# STAR OPERATIONS RELATED TO POLYNOMIAL CLOSURE

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Abstract. We study polynomial closure of ideals from the point of view of star operations. We show that, if $D$ is an integrally closed domain or a domain of residue characteristic 0, the polynomial closure of an ideal coincides with the divisorial closure. We also analyze what happens in characteristic $p$.

## 1. Introduction

Throughout the paper, $D$ is an integral domain with quotient field $K$.

Let $E$ be a subset of $K$. A polynomial $f(X) \in K[X]$ is *$D$-integer-valued* over $E$ if $f(E) \subseteq D$, and the set $\mathrm{Int}(E, D)$ of such polynomials is a subring of $K[X]$. When $E = D$ the set $\mathrm{Int}(D, D) := \mathrm{Int}(D)$ is called the *ring of integer-valued polynomials* over $D$. The *polynomial closure of $E$ in $D$* is the largest subset $F$ of $K$ such that $\mathrm{Int}(E, D) = \mathrm{Int}(F, D)$ ([2]). In the following, since $D$ is fixed, we will just write polynomial closure.

Such a closure has been studied in several contexts from a topological point of view: for example, if $D$ is a valuation domain, the polynomial closure is a topology if and only if $D$ has dimension 1 (see [3, Theorem 5.3] and [8, Theorem 2.7]); when $D$ is also a rank-one discrete valuation domain (DVR) this topology coincides with the $v$-adic topology (see [2, Proposition 4.5]).

Following [5] and [7], we study the polynomial closure of ideals as a star operation. Star operations are a particular class of closure operations for ideals in commutative rings ([6, §32]). Recall that a *fractional ideal* $I$ of $D$ is a $D$-submodule of $K$ such that $dI \subseteq D$ for some $d \in D$, $d \neq 0$. We denote by $\mathcal{F}(D)$ the set of fractional ideals of $D$. A *star operation* on $D$ is a map $\star : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(D)$, $d \in K$:

- $I \subseteq I^\star$;
- if $I \subseteq J$, then $I^\star \subseteq J^\star$;
- $(I^\star)^\star = I^\star$;

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- $(dI)^\star = d \cdot I^\star$;
- $D^\star = D$.

The set $\mathrm{Star}(D)$ of star operations on $D$ has a natural order given by $\star_1 \leq \star_2$ if $I^{\star_1} \subseteq I^{\star_2}$, for every fractional ideal $I$. Under this order, $\mathrm{Star}(D)$ is a complete lattice whose minimum is the identity and whose maximum is the *$v$-operation* (or *divisorial closure*) $I^v := (D : (D : I))$, where $(J : L) := \{x \in K \mid xL \subseteq J\}$ if $J, L$ are $D$-submodules of $K$. We also use the notation $I^{-1}$ to denote $(D : I)$, thus $I^v := (I^{-1})^{-1}$.

Polynomial closure is a star operation in any domain $D$, as shown in [5, Lemma 1.2]. More precisely, for every fractional ideal $I$ of $D$, we define $I^{\mathrm{pc}}$ to be the polynomial closure of $I$. It has been proven that pc coincides with the divisorial closure for valuation domains [5, Proposition 1.8] and, more generally, for all essential domains [7, Proposition 2.4] (we recall that a domain $D$ is essential if it is intersection of valuation overrings that are localizations of $D$ itself at some prime ideal). This last result is obtained by expressing the polynomial closure of $I$ through the divisorial closure of a special type of powers of $I$ (see the beginning of Section 2).

In this paper we show that the polynomial closure and the $v$-operation coincide for two very wide classes of domains, namely the integrally closed domains (Theorem 3.3) and the domains of residue characteristic 0 (Theorem 3.8); this is obtained by strengthening the method used in [7], considering together with the polynomial closure a whole chain of star operations constructed through powers of $I$.

Next, we analyze what happens in characteristic $p$: we show that, if $D$ contains an infinite field, then it is enough to consider $p$-powers of the ideals $I$ (Theorem 4.7).

## 2. An infinite chain of star operations

Let $I$ be a fractional ideal of $D$ and $n \in \mathbb{N}$. We define $I(n)$ to be the $D$-module generated by the $n$-th powers of the elements of $I$; i.e., $I(n) = (u^n \mid u \in I)$. We also define

$$D[X/I] := \bigcap_{u \in I \setminus \{0\}} D\left[\frac{X}{u}\right] = \bigoplus_{n \in \mathbb{N}} \left( \bigcap_{u \in I \setminus \{0\}} \frac{1}{u^n} D \right) X^n;$$

note that $\displaystyle\bigcap_{u \in I \setminus \{0\}} \frac{1}{u^n} D = (D : I(n)) (= I(n)^{-1})$. If $\mathrm{Int}(D) = D[X]$ (this happens, for example, if the residue fields of $D$ are infinite) then $\mathrm{Int}(I, D) = D[X/I]$ [4, Lemma 4.5], and

$$I^{\mathrm{pc}} = \{x \in K \mid x^n \in I(n)^v \text{ for all } n \in \mathbb{N}\}$$

for all fractional ideals $I$ [7, Proposition 1.4].

To refine the study of polynomial closure, we define

$$\begin{aligned}\star_n\colon \mathcal{F}(D) &\longrightarrow \mathcal{F}(D),\\ I &\longmapsto \{x\in K \mid x^t\in I(t)^v \text{ for all } t\leq n\}\end{aligned}$$

and

$$\begin{aligned}\star_\infty\colon \mathcal{F}(D) &\longrightarrow \mathcal{F}(D),\\ I &\longmapsto \bigcap\{I^{\star_n}\mid n\in\mathbb{N}\}.\end{aligned}$$

Note that, when $\mathrm{Int}(D)=D[X]$, then $\star_\infty=\mathrm{pc}$ [7, Proposition 1.4].

**Proposition 2.1.** *Let $n\geq 1$ be a fixed natural number. Then, $x\in I^{\star_n}$ if and only if $f(x)\in D$ for every $f(X)\in D[X/I]$ of degree at most $n$.*

*Proof.* Suppose $x\in I^{\star_n}$, and let $f(X)\in D[X/I]$ of degree $m\leq n$. Then, $f(X)=\sum_i a_iX^i$ with $a_i\in I(i)^{-1}$; hence, $a_ix^i\in I(i)^vI(i)^{-1}\subseteq D$ and thus $f(x)=\sum_i a_ix^i\in D$.

Conversely, suppose $x\notin I^{\star_n}$. Then, there is a $t\leq n$ such that $x^t\notin I(t)^v$, i.e., $x^tI(t)^{-1}\nsubseteq D$. Take $s\in I(t)^{-1}$ such that $sx^t\notin D$: then, $f(X):=sX^t$ is a polynomial in $D[X/I]$ of degree $t\leq n$ such that $f(x)=sx^t\notin D$. The claim is proved. □

**Proposition 2.2.** *The $\star_n$ are star operations on $D$, and $v=\star_1\geq\star_2\geq\star_3\geq\cdots\geq\star_\infty$.*

*Proof.* Clearly, $I^{\star_n}\subseteq I^{\star_m}$ if $n\geq m$; moreover, it follows easily from the definition that $v=\star_1$. Therefore, we only need to prove the each $\star_n$ is a star operation.

We proceed by induction on $n$: if $n=1$ then $\star_1=v$ is a star operation. Suppose the claim holds up to $n-1$.

We first show that $I^{\star_n}$ is always a fractional ideal. Suppose $x,y\in I^{\star_n}$ and $c\in D$: we need to show that $cx,x+y\in I^{\star_n}$. For all $t\leq n$, we have $(cx)^t=c^tx^t\in c^tI(t)^v\subseteq I(t)^v$, and thus $cx\in I^{\star_n}$. Let now $x,y\in I^{\star_n}$: by induction, $x+y\in I^{\star_{n-1}}$ and thus $(x+y)^t\in I(t)^v$ for all $t<n$. To show that $(x+y)^n\in I(n)^v$, it is enough to prove that $x^iy^j\in I(n)^v$ for all $i+j=n$; if $i=0$ or $j=0$ this follows directly from $x\in I^{\star_n}$ or $y\in I^{\star_n}$.

Suppose $i,j\neq 0$. We have:

$$(D:I(n))=\left(D:\sum_{u\in I}u^nD\right)=\bigcap_{u\in I}(D:u^n)=\bigcap_{u\in I\setminus\{0\}}u^{-n}D.$$

Therefore,

$$x^iy^j(D:I(n))=x^iy^j\bigcap_{u\in I\setminus\{0\}}u^{-n}D=\bigcap_{u\in I\setminus\{0\}}\frac{x^iy^j}{u^n}D=\bigcap_{u\in I\setminus\{0\}}\frac{x^i}{u^i}\frac{y^j}{u^j}D.$$

Since $x^i\in I(i)^v$, we have $x^i(D:I(i))\subseteq D$, and thus $x^i/u^i\in D$ for every $u\in I$; similarly, $y^j/u^j\in D$. Hence, $x^iy^j(D:I(n))\subseteq D$, that is,

$x^iy^j \in (D:(D:I(n))) = I(n)^v$. Thus $(x+y)^n \in I(n)^v$ and $x+y \in I^{\star_n}$, so that $I^{\star_n}$ is an ideal.

We now prove that $\star_n$ is a star operation. Clearly, if $x \in I$ then $x^t \in I(t) \subseteq I(t)^v$ and thus $x \in I^{\star_n}$, i.e. $I \subseteq I^{\star_n}$: likewise, if $I \subseteq J$, then $I(t) \subseteq J(t)$ and thus $I(t)^v \subseteq J(t)^v$; hence also $I^{\star_n} \subseteq J^{\star_n}$. To prove idempotence, we first claim that $(D:I(t)) = (D:I^{\star_n}(t))$ for all $t \leq n$. Indeed, the $(\supseteq)$ containment follows from $I \subseteq I^{\star_n}$. On the other hand, if $s \in (D:I(t))$ and $u \in I^{\star_n}$, then $u^t \in (D:(D:I(t)))$, and thus

$$su^t \in s(D:(D:I(t))) = (D:(D:sI(t))) = (sI(t))^v \subseteq D$$

as $sI(t) \subseteq D$. Hence $s \in (D:I^{\star_n})$ and $(D:I(t)) = (D:I^{\star_n}(t))$. In particular, if $x \in (I^{\star_n})^{\star_n}$ and $t \leq n$ then $x^t \in I^{\star_n}(t)^v$; however,

$$x^t(D:I^{\star_n}(t)) = x^t(D:I(t))$$

and so $x^t \in I(t)^v$. Since this holds for all $t \leq n$, we have $x \in I^{\star_n}$. Therefore $\star_n$ is idempotent and thus a closure operation.

Finally, $D^{\star_n} \subseteq D^v = D$ and thus $D^{\star_n} = D$; on the other hand, the equality $cI^{\star_n} = (cI)^{\star_n}$ follows from the fact that $(cI)(t) = c^tI(t)$ and that the $v$-operation is a star operation.

By definition, $\star_\infty$ is the infimum of the $\star_n$ in the lattice of star operations [REF], and thus it is itself a star operation. □

**Remark 2.3.** Given an ideal $I$ of $D$, the set of polynomials of $D[X/I]$ of degree at most $n$ is exactly $\bigoplus_{i=0}^n I(i)^{-1}X^i =: D[X/I]_n$.

By the proof of Proposition 2.2, we have that $(D:I(i)) = (D:I^{\star_n}(i))$, for each $i \leq n$; therefore, $D[X/I]_n = D[X/I^{\star_n}]_n$. On the other hand, if $x \notin I^{\star_n}$, then by Proposition 2.1 there is an $f \in D[X/I]_n$ such that $f(x) \notin D$. If $f = \sum_i f_iX^i$, it follows that $f_ix^i \notin D$ for some $i$; setting $J := I \cup \{x\}$, we thus have $f \notin D[X/J]_n$, and so $D[X/J]_n \neq D[X/I]_n$.

It follows that $I^{\star_n}$ can be thought of as the largest set such that $D[X/I]_n = D[X/I^{\star_n}]_n$, just like the polynomial closure of $E$ is the largest set $F$ such that $\mathrm{Int}(E,D) = \mathrm{Int}(F,D)$.

This was already noticed in [7] for the divisorial closure: in fact, $I^v$ is the biggest ideal such that $\mathrm{Hom}_D(I,D) = \mathrm{Hom}(I^v,D)$ and it is easy to check that $\mathrm{Hom}_D(I,D) \cong I^{-1}X$.

## 3. Collapsing the chain

Proposition 2.2 shows that the $\star_n$'s form a descending chain of star operations, but it does not give any information about equalities among these operations. If $D$ is an essential domain, by [7, Proposition 2.4] we have that $\star_\infty = \mathrm{pc} = v$ and so all $\star_n$ are actually equal to $v$.

In this section we generalize this result to the cases when $D$ is integrally closed (Theorem 3.3) and when $D$ has residual characteristic zero (Theorem 3.8).

**Lemma 3.1.** *Let $I, J$ be ideals of $D$. If $I^\star = J^\star$ for some star operation $\star$, then $I^v = J^v$.*

*Proof.* Since $I \subseteq I^\star \subseteq I^v$, we have $(I^\star)^v = I^v$. Analogously, $(J^\star)^v = J^v$; since $I^\star = J^\star$, it follows that $I^v = J^v$. □

**Theorem 3.2.** *Let $D$ be an integral domain and suppose that, for every ideal $I$ and every integer $n$, there is a star operation $\star$ such that $I(n)^\star = (I^n)^\star$. Then, $\star_\infty = v$.*

*Proof.* By the previous lemma, the condition $I(n)^\star = (I^n)^\star$ implies that $I(n)^v = (I^n)^v$ for every ideal $I$ and every $n$.

Clearly $I^{\star_\infty} \subseteq I^v$ since $\star_\infty$ is a star operation. Conversely, if $x \in I^v$ then, for all $n$,

$$x^n \in (I^v)^n \subseteq ((I^v)^n)^v = (I^n)^v = (I(n))^v$$

where the equality $((I^v)^n)^v = (I^n)^v$ holds by [6, Proposition 32.2 (c)]. Thus $x \in I^{\star_\infty}$. Hence $I^{\star_\infty} = I^v$, as claimed, and $\star_\infty = v$. □

**Theorem 3.3.** *Let $D$ be an integrally closed domain. Then $\star_\infty = v$.*

*Proof.* Let $b$ be the $b$-operation on $D$, i.e., $I^b = \bigcap\{IV \mid V \in \mathrm{Zar}(D)\}$, where $\mathrm{Zar}(D)$ is the set of valuation overrings of $D$. Since $D$ is integrally closed, $b$ is a star operation on $D$; we claim that $I(n)^b = (I^n)^b$ for every ideal $I$ and integer $n$, and to do so it is enough to show that $I(n)V = I^nV$ for all valuation overrings $V$. Clearly $I(n)V \subseteq I^nV$ since $I(n) \subseteq I^n$. Let $x \in I^nV$: then, $x = i_{1,1} \cdots i_{1,n}v_1 + \cdots + i_{k,1} \cdots i_{k,n}v_k$ for some $i_{j,l} \in I$, $v_1, \ldots, v_k \in V$. Since $V$ is a valuation domain, there is an $i_{a,b}$ dividing all of the $i_{j,l}$: hence, $x \in i_{a,b}^nV \subseteq I(n)V$. Therefore $I^nV \subseteq I(n)V$, so that $I^nV = I(n)V$. Hence $I(n)^b = (I^n)^b$ and the claim follows from Theorem 3.2. □

**Corollary 3.4.** *Let $D$ be an integrally closed domain such that $\mathrm{Int}(D) = D[X]$. Then $\mathrm{pc} = v$.*

*Proof.* By [7, Proposition 1.4], the hypothesis $\mathrm{Int}(D) = D[X]$ implies that $\star_\infty = \mathrm{pc}$. The claim now follows from Theorem 3.3. □

We now analyze what happens when $D$ contains $\mathbb{Q}$; this condition is equivalent to $D$ having all residue fields of characteristic 0. The main point of the proof is to show that $I(n) = I^n$.

**Lemma 3.5.** *Let $X, Y$ be indeterminates over $\mathbb{Q}$. For every $n$, the sets $\{X^n, (X+1)^n, (X+2)^n, \ldots, (X+n)^n\}$ and $\{X^n, (X+Y)^n, (X+2Y)^n, \ldots, (X+nY)^n\}$ are linearly independent over $\mathbb{Q}$.*

*Proof.* Let $f_i(X) := (X+i)^n$. Then, the Wronskian of $f_0,\dots,f_n$ is the determinant of

$$\begin{pmatrix} X^n & (X+1)^n & \cdots & (X+n)^n \\ nX^{n-1} & n(X+1)^{n-1} & \cdots & n(X+n)^{n-1} \\ \vdots & & & \\ n! & n! & \cdots & n! \end{pmatrix}.$$

Calculated for $X=1$, the Wronskian is equal to a $\frac{n!}{(n-1)!}\cdot\frac{n!}{(n-2)!}\cdots\frac{n!}{1}$ (which is nonzero since $\mathbb{Q}$ has characteristic 0) multiplied by the Vandermonde determinant relative to $1,2,\dots,n+1$, which again is nonzero. Hence the Wronskian is nonzero and thus $f_0,\dots,f_n$ are linearly independent [REF].

For $\{X^n,(X+Y)^n,(X+2Y)^n,\dots,(X+nY)^n\}$, suppose that $\lambda_0X^n+\lambda_1(X+Y)^n+\cdots+\lambda_n(X+nY)^n=0$. Then, by dividing by $Y^n$, and setting $T:=X/Y$, we have $\lambda_0T^n+\lambda_1(T+1)^n+\cdots+\lambda_n(T+n)^n=0$; by the previous part of the proof, $T^n,(T+1)^n,\dots,(T+n)^n$ are linearly independent, and thus so are $X^n$, $(X+Y)^n$, $(X+2Y)^n,\dots,(X+nY)^n$. □

**Proposition 3.6.** *Let $D$ be an integral domain with $\mathbb{Q}\subseteq D$. Let $I\in\mathcal{F}(D)$. Then, $I(n)=I^n$ for all $n\geq 0$.*

*Proof.* It is enough to show that $I\cdot I(n-1)=I(n)$ for all $n\geq 1$: indeed, if this equality is true, then $I(n)=I\cdot I(n-1)=I^2\cdot I(n-2)=\cdots=I^{n-1}\cdot I(1)=I^{n-1}\cdot I=I^n$.

The containment $I(n)\subseteq I\cdot I(n-1)$ is obvious. On the other hand, the ideal $I\cdot I(n-1)$ is generated by the elements of the form $xy^{n-1}$, for $x,y\in I$, and thus it is enough to show that these are in $I(n)$. If $x=y$ this is obvious.

Suppose $x\neq y$. By Lemma 3.5, the polynomials $X^n,(X+Y)^n,(X+2Y)^n,\dots,(X+nY)^n$ are linearly independent over $\mathbb{Q}$, and thus they span a $\mathbb{Q}$-linear subspace of $\mathbb{Q}[X]$ of dimension $n+1$; since they belong to the subspace generated by the monomials of degree $n$, which has dimension $n+1$, they must generate exactly the latter subspace. Therefore, for every $a,b$ there are $\lambda_0,\dots,\lambda_n\in\mathbb{Q}$ such that $X^aY^b=\lambda_0X^n+\lambda_1(X+Y)^n+\cdots+\lambda_n(X+nY)^n$. In particular, since $\mathbb{Q}\subseteq D$, we have $x^ay^b=\lambda_0x^n+\lambda_1(x+y)^n+\cdots+\lambda_n(x+ny)^n\in I(n)$; hence, in particular, $xy^{n-1}\in I(n)$. The claim is proved. □

**Remark 3.7.** Proposition 3.6 does not work in general for rings of characteristic 0 not containing $\mathbb{Q}$. For example, if $=\mathbb{Z}[x,y]$ and $I=(x,y)$, then $xy\in I^2$ but $xy\notin I(2)$.

**Theorem 3.8.** *Let $D$ be a domain such that $D/\mathfrak{m}$ has characteristic 0, for each maximal ideal $\mathfrak{m}$. Then $\star_\infty=\mathrm{pc}=v$.*

*Proof.* Since for each maximal ideal $\mathfrak{m}$, $(D_\mathfrak{m},\mathfrak{m}D_\mathfrak{m})$ is a local domain and $D_\mathfrak{m}/\mathfrak{m}D_\mathfrak{m}$ has characteristic 0, the ring $D_\mathfrak{m}$ contains $\mathbb{Q}$ and so $D$

contains $\mathbb{Q}$. By Proposition 3.6 $I(n) = I^n$ for every ideal $I$ and, by Theorem 3.2, $\star_\infty = v$. By [1, Corollary I.3.7] $\mathrm{Int}(D) = D[X]$, and by [7, Proposition 1.4] $\star_\infty = \mathrm{pc}$. The claim follows. $\square$

**Corollary 3.9.** *Let $(D, \mathfrak{m})$ be a local domain such that $D/\mathfrak{m}$ has characteristic* 0. *Then $\star_\infty = \mathrm{pc} = v$.*

For example, the previous results hold for polynomial rings of fields of characteristic 0 like $\mathbb{Q}[X]$ or $\mathbb{R}[X]$.

## 4. Characteristic $p$

In characteristic $p$ it is not always true that all the $\star_n$ are equal.

**Example 4.1.** Let $F \subseteq K \subseteq L$ be a tower of purely inseparable extension of degree $p$, with $L = F(y)$ simple over $F$. Consider the ring $D := F + XL[[X]]$ and its fractional ideal $I := K + XL[[X]]$: then, $I^{\star_1} = I^v = L[[X]]$. On the other hand, $I(p) = K(p) + XL[[X]] = K^p + XL[[X]] = D$, and thus $I(p)^v = D$; therefore, $y^p \notin I(p)$ since $y^p \notin F$. It follows that $I^{\star_p} \neq L[[X]]$ and thus $\star_p \neq \star_1$.

The main difference from the previous case is that Lemma 3.5 does not hold, as the matrix in the proof may have determinant equal to a multiple of $p$ and thus equal to zero in the ring $D$. However, under the assumption that the ring contains an infinite field, we can show that only the powers of $p$ matter.

**Lemma 4.2.** *Let $F$ be an infinite field, and let $e_1, \ldots, e_t \in \mathbb{N}$ be distinct. There are $a_1, \ldots, a_t \in F$ such that the determinant of $(a_i^{e_j})_{i,j}$ is nonzero.*

*Proof.* By induction on $t$. If $t = 1$ it is enough to take $a_1 \neq 0$. Suppose that the claim holds up to $t-1$, and choose $a_1, \ldots, a_{t-1}$ such that the claim holds for the exponents $e_1, \ldots, e_{t-1}$. Consider the matrix $M$ obtained from $(a_i^{e_j})_{i,j}$ by substituting $a_t$ with an indeterminate $X$; then, the determinant of $M$ is a nonzero polynomial (since the coefficient of $X^{e_t}$ is the determinant of $(a_i^{e_j})_{i,j \leq t-1}$, which is nonzero by inductive hypothesis). Hence, there are only finitely many elements of $F$ such that the determinant of $(a_i^{e_j})_{i,j}$ is zero; since $F$ is infinite, we can choose an $a_t$ that makes the determinant nonzero. The claim is proved. $\square$

**Lemma 4.3.** *Let $F$ be an infinite field of characteristic $p$, and let $n \in \mathbb{N}$ be positive. Let $e_1, \ldots, e_t$ be the natural numbers $k$ such that $p$ does not divide $\binom{n}{k}$. Then, there are $a_1, \ldots, a_t \in F$ such that the $F$-linear space generated by $(X + a_1)^n, \ldots, (X + a_t)^n$ is equal to the space generated by $X^{e_1}, \ldots, X^{e_t}$.*

*Proof.* By construction, each $(X + a_i)^n$ is contained in the space generated by $X^{e_1}, \ldots, X^{e_t}$ for every choice of $a_1, \ldots, a_t$; hence, we only need to find $a_1, \dots, a_t$ such that $(X+a_1)^n, \dots, (X+a_t)^n$ are linearly independent.

Since $\binom{n}{e_j} = \binom{n}{n-e_j}$ for all $j$, we can consider instead the basis $X^{n-e_1}, \dots, X^{n-e_t}$. Choose $a_1, \dots, a_t$ satisfying the property of Lemma 4.2: then, the coefficients of $(X+a_i)^n$ with respect to $X^{n-e_j}$ is $\binom{n}{e_j} a_i^{e_j}$, and thus the transition matrix of $(X+a_1)^n, \dots, (X+a_t)^n$ with respect to $X^{e_1}, \dots, X^{e_t}$ is $M := \left(\binom{n}{e_j} a_i^{e_j}\right)_{i,j}$, whose determinant is the product of all the binomial coefficients $\binom{n}{e_j}$ (which is nonzero in $F$ by definition of the $e_j$) and the determinant of $(a_i^{e_j})_{i,j}$, which is nonzero by definition of the $a_i$. Hence the determinant of $M$ is nonzero, and so $(X+a_1)^n, \dots, (X+a_t)^n$ are linearly independent, as claimed. □

**Lemma 4.4.** *Let $D$ be a ring of characteristic $p$ containing an infinite field, and let $n, m \in \mathbb{N}$. If $p$ does not divide $\binom{n+m}{n}$, then $I(n+m) = I(n)I(m)$.*

*Proof.* We always have that $I(n+m) \subseteq I(n)I(m)$. Let now $x, y \in I$: we need to show that $x^n y^m \in I(n+m)$. Find $a_1, \dots, a_t \in F$ as in Lemma 4.3: then, all of $(x + a_i y)^{n+m}$ are in $I(n+m)$. Since the space generated by $(X+a_1)^{n+m}, \dots, (X+a_t)^{n+m}$ is equal to the space generated by $X^{e_1}, \dots, X^{e_t}$, and since $n = e_i$ for some $i$ (by definition), we have $X^n = \sum_i \lambda_i (X+a_i)^{n+m}$ for some $\lambda_i \in F \subseteq D$; therefore, $x^n y^m = \sum_i \lambda_i (x + a_i y)^{n+m} \in I(n+m)$, as claimed. □

When the ring has characteristic $p$, the ideal $I(p)$ is called the *Frobenius power* of $I$ and is usually denoted by $I^{[p]}$. We keep the notation $I(p)$ to be consistent with the rest of the paper.

**Lemma 4.5.** *Let $D$ be a ring of characteristic $p$, and let $n \in \mathbb{N}$ be positive. Then $I(pn) = I(n)(p)$.*

*Proof.* If $x \in I$, then $x^{pn} = (x^n)^p \in I(n)(p)$, and so $I(pn) \subseteq I(n)(p)$. Conversely, if $x \in I(n)(p)$ then $x = \sum_i \lambda_i y_i^p$ for some $y_i \in I(n)$; thus we can write $y_i = \sum_j \mu_{ij} z_{ij}^n$ for some $z_{ij} \in I$. Hence,

$$x = \sum_i \lambda_i \left( \sum_j \mu_{ij} z_{ij}^n \right)^p = \sum_{i,j} \lambda_i \mu_{ij}^p z_{ij}^{pn}$$

using the fact that the ring has characteristic $p$. Since $z_{ij}^{pn} \in I(pn)$ by hypothesis, we have $x \in I(pn)$, and so $I(n)(p) \subseteq I(np)$. Thus the two ideals are equal. □

**Theorem 4.6.** *Let $D$ be a ring of characteristic $p$ containing an infinite field. Let $n = t_0 + t_1 p + \cdots + t_k p^k$, with $0 \leq t_i < p$ for every $i$. Then,*

$$I(n) = I^{t_0} \cdot I(p)^{t_1} \cdot I(p^2)^{t_2} \cdots I(p^k)^{t_k}.$$

*Proof.* By induction on $n$. If $n = 1$ then $I(1) = I$ and the claim is proved. Suppose that the claim holds up to $n-1$. If $p$ does not divide $n$, then it doesn't divide $\binom{n}{1}$ either, and thus by Lemma 4.4 $I(n) = I(1)I(n-1) = I \cdot I(n-1)$ and the claim follows by induction (since the $p$-expansion of $n-1$ is $(t_0-1)+t_1p+\cdots+t_kp^k$).

Suppose $n = pn'$: then $I(n) = I(pn') = I(n')(p)$. Applying the inductive hypothesis on $n'$, we have $I(n') = I^{t_1} \cdot I(p)^{t_2} \cdots I(p^{k-1})^{t_k}$, and thus $I(n) = (I^{t_1} \cdot I(p)^{t_2} \cdots I(p)^{t_k})(p)$. The claim now follows since $(AB)(p) = A(p)B(p)$ for all ideals $A, B$. □

**Theorem 4.7.** *Let $D$ be a ring of characteristic $p$ containing an infinite field. Then $I^{\mathrm{pc}} = \{x \in K \mid x^{p^e} \in I(p^e)^v$ for every $e \geq 0\}$.*

*Proof.* Since $D$ contains an infinite field, $\mathrm{Int}(D) = D[X]$ and thus $\star_\infty = \mathrm{pc}$. For every $x \in I^{\star_\infty}$, we have $x^{p^e} \in I(p^e)^v$ and so $I^{\star_\infty} \subseteq \{x \in K \mid x^{p^e} \in I(p^e)^v$ for every $e \geq 0\}$.

Conversely, suppose that $x^{p^e} \in I(p^e)^v$, and let $n = t_0+t_1p+\cdots+t_kp^k$ be its expansion in base $p$. Then, for every $r$,

$$(x^{p^r})^{t_r} \in (I(p^e)^v)^{t_r} \subseteq ((I(p^e)^v)^{t_r})^v = (I(p^e)^{t_r})^v$$

and thus

$$\begin{aligned} x^n = x^{t_0+t_1p+\cdots+t_kp^k} &\in (I^{t_r})^v \cdot (I(p)^{t_1})^v \cdots (I(p^k)^{t_k})^v \subseteq \\ &\subseteq ((I^{t_r})^v (I(p)^{t_1})^v \cdots (I(p^k)^{t_k})^v)^v = \\ &= (I^{t_0} I(p)^{t_1} \cdots I(p^k)^{t_k})^v = \\ &= I(t_0 + pt_1 + \cdots + p^k t_k)^v = I(n)^v \end{aligned}$$

using Theorem 4.6. Hence $x \in I^{\star_\infty}$, and thus $I^{\star_\infty} = \{x \in K \mid x^{p^e} \in I(p^e)^v\}$. □

To conclude this section, we show what happens when the domain $D$ satisfies a strong form of root closedness.

**Proposition 4.8.** *Let $D$ be an integral domain and $n$ a positive integer. Suppose that every elements of $D$ has an $n$-th root in $D$. Then:*

*(a) $(D : I)(n) = (D : I(n))$ for every ideal $I$;*
*(b) if $x \in I^v$, then $x^n \in I(n)^v$;*
*(c) $\star_{n-1} = \star_n$.*

*Proof.* (a) To prove that $(D : I)(n) \subseteq (D : I(n))$, it is enough to show that $x^n \in (D : I(n))$ for every $x \in (D : I)$: this is equivalent to $x^n I(n) \subseteq D$, or $x^n y^n \in D$ for every $y \in I$. However, $x^n y^n = (xy)^n \in D$ since $xy \in xI \subseteq D$, and thus $x^n \in (D : I(n))$.

Conversely, suppose that $x \in (D : I(n))$, that is, $xI(n) \subseteq D$. Then, $x = z^n$ for some $y \in K$, and thus $z^n y^n \in D$ for every $y \in I$. However, $z^n y^n = (zy)^n$, which belongs to $D$ by the hypothesis, and thus $zI \subseteq D$, i.e., $z \in (D : I)$. Therefore, $x \in (D : I(n))$.

(b) We have

$$I(n)^v = (D : (D : I(n))) = (D : (D : I)(n)) = (D : (D : I))(n) = I^v(n)$$

applying twice the previous point. Thus, if $x \in I^v$ then $x^n \in I^v(n) = I(n)^v$, as claimed.

(c) By definition,

$$I^{\star_n} = I^{\star_{n-1}} \cap \{x \in K \mid x^n \in I(n)^v\}.$$

However, as $I^{\star_n} \subseteq I^v$, the last set of the previous equation is actually equal to $\{x \in I^v \mid x^n \in I(n)^v\}$, which contains $I^v$ (and thus $I^{\star_{n-1}}$) by the previous point. Hence $I^{\star_n} = I^{\star_{n-1}}$, as claimed. □

In characteristic $p$, we have the following.

**Corollary 4.9.** *Let $D$ be an integral domain of characteristic $p$ that contains an infinite field and such that every element of $D$ has a $p$-th root in $D$. Then,* $\mathrm{pc} = v$.

*Proof.* We always have $I^{\mathrm{pc}} \subseteq I^v$. If $x \in I^v$, then $x^{p^e} \in I(p^e)^v$, by Proposition 4.8(b) (note that, if every element has a $p$-th root, then every element also has a $p^e$-th root). By Theorem 4.7, this is enough to show that $x \in I^{\mathrm{pc}}$; thus $\mathrm{pc} = v$. □

**Example 4.10.** Let $F$ be a perfect infinite field, and let $L$ be an algebraic extension of $F$. Consider the ring

$$D := \bigcup_{n \geq 1} F + X^{1/p^n} L[[X^{1/p^n}]]$$

Then, $D$ contains an infinite field (namely, $F$) and every element has a $p$-th root: indeed, if $x \in D$ then $x \in F + X^{1/p^n} L[[X^{1/p^n}]]$ for some $n$, that is, we can write

$$x = \sum_{i \geq 0} a_i X^{i/p^n}$$

with $a_0 \in F$ and $a_i \in L$ for all $i > 0$. Since both $F$ and $L$ are perfect, there are $b_0 \in F$ and $b_i \in L$ (for $i > 0$) such that $b_j^p = a_j$ for all $j$. Setting $y := \sum b_i X^{i/p^{n+1}}$, then $y \in D$ and $y^p = x$; therefore, $D$ satisfies the hypothesis of Corollary 4.9 and thus $\mathrm{pc} = v$.

However, if $F \neq L$, then $D$ is not integrally closed, since every $t \in L \setminus F$ is integral over $D$ and belongs to its quotient field, but it is not in $D$.

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