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Convergence Education Review

Solving Challenging Problems using GeoGebra...at a Distance*

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Abstract

Convergence in education can be seen as a way to teach a subject by integrating knowledge, methods, and expertise from different disciplines for scientific discovery and innovation. The use of problem solving, inspired by computer aided algorithms and visualization, has become a common example of convergence in geometry and its applications. As an example, we pose several mathematical problems and indicate possible solution processes using GeoGebra. The use of numeric or symbolic calculus and interactive geometric software provide approximate, exact or graphical solutions allowing to go back to the abstract nature of the problem, generalizing it and posing new questions.

Themes range from the approximation of π to solid sections, from Penrose tessellations to Escher's Circle Limit and hyperbolic geometry; problems can be very general, well or ill posed, direct or inverse, global or local.

For example, find a family of cones sharing the same XY plane section. What is the minimum value of n so that Pn, a regular polygon with n sides, it is not distinguishable from a circle? What is the minimum number n so that the ratio between the perimeter of Pn and the diameter of the circumscribed circle is at least 3.14? How many different types of regular cube sections are there? How to make the tiles of an aperiodic tessellation using a numerical cutting machine? Which kind of symmetry guided some of Escher's work?

When the problems are challenging, they are also suitable to be organized in cooperative or distance learning. Most of these subjects have been used in a course for future teachers even in this Covid year.

Keyword: GeoGebra software, Penrose aperiodic tessellation, problem solving, visualization

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Ⅰ. Introduction

Convergence in high school education(Yeonghae Ko, Jaeho An and Namje Park, 2012; Daniel J.C. Herr et al, 2019; Min Kyeong Kim, 2019). integrating knowledge, methods, and expertise from different disciplines can lead to new discoveries and innovations. The use of problem solving in geometry and its applications, often with the help of computer aided algorithms and visualization, is now a rather common teaching method at different levels.

As an example, we pose several mathematical problems and indicate possible solution processes using the software GeoGebra. The use of numeric or symbolic calculus and interactive geometric software provide approximate, exact or graphical solutions allowing to go back to the abstract nature of the problem, generalizing it and posing new questions.

When the problems are challenging, they are also suitable to be divided in steps, formulating several propositions and hypothesis, assigning different goals and organizing the solution process in cooperative or distance learning.

Problems can be very general, well or ill posed, direct or inverse, global or local. In §2 we address the problem of regular cube sections, in §3 the approximation of π via regular polygons and in §4 we discuss the possibility of modelling real objects, using thousands of points, in GeoGebra. Symmetry in some of Escher's work is presented in §5, the problem of finding a cone with a given plane section in §6 and Penrose tessellations in §7.

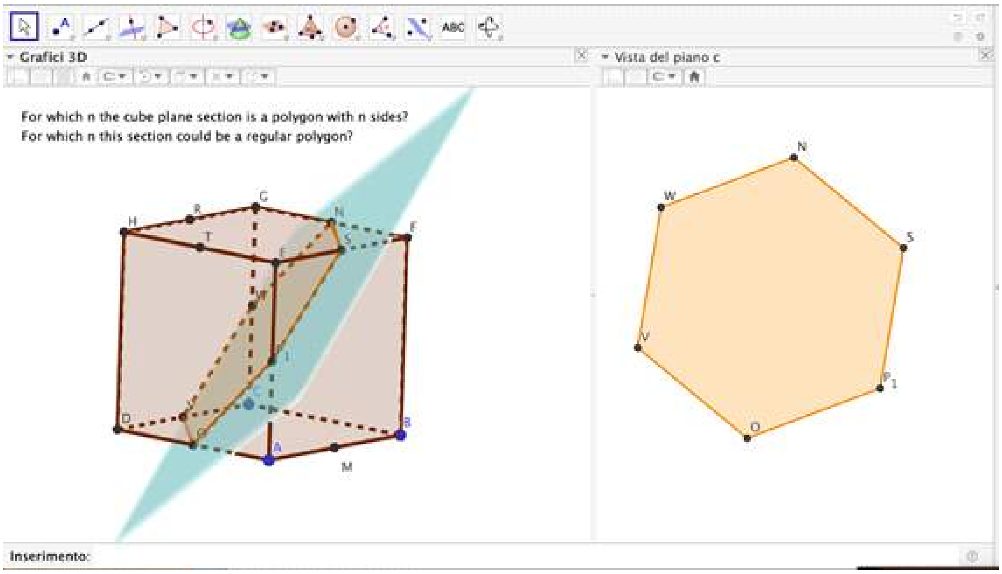
Most of these subjects have been used in a course for future teachers even in this Covid year.

Ⅱ. CUBE SECTION

How many different types of regular cube sections are there? If we cut a cube with a given plane, the generic section is a polygon whose sides are segments on the cube faces. A cube section has therefore at most 6 sides: could you imagine a picture with a cube section of 3, 4, 5 or 6 sides? Are all these sections possibly regular polygons?

Using GeoGebra, it is possible to let the students draw a fixed cube, an intersecting plane and the corresponding plane section: it is simple then to let the plane move and visualize all the possibilities in the 3D-Graphics window.

The assignments could also be to predict and describe the solutions without the help of the computer: it is easy to imagine a square section (with a plane parallel to one of the faces), an equilateral triangle section (with a plane cutting out a cube vertex) or even a regular hexagon section (with a plane which cuts two opposite faces on the midpoints of its sides, see Figure 1).



[Figure 1] A plane passing through side midpoints of two opposite faces of the cube cuts out a regular hexagon section.

What about a pentagon section, could it be regular? The answer is negative. No matter how hard you try, even with the help of computer visualization, you will not be able to construct a regular pentagonal section from a cube. The reason is geometrical: in a polygonal cube section, with more than 3 sides, opposite sides are parallel. This is not the case for a regular pentagon!

Ⅲ. APPROXIMATION OF π

What is the minimum number n of sides of a regular polygon:

- which is not distinguishable from its circumscribed circle?
- such that the ratio of its perimeter to the diameter of the circumscribed circle is at least 3.14?

The problem has been presented in the Second International Conference of Mathematics at Erbil (Iraqi Kurdistan) (Corrado Falcolini, 2019).

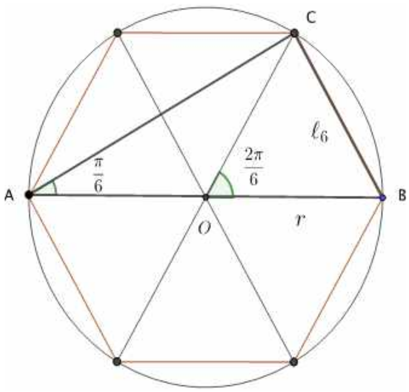
The first question, obviously, it is not well posed: we have to fix the polygon measures and its distance from the observer so that, for instance, it fits on a sheet of paper or a computer screen.

The students are then asked to use GeoGebra to construct regular polygons, with an increasing number of sides, all inscribed in a fixed circle and look for their personal answer. Usually, n is found between 50 and 60.

The second question is more complicated, but it is a well posed problem: the ratio , between the perimeter of a regular polygon and the diameter of the circumscribed circle, is independent from the diameter size, increases with n and converges to the ratio between the measures of the circumference and its diameter which, by definition, is π.

The solution is given by the formula for the length of the side of the regular polygon as a function of the radius r of the circumscribed circle (see Figure 2):

$$l_n = 2r\sin\frac{\pi}{n}, \qquad \pi_n = \frac{n \cdot l_n}{2r} = n \cdot \sin\frac{\pi}{n}$$



[Figure 2] In a regular n-sided polygon (n=6 in this picture) the angle $\angle BAC = \frac{\angle BOC}{2} = \frac{\pi}{n}$ and $\overline{BC} = l_n = 2r\sin\frac{\pi}{n}$. For the regular hexagon $l_6 = 2r\sin\frac{\pi}{6} = r$ and $\pi_6 \equiv \frac{6 \cdot l_6}{2r} = 3$.

Since the sequence π_n is increasing and bounded by π, there exists a first value of n such that π_n > 3.14: looking at $\pi_{56} \equiv 56\sin\frac{\pi}{56} \cong 3.13994$ and $\pi_{57} \equiv 57\sin\frac{\pi}{57} \cong 3.140002$ one can check that the second answer is n=57. Strangely enough, the answers to both questions are related: on the scale of a school homework paper, the usual approximation of π by 3.14 is a very good one, as geometrical and visualization problems are concerned, since it corresponds to estimate circle properties using a 57-sides polygon which is barely distinguishable from a circle.

This result can be generalized to get the sequence $n_k$of minimal n such that exceeds the first k digits of π: $\{n_k\}$ = {6, 12, 57, 94, 237,1396,2812,9820,37942,⋯ }.

Ⅳ. MODELING OF OBJECT

Is it possible to use point clouds for modelling a given real life object? Could GeoGebra enter in the process?

A point cloud of an object is a list of point coordinates, in a three-dimensional space, obtained from a laser scanner survey or a series of pictures uploaded in a photogrammetric program. The difficulty of the problem is practical, since the list of points is usually very long: I tried to answer the questions for the point cloud of a fluted column fragment, uploading a set of few cell phone photos in the program Metashape. Then I uploaded the list of coordinates in Geogebra. It works (slowly) up to 10000 points.

The first issue is to modify the output file, usually a file with several space separated data, to be read as input in GeoGebra with only 3 comma separated numbers (the coordinates x, y, z): a simple line command, using a terminal window, applied to the Metashape output file “name.txt” produce an input file for GeoGebra “in.txt” with only the first 10000 points coordinate:

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head -10000 name.txt | awk '{print $1 "," $2 "," $3}' > in.txt
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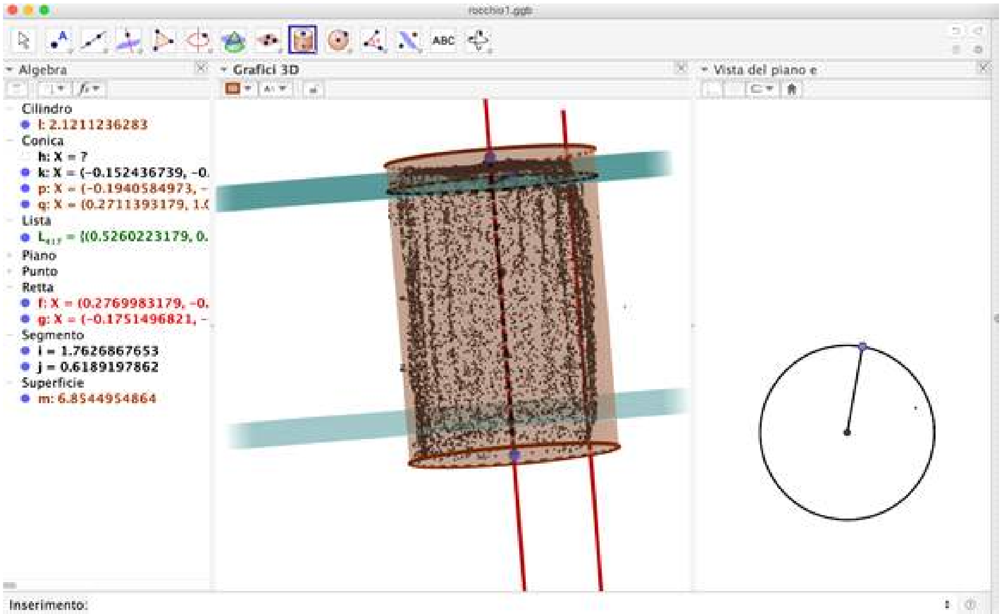
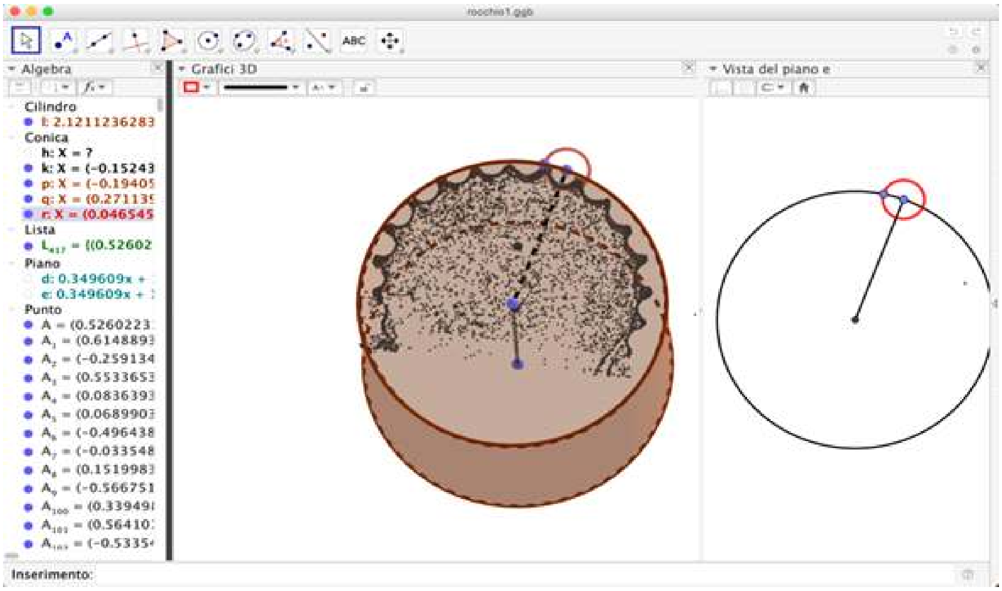
With GeoGebra is then possible to import the data file “in.txt” in the Spreadsheet View and use the tool to create and visualize the associated list of points.



[Figure 3] A list of 10000 points of a surveyed fluted column fragment is imported in GeoGebra to detect its best fitting cylinder

The points are then visualized in the 3D Graphics View, and it is possible to model simple geometric shapes to analyze the symmetry or metric properties of the object (see Figure 3): in this example it is interesting to look for the best fitting cylinder and the position and size of the flutes (see Figure 4).



[Figure 4] A top view of the surveyed fluted column fragment: the position and size of one flute is investigated

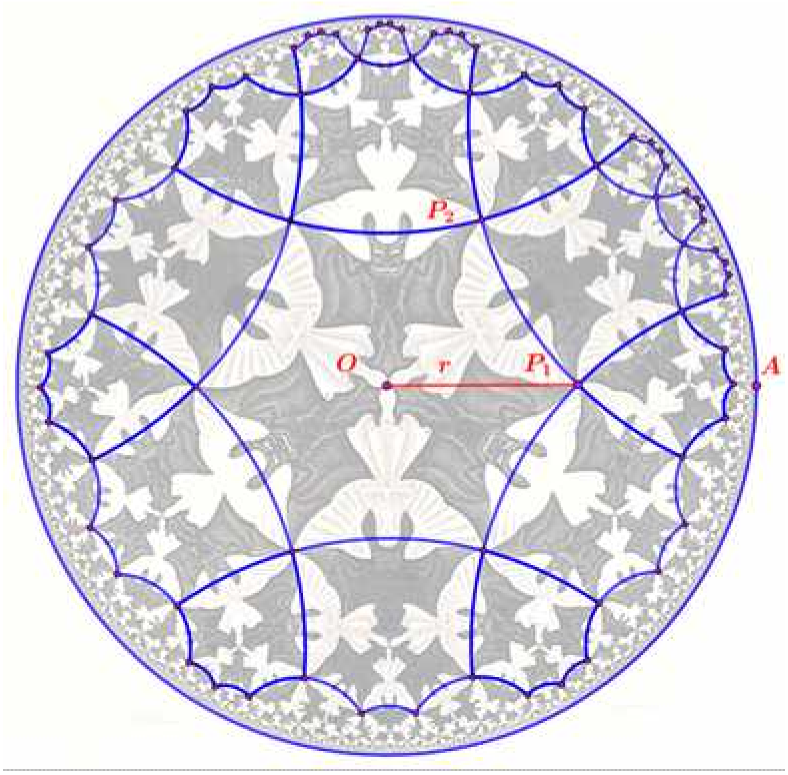
The orientation of the point cloud in itself is another nice problem to address for students work. Similar analysis, to settle interesting questions or conjectures, can be proposed on different objects, even buildings or natural shapes(Marco Abate, 2007). chosen by the students.

Ⅴ. ESCHER'S CIRCLE LIMIT SYMMETRY

Which kind of symmetry inspired Escher's Circle Limit work? It is a hyperbolic symmetry in the so called Poincaré Disk: it is interesting that, in this geometry, the construction of a segment is based on the Euclidean circular inversion transformation, which is a standard tool in GeoGebra.

First fix the unit circle as the boundary of the Poincaré disk: in this geometry the points are taken only inside the disk and the lines are circular arcs perpendicular to it; angles are defined with respect to Euclidean arc tangents at any point.

Let the students construct a new GeoGebra tool to draw an hyperbolic segment through any two points of the disk: they will need the Euclidean tools for the perpendicular bisector of a segment and for the circular inversion transformation with respect to the boundary.

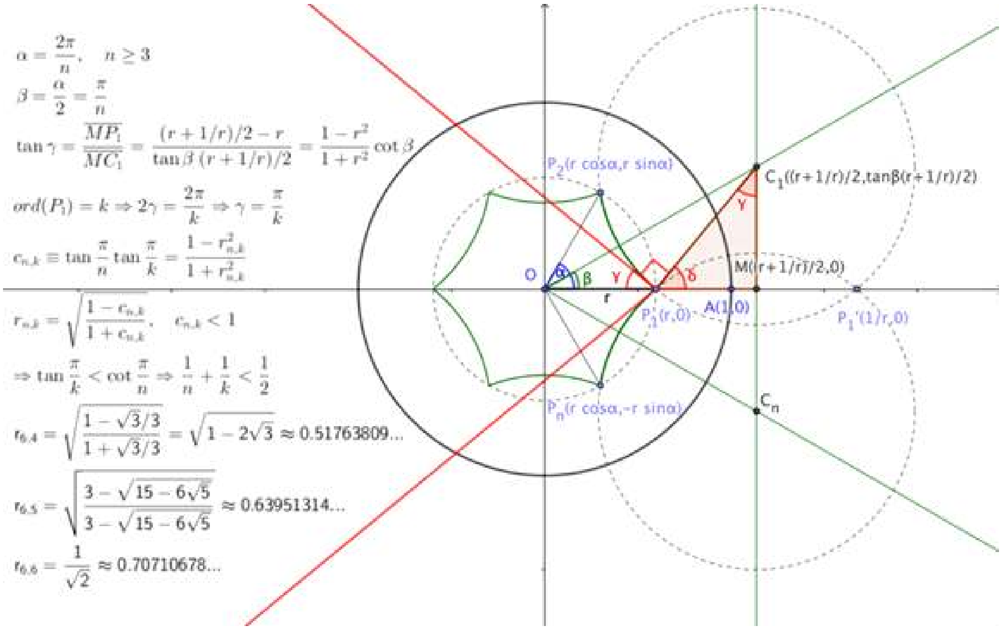


[Figure 5] Escher's Circle Limit IV: the regular hyperbolic hexagon centered in O, which produces a tessellation of the unit circle with four convergent hexagons at any point, has radius $r = r_{6.4} \equiv \overline{OP_1} \cong 0.51$

Using the new tool, it is easy to construct a regular hexagon centered in O: its vertices are on a circle of radius r. The other hexagons, in the regular (6,4)tessellation of Figure 5, are obtained from the first one by reflection along its sides: it can be proven that it is again a circular inversion with respect to the sides. The tessellation is indeed regular: all the hexagons are equilateral and equiangular. Different size and shape of the hexagons are related to the metric properties of this geometry, where distances tend to infinity when points come closer and closer to the boundary.

Another peculiar property of polygons, and thus of the hexagon centered in O, is that its inner angles change continuously with r: in fact, the (6,4) tessellation (a single regular tile with 6 sides and 4 convergent tiles at any vertex) of this example is possible because the interior angle in P_1 is a right angle precisley when the radius $r = r_{6.4} \equiv \overline{OP_1} \cong 0.51$.

A useful exercise, on the GeoGebra file used to draw Figure 5, could be to let r vary and see how the constructed hexagons could overlap: their interior angles would in fact change, ranging from 120°, in the limit of r → 0 as in the Euclidean case, and 0° when r = 1.



[Figure 6] The relation between the angles $\gamma = \angle P_2P_1O$ and $\beta = \frac{\angle P_2OP_1}{2}$ is $(1+r^2)\tan\gamma$ = $(1-r^2)\cot\beta$. Given β, the value of γ depends on the distance $r = \overline{OP_1}$.

This observation implies that there are exact values of $r = r_{n,k}$ at which a regular tessellation of n-sided polygons is possible for k convergent tiles at any vertex. All the properties discussed here can be proven with elementary geometry and some trigonometry: ask again the students to find an optimal visualization of the problem relating the angles $\gamma = \angle P_2P_1O$ and $\beta = \frac{\angle P_2OP_1}{2}$ as $(1+r^2)\tan\gamma$ = $(1-r^2)\cot\beta$ (see Figure 6).

Using the relation between β and γ, it is possible to calculate the exact value of $r_{6.4}$ = $\sqrt{1-2\sqrt{3}}$ and the condition in this geometry for the existence of a (n, k) tessellation: $\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$.

6. CONIC SECTIONS

Which family of right circular cones share the same xy-plane section $y = x^2$?

What is the locus of all possible apex positions?

Imagine a common exercise with the usual graph of the parabola $y = x^2$ on a sheet of paper over a table: since it is a conic section, have you ever thought at the position of the cone that has this parabola, or another conic curve, as a plane section with the table? The use of GeoGebra allows to connect a 2D Graphics View, usually the xy plane, with a 3D Graphics View in a simple and intuitive way. But the connection is with Geometry and Algebra as well, in an inverse problem.

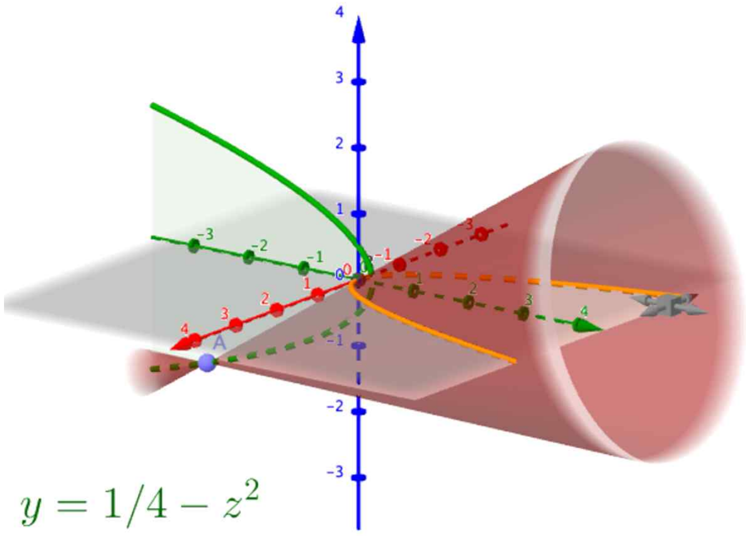
Let the students practice with the problem using GeoGebra tools: plot the parabola and define a changing cone which could depend on some points, lines or other geometrical objects; then look at his section with the xy plane and control the superposition with the given parabola.

Some intuition on 3D geometry and the role of symmetry in this context are of great help. Try to fix the inclination of a generatrix of the cone and find the apex positions: if the aperture is fixed there are two (symmetric) such cones. Are there other cones and how are they related to their aperture and apex position?

The apex must be in the yz plane, but to answer the second question we want to write the equation of the locus of all apex positions. The solution is shown in Figure 7 and, strangely enough, it is itself a parabola: the equation is $y = \frac{1}{4} - z^2$. It can of course be checked with a computer animation, seeing how the aperture of the cone changes with the apex position: GeoGebra will do it for free, if you define the relation among points and curves in the proper way.

The complete solution relates the canonical right circular cone equation, with the axis of symmetry on the y axis, aperture 2θ and apex in the origin $x^2 + z^2 = y^2 \tan^2\theta$ with a rotation, a translation and again some trigonometry. The answer to the first question, parametrized by the semi-aperture, is finally:

$$x^2 + (1 - tan^2\theta)\, z^2 - 2\sin\theta\, y\, z = 0$$



[Figure 7] An example of right circular cone with the given $y = x^2$ plane section. The apex of such family of cones lies on the yz-plane curve $y = \frac{1}{4} - z^2$

To prove the second answer, we need the relation between the apex $A = (0, b, c)$ and the aperture: $b = c\tan 2\theta$. Now we substitute and find the equation in terms of the apex coordinates

$$x^2 + \frac{2}{b - \sqrt{b^2 + c^2}}(z - c)(bz - cy) = 0.$$

Inserting $z = 0$ gives the equation of the section:

$$x^2 + \frac{2c^2}{b - \sqrt{b^2 + c^2}}\, y = 0$$

which is the parabola $y = x^2$ if $\frac{2c^2}{b - \sqrt{b^2 + c^2}} = -1$.

This condition is satisfied precisely when $b = \frac{1}{4} - c^2$.

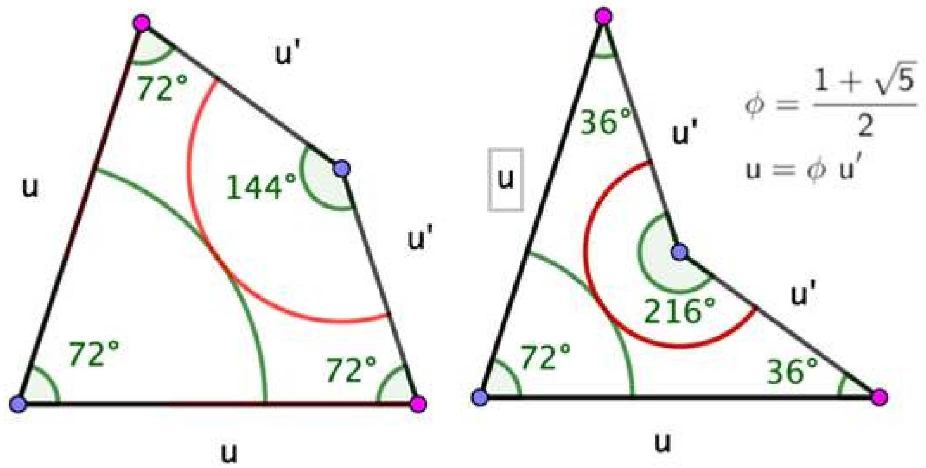
If we substitute the last condition in the equation of the family of cones, which share the same parabola section, we get a one parameter family parametrized by $c \neq 0$:

$$x^2 + (z - c)(z + \frac{y}{c} - \frac{z}{4c^2}) = 0.$$

7. APERIODIC TILINGS

How to construct an aperiodic Penrose tiling (using a cutting machine)? Could the aperiodic tessellation be symmetric?

I report here an interdisciplinary high school class laboratory on Penrose tiling(C. Falcolini, 2021). In the occasion of the 2020 Rome Maker Faire, on-line edition, we planned and constructed (with B.L. Mauti) the tiles using a laser cutting machine in our Department of Architecture and wrote a GeoGebra application to simulate the game (downloading it is also possible at http://www.formulas.it/sito/software/). The idea was to let the students in a class plan the construction of the tiles, dialogue with the technical laboratory to learn how to produce many low-cost tiles and play with the game in presence or at a distance, with the help of a computer.



[Figure 8] The two Penrose tiles: the dart and the arrow. Their geometric construction is related to a regular pentagon and the golden ratio. The colored arcs indicate the admissible junction of two adjacent tiles.

We started with the geometric construction of the two Penrose tiles. The use of GeoGebra has helped in the understanding of the process. In particular the drawing of the arcs had to be very precise, since the colored arcs provide the contact rule: join two tiles matching arcs of the same color and sides of the same length.

Then we learned how to communicate the corresponding instructions to a cutting machine: geometry was the common language for communications.

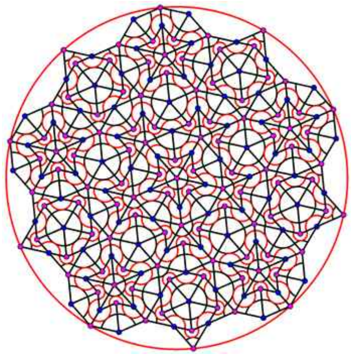
At the end we made several examples of the tessellation using the produced tiles(C. Falcolini, B.L. Mauti, L. Bonaguidi, G. Campagna and S. Fiacco, 2021). Despite the fact that the possible configurations around any vertex of a Penrose tiling are only seven, there are millions of possible combinations, and they never repeat exactly. This tiling, unlike a jigsaw puzzle, has all the pieces which are copies of only two tiles (dart and arrow) and a final picture which depends on multiple choices at any step.

The answer to the second question is positive (see Figure 9 for a small-scale example): there is a striking appearance of local five-fold symmetries at any scale.

We also simulated the tessellation on a computer using GeoGebra and the creation of many new tools: the application is downloadable at the site of the math laboratory www.formulas.it (http://www.formulas.it/sito/software/).

8. CONCLUSIONS

Convergence in high school education, as well as STEAM activities [3], are changing our way of teaching and learning. Here we considered challenging problems, on very different subjects: to give an idea on why they are suitable to be organized in cooperative or distance learning, and how the use of computer programs can suggest approximate, exact or graphical solutions keeping in mind the abstract nature of the problem, generalizing it and posing new questions.



[Figure 9] A portion of an aperiodic Penrose tiling composed by two tiles: the dart and the arrow. This example has a five-fold symmetry.

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