Bounded Control Law for Global Connectivity Maintenance in Cooperative Multi-Robot Systems

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***Abstract*—In this work, we address the connectivity maintenance problem for a multi-robot system which moves according to a given bounded collective control objective. We assume that the interaction among the robotic units is limited by a given visibility radius both in terms of sensing and communication capabilities. For this scenario, we propose a decentralized bounded control law which can provably preserve the connectivity of the multi-robot system over time. We characterize the effect of the connectivity control term on the achievement of the collective control objective by resorting to an Input-to-State Stability (ISS)-like analysis. We provide numerical and experimental results to corroborate the theoretical findings and assess the effectiveness of the proposed bounded connectivity maintenance control law.**

***Index Terms*—Distributed Robot Systems, Networked Robots, Global Connectivity Maintenance, Nonsmooth Analysis**

I. Introduction

Multi-Robot Systems (MRSs) have been a very active research field over the last three decades. The relevance of this research is motivated by the wide range of applications which can be carried out by a team of robots, such as environmental exploration, search and rescue operations, area coverage, collaborative transportation. Notably, the majority of these approaches requires the robotic units to exchange information in order to perform a collaborative task: see, for instance, [1]–[6] and references therein. Therefore, the capability to preserve the connectivity of the communication network over time is of great importance.

Indeed, the connectivity maintenance problem has been widely investigated in the last decade. Briefly, the objective is to develop (possibly decentralized) control strategies to guarantee that, if the communication graphs is initially connected, then it can be kept so over time. Originally this problem was approached in a *local* sense, i.e., by ensuring that if a communication link was activated to start with, then the control law would ensure such a link to remain active over time. Remarkable examples of local connectivity maintenance control strategies are [7]–[12] and references therein.

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However, keeping every single communication link of the network topology is often a very restrictive requirement. Indeed, a much more flexible strategy can be derived if approaching the problem in a *global* sense, that is by dynamically allowing the deletion and addition of links as long as the overall connectedness of the graph is preserved. Remarkable examples of global connectivity maintenance control strategies are [13]–[16].

As the most recent results in this domain, our work follows such a global approach. In particular, we propose a bounded control law to preserve the global connectivity of a multi-robot system which moves according to a given bounded collaborative control objective. Furthermore, we investigate how the connectivity control term affects the achievement of the collective control objective. In this regard, we consider the widely used potential-based control as a case study of collective control objective and resort to an Input-to-State Stability (ISS)-like analysis to characterize the disturbance introduced by the connectivity control term on the achievement of the objective.

To the best of the authors knowledge, this is the first work addressing the global connectivity maintenance problem with bounded control inputs and providing a theoretical analysis of the disturbance introduced by the connectivity control term on the objective control one. A preliminary version of this work was presented in [17]. To summarize, the following contributions are made in this work:

- A bounded control law that can provably preserve the connectivity of a multi-robot system which moves according to a bounded collective control objective;
- A theoretical analysis to evaluate the effects of the connectivity control term on the collective control objective carried out considering potential-based objectives as a case study;
- A theoretical analysis to evaluate the robustness of the proposed connectivity control law against errors on the knowledge of the actual value of the algebraic connectivity due to the several factors such as the availability of noisy pose measurements, or the transient of the connectivity distributed estimation process;
- A numerical validation to corroborate the theoretical results along with an experimental validation to assess the effectiveness of the proposed technique in a realistic context.

The rest of the paper is organized as follows. In section II the related work is reviewed. In section III preliminary results concerning both the multi-robot graph theoretic modeling and the elementary machinery of the nonsmooth analysis are provided. In section IV the proposed bounded connectivity maintenance control law is described and the theoretical analysis is carried out. In section V the experimental and numerical validation is described. Finally, in section VI conclusion are drawn and future work is discussed.

II. Related Work

Connectivity maintenance has been widely investigated, in the last decade. Without aiming completeness, in this section we provide a review of the most relevant and recent works, that approach the connectivity maintenance issue from a global perspective, that is without necessarily requiring to keep every initially active communication link.

Along these lines, a methodology for guaranteeing k-connectivity is presented in [18]. In this paper, authors propose a methodology for initially deploying a fault tolerant topology. For this purpose, the group of robots is divided into two groups, *rangers* and *scouts*: rangers are in charge of managing, in a decentralized manner, the initial deployment and, subsequently, they control their movements in order to guarantee a sufficient level of redundancy, thus reaching k-connectivity. Eigenvalue conditions are presented in [19], [20] for assessing the biconnectivity of a network, thus defining a gradient-based control law to enforce such property.

Optimization methodologies are a popular framework for addressing complex tasks. In [21] authors consider a multi-robot system where the motion of the robots is controlled in such a way that a given objective function is optimized, in a decentralized manner. The objective function is utilized to formalize all the objectives of the multi-robot system, including connectivity maintenance. In [22], [23] authors define an objective function, to be optimized in a decentralized manner. This objective function is utilized for keeping connectivity of the multi-robot system, while minimizing the energy consumption. The proposed optimization strategy is based on non-cooperative game theory.

A coverage problem is considered in [24]. A number of finite sensing range robots are spread as much as possible in the environment, in order to achieve coverage. Simultaneously, in order to guarantee connectivity maintenance, the minimum spanning tree is periodically recomputed and kept. A similar problem is considered in [25], where artificial neural network techniques are used for defining the motion of the robots.

A path planning algorithm was developed in [26], where the path of each robot is computed on-line in such a way that collisions are avoided and, simultaneously, connectivity of the graph is guaranteed. The main objective is to ensure these properties not only at the via points, but also in between.

A heuristic methodology is proposed in [27], with the purpose of improving the robustness of the communication topology in a multi-robot systems. The main idea consists in utilizing locally available information for assessing the existence of weakly connected robots, and subsequently control the position of the neighbors in order to mitigate such conditions. A message passing algorithm is developed in [28] for assessing the level of *criticality* of a node in a network, and subsequently mitigating this effect by moving the neighbors towards a bi-connected network, namely a network where at least two separate paths exist that connect each pair of nodes.

Heterogeneous robots are considered in [29], where connectivity is preserved for ensuring cooperation capabilities. A hierarchical, multi-layer, control architecture is developed: each layer considers a different control objective (reaching the destination, keeping all the edges active, keeping global connectivity, keeping k-connectivity, etc.), and each class of robots implements one (or a set) of the layers, based on its mechanical characteristics and objectives.

As it is well known from algebraic graph theory, the algebraic connectivity is a parameter that indicates if a graph is connected. Defined as the second smallest eigenvalue of the Laplacian matrix, the algebraic connectivity is positive if the graph is connected, and zero if the graph is not connected. Being an eigenvalue of the Laplacian matrix, it represents a global quantity, that cannot be directly computed by the robots, based only on locally available information.

Decentralized estimation procedures have been however developed, for overcoming this issue. In [30], authors developed a methodology for estimating the algebraic connectivity in a decentralized manner. The proposed method, based on the computation of powers of the adjacency matrix, provides, at each iteration, upper and lower bounds for the actual value of the algebraic connectivity.

A methodology based on signal processing methods was proposed in [31] for the decentralized estimation of the spectrum of the Laplacian matrix. This method was then utilized for achieving an estimate of the algebraic connectivity with arbitrarily small estimation error, in finite time.

A decentralized version of the standard power iteration algorithm was utilized in [13], for simultaneously computing, in a decentralized manner, estimates of the second smallest eigenvalue of the Laplacian matrix, and its corresponding eigenvector. A slightly modified power iteration algorithm was developed in [14], where an estimator for the algebraic connectivity was introduced, and its convergence properties were studied. Based on this convergence study, authors were able to define a criterion for each robot to obtain a tradeoff between number of iterations of the estimation algorithm and achieved precision.

A decentralized methodology for estimation of the algebraic connectivity and subsequent connectivity preservation was introduced in [15], [16]. The proposed estimation procedure, built on the formulation first introduced in [13], provides a local estimate of the algebraic connectivity with provably bounded estimation error. This estimate was then utilized for designing a gradient descent control strategy for guaranteeing connectivity maintenance.

A decentralized methodology for connectivity maintenance in multi-robot systems with unicycle kinematics was introduced in [32]. The proposed approach provides global connectivity maintenance under nonholonomic constraints. In addition, it only requires intermittent estimation of algebraic connectivity, and accommodates discontinuous spatial potential-based interactions among robots.

Most of the aforementioned methodologies for connectivity maintenance inherently define unbounded control actions, which generally cannot be applied on real robotic platforms due to actuators saturation. To overcome this issue, recently a few control strategies have been developed that consider *bounded* control inputs. However, the majority of these strategies solves the connectivity maintenance issue in a local manner, that is preserving all (or a subset of) the initially active communication links.

In [33], the authors address path planning for a multi-robot system as a constrained routing optimization problem, where connectivity constraints are formalized and included into the optimization problem, that provides a bounded control input for the robots. Constraints on the allowed motion are also considered in [34], where authors provide a methodology for cooperation among robots in which communication links are enforced among those robots that need to directly cooperate.

A related approach is proposed in [35], where a bounded control strategy that enables a group of robots to achieve a desired graph topology is presented. Constraints on the motion of the robots are imposed for guaranteeing avoidance of collisions, and preservation of the neighborhood. A similar objective is achieved in [36], where bounded artificial potential fields are utilized with the purpose of preserving the initially connected communication graph while tracking a leader's trajectory.

We reiterate that all these methodologies based on bounded control actions can only solve the connectivity control problem from a local perspective, that is by preserving over time the set of links initially activated. Thus, the design of a technique to address the connectivity control problem from a global perspective when considering bounded control actions is still an open problem.

Our objective in this work is exactly to fill such a gap, that is by developing a distributed control framework based on bounded control actions, which can provably solve the connectivity problem from a global perspective.

III. Preliminaries

A. Multi-Robot Graph Theoretic Modeling

Consider a system composed of N robots moving in a d-dimensional space and let us denote with $p_i \in \mathbb{R}^d$ the position of the i-th robot and with $\mathbf{p} = [p_1^T, \ldots, p_N^T]^T$ the stacked vector of robots positions.

Let us encode the multi-robot interactions by means of a time-varying graph $\mathcal{G}(t) = \{V, E(t)\}$, where the set of vertexes $V = \{1, \ldots, N\}$ denotes the indexes of the robots and the set of edges $E(t) = \{(i, j) : i \in V, j \in V, i \neq j\}$ denotes the communication availability between pairs of robots i and j at time t, that is we have $e_{ij}(t) = 1$ if the two robots are within their visibility radius R, $e_{ij}(t) = 0$ otherwise. Note that, since the graph $\mathcal{G}(t)$ is undirected, the existence of an edge (i, j) implies the existence of an edge (j, i), thus they will be used interchangeably. Let $\mathcal{N}_i(t)$ be the neighborhood of the i-th robot at time t, i.e. the set of robots j that can exchange information with robot i at time t, that is $\mathcal{N}_i(t) = \{j : (i, j) \in E(t) : e_{ij}(t) = 1\}$.

The time-varying communication graph can be described by means of the adjacency matrix $A_{\mathcal{G}(t)} \in \mathbb{R}^{N\times N}$. Each element $a_{ij}(t)$ is defined as the weight of the edge $e_{ij}(t)$ between the i-th and the j-th robot at time t, and is a positive number if $j \in \mathcal{N}_i$, zero otherwise. Again, since we are considering undirected graphs, we assume $a_{ij}(t) = a_{ji}(t)$. Let us denote with $D_{\mathcal{G}(t)} \in \mathbb{R}^{N\times N}$ the degree matrix of the graph, which is a diagonal matrix defined as $D_{\mathcal{G}(t)} = \mathrm{diag}\{d_{ii}(t)\}$, where $d_{ii}(t)$ is the degree of the i-th node of the graph, i.e. $d_{ii}(t) = \sum_{j=1}^{n} a_{ij}(t)$.

The (weighted) Laplacian matrix of the graph is defined as $L_{\mathcal{G}(t)} = D_{\mathcal{G}(t)} - A_{\mathcal{G}(t)}$. Interestingly, the second smallest eigenvalue $\lambda_2(t)$ of the Laplacian matrix describes the algebraic connectivity of the time-varying undirected graph and in particular we have that $\lambda_2(t) > 0$ if and only if the graph is connected, see [37] for further information. Therefore, $\lambda_2(t)$ provides a natural metric to measure the connectivity of the network topology.

B. Nonsmooth Analysis

In this section, we review the Filipov solution concept for differential equations with discontinuous right-hand side, the nonsmooth analysis of Clarke's Generalized Gradient, the chain-rule for differentiating regular functions along Filipov solution trajectories, and a nonsmooth version of the LaSalle's stability theorem. The reader is referred to [38]–[40] and references therein for a comprehensive overview of the topic.

Consider the following differential equation

$$\dot{x} = f(x) \tag{1}$$

with $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ a measurable and essentially locally bounded function. First, we need to clarify what it means to be a solution of this equation.

Definition 1 (Filipov Solution): A vector function $x(\cdot)$ is called solution of (1) on a time interval $[t_0, t_i]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_i]$ and for almost all $t \in [t_0, t_i]$

$$\dot{x} \in K[g](\tau) \tag{2}$$

where $K[f](x) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined as

$$K[f](x) \equiv \bigcap_{\delta>0} \bigcap_{\mu\{H\}=0} \overline{\mathrm{co}}\{f(B(x, \delta) \setminus H)\} \tag{3}$$

where $\bigcap_{\mu\{H\}=0}$ denotes the intersection over all sets H of Lebesgue measure zero, $B(x, \delta)$ denotes the ball of radius δ centered at x, $\overline{\mathrm{co}}$ the convex closure and $2^{\mathbb{R}^n}$ the set of subsets of $\mathbb{R}^n$. □

Briefly, the idea of the Filipov's solution is that the tangent vector to a solution, where it exists, must lie in the convex closure of the values of the vector field in progressively smaller neighborhoods around the solution point. A very important aspect of this definition is given by the possibility of discarding sets of measure zero. Indeed, this technical detail allows solutions to be defined even at points where the vector field itself is not defined.

We now introduce the concept of Clarke's Generalized Gradient, an essential tool in the machinery of nonsmooth analysis.

Definition 2 (Clarke's Generalized Gradient): Consider a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$. Then the generalized gradient at x is defined as

$$\partial V(x) = \overline{\text{co}} \left\{ \lim_{i \to \infty} \nabla V(x_i) \,|\, x_i \to x,\, x_i \notin \Omega_V \right\} \tag{4}$$

where Ω_V is the set of measure zeros where the gradient of V is not defined. □

We now review the chain rule which allows to differentiate Lipschitz regular functions along the Filipov's solution trajectories.

Theorem 1 (Chain Rule [39]): Let $x(\cdot)$ be a Filipov solution to (1) on an interval containing t and $V : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and, in addition, regular function. Then $V(x)$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and

$$\frac{d}{dt} V(x(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(x) \tag{5}$$

where the generalized time derivative $\dot{\tilde{V}}(x)$ is defined as

$$\dot{\tilde{V}}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x) \tag{6}$$

□

So far, we have introduced the essential tools constituting the machinery of the nonsmooth analysis, where the right-hand side of differential equations may be discontinuous and the Lyapunov function may be non-differentiable. Interestingly, this machinery can be simplified under the assumption of continuous differentiability of the Lyapunov function.

First, let us notice that for a continuously differentiable function the generalized derivative becomes a singleton containing the actual gradient of the function, that is

Corollary 1 ([38]): Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then

$$\partial V(x) = \{\nabla V(x)\} \tag{7}$$

□

Then, by exploiting this information we can give a simplified version of the chain rule stated in Theorem 1 as:

Theorem 2 (Simplified Chain Rule): Let $x(\cdot)$ be a Filipov solution to (1) on an interval containing t and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. Then $V(x)$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and

$$\frac{d}{dt} V(x(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(x) \tag{8}$$

where the generalized time derivative $\dot{\tilde{V}}(x)$ is defined as

$$\dot{\tilde{V}}(x) := (\nabla V(x))^T K[f](x) \tag{9}$$

□

We now review a nonsmooth version of LaSalle's theorem. This will prove useful to establish stability results for dynamical systems described by differential equations with discontinuous right-hand side.

Theorem 3 (Nonsmooth LaSalle Invariance Principle [39]): Let Ω be a compact set such that every Filipov solution to (1) starting in Ω is unique and remains in Ω for all $t \geq 0$. Let $V : \Omega \to \mathbb{R}$ be a time independent regular function such that $v \leq 0$ for all $v \in \dot{\tilde{V}}(x)$ (if $\dot{\tilde{V}}(x)$ is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega : 0 \in \dot{\tilde{V}}(x)\}$. Then every trajectory in Ω converges to the largest invariant set M, in the closure of S. □

Finally, we review a calculus for computing the Filipov's differential inclusions, originally developed in [38] (and further extended in [40]).

Theorem 4 (Calculus for K [38]): The map $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ has the following properties

1) Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded. Then $\exists\, H_f \subset \mathbb{R}^n$, $\mu\{H_f\} = 0$, such that $\forall\, H \in \mathbb{R}^n$, $\mu\{H\} = 0$,
$$K[f](x) = \text{co} \left\{ \lim_{i \to \infty} f(x_i) \,|\, x_i \to x, x_i \notin H_f \cup H \right\} \tag{10}$$
2) Assume that $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are locally bounded; then
$$K[f + g](x) \subset K[f](x) + K[g](x) \tag{11}$$
3) Assume that $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, $j \in \{1, \ldots, N\}$, are locally bounded; then
$$K\left[\times_{j=1}^{N} f_j(x) \right] \subset \times_{j=1}^{N} K[f_j](x) \tag{12}$$
where the cartesian product notation and the column vector notation are used interchangeably.
4) Let $g : \mathbb{R}^m \to \mathbb{R}^{p \times n}$ (i.e., matrix valued) be continuous and $f : \mathbb{R}^m \to \mathbb{R}^n$ be locally bounded; then
$$K[g\, f](x) = g(x)\, K[f](x) \tag{13}$$
where $g\, f \triangleq g(x)\, f(x) \in \mathbb{R}^p$.
5) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous; then
$$K[f](x) = \{f(x)\} \tag{14}$$

□

IV. Bounded Connectivity Maintenance

Let us consider a multi-robot system composed of N robots interacting according to a time-varying graph $\mathcal{G}(t) = \{V, E(t)\}$, and let us assume that the dynamics of each robot i is described as follows

$$\dot{p}_i = f_i(\mathbf{p}) + u_i \tag{15}$$

where $f(\cdot) : \mathbb{R}^{Nd} \to \mathbb{R}^d$ is the desired control objective, $\mathbf{p} = [p_1^T, \ldots, p_N^T]^T$ is the stacked vector of per-agent state with $p_i \in \mathbb{R}^d$, and $u_i \in \mathbb{R}^d$ is the local control input[1]. It follows that the dynamics of the multi-robot system can be written in a compact form as

$$\dot{\mathbf{p}} = f(\mathbf{p}) + \mathbf{u} \tag{16}$$

where $f(\mathbf{p}) = [f_1(\mathbf{p})^T, \ldots, f_N(\mathbf{p})^T]^T$ is the stacked vector of the per-agent desired objective functions and

[1]It is worth noting that, even though this is a very simplified model, it can still be effectively used to control real robotic systems. In particular, by endowing a robot with a sufficiently good Cartesian trajectory tracking controller, it is possible to use single integrators to generate velocity-reference for several types of robotic platforms such as wheeled mobile robots [41], and unmanned aerial vehicles [42].

$\mathbf{u} = [u_1^T, \ldots, u_N^T]^T$ is the stacked vector of per-agent control inputs.

The following assumption is taken on the dynamics (15).

Assumption 1: There exist $U_o > 0 \in \mathbb{R}$ and $U_c > 0 \in \mathbb{R}$ such that

$$\begin{aligned} \|f_i(\mathbf{p})\| \leq U_o, \quad \forall i \in [1, \ldots, N], \\ \|u_i\| \leq U_c, \quad \forall i \in [1, \ldots, N]. \end{aligned} \tag{17}$$

with $U_o + U_c \leq U$, where $U > 0 \in \mathbb{R}$ might represent some physical limitations on the actuator capabilities. □

Similarly to [43], let us now consider a continuously differentiable version of the edge weights originally introduced in [13] as follows

$$a_{ij}(t) = \begin{cases} e^{\dfrac{(R^2 - \|p_i - p_j\|^2)^2}{2\,\sigma^2}} - 1 & \text{if } \|p_i - p_j\| \leq R \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

with R the visibility radius, σ a tuning parameter, and for which the $\nabla_{p_i}\lambda_2(\mathbf{p})$ can be computed as

$$\nabla_{p_i}\lambda_2(\mathbf{p}) = \sum_{j \in \mathcal{N}_i(t)} \nabla_{p_i} a_{ij}(t)(v_i - v_j)^2 \tag{19}$$

where v_i and v_j are the i-th and j-th component of the eigenvector $v = [v_1, \ldots, v_n]^T$ associated with algebraic connectivity λ_2 and the gradient $\nabla_{p_i} a_{ij}(t)$ is computed as

$$\nabla_{p_i} a_{ij}(t) = -a_{ij}(t)\left(R^2 - \|p_i - p_j\|^2\right)\frac{(p_i - p_j)}{\sigma^2} \tag{20}$$

We now propose the following connectivity maintenance control term for each robot i described by the dynamics (15)

$$u_i = \begin{cases} k_c\, e^{\dfrac{(-\lambda_2(\mathbf{p}) + \epsilon)}{c}} \dfrac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2\|} & \text{if } \|\nabla_{p_i}\lambda_2(\mathbf{p})\| \neq 0 \\ 0 & \text{if } \|\nabla_{p_i}\lambda_2(\mathbf{p})\| = 0 \end{cases} \tag{21}$$

where $\epsilon > 0$ is the desired lower-bound for the algebraic connectivity $\lambda_2(\mathbf{p})$, $\nabla_{p_i}\lambda_2(\mathbf{p})$ is the gradient of $\lambda_2(\mathbf{p})$ with respect to p_i as defined in (19), and $c > 0$ and $k_c \in (0, U_c]$ are tunable parameters. Details concerning the tuning of the parameters c and k_c will be provided in the sequel.

Note that a decentralized computation of (21) requires the estimation of the global terms $\lambda_2(\mathbf{p})$ and v. This problem will however not be addressed in this paper, since several procedures to attain this objective can be found in the literature. In particular, [13], [15] utilize the power iteration algorithm, implemented in a decentralized manner utilizing consensus-based distributed averaging. Both power iteration [44] and consensus [45] provide only asymptotic convergence to the desired value. However, convergence speed can be arbitrarily increased by appropriately selecting tuning gains. In a similar way, a consensus-based decentralized estimation protocol is proposed in [46] for simultaneously computing Laplacian eigenvalues and eigenvectors. A different approach is utilized in [31], where local interaction rules are defined among the robots in such a way that their state trajectory converge to a linear combination of sinusoidal signals, whose characteristic frequencies are function of the eigenvalues of the Laplacian matrix. Convergence is guaranteed in finite time, and eigenvalues are then found using standard signal processing techniques.

In addition, it should be noticed that (21) is discontinuous at $\|\nabla_{p_i}\lambda_2(\mathbf{p})\| = 0$. Thus, it necessitates the application of the tools coming from the nonsmooth analysis reviewed in Section III-B.

We now demonstrate that the proposed control term (21) can maintain the connectivity of the time-varying interaction graph $\mathcal{G}(t)$ over time. Successively, we investigate the effects of the connectivity control term on the collective control objective. Then, we discuss the robustness of the proposed control law against error affecting the estimation of the algebraic connectivity. The dependence of λ_2 from the stacked vector of agent positions $\mathbf{p}$, as in (21), will be dropped in the sequel when not strictly required for the sake of readability.

A. Theoretical Analysis of Connectivity Maintenance

The following theorem establishes conditions for which the connectivity of the multi-robot system can be preserved over time.

Theorem 5: Consider the dynamics (15) with the local connectivity control input (21) under Assumption 1. Let the initial value of the algebraic connectivity be $\lambda_2 > \epsilon$. Then, if $k_c > U_o$ the algebraic connectivity will necessarily increase if

$$\lambda_2 < \epsilon + c\,\log\left(\frac{k_c}{U_o}\right), \tag{22}$$

thus ensuring that λ_2 never goes below ϵ as the system evolves. □

Proof: Consider a continuously differentiable function $V(\cdot) : \mathbb{R}^{Nd} \to \mathbb{R}^+$ defined as follows

$$V(\mathbf{p}) = e^{\dfrac{(-\lambda_2(\mathbf{p}) + \epsilon)}{c}} \tag{23}$$

along with the set $\Omega_\beta = \left\{\mathbf{p} \in \mathbb{R}^{Nd} : V(\mathbf{p}) \leq \beta\right\}$. Arguments concerning the compactness of the set Ω_β with respect to relative distances can be found for instance in [47]–[49].

Since the Lyapunov candidate $V(\mathbf{p})$ is a smooth function, we can resort to the (simplified) chain rule given in Theorem 2 to compute the generalized time-derivative as

$$\begin{aligned} \dot{\tilde{V}}(\mathbf{p}) &= (\nabla_{\mathbf{p}} V)^T\ K[\dot{\mathbf{p}}] \subset \sum_{i=1}^{N} (\nabla_{p_i} V)^T\ K[\dot{p}_i] \\ &= \sum_{i=1}^{N}\left(-\frac{1}{c} e^{\dfrac{(-\lambda_2 + \epsilon)}{c}} \nabla_{p_i}\lambda_2\right)^T K[\dot{p}_i] \\ &= -\frac{1}{c} e^{\dfrac{(-\lambda_2 + \epsilon)}{c}} \sum_{i=1}^{N} (\nabla_{p_i}\lambda_2)^T\ K[\dot{p}_i] \end{aligned} \tag{24}$$

where the third property of the calculus given in Theorem 4 has been used.

Let us now evaluate the i-th term of the sum in (24) as

$$\begin{aligned} (\nabla_{p_i}\lambda_2)^T K[\dot{p}_i] &= (\nabla_{p_i}\lambda_2)^T K\left[f_i(\mathbf{p}) + u_i\right] \\ &\subset (\nabla_{p_i}\lambda_2)^T K\left[f_i(\mathbf{p})\right] + (\nabla_{p_i}\lambda_2)^T K\left[u_i\right] \end{aligned} \tag{25}$$

where the fourth property of the calculus given in Theorem 4 has been used. At this point, by plugging (21) into (25) we obtain

$$(\nabla_{p_i}\lambda_2)^T K[\dot{p}_i] \subset (\nabla_{p_i}\lambda_2)^T K[f_i(\mathbf{p})] + (\nabla_{p_i}\lambda_2)^T K\left[\left(k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \frac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2\|}\right)\right] \tag{26}$$

From (26), by recalling that $\|\nabla_{p_i}\lambda_2\| = 0$ implies $\nabla_{p_i}\lambda_2 = 0$, it follows that

$$(\nabla_{p_i}\lambda_2)^T K[\dot{p}_i] \subset \{0\} \quad \text{if } \|\nabla_{p_i}\lambda_2\| = 0 \tag{27}$$

Let us now consider the case $\|\nabla_{p_i}\lambda_2\| \neq 0$. In this regard, according to the first and fourth properties of Theorem 4 we have that

$$K[f_i(\mathbf{p})] = \text{co}\left\{\lim_{h\to\infty} f_i(\mathbf{p}^h) \,|\, \mathbf{p}^h \to \mathbf{p}, \mathbf{p} \notin H_{f_i}\right\} \tag{28}$$

with H_{f_i} the set of points for which $f_i(\mathbf{p})$ fails to be the differentiable. It follows that (26) can be simplified as

$$(\nabla_{p_i}\lambda_2)^T K[\dot{p}_i] \subset (\nabla_{p_i}\lambda_2)^T \text{co}\left\{\lim_{h\to\infty} f_i(\mathbf{p}^h) \,|\, \mathbf{p}^h \to \mathbf{p}, \mathbf{p} \notin H_{f_i}\right\} + (\nabla_{p_i}\lambda_2)^T \left\{\left(k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \frac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2\|}\right)\right\} \tag{29}$$

In particular, an element ζ_i^h of this intersection looks like

$$\begin{aligned}\zeta_i^h &= (\nabla_{p_i}\lambda_2)^T f_i(\mathbf{p}^h) + (\nabla_{p_i}\lambda_2)^T k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \frac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2)\|} \\ &\geq \|\nabla_{p_i}\lambda_2\| \left[-\left\|f_i(\mathbf{p}^h)\right\| + k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \frac{\|\nabla_{p_i}\lambda_2\|}{\|\nabla_{p_i}\lambda_2\|}\right] \\ &\geq \|\nabla_{p_i}\lambda_2\| \left[-U_o + k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}}\right]\end{aligned} \tag{30}$$

Therefore, a sufficient condition for the term ζ_i^h to be positive definite is that

$$-U_o + k_c\, e^{\frac{(-\lambda_2+\epsilon)}{c}} > 0 \tag{31}$$

From Theorem 2 we know that

$$\frac{d}{dt}V(\mathbf{p}(t)) \in^{\text{a.e.}} \dot{\tilde{V}}(\mathbf{p}) \tag{32}$$

and by recalling (24) we have

$$\frac{d}{dt}V(x(t)) \in^{\text{a.e.}} -\frac{1}{c}e^{\frac{(-\lambda_2+\epsilon)}{c}} \sum_{i=1}^{N} (\nabla_{p_i}\lambda_2)^T K[\dot{p}_i] \tag{33}$$

At this point, by exploiting (31), it follows that the Lyapunov time derivative is negative definite, and thus the algebraic connectivity λ_2 increases, if

$$\lambda_2 < \epsilon + c \log\left(\frac{k_c}{U_o}\right) \tag{34}$$

and finally, since $\log\left(\frac{k_c}{U_o}\right) > 0$ the thesis follows. ∎

Notably, from Theorem 5, it follows that $U_o < U_c$ with $U_o + U_c < U$. Therefore the boundedness condition on the local control inputs implies that some design constraints must be taken when defining the desired collective control objective.

Interestingly, what Theorem 5 tells us is that the proposed control law renders the set $\Omega_\epsilon = \{\mathbf{p} \in \mathbb{R}^{Nd} : \lambda_2(\mathbf{p}) > \epsilon\}$ positively invariant, that is by assuming the multi-robot system starts with a configuration that belongs to such a set, then it will remain there as time goes to infinity, thus preserving the connectedness of the multi-robot interaction graph over time.

Furthermore, it should be pointed out that the result given in Theorem 5 could be generalized by introducing into the dynamics (15) a bounded additive uncertainty term d_i. Briefly, as for the term $f_i(\mathbf{p})$ for which we introduced U_o as an upper bound, it would suffice to consider an upper bound D_o for the disturbance d_i, and repeat the analysis by considering both terms. Notably, by following the same reasoning as in Theorem 5, the final result would be that the Lyapunov time derivative is negative definite, and thus the algebraic connectivity λ_2 increases, if

$$\lambda_2 < \epsilon + c \log\left(\frac{k_c}{U_o + D_o}\right) \tag{35}$$

and thus the thesis would follow under the assumption $\log\left(\frac{k_c}{U_o+D_o}\right) > 0$. Thus, clearly imposing a more restrictive condition on the value of the algebraic connectivity due to the presence of an additional term.

B. Theoretical Analysis of Control Objective Perturbation

Theorem 5 states that the proposed control law can guarantee the connectivity maintenance over time in the presence of any bounded control objective such that $U_o < k$. We now change perspective and investigate how the connectivity control term may affect the achievement of the collective control objective.

For the sake of the analysis and with no lack of generality, we will assume that the collective control objective can be obtained through a potential-based design approach. Indeed, the potential-based control design is a very popular framework widely exploited in the robotics and control communities for controlling multi-robot systems [7], [32], [47]–[52]. Briefly, the idea is to define a potential function $\Phi(\mathbf{p})$ encoding the energy of the system in such a way that the desired configurations of the multi-robot system correspond to those points of the potential for which the energy is minimized. Then, a natural way to design a control law to achieve these configurations is to let the multi-robot system move along the anti-gradient $-\nabla_{\mathbf{p}}\Phi$ of the potential $\Phi(\mathbf{p})$.

A very common approach to design distributed potential-based control objectives is to consider, for each pair of robots i and j, a (continuously differentiable[2]) potential func-

[2]A generalization to handle pairwise non-smooth potential functions can be found in [32]. Here, for the sake of simplicity, since the focus of the paper is on the design of a bounded connectivity control term, we limit ourself on smooth pairwise potentials.

tion $\Phi_{ij}(\mathbf{p}) \triangleq \Phi_{ij}(p_i, p_j)$ for which the following properties hold

$$\begin{aligned} \nabla_{p_i}\Phi_{ij} &= \nabla_{p_i}\Phi_{ji} \\ \nabla_{p_i}\Phi_{ij} &= -\nabla_{p_j}\Phi_{ij} \end{aligned} \tag{36}$$

Then, the following global potential function for the multi-robot system can be considered

$$\Phi(\mathbf{p}) = \sum_{i=1}^{N}\sum_{j\neq i}\Phi_{ij}(p_i, p_j) \tag{37}$$

and the following per-agent anti-gradient control term can be plugged into (15) as the desired control objective (with $u_i = 0$) to ensure that a (local) minimum of the potential function $\Phi(\mathbf{p})$ is achieved

$$f_i(\mathbf{p}) = -\nabla_{p_i}\Phi(\mathbf{p}) \quad i \in \{1, \ldots, N\} \tag{38}$$

with $\nabla_{p_i}\Phi(\mathbf{p})$ defined according to (36) and (37) as

$$\nabla_{p_i}\Phi(\mathbf{p}) = \sum_{j\in\mathcal{N}_i(t)}\nabla_{p_i}\Phi_{ij}. \tag{39}$$

Our objective is to consider this control objective as a case study to investigate how the connectivity control term, while preserving the connectedness of the graph, may affect the achievement of the desired objective. Note that in order to fulfill Assumption 1 we are going to consider the following normalized version of (38)

$$f_i(\mathbf{p}) = -k_o\frac{\nabla_{p_i}\Phi(\mathbf{p})}{\|\nabla_{p_i}\Phi(\mathbf{p})\|} \quad i \in \{1, \ldots, N\} \tag{40}$$

with $0 < k_o \leq U_o$ the gain of the collective control objective. Indeed, it can be shown that the same equilibria could be reached by the multi-robot system if each agent i were controlled to run (40) instead of (38), see [53] for further details.

However, it should be noticed that these control laws (38) and (40) would not ensure that the connectedness of the communication graph is preserved over time for any set of initial conditions, i.e., the stacked vector $\mathbf{p}$ describing the robot locations, as numerically demonstrated also in Section V-A.

The following theorem provides an Input-to-State Stability (ISS)-like result in the algebraic connectivity domain (see [54] for a comprehensive overview of the ISS framework) for the proposed connectivity maintenance control law. As a matter of fact, while acting primarily as a mathematical formalization of a quite intuitive theoretical result, that is with a disturbance in general one cannot achieve exactly the desired objective, this theorem will also prove useful to narrow down scenarios for which the objective could be indeed achieved, even in the presence of the connectivity control term.

Theorem 6: Consider the dynamics (15) with the desired control objective (40) and the local connectivity control input (21). Let the global potential function $\Phi(\mathbf{p})$ be defined as in (37). Then, the multi-robot system will move towards a local minimum of such a global potential function $\Phi(\mathbf{p})$ until

$$\lambda_2(\mathbf{p}) > \epsilon + c\,\log\left(\frac{k_c}{k_o}\right), \quad k_o \in (0, U_o] \tag{41}$$

with $\mathbf{p} = [p_1, \ldots, p_N]^T$. □

Proof: Consider the dynamics (15) of the i-th robot when the desired objective (40) and the connectivity control law (21) are used

$$\dot{p}_i = -k_o\frac{\nabla_{p_i}\Phi(\mathbf{p})}{\|\nabla_{p_i}\Phi(\mathbf{p})\|} + k_c\,e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}}\frac{\nabla_{p_i}\lambda_2(\mathbf{p})}{\|\nabla_{p_i}\lambda_2(\mathbf{p})\|} \tag{42}$$

Consider the global potential function $\Phi(\mathbf{p})$ in (37) as the Lyapunov candidate function along with the set of $\Omega_\beta = \{\mathbf{p} \in \mathbb{R}^{Nd} : \Phi(\mathbf{p}) \leq \beta\}$. As for (23), arguments concerning the compactness of the set Ω_β with respect to relative distances can be found for instance in [47]–[49]. Then, by recalling that $\Phi(\mathbf{p})$ is continuously differentiable we can resort again to the (simplified) chain rule given in Theorem 2 and thus the generalized time-derivative can be computed as

$$\dot{\tilde{\Phi}}(\mathbf{p}) = (\nabla_{\mathbf{p}}\Phi)^T\,\dot{\mathbf{p}} \subset \sum_{i=1}^{N}(\nabla_{p_i}\Phi)^T\,K[\dot{p}_i] \tag{43}$$

where again the third property of the calculus given in Theorem 4 has been used. Let us now evaluate the i-th term of the sum in (43) as

$$\begin{aligned} (\nabla_{p_i}\Phi)^T K[\dot{p}_i] &= (\nabla_{p_i}\Phi)^T K\left[f_i(\mathbf{p}) + u_i\right] \\ &\subset (\nabla_{p_i}\Phi)^T K\left[f_i(\mathbf{p})\right] + (\nabla_{p_i}\Phi)^T K\left[u_i\right] \\ &= (\nabla_{p_i}\Phi)^T K\left[\left(-k_o\frac{\nabla_{p_i}\Phi}{\|\nabla_{p_i}\Phi\|}\right)\right] \\ &\quad + (\nabla_{p_i}\Phi)^T K\left[k_c\,e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}}\frac{\nabla_{p_i}\lambda_2(\mathbf{p})}{\|\nabla_{p_i}\lambda_2(\mathbf{p})\|}\right] \end{aligned} \tag{44}$$

From (44), by recalling that $\|\nabla_{p_i}\Phi\| = 0$ implies $\nabla_{p_i}\Phi = 0$, it follows that

$$(\nabla_{p_i}\Phi)^T K[\dot{p}_i] \subset \{0\} \quad \text{if } \|\nabla_{p_i}\Phi\| = 0 \tag{45}$$

Let us now consider the case $\|\nabla_{p_i}\Phi\| \neq 0$. In particular, (44) simplifies to

$$\begin{aligned} (\nabla_{p_i}\Phi)^T K[\dot{p}_i] &\subset (\nabla_{p_i}\Phi)^T\left\{-k_o\frac{\nabla_{p_i}\Phi}{\|\nabla_{p_i}\Phi\|}\right\} \\ &\quad + (\nabla_{p_i}\Phi)^T K\left[k_c\,e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}}\frac{\nabla_{p_i}\lambda_2(\mathbf{p})}{\|\nabla_{p_i}\lambda_2(\mathbf{p})\|}\right] \end{aligned} \tag{46}$$

First, let us consider the case $\|\nabla_{p_i}\lambda_2\| \neq 0$ for which we have

$$\begin{aligned} (\nabla_{p_i}\Phi)^T K[\dot{p}_i] &\subset (\nabla_{p_i}\Phi)^T\left\{-k_o\frac{\nabla_{p_i}\Phi}{\|\nabla_{p_i}\Phi\|}\right\} \\ &\quad + (\nabla_{p_i}\Phi)^T\left\{k_c\,e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}}\frac{\nabla_{p_i}\lambda_2(\mathbf{p})}{\|\nabla_{p_i}\lambda_2(\mathbf{p})\|}\right\} \\ &= \left\{-\|\nabla_{p_i}\Phi\|\left(k_o - k_c\,e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}}\cos(\alpha_i)\right)\right\} \end{aligned} \tag{47}$$

with α_i the angle between the two gradients $\nabla_{p_i}\Phi$ and $\nabla_{p_i}\lambda_2$, that is

$$\alpha_i = \angle(\nabla_{p_i}\Phi, \nabla_{p_i}\lambda_2) \tag{48}$$

In particular, since the following bound holds for (47)

$$(\nabla_{p_i}\Phi)^T K[\dot{p}_i] \leq \left\{ -\|\nabla_{p_i}\Phi\| \left(k_o - k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} \right) \right\} \tag{49}$$

a sufficient condition for its negative-definiteness is that

$$k_o - k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} > 0 \tag{50}$$

which, by means of some manipulations, can be rewritten as

$$\lambda_2(\mathbf{p}) > \epsilon + c\, \log\left(\frac{k_c}{k_o}\right) \tag{51}$$

Let us now consider the case $\|\nabla_{p_i}\lambda_2\| = 0$. In this regard, let us recall that the generalized gradient for the discontinuous right-hand side of (21) according to the first and the fourth properties of Theorem 4 takes the form

$$\begin{aligned} &K\left[k\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \frac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2\|}\right] \\ &= k\, e^{\frac{(-\lambda_2+\epsilon)}{c}} K\left[\frac{\nabla_{p_i}\lambda_2}{\|\nabla_{p_i}\lambda_2\|}\right] \\ &= k\, e^{\frac{(-\lambda_2+\epsilon)}{c}} \operatorname{co}\left\{ \lim_{h\to\infty} \frac{\nabla_{p_i^h}\lambda_2}{\|\nabla_{p_i^h}\lambda_2\|} \,|\, p_i^h \to p_i,\ \|\nabla_{p_i^h}\lambda_2\| \neq 0 \right\} \end{aligned} \tag{52}$$

At this point, by following a similar analysis as before we obtain

$$\begin{aligned} (\nabla_{p_i}\Phi)^T K[\dot{p}_i] \subset (\nabla_{p_i}\Phi)^T &\left\{ -k_o \frac{\nabla_{p_i}\Phi}{\|\nabla_{p_i}\Phi\|} \right\} \\ &+ k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} (\nabla_{p_i}\Phi)^T \\ &\operatorname{co}\left\{ \lim \frac{\nabla_{p_i^h}\lambda_2}{\|\nabla_{p_i^h}\lambda_2\|} \,|\, p_i^h \to p_i,\ \left\|\nabla_{p_i^h}\lambda_2\right\| \neq 0 \right\} \end{aligned} \tag{53}$$

In particular, an element ζ_i^h of this intersection looks like

$$\begin{aligned} \zeta_i^h &= -k_o \|\nabla_{p_i}\Phi\| + k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} (\nabla_{p_i}\Phi)^T \frac{\nabla_{p_i^h}\lambda_2}{\|\nabla_{p_i^h}\|} \\ &= -\|\nabla_{p_i}\Phi\| + k\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} \|\nabla_{p_i}\Phi\| \cos(\alpha_i^h) \end{aligned} \tag{54}$$

where α_i^h is defined as

$$\alpha_i^h = \angle(\nabla_{p_i}\Phi, \nabla_{p_i^h}\lambda_2) \tag{55}$$

that is, the angle between the two gradients $\nabla_{p_i}\Phi$ and $\nabla_{p_i^h}\lambda_2$.

In particular, the following bound holds for the element ζ_i^h

$$\begin{aligned} \zeta_i^h &\leq -k_o \|\nabla_{p_i}\Phi\| + k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} \|\nabla_{p_i}\Phi\| \\ &\leq -\|\nabla_{p_i}\Phi\| \left[k_o - k_c\, e^{\frac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} \right] \end{aligned} \tag{56}$$

and thus, also in this case, a sufficient condition for negative-definiteness of ζ_i^h is that

$$\lambda_2(\mathbf{p}) > \epsilon + c\, \log\left(\frac{k_c}{k_o}\right) \tag{57}$$

From Theorem 2 we know that

$$\frac{d}{dt}\Phi(\mathbf{p}(t)) \in^{\text{a.e.}} \dot{\tilde{\Phi}}(\mathbf{p}) \tag{58}$$

and by recalling (43) we have

$$\frac{d}{dt}\Phi(\mathbf{p}(t)) \in^{\text{a.e.}} \sum_{i=1}^{N} (\nabla_{p_i}\Phi)^T\ K[\dot{p}_i] \tag{59}$$

Therefore, it follows that a sufficient condition for the Lyapunov time derivative to be negative definite is that conditions (51) and (57) hold for each $i \in \{1, \ldots, N\}$. Thus the thesis follows. ■

A few remarks are now in order.

- In general, the connectivity control term does not allow to reach exactly a configuration corresponding to a local minimum of the global potential $\Phi(\mathbf{p})$.
- If the algebraic connectivity is sufficiently large in the neighborhood of the critical points of the global potential $\Phi(\mathbf{p})$, then the connectivity control term does not affect (at all) the control objective.
- The disturbance introduced by the connectivity control term can be (partially) mitigated through the tuning of the parameter c.

We reiterate that Theorem 6 provides a mathematical formalization of a quite intuitive theoretical result. As a matter of fact, such a result becomes even more clear when looking at this problem from the perspective of the Input-to-State Stability (ISS) framework, that is the connectivity control term can be thought as a disturbance acting on the actual desired objective, and thus (in general) perturbing the nominal equilibria.

Furthermore, due to the generality of the problem formulation where the potential design was only required to satisfy the (commonly assumed) properties (36), conditions for the actual convergence of the system to the desired objective could not be derived. Indeed, as shown by the following corollary, better convergence results can be established by imposing further (and in some application contexts reasonable) assumptions on the design of the global potential function $\Phi(\mathbf{p})$.

Corollary 2: Consider the dynamics (15) with the desired control objective (40) and the local connectivity control input (21). Let the global potential function $\Phi(\mathbf{p})$ be defined as in (37) and assume that the following hold

$$(\nabla_{p_i}\Phi)^T \nabla_{p_i}\lambda_2 \geq 0 \quad \text{if} \quad \lambda_2(\mathbf{p}) \leq \epsilon + c\, \log\left(\frac{k_c}{k_o}\right) \tag{60}$$

and

$$\|\nabla_{p_i}\lambda_2\| = 0 \quad \Rightarrow \quad \|\left(\nabla_{p_i}\Phi\right)\| = 0 \tag{61}$$

with $i \in \{1, \ldots, N\}$. Then the multi-robot system reaches a local minimum of the global potential function $\Phi(\mathbf{p})$.

Proof: The result follows from the proof of Theorem 6. In particular, we know that if (41) holds then each term $(\nabla_{p_i}\Phi)^T K[\dot{p}_i]$ is negative definite, that is

$$(\nabla_{p_i}\Phi)^T K[\dot{p}_i] \leq \Big\{ -\gamma_i^a(\mathbf{p})\|\nabla_{p_i}\Phi\| \Big\} \tag{62}$$

with $\gamma_i^a(\mathbf{p}) > 0$ defined as

$$\gamma_i^a = \left(k_o - k_c\, e^{\dfrac{(-\lambda_2(\mathbf{p})+\epsilon)}{c}} \right). \tag{63}$$

Furthermore, when (41) no longer holds, by means of (60), we can rewrite (47) as

$$(\nabla_{p_i}\Phi)^T K[\dot{p}_i] \leq \Big\{ -k_o\,\|\nabla_{p_i}\Phi\| \Big\} \tag{64}$$

Furthermore, (61) allows to simplify (53) as

$$(\nabla_{p_i}\Phi)^T K[\dot{p}_i] = \{0\} \tag{65}$$

Therefore, by recalling Theorem 2 and by combining (62), (64) and (65) , the negative semi-definiteness of the Lyapunov derivative follows. Thus, by resorting to the nonsmooth version of the LaSalle's invariance principle given in Theorem 3 the result follows. ■

Notably, Corollary 2 provides us with guidelines to understand whether in principle a desired collective objective can be (completely) achieved or not. In particular, (60) tells us that anytime the multi-robot system is approaching a "critical" configuration for the connectivity maintenance, the potential encoding the desired objective function must be "coherent" with the potential encoding the algebraic connectivity, where by means of "coherence" we mean that the direction of motion required to minimize the objective function should not further reduce the algebraic connectivity λ_2 if it is already close to its lower bound ϵ. Furthermore, (61) tells us that the equilibria of the system (corresponding to the critical points of the desired objective function) should match the critical points of the algebraic connectivity.

C. Discussion on the Robustness to the Estimation Error

So far, an exact knowledge of the algebraic connectivity λ_2 was assumed to be available to carry out the theoretical analysis. However in a real scenario where the proposed control law is run onboard each robot, we are likely to have only an estimate of the algebraic connectivity $\tilde{\lambda}_2$ (see for example [15], [31]), due to several factors such as, for example, robots pose estimation errors, communication delays, packets drop, distributed estimation process transient and so on. Therefore, it is mandatory to evaluate whether such a control law can be still used when only an estimate of the algebraic connectivity is available.

In this regard, let us consider (21) where the actual value of the algebraic connectivity λ_2 is replaced by an available estimate $\tilde{\lambda}_2$ as follows

$$u_i = \begin{cases} k_c\, e^{\dfrac{\left(-\tilde{\lambda}_2+\epsilon\right)}{c}} \dfrac{\nabla_{p_i}\tilde{\lambda}_2}{\|\nabla_{p_i}\tilde{\lambda}_2\|} & \text{if } \|\nabla_{p_i}\tilde{\lambda}_2(\mathbf{p})\| \neq 0 \\ 0 & \text{if } \|\nabla_{p_i}\tilde{\lambda}_2(\mathbf{p})\| = 0 \end{cases} \tag{66}$$

The following assumption is taken on the estimation process of the algebraic connectivity.

Assumption 2: The estimation process of the algebraic connectivity is such that at each time step there is a bound on the estimation error as follows

$$|\lambda_2 - \tilde{\lambda}_2| \leq \Psi \tag{67}$$

□

Let us now evaluate the "robustness" of the proposed control law, where by robustness we mean the capability to continue preserving the connectivity over time, more specifically above a certain threshold ϵ, even in the presence of a (bounded) estimation error, as in Assumption 2. To this end, we now review the results of Theorem 5 when the control term (66) is considered instead of (21).

Corollary 3: Consider the dynamics (15) with the local connectivity control input (66) under Assumptions 1 and 2. Let the initial value of the estimate of the algebraic connectivity be $\tilde{\lambda}_2 > \epsilon + \Psi$. Then, if $k_c > U_o$ the algebraic connectivity will necessarily increase if

$$\lambda_2 < \epsilon + c\,\log\left(\frac{k_c}{U_o}\right) - \Psi, \tag{68}$$

thus ensuring that λ_2 never goes below ϵ as the system evolves.

Proof: The result follows the same reasoning as in the proof of Theorem 5, where the continuously differentiable function (23) is replaced by

$$V(\mathbf{p}) = e^{\dfrac{\left(-\tilde{\lambda}_2+\epsilon\right)}{c}} \tag{69}$$

for which the generalized time-derivative becomes

$$\begin{aligned} \dot{\tilde{V}}(\mathbf{p}) &= (\nabla_{\mathbf{p}}V)^T\; K[\dot{\mathbf{p}}] \subset \sum_{i=1}^{N} (\nabla_{p_i}V)^T\; K[\dot{p}_i] \\ &= -\frac{1}{c} e^{\dfrac{(-\tilde{\lambda}_2+\epsilon)}{c}} \sum_{i=1}^{N} \left(\nabla_{p_i}\tilde{\lambda}_2\right)^T\; K[\dot{p}_i] \end{aligned} \tag{70}$$

In particular, when $\left\|\nabla_{p_i}\tilde{\lambda}_2\right\| \neq 0$, a sufficient condition for a term ζ_i^h of the intersection $\left(\nabla_{p_i}\tilde{\lambda}_2\right)^T\; K[\dot{p}_i]$ to be positive definite is that

$$\tilde{\lambda}_2 < \epsilon + c\,\log\left(\frac{k_c}{U_o}\right) \tag{71}$$

Therefore, by exploiting Assumption 2, it follows that the Lyapunov time derivative is negative definite, and thus the algebraic connectivity λ_2 necessarily increases, if

$$\lambda_2 < \epsilon + c\,\log\left(\frac{k_c}{U_o}\right) - \Psi. \tag{72}$$

■

Note that an interesting consequence of this analysis is that in the presence of an estimation error to keep the algebraic connectivity λ_2 above a certain threshold ϵ, we are now forced to pose a further constraint on the choice of the tuning gains c and k_c, that is we now need to ensure that

$$c\,\log\left(\frac{k_c}{U_o}\right) > \Psi. \tag{73}$$

Indeed, this additional constraint reflects also on the capability to mitigate the effect that such a control law has on a potential-based desired objective as further detailed in the following.

Corollary 4: Consider the dynamics (15) with the desired control objective (40) and the local connectivity control input (66). Let the global potential function $\Phi(\mathbf{p})$ be defined as in (37). Then, the multi-robot system will move towards a local minimum of such a global potential function $\Phi(\mathbf{p})$ until

$$\lambda_2(\mathbf{p}) > \epsilon + c\,\log\left(\frac{k_c}{k_o}\right) + \Psi, \quad k_o \in (0, U_o] \tag{74}$$

with $\mathbf{p} = [p_1, \ldots, p_N]^T$.

Proof: The result follows the same reasoning as in the proof of Theorem 6, where the condition for negative definiteness in (51) and (57) now becomes

$$\tilde{\lambda}_2(\mathbf{p}) > \epsilon + c\,\log\left(\frac{k_c}{k_o}\right). \tag{75}$$

Therefore, by exploiting Assumption 2 a sufficient condition for the Lyapunov time derivative to be negative definite is that

$$\lambda_2(\mathbf{p}) > \epsilon + c\,\log\left(\frac{k_c}{k_o}\right) + \Psi. \tag{76}$$

■

An important remark is now in order. According to Corollaries 3 and 4, in the presence of an estimate of the algebraic connectivity, the connectivity control term dominates if (68) holds, while the system moves towards a local minimum of the desired objective if (74) holds. Compared to the case of exact knowledge of the algebraic connectivity, where the conditions were (22) and (41) respectively, two major differences arises:

- We now loose the capability to arbitrarily mitigate the effects of the connectivity control term through the parameter c due to the presence of the additional constraint (73);
- There exists now a gap between the inequalities (68) and (74), which can be explained by the necessity to be conservative for handling the fact that the estimate of the algebraic connectivity $\tilde{\lambda}_2$ may deviate from the actual value of the algebraic connectivity λ_2 at most by Ψ.

V. Experimental and Numerical Validation

A. Simulations

Simulations have been carried out by exploiting a framework developed by the authors in Matlab©. In particular, four different simulation scenarios have been considered. First, by taking inspiration from [7], we numerically demonstrate that for an initially connected interaction graph, a lost of connectivity can be experienced in the case of bounded control objective. Successively, to support the theoretical results, under the assumption of perfect knowledge of the algebraic connectivity, we consider a simple aggregative potential as an example of "coherent" collaborative objective for which a local minimum can be reached, and a simple dispersive potential as an example of "incoherent" collaborative objective for which a local minimum cannot be reached. In particular, given the switching nature of the interactions arising in a proximity-limited setting, we adopt continuously differentiable pairwise potential functions inspired by [55], which are smooth over neighborhood transitions in order to comply with the theoretical assumptions made in this work. Then, by assuming that only an estimate of the actual value of the algebraic connectivity is available, we consider again a dispersive scenario to demonstrate the robustness of the proposed control law against a bounded estimation error. Finally, to validate the effectiveness of the proposed connectivity control strategy in a typical formation control setting, we consider a bounded version of a well-known distance-based gradient-descent control framework, which takes the form of (37). Notably, as discussed in a recent survey on multi-agent formation control [56], different approaches are available at the state of the art for defining the pairwise potential function. As already discussed above, we adopt continuously differentiable pairwise potential functions as in [57], which are smooth over neighborhood transitions in order to comply with the theoretical assumptions made in this work.

Note that, for the numerical evaluation where our main objective is to corroborate the theoretical findings, we do not consider an obstacle avoidance as it will be done for the experimental validation.

Table I summarizes the parameters setting used in the simulations for the proposed control law (21).

Symbol	N	R	σ	k_o	k_c	c	ϵ	Ψ
Value	9	1.5	5	0.6	1	1	0.5	0.4

TABLE I
Parameters Setting for Simulations

Figure 1 depicts the behavior of a multi-robot system composed of 9 robots running a bounded aggregative control objective of the form (40) with no algebraic connectivity control. In particular, Figure (1a) represents the initial configuration of the multi-robot system where it can be noticed that the interaction graph is initially connected, Figure (1b) depicts the instant at which the graph connectivity is lost, and Figure (1c) shows how the multi-robot system keeps evolving into connected components of robots. Indeed, as discussed in [7], also in our setting with no algebraic connectivity control, the multi-robot system may experience a loss of the graph connectivity even tough the interaction graph was initially connected. This demonstrate that the same connectivity maintenance issues may arise even when considering objective functions encoded by bounded control laws, as for the case of unbounded control laws discussed in [7]. Thus, we reiterate that the connectivity maintenance problem is relevant also under the assumption of bounded control terms.

Figure 2 depicts the behavior of a multi-robot system composed of 9 robots running a bounded aggregative control objective of the form (40) along with the algebraic connectivity control term (21) under the ideal case of perfect knowledge of the algebraic connectivity. Figure (2a) depicts the initial configuration of the multi-robot system, while Figure (2b) shows an intermediate configuration, and Figure (2c) illustrates the final configuration of the multi-robot system. It can be seen that the system is able to achieve a local minimum of the desired collective objective function. This can be

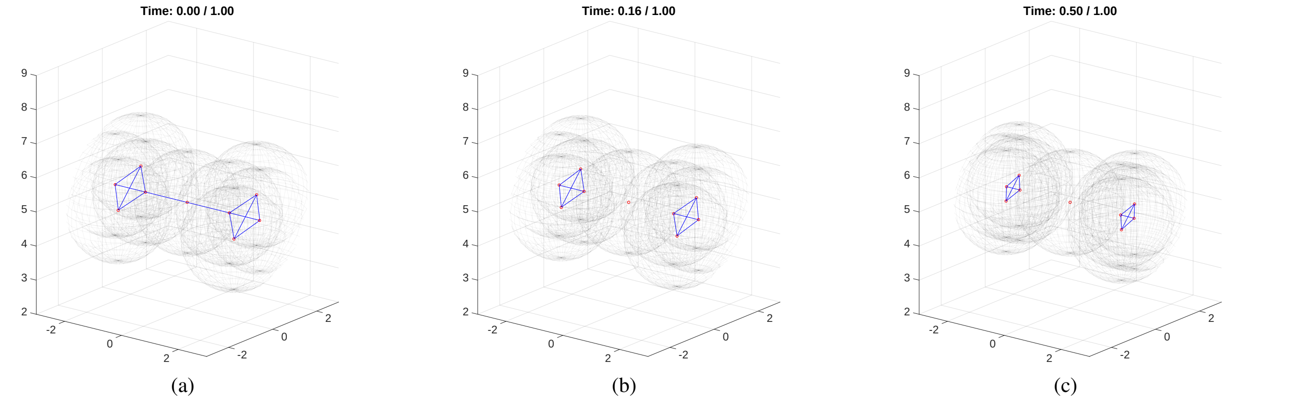


Fig. 1. Multi-robot system composed of 9 robots running a bounded aggregative control objective with no algebraic connectivity control losing graph connectivity over time even though the interaction graph was initially connected.

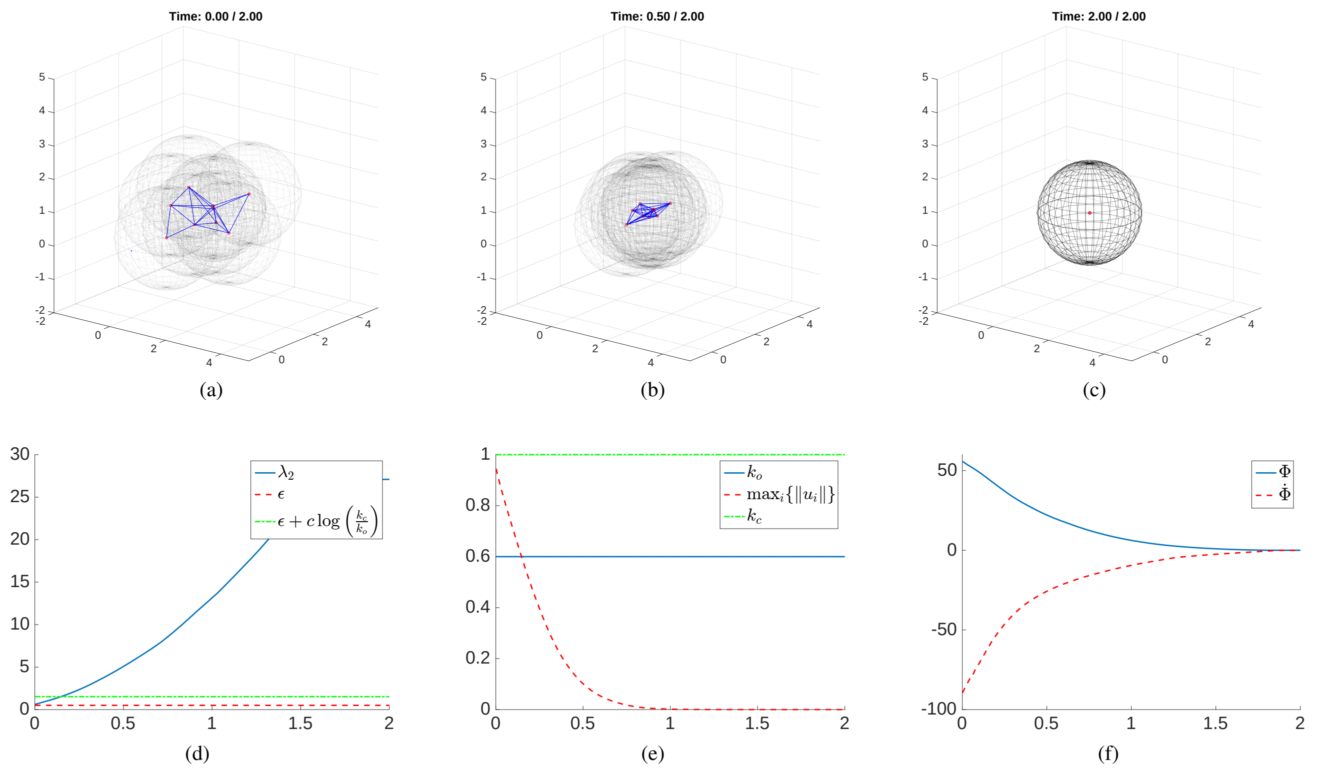


Fig. 2. Multi-robot system composed of 9 robots running a bounded aggregative control objective with algebraic connectivity control under the ideal case of perfect knowledge of the algebraic connectivity. The system is able to reach a local minimum of the collective objective function as both the conditions of Theorem 6 and of Corollary 2 are satisfied.

explained by the fact that both the conditions of Theorem 6 and Corollary 2 are satisfied. Figure (2d) illustrates how the algebraic connectivity changes over time. In particular, it can be noticed that the initial configuration of the system does not satisfy the condition (41) of Theorem 6. However, since the algebraic connectivity monotonically increases when the inter-robot distance decreases, it follows that the condition (60) of Corollary 2 must necessarily hold, and thus the system is moving toward a local minimum of the aggregative potential even when condition (41) does not hold. Figure (2e) shows the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term. In particular, it can be noticed how the connectivity control term dominates the collaborative objective control term in magnitude while condition (41) is not satisfied. Furthermore, it can be noticed that $\|\nabla_{p_i}\lambda_2\| = 0$ for each $i = \{1, \dots, N\}$ implies that $p_1 = p_2 = \dots = p_N$. Indeed, by construction this represents a critical point of the aggregative potential as well, and thus also the condition (61) of Corollary 2 is satisfied. Therefore, it follows that the system can reach a local minimum of the collective objective even in the presence of the connectivity maintenance control term. Figure (2f) supports this reasoning by numerically demonstrating that the potential reaches zero as time goes by, thus showing that a critical point has been reached.

Figure 3 depicts the behavior of a multi-robot system composed of 9 robots running a bounded dispersive control objective of the form (40) along with the algebraic connectivity control term (21) under the ideal case of perfect knowledge of the algebraic connectivity. Figure (3a) depicts the initial configuration of the multi-robot system, while Figure (3b) shows an intermediate configuration, and Figure (3c) illustrates the final configuration of the multi-robot system. It can be noticed that the system cannot achieve a local minimum of the desired collective objective function as this would imply letting each robot move arbitrarily far away from each other, a behavior that would eventually break the connectedness of the interaction graph. As proven in Theorem 6 the system keeps moving towards a local minimum of the objective function until condition (41) holds. In this regard, Figure (3d) shows how the algebraic connectivity changes over time, while Figure (3e) shows the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term, and Figure (3f) illustrates how the potential changes over time. More specifically, from Figure (3d) it can be noticed that the algebraic connectivity, which initially satisfies (41), approaches the value $\epsilon + c \log\left(\frac{k_c}{k_o}\right)$. This ensures that the system is temporarily moving towards a local minimum of the potential function. However, from Figure (3e) it can be noticed how the connectivity control effort matches the collective objective control term to preserve the connectedness of the interaction graph while the algebraic connectivity is approaching such a value. As a matter of fact, from Figure (3e) it can be noticed that this control action prevents the potential function encoding the collective objective to keep decreasing, thus indicating a local minimum cannot be reached.

Figure 4 illustrates the results obtained for a multi-robot system composed of 9 robots running the same dispersive objective as above, where only an estimate of the algebraic connectivity is assumed to be available under Assumption 2, and thus the connectivity control term (21) is replaced with (66). In particular, we recall that in this setting a further constraint must be enforced to ensure that the algebraic connectivity remains above a desired threshold. In this regard, we point out that the choice of the parameters is such that $c\ \log\left(\frac{k_c}{k_o}\right) = 0.5108$ while $\Psi = 0.4$, and thus (73) holds. Note that, due to space limitations, in this case we do not report a screenshot describing the sequence of configurations of the multi-robot system, as it turns out to be compatible with the nominal case described in Figures (3a), (3b), and (3c). Furthermore, two different scenarios representing the boundary cases were considered, that is in the first case we assume $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) - \Psi$, while in the second we assume $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) + \Psi$. Indeed, the behavior of any other possible estimation scenario satisfying Assumption 2 is confined within the evolution of these two. Figure (4a) shows how the algebraic connectivity and its estimate change over time, while Figure (4b) shows the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term, and Figure (4c) illustrates how the potential changes over time, for the case $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) - \Psi$. Similarly, Figure (4d) shows how the algebraic connectivity and its estimate change over time, while Figure (4e) shows the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term, and Figure (4f) illustrates how the potential changes over time, for the case $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) + \Psi$. In particular, from Figure (4a) it can be noticed how in the first case the algebraic connectivity approaches the value $\epsilon + c \log\left(\frac{k_c}{k_o}\right) - \Psi$, while from Figure (4d) it can be noticed how in the second case the algebraic connectivity approaches the value $\epsilon + c \log\left(\frac{k_c}{k_o}\right) + \Psi$. Indeed this supports the fact that the behavior of any other possible estimation scenario satisfying Assumption 2 is confined within the evolution of these two boundary cases. Furthermore, it can also be noticed that since in both cases the conditions of Corollary 3 are satisfied, then the system is able to preserve the connectedness of the interaction graph even though only an estimate of the algebraic connectivity is available. Furthermore, according to Corollary 4 the system in both cases keeps moving towards a local minimum of the objective function until condition (74) holds.

Figure 5 depicts the behavior of a multi-robot system composed of 5 robots running a bounded version of a distance-based formation control of the form (40), for which pairwise potentials are derived from [57], with and without the algebraic connectivity control term (21). For this scenario, the objective formation was a square pyramid encoded by a set of desired inter-robot distances. Note that, to properly encode the desired shape, a constraint among each pair of robots was required, and the resulting graph will be denoted in the following as $\mathcal{G}^*$. Indeed, this is a constraint that generally speaking cannot be satisfied by our connectivity control law, which instead aims only at ensuring the overall connectedness of the graph topology, rather than controlling the existence of individual edges. For this reason, and with no lack of generality for the validation of the effects of the proposed connectivity control law in a formation control setting, a suitable set of initial conditions has been chosen. Figure (5a) depicts the initial configuration of the multi-robot system, while Figure (5b) illustrates the final configuration of the multi-robot system for the case with connectivity maintenance, and Figure (5c) shows the final configuration of the multi-robot system for the case without connectivity control. For both scenario with and without connectivity control term respectively, Figures (5d) and (5g) show how the algebraic connectivity changes over time, while Figures (5e) and (5h) show the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term, and Figures (5f) and (5i) illustrate the overall formation error computed as the sum of per-pair squared error, that is $\sum_{(i,j)\in\mathcal{G}^*} \left(\|p_i - p_j\| - \|p_i^* - p_j^*\|\right)^2$. In particular, it can be noticed how the algebraic connectivity never goes below the value $\epsilon + c \log\left(\frac{k_c}{k_o}\right)$, and thus the system can reach the desired formation shape, since the objective control term always dominates over the connectivity control term. Indeed, as it can be observed by comparing the formation control

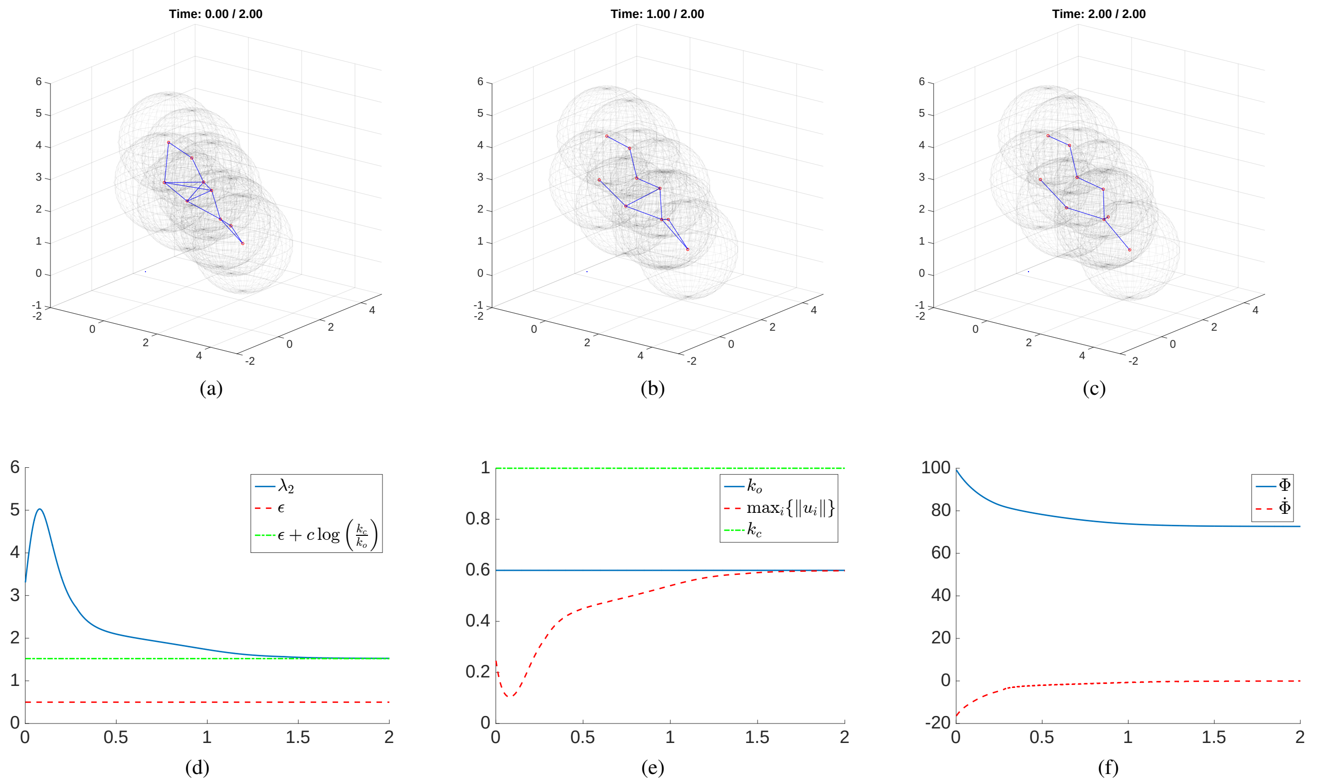


Fig. 3. Multi-robot system composed of 9 robots running a bounded dispersive control objective with algebraic connectivity control under the ideal case of perfect knowledge of the algebraic connectivity. The system is only able to get arbitrarily close to a local minimum of the collective objective function as only the conditions of Theorem 6 are satisfied.

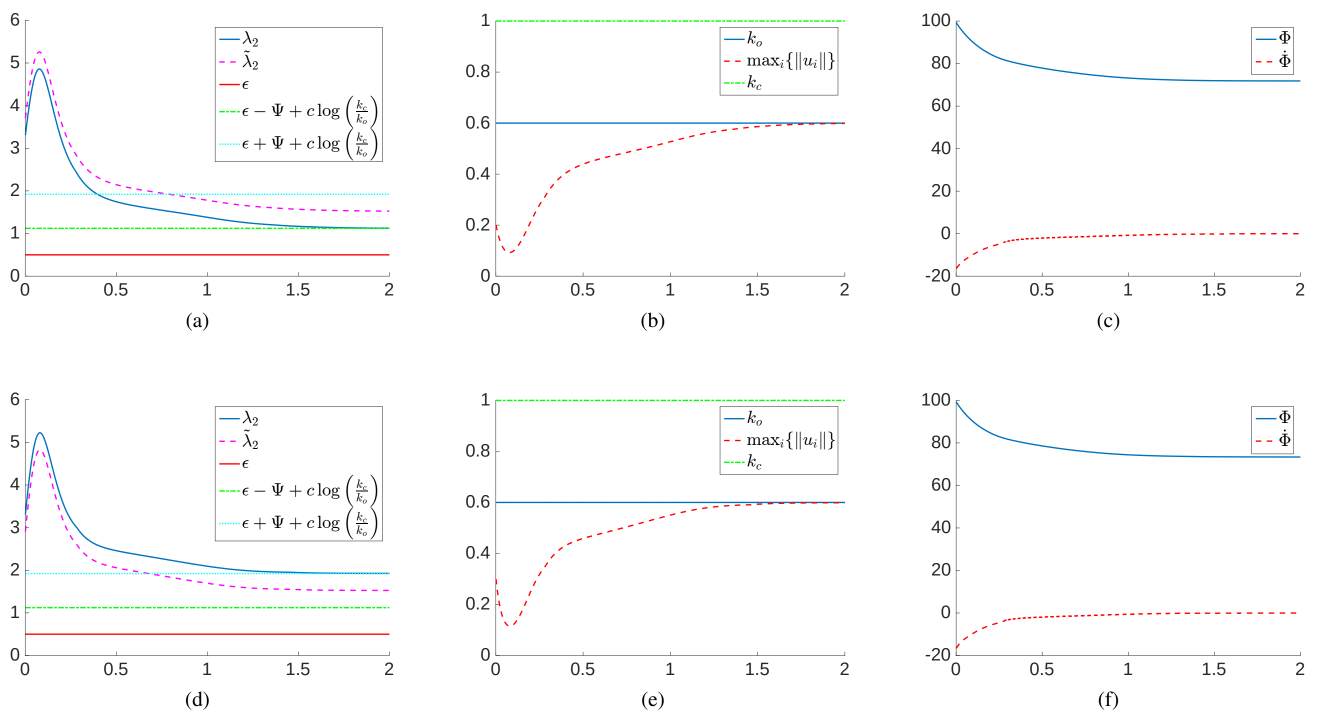


Fig. 4. Multi-robot system composed of 9 robots running a bounded dispersive control objective with algebraic connectivity control under the case of estimation error affecting the knowledge of the algebraic connectivity. The two boundary cases are analyzed, that is in the first case the estimate is assumed to be $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) - \Psi$, while in the second case the estimate is assumed to be $\tilde{\lambda}(\mathbf{p}) = \lambda(\mathbf{p}) + \Psi$. Note that the behavior of any other possible estimation scenario satisfying Assumption 2 is confined within their evolution. Indeed, the system is able to preserve the connectedness of the interaction graph even though only an estimate of the algebraic connectivity affected by (bounded) estimation error is available.

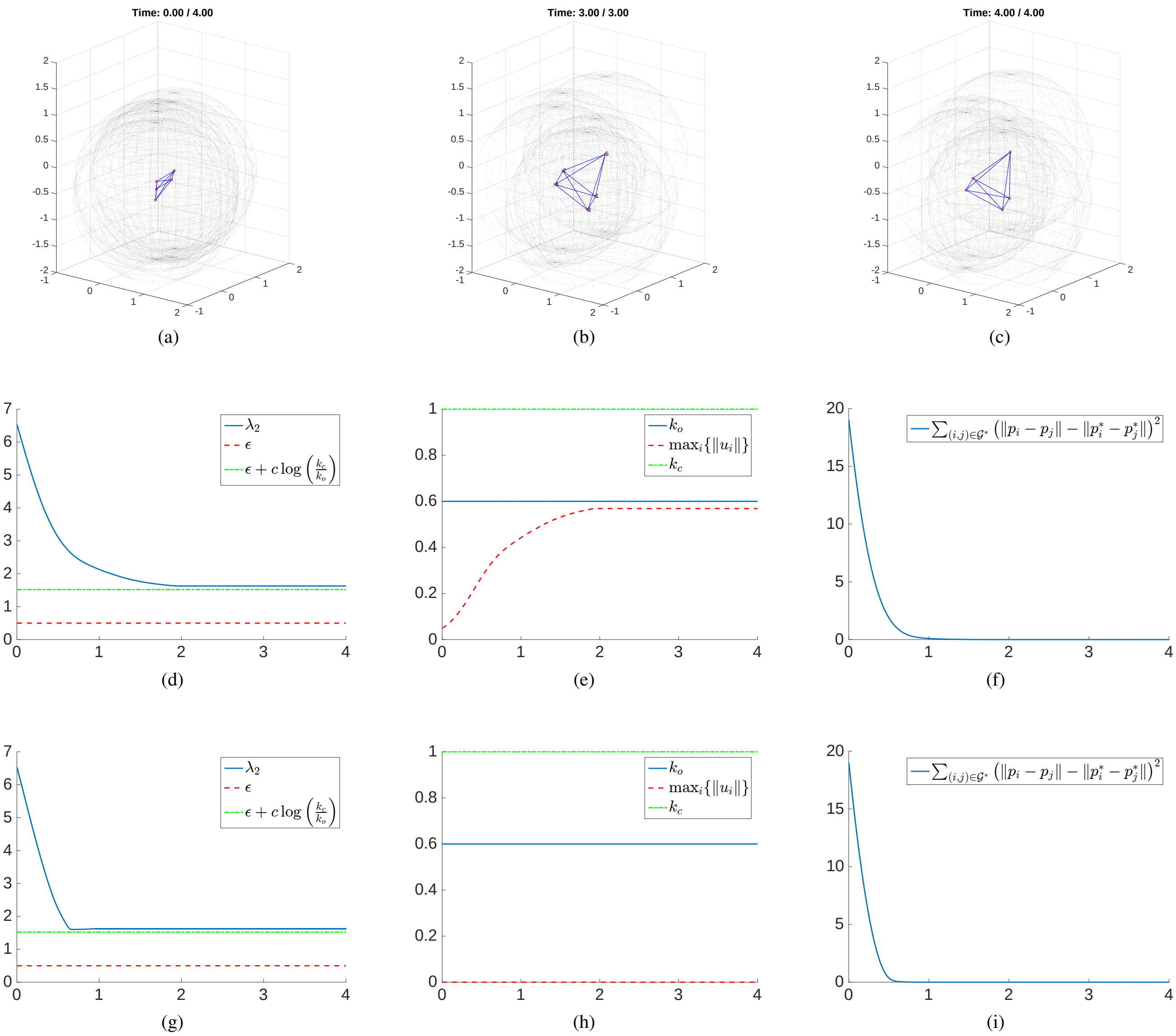


Fig. 5. Multi-robot system composed of 5 robots running a bounded version of the gradient-descent control framework for achieving distance-based formation control defined in [57], with and without the algebraic connectivity control term (21). The desired formation is a square pyramid encoded by a set of desired inter-robot distances for which the related algebraic connectivity is above the threshold $\epsilon + c\log\left(\frac{k_c}{k_o}\right)$.

error in Figure (5f) and in Figure (5i), the major difference introduced by the presence of the connectivity control term is the reduction of the convergence speed, as it takes longer for the system to achieve the desired formation when the connectivity control term is present.

Figure 6 depicts the behavior of a multi-robot system composed of 5 robots running again a bounded version of a distance-based formation control of the form (40), for which pairwise potentials are derived from [57], with and without the algebraic connectivity control term (21). Also for this scenario the objective formation was a square pyramid encoded by a set of desired inter-robot distances. However, compared to the previous setting, in this scenario the desired inter-robot distances were purposely chosen such that the final configuration would require the algebraic connectivity to go below the value $\epsilon + c\log\left(\frac{k_c}{k_o}\right)$. Figure (6a) depicts the initial configuration of the multi-robot system, while Figure (6b) illustrates the final configuration of the multi-robot system for the case with connectivity maintenance, and Figure (6c) shows the final configuration of the multi-robot system for the case without connectivity control. For both scenarios with and without connectivity control term respectively, Figure (6d) and (6g) show how the algebraic connectivity changes over time, while Figure (6e) and (6h) show the per-agent (largest) control effort along with the maximum effort for both the collective objective control term and algebraic connectivity control term, and Figure (6f) and (6i) illustrate the overall formation error computed as the sum of per-pair squared error, that is again $\sum_{(i,j)\in\mathcal{G}^*}\left(\|p_i - p_j\| - \|p_i^* - p_j^*\|\right)^2$. In particular, it can be noticed how the system reaches different equilibria with and without the connectivity control term, that is the system cannot reach the desired formation shape with

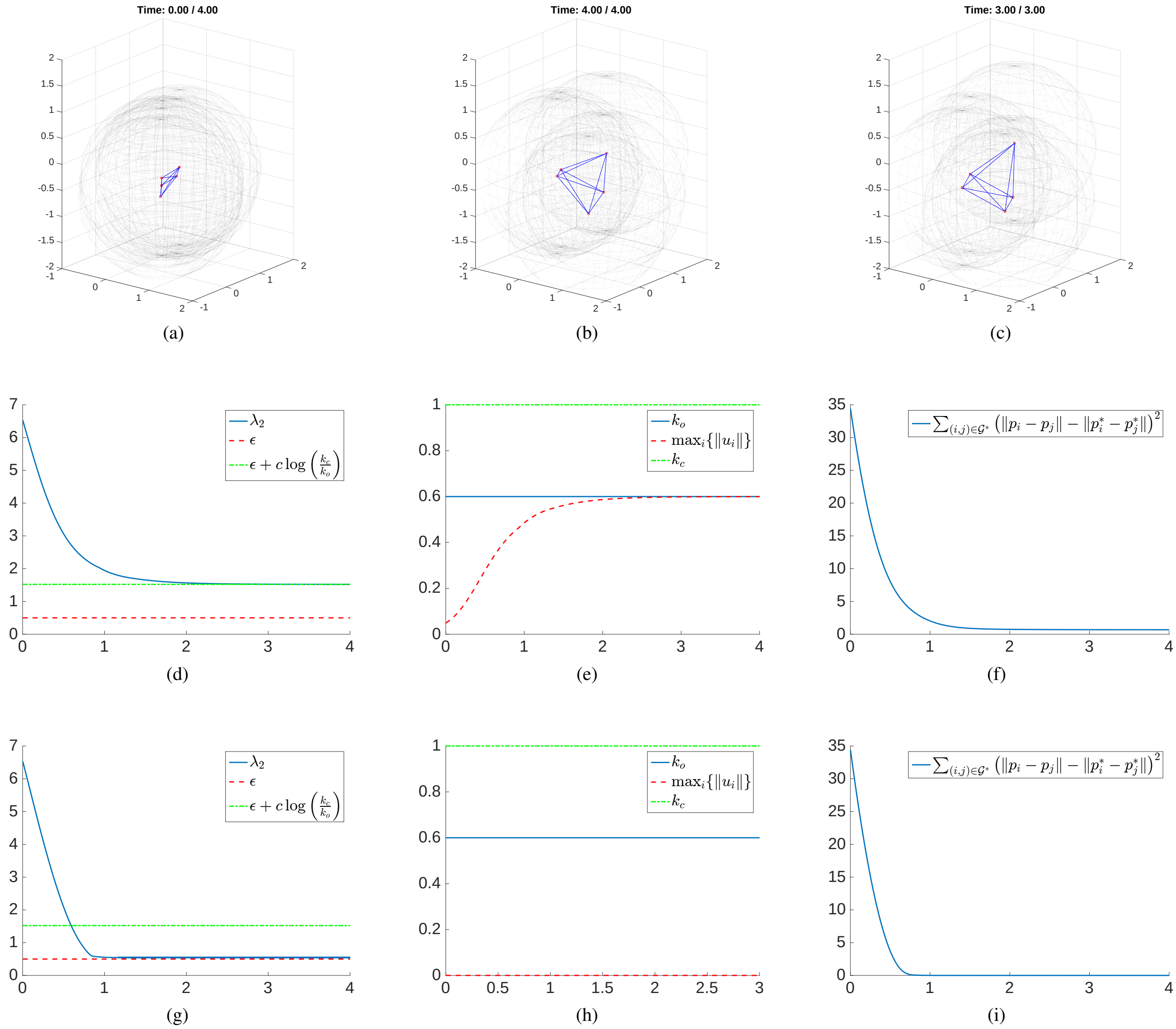


Fig. 6. Multi-robot system composed of 5 robots running a bounded version of the gradient-descent control framework for achieving distance-based formation control defined in [57], with and without the algebraic connectivity control term (21). The desired formation is a square pyramid encoded by a set of desired inter-robot distances for which the related algebraic connectivity is below the threshold $\epsilon + c\log\left(\frac{k_c}{k_o}\right)$.

the connectivity control term, while the system can reach it without the connectivity control term. This can be explained by the fact that in the case without connectivity control the algebraic connectivity is allowed to go below the theoretical bounds for which the connectivity control term dominates to ensure the graph connectedness. On the contrary, in the case with connectivity control, while the algebraic connectivity is approaching $\epsilon + c\log\left(\frac{k_c}{k_o}\right)$ the connectivity control effort matches the collective objective control term (see Figures (6d) and (6e)). On the one hand, this ensures that the connectedness of the interaction graph is preserved over time, on the other hand this prevents the multi-robot system to fully achieve the desired formation as it corresponds to a configuration for which the algebraic connectivity is below the activation threshold of the connectivity control.

B. *Experiments*

Experiments have been carried out by exploiting 4 units of the SAETTA mobile robotic platform along with a low-cost vision tracking system composed of commercial wide-angle webcams both developed at the Robotics and Sensor Fusion Lab of the Department of Engineering at the University of Rome "Roma Tre" and integrated within a ROS network. Briefly, the Robot Operating System (ROS) is a flexible framework for writing robot software. Further details regarding ROS can be found in [58].

The SAETTA mobile robot is a small low-cost robotic platform which features a complete sensorial system, a very accurate traction in indoor environment, and a wireless communication channel for multi-robot applications. Further details regarding the SAETTA platform can be found in [59]. Relative distance information was provided by such a low-cost vision tracking system with a resolution of the order of about 5 cm, sufficiently accurate considering the size 14×29 cm of the SAETTA robot platform.

The main objective of this experimental validation was to test the proposed control law in a real context against not modeled factors such as pose measurements affected by noise, communication affected by packets delay (or packets loss), or inherent asynchronism in the robot-to-robot interaction. For the sake of comparison we considered again an aggregative and dispersive scenarios as for the numerical evaluation described in Section V-A with the same parameters setting used for the simulations (see Table I).

Figure 7 illustrates the results obtained for the experimental validation of the aggregative scenario, where also a repulsive potential-based control term was considered for avoiding inter-robot collisions. In particular, it can be observed that a similar behavior compared to the simulation results is experienced during the transient. More specifically, first the connectivity control term dominates the collective objective until the algebraic connectivity crossed the theoretical bound denoting its activation, then it starts fading out as the algebraic connectivity increases due to the fact that the robots get closer and closer. Furthermore, it can be noticed that, once a certain inter-robot distance close to the equilibrium has been reached, the robots motion exhibits a sustained oscillation. Although such an oscillation can be reasonably related to the non-holonomic nature of the robots kinematics, we believe it to be mainly induced by the (quite) poor resolution of the developed low-cost vision tracking system, which in turn highly emphasizes the discontinuous nature of the controller adopted for achieving aggregation and for avoiding collisions. Note that this behavior could be significantly mitigated by properly tuning the control gains and introducing small code workarounds, but we decided to keep it as it gives an idea of the entity of the estimation error affecting the experimental validation and thus in turn of the robustness of the proposed control law.

Figure 8 illustrates the results obtained for the experimental validation of the dispersive scenario. In particular, it can be seen that once the algebraic connectivity is approaching the theoretical bound for which the connectivity control term has to dominate the control objective in order to guarantee the connectedness of the interaction graph, a similar (even though smaller) sustained oscillation can be noticed. Also in this case, this can be explained both by the non-holonomic nature of the robots kinematics and the (quite) poor resolution of the developed low-cost vision tracking system and its effects on the discontinuous nature of the adopted dispersive controller. Nevertheless, the system is still capable to preserve the connectedness of the interaction graph even in the presence of (significantly) noisy pose measurements.

VI. Conclusion

In this paper we addressed the connectivity maintenance problem for a team of mobile robots which move according to a given bounded collective control objective. We proposed a bounded connectivity control law which can provably preserve the connectedness of the multi-robot system over time. We characterized the effects of the connectivity control term on the collective control objective by resorting to the ISS-like analysis framework. We investigated its robustness against estimation error of the algebraic connectivity. We validated the effectiveness of the proposed connectivity control term by considering both simulations and experiments results. Future work will be focused on extending the proposed approach to systems with higher-order dynamics.

References

[1] D. Calisi, A. Farinelli, L. Iocchi, and D. Nardi, "Multi-objective exploration and search for autonomous rescue robots: Research articles," *J. Field Robot.*, vol. 24, no. 8-9, pp. 763–777, Aug. 2007.

[2] Y. Kim and M. Minor, "Coordinated kinematic control of compliantly coupled multirobot systems in an array format," *Robotics, IEEE Transactions on*, vol. 26, no. 1, pp. 173–180, 2010.

[3] D. Carboni, R. Williams, A. Gasparri, G. Ulivi, and G. Sukhatme, "Rigidity-preserving team partitions in multiagent networks," *IEEE Transactions on Cybernetics*, vol. 45, no. 12, pp. 2640–2653, Dec 2015.

[4] S. G. Lee, Y. Diaz-Mercado, and M. Egerstedt, "Multirobot control using time-varying density functions," *IEEE Transactions on Robotics*, vol. 31, no. 2, pp. 489–493, April 2015.

[5] D. Di Paola, A. Gasparri, D. Naso, and F. Lewis, "Decentralized dynamic task planning for heterogeneous robotic networks," *Autonomous Robots*, vol. 38, no. 1, pp. 31–48, 2015.

[6] L. Sabattini, C. Secchi, M. Cocetti, A. Levratti, and C. Fantuzzi, "Implementation of coordinated complex dynamic behaviors in multi-robot systems," *IEEE Transactions on Robotics*, vol. 31, no. 4, pp. 1018–1032, aug. 2015.

[7] M. Ji and M. Egerstedt, "Distributed coordination control of multiagent systems while preserving connectedness," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, Aug 2007.

[8] Y. Cao and W. Ren, "Distributed coordinated tracking via a variable structure approach – part I: consensus tracking. part II: swarm tracking," in *Proceedings of the American Control Conference*, 2010.

[9] M. A. Hsieh, A. Cowley, V. Kumar, and C. J. Talyor, "Maintaining network connectivity and performance in robot teams," *Journal of Field Robotics*, vol. 25, no. 1, pp. 111–131, 2008.

[10] A. Ajorlou, A. Momeni, and A. G. Aghdam, "A class of bounded distributed control strategies for connectivity preservation in multi–agent systems," *IEEE Transactions on Automatic Control*, vol. 55, pp. 2828–2833, 2010.

[11] D. V. Dimarogonas and K. H. Johansson, "Bounded control of network connectivity in multi–agent systems," *IET Control Theory & Applications*, vol. 4, pp. 1751–8644, 2010.

[12] F. Morbidi, A. Giannitrapani, and D. Prattichizzo, "Maintaining connectivity among multiple agents in cyclic pursuit: A geometric approach," in *Proceedings of the IEEE International Conference on Decision and Control*, 2010, pp. 7461–7466.

[13] P. Yang, , R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity for mobile sensor networks," *Automatica*, vol. 46, no. 2, pp. 390–396, Feb. 2010.

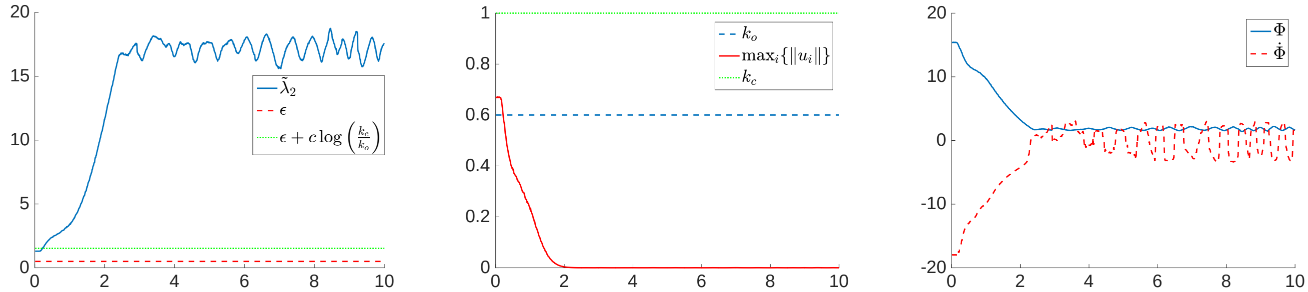
[14] H. A. Poonawala and M. W. Spong, "Decentralized estimation of the algebraic connectivity for strongly connected networks," in *American Control Conference (ACC)*. IEEE, 2015, pp. 4068–4073.

[15] L. Sabattini, N. Chopra, and C. Secchi, "Decentralized connectivity maintenance for cooperative control of mobile robotic systems," *I. J. Robotic Res.*, vol. 32, no. 12, pp. 1411–1423, 2013.

[16] L. Sabattini, C. Secchi, N. Chopra, and A. Gasparri, "Distributed control of multi-robot systems with global connectivity maintenance," *IEEE Transactions on Robotics*, vol. 29, no. 5, pp. 1326–1332, October 2013.

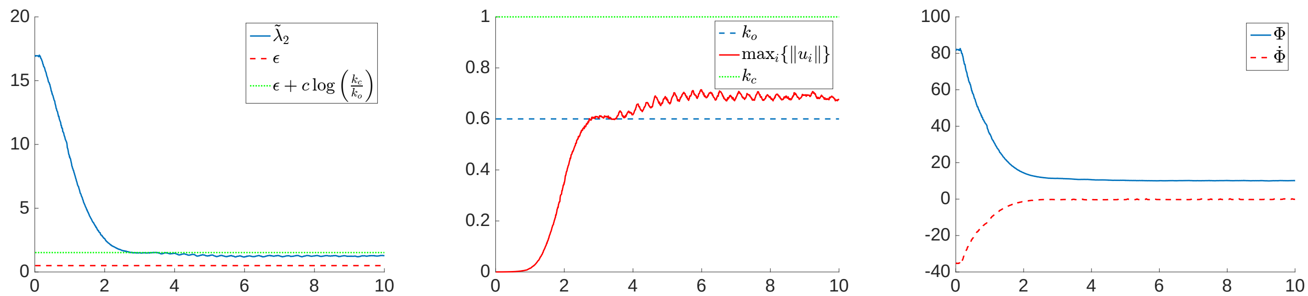
[17] A. Gasparri, A. Leccese, L. Sabattini, and G. Ulivi, "Collective control objective and connectivity preservation for multi-robot systems with bounded input," in *American Control Conference (ACC)*, 2014.

[18] M. S. Couceiro, C. M. Figueiredo, R. P. Rocha, and N. M. F. Ferreira, "Darwinian swarm exploration under communication constraints: Initial deployment and fault-tolerance assessment," *Robotics and Autonomous Systems*, vol. 62, no. 4, pp. 528–544, 2014.



(a) (b) (c)

Fig. 7. Experimental validation carried out with a multi-robot system composed of 4 SAETTA units. The multi-robot system is running a bounded aggregative potential along with the connectivity control term.



(a) (b) (c)

Fig. 8. Experimental validation carried out with a multi-robot system composed of 4 SAETTA units. The multi-robot system is running a bounded dispersive potential along with the connectivity control term.

[19] M. Zareh, L. Sabattini, and C. Secchi, "Decentralized biconnectivity conditions in multi-robot systems," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, Las Vegas, NV, USA, dec. 2016.

[20] ——, "Enforcing biconnectivity in multi-robot systems," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, Las Vegas, NV, USA, dec. 2016.

[21] N. Carlési and P. Bianchi, "Distributed coordination of a formation of heterogeneous agents with individual regrets and asynchronous communications," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2012, pp. 3504–3511.

[22] M. Xu, Q. Yang, and K. S. Kwak, "Algebraic connectivity aided energy-efficient topology control in selfish ad hoc networks," *Wireless Networks*, pp. 1–11, 2016.

[23] ——, "Distributed topology control with lifetime extension based on non-cooperative game for wireless sensor networks," *IEEE Sensors Journal*, vol. 16, no. 9, pp. 3332–3342, May 2016.

[24] R. Aragues, C. Sagues, and Y. Mezouar, "Triggered minimum spanning tree for distributed coverage with connectivity maintenance," in *European Control Conference (ECC)*. IEEE, 2014, pp. 1881–1887.

[25] C. Razafimandimby, V. Loscri, and A. M. Vegni, "A neural network and iot based scheme for performance assessment in internet of robotic things," in *I4T-1st International Workshop on Interoperability, Integration, and Interconnection of Internet of Things Systems*, 2016.

[26] T. Murayama, "Online trajectory planning method for multi-vehicle system considering network connectivity and collision avoidance simultaneously," *SICE Journal of Control, Measurement, and System Integration*, vol. 8, no. 1, pp. 15–21, 2015.

[27] C. Ghedini, C. Secchi, C. H. C. Ribeiro, and L. Sabattini, "Improving robustness in multi-robot networks," in *Proceedings of the IFAC Symposium on Robot Control (SYROCO)*, Salvador, Brazil, aug. 2015.

[28] D. Aksaray, A. Y. Yazicioglu, E. Feron, and M. D. N., "Message-passing strategy for decentralized connectivity maintenance in multiagent surveillance," *Journal of Guidance, Control, and Dynamics*, vol. 39, no. 3, pp. 542–555, 2016.

[29] T. D. Ngo, P. D. Hung, and M. T. Pham, "A kangaroo inspired heterogeneous swarm of mobile robots with global network integrity for fast deployment and exploration in large scale structured environments," in *IEEE International Conference on Robotics and Biomimetics (ROBIO)*. IEEE, 2014, pp. 1205–1212.

[30] R. Aragues, G. Shi, D. V. Dimarogonas, C. Sagüés, K. H. Johansson, and Y. Mezouar, "Distributed algebraic connectivity estimation for undirected graphs with upper and lower bounds," *Automatica*, vol. 50, no. 12, pp. 3253–3259, 2014.

[31] M. Franceschelli, A. Gasparri, A. Giua, and C. Seatzu, "Decentralized estimation of laplacian eigenvalues in multi-agent systems," *Automatica*, vol. 49, no. 4, pp. 1031 – 1036, 2013.

[32] R. K. Williams, A. Gasparri, G. S. Sukhatme, and G. Ulivi, "Global connectivity control for spatially interacting multi-robot systems with unicycle kinematics," in *2015 IEEE International Conference on Robotics and Automation (ICRA)*, May 2015, pp. 1255–1261.

[33] S. Chopra and M. Egerstedt, "Spatio-temporal multi-robot routing," *Automatica*, vol. 60, pp. 173–181, 2015.

[34] B. Wu, J. Dai, and H. Lin, "Combined top-down and bottom-up approach to cooperative distributed multi-agent control with connectivity constraints," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 224 – 229, 2015, analysis and Design of Hybrid Systems ADHSAtlanta, GA, USA.

[35] Z. Kan, L. Navaravong, J. M. Shea, E. L. Pasiliao, and W. E. Dixon, "Graph matching-based formation reconfiguration of networked agents with connectivity maintenance," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 1, pp. 24–35, 2015.

[36] Z. Feng, C. Sun, and G. Hu, "Robust connectivity preserving rendezvous of multi-robot systems under unknown dynamics and disturbances," in *IEEE 54th Conference on Decision and Control (CDC)*. IEEE, 2015.

[37] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

[38] B. Paden and S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Transactions on Circuits and Systems*, vol. 34, no. 1, pp. 73–82, Jan 1987.

[39] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1910–1914, Sep 1994.

[40] J. Cortes, "Discontinuous dynamical systems," *IEEE Control Systems*, vol. 28, no. 3, pp. 36–73, June 2008.

[41] R. Soukieh, I. Shames, and B. Fidan, "Obstacle avoidance of non-

holonomic unicycle robots based on fluid mechanical modeling," in *Proceedings of the European Control Conference*, 2009.

[42] D. Lee, A. Franchi, H. Son, C. Ha, H. Bulthoff, and P. Robuffo Giordano, "Semiautonomous haptic teleoperation control architecture of multiple unmanned aerial vehicles," *IEEE/ASME Transactions on Mechatronics*, vol. 18, no. 4, pp. 1334–1345, Aug 2013.

[43] A. Gasparri, R. Williams, A. Leccese, and G. Ulivi, "Set input-to-state stability for spatially interacting multi-agent systems," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, Dec 2014.

[44] L. N. Trefthen and D. Bau, *Numerical Linear Algebra*. SIAM, 1997.

[45] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[46] M. Zareh, L. Sabattini, and C. Secchi, "Distributed laplacian eigenvalue and eigenvector estimation in multi-robot systems," in *Proceedings of the International Symposium on Distributed Autonomous Robotic Systems (DARS)*, London, UK, nov. 2016.

[47] D. V. Dimarogonas and K. J. Kyriakopoulos, "Connectedness preserving distributed swarm aggregation for multiple kinematic robots," *IEEE Transactions on Robotics*, vol. 24, no. 5, pp. 1213–1223, Oct 2008.

[48] M. M. Zavlanos, H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Hybrid control for connectivity preserving flocking," *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2869–2875, Dec 2009.

[49] R. K. Williams and G. S. Sukhatme, "Constrained Interaction and Coordination in Proximity-Limited Multi-Agent Systems," *IEEE Transactions on Robotics*, vol. 29, pp. 930–944, 2013.

[50] D. E. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Advances in Applied Mathematics*, vol. 11, no. 4, pp. 412 – 442, 1990.

[51] M. M. Zavlanos and G. J. Pappas, "Potential fields for maintaining connectivity of mobile networks," *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 812–816, Aug 2007.

[52] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[53] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[54] E. D. Sontag, *Nonlinear and Optimal Control Theory: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy June 19–29, 2004*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, ch. Input to State Stability: Basic Concepts and Results, pp. 163–220.

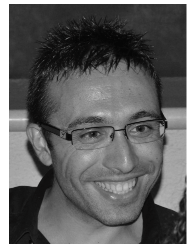
[55] V. Gazi and K. M. Passino, "A class of attractions/repulsion functions for stable swarm aggregations," *International Journal of Control*, vol. 77, no. 18, pp. 1567–1579, 2004. [Online]. Available: http://dx.doi.org/10.1080/00207170412331330021

[56] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424 – 440, 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109814004038

[57] D. V. Dimarogonas and K. H. Johansson, "Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control," *Automatica*, vol. 46, no. 4, pp. 695 – 700, 2010. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109810000324

[58] M. Quigley, K. Conley, B. Gerkey, J. Faust, T. B. Foote, J. Leibs, R. Wheeler, and A. Y. Ng, "ROS: an open-source robot operating system," in *ICRA Workshop on Open Source Software*, 2009.

[59] M. D. Rocco, F. L. Gala, and G. Ulivi, "Testing multirobot algorithms: SAETTA: A small and cheap mobile unit," *IEEE Robot. Automat. Mag.*, vol. 20, no. 2, pp. 52–62, 2013.



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