Exponential and sub-exponential stability times for the NLS on the circle

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Abstract

In this note we study stability times for a family of parameter dependent nonlinear Schrödinger equations on the circle, close to the origin. Imposing a suitable Diophantine condition (first introduced by Bourgain), we state a rather flexible Birkhoff Normal Form theorem, which implies, e.g., exponential and sub-exponential time estimates in the Sobolev and Gevrey class respectively. Complete proofs are given elsewhere (see [BMP18]).
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1 Introduction and main results

In this note we consider families of NLS equations on the circle with external parameters of the form:

$$\mathrm{i}u_t + u_{xx} - V * u + f(x, |u|^2)u = 0\,, \tag{1.1}$$

where $\mathrm{i} = \sqrt{-1}$ and $V*$ is a Fourier multiplier

$$V * u = \sum_{j\in\mathbb{Z}} V_j u_j e^{\mathrm{i}jx}\,, \quad (V_j)_{j\in\mathbb{Z}} \in \mathtt{w}_q^\infty\,,$$

living in the weighted ℓ^∞ space

$$\mathtt{w}^\infty_q := \{V = (V_j)_{j\in\mathbb{Z}} \in \ell^\infty \quad | \quad |V|_q := \sup_{j\in\mathbb{Z}} |V_j|\langle j\rangle^q < \infty\}\,, \qquad q \geq 0\,,$$

where $\langle j\rangle := \max\{|j|, 1\}$, while $f(x, y)$ is 2π periodic and real analytic in x and is real analytic in y in a neighborhood of $y = 0$. We shall assume that $f(x, y)$ has a zero in $y = 0$. By analyticity, for some $\mathtt{a}, R > 0$ we have

$$f(x, y) = \sum_{d=1}^{\infty} f^{(d)}(x)y^d\,, \quad |f|_{\mathtt{a},R} := \sum_{d=1}^{\infty} |f^{(d)}|_{\mathbb{T}_\mathtt{a}} R^d < \infty\,, \tag{1.2}$$

where, given a real analytic function $g(x) = \sum_{j\in\mathbb{Z}} g_j e^{\mathrm{i}jx}$, we set[1] $|g|^2_{\mathbb{T}_\mathtt{a}} := \sum_{j\in\mathbb{Z}} |g_j|^2 e^{2\mathtt{a}|j|}$.
Note that if f is independent of x (1.2) reduces to

$$|f|_R := \sum_{d=1}^{\infty} |f^{(d)}|R^d < \infty\,. \tag{1.3}$$

Equation (1.1) is at least locally well-posed (say in a neighborhood of $u = 0$ in H^1) and has an elliptic fixed point at $u = 0$, so that an extremely natural question is to understand *stability times* for small initial data. One can informally state the problem as follows: let $E \subset H^1$ be some Banach space and consider (1.1) with initial datum u_0 such that $|u_0|_E \leq \delta \ll 1$. By local well posedness, the solution $u(t, x)$ of (1.1) with such initial datum exists and is in H^1.

Definition 1.1. We call *stability time* $T = T(\delta)$ the supremum of the times t such that *for all* $|u_0|_E \leq \delta$ *one has* $u(t, \cdot) \in E$ *with* $|u(t, \cdot)|_E \leq 4\delta$.

Computing the stability time $T(\delta)$ is out of reach, so the goal is to give lower (and possibly upper) bounds.
A good comparison is with the case of a finite dimensional Hamiltonian system with a non-degenerate elliptic fixed point, which in the standard complex symplectic coordinates $u_j = \frac{1}{\sqrt{2}}(q_j + \mathrm{i}p_j)$ is described by the Hamiltonian

$$\sum_{j=1}^{n} \omega_j|u_j|^2 + O(u^3)\,, \quad \text{where } \omega_j \in \mathbb{R} \text{ are the } \textit{linear frequencies.} \tag{1.4}$$

Here if the frequencies ω are sufficiently non degenerate, say Diophantine[2], then one can prove exponential lower bounds on $T(\delta)$ and, if the nonlinearity satisfies some suitable hypothesis (e.g. convexity or steepness), even super-exponential

[1] Namely g is a holomorphic function on the domain $\mathbb{T}_a := \{x \in \mathbb{C}/2\pi\mathbb{Z} : |\mathrm{Im}\, x| < a\}$ with L^2-trace on the boundary.

[2] A vector $\omega \in \mathbb{R}^n$ is called Diophantine when it is badly approximated by rationals, i.e. it satisfies, for some $\gamma, \tau > 0$, $|k \cdot \omega| \geq \gamma|k|^{-\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}$.

ones, see for instance [MG95], [BFN15] and reference therein.
The strategy for obtaining exponential bounds is made of two main steps. The first one consists in the so-called Birkhoff normal form procedure: after $\mathtt{N} \geq 1$ steps the Hamiltonian (1.4) is transformed into

$$\sum_{j=1}^{n} \omega_j |u_j|^2 + Z + R \,, \tag{1.5}$$

where Z depends only on the actions $(|u_i|^2)_{i=1}^n$ while $R = O(|u|^{2\mathtt{N}+3})$ contains terms of order at least $2\mathtt{N}+3$ in $|u|$.
It is well known that this procedure generically diverges in $\mathtt{N}$, so the second step consists in finding $\mathtt{N} = \mathtt{N}(\delta)$ which minimizes the size of the remainder R.
The problem of *long-time* stability for equations (1.1) has been studied by many authors. In the context of infinite chains with a finite range coupling, we mention [BFG88]. Regarding applications to PDEs (and particularly the NLS) the first results were given in [Bou96] by Bourgain, who proved polynomial bounds for the stability times in the following terms: for any M there exists $s = s(M)$ such that initial data which are δ-small in the H^{r+s} norm stay small in the H^r norm, for times of order δ^{-M}. Afterwards, Bambusi in [Bam99b] proved that superanalytic initial data stay small in analytic norm, for times of order $e^{(\ln(\delta^{-1})^{1+b})}$, where $b > 1$.
Bambusi and Grebert in [BG06] proved polynomial bounds for a class of *tame-modulus* PDEs, which includes (1.1). More precisely, they proved that for any $\mathtt{N} \gg 1$ there exists $p(\mathtt{N})$ (tending to infinity as $\mathtt{N} \to \infty$) such that for all $p \geq p(\mathtt{N})$ and initial datum in H^p one has $T \geq C(\mathtt{N}, p)\delta^{-\mathtt{N}}$. For an application to the present model we refer also to [ZG17].
Similar results were also proved for the Klein Gordon equation on Zoll manifolds in [BDGS]. Successively Faou and Grebert in [FG13] considered the case of analytic initial data and proved subexponential bounds of the form $T \geq e^{c\ln(\frac{1}{\delta})^{1+\beta}}$ for classes of NLS equations in $\mathbb{T}^d$ (which include (1.1) by taking $d = 1$). Finally, Feola and Iandoli in [FI] prove polynomial lower bounds for the stability times of reversible NLS equations with two derivatives in the nonlinearity.

A closely related topic is the study of orbital stability times close to periodic or quasi-periodic solutions of (1.1). In the case $E = H^1$, Bambusi in [Bam99a] proved a lower bound of the form $T \geq e^{c\delta^{-\beta}}$ for perturbations of the integrable cubic NLS close to a quasi-periodic solution. Regarding higher Sobolev norms, most results are in the periodic case. See [FGL13] (polynomial bounds for Sobolev initial data) and the preprint [MSW18] (subexponential bounds for subanalytic initial data).
A dual point of view is to construct special orbits for which the Sobolev norms grow as fast as possible (thus giving an upper bound on the stability times). As far as we are aware such results are mostly on $\mathbb{T}^2$ and in parameterless cases (for instance [CKS$^+$10], [GK15], [GHP16]) and the time scales involved are much longer

than our stability times (see [Gua14] for the instability of (1.1) on $\mathbb{T}^2$ and [Han14] for the istability of the plane wave in H^s with $s < 1$).

1.1 The stability results

In this paper we recover and improve the results in [BG06] (*Sobolev* initial data) and [FG13] (*analytic and subanalytic* initial data) under a different Diophantine non-resonance condition on the linear frequencies, by application of a different Birkhoff normal form approach (see the comments after Theorem 1.4). More precisely, following Bourgain [Bou05], we set

$$\Omega_q := \left\{ \omega = (\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}, \quad \sup_j |\omega_j - j^2|\langle j\rangle^q < 1/2 \right\} \tag{1.6}$$

and, for $\gamma > 0$ we define the set of "good frequencies" as

$$\mathtt{D}_{\gamma,q} := \left\{ \omega \in \Omega_q \, : \, |\omega \cdot \ell| > \gamma \prod_{n\in\mathbb{Z}} \frac{1}{(1 + |\ell_n|^2 \langle n\rangle^{2+q})} . \quad \forall \ell \in \mathbb{Z}^{\mathbb{Z}} : |\ell| < \infty \right\}, \tag{1.7}$$

Note that $\mathtt{D}_{\gamma,q}$ is large with respect to a natural probability product measure on Ω_q (see e.g. [Bou05]).

Remark 1.1. From now on we shall fix $\gamma > 0$, $q \geq 0$ and assume that V in (1.1) is such that $|V|_q < 1/2$ and $\omega = (\omega_j)_{j\in\mathbb{Z}} \in \mathtt{D}_{\gamma,q}$ where $\omega_j = j^2 + V_j$.

We note that some non-resonance condition on the frequencies is inevitable if one wants to prove *long-time* stability, indeed if one takes $V = 0$ and $f(x, |u|^2) = |u|^4$ then one can exhibit orbits in which the Sobolev norm is unstable in times of order δ^{-4}, see [GT12], [HP17].

Sobolev initial data. In the case of Sobolev initial data it is fundamental to have a good control on the dependence of the stabiliy time T on the the regularity p. This means that results are very sensitive to which (of various equivalent) Sobolev norms one considers. Recalling that the L^2-norm is invariant for the equation 1.1, we will consider two cases:

- In the first case we deal with the usual norm $|u_0|_{L^2} + |\partial_x^p u_0|_{L^2}$, for $p > 1$. We denote this case as $\mathtt{S}$ (Sobolev case) and, by fixing $p = p(\delta)$, we prove *sub-exponential lower bound* for the stability time $T(\delta)$.

- In the second case, denoted by $\mathtt{M}$ (Modified-Sobolev case), we consider the equivalent norm $2^p|u_0|_{L^2} + |\partial_x^p u_0|_{L^2}$. In order to simplify the exposition and obtain better bounds, in this case we consider (1.1) with f independent of x (translation invariance). Again, fixing $p = p(\delta)$, we prove *exponential lower bound* on the stability time $T(\delta)$.

Of course, the norms in $\mathtt{S}$ and $\mathtt{M}$ are equivalent with constants depending on p. Note that when p depends on δ such constants become very important.
The main qualitative difference between $\mathtt{S}$ and $\mathtt{M}$ is that in the latter we are requiring that the Fourier modes $0, 1, -1$ of the initial datum have *very little energy*. Indeed, passing to the Fourier side $u_0(x) = \sum_{j\in\mathbb{Z}} u_{0,j} e^{\mathrm{i}jx}$, if both $|u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \le \delta/2$ and the extra condition $|u_{0,0}|^2 + |u_{0,1}|^2 + |u_{0,-1}|^2 \le \delta^2 2^{-2p-2}$ hold, then one has $2^p |u_0|_{L^2} \le \delta$.

Below we formally state our first result, which depends on some constants, denoted by $\tau_{\mathtt{S}}, \delta_{\mathtt{S}}, \mathtt{k}_{\mathtt{S}}, \mathtt{T}_{\mathtt{S}}, \mathtt{K}_{\mathtt{S}}, \tau_{\mathtt{M}}, \delta_{\mathtt{M}}, \mathtt{T}_{\mathtt{M}}$, which depend only on $\gamma, q, \mathtt{a}, R, |f|_{\mathtt{a},R}$ in the case $\mathtt{S}$ and on $\gamma, q, R, |f|_R$ in the case $\mathtt{M}$.

Theorem 1.1 (Sobolev stability)**.** *Consider equation* (1.1) *with f satisfying* (1.2) *for* $\mathtt{a}, R > 0$.
($\mathtt{S}$) *For any $p > 1$ such that $(p-1)/\tau_{\mathtt{S}} \in \mathbb{N}$ and any initial datum $u(0) = u_0$ satisfying*

$$|u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \le \delta \le \min\left\{ \delta_{\mathtt{S}} (\mathtt{k}_{\mathtt{S}} p)^{-3p} \,,\; \frac{\sqrt{R}}{20} \right\}, \tag{1.8}$$

the solution $u(t)$ of (1.1) *with initial datum $u(0) = u_0$ exists for all times*

$$|t| \le \frac{\mathtt{T}_{\mathtt{S}}}{\delta^2} (\mathtt{K}_{\mathtt{S}} p)^{-5p} \left(\frac{\delta_{\mathtt{S}}}{\delta} \right)^{\frac{2(p-1)}{\tau_{\mathtt{S}}}} \quad \text{and satisfies} \quad |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \le 4\delta\,. \tag{1.9}$$

($\mathtt{M}$) *Assume that f in* (1.1) *is independent of x. For any $p > 1$ such that $(p-1)/\tau_{\mathtt{M}} \in \mathbb{N}$ and for any initial datum $u(0) = u_0$ satisfying*

$$2^p |u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \le \delta \le \min\left\{ \frac{2\sqrt{\tau_{\mathtt{M}}}\delta_{\mathtt{M}}}{\sqrt{p}} \,,\; \frac{\sqrt{R}}{4\sqrt{10}} \right\}, \tag{1.10}$$

the solution $u(t)$ of (1.1) *exists for all times*

$$|t| \le \frac{\mathtt{T}_{\mathtt{M}}}{\delta^2} \left(\frac{4\tau_{\mathtt{M}} \delta_{\mathtt{M}}^2}{(p-1)\delta^2} \right)^{\frac{p-1}{\tau_{\mathtt{M}}}} \quad \text{and satisfies} \quad 2^p |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \le 4\delta\,. \tag{1.11}$$

Remark 1.2. Some remarks on the optimality of Theorem 1.1 are in order.

1. We stress the fact that estimates (1.8) of case $\mathtt{S}$ is optimal in some sense. The simplest way of showing this fact is to construct a Hamiltonian which does not preserve momentum and exhibits fast drift. In fact, if we take $\delta > (e^{-1} p)^{-p/2}$ then orbits performing "fast drift" in a time of order 1 may occur. Indeed consider e.g. , for $2 \le j \in \mathbb{N}$ the family of Hamiltonians:

$$H^{(j)}(u_1, u_j) := |u_1|^2 + (j^2 + V_j)|u_j|^2 + e^{-\mathtt{a}j} \operatorname{Re}(|u_1|^2 u_1 \bar{u}_j)\,.$$

Passing to action-angle variables $u_i = \sqrt{I_i} e^{\mathrm{i}\vartheta_i}$ we get the new Hamiltonian

$$I_1 + \omega I_j + e^{-\mathtt{a}j} I_1^{3/2} \sqrt{I_j} \cos(\vartheta_1 - \vartheta_j) = J_1 + \omega(J_2 - J_1) + e^{-\mathtt{a}j} J_1^{3/2} \sqrt{J_2 - J_1} \cos\varphi_1$$

in the new symplectic variables $J_1 = I_1$, $J_2 = I_1 + I_j$, $\varphi_1 = \vartheta_1 - \vartheta_j$, $\varphi_2 = \vartheta_j$. Note that this Hamiltonian has J_2 as constant of motion while

$$\dot{J}_1 = e^{-\mathtt{a}j} J_1^{3/2} \sqrt{J_2 - J_1} \sin \varphi_1 .$$

In this case the norm in (1.12) reads

$$\sqrt{|u_1|^2 + |u_j|^2} + \sqrt{|u_1|^2 + j^{2p}|u_j|^2} = \sqrt{J_2} + \sqrt{(1 - j^{2p})J_1 + j^{2p} J_2} .$$

Taking the initial datum $u(0) = (u_1(0), u_j(0))$ with $u_1(0) = \delta/4$, $u_j(0) = j^{-p}\delta/4$, we have that its norm is smaller than δ, while J_1 can have a drift of order $\delta^4 j^{-p} e^{-\mathtt{a}j}$ in a time T of order 1. This means that the Sobolev norm of $u(T)$ is of order $\delta^3 e^{-\mathtt{a}j} j^p$ hence greater than 4δ if $\delta^2 e^{-\mathtt{a}j} j^p$ is large. Maximizing on j we get a constraint of the form $\delta^2 e^{-p} (\mathtt{a}^{-1} p)^p < 1$.
Of course this pathological "fast diffusion" phenomenon comes from the non conservation of momentum[3], and would appear (with similar constants) also in the case $\mathtt{M}$.

2. It is very important to stress that in the case $\mathtt{S}$ restricting to translation invariant Hamiltonians would not result in signficantly weaker constraints on the smallness of δ w.r.t. p. This can be seen in the following example. Consider the familiy of Hamiltonians (in three degrees of freedom)

$$K^{(j)} := |u_1|^2 + j^2 |u_j|^2 + \mathrm{Re}(\bar{u}_0^{j-1} u_1^j \bar{u}_j)$$

with the constants of motion

$$L = |u_0|^2 + |u_1|^2 + |u_j|^2 , \quad M = |u_1|^2 + j|u_j|^2 .$$

Following the same approach as in the previous example one shows that $|u_j|^2$ can have a drift of order $j^{-p}\delta^{2j}$ in a time T of order 1. This means that the Sobolev norm of $u(T)$ is of order $\delta^{2j} j^p$. Maximizing on j we get a constraint of the form $\delta e^{p^{1^-}} < 1$. We point out that the Hamiltonian discussed above is stable in the $\mathtt{M}$ norm for all times and for δ small independent of p. This is the main reason for restricting in $\mathtt{M}$ to translation invariant Hamiltonians.

From Theorem 1.1 it is straightforward to maximize over p and find an optimal regularity. We stress that in the case $\mathtt{S}$ our estimate on the stability time is an increasing function of p, so the maximum is obtained by just fixing p so that $\delta = (\mathtt{C}_\mathtt{S} p)^{-3p}$. On the other hand in the case $\mathtt{M}$ there is a proper maximum.
We thus have the following result. As before our statements depend on some constants, denoted by $\bar{\delta}_\mathtt{S}, \bar{\delta}_\mathtt{M}$, which depend only on $\gamma, q, \mathtt{a}, R, |f|_{\mathtt{a},R}$ in the case $\mathtt{S}$ and on $q, R, |f|_R$ in the case $\mathtt{M}$. By $[\cdot]$ we denote the integer part.

[3]indeed the term $e^{-\mathtt{a}j}$ is added in order to ensure that monomials with very high momentum give an exponenially small contribution to the Hamiltonian

Theorem 1.2 (Sobolev stability: optimization)**.**
($\mathtt{S}$) *For any* $0 < \delta \le \bar\delta_{\mathtt{S}}$ *and any* u_0 *such that*

$$|u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \le \delta\,, \qquad\qquad p = p(\delta) := 1 + \tau_{\mathtt{S}}\left[\frac{1}{6\tau_{\mathtt{S}}}\frac{\ln(\delta_{\mathtt{S}}/\delta)}{\ln\ln(\delta_{\mathtt{S}}/\delta)}\right]\,, \tag{1.12}$$

the solution $u(t)$ *of* (1.1) *with initial datum* $u(0) = u_0$ *exists for all times*

$$|t| \le \frac{\mathtt{T}_{\mathtt{S}}}{\delta^2} e^{\frac{\ln^2(\delta_{\mathtt{S}}/\delta)}{4\tau_{\mathtt{S}}\ln\ln(\delta_{\mathtt{S}}/\delta)}} \quad \textit{and satisfies} \quad |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \le 4\delta\,. \tag{1.13}$$

($\mathtt{M}$) *Assume that* f *in* (1.1) *is independent of* x*. For any* $0 < \delta \le \bar\delta_{\mathtt{M}}$ *and*

$$\forall\, p \ge p(\delta) := 1 + \tau_{\mathtt{M}}\left[\frac{\delta_{\mathtt{M}}^2}{\delta^2}\right]\,, \quad \forall u_0 \quad \textit{s.t.} \quad 2^p |u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \le \delta\,, \tag{1.14}$$

the solution $u(t)$ *of* (1.1) *with initial datum* $u(0) = u_0$ *exists for all times*

$$|t| \le \frac{\mathtt{T}_{\mathtt{M}}}{\delta^2} e^{(\delta_{\mathtt{M}}/\delta)^2} \quad \textit{and satisfies} \quad 2^p |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \le 4\delta\,. \tag{1.15}$$

Remark 1.3. Some remarks on Theorem 1.2 are in order.

Note that (1.13) is the stability time computed in [BFG88] for short range couplings.

1. In our study we have only considered *Gauge preserving* equations, that is PDEs which preserve the L^2 norm. We believe that this is just a technical question and that we could deal with more general cases. Similarly in the case $\mathtt{M}$ we have assumed that f in (1.1) is independent of x, namely *momentum preserving.* Not only this simplifies the proof but as explained after Theorem 1.1 allows us much better estimates. Of course we could prove the theorem (with different constants) also for x-dependent f, as in the case $\mathtt{S}$.

2. We will prove the case $\mathtt{M}$ only for $p = p(\delta)$, the general case being analogous[4] (with the same constants!) also if $p \ge p(\delta)$.

3. One can easily restate Theorem 1.2 in terms of the Sobolev exponent p, instead of δ, since the map $\delta \to p(\delta)$ is injective.

In this paper we have considered the simplest possible example of dispersive PDE on the circle. One can easily see that the same strategy can be followed word by word in more general cases provided that the non-linearity does not contain derivatives. A much more challenging question is to consider NLS models with derivatives in the non-linearity. As we have mentioned a semilinear case was discussed by [CMW]. A very promising approach to Birkhoff normal form for quasilinear PDEs is the one of [Del12]- [BD18] which was applied to fully-nonlinear

[4]Indeed, thanks to the monotonicity property of our norms the canonical transformation putting the system in Birkhoff Normal Form (see Theorem 1.4 below) in the p-case is simply the restriction to H^p of the one of the $p(\delta)$-case.

reversible NLS equations in [FI]. It seems very plausible that one can adapt their methods (based on paralinearizations and paradifferential calculus) to our setting, however it seems that in this case one must give up the Hamiltonian structure.

Analytic and Gevrey initial data
In this case our result is similar to [FG13] in the sense that we also prove *subexponential bounds* on the time. We mention however that in [FG13] the control of the Sobloev norm in time is in a lower regularity space w.r.t. the initial datum. Recently we have been made aware of a preprint by Cong, Mi and Wang [CMW] in which the authors give subexponential bounds for subanalytic initial data of a model like (1.1), very similar to ours. A difference is that in their case the non linearity contains a derivative (see the comments after Theorem 1.3) but satisfies momentum conservation.
Let us fix $0 < \theta < 1$, and define the function spaces

$$\mathtt{H}_{p,s,a} := \left\{ u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \in L^2 \ : \ |u|^2_{p,s,a} := \sum_{j\in\mathbb{Z}} |u_j|^2 \langle j\rangle^{2p} e^{2a|j|+2s\langle j\rangle^\theta} < \infty \right\}. \tag{1.16}$$

with the assumption $a \geq 0, s > 0, p > 1/2$. We remark that if $a > 0$ this is a space of analytic functions, while if $a = 0$ the functions have Gevrey regularity. Note that for technical reasons connected to the way in which we control the small divisors, we *cannot* deal with the purely analytic case $\theta = 1$. For this reason we denote this result as $\mathtt{G}$ (Gevrey case). The main important difference with the cases $\mathtt{S}, \mathtt{M}$ is that now the regularity p, s, a is *independent* of δ.
As before our result, stated below, depends on some constants $\bar{\delta}_{\mathtt{G}}, \delta_{\mathtt{G}}, \mathtt{T}_{\mathtt{G}}$, depending only on $\gamma, q, \mathtt{a}, R, |f|_{\mathtt{a},R}, p, s, a, \theta$.

Theorem 1.3 (Gevrey Stability)**.** *Fix any $a \geq 0$, $s > 0$ such that $a + s < \mathtt{a}$ and any $p > 1/2$. For any $0 < \delta \leq \bar{\delta}_{\mathtt{G}}$ and any u_0 such that*

$$|u_0|_{p,s,a} \leq \delta \,,$$

the solution $u(t)$ of (1.1) with initial datum $u(0) = u_0$ exists for all times

$$|t| \leq \frac{\mathtt{T}_{\mathtt{G}}}{\delta^2} e^{\left(\ln \frac{\delta_{\mathtt{G}}}{\delta}\right)^{1+\theta/4}} \quad \text{and satisfies} \quad |u(t)|_{p,s,a} \leq 2\delta \,.$$

Remark 1.4. Some comments on Theorem 1.3 are in order.

1. We did not make an effort to maximize the exponent $1+\theta/4$ in the stability time. In fact, by trivially modifying the proof, one could get $1+\theta/(2^+)$. We remark that in [CMW], in which $\theta = 1/2$, the exponent is better, i.e. it is $1 + 1/(2^+)$.

2. As we mentioned before, the main difference w.r.t. the cases $\mathtt{S}, \mathtt{M}$ is that now the regularity p, s, a is independent of δ, with the only requirement that $p > 1/2$ and $s > 0$. If instead we took s appropriately large with δ we would get an exponential bound just like in case $\mathtt{M}$.

3. One could consider initial data with an *intermediate* regularity between Sobolev and Gevrey and compute stability times. A good example (suggested to us by Z. Hani) could be the space

$$\mathtt{H}_c := \left\{ u = \sum_j u_j e^{\mathrm{i}jx} \in L^2 : \quad \sum_j |u_j|^2 e^{c(\ln(\lfloor j \rfloor)^2)} < \infty \right\}$$

where $c > 0$ and $\lfloor j \rfloor := \max\{|j|, 2\}$. Following the proof of Theorem 1.3 almost verbatim one can get an estimate of the type $T \geq C\delta^{-3+\ln(\ln(1/\delta))}$.

1.2 The Birkhoff Normal Form

Our results are based on a Birkhoff normal form procedure, which we now describe. Let us pass to the Fourier side via the identification

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \;\mapsto\; u = (u_j)_{j\in\mathbb{Z}}\,, \tag{1.17}$$

where u belongs to some complete subspace of ℓ^2. More precisely, given a real sequence $\mathtt{w} = (\mathtt{w}_j)_{j\in\mathbb{Z}}$, with $\mathtt{w}_j \geq 1$ we consider the Hilbert space[5]

$$\mathtt{h}_\mathtt{w} := \left\{ u := (u_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{C}) : \quad |u|^2_\mathtt{w} := \sum_{j\in\mathbb{Z}} \mathtt{w}_j^2 |u_j|^2 < \infty \right\}, \tag{1.18}$$

and fix the symplectic structure to be

$$\mathrm{i}\sum_j du_j \wedge d\bar{u}_j\,. \tag{1.19}$$

In this framework the Hamiltonian of (1.1) is

$$\begin{aligned} &H_{\mathrm{NLS}}(u) := D_\omega + P\,, \qquad \text{where} \\ &D_\omega := \sum_{j\in\mathbb{Z}} \omega_j |u_j|^2\,, \quad P := \int_\mathbb{T} F(x, |u(x)|^2)dx\,, \quad F(x,y) := \int_0^y f(x,s)ds\,. \end{aligned} \tag{1.20}$$

As examples of $\mathtt{h}_\mathtt{w}$ we consider:

S) (Sobolev case) $\mathtt{w}_j = \langle j \rangle^p$, which is isometrically isomorphic, by Fourier transform, to $\mathtt{H}_{p,0,0}$ defined in (1.16) and is equivalent to H^p equipped with the norm $|\cdot|_{L^2} + |\partial_x^p \cdot|_{L^2}$ with equivalence constants independent of p

M) (Modified-Sobolev case) $\mathtt{w}_j = \lfloor j \rfloor^p$, where $\lfloor j \rfloor := \max\{|j|, 2\}$; this space is equivalent to H^p equipped with the norm $2^p|\cdot|_{L^2} + |\partial_x^p \cdot|_{L^2}$ with equivalence constants independent of p

[5] Endowed with the scalar product $(u,v)_{\mathtt{h}_\mathtt{w}} := \sum_{j\in\mathbb{Z}} \mathtt{w}_j^2 u_j \bar{v}_j$.

G) (Gevrey case) $\mathtt{w}_j = \langle j\rangle^p e^{a|j|+s\langle j\rangle^\theta}$, which is isometrically isomorphic, by Fourier transform, to $\mathtt{H}_{p,s,a}$ defined in (1.16).

Here and in the following, given $r>0$, by $B_r(\mathtt{h}_\mathtt{w})$ we mean the closed ball of radius r centered at the origin of $\mathtt{h}_\mathtt{w}$.

Definition 1.2 (majorant analytic Hamiltonians)**.** For $r>0$, let $\mathcal{A}_r(\mathtt{h}_\mathtt{w})$ be the space of Hamiltonians

$$H : B_r(\mathtt{h}_\mathtt{w}) \to \mathbb{R}$$

such that there exists a pointwise absolutely convergent power series expansion[6]

$$H(u) = \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^{\mathbb{Z}}, \\ |\boldsymbol{\alpha}|+|\boldsymbol{\beta}|<\infty}} H_{\boldsymbol{\alpha},\boldsymbol{\beta}} u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}, \qquad u^{\boldsymbol{\alpha}} := \prod_{j\in\mathbb{Z}} u_j^{\boldsymbol{\alpha}_j}$$

with the following properties:

(i) Reality condition:
$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \overline{H}_{\boldsymbol{\beta},\boldsymbol{\alpha}}; \tag{1.21}$$

(ii) Mass conservation:
$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{if } |\boldsymbol{\alpha}| \neq |\boldsymbol{\beta}|, \tag{1.22}$$
namely the Hamiltonian Poisson commutes with the *mass* $\sum_{j\in\mathbb{Z}} |u_j|^2$;

Finally, given H as above, we define its majorant $\underline{H} : B_r(\mathtt{h}_\mathtt{w}) \to \mathbb{R}$ as

$$\underline{H}(u) = \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^{\mathbb{Z}}, \\ |\boldsymbol{\alpha}|+|\boldsymbol{\beta}|<\infty}} |H_{\boldsymbol{\alpha},\boldsymbol{\beta}}| u^{\boldsymbol{\alpha}} \bar{u}^{\boldsymbol{\beta}}. \tag{1.23}$$

We also define the subspace of normal form Hamiltonians

$$\mathcal{K} := \left\{ Z \in \mathcal{A}_r(\mathtt{h}_\mathtt{w}) \, : \, Z(u) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^{\mathbb{Z}}} Z_{\boldsymbol{\alpha},\boldsymbol{\alpha}} |u|^{2\boldsymbol{\alpha}} \right\}. \tag{1.24}$$

Note that $Z_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathbb{R}$ for every $\boldsymbol{\alpha} \in \mathbb{N}^{\mathbb{Z}}$ by condition (1.21).

In the following we will also deal with a smaller class of Hamiltonians, namely the ones which have the *momentum* $\sum_{j\in} j|u_j|^2$ as additional first integral.

Definition 1.3. We say that a Hamiltonian $H \in \mathcal{A}_r(\mathtt{h}_\mathtt{w})$ preserves momentum when

$$H_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{if} \quad \sum_{j\in\mathbb{Z}} j(\alpha_j - \beta_j) \neq 0,$$

namely the Hamiltonian H Poisson commutes with $\sum_{j\in} j|u_j|^2$.

[6]As usual given a vector $k \in \mathbb{Z}^{\mathbb{Z}}$, $|k| := \sum_{j\in\mathbb{Z}} |k_j|$.

Note that if the nonlinearity f in equation (1.1) does not depend on the variable x, then the Hamiltonian P in (1.20) preserves momentum.

We now state a Birkhoff Normal Form Theorem for the Hamiltonian in (1.20). Fix any $\mathtt{N} \geq 1$ and consider the space $\mathtt{h_w}$ where $\mathtt{w}$ is one of the following three cases:

S) (Sobolev case) $\mathtt{w}_j = \langle j \rangle^{1+\tau_{\mathtt{S}}\mathtt{N}}$;

M) (Modified-Sobolev case) $\mathtt{w}_j = \lfloor j \rfloor^{1+\tau_{\mathtt{M}}\mathtt{N}}$, where $\lfloor j \rfloor := \max\{|j|, 2\}$;

G) (Gevrey case) $\mathtt{w}_j = e^{a|j|+s\langle j \rangle^\theta} \langle j \rangle^p$ with $p > 1/2, s > 0, 0 \leq a < \mathtt{a}$.

The constants $\mathtt{r}, \mathtt{C}_1, \mathtt{C}_2, \mathtt{C}_3$, below, corresponding to the cases $\mathtt{S}, \mathtt{M}, \mathtt{G}$ respectively, depend on $\mathtt{N} \geq 1$.

Theorem 1.4 (Birkhoff Normal Form). *Fix any* $\mathtt{N} \geq 1$ *and consider the space* $\mathtt{h_w}$ *where* $\mathtt{w}$ *is one of the three above cases:* $\mathtt{S}, \mathtt{M}, \mathtt{G}$. *Consider the Hamiltonian* (1.20), *assuming, only in the case* $\mathtt{M}$, *that* f *does not depend on* x *(momentum conservation). Then for any* $0 < r \leq \mathtt{r}$ *there exists two close to identity invertible symplectic change of variables*

$$\Psi, \Psi^{-1} : \quad B_r(\mathtt{h_w}) \mapsto \mathtt{h_w}, \quad \sup_{|u|_{\mathtt{w}} \leq r} |\Psi^{\pm 1}(u) - u|_{\mathtt{w}} \leq \mathtt{C}_1 r^3 \leq \tfrac{1}{8} r,$$
$$\Psi \circ \Psi^{-1} u = \Psi^{-1} \circ \Psi u = u, \quad \forall u \in B_{\frac{7}{8}r}(\mathtt{h_w}) \tag{1.25}$$

such that in the new coordinates

$$H \circ \Psi = D_\omega + Z + R,$$

for suitable majorant analytic Hamiltonians $Z, R \in \mathcal{A}_r(\mathtt{h_w})$, $Z \in \mathcal{K}$, *satisfying the estimate*

$$\sup_{|u|_{\mathtt{w}} \leq r} |X_{\underline{Z}}|_{\mathtt{w}} \leq \mathtt{C}_2 r^3, \quad \sup_{|u|_{\mathtt{w}} \leq r} |X_{\underline{R}}|_{\mathtt{w}} \leq \mathtt{C}_3 r^{2\mathtt{N}+3}, \tag{1.26}$$

$X_{\underline{Z}}$ *(resp.* $X_{\underline{R}}$*), being the hamiltonian vector field generated by the the majorant of* Z *(resp.* R*). Moreover, in the case* $\mathtt{M}$, R *preserves momentum.*

The proof of our Birkhoff normal form result (contained in [BMP18]) is based on a procedure which, while following the line of previous works such as [BG06] and [FG13], it takes a slightly different point of view. Broadly speaking the core is the following: as already noticed in [FG13] small divisor estimates and hence stability are simpler to prove for traslation invariant PDEs (i.e. Hamiltonian systems which preserve the momentum). Considering this fact we introduce in [BMP18] an appropriate norm, which weights non-momentum preserving monomial exponentially. This norm is rather cumbersome and depends on many parameters but we show that it is very well suited for performing Birkhoff normal form steps for dispersive PDEs on the circle. This rather simple idea, allows us a very good control of the small divisors by generalizing the estimates by Bourgain in [Bou05]. As a byproduct our normal forms are *simpler*, in the sense that they are functions only of the linear actions, and it is relatively easy to compute all the constants.
Above we stated Theorem 1.4 only in the cases $\mathtt{S}, \mathtt{M}, \mathtt{G}$, but our method is quite versatile and one can formulate a Birkhoff Norma Form result in the general contest of weighted Hilbert spaces.

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