjecture that the $(\log\log(1/q))^{O(1)}$ correction is in fact $\Theta(1)$, except, possibly, for models with exactly one direction $u \in S^1$ with $\alpha(u) > \alpha$.

It should be noted that such sharp results are not known for any critical model, so Conjecture 6.2 provides the highest precision currently feasible. In fact, the level of precision of the conjecture is not attained for any model, including the most classical FA2f model falling in the first class.[3]

Since the present work was submitted, in two companion papers Marêché and the first author [13] and the first author [12], proved Conjecture 6.2. Moreover, they established that the peculiar log log correction left uncertain in the conjecture is indeed present. We refer the reader to those works for more detailed discussions of the bottlenecks and mechanisms involved, particularly for the anomalous case with exactly one direction of difficulty larger than α dubbed *semidirected* there.

[3]Since the submission of the present work, a much sharper result was proved for FA2f by the authors [15].

APPENDIX

PROOF OF LEMMA 3.12. We will consider the linear case—the periodic one is treated identically. For simplicity, we assume that $2k$ divides n. Partition $[n]$ into blocks $I_0, \dots, I_{N-1}$ where $I_i := \{i\kappa, \dots, (i+1)\kappa - 1\}$ and $N = n/\kappa$. Let $\mathcal{H}^{(\ell)}$ be the event that there exists i in the *left-half* $[N]^{(\ell)} := [N/2]$ of $[N]$ such that $\omega_j \in \mathcal{H}$ for all $j \in I_i$. Let $\mathcal{H}^{(r)}$ be defined similarly but for the blocks with index in the right-half $[N]^{(r)} := [N] \setminus [N]^{(\ell)}$. Using the assumption of the lemma $\nu(\mathcal{H}^{(\ell)}) = \nu(\mathcal{H}^{(r)}) > 15/16$ and [4], Lemma 6.5, we get

$$\operatorname{Var}_\nu(f \mid \Omega_\mathcal{H}) \le 24\nu\big(\mathbb{1}_{\mathcal{H}^{(r)}} \operatorname{Var}^{(\ell)}(f) + \mathbb{1}_{\mathcal{H}^{(\ell)}} \operatorname{Var}^{(r)}(f) \mid \Omega_\mathcal{H}\big),$$

where $\operatorname{Var}^{(\ell)}$ denotes the variance computed w.r.t. the variables corresponding to the blocks in the left-half and similarly for $\operatorname{Var}^{(r)}$.

Given $\mathcal{H}^{(\ell)}$, let ξ be the smallest label in $[N]^{(\ell)}$ such that $\omega_j \in \mathcal{H}$ for all $j \in I_\xi$. Using Lemma 3.9 and the fact that the event $\{\xi = i\}$ is independent of the variables $(\omega_j)_{j \ge (i+1)\kappa}$, we get that

$$\begin{aligned} \nu\big(\mathbb{1}_{\mathcal{H}^{(\ell)}} \operatorname{Var}^{(r)}(f) \mid \Omega_\mathcal{H}\big) &\le \sum_{i \in [N]^{(\ell)}} \nu\big(\mathbb{1}_{\{\xi=i\}} \operatorname{Var}_{\ge(i+1)\kappa}(f) \mid \Omega_\mathcal{H}\big) \\ &\le \frac{1}{\widehat{\nu}(\mathcal{H})^\kappa} \sum_{i \in [N]^{(\ell)}} \nu\big(\mathbb{1}_{\{\xi=i\}} \operatorname{Var}_{\ge(i+1)\kappa}(f)\big), \end{aligned} \tag{37}$$

where $\operatorname{Var}_{\ge(i+1)\kappa}(f)$ is the variance w.r.t. the variables $(\omega_j)_{j \ge (i+1)\kappa}$. The right-hand side above can now be bounded above using [20], Proposition 3.4. If $\mathcal{H}_{I_j}$ is the event that $\omega_l \in \mathcal{H}$ for all $l \in I_j$, with the convention that $\mathcal{H}_{I_N}$ and $\mathcal{H}_{I_{-1}}$ do not occur, we get that

$$\mathbb{1}_{\{\xi=i\}} \operatorname{Var}_{j \ge (i+1)\kappa}(f) \le \frac{1}{\widehat{\nu}(\mathcal{H})^{O(\kappa)}} \sum_{j=i+1}^{N-1} \nu_{\ge(i+1)\kappa}\big(\mathbb{1}_{\{\xi=i\}} \mathbb{1}_{\mathcal{H}_j^\pm} \operatorname{Var}_{I_j}(f)\big),$$

where $\mathcal{H}_j^\pm = \mathcal{H}_{I_{j-1}} \cup \mathcal{H}_{I_{j+1}}$ and Var_{I_j} is the variance w.r.t. the variables in I_j. By inserting the right-hand side above into the right-hand side of (37), we obtain that $\nu(\mathbb{1}_{\mathcal{H}^{(\ell)}} \operatorname{Var}^{(r)}(f) \mid \Omega_\mathcal{H})$ is smaller than

$$\begin{aligned} &\frac{1}{\widehat{\nu}(\mathcal{H})^{O(\kappa)}} \nu\Bigg(\sum_{j=1}^{N-1} \sum_{i=0}^{j-2} \mathbb{1}_{\{\xi=i\}} \mathbb{1}_{\mathcal{H}_j^\pm} \operatorname{Var}_{I_j}(f) + \sum_{i \in [N]^\ell} \mathbb{1}_{\{\xi=i\}} \mathbb{1}_{\mathcal{H}_{i+1}^\pm} \operatorname{Var}_{I_{i+1}}(f)\Bigg) \\ &\quad \le \frac{2}{\widehat{\nu}(\mathcal{H})^{O(\kappa)}} \sum_{j=1}^{N-1} \nu\big(\mathbb{1}_{\mathcal{H}_j^\pm} \operatorname{Var}_{I_j}(f)\big), \end{aligned}$$

where we have isolated the term $j = i+1$ and used $\sum_i \mathbb{1}_{\{\xi=i\}} \le 1$ and $\mathbb{1}_{\{\xi=i\}} \le 1$ for the two terms, respectively. Exactly the same argument can be applied to the term $\nu(\mathbb{1}_{\mathcal{H}^{(r)}} \operatorname{Var}^{(\ell)}(f) \mid \Omega_\mathcal{H})$ to conclude that

$$\operatorname{Var}_\nu(f \mid \Omega_\mathcal{H}) \le \frac{96}{\widehat{\nu}(\mathcal{H})^{O(\kappa)}} \sum_{j=0}^{N-1} \nu\big((\mathbb{1}_{\mathcal{H}_{I_{j+1}}} + \mathbb{1}_{\mathcal{H}_{I_{j-1}}}) \operatorname{Var}_{I_j}(f)\big). \tag{38}$$

We finally bound from above a generic term, considering $\nu(\mathbb{1}_{\mathcal{H}_{I_1}} \operatorname{Var}_{I_0}(f))$ for concreteness.

We apply Lemma 3.10 with $X_1 = \omega_{\kappa-1}$, $X_2 = (\omega_0, \omega_1, \dots, \omega_{\kappa-2})$ and facilitating event $\{\omega_{\kappa-1} \in \mathcal{H}\}$ to $\operatorname{Var}_{I_0}(f)$ in order to get

$$\operatorname{Var}_{I_0}(f) \le \frac{2}{\widehat{\nu}(\mathcal{H})} \nu_{I_0}\big(\operatorname{Var}_{\kappa-1}(f) + \mathbb{1}_{\{\omega_{\kappa-1} \in \mathcal{H}\}} \operatorname{Var}_{I_0 \setminus \{\kappa-1\}}(f)\big). \tag{39}$$

Thus, we obtain

$$\nu\big(\mathbb{1}_{\mathcal{H}_{I_1}}\operatorname{Var}_{I_0}(f)\big) \le \frac{2}{\widehat{\nu}(\mathcal{H})}\big(\nu\big(\mathbb{1}_{\mathcal{H}_{I_1}}\operatorname{Var}_{\kappa-1}(f)\big) + \nu\big(\mathbb{1}_{\mathcal{H}_{I_1}}\mathbb{1}_{\{\omega_{\kappa-1}\in\mathcal{H}\}}\operatorname{Var}_{I_0\setminus\{\kappa-1\}}(f)\big)\big)$$
$$\le \frac{2}{\widehat{\nu}(\mathcal{H})}\big(\nu\big(\mathbb{1}_{\mathcal{H}_{\kappa-1}}\operatorname{Var}_{\kappa-1}(f)\big) + \nu\big(\mathbb{1}_{\mathcal{H}_{\kappa-2}}\operatorname{Var}_{I_0\setminus\{\kappa-1\}}(f)\big)\big).$$

We can repeat the step leading to (39) with $X_1 = \omega_{\kappa-2}$, $X_2 = (\omega_0, \dots, \omega_{\kappa-3})$ and facilitating event $\{\omega_{\kappa-2} \in \mathcal{H}\}$ and so on. At the end of the iteration, we finally get that

$$\nu\big(\mathbb{1}_{\mathcal{H}_{I_1}}\operatorname{Var}_{I_0}(f)\big) \le \left(\frac{2}{\widehat{\nu}(\mathcal{H})}\right)^{\kappa}\sum_{i\in I_0}\nu\big(\mathbb{1}_{\mathcal{H}_i}\operatorname{Var}_i(f)\big).$$

Putting all together, we have finally proved that

$$\operatorname{Var}_\nu(f) \le \frac{96}{\widehat{\nu}(\mathcal{H})^{O(\kappa)}}\sum_{j\in[N]}\nu\big((\mathbb{1}_{\mathcal{H}_{I_{j+1}}} + \mathbb{1}_{\mathcal{H}_{I_{j-1}}})\operatorname{Var}_{I_j}(f)\big)$$
$$\le \left(\frac{2}{\widehat{\nu}(\mathcal{H})}\right)^{O(\kappa)}\sum_j\sum_{i\in I_j}\nu\big(\mathbb{1}_{\mathcal{H}_i}\operatorname{Var}_i(f)\big).$$

□

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