Infinitely many cyclic solutions to the Hamilton-Waterloo problem with odd length cycles *

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Abstract

It is conjectured that for every pair (ℓ, m) of odd integers greater than 2 with $m \equiv 1 \pmod{\ell}$, there exists a cyclic two-factorization of $K_{\ell m}$ having exactly $(m-1)/2$ factors of type ℓ^m and all the others of type m^ℓ. The authors prove the conjecture in the affirmative when $\ell \equiv 1 \pmod 4$ and $m \geq \ell^2 - \ell + 1$.

Keywords: two-factorization; Hamilton-Waterloo problem; Skolem sequence; group action.

1 Introduction

A 2-*factorization* of order v is a set $\mathcal{F}$ of spanning 2–regular subgraphs of K_v (the *complete graph* of order v) whose edges partition the edge set of K_v or $K_v - I$ (the complete graph minus a 1-factor I) according to whether v is odd or even. We refer the reader to [34] for the standard terminology and notation of elementary graph theory.

Note that every spanning 2-regular subgraph F of K_v determines a partition $\pi = [\ell_1^{n_1}, \ell_2^{n_2}, \ldots, \ell_t^{n_t}]$ of the integer v where $\ell_1, \ell_2, \ldots, \ell_t$ are the distinct lengths of the cycles of F and n_i is the number of cycles in F of length ℓ_i (briefly, ℓ_i-cycles). We will refer to F as a 2–*factor of* K_v of *type* π. Of course, 2–factors of the same type are pairwise isomorphic, and viceversa.

In this paper we deal with the well-known Hamilton-Waterloo problem (in short, HWP) which can be formulated as follows: given two non-isomorphic 2-factors F, F' of K_v, and two positive integers r, r' summing up to $\lfloor (v-1)/2 \rfloor$, then HWP asks for a 2-factorization of order v consisting of

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r copies of F and r' copies of F'. Denoted by π and π' the types of F and F', respectively, this problem will be denoted by $HWP(v;\pi,\pi';r,r')$.

The case where F and F' are 2-factors of the same type π is known as the Oberwolfach problem, $OP(v;\pi)$. This has been formulated much earlier by Ringel in 1967 for v odd, while the case v even was later considered in [26]. Apart from OP$(6;[3^2])$, OP$(9;[4,5])$, OP$(11;[3^2,5])$,OP$(12,[3^4])$, none of which has a solution, the problem is conjectured to be always solvable: evidence supporting this conjecture can be found in [8]. We only mention some of the most important results on the Oberwolfach problem recently achieved: OP$(v;\pi)$ is solvable for an infinite set of primes $v \equiv 1 \pmod{96}$ [9], when π has exactly two terms [32], and when every term of π is even [7, 24]. However, although the literature is rich in solutions of many infinite classes of OP$(v;\pi)$, these only solve a small fraction of the problem which still remains open.

As one would expect, there is much less literature on the Hamilton-Waterloo problem. Apart from some non-solvable instances of small order [8], the Hamilton-Waterloo problem HWP$(v;\pi,\pi';r,r')$ is known to have a solution when v is odd and ≤ 17 [2, 21, 22], and when v is even and ≤ 10 [2, 4]. When all terms of π and π' are even, with $r, r' > 1$, a complete solution has been given in [7]. Surprisingly, very little is known when π or π' contain odd terms, even when all terms of π and all terms of π' coincide. For example, HWP$(v;[4^{v/4}],[\ell^{v/\ell}];r,r')$ has been dealt with in [28, 29] and completely solved only when $\ell = 3$ in [5, 17, 33], while HWP$(v;[3^{v/3}],[v];\lfloor v/2\rfloor - 1, 1)$ is still open (see, [19, 20, 25]). Other results can be found in [3, 13]. In this paper we make significant headway with the most challenging case of the Hamilton-Waterloo problem, that is, the one in which the two partitions contain only odd terms. Further progress on this case has been recently made in [16].

An effective method to determine a 2-factorization $\mathcal{F}$ solving a given Oberwolfach or Hamilton-Waterloo problem is to require that $\mathcal{F}$ has a suitable *automorphism group* G, that is, a group of permutation on the vertices which leaves $\mathcal{F}$ invariant. One usually requires that G fixes k vertices and has $r \geq 1$ regular orbits on the remaining vertices. If $k = 1$, then $\mathcal{F}$ is said to be *r-rotational* (under G) [14]; if $r = 1$ and $k \geq 1$, then $\mathcal{F}$ is k-pyramidal [6] – note that 1-pyramidal means 1-rotational. Finally, $\mathcal{F}$ is *sharply transitive* or *regular* (under G) if $(k,r) = (0,1)$.

The 1-rotational approach and the 2-pyramidal one have proved to be successful [14, 15, 32] in solving infinitely many cases of the Oberwolfach problem. On the other hand, all solutions of small order given in [18] turn out to be r-rotational for some suitable small r.

A 2-factorization $\mathcal{F}$ of order v is regular under G if we can label the vertices with the elements of G so that for any 2–factor $F \in \mathcal{F}$ we have that $F + g$ is also in $\mathcal{F}$. One usually speaks of a *cyclic* 2-factorization when G is the cyclic group. The few known facts on regular solutions to the Oberwolfach problem only concern cyclic groups [11, 12, 27], elementary abelian groups and Frobenius groups [13]. Concerning the Hamilton-Waterloo problem, there are only two known infinite classes of solutions having a cyclic automorphism group; specifically, there exists a cyclic solution to HWP$(18n+3; [3^{6n+1}], [(6n+1)^3]; 3n, 6n+1)$ [13] and HWP$(50n+5; [5^{10n+1}], [(10n+1)^5]; 5n, 20n+2)$ [10] for any $n \geq 1$. These are instances of the following much more general problem.

Problem 1.1. *Given $\ell \geq 3$ odd and given $n > 0$, establish if there exists a cyclic solution to HWP$(\ell(2\ell n+1); [\ell^{2\ell n+1}], [(2\ell n+1)^\ell]; \ell n, \frac{(\ell-1)(2\ell n+1)}{2})$.*

The above mentioned solutions settle the problem for $\ell = 3$ and $\ell = 5$.

In this paper, we build on the techniques of [10] and consider the case $\ell = 4k+1$; we manage to solve the problem for *all* ℓ and *all* $n \geq (\ell-1)/2$ by giving always "cyclic" solutions. Our main result is the following.

Theorem 1.2. *If $\ell \equiv 1 \pmod 4$, then Problem 1.1 admits a solution provided that $n \geq (\ell-1)/2$.*

We believe that techniques similar to those developed in this paper may be used to settle the case $\ell \equiv 3 \pmod 4$. However, in view of the complexity of this article, we postpone the study of this case to a future work.

We finally point out that regular solutions under non-cyclic groups can be found in [13]: for example, given a positive integer j and two odd primes ℓ, m such that $m^j \equiv 1 \pmod \ell$, there exists a regular solution to HWP$(\ell m^j; [\ell^{(m^j)}], [m^{\ell m^{j-1}}]; (\ell-1)m^j/2, (m^j-1)/2)$ under a Frobenius group of order ℓm^j.

2 Some preliminaries

The techniques used in this paper are based on those in the papers by Buratti and Rinaldi [13] and Buratti and Danziger [10]; we collect here some preliminary notation and definitions, more details can be found in these references.

In what follows, we shall label the vertices of the complete graph K_v with the elements of $\mathbb{Z}_v$; if Γ is a subgraph of K_v, we define the list of differences of Γ to be the multiset $\Delta\Gamma$ of all possible differences $\pm(a-b)$ between a pair (a,b) of adjacent vertices of Γ. More generally, the list of differences of a collection $\{\Gamma_1, \ldots, \Gamma_t\}$ of subgraphs of Γ is the multiset union of the lists of differences of all the Γ_is.

A cycle C of length d in K_v is called *transversal* if $d|v$ and the vertices of C form a complete system of representatives for the residue classes modulo d, namely a transversal for the cosets of the subgroup of $\mathbb{Z}_v$ of index d.

A fundamental result we shall use in our construction is the following theorem, a consequence of a more general result proved in [13] (see also [10]).

Theorem 2.1. *There exists a cyclic solution to HWP($\ell m; [\ell^m], [m^\ell]; r, r'$) if and only if the following conditions hold:*

- $(r, r') = (\ell x, my)$ *for a suitable pair* (x, y) *of positive integers such that* $2\ell x + 2my = \ell m - 1$;
- *there exist* x *transversal* ℓ*-cycles* $\{A_1, \ldots, A_x\}$ *of* $K_{\ell m}$ *and* y *transversal* m*-cycles* $\{B_1, \ldots, B_y\}$ *of* $K_{\ell m}$ *whose lists of differences cover, altogether,* $\mathbb{Z}_{\ell m} \setminus \{0\}$ *exactly once.*

The reason for looking for a cyclic solution to the HWP with the particular parameter set of Problem 1.1 stems from the fact that for ℓ an odd integer and $m = 2\ell n+1$ with n a positive integer, taking $x = n$ and $y = \frac{\ell-1}{2}$ will give $2\ell x + 2my = \ell m - 1$; thus Theorem 2.1 ensures that the problem will be solved if we construct n transversal ℓ-cycles of $K_{\ell(2\ell n+1)}$ and $(\ell - 1)/2$ transversal $(2\ell n + 1)$-cycles of $K_{\ell(2\ell n+1)}$ whose list of differences cover $\mathbb{Z}_{\ell(2\ell n+1)} \setminus \{0\}$ exactly once.

For the sake of brevity, throughout the paper we will refer to an ℓ–cycle or a $(2\ell n+1)$–cycle as a *short* cycle or a *long* cycle, respectively. Also, using the Chinese remainder theorem, we will identify $\mathbb{Z}_{\ell(2\ell n+1)}$ with $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$. We point out to the reader that, after such an identification, a long (short) cycle C of $K_{\ell(2\ell n+1)}$ is transversal if and only if the first (second) components of the vertices of C are all distinct.

By $(c_0, c_1, \ldots, c_\ell)$ we will denote both the path of length ℓ with edges $(c_0, c_1), (c_1, c_2), \ldots, (c_{\ell-1}, c_\ell)$ and the cycle that we get from it by joining c_0 and c_ℓ. To avoid misunderstandings, we will always specify whether we are dealing with a path or a cycle. Finally, given two integers $a \le b$, we will denote by $[a, b]$ the *interval* containing the integers $a, a + 1, a + 2, \ldots, b$. Of course, if $a < b$, then $[b, a]$ will be the empty set.

3 The general construction

In this section we shall sketch the steps needed to prove the main result of this paper, Theorem 1.2. Let us start with some notation.

From now on ℓ will be an odd positive integers with $\ell \equiv 1 \pmod 4$. We will usually write $\ell = 4k + 1$: we shall assume that $k > 1$, since the case $k = 1$, that is $\ell = 5$, has been covered in [10]. For n a positive integer, we denote by $\mathbb{Z}^*_{2\ell n+1}$ and $\mathbb{Z}^*_\ell$ the set of nonzero-elements of $\mathbb{Z}_{2\ell n+1}$ and $\mathbb{Z}_\ell$, respectively. Also, $\mathbb{Z}^-_{2\ell n+1}$ will denote the set $\mathbb{Z}^*_{2\ell n+1} \setminus \{\pm 1, \pm \ell n\}$.

We point out to the reader that given a subset S of $\mathbb{Z}_m$ $(m > 1)$ and a subset T of $\mathbb{Z}$, we write $S = T$ whenever T is a complete set of representatives for the residue classes $\pmod m$ in S. For example, $\mathbb{Z}_m = [0, m - 1]$ or $\mathbb{Z}_m = [-\frac{m-1}{2}, \frac{m-1}{2}]$ when m is odd.

To explain the construction used in what follows we first need to define a specific set D of $2k-2$ positive integers which will play a crucial role in building the cycles we need.

Definition 3.1. For $\ell = 4k+1$ and for an integer $n > 0$, we denote by $D = \{d_{i1}, d_{i2} | i = 1, \ldots, k-1\}$ the set of positive integers defined as follows:

$$d_{i1} = \begin{cases} 4i-2 & \text{if } n \text{ is odd,} \\ 4i & \text{if } n \text{ is even,} \end{cases} \quad \text{and} \quad d_{i2} = 4i+1.$$

For the case $n - 2k \equiv 2, 3 \pmod 4$, we replace the last pair $(d_{k-1,1}, d_{k-1,2})$ with

$$d_{k-1,1} = (4k+1)n - 2k - 1 \quad \text{and} \quad d_{k-1,2} = (4k+1)n - 2k + 3.$$

Also, we set $\overline{D} = [2, \ell n - 1] \setminus D$.

Example 3.2. All through this work, we shall use our construction to build explicitly a solution for $\ell = 9$ and $n = 5$; in this case we have $D = \{2, 5\}$ and $\overline{D} = \{3, 4\} \cup [6, 44]$.

Let us now look at how to prove Theorem 1.2; we need to produce a set $\mathcal{B}$ of base cycles as required in Theorem 2.1, so we will build $(\ell-1)/2$ transversal long cycles and n transversal short cycles whose list of differences will cover $\mathbb{Z}_{\ell(2\ell n+1)} \setminus \{0\}$ exactly once.

First, in Section 4 we shall construct a set $\mathcal{L}$ consisting of $(\ell-5)/2$, that is all but two, of the transversal long cycles of length $2\ell n + 1$. The list of differences provided by $\mathcal{L}$ will be

$$\Delta\mathcal{L} = \ (\mathbb{Z}^*_{2\ell n+1} \times (\mathbb{Z}_\ell \setminus \{0, \pm 1, \pm 2\})) \ \cup \ \{(d, f(d)) \mid d \in D \ \cup \ -D\} \quad (3.1)$$

with $f(d) = \pm 1$ for any $d \in D \ \cup \ -D$. We point out that the differences not covered by $\mathcal{L}$ are chosen in such a way as to facilitate the construction of the transversal short cycles.

Then in Section 5 we shall construct a set $\mathcal{S}$ of n transversal short cycles such that

$$\Delta\mathcal{S} = (\{0\} \times \mathbb{Z}^*_\ell) \ \cup \ \{(x, \varphi(x)) \mid x \in Z^-_{2\ell n+1} \setminus (D \ \cup \ -D)\} \quad (3.2)$$

where $\varphi : \mathbb{Z}^-_{2\ell n+1} \setminus (D \ \cup \ -D) \to \{\pm 1, \pm 2\}$ is a map with some additional properties. More precisely,

1. we construct (Lemma 5.4) a set $\mathcal{A} = \{A_1, \ldots, A_{n-2k}\}$ of ℓ-cycles and a set $\mathcal{B} = \{B_1, \ldots, B_{2k}\}$ of $(\ell-1)$-cycles with vertices in $\mathbb{Z}_{2\ell n+1}$ such that $\Delta\mathcal{A} \ \cup \ \Delta\mathcal{B} = \mathbb{Z}^-_{2\ell n+1} \setminus (D \ \cup \ -D)$. The construction of the set $\mathcal{A}$ requires the crucial use of Skolem sequences. Also, the cycles in $\mathcal{B} \setminus \{B_1\}$ will have a particular structure called alternating (see Definition 5.1).

2. we "lift" (Proposition 5.11) the cycles in $\mathcal{A} \cup \mathcal{B}$ to obtain a set $\mathcal{S}$ of transversal ℓ-cycles with vertices in $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$; to lift these cycles to $K_{\ell(2\ell n+1)}$ means first to add a vertex to the cycles in $\mathcal{B}$ (so that they become ℓ-cycles) and then to add a second coordinate to the vertices of each cycle, in such a way that $\Delta\mathcal{S}$ satisfies (3.2).

Let us point out that in Proposition 5.11 we construct a set of transversal short cycles whose list of differences covers, in particular, $\{0\} \times \mathbb{Z}^*_{4k+1}$. More precisely, the differences of the form $\pm(0,1), \pm(0,2), \ldots, \pm(0,2k)$ are covered by $2k$ distinct transversal short cycles. Thus our construction requires at least $2k$ transversal short cycles, and this is the reason why in Theorem 1.2 we require $n \geq 2k = (\ell-1)/2$.

To complete the set of base cycles, we need to provide two more transversal long cycles C and C'. The construction of these two missing $(2\ell n+1)$-cycles is described in Section 6; clearly C and C' are built in so as to cover the set of the remaining differences, that is, the union of the following disjoint sets:

$$\mathbb{Z}^*_{2\ell n+1} \times \{0\}; \quad \{\pm 1, \pm \ell n\} \times \{\pm 1, \pm 2\}; \quad \bigcup_{i \in \mathbb{Z}^-_{2\ell n+1}} \{i\} \times (\{\pm 1, \pm 2\} \setminus \{F(i)\}).$$

where $F : \mathbb{Z}^-_{2\ell n+1} \to \{\pm 1, \pm 2\}$ is the map obtained by glueing together the maps f and φ, that is, $F(x) = f(x)$ or $F(x) = \varphi(x)$ according to whether $x \in D \cup -D$ or not. For example, to cover the differences of the form $(i, -F(i))$, the cycle C will contain the path $P = (c_1, c_2, \ldots, c_{\ell n-1})$ where $c_i = (x_i, y_i)$ with $(x_1, x_2, \ldots, x_{\ell n-1}) = (1, -1, 2, -2, 3, -3, \ldots)$ and $(y_1, y_2, \ldots, y_{\ell n-1}) = (1, 1+F(2), 1+F(2)-F(3), 1+F(2)-F(3)+F(4), \ldots)$. The reader can check that $\Delta P = \pm\{(i, -F(i)) \mid 2 \leq i \leq \ell n - 1)\}$.
In order to obtain the remaining differences, we need $y_{\ell n-1} \equiv 1 \pmod{\ell}$, and since $y_{\ell n-1} = 1 + \sum F$ with $\sum F = \sum_{j=2}^{\ell n-1} (-1)^j F(j)$, we must have $\sum F \equiv 0 \pmod{\ell}$. Recall that F is obtained from the maps f and φ; we will show that $\sum F \equiv 0 \pmod{\ell}$ by choosing suitably the map φ, relying on the fact, proved in Proposition 5.11.1, that the alternating sums for φ can take any value in $\mathbb{Z}_\ell$.

Putting it all together, the set $\mathcal{B} = \mathcal{L} \cup \mathcal{S} \cup \{C, C'\}$ is a set of base cycles as required in Theorem 2.1.

4 Transversal long cycles

In this section we construct (Proposition 4.4) a set $\mathcal{L}$ of $(\ell-5)/2$ transversal long cycles whose list of differences $\Delta\mathcal{L}$ has no repeated elements. All the remaining differences not lying in $\Delta\mathcal{L}$ will be covered by the two transversal long cycles given in Section 6 together with the transversal short cycles constructed in Section 5.

The transversal long cycles that we are going to build will be obtained as union of paths with specific vertex-set and list of differences. More precisely, given four integers $a \leq b < c \leq d$ with $|(b-a)-(d-c)| \leq 1$, we denote by $\mathcal{P}([a,b],[c,d])$ the family of all paths P with vertex-set $[a,b] \cup [c,d]$ which satisfy the following properties:

(1) for each edge $\{u,u'\}$ of P, with $u < u'$, we have that $u \in [a,b]$ and $u' \in [c,d]$;

(2) $\Delta P = \pm[c-b, d-a]$.

In other words, $\mathcal{P}([a,b],[c,d])$ contains all paths whose two partite sets are $[a,b]$ and $[c,d]$, and having the interval $\pm[c-b,d-a]$ as list of differences. We point out the connection to α-labelings: a member of $\mathcal{P}([0,b],[b+1,d])$ can be seen as an *α-labeling* of a path on $d+1$ vertices (see [23]).

The following lemma generalizes a result by Abrham [1] concerning the existence of an α-labeling of a path with a given end-vertex.

Lemma 4.1. *Let $a \leq b < c \leq d$ with $|(b-a)-(d-c)| \leq 1$; also, set $\gamma_1 = b-a$ and $\gamma_2 = d-c$. Then, there exists a path of $\mathcal{P}([a,b],[c,d])$ with end vertices w and w' for each of the following values of the pair (w,w'):*

1. *$(w,w') = (a+i, b-i)$ if $\gamma_1 = \gamma_2+1$ and $i \in [0,\gamma_1] \setminus \{\gamma_1/2\}$;*
2. *$(w,w') = (a+i, c+i)$ if $\gamma_1 = \gamma_2$ and $i \in [0,\gamma_1]$;*
3. *$(w,w') = (c+i, d-i)$ if $\gamma_1 = \gamma_2-1$ and $i \in [0,\gamma_2] \setminus \{\gamma_2/2\}$.*

Proof. Set $I_1 = [0,\gamma_1]$ and $I_2 = [\gamma_1+1, \gamma_1+\gamma_2+1]$. The existence of a path $P \in \mathcal{P}(I_1, I_2)$ whose end-vertices u_i and v_i satisfy the assertion is proven in [1] by using the terminology of α-labelings (see also [32]).

Now, let $f : V(P) \to \mathbb{Z}$ be the map defined as follows:

$$f(x) = \begin{cases} x+a & \text{if } x \in I_1, \\ x+c-(b-a+1) & \text{if } x \in I_2 \end{cases}$$

and let $Q = f(P)$ be the path obtained from P by replacing each vertex, say x, with $f(x)$. It is clear that the two partite sets of Q are $f(I_1) = [a,b]$ and $f(I_2) = [c,d]$. By recalling the properties of a path in $\mathcal{P}(I_1,I_2)$, we have that for any $d \in [1, \gamma_1+\gamma_2+1]$, there exists an edge $\{u,u'\}$ of P, with $u \leq \gamma_1 < u'$ such that $u'-u = d$. By construction, the edge $\{f(u), f(u')\} = \{u+a, u'+c-(b-a+1)\}$ lies in Q and it gives rise to the differences $\pm(d+c-b-1)$; hence, $\Delta Q = \pm[c-b, d-a]$. In other words, $Q \in \mathcal{P}([a,b],[c,d])$ and it is not difficult to check that its end-vertices, $f(w_i)$ and $f(w_i')$, satisfy the assertion. □

We use the above result to construct pairs of $(2t+1)$-cycles with vertices in $\mathbb{Z} \times \mathbb{Z}$ satisfying the conditions of the following lemma.

Lemma 4.2. *Let d_1, d_2, and $t \geq 2$ be integers having the same parity, with $d_1, d_2 \in [1, t-1] \setminus \{\lceil t/2 \rceil\}$. For any $x, y \in \mathbb{Z}$, there exist two $(2t+1)$-cycles C_1 and C_2 with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that*

(*i*) *the projection of $V(C_i)$ on the first components is $[0, 2t]$ for $i = 1, 2$,*

(*ii*) $\Delta C_1 \cup \Delta C_2 = \pm([1, 2t] \times \{x, y\}) \cup \{\pm(d_1, x-y), \pm(d_2, y-x)\}$.

Proof. Let d_1, d_2, t, x, y be integers satisfying the assumptions. Also, set $J = [1, 2t] \setminus I$ where $I = [1, t]$ or $[1, t+1]$ according to whether t is odd or even. We are going to show that for any integer $d \in [1, t-1] \setminus \{\lceil t/2 \rceil\}$ with $d \equiv t \pmod 2$, there exists a cycle $C(d)$ with vertices in $\mathbb{Z} \times \mathbb{Z}$ satisfying the following two properties:

(1) the projection of $V(C(d))$ on the first components is $[0, 2t]$,

(2) $\Delta C(d) = \pm(I \times \{x\}) \cup \pm(J \times \{y\}) \cup \{\pm(d, y-x)\}$.

By setting $C_1 = C(d_1)$ and letting C_2 be the cycle obtained from $C(d_2)$ by exchanging the roles of x and y, we obtain a pair $\{C_1, C_2\}$ of cycles which clearly satisfy the assertion.

We first deal with the case $t \geq 3$ odd. Thus, set $t = 2m - 1$ and let $d = 2k + 1$ with $0 \leq k \leq m - 2$.
By Lemma 4.1 with $[a, b] = [m, 2m-2]$, $[b, c] = [2m-1, 3m-2]$ and $i = k$, we obtain a path $U = (u_1, \ldots, u_{2m-1})$ of $\mathcal{P}([a, b], [c, d])$ whose end-vertices are $u_1 = 3m - 2 - k$ and $u_{2m-1} = 2m - 1 + k$. Again by Lemma 4.1 with $[a, b] = [0, m-1]$, $[c, d] = [3m-1, 4m-2]$ and $i = k$, there exists a path $W = (w_0, w_1, \ldots, w_{2m-1})$ of $\mathcal{P}([a, b], [c, d])$ whose end-vertices are $w_0 = k$ and $w_{2m-1} = 3m - 1 + k$. Now, let U' and W' be the following paths:

$$U' = ((u_1, x), (u_2, 0), (u_3, x), \ldots, (u_{2m-3}, x), (u_{2m-2}, 0), (u_{2m-1}, x)),$$
$$W' = ((w_0, 0), (w_1, y), (w_2, 0), \ldots, (w_{2m-3}, y), (w_{2m-2}, 0), (w_{2m-1}, y)).$$

and let $C(d) = U' \cup ((w_{2m-1}, y), (u_1, x)) \cup W_y \cup ((u_{2m-1}, x), (w_0, 0))$ be the $(4m-1)$-cycle obtained by joining U' and W'. Since the projection of $V(C(d))$ on the first coordinates is $V(U) \cup V(W) = [0, 2t]$, condition (1) holds.
Now note that $u_i - u_j > 0$ ($w_i - w_j > 0$) for any i odd and j even. Therefore, by recalling the list of differences of a path in $\mathcal{P}([a, b], [c, d])$, we have that

$$\Delta U' = \pm([1, 2m-2] \times \{x\}), \quad \text{and} \quad \Delta W' = \pm([2m, 4m-2] \times \{y\}).$$

Since $\Delta C(d) = \Delta U' \cup \Delta W' \cup \{\pm(w_{2m-1} - u_1, y - x), \pm(u_{2m-1} - w_0, x)\}$ where $w_{2m-1} - u_1 = d$ and $u_{2m-1} - w_0 = 2m - 1$, then (2) is satisfied.

We proceed in a similar way when $t \geq 2$ is even. So, let $t = 2m$ and let $d = 2k$ with $1 \leq k \leq m - 1$ and $k \neq m/2$.
We apply Lemma 4.1 with $[a, b] = [m, 2m-1]$, $[c, d] = [2m, 3m]$ and $i = k$

to obtain a path $U = (u_1, \dots, u_{2m+1})$ of $\mathcal{P}([a,b],[c,d])$ whose end-vertices are $u_1 = 3m-k$ and $u_{2m+1} = 2m+k$. Again, we apply Lemma 4.1 with $[a,b] = [0, m-1]$, $[c,d] = [3m+1, 4m]$ and $i = k-1$ and obtain a path $W = (w_0, w_1, \dots, w_{2m-1})$ of $\mathcal{P}([a,b],[c,d])$ whose end-vertices are $w_0 = k-1$ and $w_{2m-1} = 3m+k$. Now, let U' and W' be the following paths:

$$U' = ((u_1, x), (u_2, 0), (u_3, x), \dots, (u_{2m-3}, x), (u_{2m-2}, 0), (u_{2m+1}, x)),$$
$$W' = ((w_0, 0), (w_1, y), (w_2, 0), \dots, (w_{2m-3}, y), (w_{2m-2}, 0), (w_{2m-1}, y)).$$

and let $C(d) = U' \cup ((w_{2m-1}, y), (u_1, x)) \cup W' \cup ((u_{2m+1}, x), (w_0, 0))$ be the $(4m+1)$-cycle obtained by joining U' and W'. As before, one can check that $C(d)$ satisfies (1) and (2). This completes the proof. □

Example 4.3. Let $(d_1, d_2, t) = (1, 43, 45)$, and let $(x, y) = (3, 4)$. Below are two 91-cycles C_1 and C_2 satisfying Lemma 4.2:

$$\begin{aligned}
C_1 = ((0,0),(90,3),(1,0),(89,3),\dots,(i,0),(90-i,3),\dots,(22,0),(68,3),\\
(67,4),(23,0),(66,4),\dots,(67-i,4),(23+i,0),\dots,(46,4),(44,0),(45,4)),\\
C_2 = ((21,0),(69,4),(22,0),(68,4),\dots,(21-2i,0),(69+2i,4),(22-2i,0),(68+2i,4),\dots,\\
(3,0),(87,4),(4,0),(86,4),(1,0),(88,4),(2,0),(90,4),(0,0),(89,4),\\
(46,3),(43,0),(45,3),(44,0),\dots,(46+2i,3),(43-2i,0),(45+2i,3),(44-2i,0),\dots,\\
(62,3),(27,0),(61,3),(28,0),(64,3),(24,0),(65,3),(26,0),(63,3),(25,0),(67,3),(23,0),(66,3))
\end{aligned}$$

It is straightforward to check that $\Delta C_1 = \pm([1,45] \times \{4\}) \cup \pm([46,90] \times \{3\}) \cup \{\pm(1,-1)\}$ and $\Delta C_2 = \pm([1,45] \times \{3\}) \cup \pm([46,90] \times \{4\}) \cup \{\pm(43,1)\}$.

We are now able to construct the required set $\mathcal{L}$ of $(\ell-5)/2$ transversal long cycles; we shall use the set D of definition 3.1.

Proposition 4.4. *For $\ell \equiv 1 \pmod 4$ and $\ell \geq 5$ there exists a set $\mathcal{L}$ of $(\ell-5)/2$ transversal $(2\ell n+1)$-cycles of $K_{\ell(2\ell n+1)}$ such that*

$$\Delta\mathcal{L} = (\mathbb{Z}^*_{2\ell n+1} \times (\mathbb{Z}_\ell \setminus \{0, \pm 1, \pm 2\})) \cup \{(d, f(d)) \mid d \in D \cup -D\}$$

where $f(d) = \pm 1$ for any $d \in D \cup -D$.

Proof. We shall construct a set of transversal long cycles $\mathcal{L}$ by applying repeatedly Lemma 4.2. We will use the following straightforward property of an integer $d \in D$: $d \equiv 1, 2 \pmod 4$ for n odd, and $d \equiv 0, 1 \pmod 4$ for n even.

Set $t = \ell n$, $k = (\ell-1)/4$ and let $D = \{d_{i1}, d_{i2} \mid i = 1, \dots, k-1\}$. For any $d \in D$ we define the integer d' as follows:

$$d' = d \text{ when } d \equiv 0, 2 \pmod 4, \text{ and } d' = 2t+1-d \text{ when } d \equiv 1 \pmod 4,$$

and set $D' = \{d' \mid d \in D\}$. Of course, $D \cup -D \equiv D' \cup -D' \pmod{2t+1}$; also, all integers in D' are congruent to 0 or 2 mod 4 according to whether t

is even or odd; it then follows that all integers in $D'/2$ have the same parity as t. Recalling how D is defined (Definition 3.1), it is easy to check that $D'/2 \subseteq [2, t-1] \setminus \{\lceil t/2 \rceil\}$.

For $i = 1, \ldots, k-1$ set $x = 2i+1, y = 2i+2$ and take $d_1 = d'_{i1}/2$ and $d_2 = d'_{i2}/2$. In view of the considerations above, all the assumptions of Lemma 4.2 are satisfied. Therefore, there exist two cycles C_{i1}, C_{i2} for $i = 1, \ldots, k-1$ with vertices in $\mathbb{Z} \times \mathbb{Z}$ such that

$$\text{the projection of } C_{i1} \text{ (resp. } C_{i2}\text{) on the first coordinates is } [0, 2\ell n], \text{and}$$

$$\Delta C_{i1} \cup \Delta C_{i2} = \pm([1, 2\ell n] \times [2i+1, 2i+2]) \cup \{\pm(d'_{i1}/2, -1), \pm(d'_{i2}/2, 1)\}.$$

Now, consider the vertices of C_{i1} and C_{i2} (and also their differences), as elements of $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$. We denote by ψ the permutation of $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$ defined as $\psi(a, b) = (2a, b)$, and set $C'_{ij} = \psi(C_{ij})$ for $i \in [1, k-1]$ and $j = 1, 2$. It is clear that C'_{ij} is a transversal $(2\ell n+1)$-cycle of $K_{\ell(2\ell n+1)}$; moreover,

$$\Delta C'_{i1} \ \cup \ \Delta C'_{i2} = \ \pm\left(\mathbb{Z}^*_{2\ell n+1} \times \{2i+1, 2i+2\}\right) \ \cup \ \{\pm(d'_{i1}, -1), \pm(d'_{i2}, 1)\}.$$

Finally, set $\mathcal{L} = \{C'_{i1}, C'_{i2} \mid i \in [1, k-1]\}$; of course, $\Delta\mathcal{L} = \bigcup_i \Delta\{C'_{i1}, C'_{i2}\}$. By recalling how the d'_{ij}s are defined, we get

$$\Delta\mathcal{L} = (\mathbb{Z}^*_{2\ell n+1} \times \mathbb{Z}^-_\ell) \ \cup \ \{\pm(d_{i1}, f(d_{i1})) \mid i \in [1, k-1], j = 0, 1\}.$$

where $f(d_{ij}) = 1$ or -1 for any $i \in [1, k-1]$ and $j = 1, 2$. This shows that $\mathcal{L}$ is the required set of transversal long cycles. □

Example 4.5. Let $\ell = 9$ and $n = 5$; then $k = 2$ and the set D is $\{2, 5\}$. In this case, the set $\mathcal{L}$ of Proposition 4.4 consists of two transversal long cycles, $\mathcal{L} = \{C'_1, C'_2\}$, that we will construct by following the proof. Note that $D' = \{2, 86\}$, and hence $D'/2 = \{1, 43\}$.

Now, set $(d_1, d_2, t) = (1, 43, 45)$, $(x, y) = (3, 4)$ and apply Lemma 4.2 to obtain two transversal 91-cycle C_1 and C_2: for example, we will take the cycles of Example 4.3. To obtain C'_1 and C'_2, it is enough to multiply by 2 the first component in the cycles C_1 and C_2 and reduce modulo 91, that is,

$$C'_1 = ((0,0), (89,3), (2,0), (87,3), \ldots, (45,3), (43,4), \ldots, (1,4), (88,0), (90,4)),$$
$$C'_2 = ((42,0), (47,4), (44,0), (45,4), \ldots, (87,4), (1,3), \ldots, (43,3), (46,0), (41,3)).$$

Taking into account the list of differences of C_1 and C_2 (Example 4.3), we can see that $\Delta\mathcal{L} = \mathbb{Z}^*_{91} \times \{\pm 3, \pm 4\} \ \cup \ \{\pm(2, -1)), \pm(5, -1)\}$.

5 Transversal short cycles

In this section we show that there exists a set $\mathcal{S}$ of n transversal short cycles for all $n \geq (\ell - 1)/2$, whose list of differences is disjoint from $\Delta\mathcal{L}$, where $\mathcal{L}$ is the set of transversal long cycles constructed in Section 4.

We first provide a set $\mathcal{A}$ of $n-2k$ cycles of length $\ell = 4k+1$ and a set $\mathcal{B}$ of $2k$ cycles of length $\ell - 1$ (Lemma 5.4), with vertices in $\mathbb{Z}_{2\ell n+1}$. After that, in Proposition 5.11 we first inflate the cycles in $\mathcal{B}$ to get ℓ–cycles; then, we lift the cycles in $\mathcal{A}$ and $\mathcal{B}$ to $\mathbb{Z}_{\ell(2\ell n+1)}$ by adding a second coordinate to the vertices of the cycles (this will be done by following Lemmas 5.8 and 5.9). All these steps will give us the required set $\mathcal{S}$.

5.1 Building the short cycles in $\mathbb{Z}_{2\ell n+1}$

First, we need the following definition which describes the structure of the cycles in the set $\mathcal{B}$ we shall build in Lemma 5.4.

Definition 5.1. Let $\ell = 4k+1$ and let $B = (b_0 = 0, b_1, \dots, b_{4k-1})$ be a $4k$-cycle with vertices in $[-\ell n, \ell n]$. For any $i \in [1, 4k-1]$ set $\delta_i = (-1)^i(b_i - b_{i-1})$ and $\delta_{4k} = b_0 - b_{4k-1}$. Then B is said to be *alternating* if the following conditions are satisfied:

1. $\delta_i \in [1, \ell n]$ for any $i \in [1, 4k]$;
2. $\delta_i \equiv i+1 \pmod 2$ for $i \in [1, 2k]$;
3. $\delta_i \equiv i \pmod 2$ for $i \in [2k+1, 4k]$.

Example 5.2. Let $\ell = 9$, $n = 5$. Denote by $B = (b_0, b_1, \dots, b_7)$ the 8-cycle with vertices in $[-45, 45]$ defined as follows: $B = (0, -30, 1, -31, 2, -33, 3, -34)$. Now, note that $(\delta_1, \delta_2, \dots, \delta_8) = (30, 31, 32, 33, 35, 36, 37, 34)$; also each δ_h lies in $[1, 45]$. Therefore, conditions 1, 2, and 3 of the above definition are satisfied, and hence B is alternating.

Note that $-B = (0, 10, -1, 11, -2, 13, -3, 14)$ has the same list of differences as B, but it is not alternating since conditions 2 and 3 are not satisfied.

We start building the short cycles. First, we need a result providing a sufficient condition for a set of $(\ell - 1)$ positive integers U to be obtained as the list of differences of an $(\ell - 1)$-cycle, possibly alternating.

Lemma 5.3. *Let $\ell \equiv 1 \pmod 4$ and let $U \subseteq [1, \ell n]$ be a set of size $(\ell - 1)$. If U can be partitioned into pairs of consecutive integers, then there exists an $(\ell - 1)$–cycle C with $V(C) \subseteq \mathbb{Z}_{2\ell n+1}$ and $\Delta C = \pm U$.*

Also, if $U = [u, u+\ell-2]$ with u even, then C is alternating.

Proof. Let $\ell = 4k+1$ and let $(\delta_1, \delta_2, \dots, \delta_{2k}, x, \delta_{2k+1}, \dots, \delta_{4k-1})$ be the increasing sequence of the integers in U. By assumption, U can be partitioned into pairs of consecutive integers, that is, $\delta_{2i-1} + 1 = \delta_{2i}$, $x + 1 = \delta_{2k+1}$, and $\delta_{2j} + 1 = \delta_{2j+1}$, for $i \in [1, k]$ and $j \in [k+1, 2k-1]$. Therefore, it is not difficult to check that $x = -\sum_{i=1}^{4k-1}(-1)^i\delta_i$.

Now, set $b_0 = 0$ and $b_i = \sum_{h=1}^{i}(-1)^h\delta_h$ for $i \in [1, 4k-1]$. Since the sequence $(\delta_1, \delta_2, \dots, \delta_{4k-1})$ is increasing, it is straightforward to check that all b_is are pairwise distinct. Then, we can consider the $4k$-cycle $C = (b_0, b_1, \dots, b_{4k-1})$. Note that $\delta_i = (-1)^i(b_i - b_{i-1})$ for $i \in [1, 4k-1]$. Also, $b_{4k-1} = -x$ hence, $x = b_0 - b_{4k-1}$. It then follows that $\Delta C = \pm U$.

The final part of the assertion follows by taking into account that $(\delta_1, \dots, \delta_{2k}, x, \delta_{2k+1}, \dots, \delta_{4k-1})$ is the increasing sequence of the integers in U. □

As in [10, 13], we also need the crucial use of *Skolem sequences*. A Skolem sequence of order n can be viewed as a sequence $S = (s_1, \dots, s_n)$ of positive integers such that $\bigcup_{i=1}^{n}\{s_i, s_i+i\} = \{1, 2, \dots, 2n\}$ or $\{1, 2, \dots, 2n+1\} \setminus \{2n\}$. One speaks of an *ordinary Skolem sequence* in the first case and of a *hooked Skolem sequence* in the second. It is well known (see [31]) that there exists a Skolem sequence of order n for every positive integer n; it is ordinary for $n \equiv 0$ or $1 \pmod 4$ and hooked for $n \equiv 2$ or $3 \pmod 4$.

Lemma 5.4. *For $\ell = 4k+1$ and $n \geq 2k$, there exist a set $\mathcal{A} = \{A_1, \dots, A_{n-2k}\}$ of ℓ-cycles and a set $\mathcal{B} = \{B_1, \dots, B_{2k}\}$ of $(\ell-1)$-cycles with vertices in $\mathbb{Z}_{2\ell n+1}$ satisfying the following properties:*

1. *$\Delta\mathcal{A} \cup \Delta\mathcal{B} = \mathbb{Z}_{2\ell n+1}^{-} \setminus (D \cup -D)$;*
2. *for $i \geq 2$, the cycle B_i is alternating and $\Delta B_i = \pm[u_i, u_i + 4k - 1]$, with $u_i = 2(i-1)n$.*

Proof. In the proof we will start by constructing the cycles of $\mathcal{A}$ so that the set of integers in $\mathbb{Z}_{2\ell n+1}^{-} \setminus (D \cup -D)$ not covered by $\Delta\mathcal{A}$ can be split up into even-length intervals. A suitable choice of $4k$-gons as in Lemma 5.3 will then make up the set $\mathcal{B}$ and take care of the remaining differences.

Let us fix a Skolem sequence $S = (s_1, ..., s_{n-2k})$ of order $n - 2k$. So S is ordinary for $n - 2k \equiv 0$ or $1 \pmod 4$ and hooked otherwise.

We start with the non-hooked case, so let $n \equiv 2k$ or $2k+1 \pmod 4$.

Let us construct the cycles of $\mathcal{A}$; for $1 \leq i \leq n-2k$, let $A_i = (a_{i0}, a_{i1}, \dots, a_{i,4k})$, where:

$$a_{ij} = \begin{cases} (4k-2-j)n & \text{for } 1 \leq j \leq 4k-3,\ j \text{ odd}, \\ jn + i - 2k & \text{for } 0 \leq j \leq 2k-2,\ j \text{ even}, \\ jn + i - 1 + 2k & \text{for } 2k \leq j \leq 4k-2,\ j \text{ even}, \\ -2k & \text{for } j = 4k-1, \\ s_i + i + (4k-1)n - 1 & \text{for } j = 4k. \end{cases}$$

It is tedious but straightforward to check that $\Delta\mathcal{A} = I_0 \cup I_1 \cdots \cup I_{2k-1}$ where

$$I_\alpha = \begin{cases} [4k, 2n-1] + 2n \cdot \alpha & \text{for } \alpha = 0, \dots, 2k-2 \\ [(4k-2)(n+1)+2, (4k+1)n - 2k - 1] & \text{for } \alpha = 2k-1. \end{cases}$$

We now apply Lemma 5.3 to construct $\mathcal{B}$ such that $\Delta\mathcal{B} = \mathbb{Z}^-_{2\ell n+1} \setminus (\pm D \,\cup\, \Delta\mathcal{A})$. Let J_β be the interval between $I_{\beta-1}$ and I_β for $1 \leq \beta \leq 2k-1$; each such J_β has even length $4k$. Also, set $J_0 = [2, 4k-1]$ and $J_{2k} = [(4k+1)n - 2k, (4k+1)n - 1]$. Note that the I_α's and J_β's are pairwise disjoint and cover altogether the integers from 2 to $(4k+1)n - 1$, namely:

$$[2, (4k+1)n - 1] = \bigcup_{\alpha=0}^{2k-1} I_\alpha \;\cup\; \bigcup_{\beta=0}^{2k} J_\beta.$$

It is easy to check that $D \subseteq J_0$ and $J_0 \setminus D$ is a set of k pairs of disjoint consecutive integers. In view of Lemma 5.3 there exists $2k$ cycles $C_0, C_1, \ldots, C_{2k-1}$ of length $4k$ and vertices in $\mathbb{Z}_{2\ell n+1}$ whose lists of differences are the following:

$$\Delta C_\beta = \begin{cases} \pm(J_0 \setminus D) \;\cup\; \pm J_{2k} & \text{for } \beta = 0 \\ \pm J_\beta & \text{for } 1 \leq \beta \leq 2k-1. \end{cases}$$

Note that for any $\beta \in [1, 2k-1]$ the smallest integer in J_β is even, hence C_β is alternating by Lemma 5.3.

We now consider the hooked case, so let $n \equiv 2k+2$ or $2k+3 \pmod 4$.

Let $\mathcal{A}$ be the cycle–set constructed earlier. Since now the Skolem sequence is hooked, then the list of differences of $\mathcal{A}$ has the following form: $\Delta\mathcal{A} = I_0 \;\cup\; I_1 \ldots \;\cup\; I_{2k-2} \;\cup\; I^*_{2k-1}$ where $I_\alpha = [4k, 2n-1] + 2n \cdot \alpha$ for $\alpha = 0, \ldots, 2k-2$ and

$$I^*_{2k-1} = [(4k-2)(n+1) + 2, (4k+1)n - 2k - 2] \;\cup\; \{(4k+1)n - 2k\}.$$

We apply Lemma 5.3 to construct $\mathcal{B}$ such that $\Delta\mathcal{B} = \mathbb{Z}^-_{2\ell n+1} \setminus (\pm D \,\cup\, \Delta\mathcal{A})$. Let J_β be the interval between $I_{\beta-1}$ and I_β for $1 \leq \beta \leq 2k-1$; each such J_β has even length $4k$. Also, set $J_0 = [2, 4k-1]$ and $J_{2k} = [(4k+1)n - 2k - 1, (4k+1)n - 1] \setminus \{(4k+1)n - 2k\}$. Note that the I_α's and J_β's are pairwise disjoint and cover altogether the integers from 2 to $(4k+1)n - 1$, namely:

$$[2, (4k+1)n - 1] = \bigcup_{\alpha=0}^{2k-1} I_\alpha \;\cup\; \bigcup_{\beta=0}^{2k} J_\beta.$$

It is easy to check that $D \subseteq J_0 \;\cup\; J_{2k}$ and that $J_0 \setminus D$ is a set of $k+1$ pairs of disjoint consecutive integers. Also, $J_{2k} \setminus D = [(4k+1)n - 2k + 1, (4k+1)n - 2k + 2] \;\cup\; [(4k+1)n - 2k + 4, (4k+1)n - 1]$; then, J_{2k} is the union of a 2–set and a $(2k-4)$–set both made of consecutive integers. In view of Lemma 5.3 there exist $2k$ cycles $C_0, C_1, \ldots, C_{2k-1}$ of length $4k$ and vertices in $\mathbb{Z}_{2\ell n+1}$ whose lists of differences are the following:

$$\Delta C_\beta = \begin{cases} \pm(J_0 \setminus D) \;\cup\; \pm(J_{2k} \setminus D) & \text{for } \beta = 0 \\ \pm J_\beta & \text{for } 1 \leq \beta \leq 2k-1. \end{cases}$$

As before, we point out that for any $\beta \in [1, 2k-1]$ the smallest integer in J_β is even, hence C_β is alternating by Lemma 5.3.

We shall obtain the set $\mathcal{B}$ required in the proof by setting, for instance, $B_{i+1} = C_i$ for $i \in [0, 2k-1]$. □

Example 5.5. Let $n = 5$ and $\ell = 9$, hence $k = 2$. In this case, $\mathbb{Z}_{91}^- = \pm[2, 44]$ and the set D of Definition 3.1 has only two elements: $D = \{2, 5\}$. Below, we provide the sets of short cycles $\mathcal{A} = \{A_1\}$ and $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ of Lemma 5.4.

$$\begin{aligned}
&A_1 = (-3, 25, 7, 15, 24, 5, 34, -4, 36),\\
&B_1 = (0, -3, 1, -5, 2, -40, 3, -41), \quad B_2 = (0, -10, 1, -11, 2, -13, 3, -14),\\
&B_3 = (0, -20, 1, -21, 2, -23, 3, -24), \quad B_4 = (0, -30, 1, -31, 2, -33, 3, -34).
\end{aligned}$$

First, note that $\Delta A_1 = \pm\{8, 9, 18, 19, 28, 29, 38, 39, 40\}$ and $\Delta B_1 = \pm\{3, 4, 6, 7, 41, 42, 43, 44\}$. Now, for $i = 2, 3, 4$, let $B_i = (b_{0,i}, b_{1,i}, \ldots, b_{4k-1,i})$, and set $\delta_{h,i} = (-1)^h(b_{h,i} - b_{h,i})$ for $h \in [1, 8]$, with $b_{4k} = b_0$. It is easy to check that

$$(\delta_{1,i}, \delta_{2,i}, \ldots, \delta_{4k,i}) = \begin{cases} (10, 11, 12, 13, 15, 16, 17, 14) & \text{for } i = 2,\\ (20, 21, 22, 23, 25, 26, 27, 24) & \text{for } i = 3,\\ (30, 31, 32, 33, 35, 36, 37, 34) & \text{for } i = 4. \end{cases}$$

It follows that the cycles B_2, B_3, B_4 are alternating. Also, $\Delta B_2 = \pm[10, 17]$, $\Delta B_3 = \pm[20, 27]$, and $\Delta B_4 = \pm[30, 37]$. Therefore, $\Delta\mathcal{A} \cup \Delta\mathcal{B} = \mathbb{Z}_{81} \setminus (\pm D)$.

5.2 Lifting the short cycles to $\mathbb{Z}_{\ell(2\ell n+1)}$

As mentioned at the beginning of this section, to obtain the set $\mathcal{S}$ of transversal short cycles we need to lift the cycles in $\mathcal{A}$ and $\mathcal{B}$ by adding a second component to each vertex. A special attention is to be devoted to the cycles of $\mathcal{B}$. Indeed, given an $(\ell-1)$-cycle $B \in \mathcal{B}$, say $B = (b_0, b_1, \ldots, b_{4k-1})$, we will consider as a lift of B any cycle B' of the following form:

$$B' = ((b_0, p_0), (b_1, p_1), \ldots, (b_{4k-1}, p_{4k-1}), (b_{4k}, p_{4k}))$$

whose vertices are elements of $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$ and either $b_{4k} = b_0$ or $b_{4k} = b_{4k-1}$. If $\Delta B = \pm[u, u']$ with $0 < u \le u'$, then $\Delta B'$ has the following form:

$$\Delta B' = \{\pm(i, \varphi(i)) \mid i \in [u, u']\} \ \cup\ \{\pm(0, x)\}$$

for a suitable map $\varphi : [u, u'] \to \mathbb{Z}_\ell$, and $x = p_{4k} - p_0$ or $x = p_{4k} - p_{4k-1}$ according to whether $b_{4k} = b_0$ or b_{4k-1}. As will become clear later on, we will need to determine the *partial sum* $\sum_{i=u}^{u'}(-1)^i\varphi(i)$ related to B'.

The following lemma shows that the partial sum related to the lift B' of an alternating cycle B depends only on some of the labelings of the second components. More precisely,

Lemma 5.6. *Let $\ell = 4k+1$ and let $B = (0, b_1, \dots, b_{4k-1})$ be a $4k$-cycle with vertices in $\mathbb{Z}_{2\ell n+1}$. Denote by B' the ℓ-cycle with vertices in $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$ obtained from B as follows: $B' = ((0,0), (b_1, p_1), \dots, (b_{4k-1}, p_{4k-1}), (b_{4k}, p_{4k}))$ where $b_{4k} = 0$ or b_{4k-1}.*

If B is alternating and $\Delta B \cap [1, \ell n] = [u, u']$, then the map $\varphi : [u, u'] \to \mathbb{Z}_\ell$ such that $(i, \varphi(i)) \in \Delta B'$ satisfies the relation:

$$\sum_{i=u}^{u'} (-1)^i \varphi(i) = \begin{cases} p_{4k} - 2p_{2k} & \text{if } b_{4k} = 0, \\ p_{4k-1} - p_{4k} - 2p_{2k} & \text{if } b_{4k} = b_{4k-1}. \end{cases}$$

Proof. For brevity, set $\sum = \sum_{i=u}^{u'} (-1)^i \varphi(i)$. Also, as in Definition 5.1, for any $h \in [1, 4k-1]$ set $\delta_h = (-1)^h (b_h - b_{h-1})$, where $b_0 = 0$, and set $\delta_{4k} = -b_{4k-1}$. Since B is alternating, $\delta_h \in [1, \ell n]$ for any $h \in [1, 4k]$; also, by assumption, $\Delta B \cap [1, \ell n] = [u, u']$. Therefore, $[u, u'] = \{\delta_1, \delta_2, \dots, \delta_{4k}\}$ hence, $\sum = \sum_{h=1}^{4k} (-1)^{\delta_h} \varphi(\delta_h)$.

For any $h \in [1, 4k-1]$ we have that $\varphi(\delta_h) = (-1)^h (p_h - p_{h-1})$, where $p_0 = 0$; also, it is not difficult to check that

$$\varphi(\delta_{4k}) = \begin{cases} p_{4k} - p_{4k-1} & \text{if } b_{4k} = 0, \\ -p_{4k} & \text{if } b_{4k} = b_{4k-1}. \end{cases}$$

Now set $\sum' = \sum_{h=1}^{2k} (-1)^{\delta_h} \varphi(\delta_h)$ and $\sum'' = \sum_{h=2k+1}^{4k-1} (-1)^{\delta_h} \varphi(\delta_h)$. Since by Definition 5.1 we have $\delta_h \equiv h+1 \pmod 2$ for $h \in [1, 2k]$, then

$$\sum{}' = \sum_{h=1}^{2k} (-1)^{h+1} \varphi(\delta_h) = -\sum_{h=1}^{2k} (p_h - p_{h-1}) = p_0 - p_{2k} = -p_{2k}.$$

On the other hand, $\delta_h \equiv h \pmod 2$ for $h \in [2k+1, 4k]$, therefore

$$\sum{}'' = \sum_{h=2k+1}^{4k-1} (-1)^h \varphi(\delta_h) = \sum_{h=2k+1}^{4k-1} (p_h - p_{h-1}) = p_{4k-1} - p_{2k}.$$

and $(-1)^{\delta_{4k}} \varphi(\delta_{4k}) = \varphi(\delta_{4k})$. Since $\sum = \sum' + \sum'' + \varphi(\delta_{4k})$, the assertion easily follows. □

Example 5.7. Let once more $n = 5$ and $\ell = 9$ (i.e. $k = 2$). Consider the 8-cycle $B_4 = (0, -30, 1, -31, 2, -33, 3, -34)$ of Example 5.5. It is alternating and $\Delta B \cap [1, 45] = [30, 37]$. Therefore, the assumption of Lemma 5.6 are satisfied. Now, denote by B' the cycle with vertices in $\mathbb{Z}_{91} \times \mathbb{Z}_9$ obtained from B by repeating the last vertex and then adding the second components $(p_0, p_1, \dots, p_8) = (0, 2, 3, 4, 5, 7, 8, 6, 1)$, that is,

$$B' = ((0,0), (-30,2), (1,3), (-31,4), (2,5), (-33,7), (3,8), (-34,6), (-34,1)).$$

It is easy to check that $\Delta B' = \pm(\{31,33,36\}\times\{1\}) \cup \pm(\{32,34\}\times\{-1\}) \cup \pm(\{30,35\}\times\{-2\}) \cup \{\pm(37,2), \pm(0,4)\}$. Therefore, we can define the map $\varphi : [30,37] \to \mathbb{Z}_\ell$ such that $(i,\varphi(i)) \in \Delta B'$. It is easy to check that

$$\sum_{i=30}^{37}(-1)^i\varphi(i) = 4 = p_7 - p_8 - 2p_4.$$

The following two lemmas tell us how to label the second components of the vertices in the cycles of Proposition 5.4, in order to get the set $\mathcal{S}$ of short cycles.

Lemma 5.8. *Let $\ell = 4k+1$; for any $i \in [1, 2k-1]$ there is an ℓ-cycle $Q_i = (0, q_1, \ldots q_{4k})$ with vertex set $\mathbb{Z}_\ell$ such that $q_j = j$ for $j = 1, 2, \ldots, 4k-i, q_{4k} = -i$, and $\Delta Q_i \setminus \{\pm i\}$ contains only ± 1s and ± 2s.*

Proof. The assertion is easily verified: consider for instance the cycle

$$\begin{aligned} Q_i = (&0, 1, 2, \ldots, 4k-i,\\ &4k-i+2, 4k-i+4, 4k-i+6, \ldots, 4k,\\ &4k-1, 4k-3, 4k-5, \ldots, 4k-i+1) \end{aligned}$$

for i even, and for i odd the cycle

$$\begin{aligned} Q_i = (&0, 1, 2, \ldots, 4k-i,\\ &4k-i+2, 4k-i+4, 4k-i+6, \ldots, 4k-1,\\ &4k, 4k-2, 4k-4, \ldots, 4k-i+1). \end{aligned}$$

□

We point out to the reader that the cycles Q_is of the above lemma are unique.

Lemma 5.9. *Let $\ell = 4k+1$; for any $m \in \mathbb{Z}_\ell$ there is an ℓ-cycle $P_m = (0, p_1, \ldots, p_{4k})$ with vertex set $\mathbb{Z}_\ell$ such that*

1. *for $m \neq \pm 2k$, we have $p_{4k} = 2k$ and $p_{4k} - 2p_{2k} = m$,*
2. *for $m = \pm 2k$, we have $p_{4k} - p_{4k-1} = m$ and $p_{2k} = -m$,*
3. *for any $m \in \mathbb{Z}_\ell$, $\Delta P_m \setminus \{\pm 2k\}$ contains only ± 1s and ± 2s.*

Proof. For a clear description of our construction, we need to work first on the integers. The required cycles will be then obtained by reducing (mod ℓ).

Let $m \in [0, 4k]$ with $m \neq 2k, 2k+1$ and let x be the integer in $[0, 4k]$ such that $x \equiv (2k - m)/2 \pmod{\ell}$; of course, $x \neq 0, 2k$. We first work with the case $0 < x < 2k$. If x is odd, we may take

$$\begin{aligned}P_m = (&0, 4k-1, 4k-3, \ldots, 2k+x+2,\\ &2k+x+1, 2k+x+3, 2k+x+5, \ldots, 4k,\\ &1, 2, 3, \ldots, x, x+1, \ldots, 2k-1,\\ &2k+1, 2k+3, 2k+5, \ldots, 2k+x,\\ &2k+x-1, 2k+x-3, 2k+x-5, \ldots, 2k).\end{aligned}$$

Similarly, for x even, we may take

$$\begin{aligned}P_m = (&0, 4k-1, 4k-3, \ldots, 2k+x+1,\\ &2k+x+2, 2k+x+4, 2k+x+6, \ldots, 4k,\\ &1, 2, \ldots, x, x+1, \ldots, 2k-1,\\ &2k+1, 2k+3, 2k+5, \ldots, 2k+x-1\\ &2k+x, 2k+x-2, 2k+x-4, \ldots, 2k).\end{aligned}$$

Suppose now that $2k+1 \leq x \leq 4k$, and say $x = 2k + x'$. If x is even, we take

$$\begin{aligned}P_m = (&0, 2, 4, \ldots, x'-2,\\ &x'-1, x'-3, x'-5, \ldots, 1,\\ &4k, 4k-1, 4k-2, \ldots, x, x-1, \ldots, 2k+1,\\ &2k-1, 2k-3, 2k-5, \ldots, x'+1,\\ &x', x'+2, x'+4, \ldots, 2k).\end{aligned}$$

If $x \neq 2k+1$ is odd, then

$$\begin{aligned}P_m = (&0, 2, 4, \ldots, x'-1,\\ &x'-2, x'-4, x'-6, \ldots, 1,\\ &4k, 4k-1, 4k-2, \ldots, x, x-1, \ldots, 2k+1\\ &2k-1, 2k-3, 2k-5, \ldots, x',\\ &x'+1, x'+3, x'+5, \ldots, 2k),\end{aligned}$$

and for $x = 2k+1$, that is for $m = 2k-1$, we take:

$$\begin{aligned}P_{2k-1} = (&0, 4k, 4k-1, \ldots, 2k+1,\\ &2k-1, 2k-3, 2k-5, \ldots, 1,\\ &2, 4, 6, \ldots, 2k).\end{aligned}$$

In all the above cases we have $p_{2k} = x$. Finally, the cycle P_{2k} is given as follows:

$$\begin{aligned} P_{2k} = (&0, 2, 3, 4, \dots, p_{2k} = 2k+1, \\ &2k+3, 2k+5, 2k+7, \dots, 4k-1, \\ &4k, 4k-2, 4k-4, \dots, 2k+2, 1), \end{aligned}$$

and $P_{-2k} = -P_{2k}$. It is straightforward to check that after reducing modulo ℓ, all cycles just defined satisfy the requirements of the lemma. $\square$

Example 5.10. For $\ell = 9$ the cycles constructed in the two previous lemmas are as follows.

$$\begin{array}{lll} Q_1 = (0,1,2,3,4,5,6,7,8) & Q_2 = (0,1,2,3,4,5,6,8,7) & Q_3 = (0,1,2,3,4,5,7,8,6) \\ P_0 = (0,7,8,1,2,3,5,6,4) & P_1 = (0,1,8,7,6,5,3,2,4) & P_2 = (0,7,6,8,1,2,3,5,4) \\ P_3 = (0,8,7,6,5,3,1,2,4) & P_4 = (0,2,3,4,5,7,8,6,1) & P_5 = (0,7,6,5,4,2,1,3,8) \\ P_6 = (0,2,3,1,8,7,6,5,4) & P_7 = (0,8,1,2,3,5,7,6,4) & P_8 = (0,2,1,8,7,6,5,3,4). \end{array}$$

We are now able to construct the required set $\mathcal{S}$ of n transversal short cycles in such a way that $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ has no repeated differences.

Proposition 5.11. *Let $\ell \equiv 1 \pmod 4$. For every integer $n \geq (\ell-1)/2$ and for every $s \in \mathbb{Z}_\ell$, there exists a set $\mathcal{S}$ of n transversal ℓ–cycles of $K_{\ell(2\ell n+1)}$ whose list of differences is of the form*

$$\Delta\mathcal{S} = (\{0\} \times \mathbb{Z}_\ell^*) \cup \{(x, \varphi(x)) \mid x \in Z_{2\ell n+1}^- \setminus (D \cup -D)\}$$

where $\varphi : \mathbb{Z}_{2\ell n+1}^- \setminus (D \cup -D) \to \{\pm 1, \pm 2\}$ is a map such that:

1. *$\sum_{i \in \overline{D}} (-1)^i \varphi(i) = s$, where $\overline{D} = [2, \ell n - 1] \setminus D$;*
2. *the set $\{i \in \overline{D} \mid \varphi(i) = (-1)^{i+1}\}$ has size $\geq (\ell-5)(\ell-1)/4$.*

Proof. Consider the sets $\mathcal{A}$ and $\mathcal{B}$, constructed in Lemma 5.4, containing cycles of $K_{2\ell n+1}$. We have to "lift" them to cycles of $K_{\ell(2\ell n+1)}$. Since the vertices of the cycles in $\mathcal{A} \cup \mathcal{B}$ lie in $\mathbb{Z}_{2\ell n+1}$, while the vertex-set of $K_{\ell(2\ell n+1)}$ has been identified with $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$, to lift these cycles to $K_{\ell(2\ell n+1)}$ means to add a second coordinate to each of their vertices. We will also add a vertex to the cycles in $\mathcal{B}$ so that they become ℓ-cycles.

This lift is easily done for the set $\mathcal{A}$; from each cycle $A \in \mathcal{A}$, $A = (a_0 = 0, a_1, \dots, a_{4k})$, we obtain the cycle $A' = (a'_0, \dots, a'_{4k})$ by setting $a'_i = (a_i, i)$, and we set $\mathcal{A}'$ to be the set $\{A' | A \in \mathcal{A}\}$. It is important to note that

$$\text{the projection of } \Delta A'_i \text{ on } \mathbb{Z}_{2\ell n+1} \text{ is } \Delta A_i, \tag{5.1}$$

$$\text{the projection of } \Delta A'_i \text{ on } \mathbb{Z}_\ell \text{ is a list of } \pm 1\text{'s}, \tag{5.2}$$

for $i = 1, \dots, n-2k$.

We will now work with the set $\mathcal{B} = \{B_1, \dots, B_{2k}\}$, where $B_i = (b_{0,i}, b_{1,i}, \dots, b_{4k-1,i})$ with $b_{0,i} = 0$ for $i = 1, \dots, 2k$. When dealing with the set $\mathcal{B}$ we need to add an extra vertex to each B_i to have an ℓ-cycle of $K_{2\ell n+1}$ *and* a second coordinate to obtain cycles in $K_{\ell(2\ell n+1)}$. The lift will work differently in the case $i = 2k$, so we shall start by describing the situation for $i = 1, \dots, 2k-1$.

Consider the cycles $Q_i = (0, q_{1,i}, \dots q_{4k,i})$ constructed in Lemma 5.8, and consider the ℓ-cycle $B'_i = ((0,0), (b_{1,i}, q_{1,i}), \dots, (b_{4k-1,i}, q_{4k-1,i}), (0, q_{4k,i}))$. By following this construction to lift B_i we have that

$$\text{the projection of } \Delta B'_i \text{ on } \mathbb{Z}_{2\ell n+1} \text{ is } \Delta B_i \cup \{0,0\}, \tag{5.3}$$

$$\text{the projection of } \Delta B'_i \text{ on } \mathbb{Z}_\ell \text{ is } \Delta Q_i. \tag{5.4}$$

By recalling that $q_{4k,i} = -i$ and $\Delta Q_i \setminus \{\pm q_{4k,i}\}$ contains only ±1s and ±2s (Lemma 5.8), we then have that for any $i = 1, \dots, 2k-1$,

$$\text{the differences of } B'_i \text{ in } \{0\} \times \mathbb{Z}_\ell \text{ are } \pm(0, q_{4k,i}) = \pm(0,i) \tag{5.5}$$

$$\text{the differences of } B'_i \text{ in } \mathbb{Z}^*_{2\ell n+1} \times \mathbb{Z}_\ell \text{ lie in } \Delta B_i \times \{\pm1, \pm2\}. \tag{5.6}$$

A similar, but slightly more complicated approach will be used when lifting B_{2k}; to obtain the result, we need ℓ possible lifts for $B_{2k} = (0, b_{1,2k}, \dots, b_{4k-1,2k})$, say $B'_{2k,m}$ for $m \in \mathbb{Z}_\ell$, corresponding to the fact that we want condition (1) to be satisfied for all $s \in \mathbb{Z}_\ell$. Once more, we add a vertex to B_{2k}, and a second coordinate using now the cycles $P_m = (0, p_{1,m}, \dots p_{4k,m})$ built in Lemma 5.9. For $m \in \mathbb{Z}_\ell$, $m \neq \pm 2k$, set

$$B'_{2k,m} = ((0,0), (b_{1,2k}, p_{1,m}), \dots, (b_{4k-1,2k}, p_{4k-1,m}), (0, p_{4k,m}));$$

on the other hand for $m = \pm 2k$, the difference with 0 as first coordinate will appear from positions $4k-1$ and $4k$; in this case set

$$B'_{2k,m} = ((0,0), (b_{1,2k}, p_{1,m}), \dots, (b_{4k-1,2k}, p_{4k-1,m}), (b_{4k-1,2k}, p_{4k,m})).$$

We first note that $B'_{2k,m}$ satisfies conditions equivalent to (5.3)-(5.4). Also, by Lemma 5.9 we have that $p_{4k,m} = 2k$ for $m \neq \pm 2k$, $p_{4k,m} - p_{4k-1,m} = m$ for $m = \pm 2k$ and $\Delta P_m \setminus \{\pm 2k\}$ contains only ±1s and ±2s. It then follows by the structure of $B'_{2k,m}$ that for any $m \in \mathbb{Z}_\ell$

$$\text{the differences of } B'_{2k,m} \text{ in } \{0\} \times \mathbb{Z}_\ell \text{ are } \pm(0, 2k) \tag{5.7}$$

$$\text{the differences of } B'_{2k,m} \text{ in } \mathbb{Z}^*_{2\ell n+1} \times \mathbb{Z}_\ell \text{ lie in } \Delta B_{2k,m} \times \{\pm1, \pm2\}. \tag{5.8}$$

In this way, taking into account (5.5)-(5.8) any set $\mathcal{S}$ of possible lifts of the B_i's that we obtain will satisfy the first part of the assertion, that is,

$$\begin{aligned} \Delta\mathcal{S} =& (\{0\} \times \mathbb{Z}^*_\ell) \cup \{(x, \varphi_{\mathcal{S}}(x)) \mid x \in Z^-_{2\ell n+1} \setminus (D \cup -D)\} \\ & \text{where } \varphi_{\mathcal{S}} : \mathbb{Z}^-_{2\ell n+1} \setminus (D \cup -D) \to \{\pm1, \pm2\}. \end{aligned} \tag{5.9}$$

Finally, consider the set $\mathcal{S}_m = \{A'_1, \ldots, A'_{n-2k}, B'_1, \ldots, B'_{2k-1}, B'_{2k,m}\}$ consisting of the lifts of the cycles in $\mathcal{A}$ and $\mathcal{B}$ just described. In view of the above considerations, for $m \in \mathbb{Z}_\ell$ the set $\mathcal{S}_m$ satisfies (5.9). For the sake of readability, we set $\varphi_m = \varphi_{\mathcal{S}_m}$ and $\sum_m = \sum_{i\in\overline{D}}(-1)^i\varphi_m(i)$ where $\overline{D} = [2, \ell n - 1] \setminus D$. We want to evaluate the contribution of the cycle $B'_{2k,m}$ to the quantity $\sum_m$; to do so, note that by Lemma 5.4 we have $\Delta B_{2k} = \pm[u, u']$ where $u = 2(2k-1)n$ and $u' = u + 4k - 1$, and set

$$S_{2k,m} = \sum_{i=u}^{u'}(-1)^i\varphi_m(i) \quad \text{for any } m \in \mathbb{Z}_\ell.$$

On the other hand, the contribution given by $\mathcal{S}_m \setminus \{B'_{2k,m}\}$ to $\sum_m$ does not depend on m, so we can write $\sum_m = S_{2k,m} + S'$ for some fixed S'. Since B_{2k} is alternating (Lemma 5.4), we have that $B'_{2k,m}$ satisfies the assumption of Lemma 5.6 for any $m \in \mathbb{Z}_\ell$. By Lemma 5.9 we have that $p_{4k} - 2p_{2k} = m$ for $m \neq \pm 2k$; also, $p_{4k-1} - p_{4k} - 2p_{2k} = m$ for $m = \pm 2k$. It then follows by Lemma 5.6 that $S_{2k,m} = m$ for any $m \in \mathbb{Z}_\ell$ hence, $\sum_m = m + S'$. As m runs over $\mathbb{Z}_\ell$, we have that $\sum_m$ covers all integers in $\mathbb{Z}_\ell$ hence, condition 1 is proven.

We are left to show that condition 2 holds. For $i \in [2, 2k-1]$ we recall that $Q_i = (0, q_{1,i}, \ldots, q_{4k,i})$ is the cycle of Lemma 5.8 hence, $q_{h,i} = h$ for $h \in [1, 2k]$. Therefore, the lift B'_i of the cycle B_i has the following form: $B'_i = ((0,0), (b_{1,i}, 1), \ldots, (b_{2k,i}, 2k), \ldots)$.

Since B_i is a cycle as in Lemma 5.4 then, B_i is alternating and for any $h \in [1, 4k]$ we have that $\delta_{h,i} = (-1)^h(b_{h,i} - b_{h-1,i}) \in \overline{D}$. By Definition 5.1, $\delta_{h,i} \equiv h + 1 \pmod 2$ and $q_{h,i} - q_{h-1,i} = 1$ for $h \in [1, 2k]$, hence $\varphi(\delta_{h,i}) = (-1)^h(q_{h,i} - q_{h-1,i}) = (-1)^{\delta_{h,i}+1}$ for $h \in [1, 2k]$. In other words, set $X = \{\delta_{h,i} \mid (h,i) \in [1, 2k] \times [2, 2k-1]\}$, we have that $\varphi(x) = (-1)^{x+1}$ for any $x \in X$, where $X \subseteq \overline{D}$ has size $(2k-2)2k$ and this proves condition 2. □

Example 5.12. Once again, let $\ell = 9$ and $n = 5$, so that $D = \{2, 5\}$. Here, we explicitly construct a set S of transversal short cycles as in Proposition 5.11 with $s = 0$.

First, we consider the sets $\mathcal{A} = \{A_1\}$ and $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ of short cycles with vertices in $\mathbb{Z}_{91}$ from Example 5.5:

$$A_1 = (-3, 25, 7, 15, 24, 5, 34, -4, 36),$$
$$B_1 = (0, -3, 1, -5, 2, -40, 3, -41), \quad B_2 = (0, -10, 1, -11, 2, -13, 3, -14),$$
$$B_3 = (0, -20, 1, -21, 2, -23, 3, -24), \quad B_4 = (0, -30, 1, -31, 2, -33, 3, -34).$$

We lift these cycles to $\mathbb{Z}_{9\cdot 91}$ according to Proposition 5.11. First we have that $A'_1 = ((-3,0), (25,1), (7,2), (15,3), (24,4), (5,5), (34,6), (-4,7), (36,8))$.
Let us "expand and lift" the first three cycles of $\mathcal{B}$. We will need the three cycles from Lemma 5.8 which in our case are $Q_1 = (0,1,2,3,4,5,6,7,8)$,

$Q_2 = (0, 1, 2, 3, 4, 5, 6, 8, 7)$ and $Q_3 = (0, 1, 2, 3, 4, 5, 7, 8, 6)$, so that the new cycles are

$$\begin{aligned}
B'_1 =&((0,0), (-3,1), (1,2), (-5,3), (2,4), (-40,5), (3,6), (-41,7), (0,8)),\\
B'_2 =&((0,0), (-10,1), (1,2), (-11,3), (2,4), (-13,5), (3,6), (-14,8), (0,7)),\\
B'_3 =&((0,0), (-20,1), (1,2), (-21,3), (2,4), (-23,5), (3,7), (-24,8), (0,6)).
\end{aligned}$$

For the cycle B_4 we need 9 different lifts $B'_{4,m}$ obtained through the cycles P_m, $m = 0, 1, \ldots, 8$, of Lemma 5.9:

$$\begin{aligned}
P_0 =&(0, 7, 8, 1, 2, 3, 5, 6, 4) & P_1 =&(0, 1, 8, 7, 6, 5, 3, 2, 4)\\
P_2 =&(0, 7, 6, 8, 1, 2, 3, 5, 4) & P_3 =&(0, 8, 7, 6, 5, 3, 1, 2, 4)\\
P_4 =&(0, 2, 3, 4, 5, 7, 8, 6, 1) & P_5 =&(0, 7, 6, 5, 4, 2, 1, 3, 8)\\
P_6 =&(0, 2, 3, 1, 8, 7, 6, 5, 4) & P_7 =&(0, 8, 1, 2, 3, 5, 7, 6, 4)\\
P_8 =&(0, 2, 1, 8, 7, 6, 5, 3, 4).
\end{aligned}$$

For example, when $m = 6$, the lift $B'_{4,6}$ of B'_4 via P_6 is:

$$B'_{4,6} = ((0,0), (-30,2), (1,3), (-31,1), (2,8), (-33,7), (3,6), (-34,5), (0,4)).$$

Finally, we set $\mathcal{S}_m = \{A'_1, B'_1, B'_2, B'_3, B'_{4,m}\}$ for any $m \in \mathbb{Z}_9$. Of course, $\Delta\mathcal{S}_m = \Delta A'_1 \cup \Delta B'_1 \cup \ldots \cup \Delta B'_{4,m}$ where:

$$\begin{aligned}
\Delta A'_1 &= \pm(\{8, 9, 28, 29, 40\} \times \{1\}) \cup \pm(\{18, 19, 38, 39\} \times \{-1\}),\\
\Delta B'_1 &= \pm(\{4, 7, 41, 43\} \times \{1\}) \cup \pm(\{3, 6, 42, 44\} \times \{-1\}) \cup \{\pm(0, 1)\},\\
\Delta B'_2 &= \pm(\{11, 13, 16\} \times \{1\}) \cup \pm(\{10, 12, 14, 15\} \times \{-1\})\\
&\quad \cup \{\pm(0, 2), \pm(17, -2)\},\\
\Delta B'_3 &= \pm(\{21, 23\} \times \{1\}) \cup \pm(\{20, 22, 25, 27\} \times \{-1\})\\
&\quad \cup \{\pm(0, 3), \pm(24, -2), \pm(26, 2)\}.
\end{aligned} \tag{5.10}$$

In view of Lemma 5.9, the reader can check that $\Delta B'_{4,m} \subset \Delta B_4 \times \{\pm 1, \pm 2\} \cup \{(0, \pm 4)\}$ for any $m \in \mathbb{Z}_9$. For example, when $m = 6$ we have

$$\begin{aligned}
\Delta B'_{4,6} = &\pm(\{31, 35, 37\} \times \{1\}) \cup \pm(\{34, 36\} \times \{-1\}) \cup \pm(\{30, 33\} \times \{-2\})\\
&\cup \{\pm(0, 4), \pm(32, 2)\}.
\end{aligned}$$

Since the projection of $\Delta\mathcal{S}_m$ on $\mathbb{Z}^*_{91}$ is the set $\Delta A_1 \cup \Delta B_1 \cup \ldots \cup \Delta B_4$ which therefore does not have repeated elements, we are guaranteed that there exists a map $\varphi_m : \mathbb{Z}^-_{91} \setminus (D \cup -D) \to \{\pm 1, \pm 2\}$, where $D = \{2, 5\}$, that allows us to describe $\Delta\mathcal{S}_m$ as follows:

$$\Delta\mathcal{S}_m =(\{0\} \times \mathbb{Z}^*_9) \cup \{(x, \varphi_m(x)) \mid x \in Z^-_{91} \setminus (D \cup -D)\}.$$

Note that $\varphi_m(-x) = -\varphi_m(x)$ for any $x \in Z_{91}^- \setminus (D \cup -D)$. In particular, for $m = 6$ the map φ_m acts on $[2, 44] \setminus D$ as follows:

$$\{(x, \varphi_6(x)) \mid x \in [3, 44] \setminus \{5\}\} = \{(3, -1), (4, 1), (6, -1), (7, 1), (8, 1), (9, 1), (10, -1), (11, 1), (12, -1), (13, 1), (14, -1), (15, -1), (16, 1), (17, -2), (18, -1), (19, -1), (20, -1), (21, 1), (22, -1), (23, 1), (24, -2), (25, -1), (26, 2), (27, -1), (28, 1), (29, 1), (30, -2), (31, 1), (32, 2), (33, 7), (34, -1), (35, 1), (36, -1), (37, 1), (38, -1), (39, 8), (40, 1), (41, 1), (42, -1), (43, 1), (44, -1)\}.$$

By (5.10), we can easily see that condition 2 of Proposition 5.11 is satisfied. We are then left to compute the sum $\sum_m = \sum_{i \in \overline{D}} (-1)^i \varphi_m(i)$ where $\overline{D} = [2, 44] \setminus D$. Note that the projection of $\Delta B'_{4,m}$ on $\overline{D}$ is $[30, 37]$. Therefore, $\sum_m = S' + S_{4,m}$ where S' and $S_{4,m}$ are the contributions given by $\mathcal{S}_m \setminus \{B'_{4,m}\}$ and $B'_{4,m}$ to $\sum_m$, respectively. In other words, for any $m \in \mathbb{Z}_9$ we have

$$S' = \sum_{i \in \overline{D} \setminus [30,37]} (-1)^i \varphi_m(i) = 3 \quad \text{and} \quad S_{4,m} = \sum_{i=30}^{37} (-1)^i \varphi_m(i).$$

Note that S' does not depend on m. According to Lemma 5.9 we have that $S_{4,m} = m$: the reader can easily check that $S_{4,6} = 6$ (check also Example 5.7 for the case $m = 4$). Therefore, $\sum_m = m + S'$ for any $m \in \mathbb{Z}_9$; in particular, $\sum_6 = 0$.

6 The main result

In this section we finish proving our main result, Theorem 1.2. We are trying to build a set of base cycles such that its list of differences is $\mathbb{Z}_{\ell \cdot (2\ell n+1)} \setminus \{0\}$; this will be done by completing the set $\mathcal{L} \cup \mathcal{S}$ of transversal cycles we have built in the two previous sections with two more transversal long cycles C and C' providing the $4 \cdot (2\ell n + 1)$ missing differences. These differences are a particular subset of elements of the form $\mathbb{Z}^*_{2\ell n+1} \times \{\pm 1, \pm 2\}$, together with the elements of the form $\mathbb{Z}^*_{2\ell n+1} \times \{0\}$.

Much of the work in this section will be to ensure that the differences from $\mathbb{Z}^*_{2\ell n+1} \times \{\pm 1, \pm 2\}$ that appear in C and C' have not been already covered in $\mathcal{L} \cup \mathcal{S}$: to this end, first we build an auxiliary function G.

Given a map $F : \mathbb{Z}^-_{2\ell n+1} \to \{\pm 1, \pm 2\}$, we will briefly denote with $\sum F$ the integer $\sum_{i=2}^{\ell n-1} (-1)^i F(i)$. Also, recall that $\overline{D} = [2, \ell n - 1] \setminus D$.

Lemma 6.1. *Let $F : \mathbb{Z}^-_{2\ell n+1} \to \{\pm 1, \pm 2\}$ be a map such that*

$$\sum F \equiv 0 \pmod{\ell}, \qquad F(-x) = -F(x) \text{ for every } x \in \mathbb{Z}^-_{2\ell n+1},$$

and assume that the set $\{x \in \overline{D} \mid F(x) = (-1)^{x+1}\}$ *has size* $\geq \ell - 1$*. Then, for every integer* ρ *there exists a map* $G : \mathbb{Z}_{2\ell n+1}^{-} \to \{\pm 1, \pm 2\}$ *such that:*

1. $\sum G \equiv \rho \pmod{\ell}$*;*
2. $|G(x)| = 1$ *(or* 2*) if and only if* $|F(x)| = 2$ *(or* 1*);*
3. $G(-x) = -G(x)$ *for every* $x \in \mathbb{Z}_{2\ell n+1}^{-}$*.*

Proof. Let $g : \mathbb{Z}_{2\ell n+1}^{-} \to \{\pm 1, \pm 2\}$ be the map defined as follows:

$$g(x) = \begin{cases} 2F(x) & \text{if } F(x) = \pm 1, \\ \frac{F(x)}{2} & \text{if } F(x) = \pm 2. \end{cases}$$

We will get the map required in the statement by slightly modifying g. Let $t \in [0, \ell - 1]$ such that $t \equiv (\sum g - \rho)(\ell - 1)/4 \pmod{\ell}$. By assumption, there is a set $X \subseteq \overline{D}$ of size t such that $F(x) = (-1)^{x+1}$ for any $x \in X$. We define a map $G : \mathbb{Z}_{2\ell n+1}^{-} \to \{\pm 1, \pm 2\}$ as follows

$$G(x) = \begin{cases} -g(x) & \text{if } x \in X \ \cup \ -X, \\ g(x) & \text{otherwise.} \end{cases}$$

Note that $g(x) = (-1)^{x+1} \cdot 2$ for $x \in X$, hence

$$-g(x) = g(x) + (-1)^x \cdot 4 \quad \text{for any} \quad x \in X. \tag{6.1}$$

It is easily seen, keeping in mind the definition of g and the fact that $|G(x)| = |g(x)|$ for $x \in \mathbb{Z}_{2\ell n+1}^{-}$, that Property 2. and 3. hold. Let us prove that also the first property is satisfied.

Let S_1, S_2, and S_3 be the partial sums defined below:

$$S_1 = \sum_{i \in \overline{X}} (-1)^i G(i), \quad S_2 = \sum_{i \in X_e} G(i), \quad \text{and} \quad S_3 = -\sum_{i \in X_o} G(i).$$

where X_e (resp. X_o) denotes the set of even (resp. odd) integers in X, and $\overline{X} = [2, \ell n - 1] \setminus X$; of course,

$$\sum G = S_1 + S_2 + S_3 \quad \text{and} \quad t = |X| = |X_e| + |X_o|.$$

Taking (6.1) into account and recalling how G is defined, it is straightforward to check that we have:

$$S_1 = \sum_{i \in \overline{X}} (-1)^i g(i), \qquad S_2 = \sum_{i \in X_e} -g(i) = \sum_{i \in X_e} (g(i) + 4) = \sum_{i \in X_e} g(i) + 4 \cdot |X_e|,$$

$$\text{and} \quad S_3 = \sum_{i \in X_o} g(i) = \sum_{i \in X_o} (-g(i) + 4) = \sum_{i \in X_o} -g(i) + 4 \cdot |X_o|.$$

By recalling how t is defined, it follows that

$$\sum G = \sum_{i=2}^{\ell n-1} (-1)^i g(i) + 4 \cdot (|X_e| + |X_o|) =$$
$$\sum g + 4t \equiv \ell(\sum g - \rho) + \rho \equiv \rho \pmod{\ell}.$$

We have therefore proven that Property 1. holds and this completes the proof. □

We are now able to prove the main result of this paper.

Theorem 1.2. *HWP*$(\ell(2\ell n+1); [\ell^{2\ell n+1}], [(2\ell n+1)^{\ell}]; \ell n, \frac{(\ell-1)(2\ell n+1)}{2})$ *admits a cyclic solution for any* $\ell \equiv 1 \pmod 4$ *and* $n \geq (\ell-1)/2$.

Proof. Let $\ell = 4k+1$ and assume $n \geq 2k$. Once more we set $k \geq 2$, since the assertion has been proven in [10] for $k = 1$.

We first take a set $\mathcal{L}$ of $2k-2$ transversal $(2\ell n+1)$-cycles as in Proposition 4.4, and set $s = -\sum_{i\in D}(-1)^i f(i)$, where f is the map from Proposition 4.4. Then we take a set $\mathcal{S}$ of n transversal ℓ–gons as in Proposition 5.11, choosing s as above: it then follows that $s = \sum_{i\in\overline{D}}(-1)^i\varphi(i)$ where $\overline{D} = [2, \ell n-1] \setminus D$ and φ is the map from Proposition 5.11.

To apply Theorem 2.1 we have to find two transversal $(2\ell n+1)$-cycles C and C' of $K_{\ell(2\ell n+1)}$ whose differences coincides with the complement of $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ in $(\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell) \setminus \{(0,0)\}$.

Let F be the map $F : \mathbb{Z}^-_{2\ell n+1} \to \{\pm 1, \pm 2\}$ obtained by glueing together the maps f and φ:

$$F(x) = \begin{cases} f(x) & \text{if } x \in D \cup -D, \\ \varphi(x) & \text{if } x \in \overline{D} \cup -\overline{D}. \end{cases} \tag{6.2}$$

Note that we can write $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ as the disjoint union of the following sets:

$$\Delta_1 = \{0\} \times \mathbb{Z}^*_\ell; \qquad \Delta_2 = \mathbb{Z}^*_{2\ell n+1} \times \mathbb{Z}_\ell \setminus \{0, \pm 1, \pm 2\}; \tag{6.3}$$
$$\Delta_3 = \cup_{i\in\mathbb{Z}^-_{2\ell n+1}} \{i\} \times \{F(i)\}.$$

One can check that F satisfies the assumptions of Lemma 6.1. In fact,

$$\sum F = s + \sum_{i\in D} (-1)^i f(i) \equiv 0 \pmod{\ell}. \tag{6.4}$$

Moreover, it is clear that the map F has the property that $F(-x) = F(x)$ for every $x \in \mathbb{Z}^-_{2\ell n+1}$ since $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ is obviously symmetric. Finally, Proposition 5.11.(2) ensures that $\{x \in \overline{D} \mid F(x) = (-1)^{x+1}\}$ has size $\geq \ell-1$. Then, by Lemma 6.1 there is a map $G : \mathbb{Z}^-_{2\ell n+1} \to \{\pm 1, \pm 2\}$ such that

$$\sum G \equiv -1 \pmod{\ell}; \tag{6.5}$$

$$|G(x)| = 1 \ \text{(or 2)} \quad \text{if and only if} \quad |F(x)| = 2 \ \text{(or 1)}; \tag{6.6}$$

$$G(-x) = -G(x) \quad \text{for every} \quad x \in \mathbb{Z}^-_{2\ell n+1}. \tag{6.7}$$

Let us construct the transversal $(2\ell n + 1)$-cycle $C = (c_0, c_1, \ldots, c_{2\ell n})$ of $K_{\ell(2\ell n+1)}$ with $c_i = (x_i, y_i) \in \mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell$ defined as follows:

$$x_i = (-1)^{i+1} \left\lfloor \frac{i+1}{2} \right\rfloor \quad \text{for } 0 \le i \le 2\ell n;$$

$$y_i = \begin{cases} 1 + \displaystyle\sum_{j=2}^{i} (-1)^j F(j) & \text{for } i \in [2, \ell n - 1], \\ 1 + \displaystyle\sum_{j=\ell n+2}^{i} (-1)^j G(j) & \text{for } i \in [\ell n + 2, 2\ell n - 1]; \end{cases}$$

$$(y_0, y_1, y_{\ell n}, y_{\ell n+1}, y_{2\ell n}) = (0, 1, 2, 1, -2).$$

Now construct another transversal $(2\ell n+1)$-cycle $C' = (c'_0, c'_1, \ldots, c'_{2\ell n})$ of $K_{\ell(2\ell n+1)}$ with $c'_i = (x'_i, y'_i)$ defined as follows.

1st case: n is even.

$$x'_i = \begin{cases} x_i & \text{for } i \in [0, \ell n], \\ -x_i & \text{for } i \in [\ell n + 1, 2\ell n]; \end{cases} \qquad y'_i = \begin{cases} 0 & \text{for } i \in [0, \ell n], \\ y_i & \text{for } i \in [\ell n + 1, 2\ell n]. \end{cases}$$

2nd case: n is odd.

$$x'_i = \begin{cases} x_i & \text{for } i \in [0, \ell n - 1], \\ -x_i & \text{for } i \in [\ell n, 2\ell n]; \end{cases} \qquad y'_i = \begin{cases} 0 & \text{for } i \in [0, \ell n - 1], \\ 1 & \text{for } i = \ell n, \\ y_i & \text{for } i \in [\ell n + 1, 2\ell n]. \end{cases}$$

Let $\pi(C)$ and $\pi(C')$ be the projections of C and C' on $\mathbb{Z}_{2\ell n+1}$. It is clear that

$$\pi(C) = (0, 1, -1, 2, -2, \ldots, \ell n, -\ell n).$$

The expression of $\pi(C')$ is similar but there is a twist in the middle, namely,

$$\pi(C') = (0, 1, -1, 2, -2, \ldots, \nu, -\nu, -(\nu+1), \nu+1, \ldots, -\ell n, \ell n), \text{ and } \nu = \left\lfloor \frac{\ell n}{2} \right\rfloor.$$

In any case the above remark guarantees that both C and C' are transversals. Now we need to calculate the lists of differences of C and C'. This is easily done, but it is important to note first that we have

$$y_{\ell n-1} = 1 \qquad \text{and} \qquad y_{2\ell n-1} = 0. \tag{6.8}$$

Indeed, by definition, we have $y_{\ell n-1} = 1 + \sum F$. Again by definition, we have $y_{2\ell n-1} = 1 + \sigma$ with $\sigma = \sum_{i=\ell n+2}^{2\ell n-1} (-1)^i G(i)$. Taking (6.7) into account, it is straightforward to check that $\sigma = \sum_{i=2}^{\ell n-1}(-1)^{i+1}G(-i) = \sum G$. Hence, in view of (6.4) and (6.5), we have that (6.8) holds.

Let us consider the following subpaths of the cycle C:

$$P_1 = (c_1, c_2, \ldots, c_{\ell n-1}); \quad P_2 = (c_{\ell n-1}, c_{\ell n}, c_{\ell n+1});$$

$$P_3 = (c_{\ell n+1}, c_{\ell n+2}, \ldots, c_{2\ell n-1}); \quad P_4 = (c_{2\ell n-1}, c_{2\ell n}, c_0, c_1).$$

Taking (6.8) into account, it is straightforward to check that we have:

$$\Delta P_1 = \pm\{(i, -F(i)) \mid 2 \le i \le \ell n - 1\};$$

$$\Delta P_2 = \pm\{(\ell n, -1), (\ell n + 1, -1)\} = \pm\{(\ell n, -1), (\ell n, 1)\};$$

$$\Delta P_3 = \pm\{(i, -G(i)) \mid \ell n{+}2 \le i \le 2\ell n{-}1\} = \pm\{(i, -G(i)) \mid 2 \le i \le \ell n{-}1\};$$

$$\Delta P_4 = \pm\{(1, -2), (\ell n, 2), (1, 1)\}.$$

Now consider the following subpaths of the cycle C':

$$P'_1 = (c'_0, c'_1, \ldots, c'_{\ell n}); \quad P'_2 = (c'_{\ell n}, c'_{\ell n+1});$$

$$P'_3 = (c'_{\ell n+1}, c'_{\ell n+2}, \ldots, c'_{2\ell n-1}); \quad P'_4 = (c'_{2\ell n-1}, c'_{2\ell n}, c'_0).$$

Also here, in view of (6.8), we can easily check that we have:

$$\Delta P'_1 = \begin{cases} \mathbb{Z}^*_{2\ell n+1} \times \{0\} & \text{for } n \text{ even;} \\ (\mathbb{Z}^*_{2\ell n+1} \setminus \{\pm \ell n\}) \times \{0\} \ \cup \ \pm\{(1, -1)\} & \text{for } n \text{ odd;} \end{cases}$$

$$\Delta P'_2 = \begin{cases} \pm\{(1, -1)\} & \text{for } n \text{ even;} \\ \pm\{(\ell n, 0)\} & \text{for } n \text{ odd;} \end{cases}$$

$$\Delta P'_3 = \pm\{(i, G(i)) \mid 2 \le i \le \ell n - 1\}; \quad \Delta P'_4 = \pm\{(1, 2), (\ell n, -2)\}.$$

It is evident that C is the union of the pairwise edge-disjoint paths P_is and that C' is the union of the edge-disjoint paths P'_is; we can therefore write:

$$\Delta\{C, C'\} = \Delta\{P_1, P_2, P_3, P_4, P'_1, P'_2, P'_3, P'_4\}.$$

In this way we see that $\Delta\{C, C'\}$ is the union of the following pairwise disjoint lists:

$$\Delta_4 = \mathbb{Z}^*_{2\ell n+1} \times \{0\}; \qquad \Delta_5 = \{1, \ell n, \ell n + 1, 2\ell n\} \times \{\pm 1, \pm 2\};$$

$$\Delta_6 = \bigcup_{i \in \mathbb{Z}^-_{2\ell n+1}} \{i\} \times \{-F(i), G(i), -G(i)\}.$$

Note that $\{1, \ell n, \ell n + 1, 2\ell n\} = \mathbb{Z}^*_{2\ell n+1} \setminus \mathbb{Z}^-_{2\ell n+1}$. Also, taking (6.6) into account we can write $\{-F(i), G(i), -G(i)\} = \{\pm 1, \pm 2\} \setminus \{F(i)\}$ for every

$i \in \mathbb{Z}^-_{2\ell n+1}$. Therefore, recalling that $\Delta_3 = \bigcup_{i\in\mathbb{Z}^-_{2\ell n+1}} \{i\} \times \{F(i)\}$, we have that

$$\Delta_3 \cup \Delta_5 \cup \Delta_6 = \mathbb{Z}^*_{2\ell n+1} \times \{\pm1, \pm2\}.$$

We then conclude that $\Delta\{C, C'\} \cup \Delta\mathcal{S} \cup \Delta\mathcal{L} = \mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell \setminus \{(0, 0)\}$, namely, $\Delta\{C, C'\}$ is exactly the complement of $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ in $\mathbb{Z}_{2\ell n+1} \times \mathbb{Z}_\ell \setminus \{(0, 0)\}$. This means that $\mathcal{S}$ and $\mathcal{L}' = \mathcal{L} \cup \{C, C'\}$ are two sets of short and long transversal cycles, respectively, as required by Theorem 2.1, and the assertion follows. □

Example 6.2. Here, we solve HWP$(9 \cdot 91; [9^{91}], [91^9]; 45, 364)$ by showing the existence of two sets of short and long transversal cycles as required by Theorem 2.1.

Let $\ell = 9$ and $n = 5$. Note that $D = \{2, 5\}$ and $\overline{D} = [2, 44] \setminus \{2, 5\}$; also, $\mathbb{Z}^-_{91} = \pm[2, 44]$. The set $\mathcal{L} = \{C'_1, C'_2\}$ built in Example 4.5 according to Proposition 4.4 is such that

$$\Delta\mathcal{L} = \mathbb{Z}^*_{91} \times \mathbb{Z}_9 \setminus \{0, \pm1, \pm2\} \cup \{\pm(2, f(2))), \pm(5, f(5))\}. \tag{6.9}$$

where $f(2) = f(5) = -1$. Now set $s = -\sum_{i\in D}(-1)^i f(i) = 0$. We need a set $\mathcal{S}$ of 5 transversal 9–gons as in Proposition 5.11, choosing s as above. It is enough to take $\mathcal{S}$ to be the set of transversal short cycles constructed in Example 5.12 for $m = 6$.

Let $F : \mathbb{Z}^-_{2\ell n+1} \to \{\pm1, \pm2\}$ be the map (6.2) obtained by glueing together the maps f and φ. Recall that $F(-i) = -F(i)$ for any $i \in \mathbb{Z}^-_{91}$; also, by collecting the values of φ from Example 5.12, we have that

$$\begin{aligned}\{(i, F(i)) \mid i \in [2, 44]\} = \{&(2, 8), (3, 8), (4, 1), (5, 8), (6, 8), (7, 1), (8, 1), (9, 1),\\ &(10, 8), (11, 1), (12, 8), (13, 1), (14, 8), (15, 8), (16, 1), (17, 7), (18, 8),\\ &(19, 8), (20, 8), (21, 1), (22, 8), (23, 1), (24, 7), (25, 8), (26, 2), (27, 8),\\ &(28, 1), (29, 1), (30, 7), (31, 1), (32, 2), (33, 7), (34, 8), (35, 1), (36, 8),\\ &(37, 1), (38, 8), (39, 8), (40, 1), (41, 1), (42, 8), (43, 1), (44, 8)\}.\end{aligned}$$

Note that $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ is the disjoint union of the following sets:

$$\Delta_1 = \{0\} \times \mathbb{Z}^*_9; \quad \Delta_2 = \mathbb{Z}^*_{91} \times \mathbb{Z}_\ell \setminus \{0, \pm1, \pm2\}; \quad \Delta_3 = \{(i, F(i) \mid i \in \mathbb{Z}^-_{91}\}.$$

We are going to build two remaining transversal 91-cycles C and C' of $K_{9\cdot91}$ whose differences coincides with the complement of $\Delta\mathcal{S} \cup \Delta\mathcal{L}$ in $(\mathbb{Z}_{91} \times \mathbb{Z}_9) \setminus \{(0, 0)\}$, that is, $\Delta C \cup \Delta C' = \Delta_4 \cup \Delta_5 \cup \Delta_6$ where:

$$\Delta_4 = \mathbb{Z}^*_{91} \times \{0\}; \quad \Delta_5 = \{\pm1, \pm45\} \times \{\pm1, \pm2\};$$

$$\Delta_6 = \bigcup_{i\in\mathbb{Z}^-_{91}} \{i\} \times (\{\pm1, \pm2\} \setminus \{F(i)\}).$$

As a result, we obtain two sets $\mathcal{S}$ and $\mathcal{L}' = \mathcal{L} \cup \{C, C'\}$ of short and long transversal cycles, respectively, as required by Theorem 2.1. This guarantees the existence of a solution to HWP$(9 \cdot 91; [9^{91}], [91^9]; 45, 364)$.

Let us start with $C = ((x_0, y_0), (x_1, y_1), \ldots, (x_{90}, y_{90}))$. By construction, $(x_0, x_1, \ldots, x_{44}) = (0, 1, -1, 2, -2, \ldots, 45, -45)$; Also, it is possible to check that the sequence $(y_2, y_3, \ldots, y_{44})$ of second components is the string

$$(0, 1, 2, 3, 2, 1, 2, 1, 0, 8, 7, 6, 5, 6, 7, 0, 8, 0, 8, 7, 6, 5, 3, 4, 6, 7, 8, 7,\\ 5, 4, 6, 8, 7, 6, 5, 4, 3, 4, 5, 4, 3, 2, 1)$$

with $y_{44} = 1$ as required in (6.8). We have that $(y_0, y_1, y_{45}, y_{46}, y_{90})$ are defined in the construction to be $(0, 1, 2, 2, 1, 7)$.

It remains to apply Lemma 6.1, with $\rho = -1$, to find the function G and thus $(y_{47}, \ldots, y_{89})$. The quantity t required in the proof of the lemma is $t \equiv 2(\sum g + 1) \pmod 9$, where $g : \mathbb{Z}_{91}^- \to \{\pm 1, \pm 2\}$ is the map defined as follows: $g(x) = 2 \cdot F(x)$ or $F(x)/2$ according to whether $F(x) = \pm 1$ or $F(x) = \pm 2$. One can check that $t = 8$. Now, a set $X \subset [2, 44]$ of size t with the property that $F(x) = (-1)^{x+1}$ can be $X = \{10, 11, 12, 13, 14, 18, 20, 21\}$. We can now obtain the required map $G : \mathbb{Z}_{91}^- \to \{\pm 1, \pm 2\}$ defined as follows: $G(x) = -g(x)$ or $g(x)$ according to whether $x \in X \cup -X$ or not. One can check that $G(-x) = -G(x)$ for $x \in Z_{91}^-$ and

$$\{(i, G(i)) \mid i \in [2, 44]\} = \{(47, 2), (48, 7), (49, 2), (50, 7), (51, 7), (52, 2), (53, 2), (54, 7),\\ (55, 2), (56, 7), (57, 2), (58, 1), (59, 8), (60, 7), (61, 1), (62, 7), (63, 7),\\ (64, 2), (65, 8), (66, 2), (67, 1), (68, 7), (69, 2), (70, 2), (71, 7), (72, 2),\\ (73, 7), (74, 1), (75, 7), (76, 2), (77, 7), (78, 2), (79, 7), (80, 2), (81, 7),\\ (82, 7), (83, 7), (84, 7), (85, 2), (86, 2), (87, 7), (88, 2), (89, 2)\}.$$

The remaining list of second components $(y_{47}, \ldots, y_{89})$ can then be computed, giving us the list

$$(8, 6, 4, 2, 4, 6, 4, 2, 0, 7, 5, 6, 7, 5, 4, 2, 4, 6, 7, 0, 8, 6, 4, 6, 8,\\ 1, 3, 4, 6, 8, 1, 3, 5, 7, 0, 7, 0, 7, 5, 7, 0, 2, 0)$$

so that $y_{89} = 0$ as needed in (6.8). Some minor tweaks are required for the cycle $C' = ((x'_0, y'_0), (x'_1, y'_1), \ldots, (x'_{90}, y'_{90}))$; here the first components $(x'_0, x'_1, \ldots, x'_{90})$ are $(0, 1 {-1}, 2, -2, \ldots, 22, -22, -23, 23, -24, 24, \ldots, -45, 45)$, and the second components are simply $y'_0 = y'_1 = \cdots = y'_{44} = 0$, $y'_{45} = 1$, and $y'_{46} = y_{46}, y'_{47} = y_{47}, \ldots, y'_{90} = y_{90}$. We can finally check that:

$$\Delta C = \{(i, -F(i)), (i, -G(i)) \mid i \in \mathbb{Z}_{91}^-\} \cup \pm\{(45, \pm 1), (1, -2), (45, 2), (1, 1)\},$$

$$\Delta C' = \mathbb{Z}_{91}^* \times \{0\} \ \cup \ \{(i, G(i)) \mid i \in \mathbb{Z}_{91}^-\} \ \cup \ \pm\{(1, -1), (1, 2), (45, -2)\}.$$

Therefore $\Delta C \ \cup \ \Delta C' = \Delta_4 \ \cup \ \Delta_5 \ \cup \ \Delta_6$, since $\pm\{F(i), G(i)\} = \pm\{1, 2\}$ for any $i \in \mathbb{Z}_{91}^-$.

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