Examples and Counterexamples 2 (2022) 100063

Contents lists available at ScienceDirect

Examples and Counterexamples

journal homepage: www.elsevier.com/locate/exco

On anti-Novák cycle systems

Marco Buratti [a], Francesca Merola [b,*]

[a] Dipartimento di Matematica e Informatica, Università di Perugia, via Vanvitelli 1, 06123 Perugia, Italy
[b] Dipartimento di Matematica e Fisica, Università Roma Tre, Largo S.L. Murialdo, 1, 00142 Rome, Italy

ABSTRACT

This note is motivated by recent work by Feng et al. (2021) which studies Novák's conjecture for Steiner Triple Systems and extends it to cyclic Steiner 2-designs, and more generally to cyclic 2-designs. Here we consider instead a generalization to cyclic k-cycle systems: we show that in this setting the generalized conjecture is false for $k \geq 5$, and construct some families of counterexamples which arise.

A Steiner triple system of order v (briefly STS(v)) consists of a pair $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V}$ a v-set of points and $\mathcal{B}$ a set of 3-sets of elements of $\mathcal{V}$ called triples, such that any two distinct points belong to exactly one triple; as is well known, Steiner triple systems exist for any $v \equiv 1, 3 \pmod 6$. More generally, a (v, k, λ) 2-design is a pair $(\mathcal{V}, \mathcal{B})$ with $\mathcal{V}$ a v-set of points and $\mathcal{B}$ a set of k-subsets of $\mathcal{V}$ called blocks, such that any two distinct points belong to exactly λ blocks; a STS(v) is then a $(v, 3, 1)$ 2-design.

We denote with K_n the complete graph on n vertices, and with C_n an n-vertex cycle. If Γ and H are finite graphs, a (Γ, H)-design is a partition of the edge set of Γ into graphs isomorphic to H; a $(v, k, 1)$ 2-design then can be thought of as a (K_v, K_k)-design. A (K_v, C_k)-design is also called a k-cycle system of order v, and since K_3 and C_3 are isomorphic, a STS(v) is also a 3-cycle system of order v.

An automorphism of a 2-design $(\mathcal{V}, \mathcal{B})$ is a permutation of $\mathcal{V}$ that leaves $\mathcal{B}$ invariant, and the set of all automorphisms is the full automorphism group of the design: if it has a cyclic subgroup acting sharply transitively on V, the design is called *cyclic*.

We call *base blocks* a set of representatives of the orbits of the action of the group on the blocks. The same notions hold for (Γ, H)-designs, with the obvious modifications; in the particular case of cycle systems, a set of *base cycles* is once more a set of representatives of the orbits on cycles.

In this note, we will discuss a conjecture of Novák [1] on cyclic STS(v).

Conjecture 1. *Any cyclic Steiner Triple System of order $v \equiv 1 \pmod 6$ can be obtained from a set of disjoint base blocks.*

Until recently, very little was known on the truth of this conjecture, but the paper [2] contains significant progress. In this work, the authors consider a generalization of the conjecture to cyclic $(v, k, 1)$ 2-designs, and prove the truth of this more general version when v is a prime. They then further generalize the conjecture to (v, k, λ) 2-designs, and prove that the conjecture holds when $\lambda \leq (k-1)/2$ for large enough v.

In view of the results in [2], and of the fact that cycle systems are another possible generalization of STSs, it might be interesting try to extend Novák's conjecture to cycle systems by considering the following problem.

Problem 2. Given a cyclic (K_{2kn+1}, C_k)-design is it always possible to find a set of base cycles of this design which are pairwise vertex-disjoint?

The answer to this problem is negative in general: there are cyclic (K_{2kn+1}, C_k)-designs for which no set of disjoint base cycles exists. We will call such a design an *anti-Novák* k-cycle system.

More precisely, we have no examples of anti-Novák k-cycle systems for $k = 3$ (indeed these would provide a counterexample to Novák's conjecture); we also do not have examples for $k = 4$, and it is easy to adapt the proof of Theorem 1 in [2] to prove that every 4-cycle system of prime order has a set of disjoint base blocks. The problem may thus have an affirmative answer in these two cases.

On the other hand, for $k \geq 5$ it is possible to find k-cycle systems with no disjoint set of base cycles. We shall present in what follows two families of anti-Novák k-cycle systems.

We recall that the list of differences of a cycle $C = (c_0, c_1, \dots, c_{k-1})$ with vertices in $\mathbb{Z}_v$ is the multiset $\Delta C = \{\pm(c_{h+1} - c_h) \mid 0 \leq h < k\}$, where the subscripts are taken modulo k.

Given a family $\mathcal{F}$ of cycles with vertices in $\mathbb{Z}_{2kn+1}$, let $\Delta\mathcal{F}$ be the union (counting multiplicities) of all multisets ΔC, where $C \in \mathcal{F}$. We have $\Delta\mathcal{F} = \mathbb{Z}_{2kn+1} \setminus \{0\}$ if and only if $\mathcal{F}$ is a set of base cycles for a cyclic k-cycle system of order $2kn+1$ (see, e.g., [3]).

The first family of counterexamples to the problem easily arises from finite projective planes. As well known, if q is a prime power, then there exists a $(q^2+q+1, q+1, 1)$-difference set in $\mathbb{Z}_{q^2+q+1}$ (called *Singer difference set*). This is a set $S = \{s_0, s_1, \dots, s_q\}$ of $q+1$ elements of $\mathbb{Z}_{q^2+q+1}$ such that its list of differences $\Delta S = \{\pm(s_i - s_j) \mid 0 \leq i < j \leq q\}$ coincides

* Corresponding author.
E-mail addresses: buratti@dmi.unipg.it (M. Buratti), merola@mat.uniroma3.it (F. Merola).

https://doi.org/10.1016/j.exco.2022.100063
Received 14 February 2022; Accepted 18 February 2022

with the set $\mathbb{Z}_{q^2+q+1}\setminus\{0\}$. Moreover, the set of all the translates of S (the so-called development of S) is the set of lines of $PG(2,q)$, the projective plane over the field $\mathbb{F}_q$.

Proposition 3. *There is an anti-Novak $(q+1)$-cycle system of K_{q^2+q+1} for every $q = 2^n > 2$.*

Proof. Take a $(q^2+q+1, q+1, 1)$ difference set S. Since $q+1$ is odd, the complete graph K_{q+1} having as vertices the elements of S can be decomposed into $q/2$ hamiltonian cycles $C_1, \dots, C_{q/2}$ (one may take, for instance, the well-known Walecki decomposition, see for instance [4]). Clearly, their list of differences coincides with the list of differences from S, that is $\mathbb{Z}_{q^2+q+1} \setminus \{0\}$. Hence these cycles form a set of base cycles of a cyclic (K_{q^2+q+1}, C_{q+1})-design. Now any two members of a set of base cycles of this design have exactly one vertex in common, since their sets of vertices are lines of $PG(2,q)$. □

Example 4. The smallest example has $q = 4$, giving an anti-Novák 5-cycle system of order 21: a difference set in $\mathbb{Z}_{21}$ is, for instance, the set $S = \{0, 1, 6, 8, 18\}$ from which we may obtain the two base cycles $C_1 = (0, 1, 6, 8, 18)$ and $C_2 = (0, 6, 18, 1, 8)$.

In what follows we describe another, less obvious family of anti-Novák cycle systems. We recall the following basic facts on cyclotomic numbers we shall need in constructing the counterexamples (see also [5]).

Given a prime $p \equiv 1 \pmod{e}$, let C^e be the subgroup of the multiplicative group of $\mathbb{Z}_p$ of index e, that is the group of eth powers in $\mathbb{Z}_p$. Take a primitive element r of $\mathbb{Z}_p$ and set $C_i^e = \{r^{eh+i} \mid 1 \le h \le (p-1)/e\}$ for $0 \le i \le e-1$. Note that $\{C_0^e, C_1^e, \dots, C_{e-1}^e\}$ is the set of cosets of C^e in the multiplicative group of $\mathbb{Z}_p$. The *cyclotomic numbers* of order e in $\mathbb{Z}_p$ are the e^2 non-negative integers $(i,j)_e$ defined as follows

$$(i,j)_e = |\{x \in C_i^e : x+1 \in C_j^e\}|.$$

These are all positive as soon as p is sufficiently large. In particular, we shall need the following.

Lemma 5. *Let $q > 3$ be a prime such that $p = 4q+1$ is also prime, then the five cyclotomic numbers $(i,i)_4, i = 1, \dots 4$ and $(1,0)_4$ in $\mathbb{Z}_p$ are positive.*

Proof. This result is possibly already known; a proof can anyway be obtained using, for instance, Application 3 in [5]. The bound there states that cyclotomic numbers of order 4 in $\mathbb{Z}_p$ are positive as soon as $p \ge 127$, and one can check explicitly that the result holds for $p = 29$ and 53, the only primes below the bound of the form $4q+1$ for $q > 3$ a prime. □

Proposition 6. *Let $q > 3$ be a prime such that $p = 4q+1$ is also prime. Then there is an anti-Novák (K_{4q+1}, C_q)-design.*

Proof. As usual, the set of points of the design is identified with $\mathbb{Z}_p$, so that the cyclotomic numbers have to be understood in $\mathbb{Z}_p$. By Lemma 5, we can find $x \in C^4$ such that $x-1 \in C^4$, and an element $y \in C^4$ such that $y-1 \in C_1^4$. Consider the q-cycles

$$X = (x, x^2, x^3, \dots, x^q), \quad Y = (y, y^2, y^3, \dots, y^q).$$

Clearly, the vertex-set of each of these cycles is C^4. Their lists of differences are the following

$$\Delta X = (x-1)C^2 = C_0^2, \quad \Delta Y = (y-1)C^2 = C_1^2$$

It is then obvious that $\{X, Y\}$ is a set of base cycles of a cyclic (K_p, C_q)-design.

Now take any non-zero element t of $\mathbb{Z}_p$ and let C_i^4 be the coset of C^4 containing t; by Lemma 5, we can find $c \in C_{4-i}^4$ such that $c+1$ is also in C_{4-i}^4. Then, multiplying by t, we get $tc+t \in tC_{4-i}^4 = C^4$. On the other hand, the element $z = tc$ also belong to $tC_{4-i}^4 = C^4$. Thus z and $z+t$ are both in C^4, that is are both vertices of X and Y. It follows that there is no translate $X+t$ of X which is vertex-disjoint from Y. □

Example 7. The smallest example comes from $q = 7$ and $p = 29$: in this case $C^4 = \{1, 16, 24, 7, 25, 23, 20\}$, $C_1^4 = \{2, 3, 19, 14, 21, 17, 11\}$: we may choose for instance $x = 24$ and $y = 16$, and obtain the following two base cycles.

$$X = (24, 25, 20, 16, 7, 23, 1), \quad Y = (16, 24, 7, 25, 23, 20, 1).$$

The existence of anti-Novák cycle systems is not surprising, and does not impinge on the truth of Conjecture 1, or its generalizations considered in [2]: the set of base blocks of a cyclic $(v, k, 1)$ 2-design has size $(v-1)/(k(k-1))$, while for a (K_v, C_k)-design one needs $(v-1)/2k$ base cycles, a much bigger number as k grows larger. Indeed note that, in the two families of counterexamples presented, the base cycles (or some suitable translates of the base cycles) have the same vertex-set. An example in which this does not happen (Feng [6]) is the following: the two cycles $C_1 = (0, 1, 3, 6, 2, 9)$ and $C_2 = (0, 5, 16, 10, 20, 12)$ are base cycles for an anti-Novák 6-cycle system of order 25.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] J. Novák, A note on disjoint cyclic steiner triple systems, in: Recent Advances in Graph Theory, Proc. Symp., Prague, Vol. 1974, Academia, Praha, 1975, pp. 439–440.
[2] T. Feng, D. Horsley, X. Wang, Novák's conjecture on cyclic Steiner triple systems and its generalization, J. Combin. Theory Ser. A 184 (2021) 1–10, Nr 105515.
[3] M. Buratti, A. Del Fra, Existence of cyclic k-cycle systems of the complete graph, Discrete Math. 261 (2003) 113–125.
[4] B. Alspach, The wonderful walecki construction, Bull. Inst. Comb. Appl. 52 (2008) 7–20.
[5] M. Buratti, A. Pasotti, Combinatorial designs and the theorem of weil on multiplicative character sums, Finite Fields Appl. 15 (2009) 332–344.
[6] T. Feng, personal communication.