WALL DIVISORS AND ALGEBRAICALLY COISOTROPIC SUBVARIETIES OF IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS

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Abstract. Rational curves on Hilbert schemes of points on $K3$ surfaces and generalised Kummer manifolds are constructed by using Brill-Noether theory on nodal curves on the underlying surface. It turns out that all wall divisors can be obtained, up to isometry, as dual divisors to such rational curves. The locus covered by the rational curves is then described, thus exhibiting algebraically coisotropic subvarieties. This provides strong evidence for a conjecture by Voisin concerning the Chow ring of irreducible holomorphic symplectic manifolds. Some general results concerning the birational geometry of irreducible holomorphic symplectic manifolds are also proved, such as a non-projective contractibility criterion for wall divisors.

0. Introduction

Rational curves play a pivotal role in the study of the birational geometry and the Chow ring of algebraic varieties. The present paper concerns a specific class of varieties, namely, *irreducible holomorphic symplectic (IHS) manifolds* and, more precisely, *Hilbert schemes of points on $K3$ surfaces* and *generalised Kummer manifolds* (cf. §1), and is focused on some special rational curves arising from the Brill-Noether theory of normalisations of curves lying on $K3$ and abelian surfaces. In order to treat the two cases simultaneously, we introduce the following notation: we set $\varepsilon = 0$ (respectively, $\varepsilon = 1$) when S is a $K3$ (resp., abelian) surface, and we denote by $S_\varepsilon^{[k]}$ the Hilbert scheme of k points on S when $\varepsilon = 0$ and the $2k$-dimensional generalised Kummer variety on S when $\varepsilon = 1$.

In the last few years, some classical results concerning (-2)-curves on $K3$ surfaces have been generalised to higher dimension and in particular it was shown that rational curves fully control the birational geometry of IHS manifolds. More precisely, Ran [Ra] proved that extremal rational curves can be deformed together with the ambient IHS manifold, and this was exploited by Bayer, Hassett and Tschinkel [BHT] in order to determine the structure of the ample cone. The same result was independently obtained by the third named author [Mo1] using intrinsic properties of IHS manifolds and a deformation invariant class of divisors, the so-called *wall divisors* (cf. Definition 2.2), which contains all divisors dual to extremal rays. This class of divisors was also studied by Amerik and Verbitsky [AV], who investigated fibres of extremal contractions. Indeed, the MBM classes in [AV] turn out to be precisely the dual curve classes to wall divisors, cf. Remark 2.4.

By results of Bayer and Macrì [BM1, BM2] and Yoshioka [Yo3], moduli spaces of stable objects in the bounded derived category of a $K3$ or abelian surface S provide examples of deformations of $S_\varepsilon^{[k]}$ and the space of stability conditions can be used towards computing their ample cones.

In this paper we use Brill-Noether theory of nodal curves on abelian and $K3$ surfaces in order to exhibit rational curves in $S_\varepsilon^{[k]}$ and describe, in many cases, the locus they cover. Our construction proceeds as follows. Let (S, L) be a general primitively polarized $K3$ or abelian surface of genus $p := p_a(L)$ and let $C \in |L|$ be a δ-nodal curve whose normalization $\widetilde{C}$ has a linear series of type $g^1_{k+\varepsilon}$. Existence of a family of such curves having the expected dimension (and satisfying certain additional properties) has been proved in [CK, KLM] under suitable conditions on the triple (p, k, δ), cf. Theorem 3.1. Any pencil of degree $k + \varepsilon$ on $\widetilde{C}$ defines a rational curve in $S_\varepsilon^{[k]}$, whose class is

$$R_{p,\delta,k} := L - (p - \delta + k - 1 + \varepsilon)\mathfrak{r}_k,$$

in terms of the canonical decomposition $N_1(S_\varepsilon^{[k]}) \simeq N_1(S) \oplus \mathbb{Z}[\mathfrak{r}_k]$, cf. (8) and Lemma 3.3. In particular, its Beauville-Bogomolov square is easily computed to be

$$q(R_{p,\delta,k}) = 2(p-1) - \frac{(p-\delta+k-1+\varepsilon)^2}{2(k-1+2\varepsilon)},$$

cf. (18). An important additional feature of the rational curves obtained in this way is that they move in a family of dimension precisely $2k-2$ in $S_\varepsilon^{[k]}$ and thus survive in all small deformations of $S_\varepsilon^{[k]}$ that keep $R_{p,\delta,k}$ algebraic.

We prove the following result concerning the dual (in the sense of the lattice duality induced by the Beauville-Bogomolov form) divisor $D_{p,\delta,k}$ to the class $R_{p,\delta,k}$.

Theorem 0.1. *(cf. Theorem 4.1) The divisor $D_{p,\delta,k}$ is a wall divisor if and only if $q(R_{p,\delta,k}) < 0$.*

Wall divisors will be introduced in Section 2; a rough idea of them is that they determine the structure of the ample cone of $S_\varepsilon^{[k]}$ and they are related also to birational transformations of $S_\varepsilon^{[k]}$. By comparison with [BM2, Yo3], we show that for every isometry orbit of wall divisors there exist p, δ and k such that $D_{p,\delta,k}$ is an element of that orbit, cf. Proposition 4.6. This is rather striking, as it shows that the birational geometry of $S_\varepsilon^{[k]}$ and its deformations can be recovered from classical Brill-Noether theory of curves on the underlying surfaces, at least when the monodromy group is maximal. By work of Bayer, Macrì and Yoshioka [BM2, Yo3], these divisors are associated to walls in the manifold of Bridgeland stability conditions, but there was no reason to believe that they could be constructed by classical means, even up to deformation. We mention that some wall divisors have also been recently constructed by Hassett and Tschinkel [HT3], using a different approach.

Under opportune assumptions on the number of nodes (small with respect to the arithmetic genus), we explicitly construct the locus in $S_\varepsilon^{[k]}$ covered by our rational curves of class $R_{p,\delta,k}$. When $D_{p,\delta,k}$ is a wall divisor this locus may be described abstractly using only lattice theoretic properties, as in [BM1, Yo3] and in the more recent [HT3]. However, our constructions only rely on the definition of our curves of class $R_{p,\delta,k}$ and are thus very concrete.

The first type of construction goes as follows. Let $\mathcal{M}$ be the component of the moduli space of (Gieseker) L-stable torsion free sheaves on S with Mukai vector $v = (2, c_1(L), \chi+2(\varepsilon-1))$ (cf. Remark 1.4 for the definition) containing the Lazarsfeld-Mukai bundle associated with the pushforward to a δ-nodal curve in S of a $g^1_{k+\varepsilon}$ on its normalization. As soon as $\chi := p-\delta-k+3-5\varepsilon \geq 2\delta+2$, we construct a variety $\mathcal{P} \to \mathcal{M} \times S^{[\delta]}$ which is generically a projective bundle. The fibre of $\mathcal{P}$ over a point $([\mathcal{E}], \tau) \in \mathcal{M} \times S^{[\delta]}$ is the projectivization of the space of global sections of $\mathcal{E}$ vanishing along τ. We then define a rational map $g : \mathcal{P} \dashrightarrow S_\varepsilon^{[k]}$ and denote by T the closure of the image of g, which is an irreducible component of the locus covered by curves of class $R_{p,\delta,k}$. We show that g is birational, thus obtaining the following:

Theorem 0.2. *(cf. Theorem 6.1) Let (S, L) be a very general primitively polarized $K3$ or abelian surface of genus $p \geq 2$. Let $k \geq 2$ and $0 \leq \delta \leq p - \varepsilon$ be integers such that*

$$\max\{2\delta+2, 4\varepsilon\} \leq \chi := p-\delta-k+3-5\varepsilon \leq \delta+k+1.$$

Then, there is a subscheme $T \subset S_\varepsilon^{[k]}$ birational to a $\mathbb{P}^{\chi-2\delta-1}$-bundle on a holomorphic symplectic manifold W of dimension $2(k+1+2\delta-\chi)$. Furthermore, the lines contained in any fibre of the rational projection $T \dashrightarrow W$ have class $R_{p,\delta,k}$.

When $q(R_{p,\delta,k}) < 0$ and hence the dual divisor $D_{p,\delta,k}$ is a wall divisor by Theorem 0.1, the contractibility of T can be studied by means of Theorem 0.5 below.

In the case where $\delta = 0$ and $R_{p,\delta,k}$ has the minimal possible Beauville-Bogomolov square, namely, $-(k+3-2\varepsilon)/2$, we use Theorem 0.2 in order to construct a Lagrangian k-plane $\mathbb{P}^k \subset S_\varepsilon^{[k]}$ such that $R_{p,\delta,k}$ is the class of its lines, cf. Example 6.5 and Proposition 6.6. This agrees with Bakker's result [Ba, Thm. 3] stating that, in the case $\varepsilon = 0$, a primitive class generating an extremal ray is the line in a Lagrangian k-plane if and only if its square is $-(k+3)/2$, and suggests that the analogous statement should hold for $\varepsilon = 1$. Note that very few examples of Lagrangian planes are explicitly described in the literature, cf. [Ba, Ex. 8, 9, 10].

Our rational curves have applications to the Chow ring of IHS manifolds, too. In the recent paper [Vo], Voisin stated the following:

Conjecture 0.3. *(cf.* [Vo, Conj. 0.4]*) Let X be a projective IHS manifold of dimension $2k$ and let $\mathbb{S}_r(X)$ be the set of points in X whose orbit under rational equivalence has dimension at least r. Then $\mathbb{S}_r(X)$ has dimension $2k-r$.*

The above sets $\mathbb{S}_r(X)$ are countable unions of closed algebraic subsets of X and endow the Chow group $\mathrm{CH}_0(X)$ of 0-cycles with a filtration $\mathbb{S}_\bullet$ which is conjecturally connected with the Bloch-Beilinson filtration and its splitting predicted by Beauville [Be2]. The question about non-emptiness of $\mathbb{S}_r(X)$ is still open and related to the existence problem for algebraically coisotropic subvarieties of X. If X has dimension $2k$ and σ is its symplectic form, a subvariety $Y \subset X$ of codimension r is *algebraically coisotropic* if there exist a $(2k-2r)$-dimensional variety B and a surjective rational map $Y \dashrightarrow B$ such that $\sigma|_Y$ is the pullback of a two-form on B. The subvarieties $T \subset S_\varepsilon^{[k]}$ of Theorem 0.2 are algebraically coisotropic by construction and they are components of $\mathbb{S}_r(S_\varepsilon^{[k]})$ of dimension $2k-r$, with $r := \chi - 2\delta - 1$ (cf. Corollary 6.2). Starting from T and then applying the natural rational map $S^{[k+\varepsilon]} \times S^{[l-k]} \dashrightarrow S^{[l+\varepsilon]}$, one obtains a component of $\mathbb{S}_r(S_\varepsilon^{[l]})$ for any $l \geq k$. We use this observation in Theorem 6.3 in order to construct components of $\mathbb{S}_r(S_\varepsilon^{[k]})$, with k fixed, for several values of r.

Our second construction of uniruled subvarieties of $S_\varepsilon^{[k]}$ is obtained by considering the Severi variety $V_{\{L\},\delta}$ parametrizing curves with precisely δ nodes in the continuous system $\{L\}$ (which is the linear system $|L|$ up to translations) with δ big enough. In particular, the assumptions on δ are set to ensure that the normalization $\widetilde{C}$ of any curve in $V_{\{L\},\delta}$ has a linear series of type $g^1_{k+\varepsilon}$, where $k \geq 2$ is a fixed integer. For any integer k' satisfying suitable conditions, the surjectivity of the Abel map $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C}) \to \mathrm{Pic}^{k'+\varepsilon}(\widetilde{C})$ yields that a general line bundle in $\mathrm{Pic}^{k'+\varepsilon}(\widetilde{C})$ is non-special and hence the symmetric product $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C})$ is generically a $\mathbb{P}^r$-bundle over $\mathrm{Pic}^{k'+\varepsilon}(\widetilde{C})$, where $r := k' + \varepsilon - p + \delta$. By varying δ and k' satisfying $r = k' + \varepsilon - p + \delta$, we exhibit $(2k-r)$-dimensional components of $\mathbb{S}_r(S_\varepsilon^{[k]})$ for any r, except $r = k$ when $\varepsilon = 1$. More precisely, we prove the following:

Theorem 0.4. *(cf. Theorem 6.4) Let (S, L) be a general primitively polarized $K3$ or abelian surface of genus $p \geq 2$ and fix an integer $k \geq 2$. Then for any integer r such that $1 \leq r \leq k - \varepsilon$, and any integer k' such that $r + \varepsilon \leq k' \leq \min\{k, p + r - \varepsilon\}$, the set $\mathbb{S}_r(S_\varepsilon^{[k]})$ has an irreducible component $W_{r,k'}$ satisfying the following:*

(i) $\dim W_{r,k'} = 2k - r$;
(ii) *$W_{r,k'}$ is birational to a $\mathbb{P}^r$-bundle and hence algebraically coisotropic;*
(iii) *the class of the lines in the $\mathbb{P}^r$-fibres is $L - [2(k'+\varepsilon) - r - 1]\mathfrak{r}_k$;*
(iv) *the maximal rational quotient of the desingularization of $W_{r,k'}$ has dimension $2(k-r)$.*

Point (iv) positively answers, in the case of $S_\varepsilon^{[k]}$, a question by Charles and Pacienza (cf. [CP, Question 1.2]) concerning existence of subvarieties of an IHS manifold whose maximal rational quotients have the minimal possible dimension.

For $\varepsilon = 0$, examples of $(2k-r)$-dimensional components of $\mathbb{S}_r(S_\varepsilon^{[k]})$ for any r were already provided in [Vo, §4.1 Ex. 1 and Lemma 4.3] by considering fibres of the Hilbert-Chow morphism $\mu_k : S^{[k]} \to \mathrm{Sym}^k(S)$. However, our components $W_{r,k'}$ are not contained in the exceptional locus of μ and are covered by rational curves whose classes are often ample, and thus provide much stronger evidence for Conjecture 0.3.

In developing techniques towards proving the above theorems, we obtain some general results on IHS manifolds. First of all, in Proposition 2.13 we provide a criterion to tell whether a deformation of $S_\varepsilon^{[k]}$ is isomorphic to $S_\varepsilon'^{[k]}$ for some surface S'. This appears to be related to ideas from [Ad] and [MW]. Secondly, we prove that wall divisors can be contracted under general assumptions:

Theorem 0.5. *(cf. Theorem 2.5) Let X be a projective IHS manifold and let D be a wall divisor on X. Then one of the following holds:*

- *There exists a curve R dual to D such that rational curves of class R cover a divisor in X and a birational map $X \dashrightarrow Y$ contracting R. Moreover Y is singular symplectic.*
- *For a general deformation (X', D') of (X, D), there is a birational map $X' \dashrightarrow X''$ with X'' IHS and a contraction $X'' \to Y$ that contracts all curves dual to D'.*

In the first item of the theorem, R is a negative curve covering a divisor E, hence running a minimal model for $K_X + E$ will contract R at some step. The second item only holds for general deformations because we must remove the closed locus inside the deformation space of pairs (X, R) where the deformation of R is not in the boundary of the Mori cone. Notice that the second item, which states contractibility after a deformation and a birational map, also applies when the locus covered by R is not divisorial, while the first item is stronger in the divisorial case.

This result holds in particular for general non-projective deformations of (X, D), where a proof of the contraction theorem was, as yet, unavailable.

The paper is organized as follows. Section 1 contains background material concerning IHS manifolds and in particular varieties of the form $S_\varepsilon^{[k]}$. In Section 2 we recall known results on the birational geometry of IHS manifolds and use them to prove Theorem 0.5. We then specialize to deformations of $S_\varepsilon^{[k]}$ and prove that $-(k + 3 - 2\varepsilon)/2$ is a lower bound for the self-intersection of a primitive generator of an extremal ray of the Mori cone, cf. Proposition 2.11; the result is new for $\varepsilon = 1$, while it had already appeared in [BHT, Mo1] for $\varepsilon = 0$.

Section 3 summarises the results from [CK, KLM] concerning the Brill-Noether theory of nodal curves on symplectic surfaces. Classes $R_{p,\delta,k}$ are computed. Proposition 3.6 proves the existence of a family of rational curves of class $R_{p,\delta,k}$ having the expected dimension and surviving in any small deformation of $S_\varepsilon^{[k]}$ that keeps the class algebraic. In Section 4 we prove Theorem 0.1 and exhibit a collection of wall divisors that we later show to be essentially "complete" in Proposition 4.6.

Section 5 proves several results concerning vector bundle techniques associated with nodal curves, which are essential in the proof of Theorem 0.2. We believe that these results are of independent interest, due to the recent activity in the study of nodal curves on $K3$ and abelian surfaces. In particular, Proposition 5.3 extends a result by Pareschi [Pa, Lemma 2] to possibly nodal curves on symplectic surfaces; Proposition 5.5 and Lemma 5.6 describe properties of general (stable) sheaves.

The main results Theorems 0.2, and 0.4 are finally proved in Section 6.

Note. After this paper was completed, a paper by H. Y. Lin [Li] appeared on the arXiv, where the author also constructs components of the locus $\mathbb{S}_r$ for generalised Kummer manifolds. Our constructions are different from Lin's and the spirit of the two papers is quite distant.

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1. Generalities on IHS manifolds

A compact Kähler manifold X is called *hyperkähler* or *irreducible holomorphic symplectic (IHS)* if it is simply connected and $H^0(\Omega_X^2)$ is generated by a symplectic form.

The symplectic form implies the existence of a canonical quadratic form $q(\,)$ on $H^2(X,\mathbb{Z})$, called the *Beauville-Bogomolov form*, and of a constant c, the *Fujiki constant*, such that for every $\alpha \in H^2(X,\mathbb{Z})$ one has:

$$q(\alpha)^n = c \cdot \alpha^{2n}, \tag{1}$$

where $\dim(X) = 2n$. We will denote by $b(\,,\,)$ the bilinear form associated with q. This endows $H^2(X,\mathbb{Z})$ with the structure of a lattice of signature $(3, b_2(X)-3)$ and provides an embedding of $H_2(X,\mathbb{Z})$ in $H^2(X,\mathbb{Q})$ as the usual lattice embedding $\mathbb{L}^\vee \hookrightarrow \mathbb{L}\otimes\mathbb{Q}$. For any $D \in H^2(X,\mathbb{Z})$ denote by $\operatorname{div}(D)$ the positive generator of the ideal $b(D, H^2(X,\mathbb{Z}))$; then the elements $D/\operatorname{div}(D)$, with D running among all primitive elements in $H^2(X,\mathbb{Z})$, generate $H_2(X,\mathbb{Z})$. The quadratic form and the symplectic form also allow to define a period domain for IHS manifolds, much as in the case of $K3$ surfaces, as follows. For any lattice $\mathbb{L}$, one defines the period domain

$$\Omega_{\mathbb{L}} := \{\omega \in \mathbb{P}(\mathbb{L}\otimes\mathbb{C}) \,|\, q(\omega) = 0,\, b(\omega,\overline{\omega}) > 0\}.$$

Any isometry $f : H^2(X,\mathbb{Z}) \to \mathbb{L}$ is called a marking and there is a natural map, the period map $\mathcal{P}$, sending a marked IHS manifold (X, f) to $\mathcal{P}(X, f) := [f(\sigma_X)] \in \Omega_{\mathbb{L}}$, where σ_X is any symplectic form on X. Let $\mathcal{M}_{\mathbb{L}}$ be the moduli space of deformation equivalent marked IHS manifolds with H^2 isometric to $\mathbb{L}$. The period map $\mathcal{P} : \mathcal{M}_{\mathbb{L}} \to \Omega_{\mathbb{L}}$ is surjective [Hu1, Thm. 8.1] and it is a local isomorphism [Be1, Thm. 5].

There are singular analogues of IHS manifolds, called symplectic varieties. A compact normal variety X is *symplectic* if it has a unique (up to scalars) nondegenerate symplectic form on its smooth locus and a resolution of singularities $\pi : \widetilde{X} \to X$ such that the pullback of this form is everywhere defined, but possibly degenerate and $\widetilde{X}$ is simply connected. (Note that these conditions do not depend on the choice of $\widetilde{X}$). Therefore, if the pullback of the two-form is nondegenerate, $\widetilde{X}$ is IHS and we say that π is a *symplectic resolution.* Symplectic varieties share many properties with IHS manifolds, especially when they admit a symplectic resolution. In this case it is indeed possible to define a quadratic form on their second cohomology group and the following results hold.

Theorem 1.1. (Namikawa [Na, Thm. 2.2]) *Let $\pi : \widetilde{X} \to X$ be a symplectic resolution of a projective symplectic variety X. Then the Kuranishi spaces $\operatorname{Def}(X)$ and $\operatorname{Def}(\widetilde{X})$ are both smooth and of the same dimension. There exists a natural map $\pi_* : \operatorname{Def}(\widetilde{X}) \to \operatorname{Def}(X)$ that is a finite covering. Moreover, X has a flat deformation to an IHS manifold. Any smoothing of X is an IHS manifold obtained as a flat deformation of $\widetilde{X}$.*

The following results deals with the space of locally trivial deformations, which are deformations of the manifold that do not change its topological structure, that is, they preserve singularities.

Theorem 1.2. (Kirchner) *Let X be a normal symplectic variety admitting a symplectic resolution of singularities and such that $\operatorname{codim}(\operatorname{Sing} X) \geq 4$. Let $\operatorname{Def}(X)_{lt}$ denote the Kuranishi space of locally trivial deformations of X. Then there is a well defined period map $\mathcal{P} : \operatorname{Def}(X)_{lt} \to \Omega_{\mathbb{L}}$, where $\mathbb{L} \simeq H^2(X,\mathbb{Z})$, having generically injective tangent map.*

Proof. Locally trivial deformations are parametrized by a locally closed subset of $\operatorname{Def}(X)$. The latter is smooth by Theorem 1.1. As $\operatorname{Def}(X)_{lt}$ might not be smooth for some X, we can suppose that $\operatorname{Def}(X)_{lt}$ is smooth after replacing X with a small locally trivial deformation. Therefore, [Ki, Cor. 3.4.2] applies and first order locally trivial deformations are parametrised by $H^1(X \setminus \operatorname{Sing} X, \Omega^1_X) \simeq H^{1,1}(X)$, where $H^{1,1}(X)$ is the pure algebraic part of the mixed Hodge structure on $H^2(X)$. Now [Ki, Thm. 3.4.4] provides the period map from $H^1(X \setminus \operatorname{Sing} X, \Omega^1_X)$ with injective tangent map as stated above. □

Remark 1.3. Keep notation as in Theorem 1.2, and let $R_1, \dots, R_i$ be the curve classes that span the classes of curves contracted by the resolution of singularities $\widetilde{X} \to X$. Then, the proof of the theorem implies that first order locally trivial deformations of X are parametrised by $H^{1,1}(\widetilde{X}) \cap \langle R_1, \dots, R_i\rangle^\perp \simeq H^{1,1}(X)$.

Very few examples of IHS manifolds are known. The present paper will focus on the two infinite families of examples introduced by Beauville [Be1], namely, *Hilbert schemes of points on* $K3$ *surfaces* and *generalised Kummer manifolds*. Let S be a $K3$ or abelian surface. Throughout the paper we will let

$$\varepsilon = \varepsilon_S := \begin{cases} 1 & \text{if } S \text{ is abelian,} \\ 0 & \text{if } S \text{ is } K3. \end{cases} \tag{2}$$

It was proved by Beauville [Be1] that the Hilbert scheme $S^{[k+\varepsilon]}$ of 0-dimensional subschemes of S of length $k+\varepsilon$, where $k \geq 2$, inherits a symplectic form from S and is smooth. This uses in an essential way general results on Hilbert schemes of surfaces proven by Fogarty [Fo]. When S is $K3$, it is simply connected and thus an IHS manifold of dimension $2k$. When S is abelian, $S^{[k+1]}$ is not simply connected, but any fibre of the Albanese map $\Sigma_k : S^{[k+1]} \to \mathrm{Alb}\, S^{[k+1]} \simeq S$ is a $2k$-dimensional IHS manifold $K^k(S)$, which is called a generalised Kummer manifold. We recall that Σ_k is the composition of the Hilbert-Chow morphism $\mu_k : S^{[k+\varepsilon]} \to \mathrm{Sym}^{k+\varepsilon}(S)$ and the summation map $+ : \mathrm{Sym}^{k+\varepsilon}(S) \to S$.

In order to handle the two families simultaneously, we set

$$S^{[k]}_\varepsilon := \begin{cases} K^k(S) & \text{if } \varepsilon = 1 \text{ (i.e., } S \text{ is abelian),} \\ S^{[k]} & \text{if } \varepsilon = 0 \text{ (i.e., } S \text{ is } K3). \end{cases} \tag{3}$$

Note that $\dim S^{[k]}_\varepsilon = 2k$ in both cases, even though $S^{[k]}_1 \subsetneq S^{[k+1]}$. By abuse of notation, in the latter case we will still use the same symbol μ_k and the same name for the restriction of the Hilbert-Chow morphism to $S^{[k]}_1$.

There are natural embeddings

$$\mathrm{NS}(S) \hookrightarrow \mathrm{Pic}(S^{[k]}_\varepsilon), \tag{4}$$

$$N_1(S) \hookrightarrow N_1(S^{[k]}_\varepsilon). \tag{5}$$

Here N_1 denotes the group generated by classes of integral curves. The former is given by associating with the class of a prime divisor D in S the divisor

$$\left\{ Z \in S^{[k]}_\varepsilon \mid \mathrm{Supp}(Z) \cap D \neq \emptyset \right\} \tag{6}$$

and the latter is given by fixing a set of general points $\{x_1, \ldots, x_{k+\varepsilon-1}\} \subset S$ and associating with the class of an effective curve $C \subset S$ the class of the curve

$$\left\{ Z \in S^{[k]}_\varepsilon \mid x_{k+\varepsilon} \in \mathrm{Supp}(Z) \cap C,\ \{x_1, \ldots, x_{k+\varepsilon-1}\} \subset \mathrm{Supp}(Z) \right\}.$$

The exceptional divisor Δ_k of the Hilbert-Chow morphism μ_k has class $2\mathfrak{e}_k$ and one has an orthogonal decomposition with respect to $b(\, , \,)$:

$$H^2(S^{[k]}_\varepsilon, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}[\mathfrak{e}_k],$$

such that $b(\, , \,)$ restricts to the usual cup product on S and $q(\mathfrak{e}_k) = -2(k-1+2\varepsilon)$. The above isometry restricts to the embedding (4) on the algebraic part, whence

$$\mathrm{Pic}(S^{[k]}_\varepsilon) \simeq \mathrm{NS}(S) \oplus \mathbb{Z}[\mathfrak{e}_k]. \tag{7}$$

Under the embedding $H_2(S^{[k]}_\varepsilon, \mathbb{Z}) \hookrightarrow H^2(S^{[k]}_\varepsilon, \mathbb{Q})$ given by lattice duality, $H_2(S^{[k]}_\varepsilon, \mathbb{Z})$ is generated by $H^2(S, \mathbb{Z})$ and $\mathfrak{r}_k := \mathfrak{e}_k / 2(k-1+2\varepsilon)$. Here $\mathfrak{r}_k$ is the class of a general rational curve lying in the exceptional divisor Δ_k of the Hilbert-Chow morphism, that is, $\mathfrak{r}_k$ is the inverse image under μ_k of a cycle in $\mathrm{Sym}^{k+\varepsilon}(S)$ supported at precisely $k-1+\varepsilon$ points. Hence, $\mathrm{div}(\mathfrak{e}_k) = 2(k-1+2\varepsilon)$ and

$$N_1(S^{[k]}_\varepsilon) \simeq N_1(S) \oplus \mathbb{Z}[\mathfrak{r}_k]. \tag{8}$$

Any smooth Kähler deformation of $S_\varepsilon^{[k]}$ is called a manifold *of Kummer type* if $\varepsilon = 1$ and *of* $K3^{[k]}$ *type* if $\varepsilon = 0$.

Remark 1.4. The manifold $S_\varepsilon^{[k]}$ can also be defined by means of moduli spaces of stable sheaves on the underlying surface. There is a natural map $\mathcal{C}oh(S) \to H^{2*}(S,\mathbb{Z})$ sending a sheaf $\mathcal{F}$ to its *Mukai vector*

$$v(\mathcal{F}) := \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}(S)} = (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \chi(\mathcal{F}) + (\varepsilon - 1)\,\mathrm{rk}\,\mathcal{F}). \tag{9}$$

We recall that the Mukai vector of a sheaf $\mathcal{F}$ on a symplectic surface S is the discrete invariant that one has to fix when constructing moduli spaces of stable sheaves; it is defined as

$$v(\mathcal{F}) := (rk(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F}) + \mathrm{rk}(\mathcal{F})(\varepsilon - 1)) \in H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}).$$

In order to construct a moduli space of sheaves, one needs also a choice of a polarization L and, for most choices of primitive v (see [Yo1, Thm. 0.1]), a general ample L gives a smooth irreducible moduli space $\mathcal{M}(v)$ of Gieseker L-stable torsion free sheaves with Mukai vector v. Moreover, the fibre of $\mathcal{M}(v)$ under the Albanese map is deformation equivalent to $S_\varepsilon^{[k]}$.

If $v := (1, 0, 1 - 2\varepsilon - k)$, every element $[\mathcal{F}] \in \mathcal{M}(v)$ can be written as $\mathcal{F} = H_0 \otimes I_Z$ with $H_0 \in \mathrm{Pic}^0(S)$ and $[Z] \in S^{[k+\varepsilon]}$. Hence, one has $\mathcal{M}(v) \simeq S^{[k]}$ in the $K3$ case, while in the abelian case $K^k(S)$ is the fibre over 0 of the Albanese map of $\mathcal{M}(v)$, cf. [Yo2, Thm. 0.1].

For $k \geq 2$, we have a canonical Hodge isometry

$$H^2(S_\varepsilon^{[k]},\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}[\mathfrak{e}_k] \simeq v^\perp \subset H^{2*}(S,\mathbb{Z}) = \Lambda := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2-2\varepsilon},$$

such that $\mathfrak{e}_k$ is sent to $(1, 0, k - 1 + 2\varepsilon)$ and the second cohomology of S is sent back to itself, cf. [Yo2, Thm. 0.2]. In particular, one has

$$\frac{v + \mathfrak{e}_k}{2} \in \Lambda \ \text{ and } \ \frac{v - \mathfrak{e}_k}{2(k - 1 + 2\varepsilon)} \in \Lambda. \tag{10}$$

2. Birational geometry and wall divisors of IHS manifolds

Having trivial canonical bundle, IHS manifolds are minimal in the sense of MMP. Therefore, maps between IHS manifolds are rather rigid, as the following shows:

Proposition 2.1. *Let X and X' be two IHS manifolds and let $f : X \dashrightarrow X'$ be a birational map. Then the following hold:*

- (i) *The manifolds X and X' are deformation equivalent and $H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$ as Hodge structures.*
- (ii) *The map f has indeterminacy locus of codimension at least 2.*
- (iii) *If X is projective, there exists a klt divisor D such that the map f is a sequence of flips obtained by running the minimal model program for the pair (X, D).*

Proof. Item (i) is the content of [Hu1, Thm. 4.6], and (ii) is proved in [Hu1, Rem. 4.4] and holds true for all manifolds with nef canonical divisor. For (iii), any (sufficiently small) multiple of an effective divisor on a IHS manifold is klt (see [HT2, Rem. 12]). Therefore, if we take an ample divisor A on X' and set $D = \epsilon f^*(A)$, for $\epsilon << 1$, we have a klt pair (X, D). As A is ample and f is well defined on divisors, D is positive on all curves C such that $\mathrm{Locus}(\mathbb{R}^+[C])$[1] is a divisor. Therefore, by running the MMP for (X, D) we do not encounter any divisorial contraction. As f_*D is ample, (X', A) is a minimal model for (X, D). □

[1]We recall that the *locus* of $V \subset N_1(X)$ is the closure of the locus in X covered by curves of class lying in V, that is, $\mathrm{Locus}(V) := \overline{\{x \in \Gamma \subset X : [\Gamma] \in V\}}$.

We remark that a termination of *all* log-minimal models for IHS manifolds has recently been proven in [LP, Thm. 4.1], however we only need that there exists a log-minimal model which terminates.

Being well-defined on divisors, any birational map between two IHS manifolds induces a pullback map between their second cohomology groups. This allows to define a birational invariant called the *birational Kähler cone* of an IHS manifold X. We recall that the *positive cone* $\mathcal{C}_X$ is the connected component containing a Kähler class of the cone of positive classes inside $H^{1,1}(X,\mathbb{R})$. It contains the *Kähler cone* $\mathcal{K}_X$, which is the cone containing all Kähler classes. The birational Kähler cone $\mathcal{BK}_X$ is the union $\cup f^{-1}\mathcal{K}_{X'}$, where f runs through all birational maps between X and any IHS manifold X'. If X is projective, then the closure of the algebraic part of the birational Kähler cone is just the movable cone, that is, the closure of the cone of divisors whose linear systems have no divisorial base components.

We recall that an isomorphism $H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(Y,\mathbb{Z})$, where X and Y are two IHS manifolds, is called a parallel transport operator if it is induced by the parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along a path of smooth deformations $\pi : \mathcal{X} \to \mathbb{D}$ over a disc $\mathbb{D}$ such that X and Y are two fibres. The group of parallel self-operators is called the monodromy group and denoted $\mathrm{Mon}^2(X)$.

Definition 2.2. ([Mo1, Def. 1.2]) *Let X be an IHS manifold and let D be a divisor on X. Then D is called a wall divisor if $q(D) < 0$ and $f(D)^\perp \cap \mathcal{BK}_X = \emptyset$ for all Hodge isometries $f \in \mathrm{Mon}^2(X)$. The set of wall divisors on X is denoted by W_X.*

The ample cone is one of the connected components of $\mathcal{C}_X - \cup_{D\in W_X} D^\perp$, which was proven in [Mo1, Prop. 1.5]. Indeed, wall divisors are closely related to extremal rays of the Mori cone, as was analyzed independently in [BHT] and [Mo1]. In particular, dual divisors to generators of rational extremal rays of negative square are wall divisors by [Mo1, Lemma 1.4]. Notice that the extremal rays needed to determine the Kähler cone are indeed rational since the part of the Mori cone of curves of negative square is locally a finite rational polyhedron [HT2, Cor. 18]. The analogy runs deeper:

Proposition 2.3. *Let D be a divisor and let R be the primitive class $D/\operatorname{div}(D) \in H_2(X,\mathbb{Z}) \subset H^2(X,\mathbb{Q})$. Then D is a wall divisor if and only if there exists a Hodge isometry $f \in \mathrm{Mon}^2(X)$ such that $f(R)$ generates an extremal ray of the Mori cone on some IHS manifold X' birational to X.*

Proof. Let D be a wall divisor. As $q(D) < 0$, we have $D^\perp \cap \mathcal{C}_X \neq \emptyset$. Therefore, if X is projective, there is a Hodge isometry $f \in \mathrm{Mon}^2(X)$ such that $f(D)^\perp \cap \overline{\mathcal{BK}}_X \neq \emptyset$ by [Ma, Thm. 6.18 (2)]. If X is not projective, the same result is a direct consequence of [Hu1, Cor. 5.2 and Rem. 5.4], where the cycle Γ in the mentioned results is of parallel transport and acts as a Hodge isometry on $H^2(X,\mathbb{Z})$. By definition of wall divisor, $f(D)^\perp$ supports a component of the boundary of $\mathcal{BK}_X$. Up to taking a different birational model X' of X, we can suppose $f(D)^\perp \cap \overline{\mathcal{K}}_X \neq \emptyset$. As $q(D) < 0$, there exists a divisor $B \in \mathcal{C}_X$ such that D pairs negatively with B. As B is big and the B negative part of the ample cone is locally rationally polyhedral by [HT2, Prop. 13], we can also suppose that $f(D)^\perp$ supports a face of this cone (again, if needed, by changing birational model). This implies that R is an extremal ray.

The converse is the content of [Mo1, Lemma 1.4] (see also [BHT, Prop. 3]). □

Remark 2.4. The above result is also implied by [BHT, Cor. 6] and can be used in order to give an equivalent definition of wall divisors, i.e., divisors dual to extremal rays up to the action of parallel transport Hodge isometries. In other words, the MBM classes defined in [AV] are exactly the classes of curves dual to wall divisors.

A different characterisation of wall divisors can be given in terms of contractions:

Theorem 2.5. *Let R be a primitive rational curve on a projective IHS manifold X such that the dual divisor D is a wall divisor. Then one the following cases occurs:*

(i) $\mathrm{Locus}(\mathbb{R}^+[R])$ *contains a divisor of class a multiple of D. Furthermore, there exists a birational map $f : X \dashrightarrow Y$ with Y singular symplectic such that f contracts R.*

(ii) *For a general small deformation* (X_t, R_t) *of* (X, R) *the locus* $\mathrm{Locus}(\mathbb{R}^+[R_t])$ *is not a divisor and there exists an IHS manifold* X'_t *along with a birational map* $f_t : X_t \dashrightarrow X'_t$ *and a morphism* $X'_t \to Y_t$ *contracting* $f_t(R_t)$*.*

Proof. Let X'' be an IHS manifold deformation of X such that the parallel transport R'' of R is an effective rational curve generating the algebraic classes of $H_2(X'', \mathbb{Z})$ (cf. [Mo1, Thm. 1.3] for the existence of such an X''). Let D'' be the dual divisor to R''.

Suppose that $\mathrm{Locus}(\mathbb{R}^+[R''])$ has codimension one (thus, the same holds for $\mathrm{Locus}(\mathbb{R}^+[R])$ by semicontinuity) and let bD'' be the class of its closure. As we deform back to X, the divisor bD'' deforms to bD, which is thus effective and is contained in $\mathrm{Locus}(\mathbb{R}^+[R])$. As $D \cdot R < 0$, the MMP for the pair (X, D) yields the existence of a birational map f as in item (i).

Let us suppose now that $\mathrm{Locus}(\mathbb{R}^+[R''])$ has codimension at least two and show that we fall in case (ii). Under this assumption X'' contains no effective divisor. Then, by the wall and chamber decomposition of the positive cone given in [Ma, §5], the closure of the birational Kähler cone of X'' coincides with its positive cone. On the other hand, as the curve R'' is effective, the Kähler cone is the intersection of the positive cone with the half space of real $(1,1)$-classes intersecting R'' positively. By the definition of the birational Kähler cone, this yields the existence of an IHS manifold Z'' along with a birational map $X'' \dashrightarrow Z''$, the indeterminacy locus of which is $\mathrm{Locus}(\mathbb{R}^+[R''])$. In particular, the class $-R''$ is effective on Z'' as proved in [Hu2, Cor. 2.4]. We now deform X'' (hence, also X) to a projective IHS manifold where the class of R'' is still effective; this is possible as, by [BHT, Prop. 3], all small deformations of X where D stays of type $(1,1)$ have R or $-R$ effective and projective deformations are dense. In particular, we can choose a projective deformation X''' where the parallel transport of R is effective and extremal; indeed, up to changing birational model, R is an extremal ray on all deformations (X_0, R_0) belonging to the Zariski open set where $\overline{\mathcal{C}_{X_0}} = \overline{\mathcal{BK}_{X_0}}$. Therefore, the Contraction Theorem yields a contraction $X''' \to Y'''$ and the conclusion follows from the next lemma. □

Lemma 2.6. *Let* Z *be a projective IHS manifold and let* R *be a curve generating an extremal ray such that* $\mathrm{Locus}(\mathbb{R}^+[R])$ *has codimension at least* 2*. Let* $Z \to Y$ *be the contraction of this extremal ray. Then for all small locally trivial deformations* Y_t *of* Y *there is a symplectic resolution* $Z_t \to Y_t$ *contracting exactly* $\mathrm{Locus}(\mathbb{R}^+[R_t])$*, where* (Z_t, R_t) *is a small deformation of* (Z, R)*.*

Proof. By [Wi, Thm. 1.3], the singular locus of Y has codimension at least four. Let Y_t be a locally trivial small deformation of Y. Then Y_t has the same Beauville-Bogomolov form of that of Y (and also the same second Betti number) and it has a symplectic resolution Z_t, which is a small deformation of Z by Theorem 1.1. Remark 1.3 ensures that the deformation $[R_t]$ of $[R]$ is algebraic. As R is extremal, small deformations $[R_t]$ of its class are represented by curves R_t [BHT, Prop. 3]; the Rigidity Lemma then implies that R_t is contracted by $Z_t \to Y_t$. By Remark 1.3, $b_2(Z) = b_2(Y) + 1$. Hence, $b_2(Z_t) = b_2(Y_t) + 1$ and the map contracts precisely $\mathrm{Locus}(\mathbb{R}^+[R_t])$. □

Remark 2.7. The first item of Theorem 2.5 is slightly stronger than [Ma, Prop. 6.1] as it ensures that *exceptional divisors*, as defined in [Ma, Def. 5.1], are contractible, up to birational equivalence. This should be regarded as the higher dimensional analogue of the contractability of effective divisors with self-intersection -2 on $K3$ surfaces. Notice that, when R is reducible, the contraction does not necessarily have relative Picard rank one. The contraction map $f : X \dashrightarrow Y$ is a composition of flops and divisorial contractions and therefore is only rational. The second item of the proposition cannot be strengthened and in particular it might not hold for (X, R). Indeed, one has to take into account the action of the subgroup W_{exc} of Mon^2 generated by the reflections on reduced and irreducible exceptional divisors. The general deformations in the statement are precisely those manifolds where W_{exc} is the identity. Note that this set strictly contains the open set of manifolds with an irreducible Hodge structure and it is Zariski open as the set of generators of W_{exc} is finite up to the monodromy action.

Wall divisors on $S_\varepsilon^{[k]}$ can be determined lattice-theoretically using results of Yoshioka [Yo3] and Bayer and Macrì [BM1]. In the following, we use the same notation as in Remark 1.4.

Remark 2.8. In [BM1] and [Yo3], Bayer, Macrì and Yoshioka determine a decomposition of the space of stability conditions $\mathrm{Stab}^0(S,v)$ given by walls and chambers. Any stability condition σ in a chamber gives a smooth moduli space $M(v,S,\sigma)$ of stable objects in $D^b(S)$ with Mukai vector v, whereas most conditions lying on a wall give a singular space and conditions on nearby chambers give its symplectic resolution. (For our purposes, the so-called "fake walls" as in [BM1, Def. 2.20(a)] do not matter.) Moreover, for every σ in a chamber of $\mathrm{Stab}^0(S,v)$, [BM1, Thm 1.2] gives a map from $\mathrm{Stab}^0(S,v)$ to the positive part of the movable cone $\overline{\mathcal{BK}_{M(v,S,\sigma)}}$, and every chamber lands in $\mathcal{BK}_{M(v,S,\sigma)}$. By Proposition 2.3, this implies that all non-fake walls of $\mathrm{Stab}^0(S,v)$ are dual to wall divisors and, up to the action of W_{exc} (defined in Remark 2.7), we obtain all wall divisors of $M(v,S,\sigma)$ in this way. By Remark 1.4 along with the fact that Gieseker stability lies in $\mathrm{Stab}^0(S,v)$ for any v, the ordinary moduli spaces of Gieseker stable sheaves with Mukai vector v is obtained as $M(v,S,\sigma)$ for a $\sigma \in \mathrm{Stab}^0(S,v)$. In particular, $S_\varepsilon^{[k]}$ is the Albanese fibre of some $M(v,S,\sigma)$.

Theorem 2.9. *Let D be a divisor of $S_\varepsilon^{[k]}$ with $q(D)<0$ and let $T \subset \Lambda := H^{2*}(S,\mathbb{Z})$ be the saturated lattice generated by $v := (1,0,1-2\varepsilon-k)$ and D. Then D is a wall divisor if and only if there is an $s \in T$ such that*

(i) *$0 \le q(s) < b(s,v) \le (q(v)+q(s))/2$; or,*
(ii) *$\varepsilon = 0$, $q(s) = -2$ and $0 \le b(s,v) \le q(v)/2$.*

Proof. Remark 2.8 implies that all wall divisors of $S_\varepsilon^{[k]}$ correspond to walls in the space $\mathrm{Stab}^0(S,v)$.

For $\varepsilon = 0$ we can thus apply [BM1, Thms. 5.7 and 12.1] with $a := s$ and $b := v - s$; our inequalities are equivalent to imposing that both a and b are in the *positive cone of* T (cf. [BM1, Def. 5.4]), i.e., $q(a) \ge 0$ and $b(v,a) > 0$ and the same for b.

For $\varepsilon = 1$ the statement follows from[Yo3, Prop. 1.3]. Indeed, the conditions in [Yo3, Def. 1.2] can be rephrased by asking that $a := s$ and $b := v - s$ are in the positive cone of T as before. The additional condition $b(s,v)^2 > q(v)q(s)$ in [Yo3, Prop. 1.3] is equivalent to the requirement that T is indefinite, which is implied by $q(D) < 0$. □

Remark 2.10. A lattice T as in the above theorem can contain several elements s satisfying (i) and (ii), and abstractly isometric lattices can even correspond to different kinds of wall divisors, as the following example illustrates (cf. also [HT3, Sec. 4]). Let $k-1+2\varepsilon = 2rt$, where r and t are relatively prime integers. Let S be a symplectic surface and let $\mathcal{M}$ be the moduli space of stable sheaves with Mukai vector $v := (r,0,-t)$. Let $\Gamma \in H^{1,1}(\mathcal{M},\mathbb{Z})$ be the image of $(r,0,t)$ under the natural Hodge isometry $H^2(\mathcal{M},\mathbb{Z}) \simeq v^\perp \subset H^{2*}(S,\mathbb{Z})$. The saturated lattice generated by v and Γ is isometric to U and contains no elements s such that $q(s)=0$ and $b(s,v)=1$, unless either r or t are 1. Note that $\frac{v+\Gamma}{2r}$ and $\frac{v-\Gamma}{2t}$ satisfy the conditions of the above theorem, and hence Γ is a wall divisor. The lattice U is also associated with the exceptional divisor Δ_k of $S_\varepsilon^{[k]}$, but in the saturated lattice generated by v and $\mathfrak{e}_k$ there is an element s such that $b(s,v)=1$ and $q(s)=0$. However, isometric lattices as in Theorem 2.9 give rise to isometric wall divisors.

Theorem 2.9 enables us to extend to manifolds of Kummer type a result obtained by Bayer, Hassett and Tschinkel, and independently by the third author, in the case of manifolds of $K3^{[k]}$ type.

Proposition 2.11. *Let R be a primitive generator of an extremal ray of the Mori cone of a manifold X deformation of $S_\varepsilon^{[k]}$. Then $q(R) \ge -(k+3-2\varepsilon)/2$.*

Proof. For $\varepsilon = 0$ this is the content of [Mo1, Cor. 2.7] or [BHT, Prop. 2].

Let $\varepsilon = 1$ and $q(R) < 0$. Then the dual divisor D to R, namely, $R = D/\operatorname{div}(D)$, is a wall divisor by Proposition 2.3. As wall divisors are invariant under deformation, we can assume $X = S_1^{[k]}$ for some abelian surface S. Let T, v, s be as in Theorem 2.9. Let $a := \mathrm{GCD}(q(v), b(s,v))$. We have $aD = b(s,v)v - q(v)s$ and $\mathrm{div}(D) = q(v)/a$. Then we have

$$\begin{aligned} q(D) &= (q(v)^2 q(s) - q(v) b(s,v)^2)/a^2 \geq \\ &\geq \frac{4q(v)^2 q(s) - q(v)^3 - q(v)q(s)^2 - 2q(v)^2 q(s)}{4a^2} \geq -\frac{q(v)^3}{4a^2} = -\frac{(k+1)\,\mathrm{div}(D)^2}{2}, \end{aligned}$$

where we have used the inequality $b(s,v) \leq (q(v)+q(s))/2$. □

The above statement in the $K3$ case is part of a conjecture by Hassett and Tschinkel [HT1, Conj. 1.2], who predicted that the class R of a primitive 1-cycle in a manifold of $K3^{[k]}$-type is effective if and only if the inequality in Proposition 2.11 holds. Counterexamples to the if part are known, cf. [BM2, Rem. 10.4] and [CK, Rem. 8.10]. The analogous conjecture for manifolds X of Kummer type was stated only in the four-dimensional case [HT1, Conj. 1.4]. Proposition 2.11 shows that the only if part holds independently of the dimension of X; on the other hand, the if part fails as soon as $\dim X > 4$, as the following example shows.

Example 2.12. Let S be an abelian surface with an order four symplectic group automorphism φ. Such an automorphism induces an automorphism φ of order four on all the generalised Kummer manifolds arising from S. There exists a primitive non-effective class $F \subset \mathrm{NS}(S)$ such that $\varphi(F) = -F$ and $F^2 = -2$, cf. [Fu, Table 15]. This class gives a 1-cycle class in $N_1(S_1^{[k]})$ that is orthogonal to any φ-invariant ample class (hence, it is not effective) and has square -2. This shows that the inequality in Proposition 2.11 is not sufficient for the effectivity of a 1-cycle.

We now state a criterion for determining whether a projective manifold of $K3^{[k]}$ or Kummer type is isomorphic to $S_\varepsilon^{[k]}$ for some S.

Proposition 2.13. *Let X be a projective manifold of $K3^{[k]}$ or Kummer type. Then X is isomorphic to $S_\epsilon^{[k]}$ for some S if and only if there is a birational map $f : S_\varepsilon^{[k]} \dashrightarrow X$ and $f^*[D] \in \mathfrak{e}_k^\perp$ for some nef divisor $D \in \mathrm{NS}(X)$.*

Proof. The only if part is trivial and we prove the converse implication.

We first claim that $\overline{\mathcal{BK}}_{S_\epsilon^{[k]}} \cap \mathfrak{e}_k^\perp = \overline{\mathcal{K}}_{S_\epsilon^{[k]}} \cap \mathfrak{e}_k^\perp$, that is, all movable divisors on $\mathfrak{e}_k^\perp$ are nef. Granting this, the divisor class $f^*[D] \in \mathrm{NS}(S_\epsilon^{[k]})$ lies in the image of (4) and is movable, hence nef. Let us take an ample class $A \in \mathrm{Pic}(X)$ and consider the ample class $D + sA$ (for small s). The pullback under f of $D + sA$ is ample on $S_\varepsilon^{[k]}$, as this class cannot be $\mathfrak{e}_k$-negative (it stays movable) and the limit of these classes as s goes to zero is nef. Thus, $X \simeq S_\varepsilon^{[k]}$ by the global Torelli Theorem [Ma, Thms. 1.2 and 1.3].

It remains to prove the claim. Let $E \in \overline{\mathcal{C}_{S_\epsilon^{[k]}}}$ be a divisor such that $b(E, \mathfrak{e}_k) = 0$. In particular, the class $[E]$ lies in the image of the restriction of (4) to the closure of the positive cone $\overline{\mathcal{C}_S}$ and we will denote by E_S an effective divisor on S representing its preimage. Let us assume that $[E]$ is not nef. Any irreducible curve $\Gamma \subset S_\epsilon^{[k]}$ such that $\Gamma \cdot E < 0$ is not contained in Δ_k. The image of such a Γ under the projection to S of the incidence variety

$$I := \left\{ (P, [Z]) \in S \times S_\epsilon^{[k]} \mid P \in \mathrm{Supp}(Z) \right\} \tag{11}$$

is an effective curve $\Gamma_S \subset S$, whose class is sent to $[\Gamma]$ by (5). Since $E_S \cdot \Gamma_S < 0$, the divisor E_S is not nef. In the abelian case this is impossible and hence $[E]$ is nef and we are done. Let us show that in the $K3$ case $[E]$ is not movable. Let $R \subset S$ be a (-2)-curve such that $E_S \cdot R < 0$ and denote by $D_R \subset S^{[k]}$ the corresponding uniruled divisor defined as in (6). Then $b(E, D_R) < 0$, whence E is not movable by [Ma, Prop. 5.6]. □

Remark 2.14. In the above proposition the condition that X is birational to $S_\varepsilon^{[k]}$ is equivalent to asking that there is a parallel transport Hodge isometry between the two manifolds, cf. [Ma, Thm.

1.3]. If S is $K3$, there is a topological way of recognizing a parallel transport Hodge isometry, cf. [Ma, Cor. 9.5]. By the computation of the monodromy group in the Kummer case [Mo2, Thm. 2.3], it is highly expected that a similar characterisation holds if S is abelian.

We end this section with a result that will be used in the proof of Theorem 0.2.

Proposition 2.15. *Let X be a holomorphic symplectic manifold, i. e., there is an étale cover $\widetilde{X} := \Pi_{i\in I} M_i \to X$, where every M_i is either IHS or abelian. For every subset $J \subset I$, denote by F_J the image in X of a general fibre of the projection $\widetilde{X} \to \Pi_{j\in J} M_j$. Let $\mathcal{P}$ be a projective variety along with a morphism $q : \mathcal{P} \to X$ that is generically a $\mathbb{P}^r$-bundle, for an integer $r \geq 1$.*

Assume that $g : \mathcal{P} \dashrightarrow Y$ is a rational map to an IHS manifold Y such that:

(i) $\dim Y = 2r + \dim X$*;*
(ii) *g is well-defined in codimension one;*
(iii) *g is injective on general fibres of q;*
(iv) *for all $J \neq \emptyset$, the map g is generically injective when restricted to $q^{-1}(F_J)$;*
(v) *the image of g is an irreducible component of the locus covered by the rational curves of class $[g(\ell)]$, where ℓ is a line in a fibre of q.*

Then g is generically finite.

Proof. Our proof consists in showing that the symplectic form on Y pulls back to a non-zero two-form on $\mathcal{P}$ and that this two-form degenerates only along fibres of q. Therefore, the statement will follow, by contradiction, from assumption (iv).

Let T denote the closure of the image of g and $h : \widetilde{T} \to T$ be its desingularization. We consider the maximal rationally connected fibration $\pi : \widetilde{T} \dashrightarrow B$ of T. We denote by $\tilde{g} : \mathcal{P} \dashrightarrow \widetilde{T}$ the rational map induced by g and assume that a general fibre of g (or, equivalently, of $\tilde{g}$) has dimension α. As T is a component of the locus covered by rational curves of class $[g(l)]$ by assumption (v), we can apply [AV, Thm. 4.4] along with assumption (i) and obtain that a general fibre F of π has dimension equal to $\mathrm{codim}_Y T = r + \alpha$ and $\tilde{g}^{-1}(F)$ has dimension $r + 2\alpha$. By hypothesis (iii), the locus $q(\tilde{g}^{-1}(F))$ is 2α-dimensional.

Let σ be a symplectic form on Y. As in [AV, Pf. of Thm. 4.4], one shows that the form $h^*(\sigma|_T)$ is degenerate precisely on the fibres of π, which are rationally connected and hence have no two-forms. By definition of $\tilde{g}$, the two-form $g^*(\sigma_{|T})$ coincides with $\widetilde{g}^*(h^*(\sigma_{|T}))$ where the latter is defined. Since $g^*(\sigma_{|T})$ is well-defined in codimension one by assumption (ii), it extends to a two-form on $\mathcal{P}$ that is degenerate along $\widetilde{g}^{-1}(F)$. On the other hand, any form on $\mathcal{P}$ is the pullback of a form on X and forms on X can be degenerate only along the F_J's. Therefore, if $\alpha > 0$, then the closure of $q(\tilde{g}^{-1}(F))$ coincides with F_J for some $J \subset I$. However, this contradicts assumption (iv). □

3. Curves on symplectic surfaces and their pencils

For a polarized surface (S, L), we denote by $\{L\}$ the continuous system of L, that is, the connected component of $\mathrm{Hilb}(S)$ containing the linear system $|L|$. If S is a $K3$ surface, then $|L| = \{L\}$. If S is an abelian surface, then $\{L\}$ is obtained translating curves in $|L|$ by points of S, whence $\dim\{L\} = \dim|L|+2$. We denote by $V_{|L|,\delta}(S)$ (respectively, $V_{\{L\},\delta}(S)$) the Severi variety parametrizing irreducible curves in $\{L\}$ (resp. $|L|$) with precisely δ nodes, and by $\{L\}^1_{\delta,d}$ (resp. $|L|^1_{\delta,d}$) the Brill-Noether locus parametrizing those nodal curves C whose normalization $\widetilde{C}$ carries a linear series of type g^1_d, that is, a pair (A, V) where A is a line bundle of degree d on $\widetilde{C}$ and V is a 2-dimensional space of global sections of A. We denote by $G^1_d(\widetilde{C})$ the *Brill-Noether variety* parametrizing all linear series of type g^1_d on $\widetilde{C}$. It has *expected dimension* $\max\{\rho(g, 1, d), 0\}$, where g is the genus of $\widetilde{C}$, and $\rho(g, r, d) := g - (r + 1)(g - d + r)$ is the classical *Brill-Noether number.*

We also recall that the ramification of a *base point free* (that is, globally generated) g^1_d is said to be *simple* if the ramification of the induced covering map onto $\mathbb{P}^1$ is, and a node of C is called *non-neutral* with respect to the g^1_d if its two preimages in $\widetilde{C}$ are not identified by the covering map.

We will make use of the following result (recall the convention (2)):

Theorem 3.1. *Let (S, L) be a general polarized $K3$ or abelian surface of genus $p := p_a(L)$. Let δ and k be integers satisfying $0 \leq \delta \leq p - 2\varepsilon$ and $k + \varepsilon \geq 2$. Then the following hold:*

(i) $\{L\}^1_{\delta,k+\varepsilon} \neq \emptyset$ *if and only if*

$$\delta \geq \alpha\Big(p - \delta - \varepsilon - (k - 1 + 2\varepsilon)(\alpha + 1)\Big), \tag{12}$$

where

$$\alpha = \Big\lfloor \frac{p - \delta - \varepsilon}{2(k - 1 + 2\varepsilon)} \Big\rfloor; \tag{13}$$

(ii) *whenever non-empty, $\{L\}^1_{\delta,k+\varepsilon}$ is equidimensional of dimension $\min\{p - \delta, 2(k - 1 + \varepsilon)\}$ and a general element in each component is an irreducible curve C with normalization $\widetilde{C}$ of genus $g := p - \delta$ such that $\dim G^1_{k+\varepsilon}(\widetilde{C}) = \max\{0, \rho(g, 1, k + \varepsilon) = 2(k - 1 + \varepsilon) - g\}$;*

(iii) *there is at least one component $Y_{\delta,k+\varepsilon}$ of $\{L\}^1_{\delta,k+\varepsilon}$ where, for C and $\widetilde{C}$ as in (ii), when $g \geq 2(k - 1 + \varepsilon)$ (respectively $g < 2(k - 1 + \varepsilon)$), any (resp. a general) $g^1_{k+\varepsilon}$ on $\widetilde{C}$ is base point free and has simple ramification and all nodes of C are non-neutral with respect to it. Furthermore, when S is abelian, for general C in this component the Brill-Noether variety $G^1_{k+1}(\widetilde{C})$ is reduced.*

Proof. This is [KLM, Thm. 1.6] when S is abelian and [CK, Thm. 0.1], combined with [KLM, Rem. 5.6], when S is $K3$. □

Remark 3.2. (i) The condition (12) is equivalent to

$$\rho(p, l, (k + \varepsilon)l + \delta) + \varepsilon l(l + 2) \geq 0 \text{ for all integers } l \geq 0. \tag{14}$$

Indeed, the left hand side of (14) attains its minimum for $l = \alpha$ as in (13) and (12) is a rewrite of (14) with $l = \alpha$.

(ii) The condition (12) is also *necessary* for the existence of an irreducible curve $C \in \{L\}$ with *partial normalization* $\widetilde{C}$ of arithmetic genus $g := p - \delta$ carrying a $g^1_{k+\varepsilon}$, regardless of its singularities. This follows from [KLM, Thm. 5.9 and Rem. 5.11] in the abelian case and [CK, Thm. 3.1] in the $K3$ case, by remarking that the proofs go through replacing the normalization of the curve with a partial normalization, as remarked in [CK, Rem. 3.2(b)]. We will however not use this fact in the present paper.

Let $\mathfrak{g}$ be a linear series of type $g^r_{k+\varepsilon}$ on the normalization $\widetilde{C}$ of a curve $C \subset S$, that is, $\mathfrak{g} = (A, V)$, where A is a line bundle of degree $k + \varepsilon$ and $V \subseteq H^0(A)$ is an $(r + 1)$-dimensional subspace. If $\mathfrak{g}$ is base point free, we have a natural rational map

$$\iota_{\mathfrak{g}} : \mathbb{P}^r := \mathbb{P}(V) \dashrightarrow S^{[k+\varepsilon]} \tag{15}$$

obtained from the composition $\mathbb{P}(V) \subseteq |A| \subset \mathrm{Sym}^{k+\varepsilon}(\widetilde{C}) \to \mathrm{Sym}^{k+\varepsilon}(C) \subset \mathrm{Sym}^{k+\varepsilon}(S)$, whose image does not lie in the exceptional locus Δ_k of the Hilbert-Chow morphism. Thus, $\mathfrak{g}$ defines a rational r-fold inside the Hilbert scheme $S^{[k+\varepsilon]}$. In particular, when $r = 1$, we obtain a rational curve.

When S is an abelian surface, the Albanese map Σ_k restricted to the image of $\iota_{\mathfrak{g}}$ is constant, because otherwise we would get a rational subvariety in S. Therefore, up to translating the curve C, we may assume that (15) lands into the generalised Kummer variety $K^k(S)$. Let us now specialise to the case $r = 1$, denote by $\nu : \widetilde{C} \to C$ the normalization map, and let $R_{C,\nu_*\mathfrak{g}} \subset S^{[k]}_\varepsilon$ be the rational curve image of $\iota_{\mathfrak{g}}$, recalling (3). (The same construction can be performed for any linear series on a partial normalization of C.)

Lemma 3.3. *Let $C \in \{L\}^1_{\delta,k+\varepsilon}$ be a curve whose normalization possesses a linear series $\mathfrak{g}$ of type $g^1_{k+\varepsilon}$ with simple ramification and such that all nodes of C are non-neutral with respect to it.*

Then the class of the rational curve $R_{C,\nu_*\mathfrak{g}}$ in $H_2(S_\varepsilon^{[k]},\mathbb{Z})$ with respect to the decomposition (8) *is*

$$R_{p,\delta,k} := L - (p-\delta+k-1+\varepsilon)\mathfrak{r}_k \tag{16}$$

and its dual divisor class is

$$D_{p,\delta,k} := L - \frac{(p-\delta+k-1+\varepsilon)}{2(k-1+2\varepsilon)}\mathfrak{e}_k, \tag{17}$$

Proof. In the $K3$ case, this is [CK, Lemma 2.1]. The proof in the abelian case is similar. □

In particular, one has:

$$q(R_{p,\delta,k}) = 2(p-1) - \frac{(p-\delta+k-1+\varepsilon)^2}{2(k-1+2\varepsilon)}. \tag{18}$$

Observe that, for fixed values of k and p, the minimum in (18), as well as the maximal "slope" $p-\delta+k-1+\varepsilon$ of the class $R_{p,\delta,k}$, is reached for a curve with the minimal number of nodes.

Remark 3.4. Under the same hypotheses as in Lemma 3.3, one may rewrite (16) as

$$q(R_{p,\delta,k}) = 2\Big(\rho+\varepsilon\alpha(\alpha+2)+\varepsilon-1\Big) - \frac{\beta^2}{2(k-1+2\varepsilon)},$$

with α as in (13),

$$\rho := \rho(p,\alpha,(k+\varepsilon)\alpha+\delta) \ \text{ and } \ \beta := (2\alpha+1)(k-1+2\varepsilon)-p+\delta+\varepsilon.$$

In particular, we have $-(k-1+2\varepsilon) < \beta \leq k-1+2\varepsilon$, and (12), or equivalently (14) with $l=\alpha$, says that $\rho+\varepsilon\alpha(\alpha+2)\geq 0$. From these inequalities one reobtains the bound from Proposition 2.11:

$$q(R_{p,\delta,k}) \geq -\frac{k+3-2\varepsilon}{2},$$

with equality if and only if

$$p = \alpha(\alpha+1)(k-1+2\varepsilon)+\varepsilon \ \text{ and } \ \delta = \alpha(\alpha-1)(k-1+2\varepsilon)$$

(see also [CK, Cor. 3.4]).

Proposition 2.11 yields the following extension of [CK, Cor. 8.6] to Kummer manifolds.

Corollary 3.5. *Assume that* $\mathrm{NS}(S)\simeq\mathbb{Z}[L]$. *Let* $n\in\mathbb{Z}^{>0}$ *and set* $p := n(n+1)(k-1+2\varepsilon)+\varepsilon$ *and* $\delta := n(n-1)(k-1+2\varepsilon)$. *Then the rational curves in* $S_\varepsilon^{[k]}$ *obtained from the component* $Y_{\delta,k}$ *of Theorem 3.1 generate extremal rays of* $S_\varepsilon^{[k]}$.

We also have:

Proposition 3.6. *The rational curves in* $S_\varepsilon^{[k]}$ *obtained from any component of the relative Brill-Noether variety* $\mathcal{G}_k^1(\{L\}^1_{\delta,k+\varepsilon})$ *parametrizing pairs* $(C,\mathfrak{g})$ *such that* $C\in\{L\}^1_{\delta,k+\varepsilon}$ *and* $\mathfrak{g}$ *is a* $g^1_{k+\varepsilon}$ *on the normalization of* C *move in a family of rational curves of dimension precisely* $2k-2$.

Any small deformation X_t *of* $X_0 = S_\varepsilon^{[k]}$ *keeping the class of the rational curves algebraic contains a* $(2k-2)$*-dimensional family of rational curves that are deformations of the rational curves in* $S_\varepsilon^{[k]}$.

Proof. Any irreducible family of rational curves in $S_\varepsilon^{[k]}$ containing our family yields, by the incidence (11), a family of pairs $(C,\mathfrak{g})$ with $C\in\{L\}$ and $\mathfrak{g}$ a linear series of type $g^1_{k+\varepsilon}$ on the normalization of C. By [KLM, Thm. 5.3], the rational curves will therefore move in a family of dimension precisely $2k-2$, which is the expected dimension of any family of rational curves on a $(2k)$-dimensional IHS manifold [Ra, Cor. 5.1]. Hence, as a consequence of [Ra, Cor. 3.2-3.3], the rational curves will deform to any X_t as in the statement, cf. [BHT, Pf. of Prop. 3]. □

4. Examples of wall divisors

Let (S, L) be a general abelian or $K3$ surface, and fix integers p, k and δ satisfying all inequalities in Theorem 3.1, in particular (12). Let $R_{p,\delta,k}$ be as in Lemma 3.3; in particular, its class is given by (16). Denote by $D_{p,\delta,k}$ its dual divisor (class).

Theorem 4.1. *The divisor $D_{p,\delta,k}$ is a wall divisor if and only if $q(R_{p,\delta,k}) < 0$.*

Proof. We only need to prove the "if" part. By Proposition 3.6, the family of rational curves with class $R_{p,\delta,k}$ has a component of dimension $2k-2$ and deforms in all small deformations X_t of $S_\varepsilon^{[k]}$ where the class $R_{p,\delta,k}$ remains algebraic. Let (X_t, R_t) be a very general such deformation. The class R_t spans $N_1(X_t)$, hence it is extremal. As it has negative square, its dual is a wall divisor. Since wall divisors are invariant under deformation, $D_{p,\delta,k}$ is a wall divisor on $S_\varepsilon^{[k]}$, too. □

Remark 4.2. If $D_{p,\delta,k}$ is a wall divisor, we can recover the lattice T associated with it in Theorem 2.9. Set $a := \mathrm{GCD}(2k-2+4\varepsilon, g+k-1+\varepsilon)$, $ab := g+k-1+\varepsilon$ and $ac := 2k-2+4\varepsilon$. The saturation of the lattice generated by v and $D_{p,\delta,k}$ is $T := \langle v, w\rangle$, where $w = \frac{b}{c}(v-\mathfrak{e}_k)+L-v$. Note that $q(w) = 2\delta-2+2\varepsilon$ and $b(w,v) = g-k+1-3\varepsilon$. The element w does not necessarily satisfy the inequalities (i) or (ii) in Theorem 2.9 for s. However, this occurs in some special cases, e.g., in the examples below.

Example 4.3. Let $p = 2k-2+5\varepsilon$ and $\delta = 0$. Then $q(R_{p,\delta,k}) = -\frac{k+3-2\varepsilon}{2}$ and the lattice T associated with $R_{p,\delta,k}$ is isometric to $\begin{pmatrix} -2+2\varepsilon & k-1+2\varepsilon \\ k-1+2\varepsilon & 2k-2+4\varepsilon \end{pmatrix}$, cf. Remark 4.2.

Example 4.4. Let $p = 2k-2+5\varepsilon-a$, $a \leq k-1+2\varepsilon$, and $\delta = 0$. Then $q(R_{p,\delta,k}) < 0$ and the lattice T associated with $R_{p,\delta,k}$ is isometric to $\begin{pmatrix} -2+2\varepsilon & k-1+2\varepsilon-a \\ k-1+2\varepsilon-a & 2k-2+4\varepsilon \end{pmatrix}$, cf. Remark 4.2.

Example 4.5. Let $p = 2k-2+5\varepsilon$ and $0 \leq \delta \leq \frac{k-1+2\varepsilon}{2}$. Then $q(R_{p,\delta,k}) < 0$ and the lattice T associated with $R_{p,\delta,k}$ is isometric to $\begin{pmatrix} 2\delta-2+2\varepsilon & k-1+2\varepsilon \\ k-1+2\varepsilon & 2k-2+4\varepsilon \end{pmatrix}$, cf. Remark 4.2.

Proposition 4.6. *Let $k \geq 2$ be an integer and set $\varepsilon = 0$ (respectively, $\varepsilon = 1$). Let $v := (1, 0, 1-2\varepsilon-k)$ and let $s \in \Lambda = U^{\oplus 4} \oplus E_8(-1)^{\oplus 2-2\varepsilon}$ be an element satisfying the inequalities (i) or (ii) in Theorem 2.9. Let $T = \langle v, s\rangle$. Then there exists a primitively polarized $K3$ (resp. abelian) surface (S, L) of genus p and an integer $0 \leq \delta \leq p-2\varepsilon$ such that p, δ, k satisfy (12) and the following hold:*

(a) *the divisor $D_{p,\delta,k}$ is a wall divisor;*
(b) *the saturation of the lattice generated by v and $D_{p,\delta,k}$ in Λ is isometric to T.*

Proof. As soon as $\{L\}^1_{\delta,k+\varepsilon}$ is non-empty, we obtain that also $\{L\}^1_{\delta+1,k+\varepsilon}$ is non-empty, as the condition (12) is satisfied and Theorem 3.1(i) applies. By a direct computation using the classes of $D_{p,\delta,k}$ and $D_{p,\delta+1,k}$, if the saturation of the lattice generated by $D_{p,\delta,k}$ and v is isometric to $\begin{pmatrix} 2\delta-2+2\varepsilon & b \\ b & 2k-2+4\varepsilon \end{pmatrix}$, the saturation of the lattice generated by $D_{p,\delta+1,k}$ and v is isometric to $\begin{pmatrix} 2\delta+2\varepsilon & b-1 \\ b-1 & 2k-2+4\varepsilon \end{pmatrix}$, for some $b \in \mathbb{Z}$. One readily checks that if inequality (12) is satisfied for a triple (p, δ, k), then it is also satisfied for the triple $(p-1, \delta, k)$. Therefore, if (S, L) has genus p and $\{L\}^1_{\delta,k+\varepsilon}$ is non-empty, then $\{L'\}^1_{\delta,k+\varepsilon}$ is non-empty for every primitively polarized (S', L') of genus $p-1$ by Theorem 3.1(i). By direct computation again, if the corresponding lattice in the genus p case is $\begin{pmatrix} 2\delta-2+2\varepsilon & b \\ b & 2k-2+4\varepsilon \end{pmatrix}$, the lattice in the genus $(p-1)$ case is $\begin{pmatrix} 2\delta-2+2\varepsilon & b-1 \\ b-1 & 2k-2+4\varepsilon \end{pmatrix}$. These remarks along with Example 4.3 give us all possible isometry classes of lattices T as in the statement. □

Remark 4.7. As explained in Remark 2.10, the above proposition does not give all wall divisors up to the monodromy action. However, when $k-1+2\varepsilon$ is a prime power, we have that T determines and is determined by the monodromy orbit of D as all isometries of $H^2(S_\varepsilon^{[k]})$ can be extended to isometries of Λ fixing v. Hence the above proposition gives a full list of wall divisors up to monodromy in these cases.

5. Lazarsfeld-Mukai bundles associated with nodal curves

In this section, we will analyze the structure of Lazarsfeld-Mukai bundles associated with linear series on the normalization of a nodal curve, with the aim of using their moduli spaces to produce interesting subvarieties of IHS manifolds in the next section. The main result of this section is Proposition 5.5, which characterizes the cohomology of Lazarsfeld-Mukai bundles twisted by ideal sheaves of points.

Let C be a nodal curve on an abelian or $K3$ surface S such that its normalization $\widetilde{C}$ possesses a *complete* $g^1_{k+\varepsilon}$, that is, a globally generated line bundle A of degree $k+\varepsilon$ such that $h^0(A)=2$. We denote by $\nu : \widetilde{C} \to C$ the normalization map, and by N the 0-dimensional subscheme of the nodes of C. By standard facts (cf., e.g., [FKP, Prop. 3.2]), ν_*A is a torsion free sheaf of rank one on C that fails to be locally free precisely at N.

Let $f : \widetilde{S} \to S$ be the blow up of S at N, so that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{C} & \subset & \widetilde{S} \\ {\scriptstyle\nu}\downarrow & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle f} \\ C & \subset & S \end{array} \tag{19}$$

We denote by $\mathcal{E}_{\widetilde{C},A}$ and $\mathcal{E}_{C,\nu_*A}$ the so-called *Lazarsfeld-Mukai bundles* associated with the line bundle A on $\widetilde{C}$ and the torsion free sheaf ν_*A on C, respectively; the duals of these bundles are the kernels of the evaluation maps regarding A and ν_*A as torsion sheaves on the surfaces, that is, we have the following short exact sequences:

$$0 \longrightarrow \mathcal{E}^\vee_{\widetilde{C},A} \longrightarrow H^0(\widetilde{C},A)\otimes\mathcal{O}_{\widetilde{S}} \xrightarrow{ev_{\widetilde{S},A}} A \longrightarrow 0, \tag{20}$$

and

$$0 \longrightarrow \mathcal{E}^\vee_{C,\nu_*A} \longrightarrow H^0(C,\nu_*A)\otimes\mathcal{O}_{S} \xrightarrow{ev_{S,\nu_*A}} \nu_*A. \tag{21}$$

The right arrow in (21) might be non-surjective, as ν_*A is not necessarily globally generated (cf. Lemma 5.1). Pushing forward (20) to $\widetilde{S}$ and using the isomorphisms $H^0(\widetilde{C},A)\simeq H^0(C,\nu_*A)$ and $f_*\mathcal{O}_{\widetilde{S}}\simeq\mathcal{O}_S$, one shows that

$$\mathcal{E}^\vee_{C,\nu_*A} \simeq f_*(\mathcal{E}^\vee_{\widetilde{C},A}). \tag{22}$$

The following result establishes when (21) is exact on the right. (Recall the definition of neutral nodes from the beginning of §3.)

Lemma 5.1. *Let C be a nodal curve and denote by $\nu : \widetilde{C} \to C$ the normalization map. Let A be a complete, base point free pencil on $\widetilde{C}$. Then the sheaf ν_*A is globally generated except precisely at the nodes of C that are neutral with respect to $|A|$.*

Proof. We can assume that C has only one node P, since the general case is analogous using partial normalizations. Let $\phi_{|A|} : \widetilde{C} \to \mathbb{P}^1$ be the morphism defined by $|A|$.

Assume that P is a neutral node with respect to $|A|$. Then $\phi_{|A|}$ factors through a morphism $\phi : C \to \mathbb{P}^1$ having the same degree as $\phi_{|A|}$. Having set $A' := \phi^*\mathcal{O}_{\mathbb{P}^1}(1)$, one has $\nu^*A' \simeq A$ and $\nu_*A \simeq \nu_*\nu^*A' \simeq A'\otimes\nu_*\mathcal{O}_C$, hence ν_*A sits in the following short exact sequence:

$$0 \longrightarrow A' \longrightarrow \nu_*A \longrightarrow \mathcal{O}_P \to 0.$$

Since $h^0(C,\nu_*A)=h^0(\widetilde{C},A)=h^0(C,A')=2$, the sheaf ν_*A cannot be globally generated.

Conversely, assume that ν_*A is not globally generated at P, that is, the evaluation map

$$ev: H^0(\nu_*A)\otimes\mathcal{O}_C\longrightarrow\nu_*A$$

is not surjective. Since A is globally generated and ν is a finite map, we have a surjection

$$H^0(\nu_*A)\otimes\nu_*\mathcal{O}_{\widetilde{C}}\simeq H^0(A)\otimes\nu_*\mathcal{O}_{\widetilde{C}}\twoheadrightarrow\nu_*A.$$

Using the standard short exact sequence

$$0\longrightarrow\mathcal{O}_C\longrightarrow\nu_*\mathcal{O}_{\widetilde{C}}\longrightarrow\mathcal{O}_P\longrightarrow 0,$$

one can easily show that the cokernel A_1 of ev sits in a short exact sequence

$$0\longrightarrow A_1\longrightarrow\nu_*A\longrightarrow\mathcal{O}_P\longrightarrow 0.$$

In particular, A_1 is a line bundle and $\nu^*A_1=A$. Hence, the morphism $\phi_{|A|}$ factors through a morphism $\phi_{|A_1|}:C\to\mathbb{P}^1$, which means that P is a neutral node with respect to $|A|$. □

The above lemma implies that if C is a general curve of the nice component $Y_{\delta,k+\varepsilon}$ in Theorem 3.1 and $|A|$ is a general $g^1_{k+\varepsilon}$ on $\widetilde{C}$, then ν_*A is globally generated and the short exact sequence (21) is exact on the right. By dualizing it, we obtain:

$$0\longrightarrow H^0(C,\nu_*A)^\vee\otimes\mathcal{O}_S\longrightarrow\mathcal{E}_{C,\nu_*A}\longrightarrow\mathfrak{ext}^1(\nu_*A,\mathcal{O}_S)\longrightarrow 0, \tag{23}$$

where $\mathfrak{ext}^1(\nu_*A,\mathcal{O}_S)$ is a rank one torsion free sheaf on C. This defines a subspace

$$V\simeq H^0(C,\nu_*A)^\vee\in G(2,H^0(S,\mathcal{E}_{C,\nu_*A}))$$

such that the evaluation map $V\otimes\mathcal{O}_S\to\mathcal{E}_{C,\nu_*A}$ is injective, drops rank along C and has rank 0 exactly at the nodes of C. As a consequence, every section in V vanishes along a 0-dimensional subscheme of length $k+\varepsilon+\delta$ always containing the subscheme N of the δ nodes of C.

We want to understand whether the pair (C,ν_*A) univocally determines the subspace V; this is equivalent to computing the dimension of $\mathrm{Hom}(\mathcal{E}_{C,\nu_*A},\mathfrak{ext}^1_{\mathcal{O}_S}(\nu_*A,\mathcal{O}_S))$. To achieve this goal we need some technical results.

For any torsion free rank one sheaf $\mathcal{A}$ on C we denote by $\mathcal{A}^D$ the dual sheaf $\mathfrak{hom}_{\mathcal{O}_C}(\mathcal{A},\mathcal{O}_C)$. We prove the following:

Lemma 5.2. *Let $\mathcal{F}$ be any rank one torsion free sheaf on a curve $C\subset S$, where S is a $K3$ or abelian surface. Then one has:*

$$\omega_C\otimes\mathcal{F}^D\simeq\mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S). \tag{24}$$

Assume furthermore that $\mathcal{F}$ is the cokernel of an injective map $V\otimes\mathcal{O}_S\to\mathcal{E}$, where $\mathcal{E}$ is a rank two bundle on S and $V\in G(2,H^0(S,\mathcal{E}))$. If Z is a zero-dimensional scheme that is the vanishing locus of $s\in V$, one obtains the isomorphisms:

$$\mathcal{F}\simeq\omega_C\otimes\mathcal{J}_{Z/C} \tag{25}$$

and

$$\mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S)\simeq\mathfrak{hom}_{\mathcal{O}_C}(\mathcal{J}_{Z/C},\mathcal{O}_C). \tag{26}$$

Proof. Applying $\mathfrak{hom}_{\mathcal{O}_S}(\mathcal{F},-)$ to the short exact sequence

$$0\longrightarrow\mathcal{J}_{C/S}\xrightarrow{\alpha}\mathcal{O}_S\longrightarrow\mathcal{O}_C\longrightarrow 0, \tag{27}$$

we obtain

$$0=\mathfrak{hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S)\longrightarrow\mathfrak{hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_C)\longrightarrow\mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{J}_{C/S})\xrightarrow{\alpha'}\mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S).$$

Since α and α' are given by multiplication with the local equation of C and $\mathcal{F}$ is supported precisely at C, the map α' is zero. Hence, we get

$$\mathfrak{hom}_{\mathcal{O}_C}(\mathcal{F},\mathcal{O}_C) \simeq \mathfrak{hom}_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_C) \simeq \mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{J}_{C/S}) \simeq \mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S(-C)) \simeq \mathfrak{ext}^1_{\mathcal{O}_S}(\mathcal{F},\mathcal{O}_S)\otimes\mathcal{O}_S(-C),$$

and (24) is obtained by tensoring with $\mathcal{O}_S(C)$.

Concerning the second part of the statement, we set $L := \det\mathcal{E}$ and consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{O}_S & & \\
 & & \downarrow & & & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{s} & \mathcal{E} & \longrightarrow & L\otimes\mathcal{J}_{Z/S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\simeq} & & \downarrow & & \\
0 & \longrightarrow & V\otimes\mathcal{O}_S & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_S & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{28}$$

The short exact sequence of ideals

$$0 \longrightarrow \mathcal{J}_{C/S} \longrightarrow \mathcal{J}_{Z/S} \longrightarrow \mathcal{J}_{Z/C} \longrightarrow 0$$

yields the isomorphism $\mathcal{J}_{Z/C} \simeq \mathcal{J}_{Z/S}\otimes\mathcal{O}_C$. Hence, (25) is obtained by restricting the vertical exact sequence on the right in (28) to C and using that i is given by multiplication with the local equation of C. Combining (25) and (24), we obtain (26). □

We now extend [Pa, Lemma 2] to possibly nodal curves on $K3$ or abelian surfaces (cf. also [LC2, Pf. of Prop. 3.2] concerning other types of irregular surfaces). In the statement of the next result, the map $\mu_{0,A}$ is the *Petri map* given by nultiplication of sections

$$\mu_{0,A} : H^0(\widetilde{C},A)\otimes H^0(\widetilde{C},\omega_{\widetilde{C}}\otimes A^\vee) \longrightarrow H^0(\widetilde{C},\omega_{\widetilde{C}}),$$

and we use the well-known fact that

$$\ker\mu_{0,A} \simeq H^0(\widetilde{C},\omega_{\widetilde{C}}\otimes(A^\vee)^{\otimes 2}) \tag{29}$$

by the *base point free pencil trick*.

Proposition 5.3. *Let C be a nodal curve on a $K3$ or abelian surface S and denote by $\nu:\widetilde{C}\to C$ the normalization map. Let A be a globally generated line bundle on $\widetilde{C}$ satisfying $h^0(\widetilde{C},A)=2$. Then there is a natural short exact sequence*

$$0 \longrightarrow \mathcal{O}_{\widetilde{C}} \longrightarrow \mathcal{E}^\vee_{C,A}\otimes\omega_{\widetilde{C}}\otimes A^\vee \longrightarrow \omega_{\widetilde{C}}\otimes(A^\vee)^{\otimes 2} \longrightarrow 0 \tag{30}$$

whose coboundary map $Q : H^0(\widetilde{C},\omega_{\widetilde{C}}\otimes(A^\vee)^{\otimes 2}) \to H^1(\widetilde{C},\mathcal{O}_{\widetilde{C}})$ coincides, up to multiplication by a non-zero scalar factor, with the composition of the Gaussian map

$$\mu_{1,A} : \ker\mu_{0,A} \longrightarrow H^0(\widetilde{C},\omega_{\widetilde{C}}^{\otimes 2})$$

and the dual of the Kodaira-Spencer map

$$\kappa : H^1(C, \nu_*\mathcal{O}_{\widetilde{C}})^\vee \simeq T_{[C]}V_{\{L\},\delta} \longrightarrow T_{[\widetilde{C}]}\mathcal{M}_{p_g(C)} \simeq H^0(\widetilde{C}, \omega_{\widetilde{C}}^{\otimes 2})^\vee,$$

via the canonical isomorphism $H^1(C, \nu_*\mathcal{O}_{\widetilde{C}}) \simeq H^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$.

Proof. The exact sequence (30) is obtained as [Pa, (4)] since $\omega_S \simeq \mathcal{O}_S$.

We denote by $\mathcal{N}'_{C/S}$ the *equisingular normal sheaf* of C in S, defined by the short exact sequence

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_S|_C \longrightarrow \mathcal{N}'_{C/S} \longrightarrow 0,$$

where $\mathcal{T}_C$ is the tangent sheaf of C and $\mathcal{T}_S|_C$ is the restriction of the tangent bundle of S to C. Similarly, we denote by $\mathcal{N}_\varphi$ the *normal sheaf to the morphism* φ in (19), which is defined by the *normal sequence*

$$0 \longrightarrow \mathcal{T}_{\widetilde{C}} \longrightarrow \varphi^*\mathcal{T}_S \longrightarrow \mathcal{N}_\varphi \simeq \omega_{\widetilde{C}} \longrightarrow 0, \tag{31}$$

where the isomorphism on the right follows from the triviality of ω_S. We recall that $\mathcal{N}'_{C/S} \simeq \nu_*\mathcal{N}_\varphi \simeq \omega_C \otimes (\nu_*\mathcal{O}_{\widetilde{C}})^D$ by, e.g., [Se, Lemma 3.4.15] and [Ta, p. 111]. In particular, we have:

$$T_{[C]}V_{\{L\},\delta} \simeq H^0(C, \mathcal{N}'_{C/S}) \simeq H^0(\widetilde{C}, \mathcal{N}_\varphi) \simeq H^1(C, \nu_*\mathcal{O}_{\widetilde{C}})^\vee. \tag{32}$$

Applying ν_* to (31) and using the isomorphism $\nu_*\omega_{\widetilde{C}} \simeq \omega_C \otimes (\nu_*\mathcal{O}_{\widetilde{C}})^D$, we obtain the exact sequence

$$0 \longrightarrow \nu_*(\omega_{\widetilde{C}}^\vee) \longrightarrow \mathcal{T}_S \otimes \nu_*\mathcal{O}_{\widetilde{C}} \longrightarrow \omega_C \otimes (\nu_*\mathcal{O}_{\widetilde{C}})^D \longrightarrow 0,$$

and its dual

$$0 \longrightarrow \omega_C^\vee \otimes (\nu_*\mathcal{O}_{\widetilde{C}}) \longrightarrow \Omega_S \otimes (\nu_*\mathcal{O}_{\widetilde{C}})^D \longrightarrow (\nu_*(\omega_{\widetilde{C}}^\vee))^D \longrightarrow 0. \tag{33}$$

The right exactness of the latter is due to the fact that $\mathfrak{ext}^1_{\mathcal{O}_C}((\nu_*\mathcal{O}_{\widetilde{C}})^D, \mathcal{O}_C) = 0$, which can easily be verified using the standard exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{\widetilde{C}} \longrightarrow \mathcal{O}_N \longrightarrow 0.$$

Tensoring (33) with ω_C, we obtain

$$0 \longrightarrow \nu_*\mathcal{O}_{\widetilde{C}} \longrightarrow \Omega_S \otimes \omega_C \otimes (\nu_*\mathcal{O}_{\widetilde{C}})^D \longrightarrow \omega_C \otimes (\nu_*(\nu_{\widetilde{C}}^\vee))^D \simeq \nu_*(\omega_{\widetilde{C}}^{\otimes 2}) \longrightarrow 0, \tag{34}$$

where the last isomorphism follows from [BP, Lemma 4.6]. By construction, the first coboundary map of (34) is $\kappa^\vee$.

We now follow [Pa, Pf. of Lemma 1]. Tensoring the derivation operator $d : \mathcal{O}_{\widetilde{C}} \to \omega_{\widetilde{C}}$ with the evaluation map $ev_{\widetilde{C},A} : H^0(A) \otimes \mathcal{O}_{\widetilde{C}} \to A$, we obtain a map $H^0(A) \otimes \mathcal{O}_{\widetilde{C}} \to \omega_{\widetilde{C}} \otimes A$, whose restriction to $\ker ev_{\widetilde{C},A} \simeq A^\vee$ is $\mathcal{O}_{\widetilde{C}}$-linear. Tensoring the restricted map with $\omega_{\widetilde{C}} \otimes A^\vee$, we obtain a map of $\mathcal{O}_{\widetilde{C}}$-modules

$$s :\ \omega_{\widetilde{C}} \otimes (A^\vee)^{\otimes 2} \longrightarrow \omega_{\widetilde{C}}^{\otimes 2}, \tag{35}$$

whose associated map at the global section level is the Gaussian map $\mu_{1,A}$, recalling (29).

Similarly, tensoring the pullback under f of the derivation operator $d : \mathcal{O}_S \to \Omega_S$ with the evaluation map $ev_{\widetilde{C},A} : H^0(A) \otimes \mathcal{O}_{\widetilde{S}} \to A$, we obtain a map $H^0(A) \otimes \mathcal{O}_{\widetilde{S}} \to f^*\Omega_S \otimes A$, whose restriction to $\mathcal{E}_{\widetilde{C},A}^\vee$ is $\mathcal{O}_{\widetilde{S}}$-linear. Tensoring the restricted map with $\omega_{\widetilde{C}} \otimes A^\vee$, we obtain a map of $\mathcal{O}_C$-modules

$$t :\ \mathcal{E}_{\widetilde{C},A}^\vee \otimes \omega_{\widetilde{C}} \otimes A^\vee \longrightarrow f^*\Omega_S \otimes \omega_{\widetilde{C}}. \tag{36}$$

The sequences and maps (30), (34), (35) and (36) combine into:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \nu_*\mathcal{O}_{\widetilde{C}} & \longrightarrow & \nu_*(\mathcal{E}^\vee_{\widetilde{C},A}\otimes\omega_{\widetilde{C}}\otimes A^\vee) & \longrightarrow & \nu_*(\omega_{\widetilde{C}}\otimes(A^\vee)^{\otimes 2}) & \longrightarrow & 0\\ & & \| & & \downarrow{\scriptstyle \nu_*t} & & \downarrow{\scriptstyle \nu_*s} & & \\ 0 & \longrightarrow & \nu_*\mathcal{O}_{\widetilde{C}} & \longrightarrow & \Omega_S\otimes\omega_C\otimes(\nu_*\mathcal{O}_{\widetilde{C}})^D & \longrightarrow & \nu_*(\omega_{\widetilde{C}}^{\otimes 2}) & \longrightarrow & 0\end{array}$$

and the result follows. □

We are now ready to prove the following:

Proposition 5.4. *If C is a general element of the component $Y_{\delta,k+\varepsilon}$ in Theorem 3.1(iii) and $|A|$ is a general $g^1_{k+\varepsilon}$ on $\widetilde{C}$, then*

$$\dim \operatorname{Hom}(\mathcal{E}_{C,\nu_*A}, \mathfrak{ext}^1_{\mathcal{O}_S}(\nu_*A,\mathcal{O}_S)) = 1.$$

Proof. By (24), we have $\operatorname{Hom}(\mathcal{E}_{C,\nu_*A}, \mathfrak{ext}^1(\nu_*A,\mathcal{O}_S)) \simeq H^0(\mathcal{E}^\vee_{C,\nu_*A}\otimes\omega_C\otimes(\nu_*A)^D)$ and, by (23) tensored with $\mathcal{E}^\vee_{C,\nu_*A}$, the latter contains $H^0(\mathcal{E}^\vee_{C,\nu_*A}\otimes\mathcal{E}_{C,\nu_*A})\simeq\mathbb{C}$, where the isomorphism follows from the fact that $\mathcal{E}_{C,\nu_*A}$ is simple when (S,L) is general as in Theorem 3.1 (this is standard, cf., e.g., [Pa, CK, KLM], and follows from the fact that $\{L\}$ does not contain reducible or non-reduced members).

If S is $K3$ the result is well-known and due to the fact that $h^1(\mathcal{O}_S)=0$ yields $h^1(\mathcal{E}^\vee_{C,\nu_*A})=0$ by (21). It remains to treat the case where S is abelian. We have $\omega_C\otimes(\nu_*A)^D\simeq\nu_*(\omega_{\widetilde{C}}\otimes A^\vee)$ by [BP, Lemma 4.6]. Hence, by (22), there is a natural morphism

$$\mathcal{E}^\vee_{C,\nu_*A}\otimes\omega_C\otimes(\nu_*A)^D\simeq f_*(\mathcal{E}^\vee_{\widetilde{C},A})\otimes\nu_*(\omega_{\widetilde{C}}\otimes A^\vee)\longrightarrow\nu_*(\mathcal{E}^\vee_{\widetilde{C},A}\otimes\omega_{\widetilde{C}}\otimes A^\vee),$$

which is injective as the left hand side is torsion free on C and the map is an isomorphism outside of N. In particular, we get an inclusion

$$H^0(C,\mathcal{E}^\vee_{C,\nu_*A}\otimes\omega_C\otimes(\nu_*A)^D)\subseteq H^0(C,\nu_*(\mathcal{E}^\vee_{\widetilde{C},A}\otimes\omega_{\widetilde{C}}\otimes A^\vee))\simeq H^0(\widetilde{C},\mathcal{E}^\vee_{\widetilde{C},A}\otimes\omega_{\widetilde{C}}\otimes A^\vee),$$

and thus it is enough to prove the injectivity of the first coboundary map Q of (30).

If $\rho(p-\delta,1,k+1)\geq 0$, then Theorem 3.1 yields $\ker\mu_{0,A}\simeq H^0(\widetilde{C},\omega_{\widetilde{C}}\otimes(A^\vee)^{\otimes 2})=0$ and Q is automatically injective.

If $\rho(p-\delta,1,k+1)<0$, then $\dim\ker\mu_{0,A}=-\rho(p-\delta,1,k+1)$ because, by Theorem 3.1(iii), A defines an isolated and reduced point of $G^1_{k+1}(\widetilde{C})$. By Proposition 5.3, we need to show that $\kappa^\vee\circ\mu_{1,A}$ is injective, or equivalently, $\mu^\vee_{1,A}\circ\kappa$ is surjective. Since A is a pencil, then the map $\mu_{1,A}$ is injective and its dual sits in the following short exact sequence:

$$0\longrightarrow T_{[\widetilde{C}]}\mathcal{M}^1_{g,k+1}\longrightarrow T_{[\widetilde{C}]}\mathcal{M}_g\xrightarrow{\mu^\vee_{1,A}}\mathcal{N}_{\mathcal{M}^1_{g,k+1}/\mathcal{M}_g}|_{[\widetilde{C}]}\longrightarrow 0,$$

cf. [ACG, pp. 807–824]. By Theorem 3.1, $Y_{\delta,k+1}$ has codimension $-\rho(p-\delta,1,k+1)$ in the Severi variety $V_{\{L\},\delta}(S)$, hence the image of the natural map $\psi: V_{\{L\},\delta}(S)\to\mathcal{M}_g$ is transversal to $\mathcal{M}^1_{g,k+1}$ around $[C]$. This forces $\mu^\vee_{1,A}\circ\kappa$ to be surjective. □

We now fix notation that will be used in the construction of a component of the locus in $S^{[k]}_\varepsilon$ covered by rational curves of class $R_{p,\delta,k}$.

Let C and A be as in Proposition 5.4. Since ν_*A is globally generated in this case (by Theorem 3.1 and Lemma 5.1), the *Mukai vector* (cf. Remark 1.4) of the Lazarsfeld-Mukai bundle $\mathcal{E}_{C,\nu_*A}$ is

(37) $$v_{p,\delta,k}:=v(\mathcal{E}_{C,\nu_*A})=(2,c_1(L),\chi+2(\varepsilon-1)),\ \text{ with } \chi:=\chi(\mathcal{E}_{C,\nu_*A})=p-\delta-k+3-5\varepsilon.$$

If $\mathrm{Pic}(S) \simeq \mathbb{Z}[L]$, which holds off a countable union of proper closed subvarieties of the moduli space of pairs (S, L), then $\mathcal{E}_{C,\nu_*A}$ is stable with respect to the polarization L (as in, e.g., [KLM, Proposition A.2]).

Let $\mathcal{M}$ be the moduli space of Gieseker L-stable torsion free sheaves on S with Mukai vector $v_{p,\delta,k}$ as in (37) that contains $\mathcal{E}_{C,\nu_*A}$. We recall (cf. Remark 1.4) that, since $\mathrm{Pic}(S) \simeq \mathbb{Z}[L]$, then $v_{p,\delta,k}$ is primitive and $\mathcal{M}$ is an IHS manifold of dimension:

$$\dim \mathcal{M} = 2p - 4\chi + (1-\varepsilon)8, \tag{38}$$

with χ as in (37).

Every torsion free sheaf $[\mathcal{E}] \in \mathcal{M}$ satisfies $h^2(\mathcal{E}) = 0$ because of Serre duality and the inequality $\mu_L(\mathcal{E}) > 0$. Furthermore, as soon as $h^0(\mathcal{E}) \geq 2$, then for all $V \in G(2, H^0(\mathcal{E}))$ the evaluation map $ev : V \otimes \mathcal{O}_S \to \mathcal{E}$ is injective. Indeed, if this were not the case, its kernel would be isomorphic to $\mathcal{O}_S(-D)$ for an effective divisor D and we would find a short exact sequence:

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{E} \longrightarrow \det \mathcal{E}(-D) \otimes I_\xi \longrightarrow 0,$$

where $\xi \subset S$ is a 0-dimensional subscheme. As $\det \mathcal{E}$ is indecomposable and $h^2(\mathcal{E}) = 0$, then $D = 0$ and this contradicts the fact that V is generated by 2 linearly independent sections of $\mathcal{E}$.

The following two results determine properties of $\mathcal{M}$ that will play a fundamental role in our construction of uniruled subvarieties of $S^{[k]}_\varepsilon$. They can also be seen as applications of Proposition 5.4 to moduli spaces of stable sheaves on S and are therefore interesting in themselves.

Proposition 5.5. *Let p, δ, k be integers such that* (12) *is satisfied and let v and χ be as in* (37)*. Then the elements of $\mathcal{M}$ satisfy the following properties:*

(i) *If $\chi \geq 2\delta + 2$, then $h^1(\mathcal{E}) = h^1(\mathcal{E} \otimes I_\tau) = 0$ for a general pair $([\mathcal{E}], \tau) \in \mathcal{M} \times S^{[\delta]}$.*

(ii) *If $\chi < 2\delta + 2$, the locus*

$$X := \left\{ ([\mathcal{E}], \tau) \in \mathcal{M} \times S^{[\delta]} \mid h^0(\mathcal{E} \otimes I_\tau) \geq 2 \right\}$$

is non-empty, with an irreducible component X_0 whose general point satisfies $h^0(\mathcal{E} \otimes I_\tau) = 2$ and $h^1(\mathcal{E} \otimes I_\tau) = 2 + 2\delta - \chi$. Furthermore, X_0 is birational to a component of the relative Brill-Noether variety

$$\mathcal{G}^1_{k+\varepsilon}\left(V^{k+\varepsilon}_{\{L\},\delta}\right) := \left\{ ([C], \mathfrak{g}) \mid [C] \in V^{k+\varepsilon}_{\{L\},\delta},\ \mathfrak{g} \in G^1_{k+\varepsilon}(\widetilde{C}),\ \text{with } \widetilde{C} \text{ the normalization of } C \right\},$$

having the expected dimension $2(k - 1 + \varepsilon)$.

Proof. Recall that $\mathcal{M}$ contains $\mathcal{E}_{C,\nu_*A}$ with C and A as in Proposition 5.4. Let $X \subset \mathcal{M} \times S^{[\delta]}$ parametrize pairs $([\mathcal{E}], \tau)$ such that $h^0(\mathcal{E} \otimes I_\tau) \geq 2$. Trivially, X coincides with $\mathcal{M} \times S^{[\delta]}$ as soon as $\chi \geq 2\delta + 2$. If instead $\chi < 2\delta + 2$, then X is a closed subscheme which can be defined using fitting ideals and its expected codimension is $2(2\delta - \chi + 2)$, whence its expected dimension is $2(k - 1 + \varepsilon)$. Furthermore, X is non-empty because $([\mathcal{E}_{C,\nu_*A}], N)$ lies in it.

Let $\mathcal{G}$ be the family of triples $([\mathcal{E}], \tau, V)$ with $([\mathcal{E}], \tau) \in X$ and $V \in G(2, H^0(\mathcal{E} \otimes I_\tau))$, and let $p : \mathcal{G} \to X \times S^{[\delta]}$ be the natural projection. Existence of $\mathcal{G}$ follows from existence of a moduli space of $\mathcal{O}_S$-stable coherent systems; indeed, since $\mathcal{M}$ parametrizes torsion free sheaves of rank 2 and indecomposable first Chern class, then any pair $([E], V)$ with $[E] \in \mathcal{M}$ and $V \in G(2, H^0(E))$ is automatically an $\mathcal{O}_S$-stable coherent system (proceed as in [LC1, §3]).

Since $([\mathcal{E}_{C,\nu_*A}], N, H^0(C, \nu_*A)^\vee)$ lies in $\mathcal{G}$, then for a general element $([\mathcal{E}], \tau, V) \in \mathcal{G}$ the evaluation map $ev : V \otimes \mathcal{O}_S \to \mathcal{E}$ is injective and drops rank along a δ-nodal curve Γ, which is singular precisely at the support of τ. Furthermore, the cokernel B of ev is torsion free of rank 1 on Γ and is not locally free exactly along $\mathrm{Sing}(\Gamma)$. We set $B_1 := \mathfrak{ext}^1(B, \mathcal{O}_S)$. If $\eta : \widetilde{\Gamma} \to \Gamma$ is the normalization map, we can write $B_1 = \eta_* A_1$ for some $A_1 \in \mathrm{Pic}^{k+\varepsilon}(\widetilde{\Gamma})$. From the short exact sequence

$$0 \longrightarrow V \otimes \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow B \longrightarrow 0, \tag{39}$$

we get that $H^0(A_1)^\vee \simeq H^0(B_1)^\vee \simeq H^1(B) \twoheadrightarrow V$ and hence $\mathfrak{g} = (A_1, V^\vee)$ defines a $g^1_{k+\varepsilon}$ on $\widetilde{\Gamma}$. We thus obtain a rational map $h : \mathcal{G} \dashrightarrow \mathcal{G}^1_{k+\varepsilon}(V^{k+\varepsilon}_{\{L\},\delta})$. Our hypotheses, along with Theorem 3.1, ensure that $(C, \nu_* A)$ lies in a component Z of the image of h of dimension:

$$\dim Z = \dim V^{k+\varepsilon}_{\{L\},\delta} + \max\{0, \rho(p-\delta, 1, k+\varepsilon)\} = 2(k-1+\varepsilon). \tag{40}$$

On the other hand, Proposition 5.4 implies that h is generically injective and hence, for a general $([\mathcal{E}], \tau) \in X$, we have:

$$\dim Z = \dim X + 2(h^0(\mathcal{E} \otimes I_\tau) - 2). \tag{41}$$

If $\chi \geq 2\delta + 2$, then $X = \mathcal{M} \times S^{[\delta]}$ and (38), (40) and (41) yield:

$$2(k-1+\varepsilon) = \dim Z \geq \dim \mathcal{M} + \dim S^{[\delta]} + 2(\chi(\mathcal{E} \otimes I_\tau) - 2) = 2(k-1+\varepsilon); \tag{42}$$

thus, equality holds and a general $([\mathcal{E}], \tau) \in \mathcal{M} \times S^{[\delta]}$ satisfies $h^1(\mathcal{E}) = h^1(\mathcal{E} \otimes I_\tau) = 0$.

If instead $\chi < 2\delta + 2$, then $\operatorname{expdim} X = 2(k-1+\varepsilon)$ and, by (40) and (41), one obtains $\dim X = \operatorname{expdim} X$ and $h^0(\mathcal{E} \otimes I_\tau) = 2$ for a general $([\mathcal{E}], \tau) \in X$. □

Lemma 5.6. *Under the same hypotheses as in Proposition 5.5, assume moreover that S is abelian and $\chi \geq 4$. Then a general $[\mathcal{E}] \in \mathcal{M}$ satisfies $h^1(\mathcal{E} \otimes \mathcal{L}_0) = 0$ for all $\mathcal{L}_0 \in \operatorname{Pic}^0(S)$.*

Proof. It is enough to show that the locus $\mathcal{F} := \{[\mathcal{E}] \in \mathcal{M} \mid h^1(\mathcal{E}) \neq 0\}$ has codimension greater than two in $\mathcal{M}$. We perform a parameter count as in [LC3, Prop. 4.2 and §7]. Let $\mathcal{G}_1$ be the parameter space of extensions

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E}' \xrightarrow{\beta} \mathcal{E} \longrightarrow 0, \tag{43}$$

with $[\mathcal{E}] \in \mathcal{F}$. Since $\operatorname{Hom}(\mathcal{E}, \mathcal{O}_S) = 0$ for all $\mathcal{E} \in \mathcal{M}$, the fibre of the natural map $\pi_1 : \mathcal{G}_1 \to \mathcal{F}$ over $[\mathcal{E}]$ is isomorphic to $\mathbb{P}(H^1(\mathcal{E}))$. A sheaf $\mathcal{E}'$ as in (43) is stable; let $v' := v_{p,\delta,k} + (1, 0, 0)$ be its Mukai vector and denote by $\pi_2 : \mathcal{G}_1 \to \mathcal{M}(v')$ the natural projection mapping (43) to $[\mathcal{E}']$. The fibre of π_2 over $[\mathcal{E}']$ is the Quot-scheme $\operatorname{Quot}_S(\mathcal{E}', P)$, where P is the Hilbert polynomial of $\mathcal{E}$. The following upper bound for the dimension of $\operatorname{Quot}_S(\mathcal{E}', P)$ at $[\beta : \mathcal{E}' \to \mathcal{E}]$ is well-known:

$$\dim_{[\beta]} \operatorname{Quot}_S(\mathcal{E}', P) \leq \dim \operatorname{Hom}(\mathcal{O}_S, \mathcal{E}) = h^0(\mathcal{E}).$$

It follows that:

$$\dim \mathcal{M} - 2\chi = \dim \mathcal{M}(v') \geq \dim \operatorname{Im} \pi_2 \geq \dim \mathcal{F} - \chi - 1,$$

and $\operatorname{codim}_{\mathcal{M}} \mathcal{F} \geq \chi - 1 > 2$ as soon as $\chi \geq 4$. □

The next lemma concerns moduli spaces of rank-2 Gieseker stable torsion free sheaves on S.

Lemma 5.7. *Let S be an abelian or $K3$ surface and let $\mathcal{M}$ be the moduli space of rank-2 Gieseker stable torsion free sheaves on S with primitive Mukai vector $v = (2, c_1, \chi + 2(\varepsilon - 1))$. For every $[\mathcal{E}] \in \mathcal{M}$, denote by $\mathfrak{S}(\mathcal{E})$ the cokernel of the injection $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$ and by $l_{\mathcal{E}}$ its length. Then every irreducible component of the locus*

$$\mathcal{M}_q := \{[\mathcal{E}] \in \mathcal{M} \mid l_{\mathcal{E}} = q\} \tag{44}$$

has codimension at least q in $\mathcal{M}$.

Proof. Assume that $\mathcal{M}_q$ is non-empty. One defines a map $\alpha : \mathcal{M}_q \to \mathcal{M}(v_q)$, where $v_q := v + (0, 0, q)$, which maps $[\mathcal{E}] \in \mathcal{M}$ to $[\mathcal{E}^{\vee\vee}] \in \mathcal{M}(v_q)$. The fibre of α over a general vector bundle $F \in \operatorname{Im}\alpha$ is isomorphic to the Quot-scheme $\operatorname{Quot}_S(F, q)$ of zero dimensional quotient sheaves of F of length q; by [OG, Prop. 6.0.1], this Quot-scheme has dimension at most $3q$. Hence,

$$\dim \mathcal{M} - 4q = \dim \mathcal{M}(v_q) \geq \dim \operatorname{Im}\alpha = \dim \mathcal{M}_q - \operatorname{Quot}_S(F, q) \geq \dim \mathcal{M}_q - 3q, \tag{45}$$

and this concludes the proof. □

6. Algebraically coisotropic subvarieties of IHS manifolds

We are now ready to prove our results concerning existence of uniruled subvarieties of $S^{[k]}_{\varepsilon}$. Let (S, L) be a *very general* primitively polarized $K3$ or abelian surface of genus p, in the sense that it satisfies Theorem 3.1 and $\mathrm{Pic}(S) \simeq \mathbb{Z}[L]$. For p, δ, k satisfying (12), let $[C] \in Y_{\delta,k+\varepsilon}$ be general and let A be a general complete $g^1_{k+\varepsilon}$ on its normalization.

Let $\mathcal{M}$ be the moduli space $\mathcal{M}(v_{p,\delta,k})$ as in the previous section, with $[\mathcal{E}_{C,\nu_*A}] \in \mathcal{M}$. Assume $\chi \geq 2\delta + 2$ and set $X := \mathcal{M} \times S^{[\delta]}$. We denote by $\mathcal{P}$ the parameter space for triples $([\mathcal{E}], \tau, [s])$ with $([\mathcal{E}], \tau) \in X$ and $[s] \in \mathbb{P}(H^0(\mathcal{E} \otimes I_\tau))$. The existence of $\mathcal{P}$ follows from the existence of a moduli space of α-stable coherent systems (cf. [He]), where $\alpha \in \mathbb{Q}[t]$ is any fixed polynomial with positive leading coefficient. Indeed, it is not difficult to show that if $\mathcal{E}$ is L-stable and the polynomial α is small enough, then the coherent system $(\mathcal{E} \otimes I_\tau, s)$ is α-stable for every $[\tau] \in S^{[\delta]}$ and for every $s \in \mathbb{P}(H^0(\mathcal{E} \otimes I_\tau))$. There is a natural forgetful map $q : \mathcal{P} \to X$. For any $([\mathcal{E}], \tau, [s]) \in \mathcal{P}$, the section s vanishes along a finite set because otherwise we would have $\mathrm{Hom}(\det \mathcal{E}, \mathcal{E}^{\vee\vee}) \neq 0$ and this would contradict the μ_L-stability of $\mathcal{E}$. Hence, we have a short exact sequence:

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\ s\ } \mathcal{E} \longrightarrow \det \mathcal{E} \otimes I_{W_s} \longrightarrow 0, \tag{46}$$

where W_s is a 0-dimensional subscheme of S of length $k + \epsilon + \delta$ containing τ. This provides a short exact sequence

$$0 \longrightarrow \eta_s \longrightarrow \mathcal{O}_{W_s} \longrightarrow \mathcal{O}_\tau \longrightarrow 0, \tag{47}$$

defining η_s, which is a torsion sheaf whose support is contained in that of W_s.

If $\mathcal{E}$ is locally free, the scheme W_s is a local complete intersection as it is the zero scheme of s. It follows that $\mathfrak{ext}^2_{\mathcal{O}_S}(\mathcal{O}_{W_s}, \mathcal{O}_S) \simeq \mathcal{O}_{W_s}$ by, e.g., [Fr, p. 36]. Applying the functor $\mathfrak{hom}_{\mathcal{O}_S}(-, \mathcal{O}_S)$ to (47) we therefore obtain a surjection

$$\mathcal{O}_{W_s} \longrightarrow \mathfrak{ext}^2_{\mathcal{O}_S}(\eta_s, \mathcal{O}_S),$$

which yields the existence of a subscheme $Z_s \subset W_s$ of length $k + \varepsilon$ such that $\mathfrak{ext}^2_{\mathcal{O}_S}(\eta_s, \mathcal{O}_S) \simeq \mathcal{O}_{Z_s}$. This defines a rational map

$$g' : \mathcal{P} \dashrightarrow S^{[k+\varepsilon]} \tag{48}$$

mapping a point $([\mathcal{E}], \tau, [s])$, with $\mathcal{E}$ locally free, to Z_s. We want to extend g' in codimension 1.

Pick a sheaf $[\mathcal{E}] \in \mathcal{M}$ that is not locally free and such that $\mathfrak{S}(\mathcal{E})$ (defined in Lemma 5.7) has length one, i.e., $\mathfrak{S}(\mathcal{E}) \simeq \mathcal{O}_P$ for some $P \in S$. A section s of $\mathcal{E}$ gives a section s' of $\mathcal{E}^{\vee\vee}$ and, having denoted by $W_{s'}$ the vanishing locus of s', one has a short exact sequence:

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_{W_s} \longrightarrow \mathcal{O}_{W_{s'}} \longrightarrow 0.$$

If both $[\mathcal{E}] \in \mathcal{M}_1$ (cf. (44)) and $s \in H^0(\mathcal{E})$ are general, then $W_{s'}$ does not contain P and P is a general point on S. Indeed, $H^0(\mathcal{E})$ is not contained in $H^0(\mathcal{E}^{\vee\vee} \otimes I_P)$ as soon as $H^1(\mathcal{E}^{\vee\vee} \otimes I_P) = 0$; the vanishing follows from the fact that a general $[F] \in \mathcal{M}(v(\mathcal{E}^{\vee\vee}))$ is generically generated by global sections. For $s \in H^0(\mathcal{E} \otimes I_\tau)$ with $\tau \in S^{[\delta]}$ such that $P \notin \mathrm{Supp}(\tau)$, we get $\mathcal{O}_{W_s} = \mathcal{O}_{W_{s'}} \oplus \mathcal{O}_P$, whence W_s is a local complete intersection, and we may find a subscheme $Z_s \subset W_s$ of length $k + \varepsilon$ as before and set $g'([\mathcal{E}], \tau, [s]) = Z_s$ also in this case.

We set

$$\mathcal{M}^\circ := \left\{ [\mathcal{E}] \in \mathcal{M} \setminus \cup_{q \geq 2} \mathcal{M}_q \,|\, h^1(\mathcal{E}^{\vee\vee} \otimes I_P) = 0 \text{ if } \mathfrak{S}(\mathcal{E}) \simeq \mathcal{O}_P \right\},$$

with $\mathcal{M}_q$ as in (44); then $\mathcal{M}^\circ$ is open in $\mathcal{M}$ and its complement has codimension at least two by Lemma 5.7. We define

$$X^\circ := \{([\mathcal{E}], \tau) \in X \,|\, [\mathcal{E}] \in \mathcal{M}^\circ, \mathrm{Supp}(\tau) \cap \mathrm{Supp}(\mathfrak{S}(\mathcal{E})) = \emptyset\};$$

the complement of X° in X has codimension at least two. Let $\mathcal{P}^\circ$ be the open subscheme of $q^{-1}(X^\circ) \subset \mathcal{P}$ consisting of those triples $([\mathcal{E}], \tau, s)$ such that either $\mathcal{E}$ is locally free, or the vanishing locus $W_{s'}$ as above does not contain the singular locus of $\mathcal{E}$. Then, the complement of $\mathcal{P}^\circ$ in $\mathcal{P}$ has codimension at least two and, by the above discussion, the rational map g' in (48) is well-defined on the whole $\mathcal{P}^\circ$. We denote by

$$g : \mathcal{P}^\circ \longrightarrow S^{[k+\varepsilon]} \tag{49}$$

the restriction of g' to $\mathcal{P}^\circ$.

We prove the following result.

Theorem 6.1. *Let (S, L) be a very general primitively polarized $K3$ or abelian surface of genus $p \geq 2$. Let $0 \leq \delta \leq p - 2\varepsilon$ and $k \geq 2$ be integers satisfying*

$$\max\{2\delta + 2, 4\varepsilon\} \leq \chi := p - \delta - k + 3 - 5\varepsilon \leq \delta + k + 1. \tag{50}$$

Then the morphism g is generically injective. In particular, the locus $\mathrm{Locus}(\mathbb{R}^+ R_{p,\delta,k}) \subset S^{[k]}_\varepsilon$*, with $R_{p,\delta,k}$ as in* (16)*, has an irreducible component that is birational to a $\mathbb{P}^{\chi-2\delta-1}$-fibration on a holomorphic symplectic manifold of dimension $2(k + 1 + 2\delta - \chi)$.*

Proof. First of all, note that when $\chi \geq 2\delta + 2$, condition (12) is equivalent to $\chi \leq \delta + k + 1$. Indeed, the latter inequality is a rewrite of (14) with $l = 1$; on the other hand, if $2\delta + 2 \leq \chi \leq \delta + k + \varepsilon$, then $\delta \leq k + \varepsilon - 2$ and $\alpha = 1$ in (12). Hence, (50) ensures that the hypotheses of Proposition 5.5 are satisfied and a general fibre of $q : \mathcal{P}^\circ \to X^\circ$ is isomorphic to $\mathbb{P}^{\chi-2\delta-1}$. We denote by T the closure of the image of the morphism g in (49). The proof proceeds by steps.

STEP I: The morphism g is injective when restricted to a general fibre of the projection q.

Let $Z = g(([\mathcal{E}], \tau, [s])) \in T$ with $([\mathcal{E}], \tau) \in X^\circ$ general, and denote by W_s the zero scheme of s. The fibre of $g|_{q^{-1}([\mathcal{E}],\tau)}$ over Z is contained in $\mathbb{P}(\mathrm{Hom}(\mathcal{E}, \det \mathcal{E} \otimes I_{W_s}))$. This projective space is a point because $\mathrm{Hom}(\mathcal{E}, \det \mathcal{E} \otimes I_{W_s}) \simeq H^0(\mathcal{E} \otimes \mathcal{E}^\vee) \simeq \mathbb{C}$ by stability, exact sequence (46) and Proposition 5.5(i), which yields $h^1(\mathcal{E}) = 0$.

STEP II: The map g is generically injective when restricted to a general fibre of the projection $p_1 : \mathcal{P}^\circ \to S^{[\delta]}$. If $\delta = 0$, then g is injective.

Let $Z = g(([\mathcal{E}_1], \tau, [s_1])) \in T$ with $([\mathcal{E}_1], \tau, [s_1]) \in \mathcal{P}^\circ$ general; in particular, $\mathrm{Supp}(\tau)$ is disjoint from $\mathrm{Supp}(Z)$ and $\mathcal{E}_1$ satisfies $h^1(\mathcal{E}_1 \otimes \mathcal{L}_0) = 0$ for all $\mathcal{L}_0 \in \mathrm{Pic}^0(S)$ by Lemma 5.6. By contradiction, assume the existence of $([\mathcal{E}_2], \tau, [s_2]) \in \mathcal{P}^\circ$ with $\mathcal{E}_2 \not\simeq \mathcal{E}_1$ such that $g(([\mathcal{E}_2], \tau, [s_2])) = Z$. In particular, one has $W_{s_1} = W_{s_2}$. We remark that $\det \mathcal{E}_1 \not\simeq \det \mathcal{E}_2$ because otherwise we would have $h^1(\det \mathcal{E}_i \otimes I_{W_{s_i}}) > 1$ and, by considering the long exact sequence in cohomology associated with (46) for $\mathcal{E}_1$, we would get a contradiction with $h^1(\mathcal{E}_1) = 0$. We tensor (46) for $\mathcal{E}_1$ with $(\det \mathcal{E}_1)^\vee \otimes \det \mathcal{E}_2$, thus obtaining:

$$0 \to (\det \mathcal{E}_1)^\vee \otimes \det \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \otimes (\det \mathcal{E}_1)^\vee \otimes \det \mathcal{E}_2 \longrightarrow \det \mathcal{E}_2 \otimes I_{W_s} \longrightarrow 0.$$

This yields the contradiction $H^1(\mathcal{E}_1 \otimes (\det \mathcal{E}_1)^\vee \otimes \det \mathcal{E}_2) \simeq H^1(\det \mathcal{E}_2 \otimes I_{W_s}) \neq 0$.

STEP III: The map g is generically injective when restricted to a general fibre of the projection $p_2 : \mathcal{P}^\circ \to \mathcal{M}^\circ \subset \mathcal{M}$.

Let $Z = g(([\mathcal{E}], \tau_1, [s_1])) \in T$ for a general $([\mathcal{E}], \tau_1, [s_1])$. By contradiction, assume the existence of a subscheme $\tau_2 \in S^{[\delta]}$ different from τ_1 and a section $s_2 \in H^0(\mathcal{E} \otimes I_{\tau_2})$ such that $g(([\mathcal{E}], \tau_2, [s_2])) = Z$. The evaluation map $ev : \langle s_1, s_2 \rangle \otimes \mathcal{O}_S \to \mathcal{E}$ is injective and drops rank along an integral curve $\Gamma \in \{L\}$ of geometric genus $\leq p - k - \varepsilon$ (indeed, Γ is singular along Z). If B is the cokernel of ev, then $B_1 := \mathfrak{ext}^1(B, \mathcal{O}_S) = n_* A_1$, where $n : \widetilde{\Gamma} \to \Gamma$ is a partial normalization of Γ with $p_a(\widetilde{\Gamma}) = p - k - \varepsilon - h$ for some $h \geq 0$ and A_1 a complete $g^1_{\delta-h}$ on $\widetilde{\Gamma}$. As a consequence, there is a subscheme $\tau_1' \subset \tau_1$ of length $\delta - h$ and a rational curve R' in $S^{[\delta-h]}$ passing through $[\tau_1']$. Starting from R', one easily constructs a rational curves in $S^{[\delta]}$ passing through τ_1 in contradiction with the generality of τ_1.

STEP IV: The map g is generically finite.

This follows from the previous steps and Proposition 2.15, which can be applied because the complement of $\mathcal{P}^\circ$ in $\mathcal{P}$ has codimension at least two and X is holomorphic symplectic.

STEP V: Let $\mathcal{P}_i = \mathbb{P}(H^0(E_i \otimes I_{\tau_i}))$ for $i = 1, 2$ with $([E_1], \tau_1), ([E_2], \tau_2) \in X^\circ$ distinct points such that $h^1(\mathcal{E}_i) = 0$ for $i = 1, 2$. Then g does not identify $\mathcal{P}_1$ and $\mathcal{P}_2$.

By contradiction, assume that $g(\mathcal{P}_1) = g(\mathcal{P}_2)$. As $g|_{\mathcal{P}_i}$ is injective for $i = 1, 2$ by Step I, it is easy to verify that, if $\ell_1 \subset \mathcal{P}_1$ is a general line, then $\ell_2 := g^{-1}(g(\ell_1)) \cap \mathcal{P}_2$ is a line, too. By generality, $\ell_1 = \mathbb{P}V_1$ corresponds to a pair $(C_1, \nu_* A_1)$, where C_1 is a δ-nodal curve and A_1 is a $g^1_{k+\varepsilon}$ on its normalization. Having set $\ell_2 = \mathbb{P}V_2$, the evaluation map $V_2 \otimes \mathcal{O}_S \to \mathcal{E}_2$ drops rank along a curve C_2 singular along τ_2. Then, $C_1 = C_2$ as both coincide with the image in S of the incidence variety

$$I := \left\{(Z, P) \in g(\ell_1) \times S \subset S^{k+\varepsilon} \times S \mid P \in \mathrm{Supp}(Z)\right\}.$$

As a consequence, $\tau_2 = \tau_1$ and $\mathcal{E}_2 \simeq \mathcal{E}_{C,\nu_* A_1} \simeq \mathcal{E}_1$.

STEP VI: The morphism g is generically injective.

By contradiction, assume that for a general $Z \in T$ there exist at least two distinct points $([\mathcal{E}_1], \tau_1), ([\mathcal{E}_2], \tau_2) \in X^\circ$ such that $Z \in g(\mathcal{P}_1) \cap g(\mathcal{P}_2)$, where $\mathcal{P}_i = \mathbb{P}(H^0(\mathcal{E}_i \otimes I_{\tau_i}))$; we may assume that $h^1(\mathcal{E}_1) = h^1(\mathcal{E}_2) = 0$. The previous step then implies that $g(\mathcal{P}_1) \neq g(\mathcal{P}_2)$. We denote by $\pi : \widetilde{T} \dashrightarrow B$ the maximal rational quotient of the desingularization $\widetilde{T}$ of T, and by $\widetilde{Z}$ the inverse image of Z in $\widetilde{T}$. Since a general fibre of π is irreducible and $\pi^{-1}(\pi(\widetilde{Z}))$ contains the strict transforms of both $g(\mathcal{P}_1)$ and $g(\mathcal{P}_2)$, then $\dim \pi^{-1}(\pi(\widetilde{Z})) \geq \chi - 2\delta > \mathrm{codim}_{S^{[k+\varepsilon]}} T$; this contradicts [AV, Thm. 4.4] (when $\varepsilon = 1$, in order to apply the mentioned result, one needs to pass to the fibers of the Albanese map and use that $g(\mathcal{P}_1) \cup g(\mathcal{P}_2)$ is contained in such a fiber).

When $\varepsilon = 0$, Step VI concludes the proof. If $\varepsilon = 1$, consider the composition of g with the Albanese map $\Sigma_k : S^{[k+1]} \to S$. Since $\Sigma_k \circ g$ is constant when restricted to any fibre of q, it induces a morphism $F : X^\circ \to S$ that factors, by the universal property of the Albanese variety, through a map $f : \mathrm{Alb}(X) \to S$. One can easily show that both F and f are surjective. The inverse image $g^{-1}(S^{[k]}_\varepsilon) \subset \mathcal{P}^\circ$ is generically a $\mathbb{P}^{\chi - 2\delta - 1}$-bundle on the inverse image $(\mathrm{alb}_X|_{X^\circ})^{-1}(\ker f)$, where alb_X is the Albanese map of X. Since $\mathrm{Alb}(X)$ is the product of copies of S and $S^\vee$, the same holds for any connected component of $\ker f$. Therefore, any component of $(\mathrm{alb}_X|_{X^\circ})^{-1}(\ker f)$ is holomorphic symplectic of dimension equal to $\dim X - 2 = 2(k + 1 + 2\delta - \chi)$, and the statement follows. □

We now give a first application to Conjecture 0.3.

Corollary 6.2. *With the same hypotheses and notation as in Theorem 6.1, set $r := \chi - 2\delta - 1$. Then, $\mathbb{S}_r(S^{[k]}_\varepsilon)$ has a $(2k - r)$-dimensional component which is an algebraic coisotropic subvariety of $S^{[k]}_\varepsilon$ covered by curves of class $R_{p,\delta,k}$.*

Proof. This follows directly from Theorem 6.1 and [Vo, Thm. 1.3], which states that a closed subvariety of an IHS manifold X contained in $\mathbb{S}_r(X)$ has codimension at least r. □

Starting from the closure of $\mathrm{Im}\, g \subset S^{[k+\varepsilon]}$, with g as in (49), and then applying the natural rational map $S^{[k+\varepsilon]} \times S^{[l-k]} \dashrightarrow S^{[l+\varepsilon]}$, one obtains subvarieties of $S^{[l]}_\varepsilon$ for any $l \geq k$. We use this observation in order to construct subvarieties of $S^{[k]}_\varepsilon$, with k fixed, of codimension r for several values of r:

Theorem 6.3. *Let (S, L) be a very general primitively polarized $K3$ or abelian surface of genus $p \geq 2$ and fix an integer $k \geq 2$.*

Then, for any integer r satisfying

$$1 \leq r \leq \min\left\{2k - 5 - \frac{p - 5\varepsilon}{2}, \frac{p - 5\varepsilon}{2} + 1\right\}, \tag{51}$$

and

$$p \geq 9 \text{ if } (\varepsilon, r) = (1,1); \quad p \geq 11 \text{ if } (\varepsilon, r) = (1,2), \tag{52}$$

there is an algebraically coisotropic subvariety of codimension r in $S_\varepsilon^{[k]}$ that is a component of $\mathbb{S}_r(S_\varepsilon^{[k]})$ and is birational to a $\mathbb{P}^r$-bundle over a holomorphic symplectic manifold.

More precisely, for any integer δ satisfying

$$\max\left\{0, \frac{p-5\varepsilon+2-r-k}{3}\right\} \leq \delta \leq \frac{p-5\varepsilon+2-2r}{4}, \quad \text{and } \delta > 0 \text{ if } r \leq 2 \text{ and } \varepsilon = 1, \tag{53}$$

there is such a subvariety $\Sigma_{r,\delta}$ whose lines have class $L - [2(p-2\delta-2\varepsilon) - r + 1]\mathfrak{r}_k$.

Proof. We note that if δ and k are integers satisfying the conditions in Theorem 6.1, then, for any integer $l \geq k$, the image of $\operatorname{Im} g \times S^{[l-k]} \subset S^{[k+\varepsilon]} \times S^{[l-k]}$ under the natural rational map $S^{[k+\varepsilon]} \times S^{[l-k]} \dashrightarrow S^{[l+\varepsilon]}$ has the same codimension r as $\operatorname{Im} g \subset S^{[k+\varepsilon]}$. The intersection of this image with any fibre of the Albanese map is birational to a $\mathbb{P}^r$-bundle over a holomorphic symplectic manifold and the coefficients in the class of the lines with respect to the canonical decomposition (8) remain unchanged. Hence, Theorem 6.1 yields that for any pair of integers (δ, k') such that

$$2 \leq k' \leq k \tag{54}$$

and

$$\max\{2\delta+2, 4\varepsilon\} \leq p - 5\varepsilon - \delta - k' + 3 \leq \delta + k' + 1, \quad \delta \geq 0 \tag{55}$$

there is a subscheme of codimension $r := p - 5\varepsilon - 3\delta - k' + 2$ satisfying the desired conditions. Rewriting (54) and (55) in terms of r instead of k' yields, respectively,

$$\frac{p-5\varepsilon+2-r-k}{3} \leq \delta \leq \frac{p-5\varepsilon+r}{3} \tag{56}$$

and

$$\frac{1}{2}\max\{0, 4\varepsilon - r - 1\} \leq \delta \leq \frac{p-5\varepsilon+2-2r}{4}, \tag{57}$$

One easily verifies that these two conditions are equivalent to (53). It only remains to prove that for each r satisfying (51) and (52), the conditions (53) are non-empty.

The inequality $r \leq 2k - 5 - \frac{p-5\varepsilon}{2}$ ensures that

$$\frac{p-5\varepsilon+2-r-k}{3} \leq \frac{p-5\varepsilon+2-2r}{4} - \frac{3}{4},$$

thus guaranteeing that the interval $\left[\frac{p-5\varepsilon+2-r-k}{3}, \frac{p-5\varepsilon+2-2r}{4}\right]$ contains an integer. The condition $r \leq \frac{p-5\varepsilon}{2} + 1$ is equivalent to $p - 5\varepsilon + 2 - 2r \geq 0$ and thus guarantees that the above interval contains a non-negative integer. We are therefore done, except for the cases $\varepsilon = 1$ and $r \leq 2$, where the requirement is that $\delta > 0$ by (53), that is, that $p - 5\varepsilon + 2 - 2r \geq 4$, which is precisely (52). □

We now perform a different construction in order to exhibit uniruled subvarieties of $S_\varepsilon^{[k]}$ of any allowed codimension, except codimension k for $\varepsilon = 1$. This provides very strong evidence for Conjecture 0.3.

Theorem 6.4. *Let (S, L) be a general primitively polarized $K3$ or abelian surface of genus $p \geq 2$ and fix an integer $k \geq 2$. Then for any integer r such that $1 \leq r \leq k - \varepsilon$ there is an algebraically coisotropic subvariety of codimension r in $S_\varepsilon^{[k]}$ that is a component of $\mathbb{S}_r(S_\varepsilon^{[k]})$ and is birational to a $\mathbb{P}^r$-bundle; furthermore, the maximal rational quotient of it desingularization has dimension $2(k-r)$.*

More precisely, for any integer k' such that $r+\varepsilon \leq k' \leq \min\{k, p+r-\varepsilon\}$, there is such a subvariety $W_{r,k'}$ whose lines have class $L - [2(k'+\varepsilon) - r - 1]\mathfrak{r}_k$.

Proof. For any r and k' as in the statement, set $g := k' - r + \varepsilon$ and $\delta := p - g$. Note that $2\varepsilon \leq g \leq p$. Since $\rho(g, 1, k + \varepsilon) \geq 0$, the Brill-Noether locus $\{L\}^1_{\delta,k+\varepsilon}$ coincides with the g-dimensional Severi variety $V_{\{L\},\delta}$ of genus g nodal curves. For any component V of $V_{\{L\},\delta}$, we denote by $\mathcal{C} \to V$ the universal family and by $\widetilde{\mathcal{C}} \to V$ the simultaneous desingularization of all curves in $\mathcal{C}$ (the latter exists, as V is smooth). Let $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{\mathcal{C}}) \to V$ be the relative $(k' + \varepsilon)$-symmetric product, with fibre over a point $[C] \in V$ equal to $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C})$, where $\widetilde{C}$ is the normalization of C. By surjectivity of the Abel map $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C}) \to \mathrm{Pic}^{k'+\varepsilon}(\widetilde{C})$, the line bundle $\mathcal{O}_{\widetilde{C}}(x_1 + \cdots + x_{k'+\varepsilon})$ is non-special if $x_1, \ldots, x_{k'+\varepsilon} \in \widetilde{C}$ are general . Therefore, $\dim |\mathcal{O}_{\widetilde{C}}(x_1 + \cdots + x_{k'+\varepsilon})| = k' + \varepsilon - g = r \geq 1$ and $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C})$ is generically a $\mathbb{P}^r$-bundle over a dense, open subset of $\mathrm{Pic}^{k'+\varepsilon}(\widetilde{C})$, which has dimension g. It follows that $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{\mathcal{C}})$ is generically a $\mathbb{P}^r$-bundle over a scheme of dimension $g + \dim V = 2g$.

Consider the natural composed morphism

$$f : \mathrm{Sym}^{k'+\varepsilon}(\widetilde{\mathcal{C}}) \to \mathrm{Sym}^{k'+\varepsilon}(\mathcal{C}) \to \mathrm{Sym}^{k'+\varepsilon}(S).$$

We first prove that f is generically injective. Clearly, for each curve $[C] \in V$ with normalization $\widetilde{C}$, the restriction of f to $\mathrm{Sym}^{k'+\varepsilon}(\widetilde{C})$ is generically injective. Hence, it is enough to show that, if $[Z] \in \mathrm{Im} f$ is general, then the scheme Z is contained in a unique curve parametrized by V. A general $[Z] \in \mathrm{Im} f$ consists of $k' + \varepsilon$ general points on a general curve C_0 parametrized by V. For a general point $x_1 \in C_0$, the set $\{[C] \in V \mid x_1 \in C\}$ has codimension one in V, as C_0 is not a common component of all curves parametrized by V. Proceeding inductively, assume that for a fixed $1 \leq j \leq g - 1$ we have chosen j distinct points $x_1, \ldots, x_j \in C_0$ such that the set $\{[C] \in V \mid x_1, \ldots, x_j \in C\}$ has codimension j in V. Again, as C_0 is not a component of any curve in this set different from C_0, for a general $x_{j+1} \in C_0$ the set $\{[C] \in V \mid x_1, \ldots, x_{j+1} \in C\}$ has codimension $j + 1$ in V. It follows that $\dim\{[C] \in V \mid x_1, \ldots, x_g \in C\} = 0$ for general points $x_1, \ldots, x_g \in C_0$. Hence, for general points $x_1, \ldots, x_{g+1} \in C_0$, we have $\{[C] \in V \mid x_1, \ldots, x_{g+1} \in C\} = \{C_0\}$, and the generic injectivity of f follows since $k' + \varepsilon = g + r \geq g + 1$.

The image of f does not lie in the singular locus of $\mathrm{Sym}^{k'+\varepsilon}(S)$. Hence, its inverse image under the Hilbert-Chow morphism is a $(k' + \varepsilon + g)$-dimensional subvariety of $S^{[k'+\varepsilon]}$ that is birational to a $\mathbb{P}^r$-bundle. As above, the natural rational map $S^{[k'+\varepsilon]} \times S^{[k-k']} \dashrightarrow S^{[k+\varepsilon]}$ maps $\mathrm{Im} f$ to a codimension r subvariety of $S^{[k+\varepsilon]}$. Since the Albanese map is constant on each rational subvariety of $S^{[k+\varepsilon]}$, we obtain a subvariety $W_{r,k'} \subset S^{[k]}_{\varepsilon}$ of codimension r that is birational to a $\mathbb{P}^r$-bundle and the maximal rational quotient of the desingularization of $W_{r,k'}$ has dimension $2(k - r)$ by [AV, Thm. 4.4]. The coefficients in the class of the lines in the $\mathbb{P}^r$-fibres of $W_{r,k'}$ with respect to the canonical decomposition (8) are the same as the ones of $R_{p,\delta,k'} = R_{p,p+r-k'-\varepsilon,k'} = L - [2(k' + \varepsilon) - r - 1]\mathfrak{r}_{k'}$. This concludes the proof. □

We conclude with an interesting example, where Theorem 6.1 provides an immersion $\mathbb{P}^k \hookrightarrow S^{[k]}_{\varepsilon}$.

Example 6.5. By Remark 3.4, when $\delta = 0$ the class $R_{p,0,k}$ has the minimal possible self-intersection (i.e., $q(R_{p,0,k}) = -(k + 3 - 2\varepsilon)/2$) if and only if $\alpha = 1$ and $p = 2(k - 1) + 5\varepsilon$. We assume these numerical conditions are satisfied and show, by explicit construction, that $R_{p,0,k}$ is the class of a line moving in a $\mathbb{P}^k \subset S^{[k]}_{\varepsilon}$. Note that in this case $\dim \mathcal{M} = 2\varepsilon$ and condition (50) is satisfied. With the same notation introduced just before Theorem 6.1, $\mathcal{P}^{\circ} = \mathcal{P}$ and any component of $g^{-1}(S^{[k]}_{\varepsilon})$ is isomorphic to $\mathbb{P}H^0(\mathcal{E})$ for some vector bundle $[\mathcal{E}] \in \mathcal{M}$. We consider the restricted morphism

$$\overline{g} := g|_{\mathbb{P}H^0(\mathcal{E})} : \mathbb{P}^k = \mathbb{P}H^0(\mathcal{E}) \longrightarrow S^{[k]}_{\varepsilon} \subseteq S^{[k+\varepsilon]},$$

which is injective by Theorem 6.1 and, more precisely, an embedding by the result below.

Proposition 6.6. *Let (S, L) be a general primitively polarized symplectic surface of genus $p = 2(k - 1) + 5\varepsilon$ for an integer $k \geq 2$. Then the map $\overline{g}$ is an embedding. In particular, the class $R_{p,0,k}$ is the class of a line in a $\mathbb{P}^k \subset S^{[k]}_{\varepsilon}$.*

Proof. It is enough to show that for all $[s] \in \mathbb{P}H^0(\mathcal{E})$ the differential

$$d\overline{g}_{[s]} : T_{[s]}\mathbb{P}(H^0(\mathcal{E})) \to T_{[Z]}S^{[k+\varepsilon]}$$

is injective, where Z is the zero scheme of s. First of all, we recall the isomorphisms

$$T_{[s]}\mathbb{P}H^0(\mathcal{E}) \simeq \mathrm{Hom}(\mathbb{C}, H^0(\mathcal{E})/\langle s\rangle) \simeq H^0(\mathcal{E})/\langle s\rangle$$

and $T_{[Z]}S^{[k+\varepsilon]} = H^0(\mathcal{N}_{Z/S})$, and use them in order to describe $d\overline{g}_{[s]}$. Given $t \in H^0(\mathcal{E})/\langle s\rangle$, the evaluation map $ev : \langle t, s\rangle \otimes \mathcal{O}_S \to \mathcal{E}$ is injective and drops rank along a curve $\Gamma_t \in \{L\}$ containing Z. We denote by B the cokernel of ev, which is supported on Γ_t. Since $h^2(\mathcal{E}) = h^1(\mathcal{E}) = 0$ by Proposition 5.5, we obtain isomorphisms:

$$\langle t, s\rangle \simeq H^1(B) \simeq H^0(B_1)^\vee, \tag{58}$$

where $B_1 := \mathfrak{ext}^1_{\mathcal{O}_S}(B, \mathcal{O}_S)$. By (26), one has $B_1 = \mathfrak{hom}_{\mathcal{O}_{\Gamma_t}}(I_{Z/\Gamma_t}, \mathcal{O}_{\Gamma_t})$. Let σ_s, $\sigma_t \in H^0(B_1)$ denote the duals of the images of s and t under (58), and consider the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\Gamma_t} \xrightarrow{\sigma_s} \mathfrak{hom}_{\mathcal{O}_{\Gamma_t}}(I_{Z/\Gamma_t}, \mathcal{O}_{X_t}) \xrightarrow{\alpha_t} \mathfrak{hom}_{\mathcal{O}_{\Gamma_t}}(I_{Z/\Gamma_t}, \mathcal{O}_Z) \longrightarrow 0.$$

Then we have $0 \neq H^0(\alpha_t)\sigma_t \in H^0(\mathfrak{hom}_{\mathcal{O}_{\Gamma_t}}(I_{Z/\Gamma_t}, \mathcal{O}_Z)) = H^0(\mathcal{N}_{Z/\Gamma_t}) \hookrightarrow H^0(\mathcal{N}_{Z/S})$, where the last inclusion is given by taking cohomology in

$$0 \longrightarrow \mathcal{N}_{Z/\Gamma_t} \xrightarrow{\iota_t} \mathcal{N}_{Z/S} \longrightarrow \mathcal{N}_{\Gamma_t/S}|_Z \longrightarrow 0. \tag{59}$$

By construction, $d\overline{g}_{[s]}(t) = H^0(\iota_t) \circ H^0(\alpha_t)\sigma_t \in H^0(\mathcal{N}_{Z/S})$.

We are now able to prove the injectivity of $d\overline{g}_{[s]}$. Let $t, t' \in H^0(\mathcal{E})/\langle s\rangle$ such that $d\overline{g}_{[s]}(t) = d\overline{g}_{[s]}(t')$. We first assume that $\Gamma_t \simeq \Gamma_{t'}$. Since $h^1(\mathcal{E}) = 0$, then the natural map $h : G(2, H^0(\mathcal{E})) \to \mathcal{G}^1_{k+1}(|L|)$ is injective; indeed, any $V \in G(2, H^0(\mathcal{E}))$ defines a short exact sequence like (39) and and it is enough to tensor it with $\mathcal{E}^\vee$ in order to conclude that $\mathbb{P}(\mathrm{Hom}(\mathcal{E}, B))$ is a point. Since $h(\langle s, t\rangle) = h(\langle s, t'\rangle) = (\Gamma_t, \mathcal{O}_{\Gamma_t}(Z))$, we have $\langle s, t\rangle = \langle s, t'\rangle$, that is, $t' = \lambda t$ for some $\lambda \in \mathbb{C}$. Then, $\sigma_{\lambda t} = \lambda\sigma_t \neq \sigma_t$ unless $\lambda = 1$. Therefore, we can assume $\Gamma_{t'} \not\simeq \Gamma_t$; it is then clear from (59) that the section $H^0(\iota_{t'}) \circ H^0(\alpha_{t'})\sigma_{t'} \in H^0(\mathcal{N}_{Z/S})$ does not lie in the image of $H^0(\iota_t)$. This concludes the proof. □

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