THE UNIVERSAL K3 SURFACE OF GENUS 14 VIA CUBIC FOURFOLDS

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ABSTRACT. Using Hassett's isomorphism between the Noether-Lefschetz moduli space $\mathcal{C}_{26}$ of special cubic fourfolds $X \subset \mathbf{P}^5$ of discriminant 26 and the moduli space $\mathcal{F}_{14}$ of polarized $K3$ surfaces of genus 14, we use the family of 3-nodal scrolls of degree seven in X to show that the universal $K3$ surface over $\mathcal{F}_{14}$ is rational.

1. INTRODUCTION

For a very general cubic fourfold $X \subseteq \mathbf{P}^5$, the lattice $A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ of middle Hodge classes contains only classes of complete intersection surfaces, so $A(X) = \langle h^2 \rangle$, where $h \in \text{Pic}(X)$ is the hyperplane class (see [V]). Hassett, in his influential paper [H1], initiated the study of Noether-Lefschetz special cubic fourfolds. If $\mathcal{C}$ is the 20-dimensional coarse moduli space of smooth cubic fourfolds $X \subseteq \mathbf{P}^5$, let $\mathcal{C}_d$ be the locus of *special* cubic fourfolds X characterized by the existence of an embedding of a saturated rank 2 lattice

$$L := \langle h^2, [S] \rangle \hookrightarrow A(X),$$

of discriminant disc$(L) = d$, where $S \subseteq X$ is an algebraic surface not homologous to a complete intersection. Hassett [H1] showed that $\mathcal{C}_d \subseteq \mathcal{C}$ is an irreducible divisor, which is nonempty if and only if $d > 6$ and $d \equiv 0, 2 (\text{mod } 6)$. The study of the divisors $\mathcal{C}_d$ for small d has received considerable attention. For instance, $\mathcal{C}_8$ consists of cubic fourfolds containing a plane, whereas $\mathcal{C}_{14}$ corresponds to cubic fourfolds containing a quintic del Pezzo surface, see [H2]. Relying on Fano's work [Fa], recently Bolognesi and Russo [BR] have shown that all fourfolds $[X] \in \mathcal{C}_{14}$ are rational.

For every $[X] \in \mathcal{C}$, we denote by $F(X) := \{\ell \in \mathbf{G}(1,5) : \ell \subseteq X\}$ the Hilbert scheme of the lines contained in X. It is well known [BD] that $F(X)$ is a hyperkähler fourfold deformation equivalent to the Hilbert square of a $K3$ surface. For discriminant $d = 2(n^2 + n + 1)$, where $n \geq 2$, it is shown in [H1] that $F(X)$ is *isomorphic* to the Hilbert scheme $S^{[2]}$ of a polarized $K3$ surface (S, H) with $H^2 = d$. If $\mathcal{F}_g$ denotes the moduli space of polarized $K3$ surfaces of genus g, the previous assignment induces a rational map

$$\mathcal{F}_{\frac{d}{2}+1} \dashrightarrow \mathcal{C}_d,$$

which is a birational isomorphism for $d \equiv 2 (\text{mod } 6)$ and a degree 2 cover for $d \equiv 0 (\text{mod } 6)$. This map, though non-explicit for it is defined at the level of moduli spaces of weight-2 Hodge structures, opens the way to the study of $\mathcal{F}_{n^2+n+2}$ via the concrete geometry of cubic fourfolds, without making a direct reference to $K3$ surfaces! The main result of this paper concerns the universal $K3$ surface $\mathcal{F}_{g,1} \to \mathcal{F}_g$.

Theorem 1.1. *The universal $K3$ surface $\mathcal{F}_{14,1}$ of genus 14 is rational.*

Nuer [Nu] proved that $\mathcal{C}_{26}$ (and hence $\mathcal{F}_{14}$ as well) is unirational. His proof relies on the fact that a general fourfold $[X] \in \mathcal{C}_{26}$ contains certain smooth rational surfaces, whose parameter space forms a unirational family. One can also show that $\mathcal{C}_{44}$ is unirational, for a general $[X] \in \mathcal{C}_{44}$ contains a Fano embedded Enriques surface and their moduli space is unirational, see [Ve2] and also [Nu]. Recently, Lai [L] showed that $\mathcal{C}_{42}$ is uniruled.

Mukai in a celebrated series of papers [M1], [M2], [M3], [M4], [M5] established structure theorems for polarized $K3$ surfaces of genus $g \le 12$, as well as $g = 13, 16, 18, 20$. In particular, $\mathcal{F}_g$ is unirational for those value of g. No structure theorem for the general $K3$ surface of genus 14 is known. A quick inspection of Mukai's methods shows that the universal $K3$ surface $\mathcal{F}_{g,1}$ is unirational for $g \le 11$ as well. On the other hand, Gritsenko, Hulek and Sankaran [GHS] have proved that $\mathcal{F}_g$ is a variety of general type for $g > 62$, as well as for $g = 47, 51, 53, 55, 58, 59, 61$. In a similar vein, recently it has been established in [TVA] that $\mathcal{C}_d$ is of general type for all d sufficiently large. As pointed out in Remark 5.4, whenever $\mathcal{F}_g$ is of general type, the Kodaira dimension of $\mathcal{F}_{g,1}$ is equal to 19.

The proof of Theorem 1.1 relies on the connection between singular scrolls and special cubic fourfolds. We fix a general point $[X] \in \mathcal{C}_{26}$ and denote by S the *associated* $K3$ surface, such that $S^{[2]} \cong F(X) \hookrightarrow \mathbf{G}(1,5)$. For each $p \in S$, we introduce the rational curve

$$\Delta_p := \bigl\{\xi \in S^{[2]} : \{p\} = \mathrm{supp}(\xi)\bigr\}.$$

Under the Plücker embedding $\mathbf{G}(1,5) \subseteq \mathbf{P}^{14}$, the degree of $\Delta_p \subseteq F(X)$ is equal to 7, which suggests that each point of $p \in S$ parametrizes a *septic* scroll $R = R_p \subseteq X$. Imposing the condition disc$\langle h^2, [R]\rangle = 26$, one obtains $R^2 = 25$. Assuming R has isolated non-normal nodal singularities, the double point formula implies that R has precisely 3 non-normal nodes. We shall prove that indeed, a general fourfold $[X] \in \mathcal{C}_{26}$ carries a 2-dimensional family of 3-nodal scrolls $R \subseteq X$ with deg$(R) = 7$. Furthermore, this family of scrolls is parametrized by the $K3$ surface S associated to X.

We now describe the moduli space of 3-nodal septic scrolls. We start with the Hirzebruch surface $\mathbf{F}_1 := \mathrm{Bl}_o(\mathbf{P}^2)$, where $o \in \mathbf{P}^2$, and denote by ℓ the class of a line and by E the exceptional divisor. The smooth septic scroll $R' = S_{3,4} \subseteq \mathbf{P}^8$ is the image of the linear system

$$\phi_{|4\ell - 3E|} : \mathbf{F}_1 \hookrightarrow \mathbf{P}^8.$$

We shall show in Section 3 that the secant variety Sec$(R') \subseteq \mathbf{P}^8$ is as expected 5-dimensional. Choose general points $a_1, a_2, a_3 \in$ Sec(R') and denote by $\Lambda := \langle a_1, a_2, a_3\rangle \in \mathbf{G}(2,8)$ their linear span. The image $R \subseteq \mathbf{P}^5$ of the projection with center Λ

$$\pi_\Lambda : R' \to \mathbf{P}^5$$

is a 3-nodal septic scroll. Conversely, up to the action of $PGL(6)$ on the ambient projective space $\mathbf{P}^5$, each such scroll appears in this way. We denote by $\mathfrak{H}_{\mathrm{scr}}$ the moduli space of unparametrized 3-nodal septic scrolls in $\mathbf{P}^5$, that is, the quotient of the corresponding Hilbert scheme under the action of $PGL(6)$. Then as showed in Proposition 3.6, the space $\mathfrak{H}_{\mathrm{scr}}$ turns out to be birationally isomorphic to the 9-dimensional unirational variety

$$\mathfrak{H}_{\mathrm{scr}} \cong \mathrm{Sym}^3\bigl(\mathrm{Sec}(R')\bigr)/\mathrm{Aut}(R').$$

Fix a general 3-nodal septic scroll $R \subseteq \mathbf{P}^5$. A general $X \in \mathbf{P}\bigl(H^0(\mathcal{I}_{R/\mathbf{P}^5}(3))\bigr) = \mathbf{P}^{12}$ is a smooth cubic fourfold. Since R has no further singularities apart from the three non-normal nodes, the double point formula implies that $[X] \in \mathcal{C}_{26}$. One sets up the following incidence correspondence between scrolls and cubic fourfolds of discriminant 26:

$$\begin{array}{ccccc} & & \mathfrak{X} := \Big\{(X,R) : R \subseteq X\Big\}/PGL(6) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathcal{C}_{26} & & & & \mathfrak{H}_{\mathrm{scr}} \end{array}$$

Thus $\mathfrak{X}$ is birational to a $\mathbf{P}^{12}$-bundle over the unirational variety $\mathfrak{H}_{\mathrm{scr}}$. We then show that the fibre over a general cubic fourfold $[X] \in \mathcal{C}_{26}$ of the projection π_1 is 2-dimensional and isomorphic to the $K3$ surface S appearing in the identification $F(X) \cong S^{[2]}$. We summarize the discussion above.

Theorem 1.2. *The universal $K3$ surface $\mathcal{F}_{14,1}$ is birational to the $\mathbf{P}^{12}$-bundle $\mathfrak{X}$ over the moduli space $\mathfrak{H}_{\mathrm{scr}}$ of 3-nodal septic scrolls $R \subseteq \mathbf{P}^5$. A general fourfold $[X] \in \mathcal{C}_{26}$ contains a two-dimensional family of such scrolls $R \subseteq X \subseteq \mathbf{P}^5$. The space of such scrolls is isomorphic to the $K3$ surface associated to X.*

Theorem 1.2 allows us to elucidate the structure of $\mathcal{F}_{14,1}$ even further and prove its rationality. We fix a 3-nodal septic scroll $R \subseteq \mathbf{P}^5$ as above and denote its nodes by p_1, p_2, p_3. The curve $\Gamma_R \subseteq \mathbf{G}(1,5)$ induced by the rulings of R is a smooth rational septic curve admitting bisecant lines L_1, L_2 and L_3 in the Plücker embedding of $\mathbf{G}(1,5)$. Precisely, L_i parametrizes the lines passing through p_i and contained in the 2-plane P_i spanned by the two rulings of R that intersect at the node p_i, for $i = 1,2,3$. Since Γ_R spans a 7-dimensional linear space in projective space $\mathbf{P}^{14}$ containing $\mathbf{G}(1,5)$, using Mukai's work [M6] on realizing canonical genus 8 curves as linear sections of the Grassmannian $\mathbf{G}(1,5)$, it follows that the intersection $\mathbf{G}(1,5) \cdot \langle \Gamma_R \rangle$ is a semi-stable curve of genus 8. We denote by $Q \subseteq \langle \Gamma_R \rangle = \mathbf{P}^7$ the residual curve defined by the following equality:

$$\mathbf{G}(1,5) \cdot \langle \Gamma_R \rangle = \Gamma_R + L_1 + L_2 + L_3 + Q. \tag{1}$$

We shall establish in Lemmas 4.1 and 4.2 that Q is a smooth rational quartic curve and $Q \cdot L_i = 1$ for $i = 1,2,3$, as well as $Q \cdot \Gamma_R = 3$. Therefore Q is the curve of rulings of a quartic scroll $R_Q \subseteq \mathbf{P}^5$, which contains three rulings ℓ_1, ℓ_2, ℓ_3, such that that $p_i \in \ell_i$ and $\ell_i \in P_i$ for $i = 1,2,3$. In particular, R_Q contains the three nodes of the septic scroll R. We can show furthermore that R_Q is smooth and isomorphic to $\mathbf{F}_0$, see Theorem 4.10.

The construction above can be reversed. Using the automorphism group of the scroll $R_Q \subseteq \mathbf{P}^5$, we fix three of its rulings $\ell_1, \ell_2, \ell_3 \in \mathbf{G}(1,5)$, as well as points $p_i \in \ell_i$. We set

$$\mathbf{P}_i^3 := \big\{P_i \in \mathbf{G}(2,5) : \ell_i \subseteq P_i\big\},$$

for $i = 1,2,3$, then define a map

$$\varkappa : \mathbf{P}_1^3 \times \mathbf{P}_2^3 \times \mathbf{P}_3^3/\mathfrak{S}_3 \dashrightarrow \mathfrak{H}_{\mathrm{scr}},$$

by reversing the above construction and using the decomposition (1). Along with the fixed point p_i, each 2-plane $P_i \in \mathbf{P}_i^3$ defines a line $L_i \subseteq \mathbf{G}(1,5)$ meeting the curve Q at the point ℓ_i. Precisely, L_i is the line of lines in P_i passing through the point p_i. To the triple (P_1, P_2, P_3) we associate the scroll $R \subseteq \mathbf{P}^5$ whose associated curve of rulings Γ_R is defined by the formula (1). The above discussion indicates that $\varkappa$ is dominant. In fact more can be proved:

Theorem 1.3. *The moduli space of scrolls $\mathfrak{H}_{\mathrm{scr}}$ is birational to $\mathbf{P}_1^3 \times \mathbf{P}_2^3 \times \mathbf{P}_3^3/\mathfrak{S}_3$ and is thus rational.*

Indeed, using the theorem on symmetric functions, see [Ma] or [GKZ] Theorem 2.8 for a recent reference, all symmetric products of projective spaces are known to be rational. It is now clear that Theorem 1.3 coupled with Theorem 1.2 implies that $\mathcal{F}_{14,1}$ is a rational variety.

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2. $K3$ SURFACES AND CUBIC FOURFOLDS

We begin by setting some notation. Let $U \subseteq |\mathcal{O}_{\mathbf{P}^5}(3)|$ be the locus of smooth cubic fourfolds and set

$$\mathcal{C} := U/PGL(6)$$

to be the 20-dimensional moduli space of cubic fourfolds. For an integer $d \equiv 0, 2 \pmod 6$, as pointed out in the Introduction, $\mathcal{C}_d$ denotes the irreducible divisor of $\mathcal{C}$ consisting of special cubic fourfolds of discriminant d. As usual, $\mathcal{F}_g$ is the irreducible 19-dimensional moduli space of smooth polarized $K3$ surfaces (S, H) of genus g, that is, with $H^2 = 2g - 2$. We denote by $u : \mathcal{F}_{g,1} \to \mathcal{F}_g$ the universal $K3$ surface of genus g in the sense of stacks. Each fibre $u^{-1}([S, H])$ is identified with the $K3$ surface S.

Using the Hodge-theoretic similarity between $K3$ surfaces of genus $g = n^2 + n + 1$ and special cubic fourfolds of degree $2g - 2$, Hassett [H1] constructed a morphism of moduli spaces

$$\varphi : \mathcal{F}_{n^2+n+2} \to \mathcal{C}_{2(n^2+n+1)},$$

which is birational for $n \equiv 0, 2 (\mathrm{mod}\ 3)$, and of degree 2 for $n \equiv 1 (\mathrm{mod}\ 3)$ respectively. In particular, for $n = 3$ there is a birational isomorphism of spaces of weight 2 Hodge structures

$$\varphi : \mathcal{F}_{14} \xrightarrow{\cong} \mathcal{C}_{26},$$

that will be of use throughout the paper. At the moment, there is no geometric construction of the polarized $K3$ surface $\varphi^{-1}([X])$ associated to a general fourfold $[X] \in \mathcal{C}_{26}$.

We recall basic facts on Hilbert squares of $K3$ surfaces and refer to [HT1] for a general reference on these matters. Let (S, H) be a polarized $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ and $H^2 = 2g-2$. We denote by $S^{[2]}$ the Hilbert scheme of length two 0-dimensional subschemes on S. Then $H^2(S^{[2]}, \mathbb{Z})$ is endowed with the *Beauville-Bogomolov* quadratic form q. We denote by $\Delta \subseteq S^{[2]}$ the diagonal divisor consisting of zero-dimensional subschemes supported only at a single point and by $\delta := \frac{[\Delta]}{2} \in H^2(S^{[2]}, \mathbb{Z})$ the reduced diagonal class. One has $q(\delta, \delta) = -2$. Note the canonical identification

$$\Delta = \mathbf{P}(T_S) = \cup\{\Delta_p : p \in S\},$$

where Δ_p is the rational curve consisting of those 0-dimensional subschemes $\xi \in \Delta$ such that $\mathrm{supp}(\xi) = \{p\}$. We set $\delta_p := [\Delta_p] \in H_2(S^{[2]}, \mathbb{Z})$.

For a curve $C \in |H|$ in the polarization class , we introduce the divisor

$$f_C := \left\{\xi \in S^{[2]} : \mathrm{supp}(\xi) \cap C \neq \emptyset\right\}$$

and set $f := [f_C] \in H^2(S^{[2]}, \mathbb{Z})$. If $p \in S$ is a general point, we also define the curve

$$F_p := \left\{\xi = p + x \in S^{[2]} : x \in C\right\}$$

and set $f_p := [F_p] \in H_2(S^{[2]}, \mathbb{Z})$. The Beauville-Bogomolov form can be extended to a quadratic form on $H_2(S^{[2]}, \mathbb{Z})$, by setting $q(\alpha, \alpha) := q(w_\alpha, w_\alpha)$, with $w_\alpha \in H^2(S^{[2]}, \mathbb{Z})$ being the class characterized by the property $\alpha \cdot u = q(w_\alpha, u)$, for every $u \in H^2(S^{[2]}, \mathbb{Z})$. Here $\alpha \cdot u$ denotes the usual intersection product.

One has the following decompositions, orthogonal with respect to q, both for the Picard group and for the group $N_1(S^{[2]}, \mathbb{Z})$ of 1-cycles modulo numerical equivalence:

$$\mathrm{Pic}(S^{[2]}) \cong \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \delta \ \text{ and } \ N_1(S^{[2]}, \mathbb{Z}) \cong \mathbb{Z} \cdot f_p \oplus \mathbb{Z} \cdot \delta_p.$$

We record, the more or less obvious relations:

$$f \cdot f_p = 2g - 2,\ \delta \cdot \delta_p = -1,\ f \cdot \delta_p = 0 \text{ and } \delta \cdot f_p = 0. \tag{2}$$

Assume now that $X \subseteq \mathbf{P}^5$ is a general special cubic fourfold of discriminant 26 and let $[S, H] = \varphi^{-1}([X]) \in \mathcal{F}_{14}$ be the associated polarized $K3$ surface such that

$$S^{[2]} \cong F(X) \subseteq \mathbf{G}(1,5) \hookrightarrow \mathbf{P}^{14}. \tag{3}$$

Following [BD], let $\gamma_S := [\mathcal{O}_{S^{[2]}}(1)]$ be the hyperplane class of $\mathbf{G}(1,5)$ restricted to the Hilbert square under the identification (3). Since $q(\gamma_S, \gamma_S) = 6$, using (2), it quickly follows that

$$\gamma_S = 2f - 7\delta \in H^2(S^{[2]}, \mathbb{Z}).$$

Proposition 2.1. *Suppose $[S, H] \in \mathcal{F}_{26}$ is a general element and let $R \subseteq S^{[2]}$ be an effective 1-cycle such that $R \cdot \gamma_S = 7$. Then R is one of the rational irreducible curves Δ_p, for $p \in S$. In particular, R is smooth.*

Proof. Assume that R is an effective 1-cycle and write $[R] = af_p - b\delta_p \in N_1(S^{[2]}, \mathbb{Z})$. Since $7 = R \cdot \gamma_S = 52a - 7b$, hence we can write $a = 7a_1$, with $a_1 \in \mathbb{Z}$, and then $b = 52a_1 - 1$. Using [BM] Proposition 12.6, we have $q(R, R) \geq -\frac{5}{2}$. We obtain $39a_1^2 - 26a_1 - 1 \leq 0$, and the only integer solution of this inequality is $a_1 = 0$, therefore $[R] = \delta_p$.

Since $[R] \cdot \delta = -1$, it follows that $R \subseteq \Delta$. We claim that R lies in one of the fibres of the $\mathbf{P}^1$-bundle $\pi : \Delta = \mathbf{P}(T_S) \to S$, which implies that $R = \Delta_p$, for some $p \in S$. Indeed, otherwise $\pi(R) \equiv mH$, for some $m > 0$. Accordingly, we write

$$mH^2 = R \cdot \pi^{-1}(H) = R \cdot f = \delta_p \cdot f = 0,$$

which is a contradiction. □

Remark 2.2. Unlike degree 26, for other values of d, a general $[X] \in \mathcal{C}_d$ may contain several types of scrolls. For instance when $d = 14$ and $\gamma_S = 2f - 5\delta$, the curves Δ_p with $p \in S$ correspond to quintic scroll, but X also contains quartic scrolls corresponding to rational curves $R \subseteq F(X)$ with $[R] = 3f_p - 16\delta_p$. Note that $q(R, R) = -2$.

We now recall the correspondence between scrolls and rational curves in Grassmannians. Following for instance [Dol] 10.4, we define a *rational scroll* to be the image $R \subseteq \mathbf{P}^n$ of a $\mathbf{P}^1$-bundle $\pi : R' = \mathbf{P}(\mathcal{E}) \to \mathbf{P}^1$ under a map $\phi : R' \to \mathbf{P}^n$ given by a linear subsystem of $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$, thus sending the fibres of π to lines in $\mathbf{P}^n$. Let $f_R : \mathbf{P}^1 \to \mathbf{G}(1, n)$ be the map

$$f_R(t) := \left[\phi(\pi^{-1}(t))\right]$$

and denote by Γ_R its image. Conversely, start with a non-degenerate map $f : \mathbf{P}^1 \to \mathbf{G}(1, n)$, then consider the pull-back under f of the projectivization of tautological rank 2 vector over $\mathbf{G}(1, n)$, that is,

$$\Xi := \left\{(t, x) : t \in \mathbf{P}^1, x \in L_{f(t)}\right\} \subseteq \mathbf{P}^1 \times \mathbf{P}^n. \tag{4}$$

Here $L_{f(t)} \subseteq \mathbf{P}^n$ denotes the line whose moduli point in $\mathbf{G}(1,n)$ is precisely $f(t)$.

The projection $\pi_2 : \Xi \to \mathbf{P}^n$ is a finite map and its image is a scroll $R \subseteq \mathbf{P}^n$ of degree

$$\deg(\Gamma_R) = \deg f^*\Big(\mathcal{O}_{\mathbf{G}(1,n)}(1)\Big).$$

Throughout the paper, we interpret scrolls in terms of their associated curves of rulings. It will be useful to determine, using this language, when a scroll is smooth.

Proposition 2.3. *Let $R \subseteq \mathbf{P}^n$ be a scroll which is not a cone and such that Γ_R is a smooth rational curve in $\mathbf{G}(1,n)$ which is not contained in a plane. Then there is a bijective correspondence between singularities of R and bisecant lines to Γ_R lying on $\mathbf{G}(1,n)$. In particular, if Γ_R admits no bisecant lines contained in $\mathbf{G}(1,n)$, then R is smooth.*

Proof. We consider the projection $\pi_2 : \Xi \to R$ defined by (4). Then Ξ is a smooth variety and the assumptions made on R imply that π_2 is a finite map. If a point $x \in R$ corresponds to a singularity, then one of the two following possibilities occur: (i) the fibre $\pi_2^{-1}(x)$ consists of more than point, or (ii) the differential of π_2 at a point of $(t,x) \in \pi_2^{-1}(x)$ is not an isomorphism.

In case (i), we choose distinct points $t_1, t_2 \in \pi_1\big(\pi_2^{-1}(x)\big)$. Denoting by $\ell_1 := f_R(t_1)$ and $\ell_2 := f_R(t_2)$ the rulings of Ξ corresponding to these points, we observe that $x \in \ell_1 \cap \ell_2$. The set L of lines in the 2-plane $\langle \ell_1, \ell_2\rangle$ passing through x is a line in $\mathbf{G}(1,n)$ such that $\Gamma_R \cap L \supseteq \{\ell_1, \ell_2\}$, that is, Γ_R possesses a secant line lying inside $\mathbf{G}(1,n)$ in its Plücker embedding. Note that L is a genuine secant line in the sense that it meets the curve Γ_R in two distinct points ℓ_1 and ℓ_2. All lines lying inside $\mathbf{G}(1,n)$ in its Plücker embedding correspond to pencils of lines in a 2-plane passing through a point in $\mathbf{P}^n$. Thus conversely, when such a line meets Γ_R in two distinct points, these will correspond to two incident rulings of R. In particular R is singular at their point of intersection.

To deal with case (ii), we carry out a local calculation. Assume $(t_0, x) \in \Xi$ is a point at which the differential of π_2 is not an isomorphism. We set $\ell_0 := f_R(t_0)$ and denote by

$$p_{ij}(t) = a_i(t)b_j(t) - a_j(t)b_i(t), \quad \text{where } 0 \le i < j \le n$$

the Plücker coordinates of the curve Γ_R in a neighborhood of ℓ_0, where $a(t) = (a_0(t), \ldots, a_n(t))$ and $b(t) = (b_0(t), \ldots, b_n(t))$.

In local coordinates, the map π_2 is given by $\mathbf{P}^1 \times \mathbb{C} \ni ([\lambda, \mu], t) \mapsto \big[(\lambda a_i(t) + \mu b_i(t))\big] =: x$. By direct calculation, the condition that $(d\pi_2)_{(t_0,x)}$ is not an isomorphism is equivalent to

$$b'(t_0) \wedge a(t_0) = 0 \in \bigwedge^2 \mathbb{C}^{n+1}.$$

Setting $a_i := a_i(t_0)$, $b_i := b_i(t_0)$, $a_i' := a_i'(t_0)$ and $b_i' := b_i(t_0)$, we then observe that the Plücker coordinates of a point on the tangent line $\mathbb{T}_{\ell_0}(\Gamma_R) \subseteq \mathbf{P}^{\binom{n+1}{2}-1}$ are given by

$$a_ib_j - a_jb_i + \mu\big(a_i'b_j + a_ib_j' - a_j'b_i - a_jb_i'\big) = b_j(a_i + \mu a_i') - b_i(a_j + \mu a_j'),$$

for some scalar μ. It follows that the tangent line to Γ_R at ℓ_0 is contained in $\mathbf{G}(1,n)$. The argument being reversible, we finish the proof. □

The scrolls $R \subseteq \mathbf{P}^n$ we consider most of the time have at worst *non-normal nodal singularities* $x \in R$, corresponding to the case $|\phi^{-1}(x)| = 2$. The tangent cone of R at x is isomorphic to the union of two 2-planes in $\mathbf{P}^4$ meeting in one point. According to Proposition 2.3, to each such singularity corresponds a line in the Plücker embedding of $\mathbf{G}(1,n)$ meeting Γ_R in two distinct points.

Suppose now that $R \subseteq X \subseteq \mathbf{P}^5$ is a rational scroll with isolated nodal singularities contained in a cubic fourfold. Using the *double point formula* [Ful] 9.3 applied to the map $\phi : R' \to X$, we find the number of singularities of $R = \phi(R')$:

$$D(\phi) = R^2 - 6h^2 - K_R^2 - 3h \cdot K_R + 2\chi_{\mathrm{top}}(R). \tag{5}$$

When $[X] \in \mathcal{C}_{26}$, assuming that $A(X) = \langle h^2, [R] \rangle$, where $h^2 \cdot [R] = \deg(R) = 7$, necessarily $R^2 = 25$. From formula (5), we compute $D(\phi) = 3$, that is, if R has only (isolated) improper nodes, then it is 3-nodal.

Before stating our next result, we recall that $\overline{\mathcal{M}}_0(F(X), 7)$ denotes the space of stable maps $f : C \to F(X)$, from a nodal curve C of genus zero such that $\deg(f^*(\mathcal{O}_{F(X)}(1)) = 7$. We denote by $\mathcal{M}_0(F(X), 7)$ the open sublocus consisting of maps with source $\mathbf{P}^1$ and denote by $\overline{\mathcal{M}}_7(X)$ the closure of $\mathcal{M}_0(F(X), 7)$ inside $\overline{\mathcal{M}}_0(F(X), 7)$.

Corollary 2.4. *Let $[X] \in \mathcal{C}_{26}$ a general special fourfold of discriminant 26 and $[S, H] \in \mathcal{F}_{26}$ its associated $K3$ surface. Then there is an isomorphism $S \cong \overline{\mathcal{M}}_7(X)$.*

Proof. Using the identification $S^{[2]} \cong F(X)$, we define the map $j : S \to \overline{\mathcal{M}}_7(X)$, by setting $j(p) := \Delta_p \subseteq F(X)$. All points in the image of j consist of embedded smooth rational curves $\mathbf{P}^1 \overset{\cong}{\hookrightarrow} \Delta_p$ and we identify Δ_p with the corresponding map $\mathbf{P}^1 \hookrightarrow F(X)$. In a neighborhood of this map, the moduli space $\overline{\mathcal{M}}_0(F(X), 7)$ is locally isomorphic to the Hilbert scheme of septic rational curves on $F(X)$.

The tangent space of $\overline{\mathcal{M}}_7(X)$ at the point $[\Delta_p]$ is canonically isomorphic to $H^0(N_{\Delta_p/F(X)})$. Using the following exact sequence on $\Delta_p \cong \mathbf{P}^1$

$$0 \longrightarrow N_{\Delta_p/\Delta} \longrightarrow N_{\Delta_p/F(X)} \longrightarrow \mathcal{O}_{\Delta_p}(\Delta) \longrightarrow 0,$$

since $N_{\Delta_p/\Delta} = \mathcal{O}_{\Delta_p}^{\oplus 2}$ and $\mathcal{O}_{\Delta_p}(\Delta) = \mathcal{O}_{\Delta_p}(-1)$, we compute $N_{\Delta_p/F(X)} = \mathcal{O}_{\Delta_p}^{\oplus 2} \oplus \mathcal{O}_{\Delta_p}(-1)$. It follows that $H^1(\Delta_p, N_{\Delta_p/F(X)}) = 0$, hence the obstruction space for deformations vanishes and

$$\dim T_{[\Delta_p]}(\overline{\mathcal{M}}_0(F(X), 7) = h^0(\Delta_p, N_{\Delta_p/F(X)}) = 2.$$

We conclude that $[\Delta_p]$ is a smooth point of expected dimension of $\overline{\mathcal{M}}_7(X)$, for every $p \in S$.

Furthermore, j is injective, because for distinct points $p, q \in S$, since $\Delta_p \cap \Delta_q = \emptyset$, the associated scrolls R_p and R_q share no rulings. We finally observe that j is an immersion. Indeed, for each $p \in S$, we have the identification $\Delta_p = \mathbf{P}\Big(T_p(S) \oplus T_p(S)/T_p(S)\Big)$, the quotient being given by the diagonal embedding. Thus the differential $dj(p)$ is essentially the identity map, via the identification $\mathbf{P}(T_S) \cong \bigcup_{p \in S} \mathbf{P}\big(N_{\Delta_p/\Delta}\big)$. Since according to Proposition 2.1, we have that $\mathcal{M}_0(F(X), 7) \subseteq \mathrm{Im}(j)$, we can conclude the proof. □

3. Nodal septic scrolls and cubic fourfolds

In this section we study in more detail the moduli space $\mathfrak{H}_{\mathrm{scr}}$ of 3-nodal septic scrolls that will be used to parametrize the universal $K3$ surface of degree 26. We fix once and for all the smooth septic scroll

$$R' := S_{3,4} \hookrightarrow \mathbf{P}^8,$$

given as the image of the map $\phi_{|4\ell-3E|}$ on the Hirzebruch surface $\mathbf{F}_1 = \mathrm{Bl}_o(\mathbf{P}^2)$. We denote by $h : R' \to \mathbf{P}^1$ the map induced by the linear system $|\ell - E|$. The fibres of h are pairwise disjoint lines in $\mathbf{P}^8$. Equivalently, we consider the vector bundle on $\mathbf{P}^1$

$$\mathcal{G} = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(4)$$

and then $R' \cong \mathbf{P}(\mathcal{G})$. One has the canonical identification between space of sections:

$$H^0\big(R', \mathcal{O}_{R'}(1)\big) \cong H^0\big(\mathbf{P}(\mathcal{G}), \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1)\big) \cong H^0\big(\mathbf{P}^1, \mathcal{G}\big).$$

Later, when computing the dimension of the parameter space of 3-nodal septic scrolls, we shall make use of the basic fact

$$\dim \mathrm{Aut}(R') = \dim \mathrm{Aut}(\mathbf{F}_1) = 6.$$

Every smooth septic scroll in $\mathbf{P}^8$ is obtained from R' by applying a linear transformation of $\mathbf{P}^8$. In particular, the Hilbert scheme of septic scrolls in $\mathbf{P}^8$ has dimension equal to

$$\dim PGL(9) - \dim \mathrm{Aut}(R') = 80 - 6 = 74.$$

Using coordinates in $\mathbf{P}^8$, if $\mathbf{P}^3_{y_0,\ldots,y_3} \subseteq \mathbf{P}^8$ is the linear span of the twisted cubic E corresponding to the exceptional divisor on $\mathbf{F}_1$ and $\mathbf{P}^4_{x_0,\ldots,x_4} \subseteq \mathbf{P}^8$ is the linear span of a rational quartic curve linearly equivalent to ℓ, then the ideal of R' in $\mathbf{P}^8$ is given by the following determinantal condition, see for instance [Ha] Lecture 9:

$$\mathrm{rk}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 \end{pmatrix} \leq 1.$$

The secant variety $\mathrm{Sec}(R') \subseteq \mathbf{P}^8$ is also determinantal, with equations given by the 3×3 minors of the following 1-*generic* matrix:

$$\mathrm{rk}\begin{pmatrix} x_0 & x_1 & x_2 & y_0 & y_1 \\ x_1 & x_2 & x_3 & y_1 & y_2 \\ x_2 & x_3 & x_4 & y_2 & y_3 \end{pmatrix} \leq 2$$

It follows from [CJ] Lemma 3.1 that, as expected, $\mathrm{Sec}(R')$ is 5-dimensional. Furthermore, applying e.g. [Ei] Corollary 3.3, it follows that the singular locus of $\mathrm{Sec}(R')$ coincides with the scroll R'.

Lemma 3.1. *Let $a_1, a_2, a_3 \in \mathrm{Sec}(R')$ be general points and set $\Lambda := \langle a_1, a_2, a_3\rangle \in \boldsymbol{G}(2, 8)$. The image R of the projection $\pi : R' \to \boldsymbol{P}^5$ with center Λ has three non-normal nodes corresponding to the three bisecant lines passing through a_1, a_2 and a_3 and no further singularities.*

Proof. The chosen points a_1, a_2, a_3 can be assumed to lie in $\mathrm{Sec}(R') - (R' \cup \mathrm{Tan}(R'))$. Since $\dim \mathrm{Sec}(R') = 5$, by using the *Trisecant lemma*, see for instance [CC] Proposition 2.6, it follows that the scheme-theoretic intersection of $\mathrm{Sec}(R')$ with Λ consists only of the points a_1, a_2, a_3. In particular, $\Lambda \cap R' = \emptyset$, hence the projection $\pi = \pi_\Lambda : R' \to R$ is a regular morphism. Furthermore, each point a_i lies on a unique bisecant line $\langle x_i, y_i\rangle$, where x_i and y_i are distinct points of R', for $i = 1, 2, 3$.

Suppose now that for $x, y \in R'$, one has $\pi(x) = \pi(y)$. This happens if and only if $\langle x, y\rangle \cap \Lambda \neq \emptyset$, hence $\emptyset \neq \langle x, y\rangle \cap \Lambda \subseteq \{a_1, a_2, a_3\}$ and then necessarily $\{x, y\} = \{x_i, y_i\}$, for $i \in \{1, 2, 3\}$. Since $\Lambda \cap \mathrm{Tan}(R') = \emptyset$, it follows that the differential of π is everywhere injective. To summarize, the only singularities of R are the three non-normal nodes $\pi(x_i) = \pi(y_i)$, for $i = 1, 2, 3$. $\square$

We now fix a general projection $\pi = \pi_\Lambda : R' \to \mathbf{P}^5$ as in Lemma 3.1. We denote by p_i the three singularities of the image scroll R. The map $\pi_{_\Lambda}$ is defined by the 6-dimensional subspace $V := H^0(\mathbf{P}^8, \mathcal{I}_{\Lambda/\mathbf{P}^8}(1))$ of $H^0(\mathbf{P}^1, \mathcal{G})$. To give Λ amounts to specifying $V \subseteq H^0(\mathbf{P}^1, \mathcal{G})$. Since $\Lambda \cap R' = \emptyset$, it follows that the evaluation map $\mathrm{ev}_V : V \otimes \mathcal{O}_{\mathbf{P}^1} \to \mathcal{G}$ is surjective. Hence ev_V defines a morphism

$$f : \mathbf{P}^1 \to \mathbf{G}(1, 5).$$

This map is induced by the ruling of the image scroll R, that is, $f_R = f$ is the map given by $f_R(t) := \big[\pi(h^{-1}(t))\big]$, for $t \in \mathbf{P}^1$. Set $\Gamma_R := \mathrm{Im}(f_R)$.

Proposition 3.2. *For a general choice of the* 3*-secant plane* Λ *to* $\mathrm{Sec}(R')$*, the following hold:*

(i) $\dim\langle p_1, p_2, p_3\rangle = 2$.
(ii) $\langle p_1, p_2, p_3\rangle \cap R = \{p_1, p_2, p_3\}$.

Proof. It suffices to consider a codimension 2 general linear section $Z \subseteq R' \subseteq \mathbf{P}^8$. Then Z is a smooth 0-dimensional scheme supported at seven distinct points $x_1, y_1, x_2, y_2, x_3, y_3$ and z, spanning a 6-dimensional linear space in $\mathbf{P}^8$. In particular, z does not lie in the 5-plane spanned by the points $\{x_i, y_i\}_{i=1}^3$ and no line intersecting the lines $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$, $\langle x_3, y_3\rangle$ exists. Pick general points $a_i \in \langle x_i, y_i\rangle$, for $i = 1, 2, 3$. Then the projection $\pi_{_\Lambda}$ defined by the plane $\Lambda = \langle a_1, a_2, a_3\rangle$ satisfies both conditions (i) and (ii). □

For a projection π_Λ satisfying the assumptions of Lemma 3.1, the map $f_R : \mathbf{P}^1 \to \mathbf{G}(1, 5)$ is an embedding, for Λ intesects no ruling of R'. We record the conclusion of Proposition 2.3 for a scroll R as above:

Proposition 3.3. *The rational curve* $\Gamma_R \subseteq \boldsymbol{G}(1, 5)$ *admits three secant lines that lie in* $\boldsymbol{G}(1, 5)$*. Conversely, a rational septic curve* $\Gamma \subseteq \boldsymbol{G}(1, 5)$ *having three secant lines lying in* $\boldsymbol{G}(1, 5)$ *is the curve of rulings of a* 3*-nodal septic scroll in* $\boldsymbol{P}^5$*.*

We establish a couple of properties concerning the linear system of cubic fourfolds containing a 3-nodal septic scroll:

Proposition 3.4. *The following statements hold for a general* 3*-nodal septic scroll* $R \subset \boldsymbol{P}^5$*:*

$$(i)\ \dim|\mathcal{I}_{R/\mathbf{P}^5}(3)| = 12 \quad \textit{and} \quad (ii)\ \ H^1(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)) = 0.$$

Proof. Recall that R is the image of a projection $\pi = \pi_\Lambda : R' \to R$ with center Λ, and denote by $p_1, p_2, p_3 \in R$ the three (non-normal) singularities of R and by $\{x_i, y_i\} = \pi^{-1}(p_i)$, for $i = 1, 2, 3$. By Proposition 3.2, the points p_1, p_2 and p_3 are in general linear position in $\mathbf{P}^5$ and thus impose independent conditions on cubic hypersurfaces, that is, $H^1(\mathbf{P}^5, \mathcal{I}_{\mathrm{Sing}(R)/\mathbf{P}^5}(3)) = 0$.

By passing to cohomology in the short exact sequence

$$0 \longrightarrow \mathcal{I}_{R/\mathbf{P}^5}(3) \longrightarrow \mathcal{I}_{\mathrm{Sing}(R)/\mathbf{P}^5}(3) \longrightarrow \mathcal{I}_{\mathrm{Sing}(R)/R}(3) \longrightarrow 0,$$

we write the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{I}_{R/\mathbf{P}^5}(3)) \longrightarrow H^0(\mathcal{I}_{\mathrm{Sing}(R)/\mathbf{P}^5}(3)) \longrightarrow H^0(\mathcal{I}_{\mathrm{Sing}(R)/R}(3)) \longrightarrow H^1(\mathcal{I}_{R/\mathbf{P}^5}(3)) \longrightarrow 0.$$

Clearly $h^0\big(\mathbf{P}^5, \mathcal{I}_{\mathrm{Sing}(R)/\mathbf{P}^5}(3)\big) = \binom{8}{3} - 3 = 53$. Furthermore, we have the following identification of linear systems:

$$\pi^*\Big(\big|\mathcal{I}_{\mathrm{Sing}(R)/R}(3)\big|\Big) = \Big|\mathcal{I}_{\{x_1,y_1,x_2,y_2,x_3,y_3\}/R'}\big(12\ell - 9E\big)\Big|. \tag{6}$$

The scroll $[R]\in \mathfrak{H}_{\mathrm{scr}}$ is obtained as a general projection like in Lemma 3.1. In particular, the points $\{x_i, y_i\}_{i=1}^3\subseteq R'$ are general as well, hence impose independent conditions on the linear system $|12\ell-9E|$ on R'. Using the identification (6), we compute:

$$h^0\bigl(R, \mathcal{I}_{\mathrm{Sing}(R)/R}(3)\bigr)=h^0\bigl(R', \mathcal{O}_{R'}(12\ell-9E)\bigr)-6=h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(12))-\binom{10}{2}-6=40.$$

Therefore $h^0\bigl(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)\bigr)=13$ if and only if $h^1\bigl(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)\bigr)=0$. This last statement can be proved via a simple *Macaulay* calculation by choosing the points a_1, a_2, a_3 randomly in the variety $\mathrm{Sec}(R')$ whose equations have been given explicitly. □

Remark 3.5. It is possible to realize the rational curve Γ_R inside the linear system $|\mathcal{O}_R(1)|$ as follows. Recall that we have denoted by $\phi:\mathbf{F}_1\hookrightarrow \mathbf{P}^8$ the embedding whose image is the smooth scroll R'. In $|4\ell-3E|\cong \mathbf{P}^8$, we consider the space of reducible hyperplane sections:

$$\Bigl\{A'+L' : A'\in|3\ell-2E|,\ L'\in|\ell-E|\Bigr\}.$$

Note that L' is a ruling of R', whereas $A'\subseteq \mathbf{P}^8$ is a sextic with $\langle A'\rangle=\mathbf{P}^6$ and with $L'\cdot A'=1$. In the linear system $|3\ell-2E|$ there exists a *unique* sextic A_0' such that $\Lambda\subseteq\langle A_0'\rangle\subseteq \mathbf{P}^8$. The curve A_0' corresponds to the unique curve in the linear system

$$\Bigl|\mathcal{I}_{\{x_1,y_1,x_2,y_2,x_3,y_3\}/R'}(3\ell-2E)\Bigr|$$

on R'. Indeed, $x_i, y_i\in A_0'$, therefore $a_i\in\langle x_i, y_i\rangle\subseteq\langle A_0'\rangle$, for $i=1,2,3$. It then follows that $\Lambda=\langle a_1,a_2,a_3\rangle\subseteq\langle A_0'\rangle$. The projection $A_0:=\pi(A_0')\subseteq \mathbf{P}^5$ is a sextic curve on R passing through the nodes p_1, p_2, p_3. One identifies Γ_R with A_0 via the map $L\mapsto L\cdot A_0$.

We denote by $\mathcal{H}_{\mathrm{scr}}$ the Hilbert scheme of 3-nodal septic scrolls in $R\subseteq \mathbf{P}^5$ and set

$$\mathfrak{H}_{\mathrm{scr}}:=\mathcal{H}_{\mathrm{scr}}/PGL(6).$$

We regard $\mathfrak{H}_{\mathrm{scr}}$ as the coarse moduli space of 3-nodal septic scrolls.

Proposition 3.6. *The parameter space $\mathfrak{H}_{\mathrm{scr}}$ is birationally isomorphic to $\mathrm{Sym}^3\bigl(\mathrm{Sec}(R')\bigr)/\mathrm{Aut}(R')$. In particular, $\mathfrak{H}_{\mathrm{scr}}$ is a unirational 9-dimensional variety.*

Proof. We identify $\mathrm{Aut}(R')$ with the group consisting of linear automorphisms $\sigma\in PGL(9)$ such that $\sigma(R')=R'$. Every $\sigma\in\mathrm{Aut}(R')$ clearly invariates $\mathrm{Sec}(R')$. Since $\mathrm{Sing}(\mathrm{Sec}(R'))=R'$, conversely, every automorphism $\sigma\in PGL(9)$ invariating $\mathrm{Sec}(R')$ belongs actually to $\mathrm{Aut}(R')$. One has a birational action of $\mathrm{Aut}(R')$ on $\mathrm{Sym}^3\bigl(\mathrm{Sec}(R')\bigr)$ given by

$$\sigma\langle a_1,a_2,a_3\rangle:=\langle\sigma(a_1),\sigma(a_2),\sigma(a_3)\rangle,$$

for $\sigma\in\mathrm{Aut}(R')$ and $a_1,a_2,a_3\in\mathrm{Sec}(R')$. We can now define a birational morphism

$$\vartheta:\mathrm{Sym}^3\bigl(\mathrm{Sec}(R')\bigr)/\mathrm{Aut}(R')\dashrightarrow\mathfrak{H}_{\mathrm{scr}},\ \text{ by setting}$$

$$\Lambda:=\langle a_1,a_2,a_3\rangle\mapsto\pi_\Lambda(R') \bmod PGL(6),$$

where $\pi_\Lambda:\mathbf{P}^9\dashrightarrow\mathbf{P}^5$ is a projection of center Λ. The assignment is clearly $\mathrm{Aut}(R')$-invariant, hence ϑ is well-defined. Applying Lemma 3.1, we obtain that ϑ is a birational isomorphism.

The secant variety $\mathrm{Sec}(R')$ being a $\mathbf{P}^1$-bundle over the rational variety $\mathrm{Sym}^2(R')$ is unirational. This implies that $\mathrm{Sym}^3\bigl(\mathrm{Sec}(R')\bigr)$ and thus $\mathfrak{H}_{\mathrm{scr}}$ are unirational as well. □

Over the Hilbert scheme $\mathcal{H}_{\mathrm{scr}}$ we consider the universal family of scrolls:

$$\mathcal{H}_{\mathrm{scr}} \xleftarrow{p_1} \mathcal{Y}_{\mathrm{scr}} \xrightarrow{p_2} \mathbf{P}^5.$$

We introduce the incidence correspondence between cubic fourfolds of discriminant 26 and nodal septic scrolls in $\mathbf{P}^5$:

$$|\mathcal{O}_{\mathbf{P}^5}(3)| \longleftarrow \mathcal{X} := \mathbf{P}\Big((p_1)_*\big(\mathcal{I}_{\mathcal{Y}_{\mathrm{scr}}/\mathcal{H}_{\mathrm{scr}}\times\mathbf{P}^5} \otimes p_2^*\mathcal{O}_{\mathbf{P}^5}(3)\big)\Big) \longrightarrow \mathcal{H}_{\mathrm{scr}}$$

The group $PGL(6)$ acts on the entire diagram. By quotienting out this action, if we set $\mathfrak{X} := \mathcal{X}/PGL(6)$, we obtain two projections:

$$\mathcal{C}_{26} \xleftarrow{\pi_1} \mathfrak{X} \xrightarrow{\pi_2} \mathfrak{H}_{\mathrm{scr}}$$

The 21-dimensional variety $\mathfrak{X}$ being a $\mathbf{P}^{12}$-bundle over the unirational variety $\mathfrak{H}_{\mathrm{scr}}$ is unirational as well. A general scroll $[R] \in \mathfrak{H}_{\mathrm{scr}}$ has precisely 3 non-normal nodes. Checking that a general cubic fourfold $X \supseteq R$ is smooth, reduces to a standard Macaulay calculation. Applying (5), we obtain that the lattice $A(X)$ contains a 2-dimensional lattice $\langle h^2, [R]\rangle$ of discriminant 26, therefore the map π_1 is well-defined. Proposition 2.1 implies dim $\pi_1^{-1}([X]) \leq 2$, for all $[X] \in \mathcal{C}_{26}$, hence $\mathfrak{X}$ dominates $\mathcal{C}_{26}$. In fact one can prove something more precise and establish in the process Theorem 1.2.

Theorem 3.7. *The incidence correspondence $\mathfrak{X}$ is birational to the universal $K3$ surface $\mathcal{F}_{14,1}$.*

Proof. We define a map $\theta : \mathfrak{X} \to \mathcal{F}_{14,1}$ as follows. We start with a pair $[X, R] \in \mathfrak{X}$ and denote by $f_R : \mathbf{P}^1 \to F(X)$ the rational curve of rulings described in Proposition 3.3. Denoting by $[S, H] := \phi^{-1}([X]) \in \mathcal{F}_{14}$ the polarized $K3$ surface provided by the identification (3), applying Proposition 2.1, there exists a uniquely determined point $p \in S$ such that $\Delta_p = \Gamma_R$.

The map θ is clearly generically injective. Since both $\mathfrak{X}$ and $\mathcal{F}_{14,1}$ are irreducible varieties of the same dimension 21, it follows that θ is birational. In particular, in the isomorphism $S \cong \overline{\mathcal{M}}_7(X)$ constructed in Corollary 2.4, the general point on both sides corresponds to a septic scroll $R \subseteq X$ which is 3-nodal and has no further singularities. □

4. THE RATIONALITY OF $\mathcal{F}_{14,1}$

In this section, using in an essential way the characterization given in Proposition 3.3 of the rational curves Γ_R of rulings of 3-nodal scrolls $R \subseteq \mathbf{P}^5$, we show that the universal $K3$ surface of genus 14 is rational.

We begin by recalling the structure of the moduli space of curves of genus 8. Consider the Grassmannian $\mathbf{G}(1,5) \subseteq \mathbf{P}^{14}$ in its Plücker embedding. Denote by

$$\mathfrak{M}_8 := \mathbf{G}\Big(7, \mathbf{P}\Big(\bigwedge^2 \mathbb{C}^6\Big)\Big)/PGL(6)$$

the space of codimension 7 linear sections of $\mathbf{G}(1,5)$. Mukai [M6] has shown that the map

$$\mathfrak{M}_8 \dashrightarrow \overline{\mathcal{M}}_8,$$

sending a general 7-plane $[\mathbf{P}(V) \hookrightarrow \mathbf{P}^{14}] \in \mathfrak{M}_8$ to the intersection $[\mathbf{G}(1,5) \cdot \mathbf{P}(V)] \in \overline{\mathcal{M}}_8$ viewed as a canonical curve of genus 8, is a birational isomorphism. For more details on how to extend Mukai's isomorphism over parts of the boundary of $\overline{\mathcal{M}}_8$, see also [FV2].

Recall that we introduced in Section 3 the smooth septic scroll $R' \cong \mathbf{F}_1 \subseteq \mathbf{P}^8$, then considered a singular scroll $R \subseteq \mathbf{P}^5$, defined as the image of a linear projection $\pi_\Lambda : R' \to \mathbf{P}^5$

whose center is a general plane $\Lambda \subset \mathbf{P}^8$, which is 3-secant to $\mathrm{Sec}(R')$. We denote by p_1, p_2, p_3 the three nodes of R and $\{x_i, y_i\} = \pi^{-1}(p_i)$. As explained in the Introduction, $P_i \subseteq \mathbf{P}^5$ denotes the 2-plane spanned by the rulings of R passing through p_i, for $i = 1, 2, 3$. The line

$$L_i \subseteq \mathbf{G}(1,5) \subseteq \mathbf{P}^{14}$$

parametrizes the lines in the plane P_i passing through the point p_i. If $\Gamma = \Gamma_R \subseteq \mathbf{G}(1,5)$ is the curve of rulings associated to R introduced in Proposition 3.3, then L_i meets Γ in two distinct points. We keep this notation throughout this section.

Due to the results of the previous section, our strategy is now to describe the family

$$\mathcal{U} \subseteq \mathrm{Hom}(\mathbf{P}^1, \mathbf{G}(1,5))$$

of smooth rational septic curves $\Gamma_R \subseteq \mathbf{G}(1,5)$ carrying three bisecant lines contained in $\mathbf{G}(1,5)$. From Proposition 3.3 it follows that $\mathcal{U}$ is birational to the Hilbert scheme $\mathcal{H}_{\mathrm{scr}}$ of 3-nodal septic scrolls in $\mathbf{P}^5$. Then we show that the quotient $\mathcal{U}/PGL(6)$ is rational. Since $\mathcal{U}/PGL(6)$ is birational to $\mathfrak{H}_{\mathrm{scr}}$ and, as proven in Theorem 1.2, the universal $K3$ surface of genus 14 is a $\mathbf{P}^{12}$-bundle over $\mathfrak{H}_{\mathrm{scr}}$, its rationality will follow.

The nodal curve $\Gamma + L_1 + L_2 + L_3 \subseteq \langle \Gamma \rangle \cdot \mathbf{G}(1,5)$ has arithmetic genus 3. It follows from Mukai's work [M1] that the intersection $\langle \Gamma \rangle \cdot \mathbf{G}(1,5)$ is a canonical curve of genus 8, provided (i) it is proper and reduced and (ii) dim $\langle \Gamma \rangle = 7$. Using the surjectivity of the period map for polarized $K3$ surfaces of genus 8, we shall show that both assumptions (i) and (ii) are satisfied. Granting both (i) and (ii) for the moment, we consider the canonically embedded curve in $\langle \Gamma \rangle = \mathbf{P}^7$, pictured also below:

$$C := \langle \Gamma \rangle \cdot \mathbf{G}(1,5) = Q + \Gamma + L_1 + L_2 + L_3. \tag{7}$$

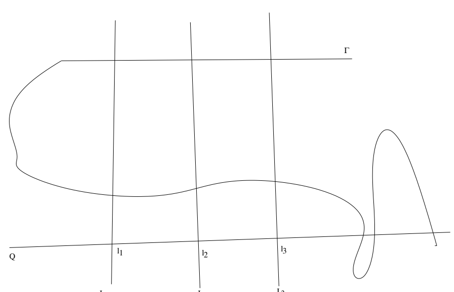

FIGURE 1. The canonical curve $C = \Gamma + Q + L_1 + L_2 + L_3$.

Bertini's Theorem implies that a general 8-dimensional space $\langle \Gamma \rangle \subseteq \mathbf{P}^8 \subseteq \mathbf{P}^{14}$ cuts out on $\mathbf{G}(1,5)$ a smooth 2-dimensional linear section T, see also [Ve1], Propositions 3.2 and 3.3. By the adjunction formula, $T \hookrightarrow \mathbf{P}^8$ is a smooth $K3$ surface (of genus 8) polarized by $\mathcal{O}_T(C)$. We now describe the Picard lattice of T:

Lemma 4.1. *One has the following intersection products on T:*

$$Q^2 = -2, \ Q \cdot \Gamma = 3, \ Q \cdot L_i = 1, \quad \Gamma \cdot L_i = 2, \ L_i \cdot L_j = -2\delta_{ij}, \quad \textit{for } i, j = 1, 2, 3.$$

Proof. The generality assumptions ensure that L_i and L_j are disjoint lines, for $i \neq j$. Else, if $L_i \cap L_j \neq \emptyset$, then $\langle p_i, p_j \rangle \subseteq P_i \cap P_j \subseteq \mathbf{P}^5$. It follows that the four rulings of R' passing through the points x_i, y_i, x_j, y_j respectively, span a 6-dimensional space in $\mathbf{P}^8$, which is impossible for

$$h^0\Big(R', \mathcal{O}_{R'}(1)(-4(\ell - E))\Big) = h^0(R', \mathcal{O}_{R'}(E)) = 1,$$

where recall that $\ell, E \in \mathrm{Pic}(R')$ denote the line class and the exceptional divisor respectively. This implies that there exists a unique hyperplane in $\mathbf{P}^8$ containing the four rulings, therefore they must span a 7-dimensional linear space.

Since $L_i^2 = -2$, by intersecting (7) with L_i, we obtain $Q \cdot L_i = 1$. Furthermore $7 = \Gamma \cdot C$ and since $\Gamma^2 = -2$, we obtain $\Gamma \cdot Q = 3$. Finally, $C \cdot Q = \deg(Q) = 4$, therefore $Q^2 + \Gamma \cdot Q + 3 = 4$, implying $Q^2 = -2$ and thus finishing the proof. □

In particular $Q \subseteq \langle T \rangle = \mathbf{P}^8$ is a reduced, connected quartic curve of arithmetic genus zero. Since $C - Q \equiv \Gamma + L_1 + L_2 + L_3$, we obtain $h^0(T, \mathcal{O}_T(C - Q)) = 4$. The next lemma summarizes the situation.

Lemma 4.2. *The span $\langle Q \rangle$ is 4-dimensional and Q is a connected nodal quartic curve with $p_a(Q) = 0$.*

In fact, we shall construct a $K3$ surface T, such that the curve Q described in Lemma 4.2 is actually smooth.

To establish the validity of the assumptions (i) and (ii) and thus the existence of the special $K3$ surface T, we use Hodge theory. We consider the following sublattice

$$\mathbb{L} := \mathbb{Z} \cdot [Q] \oplus \mathbb{Z} \cdot [\Gamma] \oplus \mathbb{Z} \cdot [L_1] \oplus \mathbb{Z} \cdot [L_2] \oplus \mathbb{Z} \cdot [L_3] \tag{8}$$

generated by the (-2) classes corresponding to Q, Γ, L_1, L_2 and L_3 respectively, and with intersection pairing as given in Lemma 4.1. We invoke the surjectivity of the period map for $K3$ surfaces. The rank 5 lattice $\mathbb{L}$ is even and has signature $(1,4)$. Applying [Mo] Corollary 2.9, there exists a smooth $K3$ surface T, such that $\mathrm{Pic}(T) \cong \mathbb{L}$. We define the following class on T

$$C := \Gamma + Q + L_1 + L_2 + L_3.$$

The genus zero curves $\Gamma, Q, L_1, L_2, L_3 \subseteq T$ cannot have multiple components, for that would make $\mathrm{Pic}(T)$ larger than $\mathbb{L}$, therefore they are all smooth, rational curves on T.

Lemma 4.3. *The linear system $|\mathcal{O}_T(C)|$ is very ample.*

Proof. We use Reider's Theorem [R], which, in the case of $K3$ surfaces, had been proven before in [SD]. It suffices to show that there exists no curve E on T with $E^2 = 0$ and $E \cdot C \in \{1, 2\}$, nor a curve F on T with $F^2 = -2$ and $F \cdot C = 0$. We prove the first statement, the second follows similarly. Assuming there is such a curve E, we express it as an integral combination $E \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ of the generators of $\mathrm{Pic}(T)$. If $C \cdot E = 1$, we obtain

$$-15x^2 - 12xy - 5y^2 + 2x + y = z_1^2 + z_2^2 + z_3^2.$$

By comparing the signs of the two sides, one concludes that this equation has no integral solutions. The case $C \cdot E = 2$ is similar. Finally, if $F \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$ is a (-2)-curve with $C \cdot F = 0$, we obtain

$$-15x^2 - 12xy - 5y^2 + 1 = z_1^2 + z_2^2 + z_3^2,$$

which implies $x = y = 0$ and, say $z_2 = z_3 = 0$ and then $z_1 = 1$. Thus $F = L_1$, but $C \cdot L_1 = 1$, hence this case does not appear. We conclude that C is very ample. □

We show that the $K3$ surface T constructed in Lemma 4.3 is a linear section of $\mathbf{G}(1,5)$. In particular, Mukai's results [M6] will apply for its hyperplane section C.

Proposition 4.4. *The $K3$ surface T carries a globally generated rank two vector bundle T with $\det(T) = \mathcal{O}_T(C)$, providing an embedding $T \hookrightarrow \mathbf{G}(1,5)$ such that*

$$\langle T \rangle \cdot \mathbf{G}(1,5) = S.$$

Proof. We use [M7] and need to show that the polarized $K3$ surface $(T, \mathcal{O}_T(C))$ is *Brill-Noether general*, that is, for all pairs of line bundles M, N on T such that $M \otimes N = \mathcal{O}_T(C)$, one has $h^0(T,M) \cdot h^0(T,N) < h^0(T,C)$. Under these circumstances, it is shown in *loc.cit.* that T carries a rigid, globally generated, stable rank 2 vector bundle E with $h^0(T,E) = 6$ and $\det(E) = \mathcal{O}_T(C)$, inducing a map $\varphi_E : T \to \mathbf{G}(1,5)$. Reasoning along the lines of [M7] Theorem 3.10, the $K3$ surface T is then a linear section of $\mathbf{G}(1,5)$ in its Plücker embedding, that is, $T = \mathbf{G}(1,5) \cdot \langle T \rangle$.

To establish the Brill-Noether generality of $(T, \mathcal{O}_T(C))$, we use for instance [GLT] Lemma 2.8. It suffices to show that in the lattice $\mathbb{L}$ there exists no vector D such that $D^2 = 2$ and $D \cdot C \in \{7, 6\}$, nor is there a vector D with $D^2 = 0$ and $D \cdot C \leq 4$.

We treat in detail only the first case, the remaining ones being similar. We write

$$D \equiv x\Gamma + yQ + z_1 L_1 + z_2 L_2 + z_3 L_3.$$

The conditions $D^2 = 2$ and $D \cdot C = 7$ translate into the equalities $z_1 + z_2 + z_3 + 7x + 4y = 7$ and $-15x^2 - 5y^2 - 12xy + 14x + 7y + 1 = z_1^2 + z_2^2 + z_3^2 \geq 0$. It is elementary to see that there are no integral solutions. $\square$

Using Proposition 4.4, we conclude that the intersection (7) corresponding to a general curve $\Gamma_R \in \mathcal{U}$ corresponds to a semistable canonical curve of genus 8.

It will be useful to have a criterion for determining when the curve Γ spans a space of maximal possible dimension in the Plücker space $\mathbf{P}^{14} \supseteq \mathbf{G}(1,5)$. To that end, recall that the Plücker embedding of the dual Grassmannian $\mathbf{G}(1,5)^\vee = \mathbf{G}(3,5) \hookrightarrow (\mathbf{P}^{14})^\vee$ assigns to a point $p \in \mathbf{G}(1,5)^\vee$ corresponding to a 3-plane $\mathbf{P}^3_p \subseteq \mathbf{P}^5$ the Schubert cycle

$$\sigma_p := \{\ell \in \mathbf{G}(1,5) : \ell \cap \mathbf{P}^3_p \neq \emptyset\}.$$

Note that $\dim\langle \Gamma \rangle + 1 = \operatorname{codim}\langle \Gamma \rangle^\perp$. Setting

$$W^1(\Gamma) := \mathbf{G}(3,5) \cap \langle \Gamma \rangle^\perp = \{p \in \mathbf{G}(3,5) : \Gamma \subseteq \sigma_p\},$$

for dimension reasons, the next lemma follows immediately:

Lemma 4.5. *Assume $W^1(\Gamma)$ is finite. Then $\dim\langle \Gamma \rangle = 7$.*

Keeping the previous notation, let $f_R : \mathbf{P}^1 \to \mathbf{G}(1,5)$ be a sufficiently general element of $\mathcal{U}$ and set again $\Gamma = \Gamma_R$. Then under the assumption $R' = S_{3,4}$, we can prove that:

Theorem 4.6. *The set $W^1(\Gamma)$ is finite. In particular $\dim\langle \Gamma \rangle = 7$ and Γ is a rational normal septic curve.*

Proof. If $p \in W^1(\Gamma)$, then $\mathbf{P}^3_p$ contains an integral curve intersecting each line of R. Its strict transform by $\pi_\Lambda : R' \to R$ is an integral section A of the ruled surface R'. Set $d := \deg(A)$, hence $A \equiv (d-3)\ell - (d-4)E \in \mathrm{Pic}(\mathbf{F}_1)$. Clearly $\langle A \rangle \subseteq \pi_\Lambda^{-1}(\mathbf{P}^3_p)$, implying dim $\langle A \rangle \leq 6$.

Let $\mathbf{I}_A := |H - A|$ be the linear system of hyperplanes in $\mathbf{P}^8$ containing the curve $A \subseteq R'$. By direct calculation, we find $\dim(\mathbf{I}_A) = \dim|H - A| = 7 - d \geq 1$ and $\dim|A| = 2d - 6$. It follows that $3 \leq d \leq 6$. Recalling that $V = H^0(\mathbf{P}^8, \mathcal{I}_{\Lambda/\mathbf{P}^8}(1))$, the condition

$$\dim(\mathbf{P}V \cap \mathbf{I}_A) \geq 1$$

is equivalent to the condition that the curve $\pi_\Lambda(A)$ be contained in a 3-space $\mathbf{P}^3_p$. For $3 \leq d \leq 6$ let $\mathbf{G}(7-d, |H|)$ denote the Grassmannian of $(7-d)$-subspaces of $|H| \cong \mathbf{P}^8$ and introduce the $(2d-6)$-dimensional variety

$$\mathbf{S}_d := \Big\{ \mathbf{I}_{A'} \in \mathbf{G}(7-d, |H|) : A' \in |(d-3)\ell - (d-4)E| \Big\}.$$

For an integer $k \geq 1$, we consider the Schubert cycle

$$\sigma_V^k := \big\{ \mathbf{I} \in \mathbf{G}(7-d, |H|) : \dim(\mathbf{P}V \cap \mathbf{I}) \geq k \big\}.$$

The cycle $\sigma_V^k \cdot \mathbf{S}_d$ is finite for $k = 1$ and empty for $k \geq 2$, provided the intersection is proper. By Kleiman's transversality of a general translate this is true for a general translate of σ_V^k in $\mathbf{G}(d-7, |H|)$, that is, for a general choice of Λ (or equivalently, of V). Hence $W^1(\Gamma)$ is finite. □

Remark 4.7. The theorem above fails for rational septic scrolls in $\mathbf{P}^8$ containing sections of degree $d \leq 2$, that is, for the scrolls $S_{a,7-a}$, where $a \neq 3$.

We turn to the smooth residual rational curve $Q \subseteq \mathbf{G}(1,5)$ defined by (7). Let

$$R_Q \subseteq \mathbf{P}^5$$

be the quartic scroll whose rulings are parametrized by the curve Q.

Lemma 4.8. *R_Q is a non-degenerate smooth rational normal scroll in $\mathbf{P}^5$.*

Proof. First, observe that R_Q cannot be a cone. Let us assume R_Q is a cone of vertex $v \in \mathbf{P}^5$. Then $\langle Q \rangle \cong \mathbf{P}^4 \subseteq \mathbf{G}(1,5)$ parametrizes the lines passing through v. This is a contradiction because $\langle Q \rangle \subseteq \langle \Gamma \rangle \cdot \mathbf{G}(1,5) = C$. Now assume that R_Q is contained in a hyperplane $H \subseteq \mathbf{P}^5$. Then Q is contained in the Grassmannian $\mathbf{G}_H := \mathbf{G}(1,H) \subseteq \mathbf{G}(1,5)$ of lines of H. Since $K_{\mathbf{G}_H} = \mathcal{O}_{\mathbf{G}_H}(-5)$, we observe that, by adjunction, the curvilinear sections of $\mathbf{G}_H$ are curves of arithmetic genus 1. Because of this fact and since $\deg(\mathbf{G}_H) = 5$, it follows that

$$\langle Q \rangle \cdot \mathbf{G}_H = Q + L \subseteq C,$$

where L is a bisecant line to Q. But the only line components in C are L_1, L_2, L_3 and none of them is bisecant to Q. Via Proposition 2.3, the same argument shows that the scroll R_Q has no incident rulings, therefore R_Q is smooth. □

Lemma 4.9. *The scroll R_Q contains no other lines except the ruling parametrized by Q.*

Proof. Assume R_Q contains a line ℓ_0 not parametrized by a point of Q. We prove that this implies that $W^1(\Gamma)$ is not finite, thus contradicting Theorem 4.6. Consider the family G of codimension 1 Schubert cycles σ_p defined by a 3-space $\mathbf{P}^3_p \supseteq \ell_0$. Note that $G \cong \mathbf{G}(1,3)$. We have $G \subseteq \langle Q \rangle^\perp$. Since $\langle Q \rangle \subseteq \langle \Gamma \rangle$, we also have $\langle \Gamma \rangle^\perp \subseteq \langle Q \rangle^\perp$. Counting dimensions it follows $\dim(G \cap \langle \Gamma \rangle^\perp) \geq 1$, which implies that $W^1(\Gamma)$ is not finite. □

There are two types of smooth quartic scrolls in $\mathbf{P}^5$, namely $S_{1,3} = \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(3)\big)$ and $S_{2,2} = \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2)\big)$. The latter case is characterized by the property that every line contained in the scroll is a ruling. Lemma 4.9 implies the following:

Theorem 4.10. *Let $\Gamma \subseteq \boldsymbol{G}(1,5)$ be a smooth septic rational curve corresponding to a general element of $\mathcal{U}$ and $Q \subseteq \boldsymbol{G}(1,5)$ the residual quartic curve. Then R_Q is isomorphic to $S_{2,2}$.*

To summarize, to a general rational curve $\Gamma = \Gamma_R \in \mathcal{U}$, we associated the quartic scroll R_Q, equipped with three rulings ℓ_1, ℓ_2, ℓ_3 corresponding to the points $L_i \cdot Q \in \mathbf{G}(1,5)$, for $i = 1,2,3$. Each ruling ℓ_i passes through the node p_i of the scroll R and is contained in the 2-plane P_i whose existence is established in Proposition 3.3.

To prove the rationality of $\mathfrak{H}_{\rm scr}$ and thus that $\mathcal{F}_{14,1}$, we reverse this construction. We denote by $\mathcal{V}$ the variety classifying elements (R_Q, p_1, p_2, p_3), where $R_Q \subseteq \mathbf{P}^5$ is a smooth quartic scroll isomorphic to $S_{2,2}$ and $p_i \in R_Q$ for $i = 1,2,3$.

Lemma 4.11. *The $PGL(6)$-stabilizer of a general point $(R_Q, p_1, p_2, p_3) \in \mathcal{V}$ is trivial. In particular, $PGL(6)$ acts transitively on $\mathcal{V}$.*

Proof. The automorphism group of $S_{2,2} \cong \mathbf{F}_0$ is the semidirect product of $PGL(2) \times PGL(2)$ with $\mathbb{Z}/2\mathbb{Z}$. The last factor corresponds to the automorphism $u \in \mathrm{Aut}(\mathbf{F}_0)$ permuting the two factors. In particular, $\mathrm{Aut}(S_{2,2})$ is 6-dimensional. This implies that the space $\mathcal{V}$ has dimension

$$\dim PGL(6) - \dim \mathrm{Aut}(S_{2,2}) + 3\dim(R_Q) = 35 = \dim\ PGL(6).$$

Choose general points $p_i = (a_i, b_i) \in \mathbf{F}_0 \cong S_{2,2}$, with $a_i \neq b_i$, for $i = 1,2,3$. Up to the action of $u \in \mathrm{Aut}(\mathbf{F}_0)$, the stabilizer $\mathrm{Stab}_{PGL(6)}(R_Q, p_1, p_2, p_3)$ corresponds to pairs of automorphism $(\sigma_1, \sigma_2) \in PGL(2) \times PGL(2)$, such that $\sigma_1(a_i) = a_i$ and $\sigma_2(b_i) = b_i$. Thus $\sigma_1 = \sigma_2 = 1$. The points p_i not lying on the diagonal of $\mathbf{F}_0$, the automorphism u does not fix any of them, thus the stabilizer in question is trivial. Since $\mathcal{V}$ and $PGL(6)$ have the same dimension, this also implies the trasitivity of the $PGL(6)$-action on $\mathcal{V}$, as claimed. $\square$

We can thus start by fixing once and for all the quartic scroll R_Q. Precisely, we embed the surface $\mathbf{F}_0 := \mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^5$ via the linear system $|\mathcal{O}_{\mathbf{F}_0}(1,2)|$ and denote by

$$R_0 \subseteq \mathbf{P}^5$$

the image quartic scroll. The rulings on R_0 are the elements of the linear system $|\mathcal{O}_{\mathbf{F}_0}(0,1)|$. Let $Q_0 \subseteq \mathbf{G}(1,5)$ be the curve of rulings of R_0. We then fix three points in $\mathbf{F}_0$, for instance

$$o_1 := \big([1:0],[0:1]\big),\ o_2 := \big([0:1],[1:0]\big) \text{ and } o_3 := \big([1:1],[-1:-1]\big),$$

which we identify with their images in R_0. As explained in Lemma 4.11, the stabilizer subgroup G of $PGL(6)$ fixing both R_0 as well as the *set* $\{o_1, o_2, o_3\}$ is isomorphic to the subgroup of $PGL(2) \times PGL(2)$ fixing the set $\{o_1, o_2, o_3\}$. Therefore $G = \mathfrak{S}_3$.

For $i = 1,2,3$, we denote by ℓ_i the ruling of R_0 passing through the point o_i. Then, let $\mathbf{P}^3_i$ be the projective space consisting of 2-planes $\Pi_i \subseteq \mathbf{P}^5$ containing the line ℓ_i. Giving a plane Π_i is equivalent to specifying a line $L_i \subseteq \mathbf{G}(1,5)$ in the Plücker embedding of the Grassmannian. Note that L_i meets Q_0 transversally at precisely one point, namely $\ell_i \in \mathbf{G}(1,5)$.

We introduce a rational map

$$\varkappa : \mathbf{P}^3_1 \times \mathbf{P}^3_2 \times \mathbf{P}^3_3/\mathfrak{S}_3 \dashrightarrow \mathfrak{H}_{\rm scr}$$

defined as follows. To a triple of planes (Π_1, Π_2, Π_3), we attach the lines $L_1, L_2, L_3 \subseteq \mathbf{G}(1,5)$. Since $Q_0 \subseteq \mathbf{G}(1,5)$ is a smooth rational quartic curve, in the Plücker embedding we have that $\langle Q_0 \rangle \cong \mathbf{P}^4$. Attaching one general 1-secant line to Q_0 increases the dimension of the linear span of the union by one, therefore by attaching three general 1-secant lines, we have

$$\langle Q_0 + L_1 + L_2 + L_3 \rangle \cong \mathbf{P}^7 \subseteq \mathbf{P}^{14}.$$

We write

$$\langle Q_0 + L_1 + L_2 + L_3\rangle \cdot \mathbf{G}(1,5) = Q_0 + L_1 + L_2 + L_3 + \Gamma,$$

where Γ is a degree 7 curve. Applying Lemma 4.1, it follows that Γ is a rational curve and $\Gamma \cdot L_i = 2$, for $i = 1, 2, 3$. We denote by ℓ_i' and ℓ_i'' the intersection points $L_i \cdot \Gamma$. From Proposition 3.3 it follows that that the scroll $R := R_\Gamma$ induced by Γ is 3-nodal, with nodes given by the intersection $\ell_i' \cap \ell_i''$ taken in the 2-plane Π_i. We set

$$\varkappa(\Pi_1 + \Pi_2 + \Pi_3) := [R].$$

We conclude the proof of the rationality of the Hilbert scheme of 3-nodal scrolls in $\mathbf{P}^5$:

Proof of Theorem 1.3. We first observe that $\varkappa$ is well-defined. To that end, we choose the polarized $K3$ surface $(T, \mathcal{O}_T(C))$ constructed in Propositions 4.3 and 4.4 and we keep the notation used there. Applying Theorem 4.10, the residual quartic rational curve $Q \subseteq \mathbf{G}(1,5)$ parametrizes the rulings of a quartic scroll $R_Q \subseteq \mathbf{P}^5$, which is isomorphic to $S_{2,2}$. Applying Lemma 4.11, there exists a unique automorphisms $\sigma \in PGL(6)$ such that $\sigma(R_Q) = R_0$ and $\sigma(p_i) = o_i$, for $i = 1, 2, 3$. Set $\sigma(P_i) =: \Pi_i \in \mathbf{P}_i^3$ and then $\varkappa(\Pi_1 + \Pi_2 + \Pi_3) = [R_\Gamma]$.

To finish the proof it suffices to observe that $\varkappa$ is generically injective. A general septic curve $\Gamma \in \mathcal{U}$ corresponding to a 3-nodal septic scroll $[R_\Gamma] \in \mathfrak{H}_{\mathrm{scr}}$ has precisely 3 bisecant lines lying in $\mathbf{G}(1,5)$. Giving Γ determines its linear span $\langle\Gamma\rangle$, hence the set $\{L_1, L_2, L_3\}$ as well. □

5. THE UNIRATIONALITY OF THE UNIVERSAL $K3$ SURFACE OF GENUS AT MOST 12

We denote by $\mathcal{F}_{g,n}$ the universal n-pointed $K3$ surface of genus g. Thus $\mathcal{F}_{g,n}$ is an irreducible variety of dimension $19 + 2n$. Similarly, one can consider the universal Hilbert scheme of 0-dimensional cycles of length n, that is, $u^{[n]} : \mathcal{F}_g^{[n]} \to \mathcal{F}_g$. We also introduce the notation $\mathcal{C}_{g,n} := \mathcal{M}_{g,n}/\mathfrak{S}_n$ for the degree n universal symmetric product over $\mathcal{M}_g$, where the symmetric group $\mathfrak{S}_n$ acts by permuting the marked points.

The aim of this short last section is to point out how Mukai's results determine the birational type of $\mathcal{F}_{g,n}$ and that of $\mathcal{F}_g^{[n]}$ for small g, and thus put our Theorem 1.1 better into context:

Theorem 5.1. *The following results on the Kodaira dimension of $\mathcal{F}_{g,n}$ hold:*

(i) *$\mathcal{F}_{g,g+1}$ is unirational for $g \leq 10$.*

(ii) *$\mathcal{F}_{11,1}$ is unirational. The Kodaira dimension of both $\mathcal{F}_{11,11}$ and $\mathcal{F}_{11}^{[11]}$ equals* 19.

Proof. For $g \leq 5$, the general $K3$ surface of genus g is a complete intersection in a projective space and the result follows easily. For details, see the table after Theorem 1.10 in [M7].

For $6 \leq g \leq 10$, Mukai [M1] has constructed a rational homogeneous variety $V_g \subseteq \mathbf{P}^{N_g}$, where $N_g = g + \dim(V_g) - 2$, such that the general $K3$ surface of genus g is obtained as a general linear section $S = V_g \cap \Lambda_g$, where $\Lambda_g \subseteq \mathbf{P}^{N_g}$ is a g-dimensional plane, with the polarization being the one induced by $\mathcal{O}_{\mathbf{P}^{N_g}}(1)$. Moreover, one has the following birational isomorphism, see [M1] Corollary 0.3:

$$\mathcal{F}_g \overset{\cong}{\dashrightarrow} \mathbf{G}(g, N_g)/\mathrm{Aut}(V_g).$$

These results imply the existence of a dominant map $\chi_g : V_g^{g+1} \dashrightarrow \mathcal{F}_{g,g+1}$ given by

$$\chi(x_1, \ldots, x_{g+1}) := \big[V_g \cap \langle x_1, \ldots, x_{g+1}\rangle, x_1, \ldots, x_{g+1}\big].$$

This proves that $\mathcal{F}_{g,g+1}$ (and hence $\mathcal{F}_{g,n}$ for $n \leq g+1$) is unirational in this range.

For $g = 11$, we use [M8], where it is shown that a general curve $[C] \in \mathcal{M}_{11}$ lies on a *unique* $K3$ surface $C \subseteq S$ as a hyperplane section, with $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$. This implies the existence of a rational map $\chi_n : \mathcal{M}_{11,n} \dashrightarrow \mathcal{F}_{11,n}$ defined by

$$\chi_n([C, x_1, \ldots, x_n]) := [S, x_1, \ldots, x_n].$$

The map χ_n is dominant for $n \leq 11$ and a birational isomorphism for $n = 11$. Indeed, in this last case, given an embedded $K3$ surface $S \overset{|H|}{\hookrightarrow} \mathbf{P}^{11}$ and general points $x_1, \ldots, x_{11} \in S$, the hyperplane $\langle x_1, \ldots, x_{11} \rangle \cong \mathbf{P}^{10}$ cuts out a canonical genus 11 curve C on S, which comes equipped with the marked points $x_1, \ldots, x_{11}$. By quotienting the action of the symmetric group $\mathfrak{S}_{11}$, the map χ_{11} induces a birational isomorphism between the universal symmetric product $\mathcal{C}_{11,11}$ and $\mathcal{F}_{11}^{[11]}$. Now we use [FV1] Theorem 0.5. Both varieties $\mathcal{M}_{11,11}$ and $\mathcal{C}_{11,11}$ have Kodaira dimension 19, hence we conclude.

We now pass on to the universal $K3$ surface $\mathcal{F}_{11,1}$. To that end we define a rational map

$$\vartheta : \mathcal{M}_{10,2} \dashrightarrow \mathcal{F}_{11,1},$$

associating to a 2-pointed curve $[C, p_1, p_2] \in \mathcal{M}_{10,2}$, the unique $K3$ surface S of genus 11 containing the curve $[X := C/p_1 \sim p_2]$ obtained from C by identifying p_1 and p_2. To show that ϑ is well-defined, that is, Mukai's construction [M8] can be also carried out for the 1-nodal curve $[X] \in \overline{\mathcal{M}}_{11}$, we use [CLM] Proposition 4.4. Observe that the $K3$ surface S has a distinguished point corresponding to the image of the singularity of X. The map ϑ is clearly dominant, for in each linear system on a $K3$ surface, the 1-nodal curves fill-up a divisor. The unirationality of $\mathcal{F}_{11,1}$ now follows from that of $\mathcal{M}_{10,2}$, which can be established in a variety of ways, see for instance [BCF] Theorem B.

$\square$

Remark 5.2. It is claimed incorrectly in [L] Table 3, that $\mathcal{M}_{11,n}$ is unirational for $n \leq 10$. The argument sketched in *loc.cit.* only establishes the uniruledness of $\mathcal{M}_{11,n}$ when $n \leq 10$, precisely using the map $\chi_n : \mathcal{M}_{11,n} \to \mathcal{F}_{11,n}$, which is birationally a $\mathbf{P}^{11-n}$-bundle. But this argument alone offers no indications concerning the birational nature of the base variety $\mathcal{F}_{11,n}$. One can establish partial results on the birational nature of $\mathcal{F}_{11,n}$, for $n \leq 10$. For instance, it is shown in [Ve1] that the universal product $\mathcal{C}_{11,6}$ is unirational, which implies that $\mathcal{F}_{11}^{[6]}$ is unirational as well.

Remark 5.3. Mukai [M4] gives an explicit orbit space realization over a projective space for the universal $K3$ surface $\mathcal{F}_{13,1}$. The unirationality of $\mathcal{F}_{13,1}$ thus follows. Presumably, a similar argument works for genus 12, when $\mathcal{F}_{12}$ is known to be birational to a $\mathbf{P}^{13}$-bundle over the rational moduli space $\mathcal{MF}_{22}$ of Fano 3-folds $V_{22} \subseteq \mathbf{P}^{13}$, see again [M1].

Remark 5.4. Since $u : \mathcal{F}_{g,1} \to \mathcal{F}_g$ is a morphism fibred in Calabi-Yau varieties, by Iitaka's easy addition formula $\kappa(\mathcal{F}_{g,1}) \leq \dim(\mathcal{F}_g) = 19$, in particular, $\mathcal{F}_{g,1}$ is never of general type. Furthermore, by [K], we also write $\kappa(\mathcal{F}_{g,1}) \geq \kappa(\mathcal{F}_g)$. In particular, when $\mathcal{F}_g$ is of general type, then $\kappa(\mathcal{F}_{g,1}) = 19$.

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