

On equitably 2-colourable odd cycle decompositions

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Abstract

An ℓ -cycle decomposition of K_v is said to be *equitably 2-colourable* if there is a 2-vertex-colouring of K_v such that each colour is represented (approximately) an equal number of times on each cycle: more precisely, we ask that in each cycle C of the decomposition, each colour appears on $\lfloor \ell/2 \rfloor$ or $\lceil \ell/2 \rceil$ of the vertices of C . In this paper we study the existence of equitably 2-colourable ℓ -cycle decompositions of K_v , where ℓ is odd, and prove the existence of such a decomposition for $v \equiv 1, \ell \pmod{2\ell}$.

KEYWORDS

colourings of designs, cycle decompositions, equitable colourings

1 | INTRODUCTION

In this paper, we consider the existence of equitably 2-colourable ℓ -cycle decompositions of K_v , where ℓ is odd. We say that a graph Γ *decomposes* into subgraphs B_1, B_2, \dots, B_t if the edge sets of the B_i partition the edges of Γ . In the case that $B_1 \simeq B_2 \simeq \dots \simeq B_t \simeq B$, we speak of a *B-decomposition* of Γ . In particular, an ℓ -cycle decomposition of Γ is a *B-decomposition* of Γ where B is an ℓ -cycle C_ℓ .

The study of cycle decompositions originates in the 19th century, with the classical results of Kirkman and Walecki on 3-cycle and Hamilton cycle decompositions of the complete graph [17, 19]. If Γ decomposes into cycles, then each vertex of Γ must have even degree. Thus, the

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complete graph K_v admits a cycle decomposition only when v is odd; for v even, it is common to instead decompose the cocktail party graph $K_v - I$, which is formed from K_v by removing the edges of a 1-factor I . Where necessary, we will use the notation K_v^* to denote K_v if v is odd and $K_v - I$ if v is even. The existence of ℓ -cycle decompositions of K_v^* was solved by Alspach and Gavlas [5] and Šajna [22]; see also [8] for an alternative proof in the odd-cycle case.

Theorem 1.1 (Alspach and Gavlas [5] and Šajna [22]). *There exists a C_ℓ -decomposition of K_v^* if and only if $3 \leq \ell \leq v$ and ℓ divides $v \lfloor \frac{v-1}{2} \rfloor$.*

Note that in particular, there exists an ℓ -cycle decomposition of K_v whenever $v \equiv 1 \pmod{2\ell}$ or ℓ is odd and $v \equiv \ell \pmod{2\ell}$.

Decompositions of K_v^* into cycles of varying lengths have also been studied. In 1981, Alspach [3] conjectured that the obvious necessary conditions for decomposing K_v^* into cycles of given lengths are also sufficient. This was finally proven in 2014 by Bryant, Horsley and Pettersson.

Theorem 1.2 (Bryant et al. [7]). *Let $n \geq 3$ be an integer and $\ell_1, \ell_2, \dots, \ell_t$ be a list of integers (not necessarily distinct) with $3 \leq \ell_i \leq v$ for all $i \in \{1, \dots, t\}$. There is a decomposition of K_v^* into cycles of lengths $\ell_1, \ell_2, \dots, \ell_t$ if and only if $\ell_1 + \ell_2 + \dots + \ell_t = v \lfloor \frac{v-1}{2} \rfloor$.*

The focus of this paper is a problem related to vertex colourings of cycle decompositions. If \mathcal{C} is a cycle decomposition of Γ , then a c -colouring of \mathcal{C} is a function $\phi : V(\Gamma) \rightarrow S$, where $|S| = c$. Informally, a c -colouring may be thought of as an assignment of c colours to the vertices of Γ . We will mainly consider the case $c = 2$, and will denote the colours by *red* and *blue*.

Of course, we generally wish such a colouring to satisfy additional properties concerning the cycles of \mathcal{C} . For example, if we require that each cycle have at least two vertices coloured differently, the colouring is called *weak*; see [11, 12, 16] for results on weak colourings of cycle systems. In this paper, we will consider equitable colourings. We say that a c -colouring of an ℓ -cycle decomposition \mathcal{C} is *equitable* if each cycle contains either $\lfloor \ell/c \rfloor$ or $\lceil \ell/c \rceil$ vertices of each colour, that is, the colours are as equally distributed as possible amongst the vertices of each cycle. Note that the term *equitable colouring* also occurs in the literature in other contexts, in particular to refer to colourings in which the sizes of the colour classes differ by at most 1 (see, e.g., [14]); our use of the word *equitable* follows the definition introduced in the Ph.D. thesis of Waterhouse [26] (see also [1, 2, 18, 27]).

Equitable colourings of cycle systems of the complete and cocktail party graphs were considered in [1, 2], while the papers [18, 27] consider equitable colourings of complete multipartite graphs. The main results of these papers restrict their attention to the case that the number of colours is 2 or 3, and the cycle length is small (at most 6). In particular, in [1, 2], the authors completely determine the existence of equitably 2- or 3-colourable ℓ -cycle systems of K_v and $K_v - I$ when the cycle length ℓ is 4, 5 or 6. In [10], the present authors consider equitable 2-colourings of even cycle systems of $K_v - I$.

In this paper, we consider equitable 2-colourings of ℓ -cycle systems of the complete graph K_v . A straightforward counting argument (see [1, 10]) shows that no equitably 2-colourable ℓ -cycle decomposition of K_v can exist if ℓ is even; hence necessarily we consider only odd cycle

length ℓ . We focus on the case where $v \equiv 1, \ell \pmod{2\ell}$, in which there exists an ℓ -cycle system of K_v for any $\ell \geq 3$. Our main result is the following.

Theorem 1.3. *Let $\ell > 5$ be odd, and $v \equiv 1$ or $\ell \pmod{2\ell}$. There is an equitably 2-colourable ℓ -cycle decomposition of K_v .*

In the case that the cycle length $\ell = p$ is prime, Theorem 1.1 asserts that a p -cycle system of K_v exists if and only if $v \equiv 1$ or $p \pmod{2p}$. Thus, Theorem 1.3, combined with the results of [1] for cycle length 5 and the nonexistence of an equitably 2-colourable Steiner triple system [21] of order greater than 3, gives a complete existence result for equitably 2-colourable p -cycle systems of K_v .

Corollary 1.4. *Let p be an odd prime. There exists an equitably 2-colourable p -cycle decomposition of K_v if and only if $p \geq 5$ and $v \equiv 1$ or $p \pmod{2p}$, or $p = v = 3$.*

Theorem 1.3 will be proved in Theorem 3.9 for $v \equiv 1 \pmod{2\ell}$, and in Theorem 3.10 for $v \equiv \ell \pmod{2\ell}$. As we will see, the proof of Theorem 3.9 relies on the existence of an equitably 2-colourable ℓ -cycle decomposition of $K_{2\ell+1}$ with colour classes of sizes ℓ and $\ell + 1$, and this will be established in Section 2.1. Note that, moreover, the existence of an equitably 2-colourable ℓ -cycle decomposition of $K_{4\ell+1}$ cannot be established via the methods used in proving Theorem 3.9, but requires a somewhat long ad hoc construction, which will be presented in Section 2.2.

2 | THE $2\ell + 1$ AND $4\ell + 1$ CASE

In this section, we prove the existence of an equitably 2-colourable ℓ -cycle decomposition of $K_{2\ell+1}$ (in Section 2.1) and $K_{4\ell+1}$ (in Section 2.2).

The constructions in Sections 2.1 and 2.2 will use 2-rotational cycle systems (see, e.g., [23]), that is, cycle systems of order $2n + 1$ admitting an automorphism having exactly one fixed point and two cycles of length n each.

The set of vertices will be identified with $(\mathbb{Z}_n \times \{0, 1\}) \cup \{\infty\}$, with ∞ being the fixed point as usual in this situation, and we will often denote the vertex (a, i) , $i = 0, 1$, with a_i . The colourings used will differ in the two cases, but share the property that vertices a_0 and a_1 will have different colours. The vertex ∞ will have colour blue.

A cycle system will be built via difference methods using a set of base cycles, which will be developed modulo $(n, -)$, that is, $(\text{mod } n)$ with respect to the first coordinate (with the usual understanding that $\infty + i = \infty$); each base cycle C used in the decompositions that follow will in this way give rise to an orbit of d cycles, $d|n$. As usual, the difference arising from an edge of a cycle $a_i \sim b_j$ will be called *mixed* if $i \neq j$ and *pure* (or *i -pure*) if $i = j$. In particular, our convention is that the edge $a_0 \sim b_1$ will give the mixed difference $b - a \pmod{n}$, and the edge $a_i \sim b_i$ will give the i -pure difference $\pm(b - a) \pmod{n}$.

We point out that the constructions used here are similar to those used in [24, 25] to obtain almost resolvable odd ℓ -cycle systems of order $2\ell + 1$ and $4\ell + 1$. In particular, a suitable colouring of the cycle systems built in [25] gives an equitably 2-colourable ℓ -cycle decomposition of $K_{2\ell+1}$ when $\ell \equiv 1 \pmod{4}$.

2.1 | Case $\nu = 2\ell + 1$

In this section we show the existence of an equitably 2-colourable ℓ -cycle decomposition of $K_{2\ell+1}$ with colour classes of sizes ℓ and $\ell + 1$. As mentioned before, this cycle system will be 2-rotational and will arise from a set of three base cycles to be developed modulo $(\ell, -)$, that is, $(\text{mod } \ell)$ with respect to the first coordinate; the first cycle contains ∞ and has an orbit of length ℓ , the second base cycle will have trivial stabilizer and give a length ℓ orbit, and the third cycle is stabilized by \mathbb{Z}_ℓ and has an orbit of length one.

To build the cycle containing ∞ we will use a result on graceful labellings. A *graceful labelling* of a path P of length k is an injective assignment of the integers $0, \dots, k$ to the vertices of P with the property that each integer $1, \dots, k$ appears as the difference of the endvertices of an edge (see [15]). For example, the ordered labelling $0, 5, 1, 4, 2, 3$ gracefully labels a path of length 5.

Lemma 2.1. *Consider a path with $h \geq 2$ vertices having labels $0, 1, \dots, h - 1$. There is a graceful labelling of this path, say P_0 , such that one leaf has label 0, and one, say P_1 , in which one leaf has label 1.*

The proof follows, for instance, from Theorem 3 in [13].

Theorem 2.2. *There is an equitably 2-colourable ℓ -cycle decomposition of $K_{2\ell+1}$ with colour classes of sizes ℓ and $\ell + 1$.*

Proof. Let $V(K_{2\ell+1}) = (\mathbb{Z}_\ell \times \{0, 1\}) \cup \{\infty\}$; as already mentioned, we will often denote the vertex (a, i) , $i = 0, 1$ with a_i for brevity. We colour the vertices by using red on the vertices with an even label, and blue otherwise in the part labelled 0, and reverse this colouring in the part with label 1. We colour ∞ blue. We will build a cycle system developing modulo $(\ell, -)$.

We can use Lemma 2.1 to construct an ℓ -cycle C_∞ having $\ell - 1$ edges of i -pure differences $\pm 1, \pm 2, \dots, \pm((\ell - 3)/2)$, $i = 0, 1$, and one of mixed difference 1, which is equitably 2-coloured with respect to the colouring given above, as follows.

Set $h = (\ell - 1)/2$ and apply Lemma 2.1 to build paths P_0 on the $h = (\ell - 1)/2$ vertices $0_0, \dots, (h - 1)_0$, with the last leaf labelled 0_0 , and P_1 on $0_1, \dots, (h - 1)_1$, with the first leaf labelled 1_1 . We obtain the cycle C_∞ by concatenating ∞, P_0 and P_1 .

Now consider the ℓ -cycle

$$C = \left(0_1, (\ell - 2)_0, 1_1, (\ell - 3)_0, 2_1, \dots, \left(\frac{\ell + 1}{2}\right)_0, \left(\frac{\ell - 3}{2}\right)_1, \left(\frac{\ell - 1}{2}\right)_0, \left(\frac{\ell - 1}{2}\right)_1 \right).$$

The cycle C contains an edge of each mixed difference except for 1 and one edge of 1-pure difference $\pm(\ell - 1)/2$.

Note that the ℓ -set of cycles we obtain by developing this cycle on first coordinates consists of equitably coloured cycles. Indeed the last two vertices will have different colours in C and all its translates; the remaining vertices form a run of $(\ell - 1)/2$ consecutive $(\text{mod } \ell)$ 1-vertices, and $(\ell - 3)/2$ consecutive $(\text{mod } \ell)$ 0-vertices, starting from, in the cycle C , $0_1, \dots, (\ell - 3)/2_1$ and $(\ell + 1)/2_0, \dots, (\ell - 2)_0$. It can be checked that

this pattern remains equitably coloured in all translates: this is fairly obvious for $C + i, i \neq 2, (\ell + 3)/2$ and can be checked explicitly in these cases.

Now C and C_∞ between them contain an edge of each mixed difference and each pure difference except for the 0-pure difference $\pm(\ell - 1)/2$, and $(\ell - 1)/2$ is a unit in \mathbb{Z}_ℓ ; hence this last difference is covered by the cycle

$$C_{(\ell-1)/2} = (0_0, ((\ell - 1)/2)_0, (2(\ell - 1)/2)_0, (3(\ell - 1)/2)_0, \dots).$$

□

Example 2.3. For $\ell = 9$, we can take the three cycles

$$\begin{aligned} C_\infty &= (\infty, 2_0, 1_0, 3_0, 0_0, 1_1, 2_1, 0_1, 3_1), \\ C &= (0_1, 7_0, 1_1, 6_0, 2_1, 5_0, 3_1, 4_0, 4_1), \\ C_4 &= (0_0, 4_0, 8_0, 3_0, 7_0, 2_0, 6_0, 1_0, 5_0). \end{aligned}$$

2.2 | Case $v = 4\ell + 1$

The aim of this section is to show an explicit construction for an equitably 2-colourable ℓ -cycle decomposition of $K_{4\ell+1}$. We will prove the following.

Theorem 2.4. *There is an equitably 2-colourable ℓ -cycle decomposition of $K_{4\ell+1}$ with colour classes of sizes 2ℓ and $2\ell + 1$.*

As above, we will construct a 2-rotational cycle system, this time developed modulo $(2\ell, -)$ from a set of six base cycles. On part 0 we will colour the vertices from 0 to $\ell - 1$ red and those from ℓ to $2\ell - 1$ blue, and reverse this colouring in part 1. The vertex ∞ has colour blue.

To help the reader, we will first present an example to clarify the general construction.

Example 2.5. Let $\ell = 17$ and $v = 69$: we identify the vertices of K_{69} with $\mathbb{Z}_{34} \times \{0, 1\}$, and denote the vertex $(a, i), i = 0, 1$ with a_i . The cycle system has six base cycles which will be developed (mod 34). The first three cycles C_p, C_0 and C_1 , have orbits of length 34 and, in our example, are as follows:

$$\begin{aligned} C_p &= (0_0, 1_0, 17_0, 2_0, 8_0, 3_0, 7_0, 4_0, 4_1, 7_1, 3_1, 8_1, 2_1, 10_1, 0_1, 1_1, 17_1) \\ C_0 &= (6_0, 9_1, 4_0, 11_1, 2_0, 13_1, 0_0, 6_1, 9_0, 4_1, 11_0, 2_1, 13_0, 0_1, 16_0, 23_0, 22_1). \\ C_1 &= (6_1, 8_0, 4_1, 12_0, 2_1, 14_0, 0_1, 6_0, 8_1, 4_0, 12_1, 2_0, 14_1, 0_0, 15_1, 22_1, 21_0) \end{aligned}$$

Note that the differences from C_0 and C_1 cover all the mixed differences except for 0 and $\ell = 17$, and in C_0 and C_1 we also have one edge giving a pure difference, in the example the 0-pure difference ± 7 in C_0 and the 1-pure difference ± 7 in C_1 . The mixed differences 0 and $\ell = 17$ appear in C_p , and the remaining differences from C_p are all pure, namely, 0-pure differences $\pm\{1, 3, 4, 5, 6, 15, 16\}$ and 1-pure differences $\pm\{1, 3, 4, 5, 6, 8, 10, 16\}$.

Two base cycles C_∞^0, C_∞^1 will contain the vertex ∞ and all remaining 0-pure (resp., 1-pure) differences, except for 0-pure difference ± 2 . These will be built using

Lemma 2.12, which states that it is possible to build cycles C_∞^i , $i = 0, 1$, that will contain ∞ , have edges of i -pure difference ℓ and any $(\ell - 3)/2$ other i -pure differences; these cycles will have orbit-length ℓ . In our case we have

$$\begin{aligned} C_\infty^0 &= (\infty, 0_0, 14_0, 1_0, 13_0, 2_0, 12_0, 3_0, 11_0, 28_0, 20_0, 29_0, 19_0, 30_0, 18_0, 31_0, 17_0), \\ C_\infty^1 &= (\infty, 0_1, 15_1, 1_1, 14_1, 2_1, 13_1, 4_1, 6_1, 23_1, 21_1, 30_1, 19_1, 31_1, 18_1, 32_1, 17_1). \end{aligned}$$

We are left with the 0-pure difference ± 2 that gets covered by the cycle

$$C_2^0 = (0_0, 2_0, 4_0, 6_0, 8_0, 10_0, 12_0, 14_0, 16_0, 18_0, 20_0, 22_0, 24_0, 26_0, 28_0, 30_0, 32_0)$$

having orbit length 2.

Proof of Theorem 2.4. Let us now describe the general construction needed to prove the theorem. As already mentioned, we will use a set of six base cycles, namely, $\{C_0, C_1, C_p, C_\infty^0, C_\infty^1, C_2^0\}$, developed modulo $(2\ell, -)$. The cycles C_0 and C_1 will cover all the mixed differences except for 0 and ℓ , and C_i will cover a single i -pure difference $m \in \pm\{(\ell - 5)/2, (\ell - 3)/2, (\ell - 1)/2, (\ell + 1)/2\}$, $i = 0, 1$. The construction of C_0 and C_1 is somewhat laborious, and it is described in Lemma 2.6.

Let $S^0 = \pm\{d_1, \dots, d_{(\ell-3)/2}\}$ be a set 0-pure differences of size $(\ell - 3)$ not containing $\pm 2, \pm m$ or ℓ , and $S^1 = \pm\{d_1, \dots, d_{(\ell-1)/2}\}$ a set 1-pure differences of size $(\ell - 1)$ not containing $\pm m$ or ℓ . The cycle C_p will have on its edges the mixed differences 0 and ℓ , the 0-pure differences in S^0 , and the 1-pure differences in S^1 . Its construction is to be found in Lemma 2.8, and in Lemma 2.10 for the case $\ell \equiv 3 \pmod{8}$.

The cycles C_0, C_1, C_p are all equitably coloured in the colouring described above.

The remaining pure differences will be found in the last three cycles. The cycles C_∞^0, C_∞^1 are built applying Lemma 2.12, which gives us the freedom to use any $\ell - 3$ 0-pure and 1-pure difference not already covered, and their development will also contain all edges through ∞ . It is easily checked that these cycles and their translates are equitably coloured.

The 0-pure difference ± 2 has not yet been used, and this difference can be found on the last base cycle $C_2^0 = (0_0, 2_0, 4_0, \dots, (2\ell - 2)_0)$, which is equitably coloured by construction, together with its translate. This completes the proof. A summary of the differences appearing in each cycle appears in Table 1. \square

Let us show first how to build two base cycles C_0 and C_1 .

The construction will be different according to the congruence class of ℓ modulo 8.

Lemma 2.6. *Let $\ell \geq 7$. There exist two ℓ -cycles, C_0 and C_1 , on vertex set $\mathbb{Z}_{2\ell} \times \{0, 1\}$ having among them the two $2\ell - 2$ edges of mixed differences in $\{0, 1, \dots, 2\ell - 1\} \setminus \{0, \ell\}$ and one edge of i -pure difference $\pm m$ in C_i , $i = 0, 1$, where m is one of $(\ell - 5)/2, (\ell - 3)/2, (\ell - 1)/2, (\ell + 1)/2$. Moreover C_i ($i = 0, 1$) and its translates when developed modulo $(2\ell, -)$ are all equitably 2-colourable under the colouring previously described.*

TABLE 1 The differences used in $C_0, C_1, C_p, C_2^0 C_\infty^0$ and C_∞^1 .

Mixed differences	$0, \ell$	C_p
	$\{0, 1, \dots, 2\ell - 1\} \setminus \{0, \ell\}$	C_0 and C_1
0-Pure differences	$\pm m$	C_0
	± 2	C_2^0
	S^0	C_p
	$\mathbb{Z}_{2\ell} \setminus (S^0 \cup \{0, \pm 2, \pm m\})$	C_∞^0
1-Pure differences	$\pm m$	C_1
	S^1	C_p
	$\mathbb{Z}_{2\ell} \setminus (S^1 \cup \{0, \pm m\})$	C_∞^1
Edges containing ∞		C_∞^0 and C_∞^1 and their translates

Proof. We will use the following notation: for $x \in \mathbb{Z}_{4\ell}, i \in \mathbb{Z}_2$ and $a \in \{0, 1, \dots, 4\ell - 1\}$, let $p_i(x; a)$ be the 2-path $x_i, (x + a)_{i+1}, (x - 2)_i$, with subscripts computed modulo 2. For instance, $p_0(5, 4) = [5_0, 9_1, 3_0]$.

Let $z_i(x; a, b)$ denote the concatenation of b 2-paths $p_i(x; a), p_i(x - 2; a + 4), p_i(x - 4, a + 8), \dots, p_i(x - 2b + 2; a + 4b - 4)$ (if $b = 0$ this is just an empty path). Note that $z_i(x; a, b)$ is a path of length $2b$ with vertex set $(\{x - 2b, x - 2b + 2, \dots, x\} \times \{i\}) \cup (\{x + a, x + a + 2, x + a + 4, \dots, x + a + 2b - 2\} \times \{i + 1\})$. Its initial vertex is x_i and terminal vertex is $(x - 2b)_i$. Moreover, $z_0(x; a, b)$ and $z_1(x; a, b)$ between them contain an edge of each mixed difference in $\pm\{a, a + 2, a + 4, \dots, a + 4b - 2\}$. For instance, $z_0(5, 4, 2) = [5_0, 9_1, 3_0, 11_1, 1_0]$.

Case. $\ell \equiv 7 \pmod{8}$

C_0 is the concatenation of paths

$$\begin{aligned} & z_1((\ell - 3)/2; 3, (\ell - 3)/4), \\ & 0_1, ((\ell - 3)/2)_0, \\ & z_0((\ell - 3)/2; 3, (\ell - 3)/4), \\ & 0_0, ((\ell - 1)/2)_0, ((\ell - 3)/2)_1. \end{aligned}$$

Note that, except for vertex $((\ell - 1)/2)_0, C_0$ either contains both of x_0 and x_1 , or else contains neither, so that C_0 and all its translates are equicoloured. Also, C_0 contains an edge of 0-pure difference $\pm(\ell - 1)/2$, and an edge of each mixed difference in

$$\{-1\} \cup \pm\{3, 5, 7, \dots, (\ell - 2)\} \cup \{-(\ell - 3)/2\}.$$

C_1 is the concatenation of paths

$$\begin{aligned}
& z_0((\ell - 3)/2; 2, (\ell - 7)/8), \\
& z_0((\ell + 1)/4; (\ell + 1)/2, (\ell + 1)/8), \\
& 0_0, ((\ell - 3)/2)_1, \\
& z_1((\ell - 3)/2; 2, (\ell - 7)/8), \\
& z_1((\ell + 1)/4; (\ell + 1)/2, (\ell + 1)/8), \\
& 0_1, ((\ell - 1)/2)_1, ((\ell - 3)/2)_0.
\end{aligned}$$

As above, C_1 and all its translates are equitably coloured. Also, C_1 contains an edge of 1-pure difference $\pm(\ell - 1)/2$, and an edge of each mixed difference in $\{1\} \cup \pm\{2, 4, 6, \dots, (\ell - 3)/2, \dots, (\ell - 1)\} \cup \{(\ell - 3)/2\}$.

Case. $\ell \equiv 1 \pmod{8}$

C_0 is the concatenation of paths

$$\begin{aligned}
& z_0((\ell - 5)/2; 3, (\ell - 5)/4), \\
& 0_0, ((\ell - 5)/2)_1, \\
& z_1((\ell - 5)/2; 3, (\ell - 5)/4), \\
& 0_1, (\ell - 1)_0, ((\ell - 1) + (\ell - 3)/2)_0, \\
& ((\ell - 1) + (\ell - 5)/2)_1, ((\ell - 5)/2)_0.
\end{aligned}$$

Note that the three vertices $(\ell - 1)_0$, $((\ell - 1) + (\ell - 3)/2)_0$ and $((\ell - 1) + (\ell - 5)/2)_1$ cannot all have the same colour, since $((\ell - 1) + (\ell - 3)/2)_0$ and $((\ell - 1) + (\ell - 5)/2)_1$ have different colours; the same holds in all but two of the translates of this 3-vertex-set, and it is readily seen that in these two cases $(\ell - 1)_0$ has a different colour. As for the remaining $\ell - 3$ vertices of the cycle say x_i , we have that C_0 either contains both of x_0 and x_1 , or else contains neither, so that C_0 and its translates are equitably coloured. Moreover, C_0 contains an edge of 0-pure difference $\pm(\ell - 3)/2$, and an edge of each mixed difference in

$$\{-1, (\ell - 5)/2\} \cup \pm\{3, 5, 7, \dots, (\ell - 4)\} \cup \pm\{(\ell - 1)\}.$$

C_1 is the concatenation of paths

$$\begin{aligned}
& z_1((\ell - 5)/2; 2, (\ell - 9)/8), \\
& z_1((\ell - 1)/4; (\ell - 1)/2, (\ell - 1)/8), \\
& 0_1, ((\ell - 5)/2)_0, \\
& z_0((\ell - 5)/2; 2, (\ell - 9)/8), \\
& z_0((\ell - 1)/4; (\ell - 1)/2, (\ell - 1)/8), \\
& 0_0, (\ell - 2)_1, ((\ell - 2) + (\ell - 3)/2)_1, \\
& ((\ell - 2) + (\ell - 5)/2)_0, ((\ell - 5)/2)_1.
\end{aligned}$$

Reasoning as for C_0 , the cycle C_1 is also equitably coloured: two different colours appear in the last three vertices, while for the remaining ones the cycle either contains both of x_0 and x_1 , or else contains neither.

The cycle C_1 contains an edge of 1-pure difference $\pm(\ell - 3)/2$, and an edge of each mixed difference in $\{1, -(\ell - 5)/2\} \cup \pm\{2, 4, 6, \dots, \cancel{(\ell - 5)/2}, \dots, (\ell - 3)\} \cup \pm\{(\ell - 2)\}$.

Case. $\ell \equiv 5 \pmod{8}$

C_0 is the concatenation of paths

$$\begin{aligned} & z_0((\ell - 5)/2; 1, (\ell - 5)/4), \\ & 0_0, ((\ell - 5)/2)_1, \\ & z_1((\ell - 5)/2; 1, (\ell - 5)/4), \\ & 0_1, (\ell - 2)_0, ((\ell - 2) + (\ell - 1)/2)_0, \\ & ((\ell - 2) + (\ell - 5)/2)_1, ((\ell - 5)/2)_0. \end{aligned}$$

Reasoning as previously, it is easy to see that C_0 is equitably coloured. It contains an edge of 0-pure difference $\pm(\ell - 1)/2$, and an edge of each mixed difference in

$$\{-2, (\ell - 5)/2\} \cup \pm\{1, 3, 5, 7, \dots, (\ell - 6)\} \cup \pm\{(\ell - 2)\}.$$

C_1 is the concatenation of paths

$$\begin{aligned} & z_1((\ell - 5)/2; 4, (\ell - 13)/8), \\ & z_1((\ell + 3)/4; (\ell - 1)/2, (\ell + 3)/8), \\ & 0_1, ((\ell - 5)/2)_0, \\ & z_0((\ell - 5)/2; 4, (\ell - 13)/8), \\ & z_0((\ell + 3)/4; (\ell - 1)/2, (\ell + 3)/8), \\ & 0_0, (\ell - 4)_1, ((\ell - 4) + (\ell - 1)/2)_1, \\ & ((\ell - 4) + (\ell - 5)/2)_0, ((\ell - 5)/2)_1. \end{aligned}$$

Reasoning as above, it is easy to see that C_1 is also equitably coloured. It contains an edge of 1-pure difference $\pm(\ell - 1)/2$, and an edge of each mixed difference in $\{2, -(\ell - 5)/2\} \cup \pm\{4, 6, \dots, \cancel{(\ell - 5)/2}, \dots, (\ell - 1)\} \cup \pm\{(\ell - 4)\}$.

Case. $\ell \equiv 3 \pmod{8}$

C_0 is the concatenation of paths

$$\begin{aligned} & z_0((\ell - 7)/2; 2, (\ell - 11)/8), \\ & ((\ell - 3)/4)_0, ((3\ell - 13)/4)_1, ((\ell - 7)/4)_0, ((3\ell - 9)/4)_1, ((\ell - 11)/4)_0, \\ & z_0((\ell - 11)/4; (\ell + 13)/2, (\ell - 11)/8), \\ & 0_0, ((\ell - 7)/2)_1, \\ & z_1((\ell - 7)/2; 2, (\ell - 11)/8), \\ & ((\ell - 3)/4)_1, ((3\ell - 13)/4)_0, ((\ell - 7)/4)_1, ((3\ell - 9)/4)_0, ((\ell - 11)/4)_1, \\ & z_1((\ell - 11)/4; (\ell + 13)/2, (\ell - 11)/8), \\ & 0_1, (2\ell - 1)_0, ((\ell - 7)/2)_0, \end{aligned}$$

C_0 contains an edge of 0-pure difference $\pm(\ell - 5)/2$, and an edge of each mixed difference in

$$\begin{aligned} & \{1, (\ell - 7)/2\} \cup \pm\{2, 4, \dots, (\ell - 11)/2\} \\ & \cup \pm\{(\ell - 5)/2, (\ell - 3)/2, (\ell - 1)/2, (\ell + 1)/2\} \\ & \cup \pm\{(\ell + 13)/2, (\ell + 17)/2, \dots, \ell - 1\}. \end{aligned}$$

C_1 is the concatenation of paths

$$\begin{aligned} & z_1((\ell - 7)/2; 3, (\ell - 11)/8), \\ & ((\ell - 3)/4)_1, ((3\ell + 3)/4)_0, ((\ell - 7)/4)_1, ((3\ell + 7)/4)_0, ((\ell - 11)/4)_1, \\ & z_1((\ell - 11)/4; (\ell + 11)/2, (\ell - 11)/8), \\ & 0_1, ((\ell - 7)/2)_0, \\ & z_0((\ell - 7)/2; 3, (\ell - 11)/8), \\ & ((\ell - 3)/4)_0, ((3\ell + 3)/4)_1, ((\ell - 7)/4)_0, ((3\ell + 7)/4)_1, ((\ell - 11)/4)_0, \\ & z_0((\ell - 11)/4; (\ell + 11)/2, (\ell - 11)/8), \\ & 0_0, (2\ell - 1)_1, ((\ell - 7)/2)_1. \end{aligned}$$

C_1 contains an edge of 1-pure difference $\pm(\ell - 5)/2$, and an edge of each mixed difference in

$$\begin{aligned} & \{-1, -(\ell - 7)/2\} \cup \pm\{3, 5, \dots, (\ell - 9)/2\} \\ & \cup \pm\{(\ell + 3)/2, (\ell + 5)/2, (\ell + 7)/2, (\ell + 9)/2\} \\ & \cup \pm\{(\ell + 11)/2, (\ell + 15)/2, \dots, \ell - 2\}. \end{aligned}$$

□

Example 2.7. Case $\ell \equiv 7 \pmod{8}$: for $\ell = 15$ the cycles are

$$\begin{aligned} C_0 &= (6_1, 9_0, 4_1, 11_0, 2_1, 13_0, 0_1, 6_0, 9_1, 4_0, 11_1, 2_0, 13_1, 0_0, 7_0) \\ C_1 &= (6_0, 8_1, 4_0, 12_1, 2_0, 14_1, 0_0, 6_1, 8_0, 4_1, 12_0, 2_1, 14_0, 0_1, 7_1) \end{aligned}$$

An example for the case $\ell \equiv 1 \pmod{8}$ is part of Example 2.5.

Case $\ell \equiv 5 \pmod{8}$: for $\ell = 21$ the cycles are

$$\begin{aligned} C_0 &= (8_0, 9_1, 6_0, 11_1, 4_0, 13_1, 2_0, 15_1, 0_0, 8_1, 9_0, 6_1, 11_0, 4_1, 13_0, 2_1, 15_0, 0_1, 19_0, 29_0, 27_1) \\ C_1 &= (8_1, 12_0, 6_1, 16_0, 4_1, 18_0, 2_1, 20_0, 0_1, 8_0, 12_1, 6_0, 16_1, 4_0, 18_1, 2_0, 20_1, 0_0, 17_1, 27_1, 25_0) \end{aligned}$$

Case $\ell \equiv 3 \pmod{8}$: for $\ell = 19$ the cycles are

$$\begin{aligned} C_0 &= (6_0, 8_1, 4_0, 11_1, 3_0, 12_1, 2_0, 18_1, 0_0, 6_1, 8_0, 4_1, 11_0, 3_1, 12_0, 2_1, 18_0, 0_1, 37_0) \\ C_1 &= (6_1, 9_0, 4_1, 15_0, 3_1, 16_0, 2_1, 17_0, 0_1, 6_0, 9_1, 4_0, 15_1, 3_0, 16_1, 2_0, 17_1, 0_0, 37_1) \end{aligned}$$

The following results are used to construct the base cycle C_p in the proof of Theorem 2.4 for $\ell \geq 9$.

Lemma 2.8. *Let $\ell \geq 7$ be odd. If $\ell \geq 9$, then for any $m \in \{(\ell - 3)/2, \dots, \ell - 3\}$, there is an ℓ -cycle C on vertex set $\mathbb{Z}_{2\ell} \times \{0, 1\}$ containing $(\ell - 3)/2$ edges of distinct 0-pure difference, $(\ell - 1)/2$ edges of distinct 1-pure difference and two edges of mixed differences 0 and ℓ , satisfying the following properties:*

1. C contains no edges of 0-pure difference $\pm 2, \pm m$ or ℓ , and no edge of 1-pure difference $\pm m$ or ℓ .
2. For every vertex x_0 in C , x_1 is also a vertex in C . In particular, C and its translates when developed modulo $(2\ell, -)$ are all equitably 2-colourable under the colouring previously described.

If $\ell = 7$, then there is a cycle satisfying properties 1 and 2 with $m = 3$.

Proof. In the case $\ell = 7$, we take

$$C = (0_0, 1_0, 5_0, 5_1, 0_1, 1_1, 7_1),$$

and note that C contains no edge of 0- or 1-pure difference $m = \pm 3 = \pm(\ell - 1)/2$. Now suppose $\ell \geq 9$. Notationally, let $y_i(x; a, b)$ be the path

$$x_i, (x + a + b)_i, (x + 1)_i, (x + a + b - 1)_i, \dots, t_i,$$

where

$$t = x + \begin{cases} a + b/2, & \text{if } b \text{ is even,} \\ (b + 1)/2, & \text{if } b \text{ is odd.} \end{cases}$$

In particular, $y_i(x; a, b)$ contains an edge of each i -pure difference $a, a + 1, \dots, a + b$. Let $y'_i(x; a, b)$ denote the path $y_i(x; a, b)$ traversed in the opposite order, that is, with initial vertex t_i and terminal vertex x_i .

If $\ell > 9$, let $Y_0 = y_0(2; 3, (\ell - 11)/2)$ and $Y_1 = y'_1(2; 3, (\ell - 11)/2)$. For $\ell = 9$, let Y_0 and Y_1 be null paths. Note that Y_i is a path containing $(\ell - 9)/2$ edges of distinct i -pure difference in $\{3, \dots, (\ell - 5)/2\}$, and vertices distinct from $0_i, 1_i$ and ℓ_i . Moreover, the vertices of Y_1 are precisely those of the form v_1 where $v_0 \in V(Y_0)$. Let the terminal vertex Y_0 (resp., the initial vertex of Y_1) be t_0 (resp., t_1).

Let u be any element of $\{(\ell + 1)/2, \dots, \ell - 2\}$ such that $m \notin \{u - 2, u\}$. Note that the number of elements in this set is $(\ell - 3)/2$, and at most two of them can equal m or $m + 2$; thus since $\ell \geq 9$, a suitable element u can be chosen.

Define the cycle C as the concatenation of the following paths:

$$\begin{aligned} &0_0 1_0 \ell_0 2_0; \\ &Y_0, \\ &t_0 t_1, \\ &Y_1, \\ &2_1 u_1 0_1 1_1 \ell_1. \end{aligned}$$

By construction, C has the required properties. □

Example 2.9. Let $\ell = 13$ and $m = 5$. Then, choosing $u = 8$, we get

$$C = (0_0, 1_0, 13_0, 2_0, 6_0, 3_0, 3_1, 6_1, 2_1, 8_1, 0_1, 1_1, 13_1).$$

Lemma 2.10. Let $\ell \equiv 3 \pmod{8}$, $\ell \geq 11$. There is an ℓ -cycle C on vertex set $\mathbb{Z}_{2\ell} \times \{0, 1\}$ containing $(\ell - 3)/2$ edges of distinct 0-pure difference, $(\ell - 1)/2$ edges of distinct 1-pure difference and two edges of mixed differences 0 and ℓ , satisfying the following properties:

1. C contains no edges of 0-pure difference ± 2 , $\pm(\ell - 5)/2$ or ℓ , and no edge of 1-pure difference $\pm(\ell - 5)/2$ or ℓ .
2. For every vertex x_0 in C , x_1 is also a vertex in C . In particular, C and its translates when developed modulo $(2\ell, -)$ are all equitably 2-colourable under the colouring previously described.

Proof. For $\ell = 11$, let

$$C = (0_0, 1_0, 11_0, 2_0, 6_0, 6_1, 2_1, 7_1, 0_1, 1_1, 11_1).$$

For $\ell > 11$, the cycle is constructed similarly to that in Lemma 2.8, except that we skip the edge of pure difference $\pm(\ell - 5)/2$. Define the notation $y_i(x; a, b)$ and $y'_i(x; a, b)$ as in the proof of Lemma 2.8. Let $Y_0 = y_0(4; 3, (\ell - 15)/2)$ and $Y_1 = y'_1(4; 3, (\ell - 15)/2)$; here $b = (\ell - 15)/2$ is even so that $t = 4 + 3 + (\ell - 15)/4 = (\ell + 13)/4$.

The cycle C is the concatenation of the following paths:

$$\begin{aligned} &0_0 1_0 \ell_0 2_0 ((\ell + 1)/2)_0 4_0; \\ &Y_0; \\ &t_0 t_1; \\ &Y_1; \\ &4_1 ((\ell + 1)/2)_1 2_1 ((\ell + 3)/2)_1 0_1 1_1 \ell_1. \end{aligned}$$

By construction, C has the required properties. \square

Example 2.11. Let $\ell = 19$. Then

$$C = (0_0, 1_0, 19_0, 2_0, 10_0, 4_0, 9_0, 5_0, 8_0, 8_1, 5_1, 9_1, 4_1, 10_1, 2_1, 11_1, 0_1, 1_1, 19_1).$$

Finally, the following lemma describes how to build the cycles containing ∞ .

Lemma 2.12. Let ℓ be odd and let $D = \{d_1, d_2, \dots, d_{(\ell-3)/2}\}$ be a set of $(\ell - 3)/2$ distinct elements of $\{1, 2, \dots, \ell - 1\}$ with $d_1 > d_2 > \dots > d_{(\ell-3)/2}$. There exists an ℓ -cycle C on vertex set $\mathbb{Z}_{2\ell} \cup \{\infty\}$ with $\partial(C) = \pm D \cup \{\ell\}$.

Proof. For $k = 1, 2, \dots, (\ell - 3)/2$, let $s_i = \sum_{i=1}^k (-1)^{i-1} d_i$. Since $d_1, d_2, \dots, d_{(\ell-3)/2}$ is a strictly decreasing sequence of positive integers and each $d_i \leq \ell - 1$, note that for each i , $0 < s_i < \ell$. Thus

$$C = (\infty, 0, s_1, s_2, \dots, s_{(\ell-3)/2}, s_{(\ell-3)/2} + \ell, s_{(\ell-5)/2} + \ell, \dots, s_1 + \ell, \ell)$$

is an ℓ -cycle with vertices in $\mathbb{Z}_{2\ell} \cup \{\infty\}$, and it is easy to check that $\partial(C)$ is as required.

□

Remark 2.13. If using the 2-colouring of $\mathbb{Z}_{2\ell}$ described at the beginning of this section, that is one colour on $0, \dots, \ell - 1$ and a different colour on $\ell, \dots, 2\ell - 1$, then the cycle built in Lemma 2.12 is equitably coloured.

3 | EQUITABLY 2-COLOURABLE ℓ -CYCLE DECOMPOSITIONS OF K_ν WHEN $\nu \equiv 1$ OR $\ell \pmod{2\ell}$

The aim of this section is to complete the proof of the existence of an equitably 2-colourable ℓ -cycle decomposition of K_ν , $\nu \equiv 1$ or $\ell \pmod{2\ell}$. The cases $\nu = 2\ell + 1$ and $\nu = 4\ell + 1$ have already been established, so if $\nu \equiv 1 \pmod{2\ell}$, we can assume $\nu \geq 6\ell + 1$.

Throughout this section, we make use of lexicographic products. Formally, the *lexicographic* product of graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$ and edges of the form $(g, h)(g', h')$, where $gg' \in E(G)$ and $h, h' \in V(H)$ or $g = g'$ and $hh' \in E(H)$. We will generally take H to be an empty graph, that is, $H \simeq \overline{K_n}$, and in this case, we denote $G[H]$ by $G[n]$; the formation of $G[n]$ can be thought of as “blowing up” the vertices of G by a factor of n .

Our approach for forming an equitably 2-colourable decomposition in the case $\nu \equiv 1 \pmod{2\ell}$ is as follows. Let $\nu = 2k\ell + 1$, and view the vertex set of K_ν as $(\mathbb{Z}_{2k} \times \mathbb{Z}_\ell) \cup \{\infty\}$. We partition $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ into $2k$ parts of size ℓ . Each part will be coloured with $(\ell + 1)/2$ red vertices and $(\ell - 1)/2$ blue vertices, while ∞ will be coloured blue. The parts are then partitioned into pairs; each pair, together with ∞ , induces a copy of $K_{2\ell+1}$, which is decomposed into equitably coloured ℓ -cycles. The remaining edges induce a subgraph isomorphic to $K_k[2\ell]$, which we decompose into copies of $C_3[\ell]$ and $C_5[\ell]$. These are further decomposed into equitably 2-coloured ℓ -cycles.

When $\nu \equiv \ell \pmod{2\ell}$, writing $\nu = (2k + 1)\ell$, we decompose K_ν into $K_{2k+1}[\ell]$ together with $2k + 1$ copies of K_ℓ . Again, each part is coloured with $(\ell + 1)/2$ red vertices and $(\ell - 1)/2$ blue vertices. Each copy of K_ℓ is decomposed into equitably coloured Hamiltonian cycles, while $K_{2k+1}[\ell]$ is again decomposed into copies of $C_3[\ell]$ and $C_5[\ell]$, which we further decompose into equitably 2-coloured ℓ -cycles.

Our first aim will be to show that the graphs $C_3[\ell]$ and $C_5[\ell]$ possess an equitably 2-colourable ℓ -cycle system; many of the ideas and constructions used here are based on the results by Alspach, Schellenberg, Stinson and Wagner in [4], and in fact, in most cases our proofs will only show that the cycle decompositions built in [4] are 2-equicolourable. Let us point out that, unlike the authors of [4], our aim is to only build a cycle system, not a 2-factorization.

When decomposing $C_5[\ell]$, it is useful to view it as a Cayley graph.

Definition 3.1. Let G be an additive group and let $\Omega \subseteq G$ be closed under negation. The *Cayley graph* $\text{Cay}[G, \Omega]$ is the graph with vertex set G , such that gh is an edge if and only if $g - h \in \Omega$.

Identifying the vertex set of $C_5[\ell]$ with $\mathbb{Z}_5 \times \mathbb{Z}_\ell$, note that $C_5[\ell] \simeq \text{Cay}[\mathbb{Z}_5 \times \mathbb{Z}_\ell, \Omega]$, with $\Omega = \{(\pm 1, \pm i), i = 0, 1, \dots, (\ell - 1)/2\}$. As in [4], to obtain an ℓ -cycle system of $C_5[\ell]$, we will partition Ω into disjoint subsets Ω_1 and Ω_2 , and decompose the graphs $\text{Cay}[\mathbb{Z}_5 \times \mathbb{Z}_\ell, \Omega_1]$

and $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega_2]$ separately into ℓ -cycles; often $\Omega_1 = \{(\pm 1, \pm i) \mid i = 0, 1, 2\}$ and $\Omega_2 = \{(\pm 1, \pm i) \mid i = 3, \dots, (\ell - 1)/2\}$.

Definition 3.2. Given an ℓ -cycle $C = ((a_0, b_0), (a_1, b_1), \dots, (a_{\ell-1}, b_{\ell-1}))$ in $C_s[\ell]$, the s -development of C is the set of cycles

$$\{((a_0 + i, b_0), (a_1 + i, b_1), \dots, (a_{\ell-1} + i, b_{\ell-1})) \mid i \in \mathbb{Z}_s\},$$

that is, the set of cycles obtained from C by developing modulo $(s, -)$.

To decompose $C_s[\ell]$, as in [4], we employ a Hamiltonian decomposition of an auxiliary Cayley graph $\text{Cay}[\mathbb{Z}_\ell, A]$ and lift these cycles to $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \{\pm 1\} \times A]$.

Definition 3.3 (Alspach et al. [4]; see also Burgess et al. [9]). Let $C = (c_0, c_1, \dots, c_{\ell-1})$ be a directed Hamiltonian cycle in K_ℓ (with vertex set identified with \mathbb{Z}_ℓ). Given the directed Hamiltonian ℓ -cycle $C = (c_0, c_1, \dots, c_{\ell-1})$, the *projection* of C onto $C_s[\ell]$ is the following ℓ -cycle in $C_s[\ell]$:

$$((0, c_0), (1, c_1), \dots, (s-1, c_{s-1}), (0, c_s), (1, c_{s+1}), \dots, (0, c_{\ell-2}), (1, c_{\ell-1})).$$

The *reverse projection* of C is the ℓ -cycle

$$((0, c_0), (s-1, c_1), (s-2, c_2), \dots, (1, c_{s-1}), (0, c_s), (s-1, c_{s+1}), (0, c_{s+2}), (1, c_{s+3}), \dots, (s-1, c_{\ell-1})).$$

Informally, we can think of the projection of C as wrapping once around the s -cycle, and then zig-zagging between the first two parts, while the reverse projection wraps in the opposite direction. An edge xy of difference $d = \pm(x - y)$ in C will yield edges of differences $\pm(1, d)$ in $C_s[\ell]$.

It is straightforward to check that the following lemma holds. For the decomposition part, see Lemma 2 in [4] and Lemma 2.6 in [9].

Lemma 3.4. *If C is a Hamiltonian decomposition for $\text{Cay}[\mathbb{Z}_\ell, A]$, then we obtain an ℓ -cycle decomposition for $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega]$ with $\Omega = \{\pm 1\} \times A$ by considering the s -development of the projection and reverse projection of C for all $C \in \mathcal{C}$.*

Moreover, if we consistently colour each ℓ -part with $(\ell + 1)/2$ red and $(\ell - 1)/2$ blue vertices, then the cycles we obtain are 2-equicoloured.

The lemma below easily follows from the main theorem of [6].

Lemma 3.5. *The graph $\text{Cay}[\mathbb{Z}_\ell, \pm\{3, 4, \dots, (\ell - 1)/2\}]$, $\ell \geq 7$ odd, has a Hamiltonian decomposition.*

Proof. From [6] we have that every 4-regular connected Cayley graph on a finite abelian group can be decomposed into two Hamiltonian cycles. When $\ell \equiv 1 \pmod{4}$, we obtain a Hamiltonian decomposition of $\text{Cay}[\mathbb{Z}_\ell, \pm\{3, 4, \dots, (\ell - 1)/2\}]$ by applying this result iteratively to $\text{Cay}[\mathbb{Z}_\ell, \pm\{3, 4\}]$, $\text{Cay}[\mathbb{Z}_\ell, \pm\{5, 6\}]$, ..., $\text{Cay}[\mathbb{Z}_\ell, \pm\{(\ell - 3)/2, (\ell - 1)/2\}]$. If

$\ell \equiv 3 \pmod{4}$, we use the same procedure to obtain a Hamiltonian decomposition for $\text{Cay}[\mathbb{Z}_\ell, \pm\{3, 4, \dots, (\ell - 3)/2\}]$ and note that $\text{Cay}[\mathbb{Z}_\ell, \pm(\ell - 1)/2]$ is a Hamiltonian cycle.

□

As in [4], the general strategy to construct ℓ -cycle systems of $C_s[\ell]$ is to separately decompose the graphs $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega_1]$ and $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega_2]$, with $\Omega_1 \dot{\cup} \Omega_2 = \Omega$, into ℓ -cycles; with one exception (a direct construction presented in Lemma 3.7) we will have that $\Omega_1 = \{(\pm 1, \pm i), i = 0, 1, 2\}$ and $\Omega_2 = \{(\pm 1, \pm i), i = 3, \dots, (\ell - 1)/2\}$. Each decomposition will be 2-equicolourable, and will have the property that each part has $(\ell + 1)/2$ red and $(\ell - 1)/2$ blue vertices.

It is easy to see that such a decomposition exists for $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega_2]$, since it is enough to combine Lemmas 3.4 and 3.5: the real work will lie in decomposing $\text{Cay}[\mathbb{Z}_s \times \mathbb{Z}_\ell, \Omega_1]$.

Lemma 3.6. *There is an equitably 2-colourable ℓ -cycle decomposition of $C_3[\ell]$ in which each part has $(\ell + 1)/2$ red and $(\ell - 1)/2$ blue vertices.*

Proof. As mentioned above, we only need to decompose $\text{Cay}[\mathbb{Z}_3 \times \mathbb{Z}_\ell, \Omega_1]$, with $\Omega_1 = \{(\pm 1, \pm i), i = 0, 1, 2\}$, into ℓ -cycles, and show that such a decomposition is equitably 2-colourable with each part having $(\ell + 1)/2$ red and $(\ell - 1)/2$ blue vertices.

We obtain the required decomposition by developing the following five base cycles modulo 3 with respect to the first components, where, in the last cycle, $a \equiv (\ell - 1)/2 \pmod{3}$.

$$\begin{aligned} C_1 &= ((0, 0), (1, \ell - 1), (0, \ell - 2), (1, \ell - 3), \dots, (1, 4), (0, 3), (1, 2), (2, 2)), \\ C_2 &= ((0, 1), (1, 0), (2, 0), (1, \ell - 2), (2, \ell - 2), (1, \ell - 4), (2, \ell - 4), \\ &\quad \dots, (1, 3), (2, 3)), \\ C_3 &= ((0, 2), (1, 1), (2, 1), (0, \ell - 1), (2, \ell - 1), (0, \ell - 3), (2, \ell - 3), \\ &\quad \dots, (0, 4), (2, 4)), \\ C_4 &= ((0, 0), (1, 1), (2, 2), (0, 3), (2, 4), (0, 5), (2, 6), \dots, (0, \ell - 2), (2, \ell - 1)), \\ C_5 &= ((0, 0), (1, 2), (2, 4), (0, 6), (1, 8), \dots, (a, \ell - 1), \\ &\quad (a + 1, 1), (a + 2, 3), (a + 1, 5), (a, 7), \dots, (0, \ell - 6), (2, \ell - 4), (1, \ell - 2)). \end{aligned}$$

Note that this is just a different presentation of the construction used in the proof of Theorem 5 in [4]. We will colour each ℓ -part of the graph $C_3[\ell]$ with the same pattern: to ensure that the colouring is equitable, then, we only need to consider the second components of the vertices in each base cycle. When $\ell \equiv 1 \pmod{4}$, we will colour the vertices in $\{(i, 0), (i, 1), \dots, (i, (\ell - 1)/2), i \in \mathbb{Z}_3\}$ red and the remaining vertices blue; when $\ell \equiv 3 \pmod{4}$, we will instead colour $\{(i, 0), (i, 1), \dots, (i, (\ell - 3)/2), i \in \mathbb{Z}_3\}$ blue and the remaining vertices red; it is easy to check that this colouring is equitable. □

When considering ℓ -cycle decompositions for $C_5[\ell]$, the case $\ell = 7$ needs an ad hoc construction, while for $\ell > 7$ we once more rely on a construction from [4].

Lemma 3.7. *There is an equitably 2-colourable 7-cycle decomposition of $C_5[7]$ in which each part has four red vertices and three blue vertices.*

Proof. In this case, we will give an explicit cycle decomposition: we colour the vertices in the set $\{(i, 2j), i \in \mathbb{Z}_5, j = 0, 1, 2, 3\}$ red, and the remaining vertices blue. Decompose $\text{Cay}[\mathbb{Z}_5 \times \mathbb{Z}_\ell, \{(\pm 1, \pm 2)\}]$ by considering the 5-development of the projection and reverse projection of the cycle $C = (0, 2, 4, 6, 1, 3, 5)$, and $\text{Cay}[\mathbb{Z}_5 \times \mathbb{Z}_\ell, \{(\pm 1, 0), (\pm 1, \pm 1), (\pm 1, \pm 3)\}]$ with the 5-development of the following five starter cycles.

$$\begin{aligned} C_1 &= ((0, 1), (1, 2), (2, 2), (3, 3), (4, 0), (0, 0), (1, 1)), \\ C_2 &= ((0, 3), (1, 2), (2, 5), (3, 2), (4, 6), (0, 6), (1, 3)), \\ C_3 &= ((0, 5), (1, 6), (2, 0), (3, 3), (4, 4), (0, 4), (1, 5)), \\ C_4 &= ((0, 1), (1, 0), (2, 4), (3, 3), (4, 6), (0, 5), (1, 4)), \\ C_5 &= ((0, 4), (1, 0), (2, 6), (3, 2), (4, 1), (0, 5), (1, 1)). \end{aligned}$$

□

Lemma 3.8. *There is an equitably 2-colourable ℓ -cycle decomposition of $C_5[\ell]$ for any odd $\ell \geq 7$ in which each part has $(\ell + 1)/2$ red and $(\ell - 1)/2$ blue vertices.*

Proof. The case $\ell = 7$ follows from the previous lemma; for $\ell > 7$, we consider once more the construction used in the proof of Theorem 5 in [4]. This construction, though lacking the compact presentation seen for $C_3[\ell]$, has the same pattern of that case in the second components of the vertices of the cycles. It is then possible to see that the same colouring presented in the proof of Lemma 3.6 will once more give an equitably 2-colourable cycle system. □

We now turn to constructing equitably 2-colourable ℓ -cycle decompositions of K_v .

Theorem 3.9. *If $v \equiv 1 \pmod{2\ell}$, then there is an equitably 2-colourable ℓ -cycle decomposition of K_v .*

Proof. The result is true when $v = 2\ell + 1$ and $v = 4\ell + 1$ by Theorems 2.2 and 2.4, so we henceforth assume $v \geq 6\ell + 1$.

Let $v = 2k\ell + 1$, where $k \geq 3$. Let the vertex set of K_v be $(\mathbb{Z}_{2k} \times \mathbb{Z}_\ell) \cup \{\infty\}$. For each element $x \in \mathbb{Z}_{2k}$, we colour $(\ell + 1)/2$ vertices of $\{x\} \times \mathbb{Z}_\ell$ red and $(\ell - 1)/2$ blue. Vertex ∞ is coloured blue.

To find an ℓ -cycle decomposition of K_v , we first decompose K_v into k subgraphs isomorphic to $K_{2\ell+1}$ (on vertex sets $(\{2i, 2i + 1\} \times \mathbb{Z}_\ell) \cup \{\infty\}$ for each $i \in \{0, \dots, k - 1\}$), along with a copy of $(K_{2k} - I)[\ell]$ (where the 1-factor I is given by the set of edges $\{\{0, 1\}, \{2, 3\}, \dots, \{2k - 2, 2k - 1\}\}$). Each copy of $K_{2\ell+1}$ has ℓ blue vertices and $\ell + 1$ red vertices. On each, we place a copy of an ℓ -cycle decomposition of $K_{2\ell+1}$ from Theorem 2.2, with the colour classes corresponding to the existing colouring.

It remains to decompose $(K_{2k} - I)[\ell]$. To this, first note that since $k \geq 3$, we can write $2k(2k - 2)/2 = 2k(k - 1) = 3m + 5n$ for some non-negative integers m and n ; thus $K_{2k} - I$ can be decomposed into m cycles of length 3 and n cycles of length 5 by Theorem 1.2. Blowing up each vertex by ℓ , we obtain a decomposition of $(K_{2k} - I)[\ell]$ into subgraphs isomorphic to $C_3[\ell]$ and $C_5[\ell]$, where each part of size ℓ has $(\ell - 1)/2$ blue

vertices and $(\ell + 1)/2$ red vertices. On each of these subgraphs, place an equitably 2-coloured ℓ -cycle decomposition using Lemma 3.6 or 3.8. \square

Theorem 3.10. *Let $v \equiv \ell \pmod{2\ell}$. There is an equitably 2-colourable ℓ -cycle decomposition of K_v .*

Proof. Let $v = 2k\ell + \ell = \ell(2k + 1)$. We form an ℓ -cycle decomposition of K_v as follows. First, writing $\binom{2k+1}{2} = k(2k + 1) = 3m + 5n$ for some non-negative integers m and n , we may decompose K_{2k+1} into m cycles of length 3 and n cycles of length 5 by Theorem 1.2. Now blow up each vertex by a factor of ℓ , so that each vertex becomes a part of size ℓ and each cycle becomes a copy of $C_3[\ell]$ or $C_5[\ell]$. Colour each part with $(\ell + 1)/2$ vertices red and $(\ell - 1)/2$ blue. On each copy of $C_3[\ell]$ or $C_5[\ell]$, place an equitably 2-coloured ℓ -cycle decomposition by Lemma 3.6 or 3.8. Finally, on each part of size ℓ , place a Hamiltonian decomposition of K_ℓ , which is clearly equitably coloured. \square

4 | CONCLUSION

In this paper, we have shown that there is an equitably 2-colourable ℓ -cycle decomposition of K_v whenever $\ell \equiv 1$ or $\ell \pmod{2\ell}$. A comparable result for even orders, namely, that there is an equitably 2-colourable ℓ -cycle decomposition of $K_v - I$ for all $\ell \equiv 0$ or $2 \pmod{\ell}$, was previously shown by the authors in [10] (along with various other existence results for even cycle length). While these results do not entirely settle the spectrum of equitably 2-colourable cycle decompositions, they do provide further evidence that equitably colourable cycle systems exist widely, in stark contrast with equitably colourable balanced incomplete block designs (see [20]). On the basis of this evidence, together with the results of [1] for small cycle lengths, we make the following conjectures.

Conjecture 4.1. *If $v \geq \ell \geq 5$ are odd integers, then there exists an equitably 2-colourable ℓ -cycle decomposition of K_v if and only if $\ell \mid \frac{v(v-1)}{2}$.*

Conjecture 4.2. *If $v \geq \ell \geq 4$ are even integers, then there exists an equitably 2-colourable ℓ -cycle decomposition of $K_v - I$ if and only if $\ell \mid \frac{v(v-2)}{2}$.*

If ℓ is even and v is odd, it is known that no equitably 2-colourable ℓ -cycle decomposition of K_v exists [1]. On the other hand, the existence of equitably 2-colourable ℓ -cycle decompositions of $K_v - I$ with ℓ odd and v even cannot be easily ruled out; indeed, if $\ell = 5$, an equitably 2-colourable cycle decomposition of $K_v - I$ exists for every admissible even order v [1]. We leave the more general existence question of equitably 2-colourable odd cycle decomposition of the cocktail party graph as an open problem.

Likewise, this paper has not considered equitable c -colourings with $c > 2$. The case $c = 3$ was considered for small cycle lengths in [2]; nevertheless, the question of when there exists an equitably c -colourable ℓ -cycle decomposition of K_v or $K_v - I$ remains wide open in general.

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